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Bifurcations in Hamiltonian systems around symmetric resonances

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Introduction and overview

Background

It is generally accepted that, with few exceptions, solution curves of non-linear dynamical systems cannot be found explicitly. However, one could try to extract qualitative information from the system, such as a description of the equilibrium configurations, stability of these configurations, existence of periodic, quasi-periodic or chaotic motion, boundness of solutions, and so on. However, even this far more restricted program turned out to be highly complex to be carried out. In order to endeavor a feasible analysis and to prove rigorous results we have to make additional restrictions on the objectives we aim at.

The first method of simplifying the analysis is to *localize* it: instead of considering the whole of phase space, one considers a small neighborhood of some special point or trajectory. Obvious candidates are equilibrium points and periodic trajectories, which are for instance investigated for this local analysis concerning their stability or, when parameters are present, the occurrence of bifurcations.

Indeed, a dynamical system depends, as a rule, on parameters. Therefore, it is important to know how the qualitative properties of the model vary as the parameters change slightly. Ideally, one would like to have a description of the dynamics for all possible parameter values, but this is sometimes too difficult. Localizing the analysis around a special point in parameter space is a widely used method for obtaining restricted but meaningful results.

Another way of making the system more "manageable" is to simplify it, in such a way that conclusions about the simplified system are relevant for the original system. Several means for simplifying systems have been developed. To start, one can try to choose new coordinates with respect to which the system assumes a "simpler" form. An example of this approach is to transform the vector field of the dynamical system ia a suitable "normal form" [28, 58]. In the case of Hamiltonian vector fields this approach can be realized by means of the *Birkhoff-Gustavson normalization* procedure [61]. It is a stepwise procedure that normalizes a given system degree-by-degree so that it has extra conserved quantities or integrals, and associated

symmetries. This symmetry can often be used to reduce the dimension of the phase space.

Since the simplified and original system are connected (or conjugated) by a smooth coordinate transformation, the conclusions for the simplified (or normalized) system are valid for the original system too. However, the Birkhoff normalization procedure generically does not converge to a smooth coordinate transformation, not because of technical difficulties, but for intrinsic reasons: the original system is generically not integrable. In order to obtain a smooth transformation, it is necessary to stop normalizing at a certain degree, resulting in a normalized system with a near-symmetry, i.e., one that is symmetric up to some degree.

This is where *perturbation theory* comes in, dealing with systems that differ from integrable ones by small perturbations [5, 16, 92]. A positive result in this area is KAM theory, which states that perturbed integrable Hamiltonian systems retain qualitatively the same dynamics as the unperturbed systems, namely quasi-periodic motion, on tori filling a fat Cantor set in phase space [4, 85, 96].

After the normalization, several methods can be used to proceed with a further reduction of the system, exploiting the acquired near-symmetries; see among others the energy-momentum map [45, 35, 80] or the planar reduction [22, 21] methods. In [45] the energy momentum map method is developed for arbitrary vector fields depending on parameters (around an equilibrium point), so for instance the generalization of the Hopf Bifurcation of [13] can be obtained.

The planar reduction method brings a certain class of Hamiltonian dynamical systems into a polynomial model. This allows to classify the dynamics using *singularity theory*, another powerful tool to reduce a system into the "simplest" one, if some suitable conditions are satisfied. A good introduction to singularity theory can be found in [56, 59, 60, 79] (see also [6, 7, 55]). The idea is to view a system as a particular member of a smooth family of systems and to work modulo coordinate transformations called equivalences, that are chosen appropriately for the problem at hand. Membership in the family is automatic if it includes all nearby systems modulo these equivalences. A family with this property is called a versal family. To find such versal families, one chooses the most degenerate parameter setting, and applies the appropriate equivalence to bring this so-called central singularity (also called organizing center) in simple form. Then one adds a (hopefully) finite number of terms to obtain a family of systems. The number of terms required for versality is called the codimension of the central singularity. To consider a generic family instead of a particular one may sound like replacing a small problem with a larger one. However, this approach often simplifies things considerably, drawing away attention from particularities while focusing on generic aspects. Moreover, this approach may actually reduce the number of parameters: although the physical model

may have many parameters, the few parameters of the family may already capture all "essential" information.

Goal and main results of the research

In this work we consider two degrees of freedom Hamiltonian systems around *symmetric* resonances. With this we mean Hamiltonian dynamical systems close to an equilibrium, invariant with respect to reflection symmetries in both configuration variables, in addition to the time reversion symmetry, and with quadratic part with unperturbed frequencies close to a resonant ratio. Our interest in these systems comes from problems of galactic dynamics, in which they appear to describe the orbital structure of elliptical galaxies. However their relevance is clearly not limited to this field, since they appear in several other areas of mathematical physics, engineering, chemistry, atomic and nuclear physics and so on.

After a (truncated) series expansion, such systems can be treated as perturbed non-linear oscillators. Our goal, in addition to a general understanding of the bifurcation sequence, is to provide *quantitative* predictions, in the form of *energy threshold values* which determine the bifurcations of the main periodic orbits.

The change in the stability of a periodic orbit is connected to frequency ratio; commensurability of low orders between frequencies is the source of non-trivial dynamics and the main trigger to stability-instability transition [3, 9, 29, 32, 101], and the interaction of resonances provides chaos. In harmonic oscillators, frequency ratios are fixed. However, the non-linear coupling between the degrees of freedom induced by the perturbation causes the frequency ratio to change. As a consequence, the system passes through resonances of order given by the integer ratios closest to the ratio of the unperturbed frequencies. This in turn is responsible for the birth of new orbit families bifurcating from the normal modes or from lower-order resonances.

Stable periodic orbits are "parents" of invariant tori on which wind a family of quasi-periodic orbits: in this way we see how periodic orbits "organize" the regular part of the phase space and justify our interest in their study. A typical situation is that in which a family of periodic orbits become unstable when a low order resonance occur between its fundamental frequency and that of a normal perturbation. The simplest case is given by an axial orbit that, depending on the specific form of the potential, can be unstable trough bifurcation of *loop* and *inclined orbits* (1 : 1 resonance), *banana* and *antibanana orbits* (1 : 2 resonance), *fish orbits* (2 : 3 resonance), etc [49, 69]. The nicknames allotted to them are due to their geometric properties.

In the following we will focus on bifurcations related with 1 : 1 and 1 : 2 resonances and we will briefly discuss higher order resonances. A

lot of work has been devoted to this study. On the 1 : 1 resonance, we recall the works of Negrini et al. [27], Kummer [72], Deprit and coworkers [40, 41, 83], Cushman and coworkers [38]. The general treatment of the symmetric 1 : 1 resonance seems to have been done by Cotter [33] in his Ph.D. thesis. We extend these works to include the phenomenon of passing through the resonance as a consequence of the nonlinear coupling. Great relevance plays the application of resonance crossing to galactic dynamics [11, 103]; a recent treatment has been given in [76]. There clearly are several other areas of application, see e.g. [1] for the semi-classical approximation of atomic nuclei.

The 1 : 2 resonance also plays a prominent role in nonlinear Hamiltonian dynamics. In galactic dynamics it appears in several fashions: to mention a few, in axisymmetric prolate systems it determines the bifurcation of the inner thin tube orbits [49]; in triaxial systems with ellipsoidal non-singular equipotentials it gives the bifurcation of banana and antibanana orbits in the symmetry planes [9, 84]. Its interest is clearly not limited to this field and its investigation in theoretical and applied nonlinear dynamics has been very active [45, 37]: a prototype system is the so-called spring-pendulum [21] and an application in satellite attitude dynamics is the tethered system [95]; in engineering it appears to exploit the aerodynamical forces generated by the flight of tethered airfoils to produce electric energy [51, 52]; in chemistry it is quite relevant in molecular vibrations [26, 70].

As we clarify later on, the presence of reflection symmetries has the effect to increase the order of the resonance. Since we are interested in systems with double reflection symmetries, we will speak more correctly of 2 : 2 and 2 : 4 resonances. This restriction makes the results of particular relevance for galactic motions [30].

To catch the main features of the orbital structure, we approximate the frequency ratio with a rational number plus a small "detuning" δ:

$$\rho = \frac{m_1}{m_2} + \delta, \qquad m_1, m_2 \in \mathbb{N}.$$

This expedient is legitimate because, as observed above, even if the unperturbed system is non-resonant, the non-linear coupling between the degrees of freedom induced by the perturbation determines a "passage through resonance" with a commensurability ratio, say m_1/m_2 with $m_1, m_2 \in \mathbb{N}$, corresponding to the local ratio of oscillations in the two degrees of freedom. This in turn is responsible for the birth of new orbit families bifurcating from the normal modes or from lower-order resonances.

2:2 Resonance

Our work starts with the study of a general detuned 1 : 1 resonance invariant under $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, that is invariant under reflection symmetries with

respect to both coordinate axes. At first, we proceed in analogy with an ordinary Birkhoff-Gustavson normalization [61], the difference being that the reducing coordinate transformations are performed through Lie transforms [28] and the detuning term is treated as a term of order two and put in the perturbation. This method will be described in detail in chapter 1. The outcome of the normalization is a "simpler" Hamiltonian function, namely a *normal form* for the system, which exhibits a (formal) $\mathbb{S}^1$ symmetry.

Afterwards, we perform a regular reduction [36] dividing out the acquired $\mathbb{S}^1$ symmetry. The reduced system has only one degree of freedom; this allows us to classify the dynamics with singularity theory [23], if some non degeneracy conditions, which depends on the physical coefficients of the system, are satisfied. Actually we have to respect the symmetries and reversibility of the system, thus we are lead in the domain of equivariant singularity theory. See chapter 2 for a short introduction to the theory. For a more general description see e.g. [59, 60, 79]. The momentum corresponding to the $\mathbb{S}^1$ symmetry serves as a parameter, usually called "distinguished" [21].

To put the one degree of freedom system in a form suitable to apply singularity theory, we turn it into a planar system using the so-called planar reduction method [22, 20], which we describe in section 3.2.

The result is a polynomial Hamiltonian model system living on the plane. This system is easy to analyze, yielding a qualitative description of the original system. In particular, bifurcation curves can be found without much effort. What is new in the current approach is that in each step towards the final polynomial model, the simplifying transformations are *computed explicitly*. This allows us to pull back the final bifurcation curves to the original parameter-energy space, so that *quantitative* results for the bifurcations of the original system are obtained. In particular, in section 3.4, we provide *energy threshold values* (depending on the original physical coefficients and on the detuning parameter) which determine the bifurcation of periodic orbits in general position (namely loop and inclined obits) from the normal modes of the original system.

For these calculations we summon the computer's help, using algorithms described and developed in chapters 1, 2 and 3.

2:4 Resonance

Hamiltonian normal forms can be used as well to get information on the bifurcations connected with the 1 : 2 resonance. We limit the analysis to systems with reflection symmetry with respect to both degrees of freedom: in this case we speak more correctly of 2 : 4 resonance. However, its peculiarities are worthy of note by themselves, since the approach followed to study systems with a single symmetry like the spring-pendulum, is not able in the case of double symmetry to unveil the generic behavior of the system

[45]. In particular, we cannot proceed as in the case of the 2 : 2 resonance, since after the planar reduction method we achieve a polynomial normal form with infinite codimension [23].

In this work we go a little bit in this direction of getting a general analysis of this by considering, in chapter 5, a generic perturbation up to the degree necessary to include resonant terms. Even if this is still not enough to deduce the general behavior for arbitrary perturbations, it allows us to gather complete information on the bifurcation structure near the resonance of the truncated system. In fact, we will see that a truncation of the normal form to the first term incorporating the resonance is able to capture the essential features of the bifurcations and allows us to obtain quantitative predictions for the appearance of banana and antibanana orbits. As for the 2 : 2 resonance, these predictions depend on the physical coefficients of the system and are expressed in terms of the energy and detuning parameter [77].

Orbit structure in systems with elliptical equipotential

In galactic dynamics (but not only, see e.g. [1]) the study of potentials with similar concentric ellipsoidal equipotentials is of great relevance. Several numerical and analytical investigations are available [9, 73, 84, 87, 94].

To shed light on the methods and to limit the algebraic complications we limit the treatment to 2 degrees of freedom non-rotating systems. Such potentials present some degeneracies which forbid the appearance of inclined orbits in the case of a 2 : 2 resonance. Instead loop orbits appear upon a critical energy threshold values [78]. See chapter 4 for more details. Explicit formulas for the bifurcation thresholds are computed in terms of the energy for a family of models with two shape parameters. They are in good agreement with numerical results available in the literature [1, 84].

If ellipsoidal symmetry-breaking perturbations are included the degeneracies of the system are removed and inclined orbits bifurcate. However the smaller the deformation, the higher the threshold value [78].

These systems are also considered around a 2 : 4 resonance, providing the critical energy values for the bifurcation of banana and antibanana orbits; and around a 4 : 6 resonance, finding conditions for the appearance of fish orbits.

The plan

The first two chapters introduces the theory which is behind the results we present in the following. In particular, chapter 1 summarizes Hamiltonian mechanics and presents the normalization procedure. Chapter 2 describes singularity theory and develops the main algorithm which allows us to reduce the 2 : 2 detuned resonance in the next chapter.

In chapter 3, exploiting the normalization procedure and singularity theory, a versal deformation of the 2 : 2 resonance is computed. Namely, if some non degeneracy conditions are satisfied, we are able to reduce the Hamiltonian function into a polynomial form. Bifurcation curves are therefore easy to compute and their expression in terms of the energy and detuning parameter is computed.

A generic perturbation of the 2 : 4 resonance is considered in chapter 5. Now, after the normalization, we arrive at a polynomial normal form of infinite codimension. Therefore we cannot classify the dynamics using singularity theory, as we did for the 2 : 2 resonance. However, if we limit ourselves to a generic perturbation up to the degree necessary to include resonant terms in the normal form, we are able to capture the essential features of the bifurcations and to obtain predictions on the appearance of banana and antibanana orbits.

In chapter 4 we analyze systems with elliptical equipotentials in the case of a 2 : 2 and 2 : 4 resonances, which are of great relevance for galactic dynamics. Such potential presents some degeneracy which forbid the appearance of inclined orbits for the 2 : 2 resonance. However if ellipsoidal symmetry-breaking deformations are included such degeneracies are removed. Energy threshold values for the bifurcation of the main periodic orbits are provided.

In the last chapter we briefly deal with higher order resonances. In particular, we consider the symmetric 2 : 3 resonance in systems with elliptic equipotentials.

Finally, we conclude by discussing some aspects of these results and some hints for further studies and possible applications.

Chapter 1

Normalization

The normalization procedure is a widely used tool for approximating a Hamiltonian system with a simpler one. This chapter starts with an introduction to Hamiltonian mechanics, followed by an explanation of the normalization procedure. In particular, we describe the Lie-transform method and show how it can be used to approximate the dynamics of Hamiltonian systems around symmetric detuned resonances.

1.1 Introduction to Hamiltonian mechanics

We give here the basics of Hamiltonian dynamics and refer to the literature for more details. See e.g [3, 4, 5, 16, 82, 92].

Let $R = \mathbb{R}^{2n}$ be an even dimensional space. The time evolution of a dynamical system on the *phase space* R is represented by the functions $(\mathbf{p}, \mathbf{q})$, with $\mathbf{q}(t) = q_1, \ldots, q_n$ and $\mathbf{p}(t) = p_1, \ldots, p_n$, where the time t varies over a real interval (finite or infinite). Heuristically, q_i is a position and p_i is the associated momentum or velocity.

The evolution of the system is determined by a function H on R, called *Hamiltonian*, through the canonical equation or *Hamilton's equations*

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad i, j = 1, \ldots n \tag{1.1}$$

where the dot denotes time derivative. In case H does not depend explicitly on t we speak of an autonomous system. Notice that a non autonomous system can be treated as an autonomous one by a trivial extension of the phase space. More precisely, we can regard the time as a new coordinate q_{n+1} (with conjugate momentum $p_{n+1} = -H$) and consider the system on a $2n + 2$ dimensional phase space. Thus, from now on we will deal with autonomous system.

More in general, one can consider the time evolution of a generic function

A on the phase space, defined by

$$A \to A(t), \quad A(t) := A(\phi_H^t(\mathbf{p}, \mathbf{q})) \tag{1.2}$$

where $\phi_H^t(\mathbf{q}, \mathbf{p})$ is the time-t flow induced by the Hamilton's equations (1.1), $\phi_H^0(\mathbf{p}, \mathbf{q}) = (\mathbf{p}, \mathbf{q})$. The time evolution of A must therefore obey the following equation

$$\dot{A} = \{A, H\} \tag{1.3}$$

where $\{.,.\}$ denotes the Poisson brackets, namely

$$\{A, H\} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial A}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial A}{\partial p_i} \right). \tag{1.4}$$

The righthand side of eq. (1.3) defines a vector field on the space of differentiable functions in the phase space variables. Given a Hamiltonian function H, this vector field is determined by the following linear operator

$$\mathcal{L}_H := \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right). \tag{1.5}$$

The operator $\mathcal{L}_H$ is nothing else than the derivative along the time flow induced by H and is also referred as *Lie derivative.*

A phase space function F is said to be a *constant of motion* or *integral of motion* if it is conserved along the flow induced by the Hamiltonian function, namely if $F(\phi_H^t(\mathbf{p}, \mathbf{q})) = F(\mathbf{p}, \mathbf{q})$ for all t. Therefore every differentiable phase space function which satisfies

$$\mathcal{L}_H F = \{F, H\} = 0 \tag{1.6}$$

is an integral of motion.

1.1.1 Canonical transformations

Sometimes coordinates changes are useful to simplify the equations of motion. However, when operating a coordinates transformation the canonical structure of the system is generally broken, in the sense that the form of Hamilton's equations is not preserved. A coordinate transformation $(\mathbf{p}, \mathbf{q}) \to (\mathbf{P}, \mathbf{Q})$ is said to be *canonical* if the equations of motion in the new variables follow from a new Hamiltonian, namely eq.s (1.1) turn in to

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \quad i, j = 1, \dots n \tag{1.7}$$

where $K(\mathbf{P}, \mathbf{Q}) = H(\mathbf{p}(\mathbf{P}, \mathbf{Q}), \mathbf{q}(\mathbf{P}, \mathbf{Q}))$.

A transformation is canonical iff it preserves the Poisson brackets. Often canonical transformation are constructed using generating functions.

It is easy to see that the flow ϕ_G^t of an arbitrary Hamiltonian function G is a canonical transformation and ϕ_G^{-t} is its inverse. It is therefore naturally to try to simplify a Hamiltonian system with Hamiltonian H by conjugating it with a canonical transformation generated by another Hamiltonian G. The function $H \circ \phi_G^t$ satisfies the differential equation

$$\frac{\partial H \circ \phi_G^t}{\partial t} = \{H \circ \phi_G^t, G\} = \mathcal{L}_G(H \circ \phi_G^t). \tag{1.8}$$

Thus, the Hamiltonian function in the new coordinates will be given by

$$H \circ \phi_G^t = \exp(\mathcal{L}_G)H := H + t\mathcal{L}_G H + \frac{t^2}{2}\mathcal{L}_G(\mathcal{L}_G H) + \dots \tag{1.9}$$

This formula is behind the normalization procedure we are going to describe in the following section.

1.2   The normalization procedure

Hamiltonian systems with more than one degree of freedom are difficult to analyze. The normalization is an iterative procedure for constructing a coordinate transformation which *normalize* the system, so that it has extra conserved quantities or *integrals*, and associated *symmetries*. Using these, the system can be reduced to less degrees of freedom, making the analysis more feasible [4].

Let us consider a N degrees of freedom dynamical system whose Hamiltonian $\mathcal{H}$ is supposed to be analytic in a neighborhood of an equilibrium point, which without loss of generality we can assume to be in the origin. Therefore, for $|\mathbf{p}|, |\mathbf{q}| \approx \varepsilon$, $\varepsilon > 0$ and sufficiently small, $\mathcal{H}$ can be expanded as a power series

$$\mathcal{H}(\mathbf{p}, \mathbf{q}) = \sum_{k=0}^{+\infty} \mathcal{H}_k(\mathbf{p}, \mathbf{q}) \tag{1.10}$$

where the terms $\mathcal{H}_k$ are homogenous polynomials of degree $k+2$ in the phase space variables. Moreover we assume H_0 to be the Hamiltonian function of an integrable system. We can perform a "blowing up" of the phase space by means of the transformation [45]

$$(\mathbf{p}, \mathbf{q}) \rightarrow \varepsilon^{-1}(\mathbf{p}, \mathbf{q}) \tag{1.11}$$

and scale the Hamiltonian (1.10) according to

$$\mathcal{H} = \varepsilon^2 H, \tag{1.12}$$

so to obtain

$$H(\mathbf{p}, \mathbf{q}) = H_0(\mathbf{p}, \mathbf{q}) + \sum_{k=1}^{+\infty} \varepsilon^k H_k(\mathbf{p}, \mathbf{q}) \tag{1.13}$$

A dynamical system with Hamiltonian function of type (1.13) is naturally apt to be treated as a small perturbation of an integrable system. To investigate the dynamics of the system, we can try to construct further *formal* independent integrals of motion around the equilibrium point. More precisely, we can look for a near identity canonical coordinate transformation which brings H into a a "simpler" Hamiltonian function, that is a *normal form*, which admits new independent integral of motions.

The coordinate transformation resulting from the normalization procedure is a *formal* transformation; generically it is given by a non convergent series. This ties in with the fact that Hamiltonian systems with two or more degrees of freedom generically are not integrable. Hence the integrals obtained by the normalization are only *approximated*, up to flat perturbations. For small excitations these perturbations are extremely small, so that the integral curves of the system stay close to those of the normalized system for a long time. For two degrees of freedom the situation is even better. By KAM theory, there exists a fat Cantor set of tori with parallel dynamics. On this part of phase space, a smooth conjugation with the integrable system does exist. The KAM tori prevent chaotic solution curves from wandering through phase space, so that even these solutions stay within a bounded distance from the integrable system's tori for ever. This provides a justification for using the integrable approximation to study the full system.

1.2.1 The Lie-transform method

As observed above, the usual approach in Hamiltonian perturbation theory is to find a canonical transformation in such a way to construct a "simpler" Hamiltonian. The idea behind the Lie-transform method [16, 28, 57, 54] is that of seeing the canonical transformation as a "flow" along the Hamiltonian vector field associated to another Hamiltonian function which serves as generating function of the transformation. It is an iterative procedure, normalizing the system degree by degree. The end result is a Hamiltonian K in normal form, which has a circle or torus symmetry, with associated conserved quantities.

Let us consider a N degrees of freedom Hamiltonian function as given in (1.13), where H_0 defines an integrable system, ε is a small perturbing parameter and, for $k > 0$, H_k is a polynomial of degree $k + 2$ in the phase space variables. We look for a canonical transformation of the type

$$(\mathbf{P},\mathbf{Q}) = M_G(\mathbf{p}, \mathbf{q}), \tag{1.14}$$

where M_G is a linear operator defined by

$$M_G \equiv e^{-\varepsilon \mathcal{L}_{G_1}} e^{-\varepsilon^2 \mathcal{L}_{G_2}} \dots e^{-\varepsilon^n \mathcal{L}_{G_n}} \dots \tag{1.15}$$

The functions G_n are the coefficients of the expansion of the generating

function of the transformation (1.14) and the differential operator $\mathcal{L}_G$ is defined through the Poisson brackets

$$\mathcal{L}_G f \equiv \{f, G\} := \sum_{i=1}^{2} \frac{\partial G}{\partial p_i} \frac{\partial f}{\partial q_i} - \frac{\partial G}{\partial q_i} \frac{\partial f}{\partial p_i}. \tag{1.16}$$

By introducing the operator M_G^{-1}, the inverse of operator (1.15),

$$M_G^{-1} = \sum_{n=0}^{+\infty} \varepsilon^n M_n, \tag{1.17}$$

with

$$M_n := \sum_{m_1 + 2m_2 + \cdots + nm_n = n} \frac{\mathcal{L}_{G_1}^{m_1} \mathcal{L}_{G_2}^{m_2} \ldots \mathcal{L}_{G_n}^{m_n}}{m_1! m_2! \ldots m_n!}, \tag{1.18}$$

it can be demonstrated that the canonical transformation (1.14) provides a new Hamiltonian given by

$$K(\mathbf{P},\mathbf{Q};\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^n K_n(\mathbf{P},\mathbf{Q}) = M_G^{-1} H(\mathbf{p},\mathbf{q}); \varepsilon). \tag{1.19}$$

By expanding the righthand side of equation (1.19) in power series of ε, we find that G_n and K_n must satisfy the following recursive set of partial derivatives equations

$$
\begin{aligned}
K_0 &= H_0, \\
K_1 &= H_1 + M_1 H_0, \\
&\vdots \\
K_n &= H_n + M_n H_0 + \sum_{m=1}^{n-1} M_{n-m} V_m, \\
&\vdots \quad .
\end{aligned}
\tag{1.20}
$$

The first equality simply states that the zero-order new Hamiltonian, coincides with the zero-order old (unperturbed) one. The second equation has to be solved to find the first order term K_1. At this point we are faced with a difficulty: we have *one* differential equation involving *two* unknown functions, K_1 and G_1. To proceed we have to make some decision about the structure the new Hamiltonian must have, that is we have to choose a normal form for it. We therefore select the new Hamiltonian in such a way that it admits a new integral of motion, that is we take a certain function, say $\mathcal{I}$, and impose that

$$\{K, \mathcal{I}\} = 0. \tag{1.21}$$

The usual choice (but not necessarily the only possible) is that of taking

$$\mathcal{I} = K - H_0 \tag{1.22}$$

and proceed in the normalization procedure imposing at each step the condition

$$\{K, H_0\} = 0. \tag{1.23}$$

Thus the unperturbed term H_0 plays the role of determining the specific form of the transformation. With this choice, the first equation of the chain, that we can also write in the form

$$K_1 = H_1 + \mathcal{L}_{G_1} H_0 = H_1 - \mathcal{L}_{H_0} G_1, \tag{1.24}$$

is solved with a trick that we illustrate in the following. With the assumption that the kernel and the range of $\mathcal{L}_{H_0}$ are in direct sum (which is always true for the applications we have in mind), we can split the term H_1 appearing in (1.24) according to the rule

$$H_1 = H_1^{\mathrm{K}} + H_1^{\mathrm{R}} \tag{1.25}$$

where H_1^{K} is the part which stays in the kernel of $\mathcal{L}_{H_0}$ and H_1^{R} is the part which stays in the range of $\mathcal{L}_{H_0}$. Since our new Hamiltonian, according to (1.23), is in normal form if and only if it stays in the kernel of $\mathcal{L}_{H_0}$, we can then solve eq.(1.24) by applying the simple prescription:

$$K_1 = H_1^{\mathrm{K}}, \quad G_1 = \mathcal{L}_{H_0}^{-1} H_1^{\mathrm{R}}. \tag{1.26}$$

We have then constructed the normal form $K_0 + K_1$ at order 1 and computed the first term of the generating function G_1: we can therefore use it in the subsequent equations of the system and go one step further repeating the procedure to compute G_2 and the normal form at order 2 and so forth. At the n-th step in the normalization procedure, we impose that K_n and G_n to be solutions of the system

$$\begin{cases} \mathcal{L}_{G_n} H_0 + R_n &= K_n \\ \mathcal{L}_{H_0} K_n &= 0. \end{cases} \tag{1.27}$$

We rewrite the first equation as

$$\mathcal{L}_{H_0} G_n + K_n = R_n \tag{1.28}$$

where the term R_n is known from the preceding $n-1$ steps.

Equation (1.28) is the so called *homological equation*. With the assumption that the kernel and the range of $\mathcal{L}_{H_0}$ are in direct sum, "in principle" the n-th homological equation can always be solved if K_n satisfies (1.23). It suffices to take

$$K_n = R_n^K \quad \text{and} \quad G_n = \mathcal{L}_{H_0}^{-1} R_n^{\mathrm{R}}. \tag{1.29}$$

However, the series which defines the new Hamiltonian K is divergent and "in practice" we have to truncate the procedure at some finite order L. We say that K and H are in *normal form to the L-th order* if $K_0, \ldots K_L$ are in the kernel of $\mathcal{L}_{H_0}$.

The transformation (1.14) yields to new coordinates which are a continuous and differentiable "deformation" of the original ones. For practical purposes, it could be convenient to express the new integral of motion $\mathcal{I}$, cfr (1.22), in terms of the original coordinates. A (formal) series expansion of $\mathcal{I}$ can be given by

$$\mathcal{I} = \sum_{n=0}^{\infty} \varepsilon^n I_n, \tag{1.30}$$

where

$$I_n = H_n - K_n + \sum_{m=1}^{n-1} M_{n-m}(H_m - I_m), \quad n \geq 1, \tag{1.31}$$

are obtained from equations (1.20), where we exploited the nice properties of the Lie transform with respect to inversion [16].

It is important to remind how the normal form K and the integral of motion $\mathcal{I}$ gives an approximation of the dynamics of the original system. Truncating the normalization procedure at order L means that in the new Hamiltonian a rest of order $O(\varepsilon^{L+1})$ has been neglected. If we are in two degrees of freedom, the two functions K and $\mathcal{I}$ produce an integrable system whose dynamics, in the new coordinates, is an approximation of $O(\varepsilon^L)$ of the dynamics of the original system. On the other hand, the power series (1.31) is an approximated integral of motion for the original system in the sense that

$$\{\mathcal{I}, H\} = O(\varepsilon^{L+1}). \tag{1.32}$$

Thus, $\mathcal{I}$ does not commute with the original Hamiltonian for higher order terms. The error committed in both cases is the same, since terms of the same order have been neglected.

As already observed, the series involved in the normalization are in general not convergent, but they are *asymptotic* in the sense of the following definition

Definition 1.1. *The power series $\sum_{n=0}^{+\infty} a_n(x-x_0)^n$ is said to be asymptotic to the function $y(x)$ as $x \to x_0$ and we write*

$$y(x) \sim \sum_{n=0}^{+\infty} a_n(x - x_0)^n \quad (x \to x_0)$$

if

$$y(x) - \sum_{n=0}^{N} a_n(x - x_0)^n \ll (x - x_0)^N \quad (x \to x_0)$$

for every $N \in \mathbb{N}$.

Thus, a power series is asymptotic to a function if the remainder after N terms is much smaller than the last retained term as $x \to x_0$. By this definition a series need not to be convergent to be asymptotic. Indeed, the definition of asymptotic series is interesting only when the series is divergent [64]. Let us contrast convergent and asymptotic series. If

$$f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n$$

is a convergent series for $|x - x_0| < R$, then the remainder $\varepsilon_N(x) \to 0$ for any fixed x such that $|x - x_0| < R$. On the other hand, if the series is asymptotic to $f(x)$,

$$f(x) \sim \sum_{n=0}^{+\infty} a_n (x - x_0)^n \quad (x \to x_0),$$

then the remainder $\varepsilon_N(x)$ goes to zero faster than $(x - x_0)^N$ as $x \to x_0$, N fixed. Typically the terms in the series get smaller for awhile, but eventually they start to increase. Since $a_{N+1}(x - x_0)^{N+1}$ is an estimate of the error, we can find the optimal order of truncation $N = N_{opt}$ determining the smallest term [12]. The optimal order depends on the interval $|x - x_0|$: the larger the interval, the smaller N_{opt} and the accuracy in the approximation. Once reached the optimal order, it can be disappointing to discard terms coming from a costly high-order computation.

1.3   Normal forms around detuned resonances

Let us consider a natural two degrees of freedom system whose Hamiltonian is given by

$$\mathcal{H}(\mathbf{p}, \mathbf{q}) = \frac{1}{2} \sum_{i=1}^{2} p_i^2 + \mathcal{V}(\mathbf{q}; \varepsilon) \tag{1.33}$$

where the potential V admits an absolute minimum point in the origin and a series expansion of type

$$\mathcal{V}(\mathbf{q}; \varepsilon) = \sum_{n=0}^{+\infty} \varepsilon^n \mathcal{V}_n(\mathbf{q}) \tag{1.34}$$

around this minimum. Moreover we assume each term V_n to be a homogeneous polynomial of degree $n + 2$:

$$\mathcal{V}_n(\mathbf{q}) = \sum_{j=0}^{n+2} a_{(j, n+2-j)} q_1^j q_2^{n+2-j}, \tag{1.35}$$

In particular, modulo a diagonalization, the zero order term $\mathcal{V}_0$ can be always taken in the form

$$\mathcal{V}_0(q_1, q_1) = \frac{1}{2}(\omega_1^2 q_1^2 + \omega_2^2 q_2^2). \tag{1.36}$$

The system is therefore naturally apt to be treated in a perturbative way as a non-linear oscillator system. The normalization procedure provides a non-resonant normal form when the two harmonic frequencies ω_1 and ω_2 are generically non-commensurable. It is customary to refer to the normal forms constructed in this case as Birkhoff normal forms [15]. A *resonant* normal form is instead assembled by assuming from the start an integer value for the ratio of the harmonic frequencies and including in the new Hamiltonian terms depending on the corresponding resonant combination of the angles. This possibility might be thought to be exceptional: it is instead *almost the rule* because, even if the unperturbed system is non-resonant with a certain real value

$$\rho = \omega_1/\omega_2 \tag{1.37}$$

of the frequency ratio, the non-linear coupling between the degrees of freedom induced by the perturbation determines a "passage through resonance" with a commensurability ratio, say m_1/m_2 with $m_1, m_2 \in \mathbb{N}$, corresponding to the local ratio of oscillations in the two degrees of freedom. This in turn is responsible of the birth of new orbit families bifurcating from the normal modes or from lower-order resonances. Moreover, the presence of terms with small denominators in the expansion forbids in general its convergence. It is therefore more effective to work from the start with a *resonant normal form* [92], which is still nonconvergent, but has the advantage of avoiding the small divisors associated with a particular resonance. Therefore we approximate the frequencies ratio with a rational number plus a small *detuning* so that

$$\rho = m_1/m_2 + \delta, \tag{1.38}$$

and speak of a **detuned $m_1 : m_2$ resonance**, implying with this a trick to deceive the formally correct but ineffective approach based on non-resonant generic frequency ratios. The detuning term is treated as a higher order term and put in the perturbation [101, 103]. For the application we have in mind, it will be convenient to assume δ to be of second order in ε, therefore $\delta = \tilde{\delta}\varepsilon^2$ (see the following subsection for more details).

In order to proceed with a resonant normalization procedure, we perform the scaling transformation

$$q_1 \to \sqrt{\omega_1}q_1, \ q_2 \to \sqrt{\omega_2}q_2, \ p_1 \to \frac{p_1}{\sqrt{\omega_1}}, \ p_2 \to \frac{p_2}{\sqrt{\omega_2}} \tag{1.39}$$

and we redefine the Hamiltonian according to

$$\tilde{H}(\mathbf{p}, \mathbf{q}) := \frac{m_2 \mathcal{H}(\mathbf{p}, \mathbf{q})}{\omega_2}, \tag{1.40}$$

so that the unperturbed term reads

$$\tilde{H}_0(\mathbf{p}, \mathbf{q}) = \frac{1}{2}(m_1(p_1^2 + q_1^2) + m_2(p_2^2 + q_2^2)). \tag{1.41}$$

Let us introduce complex coordinates

$$\begin{cases} z_\ell = & p_\ell + iq_\ell \\ w_\ell = & p_\ell - iq_\ell \end{cases} \quad \ell = 1, 2, \tag{1.42}$$

which turn the Hamiltonian function into

$$H(\mathbf{z}, \mathbf{w}) = \sum_{n=0}^{+\infty} \varepsilon^n H_n(\mathbf{z}, \mathbf{w}) \tag{1.43}$$

with

$$H_0(\mathbf{z}, \mathbf{w}) = \frac{1}{2}(m_1 w_1 z_1 + m_2 w_2 z_2). \tag{1.44}$$

Notice that the Poisson brackets change according to $\{z_k, w_\ell\} = 2i\delta_{k,\ell}$. The reason to introduce complex coordinates lies in the fact that the operator $\mathcal{L}_{H_0}$, cfr. (1.16), assumes the simple form

$$\mathcal{L}_{H_0} = -2i \sum_{\ell=1}^{2} m_\ell \left(w_\ell \frac{\partial}{\partial w_\ell} - z_\ell \frac{\partial}{\partial z_\ell} \right). \tag{1.45}$$

Since the kernel and the range of $\mathcal{L}_{H_0}$ are in direct sum over the space $\mathcal{P}_m$ of polynomials of degrees $m + 2$ in the complex coordinates (1.42), the n-th homological equation (1.28) can always be solved if we look for a normal form satisfying the condition (1.23). Moreover, thanks to its simple expression, the kernel of operator (1.45) over the polynomial space $\mathcal{P}_m$ can be easily found. Let us denote a generic monomial in $\mathcal{P}_m$ by

$$\tau = z_1^{\lambda_1} z_2^{\lambda_2} w_1^{k_1} w_2^{k_2} \tag{1.46}$$

where λ_1, λ_2, k_1, k_2 are nonnegative integers, $\lambda_1 + \lambda_2 + k_1 + k_2 = m + 2$. The operator $\mathcal{L}_{H_0}$ acts on τ as

$$\mathcal{L}_{H_0}(\tau) = 2i(m_1\lambda_1 + m_2\lambda_2 - m_1 k_1 - m_2 k_2)\tau. \tag{1.47}$$

Hence, the condition for a monomial in $\mathcal{P}_m$ to be in the kernel of $\mathcal{L}_{H_0}$ is

$$m_1(\lambda_1 - k_1) + m_2(\lambda_2 - k_2) = 0. \tag{1.48}$$

Therefore, if a monomial τ belongs to the term R_n of equation (1.28) and satisfies (1.48) it does not contribute to the generating function and stays in the normal form, otherwise its contribution to the normalized Hamiltonian is zero and the term

$$-\frac{i\tau}{2(m_1\lambda_1 + m_2\lambda_2 - m_1 k_1 - m_2 k_2)}$$

is added to the generating function G_n. This provides an effective strategy to implement algorithm to solve the homological equation (1.28) at each step in the normalization procedure. For the application we have in mind, the normal forms are computed by Mathematica®.

However, as observed above, the series are divergent, thus we must truncate the procedure at some finite order.

The monomials which give a non trivial solution of eq. (1.48), i.e. $\lambda_i \neq k_i$, are said *resonant monomials* or proper terms of the resonance. Generally, given a resonance ratio m_1/m_2, the procedure must be pushed at least to order $m_1 + m_2 - 2$, when the first resonant term appears [30].

Introducing action-angle like variables

$$\begin{cases} z_\ell = & i\sqrt{2J_\ell}e^{-i\phi_\ell} \\ w_\ell = & -i\sqrt{2J_\ell}e^{i\phi_\ell}, \ l = 1, 2 \end{cases} \tag{1.49}$$

the typical structure of the resonant normal form truncated at order $m_1 + m_2 - 2$ is

$$\begin{aligned} K &= m_1 J_1 + m_2 J_2 + \varepsilon(A_1 J_1 J_2^{\frac{1}{2}} + A_2 J_1^{\frac{1}{2}} J_2) + \ \ldots \\ &+ \ \ldots\ldots + \varepsilon^{m_1+m_2-2} C J_1^{m_2/2} J_2^{m_1/2} \cos(m_2\phi_1 - m_1\phi_2). \end{aligned} \tag{1.50}$$

The resonant term is responsible of the appearance of the angular combination $m_2\phi_1 - m_1\phi_2$.

1.3.1 Symmetric detuned resonances

It could happen that the system under study exhibits some particular symmetries. For example, the potential we are interested in the following are of the type $V = V(q_1^2, q_2^2)$ and are therefore symmetric under reflections with respect to both coordinate axes.

The presence of symmetries could affect the structure of the normal form and the minimal order of the resonant terms. As a consequence, the minimal order required to arrest the normalization procedure increases.

In case the system is symmetric under reflection with respect to both coordinate axes, the odd degree terms in the Hamiltonian expansion (1.33) are all zero. In order to construct a resonant normal form, it is therefore convenient to assume the detuning term in (1.38) to be of second order in the perturbation, namely $\delta = \tilde{\delta}\varepsilon^2$. This translates into $\varepsilon \approx \sqrt{\delta}$. Further, if we want the normal form to respect the symmetries, the action of the linear operator $\mathcal{L}_{H_0}$ has to be restricted on the space of symmetric polynomials. Thus, the resonant monomials which are not invariant under the reflection symmetries are no more allowed in the normal form. As a consequence, the odd degree terms in the normal form are all zero and the minimal normalization order increases to $2(m_1 + m_2 - 1)$ [30]. To keep into account this fact we speak of a *symmetric $m_1 : m_2$ resonance* or of a $2m_1 : 2m_2$ resonance.

After the scalings (1.39) and (1.40), the Hamiltonian function (1.33) turns into

$$H(\mathbf{z}, \mathbf{w}; \tilde{\delta}) = \sum_{i=0}^{+\infty} \varepsilon^{2i} H_{2i}(\mathbf{z}, \mathbf{w}; \tilde{\delta}) \tag{1.51}$$

where complex coordinates (1.42) have been introduced. Since the frequencies have been approximated according to (1.38) with a second order detuning parameter, the non vanishing terms read

$$H_0(\mathbf{z}, \mathbf{w}) = \frac{1}{2}(m_1 w_1 z_1 + m_2 w_2 z_2) \tag{1.52}$$

$$H_{2i}(\mathbf{z}, \mathbf{w}; \tilde{\delta}) = \sum_{j=0}^{i} b_j \tilde{\delta}^j h_{2(i-j)}(\mathbf{z}, \mathbf{w}), \quad i > 0 \tag{1.53}$$

where $h_{2(i-j)} \in \mathcal{P}_{2(i-j)}$, $i = 0, \ldots, j$, the b_j are constants which depend on the coefficients $a_{(k,\ell)}$ of the original potential and could be zero.

We can therefore proceed with an ordinary Birkhoff-Gustavson normalization [61], with two variants:

1. the coordinate transformations are performed through Lie transformations according to the method described in section 1.2.1;

2. the detuning term is treated as a term of order two and put in the perturbation.

To apply the Lie-transform method with the series expansion given in (1.51), we look for a generating function of the transformation (1.14) in the form

$$G(\mathbf{z}, \mathbf{w}; \tilde{\delta}) = \sum_{n=0}^{+\infty} G_n(\mathbf{z}, \mathbf{w}; \tilde{\delta}) \tag{1.54}$$

where G_0 equals to the identity transformation. Since we are looking for a symmetric normal form K,

$$K(\mathbf{z}, \mathbf{w}; \tilde{\delta}) = \sum_{i=0}^{+\infty} K_{2i}(\mathbf{z}, \mathbf{w}; \tilde{\delta}), \tag{1.55}$$

the odd degree terms in the generating function must be all zero, namely $G_{2i+1} = 0$ in (1.54), and we look for even degree terms in the form

$$G_{2i}(\mathbf{z}, \mathbf{w}; \tilde{\delta}) = \sum_{j=0}^{i} \tilde{\delta}^j g_{2(i-j)}(\mathbf{z}, \mathbf{w}), \quad g_{2(i-j)} \in \mathcal{P}_{2(i-j)}, \quad j = 0, \ldots, i, \quad i > 0 \tag{1.56}$$

Imposing the normalization condition (1.23), to find $G_{2\ell}$ and $K_{2\ell}$ we have to solve the homological equation (1.28), which in our case reads

$$\sum_{j=0}^{i} \tilde{\delta}^j \mathcal{L}_{H_0} g_{2(i-j)} + \sum_{j=0}^{i} \tilde{\delta}^j k_{2(i-j)} = \sum_{j=0}^{i} \tilde{\delta}^j r_{2(i-j)}, \qquad (1.57)$$

with $g_{2(i-j)}, k_{2(i-j)}$ and $r_{2(i-j)} \in \mathcal{P}_{2(i-j)}, j = 0, \ldots i$.
Since $\mathcal{L}_{H_0} : \mathcal{P}_m \to \mathcal{P}_m$, solving equation (1.57) is equivalent to solving $i+1$ equations of the type

$$\mathcal{L}_{H_0} g_{2(i-j)} + k_{2(i-j)} = r_{2(i-j)}, \quad j = 0, \ldots, i. \qquad (1.58)$$

The monomial terms in $g_{2(i-j)}$ and $k_{2(i-j)}$ can be easily found exploiting the simple expression of $\mathcal{L}_{H_0}$ over the polynomial spaces $\mathcal{P}_{2(i-j)}$, as explained above.

Introducing action-angle like variables (1.49), the normal form of the system, truncated when the first resonant term appears, has the following form

$$\begin{aligned}
K &= m_1 J_1 + m_2 J_2 + \\
&+ \varepsilon^2 (\tilde{\delta} J_1 + B_1 J_1^2 + B_2 J_1 J_2 + B_3 J_2^2) + \ldots \\
&+ \ldots\ldots + \varepsilon^{2(m_1+m_2-1)} C J_1^{m_2} J_2^{m_1} \cos(2m_2 \phi_1 - 2m_1 \phi_2). \quad (1.59)
\end{aligned}$$

As a consequence of the symmetries, only even degree terms in the perturbation parameter are present and the angular combination is now given by $2(m_2\phi_1 - m_1\phi_2)$. In the following, we will focus on systems around $2:2$ and $2:4$ resonances.

Chapter 2

Equivariant singularity theory

This chapter collects the results on singularity theory that we need in the following. In particular, in chapter 3 we show how the results presented in section 2.4 can be applied to the planar system obtained after the normalization and reduction of a $2:2$ detuned resonance.

For our purposes, singularity theory is a tool, not a goal in itself. This chapter therefore aims only to give the basics of the theory and refers to the literature for more details [18, 59, 60, 79, 88, 97].

2.1 Germs of functions

The theory we are presenting is local, i.e. the results are valid only in a sufficiently small neighborhood of some fixed point. The terminology of germs provides a convenient way of formulating results in a local theory which avoids infinite repetitions of the phrase "in a sufficiently small neighborhood of the origin".

Let us denote by $\mathcal{U}_n$ the space of all functions $f\colon \mathbb{R}^n \to \mathbb{R}$ which are defined and C^∞ in some neighborhood of the origin. We say that two functions in $\mathcal{U}_n$ are *equal as germs* if there is some neighborhood of the origin on which they coincide. We shall identify two functions which are equal as germs and call the elements of $\mathcal{U}_n$ *germs of functions*. In a more technical language a germ is an equivalence class with respect to this identification: namely, we give the following definition

Definition 2.1. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$. We denote by $\sim$ the germ equivalence $f \sim g$ if there exists $B \subset \mathbb{R}^n$, $0 \in B$, such that $f(x) = g(x)$ for all $x \in B$. The equivalence classes with respect to $\sim$ are called germs of functions. We denote by $\mathcal{U}_n$ the ring of germs of functions on $\mathbb{R}^n$.*

Let us denote by Γ a compact group with linear action on $\mathbb{R}^n$. Groups

elements $\gamma \in \Gamma$ are identified with their corresponding linear action. For example in the following we shall consider $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ with action on $\mathbb{R}^2$ given by $(x, y) \to (\varepsilon_1 x, \varepsilon_2 y)$, $\varepsilon_i = \pm 1$.

Definition 2.2. *We say that a germ f has Γ symmetry or is Γ- invariant if $f(\gamma x) = f(x) \; \forall \gamma \in \Gamma$. We denote by $\mathcal{U}_n^\Gamma$ the set of equivalence classes of Γ-invariant functions with respect to the germ equivalence, $\mathcal{U}_n^\Gamma := \{ f : \mathbb{R}^n \to \mathbb{R} : f(\gamma x) = f(x) \; \forall \gamma \in \Gamma \} / \sim$. The elements of $\mathcal{U}_n^\Gamma$ are called Γ- invariant germs of functions.*

Notice that in the trivial case $\Gamma = \{id\}$, that is Γ contains only the identity element, $\mathcal{U}_n^\Gamma = \mathcal{U}_n$.

2.2 Deformations

Almost every concept used in the singularity theory of functions and maps, has a direct counterpart in the finite dimensional context of Lie groups acting on smooth finite dimensional manifolds.

Let M be a manifold, G a Lie group, and let us start supposing both M and G to be smooth, finite dimensional manifolds. We denote by $\xi : G \times M \to M$ a smooth action of G on M. Instead of $\xi(\varphi, f)$ we sometimes simply write φf. For a given point $f \in M$, the action ξ gives rise to an orbit, in this notation given by Gf and let $T_f(Gf)$ be the tangent space to this orbit at the point f. The codimension of $T_f(Gf)$ in $T_f(M)$ is also called the codimension of f. If this codimension is 0 the inverse function theorem assures that for every g in some neighborhood of f, there exist $\varphi \in G$ such that $g = \xi(\varphi, f)$. If this is the case, f is called a *stable* element and f and g are said to be equivalent according to the following definition

Definition 2.3. *We say that f and g are equivalent, $f \simeq g$, iff there exists $\varphi \in G$ such that $\varphi f = g$.*

Stable elements, with codimension 0, form the simplest case. Now let us proceed, and suppose that the codimension of f is nonzero, say equal to d. Small changes to f along orbits of G will not change f's equivalence class; however changes transversal to its G-orbit will. A catalog of representatives of all equivalence classes that occur in a neighborhood of f is given by a transversal section of the orbit Gf at f. Such a transversal section, which is a submanifold in M of dimension d, can be parameterized as a d- parameter family of elements in M. Families depending on parameters are also called *deformations*:

Definition 2.4. *A $d-$deformation or unfolding of f in M is a map $F(u) : \mathbb{R}^d \to M$ such that $F(0) = f$.*

Now, the deformation F is a transversal section if its deformation directions complement the tangent space, or symbolically (here and alsewhere in the following D_{x_0} denotes the total differential at x_0):

$$T_f(Gf) \oplus D_0 F(\mathbb{R}^d) = T_f(M). \tag{2.1}$$

Deformations for which (2.1) holds are called *transversal deformations*. Now let $H(v) : \mathbb{R}^q \to M$ be some deformation of f. It is said to be *induced* from F if there exists $h : \mathbb{R}^q \to \mathbb{R}^d$ with $h(0) = 0$ and a deformation $I : \mathbb{R}^q \to G$ of the identity element in G such that

$$H(v) = I(v)F(h(v)).$$

If it happens that every deformation G of f can be induced from F in this way, then F is called a *versal* deformation. Again by the inverse function theorem it can be shown that a deformation is versal iff it is transversal, see e.g. [79].

We now turn to the case in which M is the set of $\Gamma-$ invariant germs of functions on $\mathbb{R}^n$, i.e. $M = \mathcal{U}_n^\Gamma$. To proceed we need the following definition

Definition 2.5. *A map ϕ on $\mathbb{R}^n$ is called Γ- equivariant if $\phi(\gamma x) = \gamma \phi(x)$ for all $\gamma \in \Gamma$.*

The group of transformations G that acts on M is the group of origin preserving Γ- equivariant C^∞ maps on $\mathbb{R}^n$ and acts on $\mathcal{U}_n^\Gamma$ by composition to the right. Now, neither the manifold M of smooth functions nor the group is finite dimensional. However, the results we stated above continue to hold in this case. The proofs are much more difficult, however, since the inverse function theorem cannot be used, see e.g. [19, 79], for proofs and more details.

The notion of equivalence and deformations are the same given above, more precisely for Γ- equivariant germs of functions they read

Definition 2.6.

i) *The ring of germs of Γ-invariant functions on $\mathbb{R}^n \times \mathbb{R}^d$, where Γ acts trivially on $\mathbb{R}^d$, is denoted by $\mathcal{U}_{n+d}^\Gamma$.*

ii) *We say that f and g are equivalent or isomorphic as Γ- invariant germs, $f \simeq g$, iff there exists a Γ equivariant and origin preserving diffeomorphisms φ such that $f \circ \varphi = g$.*

iii) *$F \in \mathcal{U}_{n+d}^\Gamma$ is called a deformation or unfolding of $f \in \mathcal{U}_n^\Gamma$ if $F(x, 0) = f(x)$.*

iv) *A deformation $H \in \mathcal{U}_{n+q}^\Gamma$ is said to be induced from $F \in \mathcal{U}_{n+d}^\Gamma$ if there exists a germ of a reparametrization $h : \mathbb{R}^d \to \mathbb{R}^q$ and a parameter dependent Γ-equivariant map $\phi(x, v)$ such that*

$$H(x, v) = F(\phi(x, v), h(v)).$$

v) A deformation F of f is said to be versal if any deformation H of F can be induced from F.

vi) A deformaton F of f is called universal if it is versal and dependents on the minimal number of parameters.

As in the finite dimensional case, the following proposition holds [60]:

Proposition 2.1. *A deformation $F \in \mathcal{E}_{n+d}^{\Gamma}$ of $f \in \mathcal{E}_n^{\Gamma}$ is versal iff*

$$T_f(Gf) \oplus D_0 F(\mathbb{R}^d) = \mathcal{U}_n^{\Gamma}. \tag{2.2}$$

From now on, we abbreviate the tangent space $T_f(G_f)$ with T_f.

Application 2.1.

In the following, the previous proposition will be very useful. In particular, it allows us to find a versal deformation of the germ $f = \varepsilon_1 x^4 + \mu x^2 y^2 + \varepsilon_2 y^4$, $\mu^2 \neq 4\varepsilon_1\varepsilon_2$, $\varepsilon_i \in \pm 1$, $i = 1, 2$. We will need this result through all next chapter in the analysis of the $2 : 2$ resonance.

To compute a versal deformation of f, we note that the tangent space T_f is generated as a $\mathcal{U}_2^{\mathbb{Z}_2 \times \mathbb{Z}_2}$- module [1] by [79]

$$g_1 = x\frac{\partial f}{\partial x} = 2x(2\varepsilon_1 x^3 + \mu x y^2) \quad \text{and} \quad g_2 = y\frac{\partial f}{\partial y} = 2y(2\varepsilon_2 y^3 + \mu x^2 y).$$

Thus,

$$T_f \oplus \text{span}_{\mathbb{R}}\{1, x^2, y^2, x^2 y^2\} = \mathcal{U}_2^{\mathbb{Z}_2 \times \mathbb{Z}_2} \tag{2.3}$$

which implies that

$$F(x, u_0, u_1, u_2) = \varepsilon_1 x^4 + \mu x^2 y^2 + \varepsilon_2 y^4 + u_3 x^2 y^2 + u_2 y^2 + u_1 x^2 + u_0$$

is a versal deformation of f. Moreover, F depends on the minimal number of parameters, therefore it is a universal deformation.

Notice that for the exceptional values $\mu^2 = 4\varepsilon_1\varepsilon_2$ the tangent space has infinite codimension [79].

---

[1]A module over a ring is a generalization of the notion of vector space, wherein the corresponding scalars are allowed to lie in an arbitrary ring. Precisely,a left R- module M over the ring R consists of an abelian group $(M, +)$ and an operation $R \times M \to M$ such that for all r, s in R and x, y in M, we have:
1. $r(x + y) = rx + ry$;
2. $(r + s)x = rx + sx$;
3. $(rs)x = r(sx)$;
4. $I_R x = x$ if R has multiplicative identity I_R.
A right R-module M is defined similarly, only the ring acts on the right. If R is commutative, then left R-modules are the same as right R-modules and are simply called R-modules.

2.3 Determinacy and germ isomorphy

Definition 2.7. *We denote with V_n^Γ the $\mathcal{U}_n^\Gamma$-module of Γ equivariant vector fields on $\mathbb{R}^n$.*

The tangent space to the group G at the identity is isomorphic to the module of vector fields V_n^Γ. Therefore the tangent space T_f to the orbit of f is given by

$$T_f := \{Xf \; : \; X \in V_n^\Gamma\}. \tag{2.4}$$

The results presented in the following are mostly taken from [79]. Let us denote with m_k^Γ the set of germs in $\mathcal{U}_n^\Gamma$ whose Taylor polynomial at 0 vanishes up to and including order $k-1$. In particular, $m^\Gamma := m_1^\Gamma = \{ f \in \mathcal{U}_n^\Gamma : f(0) = 0 \}$. In case $\Gamma = \{id\}$ we drop the Γ from the notation. We call *jet map* the projection $j^k(f)\colon \mathcal{U}_n^\Gamma \to \mathcal{U}_n^\Gamma/m_{k+1}^\Gamma$. In words, $j^k(f)$ is nothing than the Taylor polynomial of f up to and including order k.

The sets m_k^Γ are ideals in $\mathcal{U}_n^\Gamma$. In case $\Gamma = \{id\}$ we have that m_k is generated by $x_1^{k_1} \cdots x_n^{k_n}$, with $k_1 + \cdots + k_n = k$, that is by homogenous polynomials of degree k and V_n is generated, as an $\mathcal{U}_n$- module, by $x_i\frac{\partial}{\partial x_j}$. This implies that the tangent space T_f is just $m \cdot J(f)$ where $J(f)$ is the Jacobian ideal of f:

$$J(f) := \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle.$$

Definition 2.8. *A germ of function $f \in \mathcal{U}_n^\Gamma$ is said to be $k-$determined if $\forall g \in \mathcal{U}_n^\Gamma$ such that $j^k f = j^k g$ we have $g \simeq f$.*

Therefore, if a germ has finite determinacy then, modulo a coordinate transformation, it can be written as a polynomial. Notice that the existence of a transversal deformation implies that f is finitely determined.

The following proposition gives conditions under which (non-symmetric) germs are isomorphic [79]:

Proposition 2.2. *Let $f, g \in \mathcal{U}_n$, and suppose that $g - f \in m_k$, i.e $j^{k-1}(g - f) = 0$. Then*

i) If $m_k \subset T_f$ then $g \simeq f$ provided that $j^k(g - f)$ is sufficiently small.

ii) If $m_k \subset m \cdot T_f$ then $g \simeq f$.

Here $j^k(g - f)$ sufficiently small means that the coefficient of the k-th order Taylor polynomial of $g - f$ is sufficiently small. The analogous result for germs with symmetry is [22]:

Proposition 2.3. *Let $f, g \in \mathcal{U}_n^\Gamma$, and suppose that $g - f \in m_k^\Gamma$. Let M denote the finite dimensional vector space $M = \frac{m_k^\Gamma}{m^\Gamma \cdot m_k^\Gamma}$ e $M_k := M \cap (m_k^\Gamma/m_{k+1}^\Gamma)$ the set of homogeneous polynomials in M of degree k.*

i) If $m_k^\Gamma \subset T_f$ then $g \simeq f$ provided that $j^k(g-f)$ is sufficiently small.

ii) If $m_k^\Gamma \subset m^\Gamma \cdot T_f$ then $g \simeq f$.

iii) Suppose that $m_k^\Gamma \subset T_f$. Suppose further that the projection of f into M is an element of M_k. Then $g \simeq f$ provided that $j^k(g-f)$ is sufficiently small.

Application 2.2.

As a consequence of the previous proposition we state the following proposition, which will be useful in the analysis of the $2:2$ resonance, as well as application 2.1:

Proposition 2.4. *The germ $g = \varepsilon_1 x^4 + \mu x^2 y^2 + \varepsilon_2 y^4 +$ h.o.t. , with $\varepsilon_i = \pm 1$, is $\mathbb{Z}_2 \times \mathbb{Z}_2$ isomorphic to $f = \varepsilon_1 x^4 + \mu x^2 y^2 + \varepsilon_2 y^4$, provided that $\mu^2 \neq 4\varepsilon_1\varepsilon_2$.*

Proof. For $\mu^2 \neq 4\varepsilon_1\varepsilon_2$ the tangent space T_f is generated as $\mathcal{U}_2^{\mathbb{Z}_2 \times \mathbb{Z}_2}$- module by $g_1 = 2x(2\varepsilon_1 x^3 + \mu x y^2)$ and $g_2 = 2y(2\varepsilon_2 y^3 + \mu x y^2)$, thus $T_f = m_4^{\mathbb{Z}_2 \times \mathbb{Z}_2}$. Now, we apply proposition (2.3) (iii) with $k = 4$ to f. Since f is homogenous, $f \in M_4$, so we conclude that $g = f$ h.o.t. $\simeq f$ for arbitrary higher order terms.

Here μ is a *module* and different values of μ gives non isomorphic germs. For $\mu = 0$, T_f is generated by $\{\varepsilon_1 x^4, \varepsilon_2 y^4\}$ and has the same codimension as in the generic case.

$\square$

Notice that for $\mu^2 = 4\varepsilon_1\varepsilon_2$ the tangent space T_f has infinite codimension and the previous proposition does not apply.

2.4 Computing $\mathbb{Z}_2 \times \mathbb{Z}_2$-equivariant deformation morphisms

Let G be a (truncated) polynomial in two variables invariant under the action of the group $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$, $(x, y) \to (\epsilon x, \epsilon y)$, $\epsilon = \pm 1$. We assume that its coefficients g_i depend on d parameters $c_1, \ldots, c_d$, that is $g_i = g_i(c_1, \ldots, c_d)$ and that there exists singular values of the parameters, which for simplicity we assume to be 0, such that we have

$$Gs := G|_{c_i=0} = s_1 x^4 + s_2 x^2 y^2 + s_3 y^4.$$

For example, we may have

$$
\begin{aligned}
G = Gc \;\; := \;\; & c_1 x^2 + c_2 y^2 + (s_1 + c_3)x^4 + (s_2 + c_4)x^2 y^2 + (s_3 + c_5)y^4 \\
& + \; c_6 x^6 + c_7 x^2 y^4 + c_8 x^4 y^2 + c_9 y^6 + O(|x, y|^8)
\end{aligned}
\tag{2.5}
$$

Suppose that $s_1, s_3 \neq 0$, then by a simple scaling transformation we achieve that Gs turns into

$$f = \varepsilon_1 x^4 + \mu x^2 y^2 + \varepsilon_2 y^4 \quad \varepsilon_i = \pm 1, \quad i = 1, 2. \tag{2.6}$$

Therefore, G is a deformation of f.

As we have seen in section 2.2, if $\mu^2 \neq 4\varepsilon_1\varepsilon_2$, the germ $\varepsilon_1 x^4 + ax^2 y^2 + \varepsilon_2 y^4$ has finite codimension with universal deformation

$$F = \varepsilon_1 x^4 + \mu x^2 y^2 + \varepsilon_2 y^4 + u_1 x^2 + u_2 y^2 + u_3 x^2 y^2 + u_0. \tag{2.7}$$

Moreover the theory assures that there exists a $\mathbb{Z}_2 \times \mathbb{Z}_2$-equivariant morphisms ϕ which induces the deformation G from F. Such a transformation can be very useful in applications, since it allows to reduce the number of parameters to the minimal.

In [21] an algorithm is presented to compute ϕ in presence of a $\mathbb{Z}_2$ symmetry, $(x, y) \to (x, \pm y)$. In the following, we adapt the algorithm to our symmetric context.

Let us denote with $\mathcal{T}(2)$ the space of all differentiable germs of two variables invariant under the action of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and vanishing at the origin. From equation (2.3), we get that for every $g \in \mathcal{T}(2)$ there exist Γ invariant germs $\mathcal{Q}_i(x, y)$ and real numbers $\mathcal{R}_i$ such that

$$g(x, y) = \sum_i \mathcal{Q}_i(x, y) T_i(x, y) + \mathcal{R}_1 x^2 + \mathcal{R}_2 y^2 + \mathcal{R}_3 x^2 y^2 \tag{2.8}$$

where T_i is a system of generators of the tangent space T_f. Equation (2.8) is the so called *infinitesimal stability equation*.

We have seen in section (2.2) that in the particular case $f = \varepsilon_1 x^4 + \mu x^2 y^2 + \varepsilon_2 y^4$, a system of generators is given by

$$T_1(x, y) = x \frac{\partial f(x, y)}{\partial x} = x(4\varepsilon_1 x^3 + 2\mu x y^2)$$

$$T_2(x, y) = y \frac{\partial f(x, y)}{\partial y} = y(2\mu x^2 y + 4\varepsilon_2 y^3).$$

Now, suppose that we are able to solve equation (2.8), then we can construct the transformation ϕ using the following iterative algorithm. We look for a transformation

$$
\begin{aligned}
\phi \quad : \quad & \mathbb{R}^2 \times \mathbb{R}^q \to \mathbb{R}^2 \times \mathbb{R}^3 \\
& (x, y, c_i) \to (\theta(x, c_i), \rho(c_i))
\end{aligned} \tag{2.9}
$$

where θ is a diffeomorphisms which acts as a (parameter depending) coordinate transformation, $\theta : \mathbb{R}^2 \times \mathbb{R}^q \to \mathbb{R}^2$, $\rho : \mathbb{R}^q \to \mathbb{R}^2$ acts as a reparametrization.

For simplicity of notation we define $c = (c_1, \ldots, c_q)$, $u = (u_1, u_2, u_3)$ and $z = (x, y)$. Suppose that we have an algorithm that solves the infinitesimal stability equation modulo terms of order $O(z^d)$, $d \geq 2$. The basic idea is to expand θ and ρ as a formal power series in the parameters c_i [71]:

$$\theta(z, c) = \sum_{i \geq 0} \theta_j(z, c), \quad \rho(c) = \sum_{j \geq 0} \rho_j(c)$$

where θ_j and ρ_j are homogenous of degree j in the parameters c. Let us denote

$$\theta^\ell(z, c) := \sum_{i=0}^{\ell} \theta_i(z, c) \quad \rho^\ell(c) := \sum_{i=0}^{\ell} \rho_i(c)$$

and set $\theta^0(z) := z$, $\rho^0(c) := 0$. Suppose that we are able to compute θ up to order ℓ in c, that is we are able to find θ^ℓ and ρ^ℓ which solve

$$G(z, c) = F(\theta^\ell(z, c), \rho^\ell(c)) + O(c^{\ell+1}) + O(z^d). \tag{2.10}$$

Then

$$
\begin{aligned}
F(\theta^{\ell+1}, \rho^{\ell+1}) &= F(\theta^\ell + \theta_{\ell+1}, \rho^\ell + \rho_{\ell+1}) \\
&= F(\theta^\ell, \rho^\ell) + D_z F(\theta^\ell, \rho^\ell)\theta_{\ell+1} \\
&+ D_u F(\theta^\ell, \rho^\ell)\rho_{\ell+1} + O(|\theta_{\ell+1}|^2) + O(|\rho_{\ell+1}|^2) \\
&= F(\theta^\ell, \rho^\ell) + D_z g(\theta^\ell, \rho^\ell)\theta_{\ell+1} \\
&+ D_c F(\theta^\ell, \rho^\ell)|_{c=0} \cdot \rho_{\ell+1} + O(c^{\ell+2}) \tag{2.11}
\end{aligned}
$$

where we obtain the last inequality using the estimates $\theta^\ell(z, c) = z + O(c)$, $\theta_{\ell+1}(z, c) = O(c^{\ell+1})$ and $F(z, c) = f(z) + O(c)$. Thus, we have

$$
\begin{aligned}
G(x, y, c) - F(\theta^\ell(x, y, c), \rho(c)) &= \theta_{\ell+1,1}(x, y, c)\frac{\partial f}{\partial x} + \theta_{\ell+1,2}(x, y, c)\frac{\partial f}{\partial y} \\
&+ \rho_{\ell+1,1}x^2 + \rho_{\ell+1,2}y^2 + \rho_{\ell+1,3}x^2y^2 \\
&+ O(c^{\ell+2}) + O(|x, y|^d). \tag{2.12}
\end{aligned}
$$

This equation has a structure similar to the following one

$$
\begin{aligned}
g(x, y, c) &= x\mathcal{Q}_{\ell+1,1}(x, y, c)\frac{\partial f(x, y)}{\partial x} + y\mathcal{Q}_{\ell+1,2}(x, y, c)\frac{\partial f(x, y)}{\partial y} + \\
&+ \mathcal{R}_{\ell+1,1}(c)x^2 + \mathcal{R}_{\ell+1,2}(c)y^2 + \mathcal{R}_{\ell+1,3}(c)x^2y^2. \tag{2.13}
\end{aligned}
$$

We can solve (2.13) by equating the coefficients of the monomials $c^\alpha = c_1^{\alpha_1} \cdots c_s^{\alpha_s}$ left and right, $\alpha_1 + \cdots + \alpha_s = l + 1$. In such a way we have to solve several equations of the form (2.8). Thus, if we are able to solve the infinitesimal stability equation, we can find $\mathcal{Q}_1(x, y, c)$ and $\mathcal{Q}_2(x, y, c)$ solving (2.13). If we take $\theta_{\ell+1,1} = x\mathcal{Q}_{\ell+1,1}$, $\theta_{\ell+1,2} = y\mathcal{Q}_{\ell+1,1}$ and $\rho_{\ell+1,i} = \mathcal{R}_{\ell+1,i}$ we

find θ ad ρ up to order $\ell+1$ in c. In particular we have an explicit expression for the parameter u_i in term of the c_i, that is

$$u_i = \sum_{j=1}^{\ell+1} \mathcal{R}_{j,i} + O(c^{\ell+2}) \quad i = 1, 2, 3.$$

An algorithm to solve the infinitesimal stability equation, the so called *division algorithm* [22] is presented in section 2.5. Using the division algorithm to solve equation (2.12) gives the transformation inducing G_c from F. Namely, the following proposition holds

Proposition 2.5. *Let G be a symmetric polynomial under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action, vanishing at the origin and depending on parameters c_i with central singularity at $c_1 = c_2 = \cdots = 0$ given by $f = \varepsilon_1 x^4 + \mu x^2 y^2 \varepsilon_2 + \varepsilon_2 y^4$, $\varepsilon_i = \pm 1$ for $i = 1, 2$ and $\mu^2 \neq 4\varepsilon_1^2 \varepsilon_2^2$. There exists a diffeomorphisms θ and a reparametrization ρ such that*

$$G(x, y, c) = F(\theta(x, y, c_i), \rho(c_i)) \tag{2.14}$$

with $\theta(x, y, 0) = (x, y), \rho(0) = (0, 0, 0)$ and

$$F(x, y, u) = f(x, y) + u_1 x^2 + u_2 y^2 + u_3 x^2 y^2.$$

To compute Φ modulo $\mathcal{O}(|x, y|^L) + \mathcal{O}(|c_i|^M)$ it suffices to know Gc modulo $\mathcal{O}(|x, y|^{L+3}) + \mathcal{O}(|c_i|^M)$; and to compute ρ modulo $\mathcal{O}(|c_i|^M)$ we have to know Gc modulo $\mathcal{O}(|c_i|^M) + \mathcal{O}(|x, y|^6)$.

For $G = Gc$ as given in (2.5), modulo $\mathcal{O}(|c_i|^2) + \mathcal{O}(|x, y|^5)$, the coordinate transformation θ reads

$$
\begin{aligned}
x \;\rightarrow\; & x + \frac{c_3 \varepsilon_1 x}{4} + \frac{c_6 \varepsilon_1 x^3}{4} + \frac{c_7 x y^2}{4}\varepsilon_1 - \frac{c_6 x y^2 \mu}{8} \\
& + \frac{(2\varepsilon_1 c_7 + 2\varepsilon_2 c_9 - c_6 - 4c_8)xy^2}{8(4\varepsilon_1\varepsilon_2 - \mu^2)}
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
y \;\rightarrow\; & y + \frac{c_5 \varepsilon_2 y}{4} + \frac{\varepsilon_2 c_9 y^3}{4} \\
& + \frac{(2c_8\varepsilon_1 - 4c_7 - 4c_9\varepsilon_1 + 2c_6)x^2 y}{8(4\varepsilon_1\varepsilon_2 - \mu^2)}
\end{aligned}
\tag{2.16}
$$

and, modulo $\mathcal{O}(|c_i|^3)$ the reparametrization ρ is given by

$$u_1 \;=\; c_1 - \frac{c_1 c_3}{2}\varepsilon_1 \tag{2.17}$$

$$u_2 \;=\; c_2 - \frac{c_2 c_5}{2}\varepsilon_2 \tag{2.18}$$

$$
\begin{aligned}
u_3 \;=\;& \frac{1}{2}(c_4 - c_3\varepsilon_1 - c_5\varepsilon_2) - \frac{1}{4\varepsilon_1\varepsilon_2 - \mu^2}\Big[2c_4 c_5\varepsilon_1 + 2c_2 c_7\varepsilon_1 + 2c_3 c_4\varepsilon_2 + 2c_1 c_8\varepsilon_2 \\
&- \left(2c_3 c_5 + 2c_1 c_7 + c_2 c_8 + 3c_5^2\varepsilon_1\varepsilon_2 + 4c_2 c_9\varepsilon_1\varepsilon_2 + c_3^2\varepsilon_1\varepsilon_2 + 4c_1 c_6\varepsilon_1\varepsilon_2\right)\frac{\mu}{2} \\
&- \left(c_3 c_4\varepsilon_1 - c_2 c_6\varepsilon_1 + c_4 c_5\varepsilon_2 - c_1 c_9\varepsilon_2\right)\frac{\mu}{2} \\
&+ \left(3c_3^2 + 2c_1 c_6 + 3c_5^2 - 2c_2 c_9 + 2\varepsilon_1\varepsilon_2 c_3 c_5\right)\frac{\mu^3}{8}\Big].
\end{aligned}
\tag{2.19}
$$

Proof. For $\mu^2 \neq 4\varepsilon_1\varepsilon_2$, since F is a versal deformation of the germ $\varepsilon_1 x^4 + \mu x^2 y^2 + \varepsilon_2 y^4$ the existence of θ and ρ follows trivially.

By applying the iterative procedure described above to compute θ and ρ, at each step we have to solve an equation of type (2.13). This can be done by exploiting the division algorithm described in section (2.5). The fact that Gc is required up to order $L+3$ in order to compute θ only up to degree L is due to the first derivatives of the singularity to be of degree 3. Similarly, in order to fix ρ, it suffices to know Gc up to degree four in (x, y) since the maximum degree of the deformation directions (namely x^2, y^2 and $x^2 y^2$) associated to $\rho_1, \ldots, \rho_3$ is four.

Finally, a little computer algebra gives the transformations (2.15), (2.16) and (2.17), (2.18), (2.19).

2.5 Solving the infinitesimal stability equation

We have seen in the previous section how to construct a transformation inducing (2.5) from the universal deformation (2.7). Our method is based on the hypothesis that we are able to solve the infinitesimal stability equation (2.8) up to a certain order in the variables (x, y). In this section we present an algorithm to solve this equation. We take the basic ideas from [21, 22].

Let us define Σ^Γ the finite-dimensional vector space of Γ-invariant power series on $\mathbb{R}^2$ truncated at order d. We can identify Σ^Γ with the ring R^Γ of symmetric polynomial in two variables of maximum degree d. Let us denote by z^γ a monomial in R^Γ of total degree γ, that is $z^\gamma = x^{\gamma_1} y^{\gamma_2}$, where $1 \leq \gamma_1 + \gamma_2 = \gamma \leq d$. We can choose an ordering $\prec$ for monomials in R such that $z^\alpha \prec z^\beta$ if either the total degree of z^α is smaller than the total degree of z^β, or the degree are equal but z^α precedes z^β in lexicographic ordering. For example $xy \prec y^2$ since $xy \prec yy$.

Definition 2.9. *Let f be a polynomial in R.*

i) $MM(f)$ *is the minimal monomial occurring in f with respect to the monomial ordering described above;*

ii) $MC(f)$ *is the coefficient associated to $MM(f)$;*

iii) $MT(f)$ *is the term associated to $MM(f)$, that is $MT(f) = MC(f)MM(f)$;*

iv) *A monomial z^α is said to divide a monomial z^β if $\beta - \alpha$ is a vector with non negative entries, then $z^\beta / z^\alpha = z^{\beta - \alpha}$.*

If $I = f_1, \ldots, f_j$ is a set of polynomials in R^Γ, we denote by $< I >$ the ideal generated by I in R^Γ. The basic idea of the algorithm is to solve the infinitesimal stability equation (2.8) through several divisions of the polynomial f in the ring R^Γ by the ideal T generated by $\{MM(T_i)\}$, where $\{T_i\}$ is a set of generators of the tangent space to the germ orbit we have described in the previous section. However in general the remainder of such a division is not unique. We need a set of generators for the ideal T which makes the output of such a division unique. This can be done if we choose as a system of generators for T a *Groebner basis* for T with respect to the monomial ordering we have described above [10, 34]. In fact, we recall that a Groebner basis is, by definition, a set of generators for a given ideal I such that multivariate division of any polynomial in the polynomial ring R^Γ gives a unique remainder.

Now, we are ready to present the algorithm.

Division algorithm

Input: integer d, power series f truncated at degree d, $\{g_1, \ldots, g_k\}$ Groebner basis for the ideal T

Output: power series $r, q_1, \ldots, q_k$ truncated at degree d such that

$$f = \sum_{i=1}^{k} q_i g_i + r \ \text{ modulo terms of degree } d \text{ and highter.}$$

Algorithm:

$h \leftarrow g$
Reduce h modulo terms of degree d and higher
$r \leftarrow 0$
$q_i \leftarrow 0$
While $h \neq 0$ do
    If $MM(g_i)|MM(h)$ for some i, then
$q_i \leftarrow q_i + MT(h)/MT(g_i)$
$\leftarrow h - (MT(h)/MT(g_i))g_i$
Reduce h modulo terms of degree d or higher

Else
$r \leftarrow r + MT(h)$
$h \leftarrow h - MT(h)$
End if
End while.

Now we have to keep in mind that we are working in the ring of symmetric polynomials, thus we have to make sure that the output of the division algorithm respects the Γ invariance. In the case we are studying this is easy to check. In fact, if $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ a polynomial in R^Γ must be of even degree both in x and y. On the other hand, if we consider the germ function $g = \varepsilon_1 x^4 + \mu x^2 y^2 + \varepsilon_2 y^4$, we know that the corresponding invariant tangent space T is generated by $\{2x(2\varepsilon_1 x^3 + 2\mu x^2 y^2), 2y(2\mu x^2 y^2 + 2\varepsilon_2 y^3)\}$ and a Groebner basis for the ideal T is $GB = \{4\varepsilon_1 x^4, 2\mu x^2 y^2, y^6\}$ (see e.g. [21]). Thus, at every step the division algorithm is nothing else but a division between monomials of even degree in both variables. This implies that the outputs of the algorithm are necessarily polynomials of even degree both in x and y an so they respect the Γ invariance. In other cases it could be not so easy and the algorithm must be modified.

Chapter 3

Bifurcation curves in the 2:2 resonance

Among low-order resonances (see e.g. [38]) the 1:1 resonance plays a prominent role. In the following we consider a Hamiltonian system describing a *detuned* $1 : 1$ resonance invariant under $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry (in this case we speak of a $2 : 2$ resonance, to keep in mind the presence of symmetry). The purpose of this chapter is to find *quantitative* predictions for the bifurcations of periodic orbits in general position (namely loop and inclined obits) from the normal modes. They are given in section 3.4 where we provide *energy threshold values* (depending on physical coefficients and on the detuning parameter) which determine the bifurcations of the system.

We start with the construction of a $2 : 2$ detuned normal form, exploiting the algorithm described in section 1.3. Afterwards, the approach is based on the use of a regular reduction [36] dividing out the $\mathbb{S}^1$ symmetry of the normal form. The reduced system lives in one degree of freedom, this allows us to classify the dynamics with singularity theory [23, 21], exploiting the results described in the previous chapter. Actually we have to respect the symmetries and reversibility of the system, thus we are led into the framework of equivariant singularity theory. What is new in the current approach is that the simplifying transformations, which induce the system from its versal deformation, are *computed* explicitly. If we truncate the normalization procedure to the minimal order required, (i.e. $N = 1$), the planar system turns out to be already reduced to the versal unfolding (2.7). Truncating at order $N > 1$, this is not true anymore and we need the algorithms described in section 2.4.

3.1 Normal form for a 2:2 detuned resonance

Let us consider a natural two degree of freedom system whose Hamiltonian is given by

$$\mathcal{H}(\mathbf{p}, \mathbf{x}) = \frac{1}{2}(p_1^2 + p_2^2) + \mathcal{V}(x_1^2, x_2^2) \tag{3.1}$$

where we assume the potential to be a smooth function with an isolated relative minimum in the origin and symmetric under reflection with respect to both coordinate axes. Thus, the Hamiltonian (3.1) exhibits two $\mathbb{Z}_2$ symmetries in space, denoted by S_1 and S_2:

$$S_1 \ : \ (x_1, x_2, p_1, p_2) \rightarrow (-x_1, x_2, -p_1, p_2) \tag{3.2}$$

$$S_2 \ : \ (x_1, x_2, p_1, p_2) \rightarrow (x_1, -x_2, p_1, -p_2) \tag{3.3}$$

and a time reversible symmetry, denoted by T

$$T \ : \ (x_1, x_2, p_1, p_2) \rightarrow (x_1, x_2, -p_1, -p_2). \tag{3.4}$$

Let us note that the Hamiltonian function (3.1) is invariant also under other transformations, such as reflections acting on the $\mathbf{x}$ and not on the $\mathbf{p}$ and viceversa. Our choice to consider reflection symmetries (3.2) and (3.3) lies in the Lagrangian description of the system, i.e. in the relation between the x and the p variables in a non- Hamiltonian description.

By expanding the potential around the origin and retaining only terms up to degree $2(N+1)$ we get

$$\mathcal{V}(x_1^2, x_2^2) \equiv \sum_{n=0}^{N} V_{2n}(x_1^2, x_2^2) \tag{3.5}$$

where V_{2n} is a homogeneous polynomial of degree $2n+2$ in (x_1, x_2), namely

$$V_{2n} = \sum_{j=0}^{n+1} a_{2j,2(n+1-j)} x_1^{2j} x_2^{2(n+1-j)}. \tag{3.6}$$

Notice that the odd degree terms are all zero in force of the reflection symmetries. The truncation order is determined by the problem under study. Since we aim to apply the results presented in the previous chapter, we truncate at order $N = 2$ (i.e. including terms up to the sixth degree), so that we consider a generic perturbation of the basic resonant normal form. Higher order terms can be treated in the same way. The zero order term in the expansion is given by

$$V_0 = \frac{1}{2}(a_{2,0} x_1^2 + a_{0,2} x_2^2). \tag{3.7}$$

Therefore, the system is naturally apt be treated in a perturbative way as a non-linear oscillator system. The two unperturbed frequencies are given by $\omega_1 := \sqrt{a_{2,0}}$ and $\omega_2 := \sqrt{a_{0,2}}$. We perform the scaling [45]

$$(x_1, x_2, p_1, p_2) \rightarrow \varepsilon^{-1}(x_1, x_2, p_1, p_2), \quad \varepsilon > 0 \tag{3.8}$$

and also rescale the Hamiltonian (3.1) according to

$$\mathcal{H} = \varepsilon^2 \widetilde{\mathcal{H}}. \tag{3.9}$$

Thus, we obtain

$$\widetilde{\mathcal{H}}(\mathbf{p}, \mathbf{x}) = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(\omega_1^2 x_1^2 + \omega_2^2 x_2^2) + \sum_{j=1}^{N} \varepsilon^{2j} V_{2j}(x_1^2, x_2^2). \tag{3.10}$$

In general, neither the original system (3.1) nor its truncated expansion (3.10) are integrable. However, in several cases the dynamics around the equilibrium are *regular*, namely their features are indistinguishable from those of an integrable system in a large fraction of phase space [65]. Therefore we proceed to construct a normal form for the system, namely a new Hamiltonian series which, in the case of 2 degrees of freedom, is an integrable approximation of the original one. As we have observed in section 1.3, the two harmonic frequencies ω_1 and ω_2 are generically non-commensurable. However even if the unperturbed system is non-resonant with a certain real value

$$\rho = \omega_1/\omega_2 \tag{3.11}$$

of the frequency ratio, the non-linear coupling between the degrees of freedom induced by the perturbation determines a "passage through resonance" with a commensurability ratio, say m_1/m_2 with $m_1, m_2 \in \mathbb{N}$, corresponding to the local ratio of oscillations in the two degrees of freedom. This in turn is responsible of the birth of new orbit families bifurcating from the normal modes or from lower-order resonances [9, 29, 32, 101]. In the following we are interested in the bifurcations of orbit families in general positions in systems around a $1:1$ symmetric resonance. Therefore we assume that the frequency ratio (3.11) is not far from 1 and then approximate it by introducing a small detuning δ, which we assume of order two in the perturbation, so that

$$\omega_1 = (1 + \tilde{\delta}\varepsilon^2)\omega_2, \quad \text{with} \quad \delta = \tilde{\delta}\varepsilon^2. \tag{3.12}$$

Hence we proceed as if the unperturbed harmonic part would be in exact 1:1 resonance by including the remaining part in the perturbation.

After a scaling transformation

$$x_1 \rightarrow \sqrt{\omega_1}x_1, \quad x_2 \rightarrow \sqrt{\omega_2}x_2, \quad p_1 \rightarrow \frac{p_1}{\sqrt{\omega_1}}, \quad p_2 \rightarrow \frac{p_2}{\sqrt{\omega_2}}, \tag{3.13}$$

and a scaling of time

$$t \longrightarrow \omega_2 t, \quad \text{so that} \quad \tilde{\mathcal{H}} = \omega_2 H, \tag{3.14}$$

the Hamiltonian (3.1) takes the form

$$H(\mathbf{p}, \mathbf{x}) = \sum_{j=0}^{N} \varepsilon^{2j} H_{2j}(\mathbf{p}, \mathbf{x}) \tag{3.15}$$

where

$$H_0 = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(x_1^2 + x_2^2) \tag{3.16}$$

$$H_2 = \frac{1}{2}\tilde{\delta}(x_1^2 + p_1^2) + 2\left(\frac{1}{3}(ax_1^4 + bx_2^4) + cx_1^2 x_2^2\right) \tag{3.17}$$

$$H_4 = -\frac{4}{3}ax_1^4\tilde{\delta} - 2cx_1^2 x_2^2\tilde{\delta} + d_1 x_1^6 + d_2 x_1^4 x_2^2 + d_3 x_1^2 x_2^4 + d_4 x_2^6 \tag{3.18}$$

with

$$a = \frac{3a_{4,0}}{2\omega_2^3}, \quad b = \frac{3a_{0,4}}{2\omega_2^3}, \quad c = \frac{a_{2,2}}{2\omega_2^3} \tag{3.19}$$

$$d_1 = \frac{a_{6,0}}{\omega_2^4}, \quad d_2 = \frac{a_{4,2}}{\omega_2^4}, \quad d_3 = \frac{a_{2,4}}{\omega_2^4}, \quad d_4 = \frac{a_{0,6}}{\omega_2^4}. \tag{3.20}$$

Now, we can proceed according to the normalization procedure described is section 1.3. Since we are dealing with a 2 : 2 resonance the normalization must be pushed at least to the second order in ε. Here we give the normal form obtained truncating to the fourth order ($N = 2$):

$$
\begin{aligned}
K_{AA}(J_1, J_2, \phi_1, \phi_2) = {}& J_1 + J_2 + \Big[\tilde{\delta}J_1 + aJ_1^2 + bJ_2^2 + 2cJ_1 J_2 \\
& + c\cos[2(\phi_1 - \phi_2)]\big]\,\varepsilon^2 + \Big[\left(\frac{5d_1}{2} - \frac{17a^2}{9}\right)J_1^3 \\
& - \left(\frac{17b^2}{9} - \frac{5d_4}{2}\right)J_2^3 - \left(4ac + \frac{9c^2}{4} - \frac{3d_2}{2}\right)J_1^2 J_2 \\
& - \left(4bc + \frac{9c^2}{4} - \frac{3d_3}{2}\right)J_1 J_2^2 - 2J_1(aJ_1 + cJ_2)\tilde{\delta} \\
& - \frac{1}{2}J_1 J_2\left(5acJ_1 + 6c^2(J_1 + J_2) - 3(d_2 J_1 + d_3 J_2)\right. \\
& + \left. +c(5bJ_2 + 3\tilde{\delta})\right)\cos(2\phi_1 - 2\phi_2)\Big]\,\varepsilon^4 \tag{3.21}
\end{aligned}
$$

where the action-angle(–like) variables have been introduced according to (1.49). We remark that in the computation of (3.21) and results thereof, the use of algebraic manipulators like Mathematica®is practically indispensable. This Hamiltonian is in normal form with respect to the quadratic unperturbed part H_0 that in these coordinates reads

$$H_0^{AA} = J_1 + J_2. \tag{3.22}$$

3.2 The planar system

After the normalization, the system has acquired an additional (formal) $\mathbb{S}^1$ symmetry. The corresponding conserved quantity is given by $H_0^{A\hat{A}} = J_1 + J_2$. This enable us to formally reduce (3.21) to a planar system.
We perform the following canonical transformation

$$
\begin{cases}
J_1 = J \\
J_2 = \mathcal{E} - J \\
\psi = \phi_1 - \phi_2 \\
\chi = \phi_2
\end{cases}
\tag{3.23}
$$

Since χ is cyclic and its conjugate action $\mathcal{E}$ is the additional integral of motion, we introduce the effective Hamiltonian

$$
\begin{aligned}
\mathcal{K} &= K_{A\hat{A}}(J, \psi; \mathcal{E}, \tilde{\delta}, \varepsilon) \\
&= \mathcal{E} + \left[\mathcal{A}(J; \mathcal{E}, \tilde{\delta}) + \mathcal{B}(J; \mathcal{E}, \tilde{\delta}) \cos 2\psi \right] \varepsilon^2 \\
&+ \left[\mathcal{C}(J; \mathcal{E}, \tilde{\delta}) + \mathcal{D}(J; \mathcal{E}, \tilde{\delta}) \cos 2\psi \right] \varepsilon^4
\end{aligned}
\tag{3.24}
$$

where

$$
\begin{aligned}
\mathcal{A}(J; \mathcal{E}, \tilde{\delta}) &= (2A - B)\mathcal{E}^2 + (-4A + 2B + 8C)\,\mathcal{E}J \\
&+ \tilde{\delta}J + 4(A - 2C)J^2
\end{aligned}
\tag{3.25}
$$

$$
\mathcal{B}(J; \mathcal{E}, \tilde{\delta}) = 4CJ(\mathcal{E} - J)
\tag{3.26}
$$

$$
\begin{aligned}
\mathcal{C}(\mathcal{R}; \mathcal{E}, \tilde{\delta}) &= \frac{1}{9}\left(68AB - 68A^2 - 17B^2 + \frac{45}{2}d_4 \right)\mathcal{E}^3 - 8C\tilde{\delta}\mathcal{E}J \\
&+ \left(+\frac{68AB}{3} - \frac{68A^2}{3} - \frac{17B^2}{3} + 32AC - 48BC + 36C^2 \right. \\
&+ \left. \frac{3d_2}{2} - 3d_3 + \frac{15d_4}{2} \right)\mathcal{E}J^2 + 2(4C - 2A - B)\tilde{\delta}J^2 \\
&+ \left(32BC + \frac{5d_1}{2} - \frac{136AB}{9} - \frac{3d_2}{2} + \frac{3d_3}{2} - \frac{5d_4}{2} \right)J^3
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
\mathcal{D}(\mathcal{R}; \mathcal{E}, \tilde{\delta}) &= -4C\tilde{\delta}\mathcal{E}J + \left(\frac{40}{3}AC - 20BC + 32C^2 + d_2 - 2d_3 \right)\mathcal{E}J^2 \\
&+ 4C\tilde{\delta}J^2 + \left(\frac{40}{3}BC - d_2 + d_3 \right)J^3
\end{aligned}
\tag{3.28}
$$

with

$$
A = \frac{a+b}{4}, \quad B = \frac{a-b}{2}, \quad C = \frac{c}{4}.
\tag{3.29}
$$

Thus, we get a one degree of freedom system; in the following we refer to it as 1 DOF system. We now perform a further reduction into a planar system, viewing $\mathcal{E}$ as a *distinguished* parameter [21, 23].

Remark 3.1 (The distinguished parameter). *Calling $\mathcal{E}$ a parameter is justified if we consider the system at low energy, so $\tilde{\mathcal{K}}$ and therefore $\mathcal{E}$ is small. The adjective distinguished refers to the fact that $\mathcal{E}$ stems from the phase space of $\mathcal{K}$ and is a parameter only for the 1 DOF system, not for the original one.*

The planar reduction is obtained via the coordinate transformation

$$\begin{cases} x = \sqrt{2J}\cos\psi \\ y = \sqrt{2J}\sin\psi \end{cases} \tag{3.30}$$

Inverting the scalings (3.8) and (3.9), the Hamiltonian function $\mathcal{K}$ is converted into the planar Hamiltonian

$$\begin{aligned} \tilde{\mathcal{K}} &= \mathcal{K}(x,y;\mathcal{E},\delta) \\ &= \sum_{i=0}^{3}\sum_{j=0}^{3-i} c_{2i,2j} x^{2i} y^{2j} \end{aligned} \tag{3.31}$$

where $c_{2i,2j} = c_{2i,2j}(\mathcal{E},\delta)$.

Remark 3.2 (Singular circle). *The coordinate transformation (1.49) is singular at the coordinate axes $J_1 = 0$ and $J_2 = 0$. After the transformation (3.23), these axes become $J = 0$ and $J = \mathcal{E}$ respectively. The first singularity can be removed by returning to cartesian coordinates in the plane. The second singularity is called singular circle. In cartesian coordinate this circle is given by*

$$x^2 + y^2 = 2\mathcal{E}. \tag{3.32}$$

At this circle $J_2 = 0$ so that the coordinate ϕ_2 is ill defined and therefore so is ψ. In particular, this implies that $\mathcal{K}$ is constant on this circle.

3.2.1 Reduction to the central singularity

Since the system is planar now, we may use general ($\mathbb{Z}_2 \times \mathbb{Z}_2$- equivariant) planar transformations for further reductions.

At this point the system depends on a distinguished parameter $\mathcal{E}$, a detuning parameter δ and several ordinary coefficients. Recall that parameters are supposed to be *small*. We now look at the "degenerate" Hamiltonian that results when $\delta = 0$ (resonance) and $\mathcal{E} = 0$ (the diameter of the singular circle vanishes). This is called the *central singularity*, also known as the *organizing center*.

At the singular values of the parameters we have that $\tilde{\mathcal{K}}$ reduces to

$$K_s(x,y) := \mathcal{K}|_{\delta=0,\mathcal{E}=0}(x,y) = s_{4,0}x^4 + s_{2,2}x^2y^2 + s_{0,4}y^4 + h.o.t, \tag{3.33}$$

where $s_{i,j} = c_{i,j}(0,0)$. In particular

$$s_{0,4} = A - 3C, \quad s_{2,2} = 2(A - 2C), \quad s_{0,4} = A - C. \tag{3.34}$$

The constant term $c_{0,0}$ can be neglected and, by a simple scaling transformation, K_s can be turned into

$$K_s'(x,y) = \varepsilon_1 x^4 + \mu x^2 y^2 + \varepsilon_2 y^4 + h.o.t, \quad \varepsilon_1, \varepsilon_2 \in \{-1, 1\} \tag{3.35}$$

where

$$\mu = \frac{2(A - 2C)}{\sqrt{|(A - 3C)(A - C)|}}. \tag{3.36}$$

Remark 3.3 (Non degeneracy conditions). *This is possible provided that the coefficients of x^4 and y^4 in K_s are not zero. This translates into the non degeneracy conditions*

$$A - 3C \neq 0 \quad and \quad A - C \neq 0. \tag{3.37}$$

The signs of ε_1 and ε_2 are determined by the sign of $A - 3C$ and $A - C$ respectively.

In this section, we look for a near identity coordinate transformation Φ which brings the system at the central singularity into the polynomial form

$$f(x,y) = \varepsilon_1 x^4 + \mu x^2 y^2 + \varepsilon_2 y^4. \tag{3.38}$$

This morphisms has to respect the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry $(x, y) \to (\pm x, \pm y)$. As a consequence of proposition 2.4, such a transformation exists provided that

$$\mu^2 \neq 4\varepsilon_1 \varepsilon_2. \tag{3.39}$$

For our system this condition is equivalent to require that $C \neq 0$. Making this assumption, we are able to compute Φ using the iterative procedure described in [21], adapted to our symmetric context.
We set $\Phi_1^{(1)}(x, y) = x$, $\Phi_1^{(2)}(x, y) = y$ and assume that for some k

$$K's \circ \Phi_k = \varepsilon_1 x^4 + \mu x^2 y^2 + \varepsilon_2 y^4 + O(|x, y|^{2(k+2)}).$$

Then we set

$$\Phi_{k+1}^{(1)} = \Phi_k^{(1)} + \sum_i \alpha_i^{(1)} P_i^{(1)} \tag{3.40}$$

$$\Phi_{k+1}^{(2)} = \Phi_k^{(1)} + \sum_i \alpha_i^{(2)} P_i^{(2)} \tag{3.41}$$

where $\{P_i^{(1)}\}$, $\{P_i^{(2)}\}$ span the space of two variables monomials of degree $k + 1$, invariant under the $\mathbb{Z}_2$ actions $(x, y) \to (x, \pm y)$ and $(x, y) \to (\pm x, y)$ respectively. The coefficients $\alpha_i^{(j)}$ are to be found in order to cancel the terms of order $O(|x, y|^{2(k+2)})$ in $K's$. This translates into a set of linear equations for the real numbers $\alpha_i^{(j)}$. By the existence of the reducing transformation, this set of equations can always be solved if (3.39) is satisfied.

Remark 3.4. *Let us note that the transformation Φ defined above (and the following θ of proposition 3.2) is in general non canonical. The resulting system is not conjugate but* equivalent *to the original one. In particular, the reducing transformations we perform here and in the following do not alter the bifurcation sequence of the system. This will allow us to obtain, from function (3.44) (which is simpler than (3.31) and depends on the minimal number of parameters), the critical energy values which determine the bifurcations of the original system.*

If we compute Φ up to order 2 in k we get the following proposition

Proposition 3.1. *Let us consider the planar Hamiltonian $\tilde{\mathcal{K}}$. If $C \neq A$, $C \neq A/3$ and $C \neq 0$, there exists a coordinate transformation $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $K_b := \tilde{\mathcal{K}} \circ \Phi$ is of the form*

$$
\begin{aligned}
K_b(x,y,\mathcal{E}) &= (a_1\mathcal{E} + a_2\mathcal{E}^2 + b_1\mathcal{E} + b_2)x^2 + (a_3\mathcal{E} + a_4\mathcal{E}^2 + b_3\mathcal{E} + b_4)y^2 \\
&+ (\varepsilon_1 + a_5\mathcal{E} + b_5)x^4 + (\mu + a_6\mathcal{E} + b_6)x^2y^2 \\
&+ (\varepsilon_2 + a_7\mathcal{E} + b_7)y^4 + O(|x,y|^6)
\end{aligned} \tag{3.42}
$$

where the a_i are coefficients, $\mathcal{E}$ and the b_i are parameters linearly depending on δ and vanishing at $\delta = 0$. They are listed in appendix A. Neglecting terms of $O(|x,y|^5)$ the following is a suitable transformation Φ:

$$
\begin{aligned}
x &\to \quad x + \frac{\varepsilon_1(2\varepsilon_2 s_{2,4} - \mu s_{0,6})\mu + 2\mu s_{6,0} - 4\varepsilon_1 s_{4,2}}{4\varepsilon_1\varepsilon_2(4\varepsilon_1\varepsilon_2 - \mu^2)}xy^2 - \frac{s_{6,0}\varepsilon_1}{4}x^3 \\
y &\to \quad y + \frac{\varepsilon_2(+2\varepsilon_1 s_{4,2} - \mu s_{6,0})\mu + 2\mu s_{0,6} - 4\varepsilon_2 s_{2,4}}{4\varepsilon_1\varepsilon_2(4\varepsilon_1\varepsilon_2 - \mu^2)}x^2y - \frac{s_{0,6}\varepsilon_2}{4}y^3
\end{aligned} \tag{3.43}
$$

Proof. The existence of Φ is a consequence of proposition 2.4. The conditions on C are consequence of the non degeneracy conditions (cfr. remark 3.3 and of condition (3.39)). The explicit expression of the transformation up to and including terms of $O(|x,y|^3)$ has been obtained by exploiting the algorithm described above up to $k = 2$.

$\square$

3.3 Bifurcation curves

From section 2.2 we know that, if condition (3.39) is satisfied,

$$
F(x,y,u_1,u_2,u_3) = \varepsilon_1 x^4 + (\mu + u_3)x^2y^2 + \varepsilon_2 y^4 + u_1 x^2 + u_2 y^2 \tag{3.44}
$$

is a universal deformation of $f = \varepsilon_1 x^4 + \mu x^2 y^2 + \varepsilon_2 y^4$. Therefore, there exists a coordinate transformation which induces K_b from F. Such a transformation can be found by exploiting the algorithm described in section 2.4. Namely, the following proposition holds:

Proposition 3.2. *Let K_b be as in (3.42) with central singularity at $\mathcal{E} = b_1 = b_2 = \cdots = 0$ given by $f = \varepsilon_1 x^4 + \mu x^2 y^2 \varepsilon_2 + \varepsilon_2 y^4$, $\varepsilon_i = \pm 1$ for $i = 1, 2$ and $\mu^2 \neq 4\varepsilon_1^2 \varepsilon_2^2$. There exists a diffeomorphisms θ and a reparametrization ρ such that*

$$K_b(x, y, \mathcal{E}) = F(\theta(x, y, \mathcal{E}, b_i), \rho(\mathcal{E}, b_i)) \tag{3.45}$$

with $\theta(x, y, 0) = (x, y)$, $\rho(0, \ldots, 0) = (0, 0, 0)$ and

$$F(x, y, u) = f(x, y) + u_1 x^2 + u_2 y^2 + u_3 x^2 y^2.$$

Modulo $\mathcal{O}(|\mathcal{E}, b_i|^3) + \mathcal{O}(|x, y|^3)$, the coordinate transformation θ reads

$$x \quad \to \quad x + \varepsilon_1 \frac{b_5 x}{4} + \varepsilon_1 \frac{a_5 \mathcal{E} x}{4} - \frac{3b_5^2 x}{32} - \frac{3a_5 b_5 \mathcal{E} x}{16} - \frac{3a_5^2 \mathcal{E}^2 x}{32} \tag{3.46}$$

$$y \quad \to \quad y + \varepsilon_2 \frac{b_7 y}{4} + \varepsilon_2 \frac{a_7 \mathcal{E} y}{4} - \frac{3b_7^2 y}{32} - \frac{3a_7 b_7 \mathcal{E} y}{16} - \frac{3a_7^2 \mathcal{E}^2 y}{32} \tag{3.47}$$

and, modulo $\mathcal{O}(|\mathcal{E}, b_i|^3)$ the reparametrization ρ is given by

$$u_1 = b_2 + \left(a_1 + b_1 - \varepsilon_1 \frac{a_5 b_2}{2} - \varepsilon_2 \frac{a_1 b_5}{2} \right) \mathcal{E} - \varepsilon_1 \frac{b_2 b_5}{2} + \left(a_2 - \varepsilon_1 \frac{a_1 a_5}{2} \right) \mathcal{E}^2 \tag{3.48}$$

$$u_2 = b_4 + \left(a_3 + b_3 - \varepsilon_2 \frac{a_7 b_4}{2} - \varepsilon_2 \frac{a_3 b_7}{2} \right) \mathcal{E} - \varepsilon_2 \frac{b_4 b_7}{2} + \left(a_4 - \varepsilon_2 \frac{a_3 a_7}{2} \right) \mathcal{E}^2 \tag{3.49}$$

$$u_3 = b_6 - \varepsilon_1 \frac{b_5 b_6}{2} - \varepsilon_2 \frac{b_6 b_7}{2} + \frac{1}{2} \left(\frac{3b_5^2}{4} - \varepsilon_1 b_5 + \frac{3b_7^2}{4} - \varepsilon_2 b_7 + \varepsilon_1 \varepsilon_2 \frac{b_5 b_7}{2} \right) \mu$$

$$+ \left[a_6 - \varepsilon_1 \frac{a_6 b_5}{2} - \varepsilon_1 \frac{a_5 b_6}{2} - \varepsilon_2 \frac{a_7 b_6}{2} - \varepsilon_2 \frac{a_6 b_7}{2} \right.$$

$$+ \left. \left(\frac{3a_5 b_5}{8} - \varepsilon_1 \frac{a_5}{2} + \frac{3a_7 b_7}{8} - \varepsilon_2 \frac{a_7}{2} + \varepsilon_1 \varepsilon_2 \frac{a_7 b_5}{4} + \varepsilon_1 \varepsilon_2 \frac{a_5 b_7}{4} \right) \mu \right] \mathcal{E}$$

$$- \left[\varepsilon_1 \frac{a_5 a_6}{2} + \varepsilon_2 \frac{a_6 a_7}{2} - \left(\frac{3a_5^2}{8} - \frac{3a_7^2}{8} - \varepsilon_1 \varepsilon_2 \frac{a_5 a_7}{4} \right) \mu \right] \mathcal{E}^2. \tag{3.50}$$

Proof. The existence of the transformation follows from proposition 2.5. From proposition 2.5 we also know that the computation can be done up to and including terms of the first order in (x, y) and the second order in the parameters $(\mathcal{E}, b_i)$ for θ and up to and including $O(|\mathcal{E}, b_i|^2)$ for ρ.

To find the explicit expression of the transformations we exploit the algorithm we described in section 2.4, setting $c_1 = \mathcal{E}$ and $c_{i+1} = b_i$. At each step of the procedure we have to solve the infinitesimal stability equation. This has been done implementing the division algorithm presented in section 2.5.

$\square$

We now examine the possible combinations of the signs of ε_1 and ε_2.

3.3.1 $\varepsilon_1\varepsilon_2 = 1$

The fixed points of the function (3.44) are given by

$$(0,0), \quad \left(\pm\sqrt{\frac{-\varepsilon_1 u_1}{2}}, 0\right), \quad \left(0, \pm\sqrt{\frac{-\varepsilon_2 u_2}{2}}\right), \tag{3.51}$$

$$\left(\pm\frac{\sqrt{-u_2 - \frac{\varepsilon_2(-\alpha u_1 + 2\varepsilon_1 u_2)}{\alpha^2 - 4\varepsilon_1\varepsilon_2}}}{\sqrt{\alpha}}, \pm\frac{\sqrt{-\alpha u_1 + 2\varepsilon_1 u_2}}{\sqrt{\alpha^2 - 4\varepsilon_1\varepsilon_2}}\right) \tag{3.52}$$

where $\alpha = \mu + u_3$. In case $\varepsilon_1\varepsilon_2 = 1$ the corresponding bifurcation curves in the parameters space are given by $u_1 = 0$, $u_2 = 0$, $2u_2 + \alpha u_1 = 0$ and $\alpha u_2 + 2u_1 = 0$. Using the expression of the parameters u_i found in (3.48), (3.49) and (3.50) we are able to express these bifurcation curves in terms of the original parameters of the reduced system, i.e the detuning parameter δ and the distinguished parameter $\mathcal{E}$. Namely, we have the following proposition

Proposition 3.3. *In the planar system K_b of proposition 3.1 bifurcations occur along the following curves in the $(\delta, \mathcal{E})$ plane:*

$$
\begin{aligned}
\mathcal{E} \;=\; \mathcal{E}_{1u} &:= \frac{\delta}{2(2A - B - 6C)} \\
&+ \frac{\left(136A^2 + 34B^2 + 280BC + 456C^2 - 8A(17B + 70C) + 15d_3 - 45d_4\right)\delta^2}{48(2A - B - 6C)^3}
\end{aligned}
\tag{3.53}
$$

$$
\begin{aligned}
\mathcal{E} \;=\; \mathcal{E}_{1l} &:= \frac{\delta}{2(2A - B - 2C)} \\
&+ \frac{\left(136A^2 + 34B^2 + 104BC + 72C^2 - 8A(17B + 26C) + 3d_3 - 45d_4\right)\delta^2}{48(2A - B - 2C)^3}
\end{aligned}
\tag{3.54}
$$

$$
\begin{aligned}
\mathcal{E} \;=\; \mathcal{E}_{2u} &:= -\frac{\delta}{2(2A + B - 6C)} \\
&- \frac{\left(56A^2 + 14B^2 + 8A(7B - 74C) - 296BC + 1272C^2 + 45d_1 - 15d_2\right)\delta^2}{48(2A + B - 6C)^3}
\end{aligned}
\tag{3.55}
$$

$$
\begin{aligned}
\mathcal{E} \;=\; \mathcal{E}_{2l} &:= -\frac{\delta}{2(2A + B - 2C)} \\
&- \frac{\left(56A^2 + 14B^2 + 8A(7B - 22C) - 88BC + 120C^2 + 45d1 - 3d2\right)\delta^2}{48(2A + B - 2C)^3}
\end{aligned}
\tag{3.56}
$$

where terms $O(\delta^3)$ are neglected.

Remark 3.5. *The fixed points of the planar system K_b correspond to fixed points for the 1 DOF Hamiltonian $\mathcal{K}$ only if they occur into the singular circle, cfr remark 3.2.*

Moreover, the distinguished parameter $\mathcal{E}$ is non negative, therefore the previous curves determine bifurcations for the 1 DOF system defined by $\mathcal{K}$

only for those values of the coefficients and of the detuning parameter which makes (at least) the first order terms non negatives.

In the following we clarify how the bifurcation curves given in proposition 3.3 have to be interpreted in terms of the 1 DOF system.

$\varepsilon_1 = \varepsilon_2 = -1$

To fix the ideas, let us consider the case $C > 0$ and $\varepsilon_1 = \varepsilon_2 = -1$, which corresponds to $A - 3C < A - C < 0$, and let us assume that the detuning parameter is non positive.

Remark 3.6. *Notice that there is no loss of generality in assuming $\delta \leq 0$ (i.e. $\omega_1 \leq \omega_2$). If in the original phase space we exchange the axes, namely we perform the transformation*

$$AE : \quad x_1 \to x_2, \quad x_2 \to x_1, \quad p_1 \to p_2, \quad p_2 \to p_1 \qquad (3.57)$$

the Hamiltonian (3.1) takes the form

$$\tilde{\mathcal{H}} = \frac{1}{2}(\omega_2 x_1^2 + \omega_1 x_2^2 + p_1^2 + p_2^2) + a_{04} x_1^4 + a_{22} x_1^2 x_2^2 + a_{40} x_2^4.$$

The detuning parameter becomes $\delta = \frac{\omega_2}{\omega_1} - 1$, which is opposite in sign with respect to δ. Thus, applying transformation (3.57), the case $\delta > 0$ can be treated straightly from $\delta < 0$.

In this case the deformation F turns into

$$\tilde{F}(x, y) = -x^4 + \mu x^2 y^2 - y^4 + u_1 x^2 + u_2 y^2 + u_3 x^2 y^2. \qquad (3.58)$$

The critical points of the planar system are therefore given by $(0, 0)$,

$$\left(\pm \sqrt{\frac{u_1}{2}}, 0 \right), \ (0, \pm \sqrt{\frac{u_2}{2}}) \text{ and } \left(\pm \frac{\sqrt{u_1 + \frac{\alpha(-\alpha u_1 - 2u_2)}{-4 + \alpha^2}}}{\sqrt{2}}, \pm \frac{\sqrt{-\alpha u_1 - 2u_2}}{\sqrt{-4 + \alpha^2}} \right).$$

The fixed points

$$\left(\pm \sqrt{\frac{u_1}{2}}, 0 \right) \text{ and } \left(0, \pm \sqrt{\frac{u_2}{2}} \right) \qquad (3.59)$$

bifurcate from the origin when $u_1 = 0$ and $u_2 = 0$. These critical values of the unfolding parameters respectively determine the bifurcation curves (3.53) and (3.54). For $C > 0$ and $\delta \leq 0$, these critical values correspond to physical acceptable values if respectively $B > 2(A - 3C)$ and $B > 2(A - C)$.

Furthermore, for $\mathcal{E} \approx 0$, both u_1 and u_2 are negative and $\mathcal{E}_{1u} < \mathcal{E}_{1l}$. Thus, the bifurcations of fixed points (3.59) occur according to the diagram given in figure 3.1, from frame 1 to 3. The gray zone corresponds to non acceptable values of the parameters.
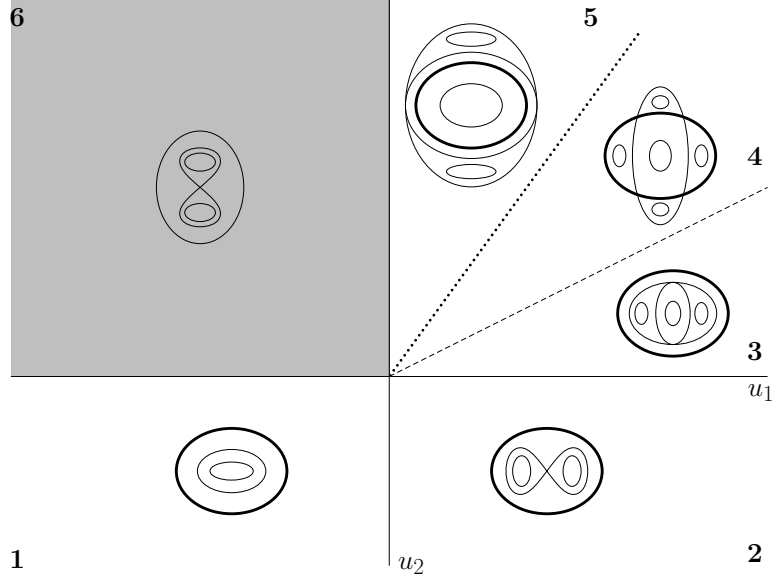
Figure 3.1: Bifurcation diagram in case $A - 3C < A - C < 0$ and $\delta < 0$.

Finally, we have to discuss the critical points

$$\left(\pm \frac{\sqrt{u_1 + \frac{\alpha(-\alpha u_1 - 2u_2)}{-4+\alpha^2}}}{\sqrt{2}}, \pm \frac{\sqrt{-\alpha u_1 - 2u_2}}{\sqrt{-4+\alpha^2}} \right). \tag{3.60}$$

which determine the bifurcation lines (dashed and dotted lines in figure 3.1 respectively)

$$\alpha u_2 = -2u_1 \quad \text{and} \quad 2u_2 = -\alpha u_1. \tag{3.61}$$

The expressions of these critical curves in the $(\delta, \mathcal{E})$ plane are respectively given in (3.55) and (3.56).

Remark 3.7. *The reduced system comes from a normalization procedure truncated to the fourth order in ε, in which both $\mathcal{E}$ and δ are assumed to be of second order. Therefore, in the computation of (3.55) and (3.56) from (3.61) we retain in α only terms $O(|\mathcal{E}, \delta|)$, since α has to multiply $x^2 y^2$ which is a fourth order term.*

The critical curves (3.55) and (3.56) correspond to acceptable values for $B > 2(C - A)$ and $B > 2(3C - A)$. However, a little computer algebra shows that the critical points (3.60) fall on the singular circle (3.32) (frame 4 in picture 3.1; the marked circle represents the singular circle), therefore in

correspondence of these points the coordinate transformation (3.30) is not invertible. On the other hand, the fixed points (3.59) could fall on the limit circle, too. At first order in the deformation parameters, this happens for

$$\frac{u_1}{\sqrt{3C - A}} = 4\mathcal{E} \quad \text{and} \quad \frac{u_2}{\sqrt{C - A}} = 4\mathcal{E}. \qquad (3.62)$$

Solving equations (3.62) gives the first order term in the detuning parameter of expressions (3.55) and (3.56). This suggests that the critical curves (3.55) and (3.56) do not determine the bifurcation of new fixed points for the reduced system defined by (3.24), but rather the disappearance of fixed points (3.59). To verify this statement, we operate a different planar reduction, according to

$$\begin{cases} x = \sqrt{2(\mathcal{E} - J)} \cos \psi \\ y = \sqrt{2(\mathcal{E} - J)} \sin \psi \end{cases} \qquad (3.63)$$

In these coordinates the singularity at $J_2 = 0$ is removed and we have a singular circle for $J_1 = 0$.

Proceeding as in the previous section we achieve the universal deformation

$$\mathcal{F}' = -x^4 + (\mu + u_3')x^2y^2 - y^4 + u_1'x^2 + u_2'y^2, \quad \varepsilon_i = \pm 1 \quad i = 1, 2. \quad (3.64)$$

where the expressions of the deformation parameters are still determined by proposition 3.2, but the values of coefficients a_i and parameters b_i have changed according to (3.63). They are listed in appendix A.

The bifurcation diagram of (3.64) in the (u_1', u_2') plane is still given by figure 3.1. However, since both u_1' and u_2' turn out to be positive for $\delta \leq 0$ and $\mathcal{E} \approx 0$, in the (u_1', u_2') plane the bifurcation diagram should be read clockwise from 3 to 1. Solving $u_1' = 0$ and $u_2' = 0$ we find the critical curves (3.55) and (3.56), which therefore must determine the disappearance of fixed points (3.59) for the reduced Hamiltonian (3.24), as we claimed.

Remark 3.8. *The bifurcation analysis of the reduced system has been deduced assuming $C > 0$. For $C < 0$ the bifurcation diagram of the germ (3.44) remains the same given in figure 3.1. However, since the distinguished parameter must be non negative and now we have $\mathcal{E}_{1l} < \mathcal{E}_{1u}$, the physical unacceptable zone would be given by panel 2 and the diagram should be read clockwise starting from frame 1. In particular the reduced system encounters the first bifurcation at $\mathcal{E} = \mathcal{E}_{1l}$ (instead of $\mathcal{E} = \mathcal{E}_{1u}$).*

Finally, we obtain the following proposition

Proposition 3.4. *Let us consider the 1 DOF system defined by $\mathcal{K}$, cfr. (3.24), with $C \neq 0$, $A - 3C < 0$, $A - C < 0$ and non positive detuning parameter δ. For sufficiently small values of δ the following statements hold.*

For $C > 0$

i) if $B > 2(A - 3C)$: pitchfork bifurcation (a pair of stable fixed point appears) at

$$\mathcal{E} = \mathcal{E}_{1u};$$

ii)if $B > 2(A - C)$: pitchfork bifurcation (a pair of unstable fixed point appears) at

$$\mathcal{E} = \mathcal{E}_{1l};$$

iii) if $B > 2(C - A)$:anti-pitchfork bifurcation (the pair of unstable fixed point disappears) at

$$\mathcal{E} = \mathcal{E}_{2l};$$

iv) if $B > 2(3C - A)$: anti-pitchfork bifurcation (the pair of stable fixed point disappears) at

$$\mathcal{E} = \mathcal{E}_{2u};$$

v) if $B \leq 2(A - 3C)$ the system does not exhibits bifurcation and has only two stable fixed points.

For $C < 0$ and $B < 2(A - C)$ the system does not exhibit any bifurcation and has only two stable fixed points. Otherwise, the bifurcations listed above occur, if the corresponding conditions on B are satisfied, but in a different sequence given by $ii) - i) - iv) - iii)$.

$\varepsilon_1 = \varepsilon_2 = 1$

The case $\varepsilon_1 = \varepsilon_2 = 1$ follows similarly through the bifurcation analysis of

$$-\tilde{F}(x, y) = x^4 + (\tilde{\mu} + \tilde{u}_3)x^2y^2 + y^4 + \tilde{u}_1x^2 + \tilde{u}_2y^2$$

where $\tilde{\mu} = -\mu$, $\tilde{u}_i = -u_i$, for $i = 1, 2, 3$. We attain the following proposition:

Proposition 3.5. *Let us consider the 1 DOF system defined by $\mathcal{K}$, with non positive and sufficiently small detuning parameter, $C \neq 0$, $A - 3C > 0$ and $A - C > 0$.*

For $C < 0$:

if $B \leq 2(A - C)$ only two stable fixed points are present;

if $B > 2(A - C)$ the full bifurcation sequence is given by $iii) - iv) - i) - ii)$ until the corresponding conditions on B are satisfied.

For $C > 0$:

if $B \leq 2(A - 3C)$ only two stable fixed points are present;

if $B > 2(A - 3C)$ the full bifurcation sequence is given by $iv) - iii) - ii) - i)$, until the corresponding conditions on B are satisfied.

3.3.2 $\varepsilon_1\varepsilon_2 = -1$

$\varepsilon_1 = -\varepsilon_2 = -1$

This case corresponds to $A - 3C < 0$ and $A - C > 0$ and the versal unfolding F turns into

$$\tilde{G} = -x^4 + (\mu + u_3)x^2y^2 + y^4 + u_1x^2 + u_2y^2 \qquad (3.65)$$

To fix the ideas, let us assume that $A - 2C < 0$ so that $\mu < 0$. The critical points of (3.44) are therefore given by

$$(0,0), \quad \left(\pm\sqrt{\frac{u_1}{2}}, 0\right), \quad \left(0, \pm\sqrt{\frac{-u_2}{2}}\right) \qquad (3.66)$$

and

$$\left(\pm\frac{\sqrt{u_1 + \frac{\alpha(-\alpha u_1 - 2u_2)}{4+\alpha^2}}}{\sqrt{2}}, \pm\frac{\sqrt{-\alpha u_1 - 2u_2}}{\sqrt{4+\alpha^2}}\right). \qquad (3.67)$$

As we can see in picture (3.2), the bifurcation diagram of the system is quite different from the previous one.

Again, we are interested in finding bifurcation curves in the $(\delta, \mathcal{E})$ plane for the one degree of freedom system defined by (3.24). Thus, we limit ourselves to consider what happens inside the singular circle (3.32), which is marked with a darker line in figure (3.2).

For $\delta \leq 0$ and small values of the distinguished parameter, we have both u_1 and u_2 negative. The physical unacceptable zone is now given by frame 7. Thus, the bifurcation sequence has to be read counter clockwise from frame 1.

Therefore the planar system exhibits the first bifurcation at $u_1 = 0$. The corresponding bifurcation for the 1 DOF system defined by (3.24) occurs for $\mathcal{E} = \mathcal{E}_{1u}$, which is acceptable only if $B > 2(A - 3C)$.

In frame 3 we see the appearance of two stable fixed points inside and four unstable points on the singular circle. By using coordinate transformation (3.63), we can easily check that, if $B > 2(C - A)$, the corresponding threshold value for the distinguished parameter is given by (3.56) and determines the bifurcation of two stable fixed point for $\mathcal{K}$.

For

$$u_2 = -\frac{1}{2}(\alpha + \sqrt{\alpha^2 + 4})u_1 \qquad (3.68)$$

(marked line in figure 3.2) a *global bifurcation* occurs. The corresponding threshold value for the distinguished parameter is given by

$$\mathcal{E} = \mathcal{E}_{gb} := -\frac{\delta}{2B} + O(\delta^2) \qquad (3.69)$$

which is an acceptable value if $B > 0$. Notice that since α multiplies a fourth order term we can consider (3.68) only up to the first order in $|\delta, \mathcal{E}|$, cfr remark 3.7. Therefore, we are able to compute the critical curve (3.69) only to the first order in the detuning parameter. Then, if $B > 2(A - C)$, we can pass through $u_2 = 0$ for $\mathcal{E} = \mathcal{E}_{1l}$ and if $B > 2(3C - A)$ a further bifurcation occurs when passing through $u_1 = 0$; the corresponding threshold value for the distinguished parameter is given by (3.55).

The case $\mu > 0$ follows similarly through the bifurcation analysis of

$$-\tilde{G}(y, x) = -x^4 + (\tilde{\mu} + \tilde{u}_3)x^2y^2 + y^4 + \tilde{u}_1x^2 + \tilde{u}_2y^2$$

where $\tilde{\mu} = -\mu < 0$, $\tilde{u}_1 = -u_2$, $\tilde{u}_2 = -u_1$ and $\tilde{u}_3 = -u_3$. Finally we have the following proposition:

Proposition 3.6. *Let us consider the 1 DOF system define by $\mathcal{K}$, cfr. (3.24) with non positive and sufficiently small detuning parameter, $C \neq 0$, $A-3C < 0$ and $A - C > 0$:*
for $A - 2C < 0 < A - C$ bifurcations might occur along the curves (3.53)-(3.56) according to the statements of proposition 3.4. However in this case they are reached in the sequence $i) - iii) - ii) - iv)$, until the corresponding conditions on B are satisfied. Furthermore, a global bifurcation might occur between $iii)$ and $ii)$:
gb) if $B > 0$ at

$$\mathcal{E} = \mathcal{E}_{gb}$$

with $\mathcal{E}_{2l} < \mathcal{E}_{gb} < \mathcal{E}_{1l}$.
For $0 < A - 2C < A - C$ bifurcations occur along the curves (3.53)-(3.56) according to the statements of proposition 3.4, until the corresponding conditions on B are satisfied. Again, if $B > 0$ the system exhibits a global bifurcation at $\mathcal{E} = \mathcal{E}_{gb}$. The full bifurcation sequence is given by $iii) - i) - gb) - iv) - ii)$.

It remains to analyze the case $\mu = 0$, corresponding to the central singularity $y^4 - x^4$, which still has finite codimension (see section 2.2). In this case (3.44) turns into

$$\mathcal{F}(x, y) = -x^4 + y^4 + u_1x^2 + u_2y^2 + u_3x^2y^2. \tag{3.70}$$

The critical points remain the same given in (3.66) and (3.67), but we now have $\alpha = u_3$. The bifurcation curves are therefore given by

$$u_1 = 0, \quad u_2 = 0 \tag{3.71}$$

and

$$u_3u_2 = 2u_1 \tag{3.72}$$
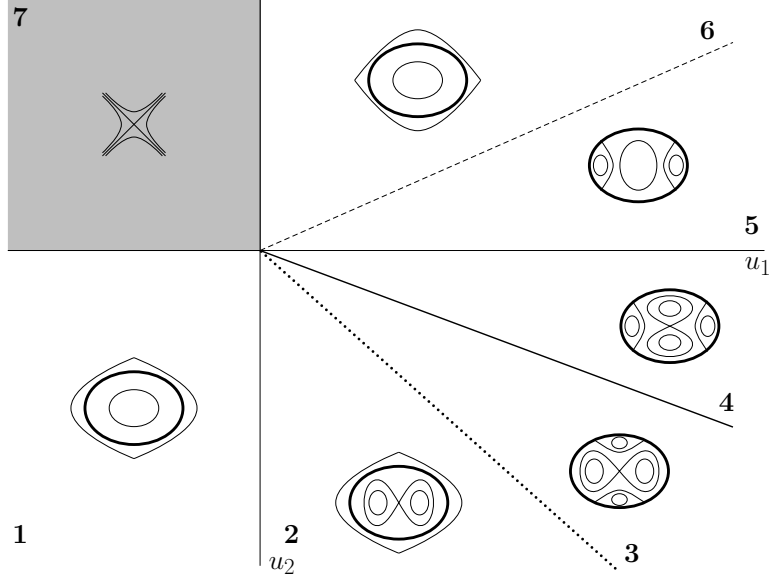
$$2u_2 = -u_1u_3. \tag{3.73}$$

Figure 3.2: Bifurcation diagram for $A - 2C < 0 < A - C$ and $\delta < 0$.

Solving (3.71), we find the critical values $\mathcal{E} = \mathcal{E}_{1u}$ and $\mathcal{E} = \mathcal{E}_{1l}$, which respectively turns out to satisfy also (3.72) and (3.73). Thus, we get

$$\mathcal{E}_{1u} = \mathcal{E}_{2l} \quad \text{and} \quad \mathcal{E}_{2u} = \mathcal{E}_{1l}. \tag{3.74}$$

For the 1 DOF system defined by (3.24), this implies that the critical points corresponding to $(\pm\sqrt{\frac{u_1}{2}}, 0)$, $(0, \pm\sqrt{\frac{-u_2}{2}})$ appear and disappear simultaneously. Furthermore, a global bifurcation occurs for

$$u_2 = -\frac{1}{2}(u_3 + \sqrt{u_3^2 + 4})u_1 \tag{3.75}$$

which give the critical curve (3.69). Finally, the following proposition holds:

Proposition 3.7. *In the 1 DOF system define by $\mathcal{K}$, cfr. (3.24), for $A = 2C > 0$ and non positive sufficiently small values of the detuning parameter, we have*

i) if $B > -2C$ two pitchfork bifurcations occur concurrently (two pairs of stable fixed points appear) at

$$\mathcal{E} = \mathcal{E}_{1u} = \mathcal{E}_{2l};$$

gb) if $B > 0$ a global bifurcation occurs at

$$\mathcal{E} = \mathcal{E}_{gb};$$

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ii) if $B > 2C$ two anti-pitchfork bifurcations occur concurrently (the two pairs of stable fixed points disappear) at

$$\mathcal{E} = \mathcal{E}_{1l} = \mathcal{E}_{2u}.$$

$\varepsilon_1 = -\varepsilon_2 = 1$

The case $\varepsilon_1 = -\varepsilon_2 = 1$ follows similarly through the bifurcation analysis of

$$-\tilde{G}(x, y) = x^4 + (\tilde{\mu} + \tilde{u}_3)x^2 y^2 - y^4 + \tilde{u}_1 x^2 + \tilde{u}_2 y^2$$

with $\tilde{\mu} = -\mu$, $\tilde{u}_i = -u_i$, for $i = 1, 2, 3$. Therefore, the following proposition holds:

Proposition 3.8. *Let us consider the 1 DOF system define by $\mathcal{K}$, cfr. (3.24) with $C \neq 0$, $A - 3C > 0$ and $A - C < 0$. For non positive and sufficiently small detuning parameter, bifurcations might occur along the curves (3.53)-(3.56) and (3.69) according to the statements of proposition 3.4, 3.6 and 3.7.*
For $A - 2C < 0 < A - 3C$ The full bifurcation sequence is given by ii) − iv) − gb) − i) − iii);
for $0 < A - 2C < A - 3C$ it is given by iv) − ii) − gb) − iii) − i);
for $A = 2C$ bifurcations occur according to the statements of proposition 3.7, but they are reached in the sequence ii) − gb) − i), until the corresponding conditions on B are satisfied.

3.4 Implications for the original system

It is known [41, 83] that if a normalization is carried far enough to obtain only isolated equilibria, we know the essential characteristics of the system; including higher orders may shift the positions of the equilibria – and may be essential for quantitative uses – but will not alter their number or stability. Therefore the isolated fixed points of the 1 DOF system defined by (3.24) correspond to periodic orbits for the original system. Of course, the results obtained are limited to low energies, in the neighborhood of a central equilibrium. Above a certain threshold, one should not expect that the formal series developed by Lie transformations would stay close for a very long time to the solutions of the original problem. Since we pushed the normalization up to including fourth order terms, the critical curves of proposition 3.4 give quantitative predictions on the bifurcation and stability of these periodic orbits in the $(\delta, \mathcal{E})$-plane up to the second order in the detuning parameter (since, we recall, this is assumed to be a second degree term).

After the coordinate transformation (3.30), the origin in the plane is a fixed point for all values of the parameters and represents the periodic orbit $J_1 = 0$, namely the normal mode along the x_2-axis. Similarly, if the

planar reduction is performed via (3.63), we find that for all values of the parameters, the origin is a fixed point again, but it corresponds in this case to the periodic orbit $J_2 = 0$, that is the normal mode along the x_1-axis. Furthermore, inverting the coordinate transformation (3.30), the critical points (3.59) turn into

$$\psi = 0, \pi \quad \psi = \frac{\pi}{2}, \frac{3}{2}\pi \qquad (3.76)$$

and represent periodic orbits in general position, namely *inclined and loop orbits*. In the previous section we found threshold values for the distinguished parameter, depending on δ and on the coefficients of the system, which determine the bifurcation of these periodic orbits in general position from the normal modes of the system. The computation of such critical curves is possible if $C \neq A/3$, $C \neq A$ and $C \neq 0$. In term of the original coefficients of the system the previous conditions turn into

$$a_{0,4} - a_{2,2} + a_{4,0} \neq 0, \quad 3a_{0,4} - a_{2,2} + 3a_{4,0} \neq 0 \quad \text{and} \quad a_{2,2} \neq 0 \qquad (3.77)$$

respectively. Furthermore, from propositions 3.4, 3.5, 3.6 and 3.8, we know that each time a periodic orbit appears (or disappears) from a normal mode, the normal mode itself changes its stability. However, it would be better to have an expression of the bifurcation curves in the (δ, E)-plane, where E is the "true" energy of the system and δ is as defined in (1.38). On the x_1−axis orbit ($J_2 = 0$, $J_1 = \mathcal{E}$), we have

$$\mathcal{K} = \mathcal{E} + \left(a\mathcal{E}^2 + \mathcal{E}\tilde{\delta}\right)\varepsilon^2 + \left(-\frac{17}{9}a^2\mathcal{E}^3 + \frac{5d_1\mathcal{E}^3}{2} - 2a\mathcal{E}^2\tilde{\delta}\right)\varepsilon^4 + O(\varepsilon^6) \ (3.78)$$

According to the rescaling (3.9) and (3.8) and (3.14), we have

$$E = \omega_2\varepsilon^2\mathcal{K}. \qquad (3.79)$$

Thus, equation (3.78) can be used to express the physical energy E in terms of $\mathcal{E}$, namely

$$E = \omega_2\mathcal{E}\varepsilon^2 + \left(a\mathcal{E}^2\omega_2 + \mathcal{E}\tilde{\delta}\omega_2\right)\varepsilon^4 + O(\varepsilon^6). \qquad (3.80)$$

Thus up to the second order in δ, for $\mathcal{E}$ satisfying equations (3.55), (3.56) and δ as defined in (3.12) we obtain the following threshold values

$$E = E_{2u} := \frac{2\delta\omega_2^4}{3(a_{2,2} - 2a_{4,0})} \qquad (3.81)$$

$$+ \frac{\omega_2^4\left(89a_{2,2}^2 - 256a_{2,2}a_{4,0} + 156a_{4,0}^2 + 120d_1\omega_2^6 - 40d_2\omega_2^6\right)}{54(a_{2,2} - 2a_{4,0})^3}\delta^2 + O(\delta^3)$$

$$E = E_{2l} := \frac{2\delta\omega_2^4}{a_{2,2} - 6a_{4,0}} \qquad (3.82)$$

$$+ \frac{\omega_2^4\left(9a_{2,2}^2 - 80a_{2,2}a_{4,0} + 156a_{4,0}^2 + 120d_1\omega_2^6 - 8d_2\omega_2^6\right)}{2(a_{2,2} - 6a_{4,0})^3}\delta^2 + O(\delta^3)$$

for the appearance (disappearance) of inclined and loop from the $x-$axis orbits respectively. They correspond to physically acceptable values, at least for small values of the detuning parameter, (which we assume to be non positive, cfr. remark 3.6) if

$$a_{2,2} - 2a_{4,0} < 0, \quad \text{and} \quad a_{2,2} - 6a_{4,0} < 0. \qquad (3.83)$$

A similar argument gives the threshold values for the bifurcation from the x_2-axis orbit of inclined and loop orbits respectively. They are given by

$$
\begin{aligned}
E \;=\; & E_{1u} := \frac{2\delta\omega_2^4}{3(2a_{0,4} - a_{2,2})} \qquad\qquad\qquad\qquad (3.84)\\
& + \frac{\omega_2^4 \left(276a_{0,4}^2 - 176a_{0,4}a_{2,2} + 19a_{2,2}^2 + 40d_3\omega_2^6 - 120d_4\omega_2^6\right)}{54(2a_{0,4} - a_{2,2})^3}\delta^2 \\
E \;=\; & E_{1l} := \frac{2\delta\omega_2^4}{6a_{0,4} - a_{2,2}} \qquad\qquad\qquad\qquad\qquad (3.85)\\
& + \frac{\omega_2^4 \left(276a_{0,4}^2 - 64a_{0,4}a_{2,2} + 3a_{2,2}^2 + 8d_3\omega_2^6 - 120d_4\omega_2^6\right)}{2(6a_{0,4} - a_{2,2})^3}\delta^2
\end{aligned}
$$

They correspond to physically acceptable values, at least for small values of the detuning parameter, if

$$a_{2,2} - 2a_{0,4} > 0, \quad \text{and} \quad a_{2,2} - 6a_{0,4} > 0. \qquad (3.86)$$

Finally, the global bifurcation might occur at

$$E = E_{gb} := \frac{2\delta\omega_2^4}{3(a_{0,4} - a_{4,0})} \qquad\qquad\qquad (3.87)$$

if

$$a_{0,4} < a_{4,0}. \qquad\qquad\qquad (3.88)$$

As we have seen in the previous section the bifurcation sequence of the system depends on the three coefficients A, B, C, with

$$A = \frac{3(a_{4,0} + a_{0,4})}{8\omega_2^3}, \quad B = \frac{3(a_{4,0} - a_{0,4})}{4\omega_2^3}, \quad C = \frac{a_{2,2}}{8\omega_2^3}. \qquad (3.89)$$

This translates into conditions on the original coefficients $a_{4,0}$, $a_{2,2}$ and $a_{0,4}$. Namely, proposition 3.4 reads now

Proposition 3.9. *Let us consider the dynamical system defined by $\mathcal{H}$, cfr. (3.1),with $a_{2,2} \neq 0$, $a_{0,4} - a_{2,2} + a_{4,0} < 0$ and $3a_{0,4} - a_{2,2} + 3a_{4,0} < 0$ and non positive detuning parameter. If the detuning parameter is sufficiently small, the following statements hold:*

for $a_{2,2} > 0$

i) if $2a_{0,4} < a_{2,2}$: a pair of stable inclined orbits appears from the x_2-normal mode (which becomes unstable) at

$$E = E_{1u};$$

ii)if $6a_{0,4} < a_{2,2}$: a pair of unstable inclined orbits appears from the x_2-normal mode (which comes back to stability) at

$$E = E_{1l};$$

iii) if $6a_{4,0} > a_{2,2}$ the pair of unstable inclined orbits disappears on the x_1-normal mode (which becomes unstable) at

$$E = E_{2l};$$

iv) if $2a_{4,0} > a_{2,2}$: the pair of stable inclined orbits disappears on the x_1-normal mode (which comes back to stability) at

$$E = E_{2u};$$

v) if $2a_{0,4} \geq a_{2,2}$ the system does not exhibits bifurcations.

For $a_{2,2} < 0$ and $a_{2,2} - 6a_{0,4} \leq 0$ the system does not exhibits any bifurcation. Otherwise, the bifurcations listed above occur, if the corresponding conditions on the coefficients are satisfied, but in a different sequence given by $ii) - i) - iv) - iii)$.

In case $a_{0,4} - a_{2,2} + a_{4,0} > 0$ and $3a_{0,4} - a_{2,2} + 3a_{4,0} > 0$, $a_{0,4} - a_{2,2} + a_{4,0} < 0$ and $3a_{0,4} - a_{2,2} + 3a_{4,0} > 0$, $a_{0,4} - a_{2,2} + a_{4,0} > 0$ and $3a_{0,4} - a_{2,2} + 3a_{4,0} < 0$ the bifurcation sequences can be easily deduced from propositions 3.5, 3.6, 3.7 and 3.8 respectively.

The following figures give a comparison between our analytical predictions (on the left) and numerical approximations (on the right). We choose $\delta = -0.025$ with coefficients $a_{4,0} = -a_{0,4} = 2/3$, $a_{2,2} = 1$ in fig.s 3.3, 3.4, and $a_{4,0} = 1/2$, $a_{0,4} = 0$, $a_{2,2} = 1$ in fig. 3.5. The pictures show (x_1, p_1) surfaces of section computed with the condition $x_2 = 0$, $p_2 > 0$.

The first set of coefficients satisfies conditions $i) - v)$ of proposition 3.9 for $a_{2,2} > 0$, which implies that the system can exhibit four bifurcations. In figure 3.3(a) we represent the dynamics on the energy level $E = 0.006$ which is below $E_{1u} = 0.007$: we see a section of the invariant tori around the x_2- axis orbit. In fig 3.3(b) we increased the energy level to $E = 0.008$, which is above E_{1u}: we see that the axial orbit is now unstable and there are two stable fixed points corresponding to the two periodic inclined orbits. Panel 3.3(c) corresponds to $E = 0.012$ which is above $E_{1l} = 0.01$: the axial orbits has come back to stability and two further fixed point are present

which correspond to unstable loop orbits. Increasing the energy values the fixed points disappear and at the end of the bifurcation sequence the x_2-axis regains its stability. Notice that both the analytical and numerical approximations of the dynamics are in very good agreement.

The second choice of coefficients satisfies the condition of proposition 3.6. Analytical and numerical surfaces of section are showed in fig.s 3.5 and 3.6. According to our analytical prediction a global bifurcation occurs in this case at $E = E_{gb} = 0.0\overline{3}$. The threshold value found numerically is 0.03365 (see fig. 3.6). The two predictions therefore agree to the third decimal number.

Figure 3.3: Analytical (on the left) and numerical (on the right) surfaces of section at energy levels (*a*) $E = 0.006$,(*b*) $E = 0.008$, (*c*) $E = 0.012$ for $\delta = -0.025$ when the original coefficients of the system are $a_{4,0} = -a_{0,4} = 2/3$, $a_{2,2} = 1$.
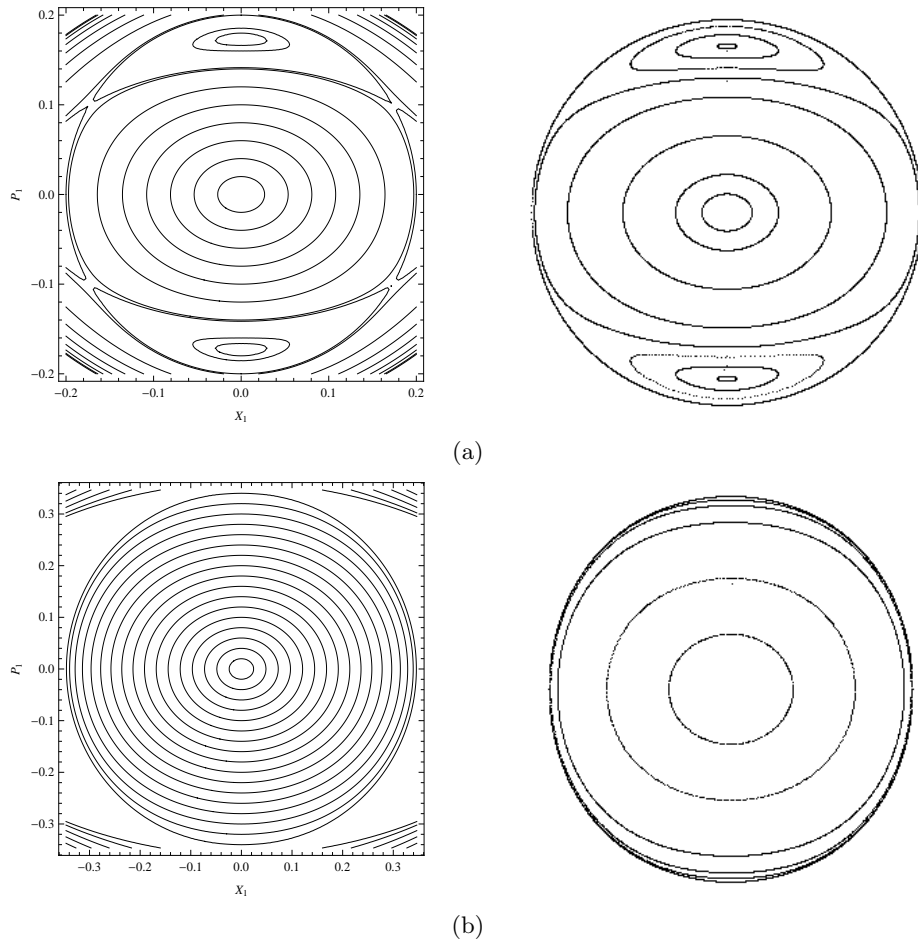
Figure 3.4: Analytical (on the left) and numerical (on the right) surfaces of section at energy levels (a) $E = 0.02$,(b) $E = 0.06$, for $\delta = -0.025$ when the original coefficients of the system are $a_{4,0} = -a_{0,4} = 2/3$, $a_{2,2} = 1$.
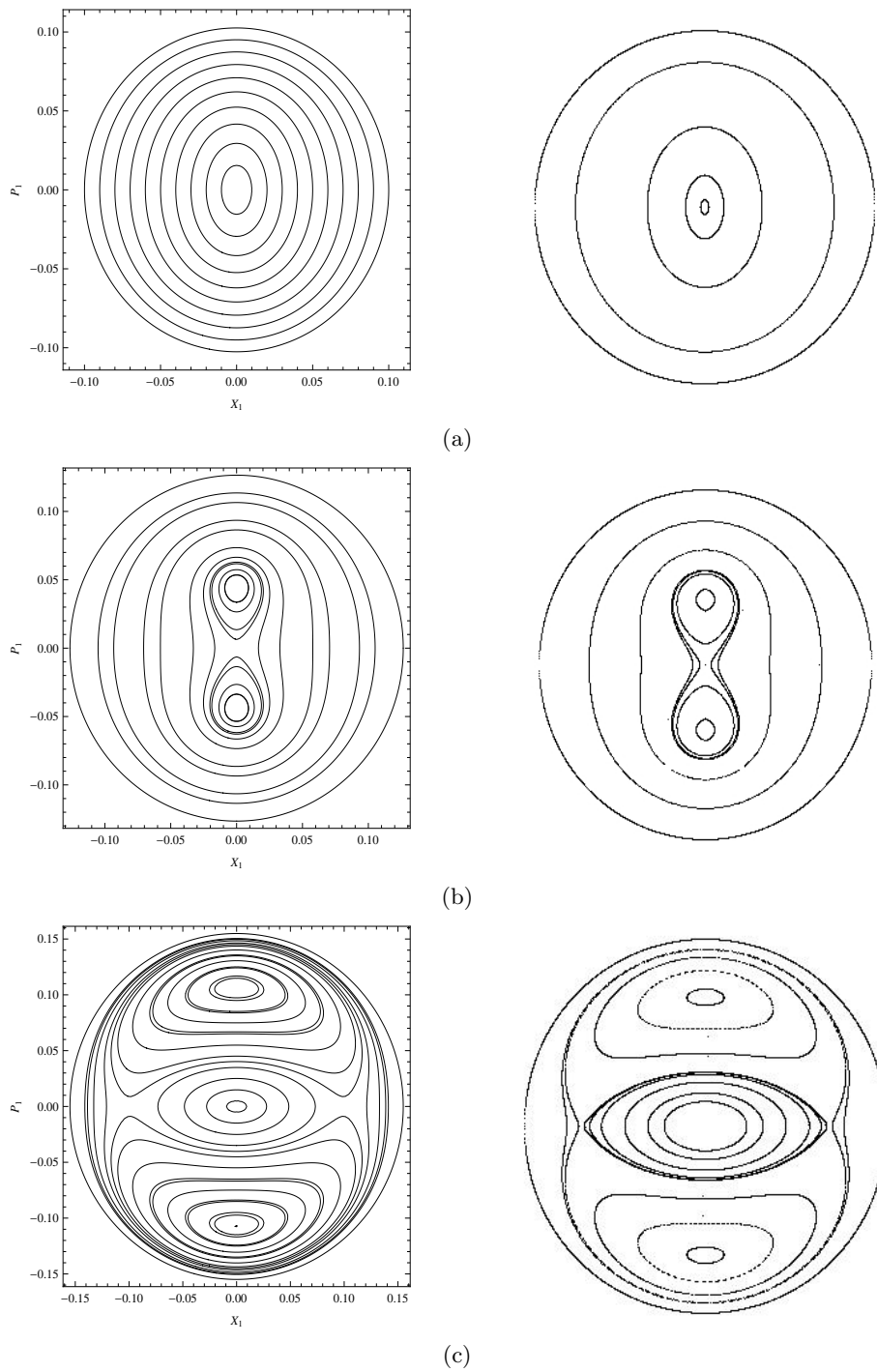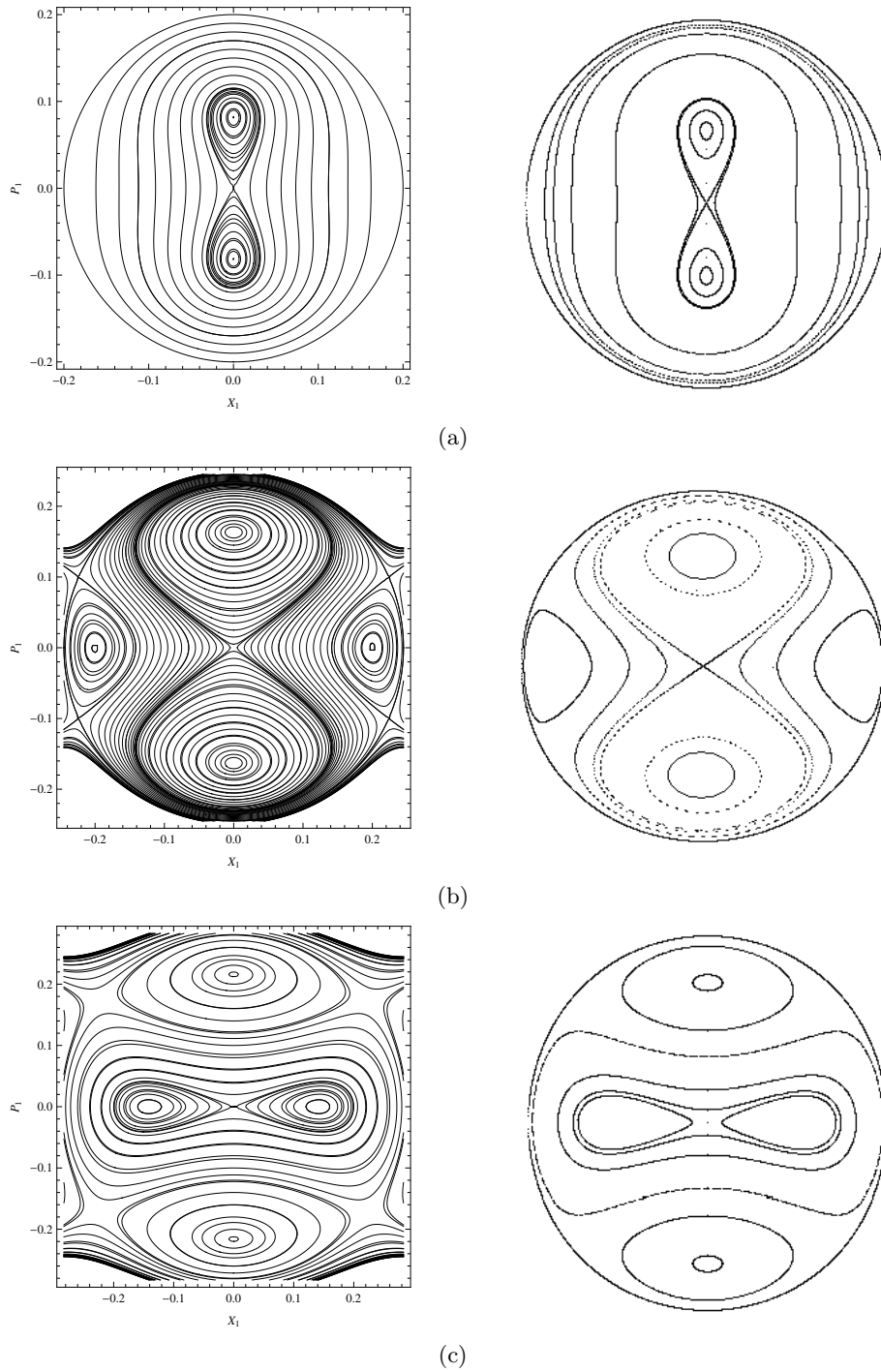
Figure 3.5: Analytical (on the left) and numerical (on the right) surfaces of section at energy levels (a) $E = 0.02$,(b) $E = 0.03$, (c) $E = 0.04$ for $\delta = -0.025$ when the original coefficients of the system are $a_{4,0} = 1/2$, $a_{0,4} = 0$, $a_{2,2} = 1$.

(a) $E = 0.0\overline{3}$ (b) $E = 0.03365$

Figure 3.6: Global bifurcation: analytical (on the left) and numerical (on the right) surfaces of section for $\delta = -0.025$ when the original coefficients of the system are $a_{4,0} = 1/2$, $a_{0,4} = 0$, $a_{2,2} = 1$.

Chapter 4

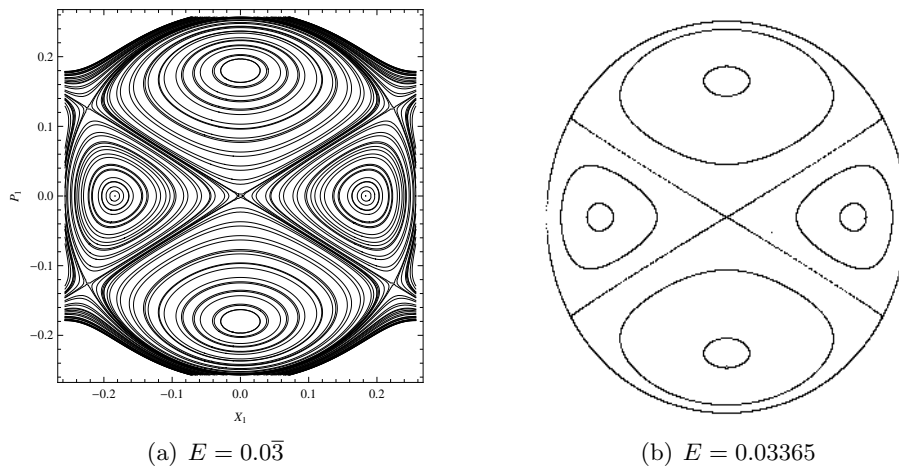# Orbit structure of systems with elliptical equipotentials

Here we present a general analysis of the orbit structure of 2D potentials with self-similar elliptical equipotentials by applying the method of Lie transform normalization. We study the most relevant resonances and related bifurcations. In the 2 : 2 resonance the first non degeneracy condition in (3.37) is not satisfied. As a consequence, inclined orbits are never present and may appear only when the equipotentials are heavily deformed. Loops bifurcate upon a critical energy value and leads to the destabilization of either one or the other normal modes, depending on the ellipticity of equipotentials. The 2 : 4 resonance determines the appearance of banana and anti-banana orbits: the first family is stable and always appears at a lower energy than the second one, which is unstable. The bifurcation sequence also produces a variation in the stability character of the major axis orbit and is modified only by very large deformations of the equipotentials.

4.1   The model

We are interested in a fairly general class of potentials with self-similar elliptical equipotentials of the form

$$
\mathcal{V}_\alpha(x_1, x_2; q) = \begin{cases} \frac{1}{\alpha}\left(1 + x_1^2 + \frac{x_2^2}{q^2}\right)^{\alpha/2}, & 0 < \alpha < 2 \\ \frac{1}{2}\log\left(1 + x_1^2 + \frac{x_2^2}{q^2}\right), & \alpha = 0. \end{cases} \tag{4.1}
$$

The corresponding Hamiltonian

$$
\mathcal{H}(\mathbf{p}, \mathbf{x}) = \frac{1}{2}(p_1^2 + p_2^2) + \mathcal{V}_\alpha(x_1, x_2; q) \tag{4.2}
$$

exhibits two $\mathbb{Z}_2$ symmetries in space, given by (3.2) and (3.3). The ellipticity of the equipotentials is determined by the parameter q: for short, we will

speak of an "oblate" figure when $q < 1$ and a "prolate" figure when $q > 1$. The "profile" parameter α determines the behavior at large radius. By expanding the potential (4.1) around the origin and retaining only terms up to degree $N + 2$ we get

$$\mathcal{V}^{(N)}(x_1^2, x_2^2; q, \alpha) \equiv \sum_{k=0}^{N} b_k(\alpha) s^{2(k+1)}(x_1, x_2; q) \tag{4.3}$$

where we have introduced the "elliptical radius"

$$s(q) = \sqrt{x_1^2 + \frac{x_2^2}{q^2}}. \tag{4.4}$$

In all cases in which we rescale the "core radius" to one we can put $b_0 = 1/2$ and, for the class (4.1), the first two coefficients of the higher-order terms are

$$b_1 = -\frac{2 - \alpha}{8}, \quad b_2 = \frac{(2 - \alpha)(4 - \alpha)}{48}. \tag{4.5}$$

Another interesting case is that of the "flattened isochrone" [50]:

$$b_1 = -\frac{1}{4}, \quad b_2 = \frac{5}{32}. \tag{4.6}$$

In force of reflection symmetries, each term in the series is given by an even power of the basic elliptical radius. Therefore the Hamiltonian (4.2) can be treated in a perturbative way as a non-linear oscillator system. The two unperturbed frequencies are given by

$$\omega_1 := \sqrt{2b_0} = 1, \quad \omega_2 := \sqrt{2b_0}/q = 1/q. \tag{4.7}$$

To find the normal form we proceed as in the previous chapter. We start by introducing a small parameter $\varepsilon > 0$ and, by performing a blowing-up of the phase-space by means of the transformation (3.8), we rescale the Hamiltonian (4.2) according to $\widetilde{H} \doteq \varepsilon^{-2}\mathcal{H}$. After a further scaling (3.13), the original Hamiltonian system (4.2) is put into the form

$$\widetilde{H}(\mathbf{p}, \mathbf{x}) = \sum_{n=0}^{N} \varepsilon^{2n} \widetilde{H}_{2n}(\mathbf{p}, \mathbf{x}). \tag{4.8}$$

We then have

$$\widetilde{H}_0 = \frac{1}{2} \left(\omega_1(p_1^2 + x_1^2) + \omega_2(p_2^2 + x_2^2) \right) \tag{4.9}$$

and $\widetilde{H}_{2j}$, $j > 0$, are essentially the higher order terms of the potential. Again, since we are interested in the behavior of the system around m_1/m_2

resonances with $m_1, m_2 \in \mathbb{N}$, we introduce a detuning parameter δ such that the frequency ratio is written as

$$\omega_1/\omega_2 = q = \frac{m_1}{m_2} + \delta. \tag{4.10}$$

The detuning parameter is treated as a term of order two in ε, $\delta = \tilde{\delta}\varepsilon^2$, and put in the perturbation. After a further scaling

$$H \equiv \frac{m_2}{\omega_2}\widetilde{H} \tag{4.11}$$

and noting that, in view of (4.10), we have

$$\frac{1}{q} = \frac{m_2}{m_1} - \frac{m_2^2}{m_1^2}\tilde{\delta}\varepsilon^2 + \frac{m_2^3}{m_1^3}\tilde{\delta}^2\varepsilon^4 + \dots, \tag{4.12}$$

by collecting the terms up to order $2N$ in ε, we finally put the Hamiltonian into the form

$$H(\mathbf{p}, \mathbf{x}) = \sum_{k=0}^{N} \varepsilon^{2k} H_{2k}(\mathbf{p}, \mathbf{x}) \tag{4.13}$$

where the unperturbed term is given by

$$H_0 = \frac{1}{2}m_1(p_1^2 + x_1^2) + \frac{1}{2}m_2(p_2^2 + x_2^2). \tag{4.14}$$

The system is now ready for the normalization procedure described in section 1.3. In the following, we are interested in the behavior of the system around $2:2$ and $2:4$ resonances.

4.2 2:2 Resonance

Let us start considering the system around a $1:1$ symmetric resonance. Therefore, we approximate the frequency ratio by

$$q = 1 + \tilde{\delta}\varepsilon^2. \tag{4.15}$$

The minimal truncation order required is $N = 1$. The terms in the expansion (4.13) are given by

$$H_0 = \frac{1}{2}(p_1^2 + x_1^2 + p_2^2 + x_2^2) \tag{4.16}$$

$$H_2 = \frac{\tilde{\delta}}{2}(x_1^2 + p_1^2) + b_1\left(x_1^2 + x_2^2\right)^2. \tag{4.17}$$

The normalization procedure described in (1.3) transforms the Hamiltonian (4.13) into the normal form

$$\begin{aligned}
K_{11} = \;& J_1 + J_2 + \varepsilon^2\tilde{\delta}J_1 + \\
& \varepsilon^2 b_1\left(\frac{3}{2}(J_1^2 + J_2^2) + J_1 J_2(2 + \cos(2\phi_1 - 2\phi_2))\right)
\end{aligned} \tag{4.18}$$

where action-angle like variables have been introduced according to (1.49). We introduce the invariants of the oscillator symmetry [40]:

$$\begin{cases} I_0 = & \frac{1}{4}(z_1\bar{z}_1 + z_2\bar{z}_2) \\ I_1 = & \frac{1}{2}\Re(z_1\bar{z}_2) \\ I_2 = & \frac{1}{2}\Im(z_1\bar{z}_2) \\ I_3 = & \frac{1}{4}(z_1\bar{z}_1 - z_2\bar{z}_2) \end{cases} \tag{4.19}$$

The set $\{I_0, I_1, I_2, I_3\}$ form a Hilbert basis of the ring of invariants and can be used as a coordinates system for the reduced phase space. The Poisson bracket is given by $\{I_i, I_j\} = \varepsilon_{ijk}I_k$, where ε_{ijk} denotes the sign of the permutation $\left(\begin{smallmatrix} 123 \\ ijk \end{smallmatrix}\right)$ and vanishes if one of the i, j, k is zero or two of them are equal.

Notice that $I_0 = H_0/2$, which is a constant of motion. If we denote this constant value with $\mathcal{E}/2$, the reduced Hamilton function takes the form

$$K(I_1, I_2, I_3) = \mathcal{E} + \left(\frac{\tilde{\delta}}{2}\mathcal{E} + b_1\mathcal{E}^2 + \tilde{\delta}I_3 + b_1(I_1^2 - I_2^2) + b_1 I_3^2\right)\varepsilon^2. \tag{4.20}$$

There is one relation between the new coordinates, namely

$$I_1^2 + I_2^2 + I_3^2 = \frac{\mathcal{E}^2}{4}. \tag{4.21}$$

Hence the sphere

$$\mathcal{S} = \left\{(I_1, I_2, I_3) \in \mathbb{R}^3 \ : \ I_1^2 + I_2^2 + I_3^2 = \frac{\mathcal{E}^2}{4}\right\} \tag{4.22}$$

is invariant under the flow defined by (4.20). This provides the reduction to a one degree of freedom system.

The two reflection symmetries of (4.2) now turn into the symmetries $I_1 \to -I_1$ and $I_2 \to -I_2$. Their composition

$$(I_1, I_2, I_3) \to (-I_1, -I_2, I_3)$$

gives a discrete symmetry of (4.20). We perform a further reduction to explicitly divide out this symmetry. This is given by the transformation [63]

$$\begin{cases} x = & I_1^2 - I_2^2 \\ y = & 2I_1 I_2 \\ z = & I_3 \end{cases} \tag{4.23}$$

which turns the sphere (4.22) into the "Lemon" space

$$\mathcal{L} = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = \left(\frac{\mathcal{E}}{2} + z\right)^2\left(\frac{\mathcal{E}}{2} - z\right)^2\right\} \tag{4.24}$$

with Poisson bracket

$$\{f, g\} = (\nabla f \times \nabla g, \nabla L)$$

where $(.,.)$ denotes the inner product and $L = x^2 + y^2 - \left(\frac{\mathcal{E}}{2} + z\right)^2 \left(\frac{\mathcal{E}}{2} - z\right)^2$. The Hamiltonian function becomes

$$\mathcal{K}(x, y, z) = \mathcal{E} + \left(\frac{\tilde{\delta}}{2}\mathcal{E} + b_1\mathcal{E}^2 + \tilde{\delta}z + b_1 x + b_1 z^2\right)\varepsilon^2. \qquad (4.25)$$

We note that both reversing symmetries of (4.20) gives the invariance of (4.25) with respect to the reflection $y \to -y$.

By varying $\mathcal{E}$ (i.e. by varying the measure of the reduced phase space (4.24)), the system could display several equilibrium configurations.

Each integral curve for the reduced system defined by (4.25), is given by the intersection between $\mathcal{L}$ and the surface

$$\{(z, x) \in \mathcal{R}^2 \ : \ \mathcal{K} = \tilde{h}\} \qquad (4.26)$$

and tangency points give equilibrium solutions. Since y does not enter in (4.25), the level sets $\{\mathcal{K} = \tilde{h}\}$ are parabolic cylinders. If a tangency point occurs between $\mathcal{L}$ and the surface (4.26), we have an (isolated) equilibrium for the reduced system. Moreover, two (degenerate) equilibria are represented by the singular points $\mathcal{Q}_1 \equiv (0, 0, -\mathcal{E}/2)$ and $\mathcal{Q}_2 \equiv (0, 0, \mathcal{E}/2)$. A tangent plane to $\mathcal{L}$ can coincide with a tangent plane to the parabolic cylinder $\{\mathcal{K} = \tilde{h}\}$ only at points where y vanishes. Hence all equilibria on $\mathcal{L}$ occur at $\{y = 0\}$: in order to study the existence and nature of the equilibria configuration of the system, it is then enough to restrict the analysis to the phase space section $\{y = 0\}$.

The contour $\mathcal{C} \equiv \mathcal{L} \cap \{y = 0\}$ in the (z, x)-plane is given by $\mathcal{C}_- \cup \mathcal{C}_+$, where

$$\mathcal{C}_- \equiv \left\{(z, x) \in \mathcal{R}^2 \ : \ x = -\left(\frac{\mathcal{E}^2}{4} - z^2\right)\right\} \qquad (4.27)$$

$$\mathcal{C}_+ \equiv \left\{(z, x) \in \mathcal{R}^2 \ : \ x = \left(\frac{\mathcal{E}^2}{4} - z^2\right)\right\} \qquad (4.28)$$

and the set $\mathcal{P} \equiv \{K = h\} \cap \{y = 0\}$ corresponds to the parabola

$$x = h - \frac{\delta}{b_1}z - z^2 \qquad (4.29)$$

where $h = \frac{1}{b_1}\left(\tilde{h}\varepsilon^2 - \left(1 + b_1\mathcal{E} + \frac{\delta}{2}\right)\mathcal{E}\right)$.

On the phase space section $\mathcal{C}$, the two degenerate equilibria $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are respectively represented by the points $Q_1 \equiv (-\mathcal{E}/2, 0)$ and $Q_2 \equiv (\mathcal{E}/2, 0)$.

It is always possible to fix h such that (4.29) intersects $\mathcal{C}$ in one of these points, say Q_1, so that

$$h = h_{Q_1} := \frac{\mathcal{E}^2}{4} - \frac{\delta}{2b_1}\mathcal{E}. \qquad (4.30)$$

In the coordinates (4.19), $\mathcal{Q}_1$ is represented by

$$\begin{cases} I_0 & = \frac{\mathcal{E}}{2} \\ I_1 & = 0 \\ I_2 & = 0 \\ I_3 & = -\frac{\mathcal{E}}{2} \end{cases}$$

The corresponding periodic orbit in action-angle variables is given by $J_1 = 0$, that is the normal mode along the x_2-axis for the two degrees of freedom system. Similarly, we recognize that the equilibrium point $\mathcal{Q}_2$ determines the periodic orbit along the x_1-axis.

(a)

(b)

(c)

(d)

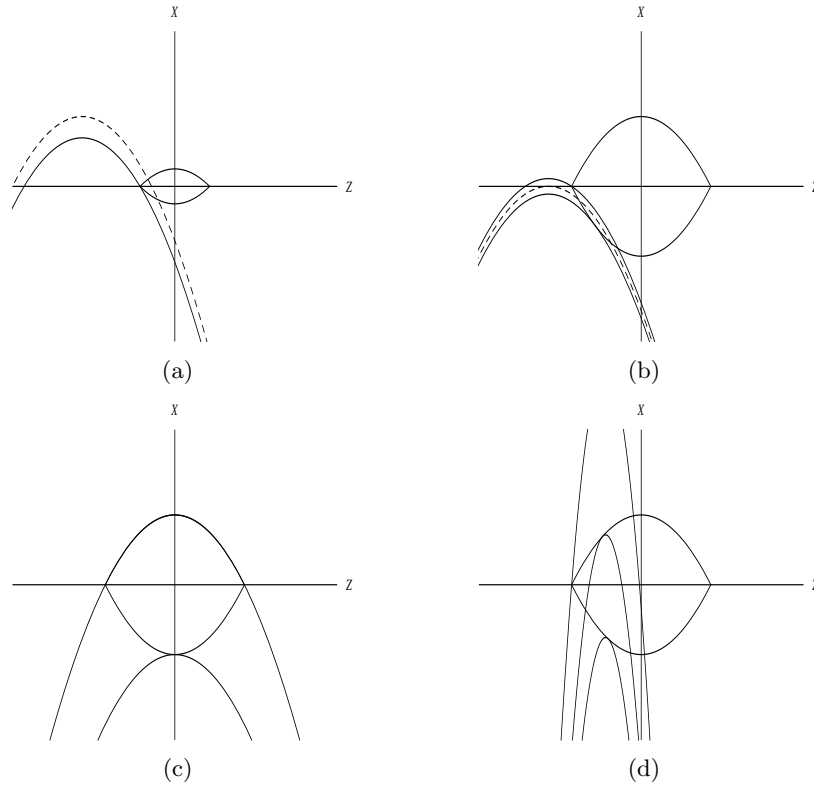Figure 4.1: Possible tangency points on the reduced phase space section $\mathcal{C} \cap \{y = 0\}$ for different values of the parameters: $a)$ $\delta = -0.2$, $\mathcal{E} = 0.3$, $\alpha = \beta = 0$; $b)$ $\delta = -0.2$, $\mathcal{E} = 0.6$, $\alpha = \beta = 0$; $c)$ $\delta = 0$, $\mathcal{E} = 0.6$, $\alpha = \beta = 0$; $d)$ $\delta = -0.2$, $\mathcal{E} = 0.6$, $\alpha = 0$, $\beta = 0.2$.

4.2.1 Bifurcation of loop orbits

Depending on the values of the constant of motion $\mathcal{E}$, the normal modes might change their stability. A stability/instability transition of a normal mode is generally associated with the bifurcation of new periodic orbits. If this is the case, one or more tangency points arise between the reduced phase space section $\mathcal{C}_+$ and the parabola (4.29).

Suppose that a tangency point occurs between $\mathcal{P}$ and the upper arc of the contour $\mathcal{C}$. Let us denote such a point by Q_U. Since its x-coordinate must be positive, if we invert the coordinate transformation (4.23), it corresponds to *two* points on the section $I_2 = 0$ of the sphere (4.22). Thus, we find two periodic orbits for the two degree of freedom system, whose angle coordinates must satisfy

$$\phi_1 - \phi_2 = 0, \pi \tag{4.31}$$

which correspond to inclined orbits.

Similarly, it can be shown that if a tangency point, say Q_L, occurs on the lower arc of the contour $\mathcal{C}$, it determines *two* loop orbits for the two degrees of freedom system given by

$$2(\phi_1 - \phi_2) = \pm\pi. \tag{4.32}$$

In the previous chapter we found critical energy values which give existence conditions for the periodic orbits described above if non degeneracy conditions are satisfied (see remark 3.3 and condition (3.39)). Notice that the (formal) constant of motion $\mathcal{E}$ plays the role of the distinguished parameter. Unfortunately, we now have $A = 3C = \frac{3}{4}b_1$, therefore such conditions are not all satisfied. Moreover, since $B = 0$, the threshold values (3.53) and (3.55) are divergent. As a consequence, we expect that inclined orbits do not appear, whereas loop orbits could bifurcate at the same critical values given in (3.54) and (3.56). Indeed, we find both critical values (3.54) and (3.56), at first order, if we perform our analysis on the reduced phase space section $\mathcal{L} \cap \{y = 0\}$. The second order terms can be computed, as well, if the normalization is pushed up to the fourth order (i.e. N=2).

We recall that only non-negative value of $\mathcal{E}$ are allowed. For $\mathcal{E} = 0$ the dynamics are trivial since the reduced phase space coincides with the origin. Let us take $\delta < 0$, so that $q < 1$. The case $\delta > 0$, i.e. $q > 1$, can be studied similarly (see remark 3.6). For $\delta = 0$, the system is in the "exact" $1 : 1$ resonance. If this is the case, we have $q = 1$ and the system turns out to be separable in polar coordinates.

Increasing the value of the distinguished parameter the area delimited by the contour phase space $\mathcal{C}$ increases. Thus, at a certain critical value for $\mathcal{E}$, the contour $\mathcal{C}$ will intersect the parabola (4.29). By varying h, $\mathcal{P}$ shifts upward or downward. Consequently, the intersection between the two curves may yield a tangency point on $\mathcal{C}$. If this is the case, the corresponding value of $\mathcal{E}$

gives a condition for the existence of an isolated equilibrium point (see Fig 4.1(b)). On the other hand, the parabola $\mathcal{P}$ achieves its maximum at

$$z = z_m := -\frac{\delta}{2b_1}, \tag{4.33}$$

is downward concave and has, at its vertex, the same curvature of $\mathcal{C}_+$ at $z = 0$. Therefore, by a simple geometrical argument, we see that a tangency point on the upper arc of reduced phase space section cannot appear. As a consequence, inclined orbits do not exist. However a tangency point on the lower arc $\mathcal{C}_-$ can occur. It can be found by imposing the system

$$\begin{cases} x = & h - \frac{\delta}{b_1}z - z^2 \\ x = & -\left(\frac{\mathcal{E}^2}{4} - z^2\right) \end{cases} \tag{4.34}$$

to have a unique solution not coinciding with $\left(\pm\frac{\mathcal{E}}{2}, 0\right)$. System (4.34) admits a unique solution if the equation

$$2z^2 + \frac{\delta}{b1}z - \frac{\mathcal{E}^2}{4} - h = 0 \tag{4.35}$$

has null discriminant. This is the case for

$$h = h_l := -\frac{1}{4}\left(\frac{\delta^2}{2b_1^2} + \mathcal{E}^2\right) \tag{4.36}$$

Notice that h is not the "true energy" of the system, therefore it is allowed to assume negative values. The corresponding solution is given by

$$z = z_l := \frac{\delta}{4b_1}$$

and determines a tangency point on $\mathcal{C}_-$ if it satisfies the constraints

$$-\frac{\mathcal{E}}{2} < z_l < \frac{\mathcal{E}}{2}. \tag{4.37}$$

For $\delta < 0$, inequalities (4.37) are verified for

$$\mathcal{E} > \frac{\delta}{2b_1} \tag{4.38}$$

which coincides with the first order term of threshold value (3.54). It determines the bifurcation of a fixed point, say $\mathcal{Q}_L$, from the equilibrium point $\mathcal{Q}_1$, which corresponds to two loop orbits for the original system. We note that values of h slightly higher than h_l correspond to closed orbits around $\mathcal{Q}_L$ on the phase space surface. (see Fig. 4.1(b)): thus, $\mathcal{Q}_L$ is a stable equilibrium point. Therefore, loop orbits turn out to be stable. As a consequence of the bifurcation of loop orbits, we expect the fixed $\mathcal{Q}_1$ to undergo a

transition to instability at the critical value (4.38). To verify this statement, we fix $h = h_{Q_1}$ so that the parabola (4.29) intersects the contour phase space $\mathcal{C}$ at the point Q_1. For $\mathcal{E}$ sufficiently small, since z_m is negative, no further intersections arise between $\mathcal{P}$ and $\mathcal{C}$. As a consequence, the equilibrium is stable. In fact, for $h > h_{Q_1}$ the parabola shifts upward and crosses twice the contour phase space. This produces closed orbits on the phase space surface around the point $\mathcal{Q}_1$. Hence, the point corresponds to a stable equilibrium (see figure 4.1(a)). Increasing the value of $\mathcal{E}$, the phase space volume increases. As a consequence, new intersections might arise between $\mathcal{P}$ and the contour phase space. If a second intersection occurs on the contour phase space, the equilibrium point $\mathcal{Q}_1$ turns out to be unstable. Its stable and unstable manifolds are given in $\mathbb{R}^3$ by the intersection curves between the phase space surface and parabolic cylinder $\{\mathcal{K} = h_{Q_1}\}$ (see figure 4.1(b)). All possible intersection points between $\mathcal{P}$ and $\mathcal{C}$ are given by the solutions of system (4.34) if we replace h with h_{Q_1}. Solving for the z-coordinate, we find

$$z = -\frac{\delta - b_1 \mathcal{E}}{2b_1} \quad \text{(on } \mathcal{C}_-\text{)}. \tag{4.39}$$

This gives acceptable values for the x-coordinates of the intersection point if it belongs to the interval $(-\mathcal{E}/2, \mathcal{E}/2)$. This translates into condition (4.38), which therefore determines the transition to instability of the x_1- axis orbit.

By a similar argument, the equilibrium point $\mathcal{Q}_2$ turns out to be stable. In fact, if the parabola $\mathcal{P}$ passes through Q_2, for geometrical reasons it cannot intersect the contour phase space in any further point. This implies that the x_1-axis orbit stays always stable.

On the other hand, by applying transformation (3.57), we easily find that in case $\delta > 0$, the bifurcation of loop orbits occurs from the x_1- normal mode, which as a consequence, suffers a transition to instability. The corresponding threshold value for $\mathcal{E}$ is now given by

$$\mathcal{E} = -\frac{\delta}{2b_1}. \tag{4.40}$$

To be concrete, we express these results in the case of the α-models (4.1). In view of the rescaling and of the expansion of the energy as a truncated series in the parameter $\mathcal{E}$, we have that $E = \omega_2 \mathcal{E} = \mathcal{E}/q$. Thus, a first order estimate of the 'true' energy of the orbital motion is given by $\mathcal{E}$. Therefore, the critical value

$$E = E_l := \frac{4}{2 - \alpha}|q - 1| + O(|q - 1|^2) \tag{4.41}$$

determines the bifurcation of loop orbits from the x_2- axis orbit in the prolate case $(q < 1)$ and from the x_1-axis orbit in the oblate case $(q > 1)$. As a consequence, in both cases, the normal mode itself suffers a transition to instability.

4.2.2 Ellipse-breaking deformations: bifurcation of inclined orbits

Let us now consider a deformation of the potentials (4.1) by introducing a small parameter β such that

$$\mathcal{V}_2 \;=\; b_1 s^4 + 2\beta x_1^2 x_2^2, \tag{4.42}$$

with 'boxy' or 'disky' shapes of the level curves when respectively $\beta < 0$ and $\beta > 0$. The presence of the parameter β affects the bifurcation of inclined orbits. The normal form of the system is the same as $\mathcal{K}_{11}$ in (4.2) except that the coefficient in front of the resonant term is replaced by

$$b_1 + \beta. \tag{4.43}$$

Now, the non degeneracy conditions in remark 3.3 and in (3.39) are satisfied, since we have

$$A = \frac{3b_1}{4}, \quad B = 0, \quad C = \frac{1}{4}(b_1 + \beta).$$

To fix the ideas, let as take $\delta < 0$. We see from proposition (3.4) that, for $\beta > 0$, unstable inclined orbits bifurcate from the x_2- normal mode of the system at

$$\mathcal{E} = \mathcal{E}_{1u} = -\frac{\delta}{3\beta} + O(\delta^2). \tag{4.44}$$

Instead, for $\beta < 0$, from proposition 3.8 we get that stable inclined orbits appear from the x_1- axis orbit at

$$\mathcal{E}_{2u} = \frac{\delta}{3\beta} + O(\delta^2). \tag{4.45}$$

Therefore, the distinction on the sign of β is relevant if one is interested in which normal mode suffers a change of stability when a new orbit arises. One could ask if for $\beta < 0$ a global bifurcation occurs. This is not the case since $B = 0$ (cfr. proposition 3.6).

Stable loop orbits still bifurcate from the x_2- normal mode and we can apply propositions 3.4 and 3.8 to find the corresponding threshold value $\mathcal{E} = \mathcal{E}_{1l}$, where now

$$\mathcal{E}_{1l} = \frac{\delta}{2b_1 - \beta} + O(\delta^2). \tag{4.46}$$

The critical values (4.44)-(4.46) comes out as well if we perform our analysis on the reduced phase space $\mathcal{L}$. In fact, in case $\beta \neq 0$, on the reduced phase space section $\mathcal{L} \cap \{y = 0\}$ the Hamiltonian corresponds to the parabola

$$x = h - \frac{\delta}{b_1 + \beta} z - \frac{b_1 - 2\beta}{b_1 + \beta} z^2 \tag{4.47}$$

which is still downward concave for small values of β (namely $\frac{b_1}{2} < \beta < -b_1$, for $b_1 < 0$), but now does not have the same curvature of the upper arc $\mathcal{C}_+$ at the vertex. As a consequence, a tangency point between the parabola and $\mathcal{C}_+$ can arise (see figure 4.1(d)). This is the case for

$$h = h_u := \left(\frac{\delta^2}{12\beta(b_1 + \beta)} + \frac{\mathcal{E}^2}{4} \right) \tag{4.48}$$

with corresponding solution given by

$$z = z_u := \frac{\delta}{6\beta}.$$

Imposing the condition

$$-\frac{\mathcal{E}}{2} < z_u < \frac{\mathcal{E}}{2}$$

we find the threshold values (4.44), if $\beta > 0$, and (4.45) if $\beta < 0$. The critical values (4.46) comes out as well from the investigation of the tangency points between parabola (4.47) and the lower contour phase space $\mathcal{C}_-$.

As observed above, the case $\delta > 0$ can be studied similarly.

Finally, let us express these results in the case of the α-models (4.1). The critical energy values at which stable loop orbits appear is now given by

$$E = E_{l\beta} := \frac{4|q - 1|}{2 - \alpha + 4\beta} + O(|q - 1|^2). \tag{4.49}$$

As in case $\beta = 0$ their bifurcation occur from the x_1 axis orbit in the oblate case and from the x_2- axis orbit in the prolate case, with a subsequent transition to instability of the normal mode.

Unstable inclined orbits bifurcate at

$$E = E_{i\beta} := \frac{|q - 1|}{|\beta|} + O(|q - 1|^2). \tag{4.50}$$

In the prolate case they appear from the x_2-normal mode (which as a consequence comes back to stability) if $\beta > 0$ and from the x_1-normal mode (which suffers a transition to instability) if $\beta < 0$. In the oblate case the roles of the normal modes in the bifurcation of inclined orbits are reversed.

Thus, if we break the ellipticity of the potential, inclined orbits appear: however the smaller the deformation, the higher the threshold value (4.44) and (4.45). Loops continue to bifurcate at a lower energy: to invert the bifurcation sequence, unreasonable high values of β are required. The phenomenon is anyway interesting because it can easily be checked that the two families are always of different stability nature: the stable one is the first to appear, therefore there is a critical value of β at which there is an exchange of stability between loops and inclined. The special value $\beta = \frac{\alpha - 2}{4}$ producing the divergence in (4.50) is associated to exact separability in rotated Cartesian coordinates which forbids the existence of the loops.

4.3 2:4 Resonance: bifurcation of banana and anti-banana orbits

Another important class of bifurcations is that of *banana* orbits [84] which usually bifurcate when the frequency of an axial orbit falls to *one half* of that of a normal perturbation. The anti-phase family are the *figure-eight* periodic orbits, or anti-banana: we will show that in the potentials (4.1) *stable* bananas bifurcates at lower energies than *unstable* anti-bananas for relevant values of the parameters.

In the case of a $m_1 = 1$, $m_2 = 2$ resonance with reflection symmetries about both axes, we know from section 1.3, that the normalization procedure must be pushed at least to order $N = 2$. The terms in the series expansion (4.13) are now given by

$$H_0 = \frac{1}{2}(p_1^2 + x_1^2) + p_2^2 + x_2^2, \tag{4.51}$$

$$H_2 = \tilde{\delta}(x_1^2 + p_1^2) + B_1(x_1^2 + 2x_2^2)^2, \tag{4.52}$$

$$H_4 = 2\tilde{\delta}B_1(x_1^4 - 4x_2^4) + B_2\left(x_1^2 + 2x_2^2\right)^3. \tag{4.53}$$

After normalization, we get the normal form

$$K_{12} = \sum_{k=0}^{2} \varepsilon^{2k} K_{2k}, \tag{4.54}$$

where

$$K_0 = J_1 + 2J_2, \tag{4.55}$$

$$K_2 = 2\tilde{\delta}J_1 + B_1\left(\frac{3}{2}J_1^2 + 4J_1 J_2 + 6J_2^2\right), \tag{4.56}$$

$$K_4 = 3\tilde{\delta}B_1\left(J_1^2 - 4J_2^2\right) - (17B_1^2 - 10B_2)\left(\frac{1}{4}J_1^3 + 2J_2^3\right)$$

$$- \frac{2}{3}(46B_1^2 - 27B_2)J_1 J_2^2 - \left(\frac{56}{3}B_1^2 - 9B_2\right)J_1^2 J_2$$

$$- \frac{3}{2}(2B_1^2 - B_2)J_1^2 J_2 \cos(4\phi_1 - 2\phi_2). \tag{4.57}$$

Due to the normalization procedure, we obtain an integrable system with dynamics generated by the Hamiltonian (4.54) with a second independent integral of motion

$$K_0 = J_1 + 2J_2. \tag{4.58}$$

To exploit the new integral of motion (5.18), we perform the quasi canonical transformation to "adapted resonance coordinates" [92]

$$\begin{cases} J_1 = \mathcal{E} + 2\mathcal{R} \\ J_2 = 2\mathcal{E} - \mathcal{R} \\ \psi = 4\phi_1 - 2\phi_2 \\ \chi = 2\phi_1 + 4\phi_2 \end{cases} \tag{4.59}$$

It can be easily seen that χ is cyclic and its conjugate action is proportional to the additional integral of motion, namely

$$K_0 = J_1 + 2J_2 = 5\mathcal{E}. \tag{4.60}$$

Thus we introduce the "effective Hamiltonian"

$$\mathcal{K}_{12}(\mathcal{R}, \psi; \mathcal{E}) \doteq K_{12}(J(\mathcal{E}, \mathcal{R}), \phi(\psi, \chi)). \tag{4.61}$$

Considering the dynamics at a fixed values of $\mathcal{E}$, we have that $\mathcal{K}$ defines a one degree of freedom system with the following equations of motion

$$\dot{\mathcal{R}} = -\frac{3}{2}\varepsilon^4(2B_1^2 - B_2)(2\mathcal{E} - \mathcal{R})(\mathcal{E} + 2\mathcal{R})^2 \sin\psi, \tag{4.62}$$

$$\dot{\psi} = 2\varepsilon^2\left(B_1(3\mathcal{E} - 4\mathcal{R}) - 2\tilde{\delta}\right) +$$

$$- \frac{1}{6}\varepsilon^4\left[5\mathcal{A}(\mathcal{R}; \mathcal{E}, \tilde{\delta}) - \mathcal{B}(\mathcal{R}; \mathcal{E})\cos\psi\right], \tag{4.63}$$

where

$$\mathcal{A} = 36B_2\mathcal{E}(-3\mathcal{E} + 4\mathcal{R})$$
$$+ B_1^2(155\mathcal{E}^2 - 276\mathcal{E}\mathcal{R} + 48\mathcal{R}^2) + 72B_1\mathcal{E}\tilde{\delta},$$
$$\mathcal{B} = 9(2B_1^2 - B_2)(7\mathcal{E}^2 + 8\mathcal{E}\mathcal{R} - 12\mathcal{R}^2).$$

The fixed points of this system give the periodic orbits of the original system. The pair of solutions with $\mathcal{R} = 2\mathcal{E}$, $\mathcal{R} = -\mathcal{E}/2$ respectively correspond to the normal modes along the x_1-axis and x_2-axis. Let us look for periodic orbits in general position. We start with setting $\psi = 0$ and looking for $\mathcal{R}$-solutions of $\dot{\psi} = 0$. Since we are dealing with a perturbation problem in ε, we look for a solution in the form [66]

$$\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1\varepsilon^2 + O(\varepsilon^4). \tag{4.64}$$

We substitute (4.64) in (4.63) with $\psi = 0$ and collect terms up to fourth order in ε. Equating to zero the coefficient of second order, we find that $\mathcal{R}_0$ has to satisfy

$$B_1(3\mathcal{E} - 4\mathcal{R}_0) - 2\tilde{\delta} = 0 \tag{4.65}$$

which gives

$$\mathcal{R}_0 = \mathcal{R}_{b0} \equiv \frac{3}{4}\mathcal{E} - \frac{\tilde{\delta}}{2B_1}. \tag{4.66}$$

Once computed $\mathcal{R}_0$ we find the coefficient of the second order term in the expansion of the fixed point

$$\mathcal{R}_b \equiv \mathcal{R}_{b0} + \mathcal{R}_{b1}\varepsilon^2, \quad \psi = 0, \tag{4.67}$$

which determines the banana orbits:

$$J_{1b} = \mathcal{E} + 2\mathcal{R}_b, \tag{4.68}$$

$$J_{2b} = 2\mathcal{E} - \mathcal{R}_b. \tag{4.69}$$

Similarly, for $\psi = 4\theta_1 - 2\theta_2 = \pi$, we find a solution of the form

$$\mathcal{R}_a \equiv \mathcal{R}_{a0} + \mathcal{R}_{a1}\varepsilon^2, \quad \psi = \pi, \tag{4.70}$$

and $J_{1a} = \mathcal{E} + 2\mathcal{R}_a$, $J_{2a} = 2\mathcal{E} - \mathcal{R}_a$, corresponding to the antibanana orbits.

In view of (4.59), the constraints $0 \leq J_1 \leq 5\mathcal{E}$, $0 \leq J_2 \leq 5\mathcal{E}/2$ applied to these solutions give the condition of existence for these periodic orbits. Non trivial existence conditions can be found by solving $J_{1,2b} \geq 0$ for the bananas and $J_{1,2a} \geq 0$ for the anti-bananas. The implicit function theorem assures that there exist unique solutions $\mathcal{E}_c = \mathcal{E}_c(\delta)$ in each cases determining the bifurcation thresholds. For the bananas, up to the second perturbation order we get

$$\mathcal{E}_{1b} = \frac{2}{5B_1}\tilde{\delta} + \frac{97B_1^2 - 36B_2}{15B_1^3}\tilde{\delta}^2\varepsilon^2, \tag{4.71}$$

$$\mathcal{E}_{2b} = -\frac{2}{5B_1}\tilde{\delta} + \frac{59B_1^2 - 27B_2}{15B_1^3}\tilde{\delta}^2\varepsilon^2, \tag{4.72}$$

which respectively determine the bifurcation from the x_2-axial normal mode in the first case, and from the x_1-axial normal mode in the second case (we discuss below which of these possibilities actually shows up). Similarly, the threshold values that gives the existence condition of anti-banana orbits are given by

$$\mathcal{E}_{1a} = \frac{2}{5B_1}\tilde{\delta} + \frac{97B_1^2 - 36B_2}{15B_1^3}\tilde{\delta}^2\varepsilon^2. \tag{4.73}$$

$$\mathcal{E}_{2a} = -\frac{2}{5B_1}\tilde{\delta} + \frac{19B_1^2 - 9B_2}{3B_1^3}\tilde{\delta}^2\varepsilon^2, \tag{4.74}$$

By comparing (4.71) with (4.73) we see a first interesting result: if the bifurcation occur from the x_2-axis, banana and anti-banana appear together. It is therefore important to discriminate between the two possibilities. Since the dominant term in the series is the first and $\mathcal{E}$ must be positive, we see that case 1 (bifurcation from the x_1-axis) or 2 (bifurcation from the x_2-axis) occur if $\tilde{\delta}$ and B_1 have different sign or not. To write the expressions of the bifurcation curves in the physical (q, E)-plane, we note that according to the rescaling (4.11) with $m_2 = 2$, on the x_1-axis orbit we have

$$E = 5\mathcal{E}\varepsilon^2 + \frac{75}{2}B_1\mathcal{E}^2\varepsilon^4 + O(\varepsilon^6), \tag{4.75}$$

so that, modulo third order terms in δ, we get

$$E_{2b} = -\frac{2}{B_1}\delta + \frac{77B_1^2 - 27B_2}{3B_1^3}\delta^2, \tag{4.76}$$

$$E_{2a} = -\frac{2}{B_1}\delta + \frac{113B_1^2 - 45B_2}{3B_1^3}\delta^2, \tag{4.77}$$

for the bifurcations from the x_1-axis. Similarly, we obtain

$$E_{1b} = E_{1a} = \frac{2}{B_1}\delta + \frac{103B_1^2 - 36B_2}{3B_1^3}\delta^2, \tag{4.78}$$

for the bifurcations from the x_2-axis. To be concrete, for our family (4.1) we have that, with $\alpha > 0$, the coefficient B_1 is negative. The ellipticity is usually $q > 1/2$ so that $\delta > 0$, therefore relevant thresholds are

$$E_{2b} = \frac{16}{2-\alpha}\left(q - \frac{1}{2}\right) + \frac{8(41\alpha - 10)}{3(2-\alpha)^2}\left(q - \frac{1}{2}\right)^2, \tag{4.79}$$

$$E_{2a} = \frac{16}{2-\alpha}\left(q - \frac{1}{2}\right) + \frac{8(53\alpha + 14)}{3(2-\alpha)^2}\left(q - \frac{1}{2}\right)^2. \tag{4.80}$$

Since the difference

$$E_{2a} - E_{2b} = 32\frac{2+\alpha}{(2-\alpha)^2}\left(q - \frac{1}{2}\right)^2 \tag{4.81}$$

is positive, we verify that, for models in the class (4.1) and with parameter ranges useful for elliptical galaxies, the bifurcation sequence is always from the major axis, with bananas appearing at lower energies than anti-bananas. By checking the nature of the two critical points (4.67,4.70), it is shown in next chapter (see also [77]) that in systems (4.1) the first family is always stable and the second unstable: (4.79,4.80) generalize the corresponding expressions for the logarithmic ($\alpha = 0$) potential reported in [11]. As long as the banana does not bifurcate the major axis is stable. It looses its stability at the first bifurcation and regains it at the second. It is natural to ask how much these results are affected by ellipse-breaking deformations: we can say that, in analogy with what we have seen for the 2 : 2 resonance, the hierarchy of bifurcations changes only for unreasonable high values of the deformation parameter.

Chapter 5

Bifurcations in the 2:4 detuned Resonance

In the following, we consider the most general (truncated) bi-dimensional potential with double reflection symmetries around a $1 : 2$ resonance (since we are in the presence of symmetries we will speak more correctly of a $2 : 4$ resonance). Its peculiarities are worth of note by themselves, since the approach followed to study systems with a single symmetry [21] still works for a $2 : 2$ resonance, but is not able to describe the generic behavior of the system in presence of a $2 : 4$ resonance [23]. In particular, the planar reduction method we applied in section 3.2 produces for a $2 : 4$ resonance a polynomial normal form with infinite codimension (namely the central singularity is still given by (3.38), but with $\varepsilon_1 = \varepsilon_2$ and $\mu^2 = 4$, cfr. proposition 2.4). Here, we consider a generic perturbation up to the degree necessary to include resonant terms. Even if this is not enough to deduce the general behavior for arbitrary perturbations, we will see that a truncation of the normal form at the first term incorporating the resonance is able to capture the essential features of the bifurcation.

5.1   The model and its normal form

Suppose the system under investigation is given by a natural Hamiltonian

$$\mathcal{H}(x_1, x_2, p_1, p_2) = \frac{1}{2}(p_1^2 + p_2^2) + \mathcal{V}(x_1, x_2) \tag{5.1}$$

where $\mathcal{V}$ is a smooth potential with an absolute minimum in the origin and symmetric under reflections with respect to each coordinate axes. We assume the potential to be expanded as a truncated power series

$$\mathcal{V}(x_1, x_2) \equiv \sum_{n=0}^{N} V_{2n}(x_1, x_2) \tag{5.2}$$

where V_{2n} is a homogeneous polynomial of degree $2n + 2$. In force of the reflection symmetries, the zero order term can be written as

$$V_0 = \frac{1}{2}(\omega_1^2 x_1^2 + \omega_2^2 x_2^2) \tag{5.3}$$

and the odd order terms are all zero. The two coefficients of the quadratic term are written so to represent the unperturbed harmonic frequencies. We therefore perform the scaling (3.8) and also rescale the Hamiltonian (5.1) according to (3.9). Thus we obtain

$$\widetilde{\mathcal{H}}(x_1, x_2, p_1, p_2) = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(\omega_1^2 x_1^2 + \omega_2^2 x_2^2) + \sum_{j=1}^{N} \varepsilon^{2j} V_{2j}(x_1, x_2). \tag{5.4}$$

In this way the terms of the expansions are ordered in powers of the small perturbation parameter.

In general, neither the original system (5.1) nor its expansion (5.4) are integrable. Let us proceed to construct a normal form for the system, namely a new Hamiltonian series which, in the case of 2 degrees of freedom, is an integrable approximation of the original one. Its structure is suitable to capture the most relevant orbital features of the system. As explained is section 1.3, we can assemble a resonant normal form by assuming from the start a rational value for the ratio of the harmonic frequencies: this assumption produces the presence in the new Hamiltonian of resonant terms, namely terms depending on a linear combination of angles with integer coefficients. The trick is then to assume that our system is such that the ratio

$$\rho = \omega_1/\omega_2, \tag{5.5}$$

is not far from a rational value and then to approximate it by introducing a small detuning δ so that

$$\rho = \frac{m_1}{m_2} + \delta. \tag{5.6}$$

Afterwards we proceed as if the unperturbed harmonic part was in exact $m_1{:}m_2$ resonance by treating the remaining part as a higher order perturbation.

In the following, let us proceed with the case of the $m_1 = 1$, $m_2 = 2$ resonance, so that (5.6) translates into

$$\delta = \frac{\omega_1}{\omega_2} - \frac{1}{2}. \tag{5.7}$$

Since we are in presence of reflection symmetries about both axes, we know that the normalization procedure must be pushed at least to the fourth degree in ε, i.e. $N = 2$. To give the system a structure suitable to apply the normalization procedure, we perform the transformation (3.13) and scale

the detuning parameter according to $\delta = \tilde{\delta}\varepsilon^2$. Afterwards, redefining the Hamiltonian according to the scaling

$$H := \frac{2}{\omega_2}\widetilde{\mathcal{H}} = \left(1 + 2\tilde{\delta}\varepsilon^2\right)\omega_1\widetilde{\mathcal{H}} \tag{5.8}$$

and collecting terms in ε, we put the rescaled Hamiltonian into the form

$$H(\mathbf{p}, \mathbf{x}; \tilde{\delta}, \varepsilon) = \sum_{j=0}^{N} \varepsilon^{2j} H_{2j}(\mathbf{p}, \mathbf{x}; \tilde{\delta}) \tag{5.9}$$

where the unperturbed term is given by

$$H_0(\mathbf{p}, \mathbf{x}) = \frac{1}{2}(x_1^2 + p_1^2) + (x_2^2 + p_2^2) \tag{5.10}$$

and the detuning term $\tilde{\delta}(x_1^2 + p_1^2)$ is considered as a term of order 2. The system is now ready for a standard resonant normalization. The higher order terms up to $N = 2$ are polynomials of the type

$$
\begin{aligned}
H_2 &= \tilde{\delta}(x_1^2 + p_1^2) + ax_1^4 + bx_2^4 + cx_1^2x_2^2 \\
H_4 &= a_1x_1^6 + b_1x_1^4x_2^2 + c_1x_1^2x_2^4 + d_1x_2^6 + (2ax_1^4 + 4cx_1^2x_2^2 + 6bx_2^4)\tilde{\delta}.
\end{aligned}
\tag{5.11}
$$
$$\tag{5.12}$$

The choice of coefficients appearing in (5.11) and (5.12) is suggested by the values the coefficients take in the most common physical cases: we remark that, in applying the results obtained in the following to specific model problems, we have to take into account that the Hamiltonian (5.10–5.12) is in the form "prepared" for normalization. Therefore the canonical variables are rescaled according to (3.13) and the frequency ratio is expanded in series of the detuning as in (5.7). These transformations affect the numerical values of the coefficients of the various terms. In case of the α-models, cfr (4.1), the coefficients read as

$$
\begin{aligned}
a &= \frac{(\alpha - 2)}{8}, \\
b &= c = \frac{(\alpha - 2)}{2} \\
a_1 &= \frac{1}{48}(2 - \alpha)(4 - \alpha), \\
b_1 &= 6a_1, \\
c_1 &= 12a_1, \\
d_1 &= 8a_1.
\end{aligned}
\tag{5.13}
$$

Here we want to study the most general potential truncated to degree six in the coordinates and complying with the enforced double reflection symmetry.

Introducing action-angle like variable (1.49), the outcome of the normalization procedure can be written as

$$K(J_1, J_2, \phi_1, \phi_2) = \sum_{j=0}^{2} \varepsilon^{2j} K_{2j}(J_1, J_2, \phi_1, \phi_2), \tag{5.14}$$

which, coherently with the truncation order $N = 2$, implies that a reminder of order six in ε is neglected. The non-vanishing terms turn out to be the following:

$$K_0 = J_1 + 2J_2 \tag{5.15}$$

$$K_2 = 2\tilde{\delta}J_1 + \frac{3}{2}aJ_1^2 + cJ_1J_2 + \frac{3}{2}bJ_2^2 \tag{5.16}$$

$$K_4 = 3a\tilde{\delta}J_1^2 + \frac{1}{2}\left(5a_1 - \frac{17a^2}{2}\right)J_1^3 + \frac{1}{2}\left(5d_1 - \frac{17b^2}{4}\right)J_2^3$$

$$+ \frac{1}{12}\left(-c(18b + 5c) + 18c_1\right)J_1J_2^2 - \left(3ac + \frac{5c^2}{12} - \frac{3b_1}{2}\right)J_1^2J_2$$

$$+ \frac{1}{4}J_1^2J_2\left((a - c)c + b_1\right)\cos(4\phi_1 - 2\phi_2) \tag{5.17}$$

5.2 Bifurcation analysis of the $2 : 4$ resonance

The essential information we need concerns the existence and stability of the periodic orbits associated to the resonance [11, 76]. We can compute the thresholds for the bifurcations sequences in terms of the parameters relying on the regular nature of the dynamics given by the normal form. In two degrees of freedom, if a Hamiltonian possesses a second independent integral of motion, the system is Liouville integrable. Due to the normalization procedure, we have obtained the Hamiltonian (5.14) with the second independent integral of motion

$$K_0 = J_1 + 2J_2. \tag{5.18}$$

We can use this integral to reduce the dimension of the problem by performing the canonical transformation to "adapted resonance coordinates" (4.59).

Since χ is cyclic and its conjugate momentum is proportional to the additional integral of motion, namely

$$\mathcal{E} = (J_1 + 2J_2)/5, \tag{5.19}$$

we introduce the reduced Hamiltonian

$$\mathcal{K}(\mathcal{R}, \psi; \mathcal{E}) = 5\mathcal{E} + \varepsilon^2 \mathcal{K}_1(\mathcal{R}) + \varepsilon^4 \mathcal{K}_2(\mathcal{R}, \psi). \tag{5.20}$$

By using Mathematica® we obtain the following expressions for $\mathcal{K}_1$ and $\mathcal{K}_2$:

$$\mathcal{K}_1(\mathcal{R};\mathcal{E}) \;=\; \frac{3}{2}b(-2\mathcal{E}+\mathcal{R})^2 + \frac{3}{2}a(\mathcal{E}+2\mathcal{R})^2 + (\mathcal{E}+2\mathcal{R})(2c\mathcal{E}-c\mathcal{R}+2\tilde{\delta})$$

(5.21)

$$\mathcal{K}_2(\mathcal{R},\psi;\mathcal{E}) \;=\; \left(3b_1 - \frac{17}{4}a^2 - 17b^2 - 6ac - 6bc - \frac{5c^2}{2} + \frac{5a_1}{2} + 6c_1 + 20d_1\right)\mathcal{E}^3$$
$$+\;\; 3a\tilde{\delta}\mathcal{E}^2 + \mathcal{A}(\mathcal{R};\mathcal{E}) + \mathcal{B}(\mathcal{R};\mathcal{E})\cos\psi$$

(5.22)

with

$$\mathcal{A}(\mathcal{R};\mathcal{E}) \;=\; \left(12ac - 34a^2 + \frac{17b^2}{8} - 3bc + \frac{5c^2}{6} + 20a_1 - 6b_1 + 3c_1 - \frac{5d_1}{2}\right)\mathcal{R}^3$$
$$+\;\; \left[\left(306b^2 - 306a^2 - 252ac - 72bc - 55c^2 + 180a_1 + 126b_1 + 72c_1\right.\right.$$
$$-\;\; \left.360d_1\right)\mathcal{E} + 144a\tilde{\delta}\right]\frac{\mathcal{E}\mathcal{R}}{12} + \left[\left(-204a^2 - 51b^2 - 48ac + 42bc + 5c^2\right.\right.$$
$$+\;\; \left.120a_1 + 24b_1 - 42c_1 + 60d_1\right)\mathcal{E} + 48a\tilde{\delta}\right]\frac{\mathcal{R}^2}{4}$$

(5.23)

$$\mathcal{B}(\mathcal{R};\mathcal{E}) \;=\; \frac{1}{4}(2\mathcal{E}-R)(\mathcal{E}+2R)^2\left((a-c)c+b_1\right).$$

(5.24)

Considering the dynamics at a fixed values of $\mathcal{E}$, we have that $\mathcal{K}$ defines a one degree of freedom system with the following equations of motion

$$\dot{\psi} \;=\; (6a\mathcal{E} - 6b\mathcal{E} + 3c\mathcal{E} + 12a\mathcal{R} + 3b\mathcal{R} - 4c\mathcal{R} + 4\tilde{\delta})\varepsilon^2$$
$$+\;\; \left(\frac{\partial\mathcal{A}}{\partial\mathcal{R}}(R;\mathcal{E}) + \frac{\partial\mathcal{B}}{\partial\mathcal{R}}(R;\mathcal{E})\right)\varepsilon^4$$

(5.25)

$$\dot{\mathcal{R}} \;=\; \frac{1}{4}\mu\left[(2\mathcal{E}-\mathcal{R})(\mathcal{E}+2\mathcal{R})^2\sin\psi\right]\varepsilon^4$$

(5.26)

where

$$\mu := (a-c)c + b_1.$$

(5.27)

The fixed points of this system give the periodic orbits of the original system so that the condition $\mu \neq 0$ must be satisfied.

The pair of fixed points with $\mathcal{R} = 2\mathcal{E}$, $\mathcal{R} = -\mathcal{E}/2$ correspond to the normal modes, that is to the periodic orbit along the x_1-axis ($J_2 = 0$) and to the periodic orbit along the x_2-axis ($J_1 = 0$) respectively. Additional periodic orbits may appear when the system passes through the resonance. These periodic orbits in general position exist only above a given threshold value for $\mathcal{E}$ when the axial orbits change their stability. This phenomenon can be seen as a bifurcation of the new family from the normal mode when it enters in 1 : 2 resonance with a normal perturbation. The phase between the

two oscillations determines the nature of the families: they are respectively given by the conditions $\psi = 0$ (banana orbits) and $\psi = \pm\pi$ (antibanana orbits). These phase conditions are solutions of $\dot{\mathcal{R}} = 0$ (when $\mathcal{R} \neq 2\mathcal{E}$ and $\mathcal{R} \neq -\mathcal{E}/2$) and determine the corresponding solutions of $\dot{\psi} = 0$.

Let us start by looking for banana orbits. Proceeding as in section 4.3, we set $\psi = 0$ in the equation $\dot{\psi} = 0$ and look for a solution in the form

$$\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1 \varepsilon^2 + O(\varepsilon^4) \tag{5.28}$$

so that the right hand side of (5.25) vanish up to fourth order in ε. For $\psi = 0$, we find that $\mathcal{R}_0$ has to satisfy

$$3(2a - 2b + c)\mathcal{E} + (3b - 4c + 12a)\mathcal{R}_0 + 4\tilde{\delta} = 0 \tag{5.29}$$

This equation admits solution only if

$$\nu := 12a + 3b - 4c \neq 0 \tag{5.30}$$

and, if this condition is satisfied, we find

$$\mathcal{R}_0 = \mathcal{R}_{B0} := \frac{-3(2a - 2b + c)\mathcal{E} - 4\tilde{\delta}}{12a + 3b - 4c}. \tag{5.31}$$

Once $\mathcal{R}_0$ is computed, we look for $\mathcal{R}_1$ such that the fourth order term of the right hand side of (4.63) vanish. Since (5.30) is satisfied, we find a solution

$$\mathcal{R}_1 = \mathcal{R}_{B1}(\mathcal{E}; \delta). \tag{5.32}$$

The corresponding fixed point is given by

$$\mathcal{R} = \mathcal{R}_B := \mathcal{R}_{B0} + \mathcal{R}_{B1}\varepsilon^2, \quad \psi = 0 \tag{5.33}$$

and determines the banana orbits:

$$J_1 = J_{1B} = \mathcal{E} + 2\mathcal{R}_B \tag{5.34}$$
$$J_2 = J_{2B} = 2\mathcal{E} - \mathcal{R}_B. \tag{5.35}$$

Similarly, if (5.30) is satisfied, for $4\phi_1 - 2\phi_2 = \pm\pi$, we find

$$J_1 = J_{1A} = \mathcal{E} + 2\mathcal{R}_A \tag{5.36}$$
$$J_2 = J_{2A} = 2\mathcal{E} - \mathcal{R}_A, \tag{5.37}$$

which correspond to the antibanana orbits. In view of (4.59), the constraints

$$0 \leq J_1 \leq 5\mathcal{E}, \quad 0 \leq J_2 \leq \frac{5\mathcal{E}}{2}, \tag{5.38}$$

applied to these solutions give the condition of existence for these periodic orbits in general position. Whether these conditions are satisfied or not, it depends on the parameters of the system.

For the banana orbits we find that at the perturbation order zero

$$J_{1B} = \frac{5(3b - 2c)\mathcal{E} - 8\tilde{\delta}}{12a + 3b - 4c},$$

$$J_{2B} = \frac{5(6a - c)\mathcal{E} + 4\tilde{\delta}}{12a + 3b - 4c}. \tag{5.39}$$

Thus, we get different existence conditions according to the sign of the constant ν defined in (5.30). If we define $M := 5(3b - 2c)$ and $N := 5(6a - c)$, so that $5\nu = M + 2N$, taking ε as small as the constraints (5.38) remain satisfied at least up to the first perturbation order, banana orbits bifurcate in the following cases:

$$\text{if } \nu < 0 \text{ and } \tilde{\delta}N < 0 \quad : \quad \begin{array}{ll} \text{if } \tilde{\delta}M < 0 & \mathcal{E} > \mathcal{E}_{2B} \\ \text{if } \tilde{\delta}M > 0 & \mathcal{E}_{2B} < \mathcal{E} < \mathcal{E}_{1B} \end{array}$$

$$\tag{5.40}$$

$$\text{if } \nu > 0 \text{ and } \tilde{\delta}M > 0 \quad : \quad \begin{array}{ll} \text{if } \tilde{\delta}N > 0 & \mathcal{E} > \mathcal{E}_{1B} \\ \text{if } \tilde{\delta}N < 0 & \mathcal{E}_{1B} < \mathcal{E} < \mathcal{E}_{2B} \end{array}$$

$$\tag{5.41}$$

where, modulo terms $O(\varepsilon^4)$ the critical values

$$\mathcal{E}_{1B} \quad := \quad -\frac{8}{5(2c - 3b)}\tilde{\delta} + \frac{4\left(153b^2 - 72bc - 20c^2 + 72c_1 - 180d_1\right)}{15(3b - 2c)^3}\tilde{\delta}^2\varepsilon^2 \tag{5.42}$$

$$\mathcal{E}_{2B} \quad \doteq \quad -\frac{4}{5(6a - c)}\tilde{\delta} + \frac{4\left(522a^2 - 69ac - 8c^2 - 180a_1 + 21b_1\right)}{15(6a - c)^3}\tilde{\delta}^2\varepsilon^2 \tag{5.43}$$

correspond to the solutions of $J_{1B} = 0$ and $J_{2B} = 0$ and respectively determine the bifurcation of banana orbits from the x_2-normal mode and the x_1-normal mode. Replacing the coefficients according to (5.13), we obtain the threshold values (4.71) and (4.72) found in section 4.3 for the $\alpha-$ models.

A similar argument provides the existence condition of antibanana orbits. Since to the first perturbative order $J_{1A} = J_{1B}$ and $J_{2A} = J_{2B}$, the birth of antibananas is given by the same conditions on the coefficients given above. However the higher order terms in J_{kA}, J_{kB}, $k = 1, 2$ are in general different: the discrimination between the thresholds for the bifurcation of the two families is possible only by going at least to second order [11]. This result was expected on the basis of the structure of the new Hamiltonian (5.14).

Nevertheless, up to the fourth order in ε we find that the threshold value corresponding to the bifurcation of antibanana orbits from the x_2-axis, $\mathcal{E}_{1A}$, coincides with the critical value $\mathcal{E}_{2B}$ of (5.43). Hence, as we found

for elliptical equipotential, cfr. section 4.3, if banana and antibanana orbits bifurcate from the x_2-normal mode, they do it simultaneously.

On the other hand the bifurcation from the x_1-axis orbit occurs for

$$\mathcal{E} = \mathcal{E}_{2A}, \qquad \text{if } \tilde{\delta}N < 0 \tag{5.44}$$

where

$$\mathcal{E}_{2A} := -\frac{4}{5(6a-c)}\tilde{\delta} + \frac{4\left(522a^2 - 75ac - 2c^2 - 180a_1 + 15b_1\right)}{15(6a-c)^3}\tilde{\delta}^2\varepsilon^2 + O(\varepsilon^4) \tag{5.45}$$

which, at second order, is different from (5.43). Thus, we obtain the following existence conditions for antibanana orbits:

$$\text{if } \nu < 0 \text{ and } \tilde{\delta}N < 0 \quad : \quad \begin{array}{ll} \text{if } \tilde{\delta}M < 0 & \mathcal{E} > \mathcal{E}_{2A} \\ \text{if } \tilde{\delta}M > 0 & \mathcal{E}_{2A} < \mathcal{E} < \mathcal{E}_{1B} = \mathcal{E}_{1A} \end{array} \tag{5.46}$$

$$\text{if } \nu > 0 \text{ and } \tilde{\delta}M > 0 \quad : \quad \begin{array}{ll} \text{if } \tilde{\delta}N > 0 & \mathcal{E} > \mathcal{E}_{1B} \\ \text{if } \tilde{\delta}N < 0 & \mathcal{E}_{1B} < \mathcal{E} < \mathcal{E}_{2A} \end{array} \tag{5.47}$$

$$\text{with } \begin{cases} \mathcal{E}_{2A} \geq \mathcal{E}_{2B}, & \text{if } \tilde{\delta}\mu > 0 \\ \mathcal{E}_{2A} < \mathcal{E}_{2B}, & \text{if } \tilde{\delta}\mu < 0 \end{cases} \tag{5.48}$$

We can write explicitly the relative magnitude of the two thresholds for bifurcation from the normal modes since we find

$$\mathcal{E}_{2A} - \mathcal{E}_{2B} = -\frac{8}{5}\frac{\mu}{(6a-c)^3}\tilde{\delta}^2\varepsilon^2 + O(\varepsilon^4), \tag{5.49}$$

$$\mathcal{E}_{1A} - \mathcal{E}_{1B} = O(\varepsilon^4). \tag{5.50}$$

The first of these expressions points out that the hierarchy of bifurcations from the x_1-normal mode is determined by the sign of the constant μ defined in (5.27). The second confirms that, at the order of our perturbation treatment, the bifurcations from the x_2-normal mode occur simultaneously. Which of the two scenarios is actually happening depend on the value of the parameters according to the conditions listed above. In case of the α-models (4.1) we find $\mu = (4 - \alpha^2)/16 > 0$, thus the bifurcation of banana orbits foregoes the appearance of antibanana orbits, as we found in section 4.3.

For practical purposes, e.g. to compare these findings with the outcomes of numerical simulations, it would be more useful to have the expression of the bifurcation curves in terms of "physical" parameters. The most natural way to represent the thresholds is by plotting curves in the (δ, E)-plane, where E is the physical energy of the system defined by (5.1)

$$\mathcal{H}(x_1, x_2, p_1, p_2) = E \tag{5.51}$$

and δ as defined in (5.6). According to the rescaling (5.8), on the x_1-axis orbit ($J_2 = 0$, $J_1 = 5\mathcal{E}$), we have

$$\varepsilon^2 K = 5\mathcal{E}\varepsilon^2 + \left(\frac{75a}{2}\mathcal{E}^2 + 10\mathcal{E}\tilde{\delta}\right)\varepsilon^4 + O(\varepsilon^6) = (1 + 2\tilde{\delta}\varepsilon^2)\omega_1 E. \tag{5.52}$$

The series from equation (5.52) can be used to express the physical energy E in terms of $\mathcal{E}$ [11, 90], namely

$$E = \frac{5\mathcal{E}}{\omega_1}\varepsilon^2 - \frac{75}{8\omega_1}a\mathcal{E}^2\varepsilon^4 + O(\varepsilon^6). \tag{5.53}$$

Thus, up to the second order in ε, for $\mathcal{E}$ satisfying equations (5.43) and (4.74) and δ as in (5.7), we obtain the following threshold values

$$E_{2B} = -\frac{4}{(6a-c)\omega_1}\delta + \frac{4\left(630a^2 - 87ac - 8c^2 - 180a_1 + 21b_1\right)}{3(6a-c)^3\omega_1}\delta^2, \tag{5.54}$$

$$E_{2A} = -\frac{4}{(6a-c)\omega_1}\delta + \frac{4\left(630a^2 - 93ac - 2c^2 - 180a_1 + 15b_1\right)}{3(6a-c)^3\omega_1}\delta^2, \tag{5.55}$$

for the bifurcation of respectively banana and antibanana orbits from the x_1-axis orbits. A similar argument gives the threshold value for the bifurcations from the x_2-axis orbit. Since $\mathcal{E}_{1A} = \mathcal{E}_{1B}$, we have $E_{1A} = E_{1B}$ with

$$E_{1B} = -\frac{8}{(2c-3b)\omega_1}\delta + \frac{4\left(99b^2 + 36bc - 68c^2 + 72c_1 - 180d_1\right)}{3(3b-2c)^3\omega_1}\delta^2. \tag{5.56}$$

These critical values generalize those found in (4.76), (4.77) and (4.78) for elliptical equipotentials.

5.3 Stability analysis of the $2:4$ resonance

Let us now consider the question of the stability of periodic orbits: this analysis complements that of the previous section allowing us to test the relation between change of nature of normal modes and bifurcation of a new family. For banana and antibanana orbits an ordinary investigation of the equations of variations of the system is enough to perform the linear stability analysis. However, in the case of axial orbits, action angle variables have singularities

on them and this also affects the adapted resonance coordinates. The remedy to this problem is quite straightforward: to use a mixed combination of action angle variables on the orbit itself and Cartesian variables for the other degree of freedom.

Let us start with the stability analysis of the periodic orbits in general position. We have to investigate the fate of a normal perturbation of the periodic orbit under test. The system of differential equations for the perturbations $(\delta\psi, \delta\mathcal{R})$ is given by

$$\frac{d}{dt}\left(\begin{array}{c} \delta\psi \\ \delta\mathcal{R} \end{array}\right) = \left(\begin{array}{cc} \mathcal{K}_{\mathcal{R}\psi} & \mathcal{K}_{\mathcal{R}\mathcal{R}} \\ -\mathcal{K}_{\psi\psi} & -\mathcal{K}_{\mathcal{R}\psi} \end{array}\right)\left(\begin{array}{c} \delta\psi \\ \delta\mathcal{R} \end{array}\right). \tag{5.57}$$

Here we again use the reduced Hamiltonian (5.20) and, with a small abuse of notation, we assume without denoting it explicitly that the entries in the Hessian matrix are evaluated on the specific orbit we are interested in. Then, the sign of the determinant

$$\Delta(\mathcal{R}, \psi; \mathcal{E}, \tilde{\delta}) = \mathcal{K}_{\mathcal{R}\mathcal{R}}\mathcal{K}_{\psi\psi} - \mathcal{K}_{\mathcal{R}\psi}^2 \tag{5.58}$$

computed on the periodic orbit determines the fate of the perturbation: if $\Delta(\mathcal{R}, \psi; \mathcal{E}, \tilde{\delta})$ is negative it gives the frequency of bounded oscillating solutions thus determining stability; a change of sign, as a consequence of varying $\mathcal{E}$, produces a stability transition.

On the banana and antibanana orbits we have respectively

$$\Delta(\mathcal{R}_B, 0) = \Delta_B := -\mu\frac{(30a\mathcal{E} - 5c\mathcal{E} + 4\tilde{\delta})(15b\mathcal{E} - 10c\mathcal{E} - 8\tilde{\delta})^2}{4(12a + 3b - 4c)^2}\varepsilon^6 + O(\varepsilon^8) \tag{5.59}$$

$$\Delta(\mathcal{R}_A, \pi) = \Delta_A := \mu\frac{(30a\mathcal{E} - 5c\mathcal{E} + 4\tilde{\delta})(15b\mathcal{E} - 10c\mathcal{E} - 8\tilde{\delta})^2}{768(12a + 3b - 4c)^2}\varepsilon^6 + O(\varepsilon^8) \tag{5.60}$$

and thus we see that the parameter μ plays an important role also for stability. Comparing with (5.42) and (5.43), for $\mu > 0$ banana orbits are stable in the case they bifurcate from the x_1-axis orbit ($\mathcal{E} > \mathcal{E}_{2B}$) and unstable in case their bifurcation occurs from the x_2-axis orbit ($\mathcal{E} > \mathcal{E}_{1B}$). Otherwise, we have instability (stability) when the bifurcation occurs from the x_1-normal mode (x_2-normal mode). Since $\Delta_A = -\Delta_B$ up to the third perturbation order, antibanana orbits turn out to be unstable when banana orbits are stable and viceversa. Actually, the fourth order terms in (5.59) and (5.60) are different, but their difference is again a multiple of μ.

We have also seen in (5.49) that the bifurcation order from the x_1-axis depends on μ. Thus, we can now state that, if μ is positive, we have at first the appearance of (stable) banana orbits followed by (unstable) antibanana

orbits (recall that the threshold values $\mathcal{E}_{2B}$ and $\mathcal{E}_{2A}$ are acceptable only for $\tilde{\delta}N < 0$). On the contrary, for negative values of μ the bifurcation order and stability nature are inverted.

Let us now study the stability of the normal modes. Considering the x_1-axis orbit, we use action-angle variables on the orbit and Cartesian variables on the normal bundle to it, namely

$$\begin{cases} X = & \sqrt{2J}\cos\theta \\ P_X = & \sqrt{2J}\sin\theta \\ Y = & Y \\ P_Y = & V \end{cases} \tag{5.61}$$

so that the periodic orbit is given by

$$Y = V = 0, \qquad J = 5\mathcal{E}. \tag{5.62}$$

In these coordinates, the system of differential equation for the perturbations of the normal mode is given by

$$\frac{d}{dt}\begin{pmatrix} \delta Y \\ \delta V \end{pmatrix} = \begin{pmatrix} \widetilde{\mathcal{K}}_{VY} & \widetilde{\mathcal{K}}_{VV} \\ -\widetilde{\mathcal{K}}_{YY} & -\widetilde{\mathcal{K}}_{YV} \end{pmatrix}\begin{pmatrix} \delta Y \\ \delta V \end{pmatrix} \tag{5.63}$$

where $\widetilde{\mathcal{K}} = \mathcal{K}(Y, V, \theta, J)$. The matrix of the second derivatives of $\widetilde{\mathcal{K}}$ on the periodic orbit depends on $\theta(t) = \omega t$, where

$$\begin{aligned} \omega \;=\;& \frac{\partial\widetilde{\mathcal{K}}}{\partial J} = 1 + \left(15a\mathcal{E} + 2\tilde{\delta}\right)\varepsilon^2 \\ +\;& \left[30a\mathcal{E}\tilde{\delta} + \frac{1}{2}\left(\frac{375}{2}a_1 - \frac{1275}{4}a^2\right)\mathcal{E}^2\right]\varepsilon^4 + O(\varepsilon^6). \end{aligned} \tag{5.64}$$

To remove the dependence on time we introduce complex coordinates

$$\begin{cases} z = & Y + iV \\ w = & Y - iV \end{cases} \tag{5.65}$$

and perform the "rotation"

$$\begin{cases} z = & Ze^{-2i\omega t}, \\ w = & We^{2i\omega t}. \end{cases} \tag{5.66}$$

In this way, the equations of variation (5.63) on the periodic orbit (5.62) give

$$\frac{d}{dt}\begin{pmatrix} \delta Z \\ \delta W \end{pmatrix} = i\begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}\begin{pmatrix} \delta Z \\ \delta W \end{pmatrix} \tag{5.67}$$

where, up to the second perturbation order,

$$
\begin{aligned}
\Lambda_{11} &= -\Lambda_{22} = -(30a\mathcal{E} - 5c\mathcal{E} + 4\tilde{\delta})\varepsilon^2 \\
&\quad - 144a\mathcal{E}\tilde{\delta} + \frac{5}{12}\left(1530a^2 - 180ac - 25c^2 - 900a_1 + 90b_1\right)\mathcal{E}^2\varepsilon^4
\end{aligned}
\tag{5.68}
$$

$$
\Lambda_{12} = -\Lambda_{21} = -\frac{25}{4}\mu\mathcal{E}^2\varepsilon^4.
\tag{5.69}
$$

By solving $\det\Lambda = 0$ we find, as expected, that the critical values of $\mathcal{E}$ which determine a change in the stability of the x_1-axis orbit are precisely given by the bifurcation values

$$
\mathcal{E} = \mathcal{E}_{2B} \quad \text{and} \quad \mathcal{E} = \mathcal{E}_{2A}
\tag{5.70}
$$

as defined by (5.43) and (5.45). Regardless of the the nature of the occurring bifurcations (this is given by the sign of μ), the first one produces a transition from stability to instability of the x_1-normal mode and the second one a return to stability.

Let us now consider the stability of the x_2-axis orbit. Since the periodic orbits in general position bifurcate simultaneously from this normal mode, we expect that the x_2-axis orbit remains stable after the bifurcation. To verify this assertion, we proceed as above by introducing the coordinates

$$
\begin{cases}
Y = & \sqrt{2J}\cos\theta \\
P_Y = & \sqrt{2J}\sin\theta \\
X = & X \\
P_X = & U
\end{cases}
\tag{5.71}
$$

so that the periodic orbit is given by

$$
X = U = 0, \qquad J = \frac{5}{2}\mathcal{E}.
\tag{5.72}
$$

The system of differential equation for the perturbation of the normal mode is given by

$$
\frac{d}{dt}\begin{pmatrix} \delta X \\ \delta U \end{pmatrix} = \begin{pmatrix} \widetilde{\mathcal{K}}_{UX} & \widetilde{\mathcal{K}}_{UU} \\ -\widetilde{\mathcal{K}}_{XX} & -\widetilde{\mathcal{K}}_{XU} \end{pmatrix}\begin{pmatrix} \delta X \\ \delta U \end{pmatrix}
\tag{5.73}
$$

where now $\widetilde{\mathcal{K}} = \mathcal{K}(X, U, \theta, J)$. Since we are dealing with a perturbation of a $1:2$ symmetric resonance, the terms proportional to $\cos(4\phi_1 - 2\phi_2)$ in K_4 are of second degree in J_1 and, as a consequence, the matrix of the second derivative of $\widetilde{K}$ computed on the x_2-normal mode does not depend on θ. Thus, we do not need to perform the transformation (5.65). The equations of variation (5.73) give

$$\frac{d}{dt}\left(\begin{array}{c}\delta X\\\delta U\end{array}\right)=\left(\begin{array}{cc}\Omega_{11}&\Omega_{12}\\\Omega_{21}&\Omega_{22}\end{array}\right)\left(\begin{array}{c}\delta X\\\delta U\end{array}\right)\tag{5.74}$$

where

$$\begin{aligned}\Omega_{11}&=&\Omega_{22}=0 &\tag{5.75}\\\Omega_{12}&=&-\Omega_{21}=1+\left(\frac{5c}{2}\mathcal{E}+2\tilde{\delta}\right)\varepsilon^2-\frac{1}{8}\left(75bc+\frac{125}{6}c^2-75c_1\right)\mathcal{E}^2\varepsilon^4. &\\&&&\tag{5.76}\end{aligned}$$

Thus,

$$\begin{aligned}\det\Omega=\Omega_{12}^2&=&1+\left(5c\mathcal{E}+4\tilde{\delta}\right)\varepsilon^2+\left[240c\tilde{\delta}\mathcal{E}\right.\\&+&\left.\frac{1}{24}\left(-450bc+25c^2+96\delta^2+450c_1\right)\mathcal{E}^2\right]\varepsilon^4+O(\varepsilon^6)\end{aligned}\tag{5.77}$$

has a positive zero order term, which implies that for ε small enough the x_2-axis orbit is always stable.

5.3.1 Application: stability analysis of the 2:4 resonance in systems with elliptical equipotentials

We now illustrate how to apply the above theory to the stability analysis of the α-models whose potential is given by (4.1). The coefficients in expansion (5.2) are listed in (5.13). The existence and stability analysis of the periodic orbits of the system follows exactly the same way of the preceding section.

Since we have $\nu=3(\alpha-2)/2<0$, the non degeneracy condition (5.30) is satisfied and the system is able to exhibit the bifurcation of periodic orbits in general position. In order to establish their existence, we look at the sign of $N=(\alpha-2)/4<0$ and/or $M=(\alpha-2)/2<0$. In case of a positive detuning parameter, these systems fall in the cases (5.40) and (5.46). Hence, as we already know from section 4.3, banana and antibanana both bifurcate from the x_1-axis orbit respectively for $E=E_{2B}=E_{2b}$ and $E=E_{2A}=E_{2a}$, where the thresholds in terms of the physical parameters are given in (4.76) and (4.77). Since in this case we have

$$\mu=\frac{4-\alpha^2}{16}>0,\tag{5.78}$$

in agreement with the general expression (5.49) the difference between the two thresholds is

$$E_{2A} - E_{2B} = 32\frac{2+\alpha}{(2-\alpha)^2}\left(q - \frac{1}{2}\right)^2. \qquad (5.79)$$

This relation and equations (5.59) and (5.60) establish the bifurcation of stable banana orbits followed by unstable antibanana orbits. These results generalize those already obtained in the work on the logarithmic potential [11, 90] and provide good approximations to the numerical investigations available in literature [1, 84, 94].

Chapter 6

Higher-order resonances

As it is well known in galactic dynamics [84], stable periodic orbits corresponding to higher-order resonances and quasi-periodic orbits parented by them give a small but not-negligible contribution to regular dynamics in systems with cores. In realistic cases with mixed (regular+chaotic) dynamics it is conjectured that these "boxlets" may become important in shaping the bulk of the density distribution [104, 105]. The main difference of these families from those seen above consists in the fact that their bifurcation is not connected with the loss (or regain in the case of a second bifurcation) of stability of the normal mode. The birth of periodic orbits in this case is rather due to the breaking of a resonant torus around the normal mode and is correctly described by applying the Poincaré-Birkhoff theorem [4]: however, the technique we applied in the previous chapters for the $1 : 2$ symmetric resonance continues to work and the conditions for the existence and stability of an m_1/m_2-resonant periodic orbit with $m_1 + m_2 > 3$ can still be found by constructing the appropriate normal form and locating fixed points of the reduced system.

6.1 Higher order symmetric resonances

In general, for a symmetric $m_1 : m_2$ detuned resonance, the normalization must be pushed at least to order $N_r = 2(m_1 + m_2 - 1)$ and produces the normal form (1.59). Proceeding as in the previous chapters, we can introduce new variables adapted to the resonance [92] by means of the quasi-canonical transformation,

$$
\begin{aligned}
\mathcal{E} &= \tfrac{\lambda}{m_1^2+m_2^2}(m_1 J_1 + m_2 J_2), \\
\mathcal{R} &= \tfrac{\lambda}{m_1^2+m_2^2}(m_2 J_1 - m_1 J_2), \\
\psi &= \mu(m_2 \phi_1 - m_1 \phi_2), \\
\chi &= \mu(m_1 \phi_1 + m_2 \phi_2).
\end{aligned} \tag{6.1}
$$

The transformation is canonical when $\lambda = 1/\mu$, but other choices can be convenient to simplify formulas: usually it is convenient to choose $\mu = 2$ and $\lambda = 1$. Under transformation to these new variables, the Hamiltonian can be expressed in the reduced form

$$\mathcal{K}(\mathcal{R}, \psi; \mathcal{E}) = \nu K(J_1(\mathcal{R}, \mathcal{E}), J_2(\mathcal{R}, \mathcal{E}), 2(m_2\theta_1 - m_1\theta_2)), \qquad (6.2)$$

with ν a scaling factor chosen to get the simplest expression from the quasi-canonical transformation. We obtain a family of 1-dof systems in the phase-plane $(\mathcal{R}, \psi)$, with equations of motion

$$\dot{\mathcal{R}} = -\frac{\partial \mathcal{K}}{\partial \psi} \qquad (6.3)$$

$$\dot{\psi} = \frac{\partial \mathcal{K}}{\partial \mathcal{R}}, \qquad (6.4)$$

parametrized by $\mathcal{E}$ that is conserved since it is proportional to the value of the integral of motion

$$H_0 = m_1 J_1 + m_2 J_2. \qquad (6.5)$$

The dynamics of the 1-dof Hamiltonian $\mathcal{K}(\mathcal{R}, \psi)$ are integrable, but this does not necessarily imply that the solution of the equations of motion can be written explicitly. However, a quite general description of the phase-space structure of the system is possible if we know the nature of the fixed points, since these turn out to be the main periodic orbits of the unreduced system. There are two types of periodic orbits that can be easily identified:

1. The *normal modes*, for which one of the J_ℓ vanishes;

2. The periodic orbits *in general position*, namely those solutions characterized by fixed relations between the two angles, $\psi_0 \equiv 2(n\theta_1 - m\theta_2)$. These are solutions of $\dot{\mathcal{R}} = 0$ (when $\mathcal{R} \neq \pm\mathcal{E}$) and determine the corresponding solutions of

$$\dot{\psi} = \left.\frac{\partial \mathcal{K}}{\partial \mathcal{R}}\right|_{\psi_0} = 0. \qquad (6.6)$$

For all cases we consider in this work they fall in two classes: $\psi_0 = 0$ (to which we refer also as the in phase oscillations) and $\psi_0 = \pm\pi$ (the anti-phase oscillations).

However, a technical issue worth to be clarified is the following: by reducing the resonant normal form (1.59) truncated at order say N_r, by means of the transformation (6.1), we obtain a polynomial of degree $N_r/2+1$ in $\mathcal{R}$. The corresponding equation of motion for ψ produces a pair of algebraic equations of degree $N_r/2+1$ which have to be solved to locate the fixed points (one for each solutions ψ_0, cfr. point 2 above). This problem is very difficult to solve if, for $N_r > 2$, we aim at general solutions depending

on the parameters of the system. However we are not interested in every
solution but only in those connected with the passage through the chosen
resonance. We can therefore resort to the perturbation method we have
described in detail in the previous section on the $1 : 2$ symmetric resonance.
In that case, with $N_r = 4$ we had to solve two equations of second degree
(cfr. the rhs side of (4.63)): this clearly does not represent a problem since
we can write explicitly the two pair of solutions. However, in each pair, *only
one* solution is geometrically acceptable because it satisfies the condition at
resonance; the other must be discarded by direct check. The perturbative
method based on the construction of the series (4.64) [66] automatically
selects the acceptable solution. The method is therefore extremely useful
for higher-order resonances: a solution of the form

$$\mathcal{R} = \sum_{k=0}^{N_r/2-1} \mathcal{R}_k \varepsilon^{2k} + O(\varepsilon^{N_r}) \tag{6.7}$$

easily allows us to select the meaningful solution without any loss in accu-
racy.

6.2   4:6 Resonance: bifurcation of fish obits

We have applied the method to the case of *fish* orbits corresponding to the
(anti-phase) 2:3 resonance. We limit to systems with elliptical equipoten-
tials, namely we assume the potential to be of the type

$$\mathcal{V}^{(N)}(x_1^2, x_2^2; q) \equiv \sum_{k=0}^{N} b_k s^{2(k+1)}(x_1, x_2; q) \tag{6.8}$$

where

$$s(x_1, x_2; q) = \sqrt{x_1^2 + \frac{x_2^2}{q^2}} \tag{6.9}$$

and q is a positive parameter which determines the ellipticity of the equipo-
tentials. In this case, we must truncate at least to order $N = 4$ ($N_r = 2N = 8$): the Hamiltonian (1.33) must be expanded up to include terms of degree

10 (b_4 in the original potential), the coefficients b_i being the following:

$$b_0 = \frac{1}{2}$$

$$b_1 = \frac{\alpha - 2}{8}$$

$$b_2 = \frac{(-4 + \alpha)(-2 + \alpha)}{48}$$

$$b_3 = \frac{(-6 + \alpha)(-4 + \alpha)(-2 + \alpha)}{384}$$

$$b_4 = \frac{(-8 + \alpha)(-6 + \alpha)(-4 + \alpha)(-2 + \alpha)}{3840}. \tag{6.10}$$

Introducing the detuning parameter, $\delta = q - \frac{2}{3}$, and after the scalings (3.8), (3.9), (3.14), the non vanishing terms in the Hamiltonian series, up to $N_r = 8$ (i.e. N=4 in series (6.8)) are given by ($\tilde{\delta} = \delta/\varepsilon^2$)

$$H_0 = p_1^2 + x_1^2 + \frac{3}{2}(p_2^2 + x_2^2) \tag{6.11}$$

$$H_2 = \frac{1}{2}b_1\sigma\left(2x_1^2 + 3x_2^2\right)^2 + \frac{3}{2}\tilde{\delta}(x_1^2 + q_1^2) \tag{6.12}$$

$$H_4 = \frac{1}{4}b_2\left(2x_1^2 + 3x_2^2\right)^3 + \frac{3}{4}b_1\left(4x_1^4 - 9x_2^4\right)\tilde{\delta} \tag{6.13}$$

$$H_6 = \frac{1}{8}b_3\left(2x_1^2 + 3x_2^2\right)^4 + \frac{3}{4}b_2\left(4x_1^6 - 27x_1^2x_2^4 - 27x_2^6\right)\tilde{\delta} + \frac{81}{8}\tilde{\delta}^2 b_1 x_2^4 \tag{6.14}$$

$$H_8 = \frac{1}{16}\left(b_4\left(2x_1^2 + 3x_2^2\right)^5 - 3\tilde{\delta}\left(b_3\left(-9x_2^2 + 2x_1^2\right)\left(2x_1^2 + 3x_2^2\right)^3\right.\right.$$
$$+ \left.\left. 81\tilde{\delta}x_2^4\left(b_1\tilde{\delta} - 2b_2x_1^2 - 3b_1x_2^2\right)\right)\right) \tag{6.15}$$

The explicit expressions of the normal form in the general class (6.8) and for the family (4.1) are a bit heavy to write here and are reported at the end of this chapter. Anyway the procedure is a straightforward extension of that illustrated in chapter 4.

The threshold for the existence of fish orbits turns out to be

$$E_f = -\frac{3}{2b_1}\delta + \frac{9}{80b_1^3}(149b_1^2 - 60b_2)\delta^2$$
$$- \frac{27(7671b_1^4 - 7840b_1^2 b_2 + 3600b_2^2 - 1500b_1 b_3)\delta^3}{1600b_1^5}$$
$$+ \frac{81}{448000b_1^7}(4852431b_1^6 - 8889450b_1^4 b_2$$
$$+9116400b_1^2 b_2^2 - 3780000b_2^3 - 3626000b_1^3 b_3$$
$$+3150000b_1 b_2 b_3 - 490000b_1^2 b_4)\delta^4. \tag{6.16}$$

This result is undoubtedly unpleasant to write (and read!) but it witnesses what is the rule with high-order expansions. However, trusting the normalization program and paying attention to write down the results without errors, the series gives us numbers we can use in specific cases. In terms of the parameters of the family (4.1), we get

$$
\begin{aligned}
E_f \;=\;& \frac{12}{2-\alpha}\delta - \frac{9(22+69\alpha)}{10(2-\alpha)^2}\delta^2 \\
& + \frac{9(4372+2508\alpha+4853\alpha^2)\delta^3}{200(2-\alpha)^3} \\
& + \frac{27(1368856+3109116\alpha+542642\alpha^2+1468293\alpha^3)\delta^4}{56000(2-\alpha)^4}.
\end{aligned}
$$

(6.17)

This result completes and generalizes the treatment of the logarithmic case presented in [11]. We may ask if it is worth the effort: in the logarithmic case ($\alpha = 0$), [84] numerically found $E_f(q = 0.7) = 0.21$ and $E_f(q = 0.9) = 2.28$ that we can consider *experimental* exact threshold values. Our analytic result predicts $E_f(q = 0.7) = 0.206$ and $E_f(q = 0.9) = 2.10$. The agreement is excellent near the resonance ($\delta = 0.7 - 2/3 \simeq 0.03$) and only moderate further away from it ($\delta = 0.9 - 2/3 \simeq 0.23$). However, we remark that the energy level $E = 2.28$ is extremely high if seen with the eye of the perturbation theorist: an error of 8% may then appear not so bad. Moreover it is possible to improve the quality of the prediction by going to still higher orders.

If one is only interested in a rough prediction around a general $m_1 : m_2$ resonance [89], from these results we can deduce the general first order expression

$$
E_{m_1:m_2} \;=\; \frac{m_2}{m_1 b_1}\delta,
$$

(6.18)

that, for the family (4.1), gives

$$
E_{m_1:m_2} \;=\; \frac{8m_2}{m_1(2-\alpha)}\left(q - \frac{m_1}{m_2}\right).
$$

(6.19)

The example of the 3:4 resonance (the *pretzel*) is a good test: for the logarithmic potential [84] numerically found $E_{3:4}(q = 0.7) = 0.25$ and $E_{3:4}(q = 0.9) = 1.22$. Eq. (6.19) with $\alpha = 0$ predicts $E_{3:4}(q = 0.7) = 0.27$ and $E_{3:4}(q = 0.9) = 0.80$. The agreement is quite good near the resonance; moreover, since for $q < 3/4$ the value provided by (6.19) is negative, accordingly with the treatment of the previous cases, we may predict that in the case $q = 0.7$ the "bifurcation" is from the x_2-axis, as actually found by [84].

Terms in the normal form for a 4:6 resonance in systems with elliptical equipotentials

$$K_0 \;=\; 2J_1 + 3J_2 \tag{6.20}$$

$$K_2 \;=\; +3\tilde{J}_1 + 3b_1 J_1^2 + 6b_1 J_1 J_2 + \frac{27}{4} b_1 J_2^2 \tag{6.21}$$

$$
\begin{aligned}
K_4 \;=\;& \frac{9}{2}\tilde{\delta} b_1 J_1^2 - \frac{17}{2} b_1^2 J_1^3 + 5b_2 J_1^3 - \frac{147}{5} b_1^2 J_1^2 J_2 + \frac{27}{2} b_2 J_1^2 J_2 \\
&- \frac{81}{8}\tilde{\delta} b_1 J_2^2 - \frac{162}{5} b_1^2 J_1 J_2^2 + \frac{81}{4} b_2 J_1 J_2^2 - \frac{459}{16} b_1^2 J_2^3 + \frac{135}{8} b_2 J_2^3
\end{aligned}
$$

$$
\begin{aligned}
K_6 \;=\;& -\frac{51}{4}\tilde{\delta} b_1^2 J_1^3 + \frac{15}{2}\tilde{\delta} b_2 J_1^3 + \frac{375}{8} b_1^3 J_1^4 - \frac{165}{4} b_1 b_2 J_1^4 + \frac{35}{4} b_3 J_1^4 \\
&- \frac{324}{25}\tilde{\delta} b_1^2 J_1^2 J_2 + \frac{11313}{50} b_1^3 J_1^3 J_2 - \frac{1809}{10} b_1 b_2 J_1^3 J_2 + 30 b_3 J_1^3 J_2 \\
&+ \frac{243}{16}\tilde{\delta}^2 b_1 J_2^2 + \frac{1701}{25}\tilde{\delta} b_1^2 J_1 J_2^2 - \frac{243}{8}\tilde{\delta} b_2 J_1 J_2^2 + \frac{11097}{25} b_1^3 J_1^2 J_2^2 \\
&- 324 b_1 b_2 J_1^2 J_2^2 + \frac{243}{4} b_3 J_1^2 J_2^2 + \frac{1377}{16}\tilde{\delta} b_1^2 J_2^3 - \frac{405}{8}\tilde{\delta} b_2 J_2^3 + \frac{63207}{200} b_1^3 J_1 J_2^3 \\
&- \frac{12069}{40} b_1 b_2 J_1 J_2^3 + \frac{135}{2} b_3 J_1 J_2^3 + \frac{30375}{128} b_1^3 J_2^4 - \frac{13365}{64} b_1 b_2 J_2^4 + \frac{2835}{64} b_3 J_2^4
\end{aligned}
$$

$$
\begin{aligned}
K_8 \;=\;& \frac{1125}{16}\tilde{\delta} b_1^3 J_1^4 - \frac{495}{8}\tilde{\delta} b_1 b_2 J_1^4 + \frac{105}{8}\tilde{\delta} b_3 J_1^4 - \frac{10689}{32} b_1^4 J_1^5 + \frac{3129}{8} b_1^2 b_2 J_1^5 \\
&- \frac{393}{8} b_2^2 J_1^5 - \frac{189}{2} b_1 b_3 J_1^5 + \frac{63}{4} b_4 J_1^5 - \frac{5103}{125}\tilde{\delta}^2 b_1^2 J_1^2 J_2 + \frac{9396}{125}\tilde{\delta} b_1^3 J_1^3 J_2 \\
&- \frac{1944}{25}\tilde{\delta} b_1 b_2 J_1^3 J_2 - \frac{577779}{280} b_1^4 J_1^4 J_2 + \frac{6353127 b_1^2 b_2 J_1^4 J_2}{2800} \\
&- \frac{146961}{560} b_2^2 J_1^4 J_2 - \frac{987}{2} b_1 b_3 J_1^4 J_2 + \frac{525}{8} b_4 J_1^4 J_2 - \frac{5103}{125}\delta^2 b_1^2 J_1 J_2^2 \\
&+ \frac{729}{16}\tilde{\delta}^2 b_2 J_1 J_2^2 - \frac{90639}{250}\tilde{\delta} b_1^3 J_1^2 J_2^2 + 486\tilde{\delta} b_1 b_2 J_1^2 J_2^2 - \frac{729}{8}\tilde{\delta} b_3 J_1^2 J_2^2 \\
&- \frac{23517}{20} b_1 b_3 J_1^3 J_2^2 + \frac{675}{4} b_4 J_1^3 J_2^2 + \frac{3645}{32}\tilde{\delta}^2 b_1 J_2^3 - \frac{12393}{64}\tilde{\delta}^2 b_1^2 J_2^3 \\
&- \frac{1420497\tilde{\delta} b_1^3 J_1 J_2^3}{1000} + \frac{216027}{200}\tilde{\delta} b_1 b_2 J_1 J_2^3 - \frac{405}{2}\tilde{\delta} b_3 J_1 J_2^3 + \frac{2025}{8} b_4 J_1^2 J_2^3 \\
&- \frac{61234299 b_1^4 J_1^2 J_2^3}{8000} + \frac{12216177 b_1^2 b_2 J_1^2 J_2^3}{1600} - \frac{1076247 b_2^2 J_1^2 J_2^3}{1280} - \frac{66339}{40} b_1 b_3 J_1^2 J_2^3 \\
&- \frac{273375}{256}\tilde{\delta} b_1^3 J_2^4 + \frac{120285}{128}\tilde{\delta} b_1 b_2 J_2^4 - \frac{25515}{128}\tilde{\delta} b_3 J_2^4 - \frac{24047199 b_1^4 J_1 J_2^4}{6400}
\end{aligned}
$$

$$\tag{6.22}$$

$$+ \quad \frac{31235949 b_1^2 b_2 J_1 J_2^4}{6400} - \frac{3355587 b_2^2 J_1 J_2^4}{5120} - \frac{20331}{16} b_1 b_3 J_1 J_2^4 + \frac{14175}{64} b_4 J_1 J_2^4$$

$$- \quad \frac{2597427 b_1^4 J_2^5}{1024} + \frac{760347}{256} b_1^2 b_2 J_2^5 - \frac{95499}{256} b_2^2 J_2^5 - \frac{45927}{64} b_1 b_3 J_2^5 + \frac{15309}{128} b_4 J_2^5$$

$$- \quad \frac{9757989 b_1^4 J_1^3 J_2^2}{1750} + \frac{16627599 b_1^2 b_2 J_1^3 J_2^2}{2800} - \frac{94041}{140} b_2^2 J_1^3 J_2^2$$

$$+ \quad \left(\frac{7065}{128} b_1^2 b_2 - \frac{2025}{64} b_1^4 - \frac{135}{16} b_2^2 J_1^3 J_2^2 - \frac{135}{8} b_1 b_3 J_1^3 J_2^2 \right) J_1^3 J_2^2 \cos\left(6\phi_1 - 4\phi_2\right)$$

$$(6.23)$$

Discussion and conclusions

In the following we discuss some implications of the results described in this work and present open problems and possible directions to cope with them.

Asymptotic expansions

Series like those described in this work are asymptotic: this means that a truncation of the series, say at order N, apparently converges in a given domain only for $N < N_{\text{opt}}$, where the *optimal* truncation order N_{opt} is linked to the extent of the domain. We remark that this semi-convergence is in general not associated to a true function: rather, it is only associated to a local geometric object we use as an invariant surface in the regular part of phase space. The optimal truncation order depends on the problem at hand and to assess it a priori is quite difficult [48]. An estimate of N_{opt} for two members of the family (4.1), $\alpha = 0, 1$, is provided in [90]. For the bifurcation of the banana orbits in the logarithmic potential, it is shown that $N_{\text{opt}} > 7$ if $q \leq 0.7$, $N_{\text{opt}} = 6$ if $q = 0.8$ and $N_{\text{opt}} = 3$ for $q = 0.9$. In this case (the worst being the furthest from exact resonance) the relative error of the prediction is 11%. However, the quality of the prediction (and the corresponding optimal order) can be further improved if different techniques of summation are employed. The paper [94] suggested to use the continued fraction method [12] to re-sum asymptotic series: applying this idea to the bifurcation threshold series in the worst case just mentioned (banana with $\alpha = 0, q = 0.9$), gives $N_{\text{opt}} = 5$ lowering the relative error to less than 4% [90]. What is indeed remarkable in this result is that the bifurcation energy is $E = 3.6$. For the logarithmic potential this corresponds to a radius of order 40 times larger than the convergence radius of the original series (4.3) so that we have an outstanding evidence of the power of asymptotic expansions.

Surfaces of section

By inverting the transformation leading to the normal form we can compute *formal* integrals of motion [30, 31] which have to be interpreted as asymptotic series as prescribed above. The most immediate use of these expansions is to construct approximations of Poincaré surfaces of sections: for the

logarithmic potential, [11] shows that, at sufficiently high energy, surfaces constructed around low-order resonances display a quite close resemblance with those numerically obtained in the scale-free limit by [84]. Moreover, by using asymptotic series as true phase-space conserved functions in a suitable domain, bifurcation curves can be computed by investigating the nature of the critical points of these functions. The results, obtained by using the formal integrals, are identical to those obtained with the normal form when expressed as series in the detuning: either approaches being effective, one can choose the one which minimizes the computational effort.

Order and chaos

The domain of "semi-convergence" of asymptotic series approximating invariant surfaces of generic systems can be taken as a measure of their regular dynamics. We have seen that, as a matter of principle, regular phase-space zones associated to resonances of any order can be adequately included and described. The approach to high-order resonances is dual: either their role is considered to be marginal [91] or they are considered as an inescapable signature of chaos [14]. However, in several interesting cases (see e.g. the scale-free models with $\alpha > 0$ treated by [98]) we have that different resonances coexist without overlapping for a large range of parameters. Resonance manifolds generate a structure that can be understood via reduction [99]. Regular dynamics are "complicated" but definitely not chaotic, so efficient tools to investigate their features are extremely useful.

3D models

The most relevant generalization is towards 3 dimensional systems. This issue has received much attention in the literature since it is linked to many problems in mechanics, optics and astronomy [46, 62, 68, 86]. The pioneering work by [102] still remains a major contribution since mathematicians, although have devoted much effort to this issue, analyzed in general only simple abstract models [92]. In [102] the orbit structure of a generic quartic potential around the 1:1:1 resonance is studied by means of the averaging method. It turns out that there may be up to 14 distinct families of simple periodic orbits, some of which are not in any of the symmetry planes of the potential.

The main problem with 3 degrees of freedom is that the normal form itself is in general not integrable. Sometimes a *renormalization* is possible [53]; namely, one can choose the generators of the coordinate changes so that

$$K = K_0 + K_1 + K_2 + \dots \qquad (6.24)$$

is a Birkhoff renormalized form, with $\{K_j, K_\ell\} = 0$, and therefore all the homogeneous terms represent constant of motion of K. Sadly, this works

only in some particular cases and, in general, the normalization procedure of resonant Hamiltonians provides only *one* formal integral [61] in addition to energy.

However, the study of the stability of the three normal modes and the bifurcations of periodic orbits in general position can be done even in the absence of a third integral. For a general quartic potential

$$V = \frac{1}{2}(\omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2) + \frac{1}{4}\sum_{i,j=1}^{3} a_{ij} x_i^2 x_j^2, \qquad (6.25)$$

near a 1:1:1 resonance, it is convenient to approximate the frequency ratios introducing *two* detuning parameters, namely

$$\frac{\omega_1}{\omega_2} = 1 + \delta_1, \quad \frac{\omega_3}{\omega_2} = 1 + \delta_2, \qquad (6.26)$$

and to treat both of them as terms of order two in the perturbation. Scaling the coordinates according to

$$x_i \rightarrow \varepsilon^{-1}\omega_i x_i, \quad i = 1, 2, 3 \qquad (6.27)$$

where ε is a small positive parameter, and dividing V by $\omega_2\varepsilon^2$, the Hamiltonian function is given by

$$\begin{aligned} H &= \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) \\ &+ \left[\tilde{\delta}_1(x_1^2 + p_1^2) + \tilde{\delta}_2(x_2^2 + p_2^2) + \frac{1}{4}\sum_{i,j=1}^{3} b_{ij} x_i^2 x_j^2\right]\varepsilon^2 \qquad (6.28) \end{aligned}$$

where $b_{ij} = \frac{a_{ij}}{\omega_1\omega_j}$ and $\tilde{\delta}_i = \delta_i/\varepsilon^2$. The system is therefore in a form suitable to apply a normalization procedure. If the reducing transformation are performed with Lie-transforms and the normalization is pushed to the minimal order required (i.e. up to second degree terms in ε, since we are in presence of symmetry with respect to all coordinate axes), we arrive at the following normal form

$$\begin{aligned} K_{1:1:1} &= \sum_{i=1}^{3} J_1 + \left[\tilde{\delta}_1 J_1 + \tilde{\delta}_3 J_3 + \frac{3}{8}\sum_{i=1}^{3} a_{ii} J_i^2\right. \\ &+ \left.\frac{1}{4}\sum_{i\neq l} a_{ij} J_i J_l(2 + \cos(2\phi_i - 2\phi_l))\right]\varepsilon^2 \qquad (6.29) \end{aligned}$$

where action angle variables $x_i = \sqrt{2J_i}\cos\phi_i$, $p_i = \sqrt{2J_i}\sin\phi_i$, have been introduced.

In galactic dynamics is relevant the analysis of 3D nearly spherical potential, which generalize the $2D$ α-models discussed previously, cfr. (4.1). They are characterized by

$$a_{11} = a_{22} = a_{33} = a_{12} = a_{13} = a_{23}. \tag{6.30}$$

An investigation of the main periodic orbits gives, in analogy with the 2D systems with elliptical equipotentials, that the only simple periodic orbits are the three normal modes and, above a certain energy, elliptic closed orbits in the three principal planes.

A further step towards a general study of relevant cases like the $1 : 2 : 2$ and $1 : 2 : 3$ symmetric resonances seems to be within the reach of the normalization method.

Non-autonomous systems

All the systems we studied in this work are autonomous, that is the Hamiltonian function H does not depend explicitly on time. In case we have an explicit dependence on the time variable t we speak of *non autonomous systems*. An example is when the equations of motion are of the type

$$\ddot{x} = f(t)x + g(t). \tag{6.31}$$

A non autonomous system can be treated as an autonomous one by a trivial extension of the phase space. It suffices to consider the time as a new coordinate variable, the conjugate momentum being given by $-H$.

Of particular interest are systems whose dynamics is governed by *Hill's equations*:

$$\ddot{x} + (a + bp(t))x = 0, \tag{6.32}$$

where p is a periodic function of time and a, b are real parameters. Much effort has been devoted in the literature to their analysis, see e.g. [8, 24, 25, 74, 75, 100]. By properly extending the phase space, a resonant Hamiltonian normal form can be introduced and used to compute the instability threshold.

Systems with indefinite quadratic part

In this work we limited the analysis to symmetric Hamiltonian functions with positive definite quadratic part. However one could consider more general systems with indefinite quadratic part, so that

$$H_0 = mJ_1 - nJ_2, \quad m, n \in \mathbb{N}. \tag{6.33}$$

In this case we speak of $m : -n$ resonances. These system differ from the previous ones in several features, even if their analysis can be performed almost in the same way. Some aspects of the $1 : -1$ resonance are discussed in [45, 81, 93]. For the $1 : -2$ see [37, 47].

Conclusions

We have presented a general analysis of the orbit structure of 2D potentials near $2:2$ and $2:4$ resonances. The main results are the following:

The $2:2$ general resonance is associated with the appearance of loops and inclined orbits upon a certain energy threshold. If some non degeneracy conditions (which depends on the physical coefficients of the potential) are satisfied, the Hamiltonian function can be reduced into a simpler polynomial form which allows to classify the dynamics using singularity theory. In this case we are able to compute explicitly the reducing transformations which bring the normal form into a versal deformation of the central singularity. However systems with elliptical equipotentials present some degeneracies which forbid the appearance of inclined orbits. Loop orbits are still present and their bifurcation leads to the destabilization of the x_2-axis orbit in the oblate case and of the x_1-axis orbit in the prolate case. Inclined orbits may appear only when the equipotentials are heavily deformed.

The $2:4$ resonance determines the appearance of banana and anti-banana orbits upon certain energy threshold values.
In case of systems with elliptical equipotentials the first family is stable and always appears at a lower energy than the second, which is unstable. The bifurcation sequence produces the change in the stability character of the major axis orbit and is modified only by very large deformations of the equipotentials.

Higher order resonances are briefly treated in the last chapter. In particular, we deal with the $4:6$ resonance in systems with elliptical equipotentials, which is associated with the bifurcation of fish orbits.

The critical energy values which determines the bifurcation of periodic orbits in general position are provided in all the cases studied, and expressed in terms of the detuning and the other physical parameters and coefficients.

Appendix A

List of coefficients and parameters

In the deformation (3.42) of proposition 3.1, if the planar reduction is performed according to (3.30), the coefficients a_i are the following

$$a_1 = \frac{12(B - 2A + 6C)}{\sqrt{|3C - A|}}$$

$$a_2 = \frac{136A^2 - 136AB + 34B^2 - 272AC + 136BC - 408C^2 + 15d_3 - 45d_4}{12\sqrt{|3C - A|}}$$

$$a_3 = \frac{12(B - 2A + 2C)}{\sqrt{|C - A|}}$$

$$a_4 = \frac{136A^2 - 136AB + 34B^2 - 112AC + 56BC - 24C^2 + 3d_3 - 45d_4}{12\sqrt{|C - A|}}$$

$$\begin{aligned}
a_5 = {} & \frac{1}{288(A - 3C)^2}(1632A^3 - 544A^2(2B + 15C) + 2A(68B^2 + 3264BC \\
& + 9(272C^2 - 5(d_1 + d_2 - 3d_3 + 5d_4))) \\
& + 3(-136B^2C - 3B(1088C^2 - 5(d_1 - d_2 + d_3 - d_4)) \\
& + 18C(272C^2 + 5(d_1 + d_2 - 3d_3 + 5d_4))))
\end{aligned}$$

$$\begin{aligned}
a_6 = {} & \frac{1}{\left(144|C - A|^{3/2}|3C - A|^{3/2}\right)}\left[-1632A^4 + 64A^3(17B + 138C)\right. \\
& - 2A^2\left(68B^2 + 3328BC + 9\left(640C^2 - 5d_1 - 3d_2 + 9d_3 - 25d_4\right)\right) \\
& + A(512B^2C - 72C(48C^2 + 5d_1 + 2d_2 - 8d_3 + 25d_4) \\
& + B(12608C^2 - 45d_1 + 27d_2 - 27d_3 + 45d_4)) \\
& - 6C(52B^2C + B(1216C^2 - 15d_1 + 6d_2 - 6d_3 + 15d_4) \\
& \left. - 3C(432C^2 + 5d_1 + 7d_2 - 25d_3 + 85d_4))\right]
\end{aligned}$$

$$a_7 = \frac{1}{288(A-C)^2} \left(1632A^3 + 32A^2(34B - 93C) \right.$$
$$+ \quad 2A\left(68B^2 - 1216BC + 528C^2 - 225d_1 + 27d_2 - 9d_3 - 45d_4\right)$$
$$+ \quad 3\left(-24B^2C + B\left(448C^2 + 3(5d_1 - d_2 + d_3 - 5d_4)\right)\right)$$
$$+ \quad \left. 6C\left(16C^2 + 25d_1 - 3d_2 + d_3 + 5d_4\right)\right).$$

The parameters b_i have the following expressions

$$b_1 = -\frac{6C\delta}{\sqrt{|3C-A|}}$$

$$b_2 = \frac{\delta}{2\sqrt{|3C-A|}}$$

$$b_3 = -\frac{2C\delta}{\sqrt{|C-A|}}$$

$$b_4 = \frac{\delta}{2\sqrt{|C-A|}}$$

$$b_5 = \frac{\left(576A^2 + 16AB - 3456AC - 48BC + 5184C^2 + 45d_1 - 45d_2 + 45d_3 - 45d_4\right)\delta}{576(A-3C)^2}$$

$$b_6 = \left(-576A^3 - 16A^2B + 3456A^2C + 32ABC - 6336AC^2\right.$$
$$+ \quad 48BC^2 + 3456C^3 - 45Ad_1 + 90Cd_1 + 27Ad_2 - 36Cd_2 - 27Ad_3 + 36Cd_3$$
$$+ \quad \left. 45Ad_4 - 90Cd_4\right)\frac{\delta}{288|C-A|^{3/2}|3C-A|^{3/2}}$$

$$b_7 = \frac{\left(576A^2 + 16A(B - 72C) + 48BC + 9\left(64C^2 + 5d_1 - d_2 + d_3 - 5d_4\right)\right)\delta}{576(A-C)^2}.$$

If the planar reduction is performed according to (3.63) the previous coefficients and parameters turn into

$$a_1 = -\frac{B + 2A - 6C}{\sqrt{|3C-A|}}$$

$$a_2 = \frac{136A^2 + 136AB + 34B^2 - 272AC - 136BC - 408C^2 + 15d_2 - 45d_1}{12\sqrt{|3C-A|}}$$

$$a_3 = -\frac{B + 2A - 2C}{\sqrt{|C - A|}}$$

$$a_4 = \frac{136A^2 + 136AB + 34B^2 - 112AC + 56BC - 24C^2 + 3d_2 - 45d_1}{12\sqrt{|C - A|}}$$

$$a_5 = \frac{1}{288(A - 3C)^2}(1632A^3 + 544A^2(2B - 15C) + 2A(68B^2 - 3264BC$$
$$+ \quad 9(272C^2 - 5(d_1 - 3d_2 + d_3 - d_4)))$$
$$+ \quad 3(-136B^2C + 3B(1088C^2 + 5(d_1 - d_2 + d_3 - d_4))$$
$$+ \quad 18C(272C^2 + 5(d_1 - 3d_2 + d_3 + d_4)))))$$

$$a_6 = \frac{1}{\left(144|C - A|^{3/2}|3C - A|^{3/2}\right)}\left[-1632A^4 - 64A^3(17B - 138C)\right.$$
$$- \quad 2A^2\left(68B^2 - 3328BC + 9\left(640C^2 - 25d_1 + 9d_2 - 3d_3 - 5d_4\right)\right)$$
$$+ \quad A(512B^2C - 72C(48C^2 + 25d_1 - 8d_2 + 2d_3 + 5d_4)$$
$$+ \quad B(-12608C^2 - 45d_1 + 27d_2 - 27d_3 + 45d_4))$$
$$+ \quad 6C(-52B^2C + B(1216C^2 + 15d_1 - 6d_2 + 6d_3 + 5d_4)$$
$$+ \quad 3C(432C^2 + 85d_1 - 25d_2 + 7d_3 + 5d_4))]$$

$$b_1 = \frac{2(2A + B - 3C)\delta}{\sqrt{-A + 3C}}$$

$$b_2 = -\frac{\delta}{2\sqrt{|3C - A|}}$$

$$b_3 = \frac{2(2A + B - C)\delta}{\sqrt{-A + C}}$$

$$b_4 = -\frac{\delta}{2\sqrt{|C - A|}}$$

$$b_5 = \frac{\left(576A^2 + 16AB - 3456AC - 48BC + 5184C^2 + 45d_1 - 45d_2 + 45d_3 - 45d_4\right)\delta}{576(A - 3C)^2}$$

$$b_6 = \left(-576A^3 - 16A^2B + 3456A^2C + 32ABC - 6336AC^2\right.$$
$$+ \quad 48BC^2 + 3456C^3 - 45Ad_1 + 90Cd_1 + 27Ad_2 - 36Cd_2 - 27Ad_3 + 36Cd_3$$
$$+ \quad 45Ad_4 - 90Cd_4)\frac{\delta}{288|C - A|^{3/2}|3C - A|^{3/2}}$$

$$b_7 = \frac{\left(576A^2 + 16A(B - 72C) + 48BC + 9\left(64C^2 + 5d_1 - d_2 + d_3 - 5d_4\right)\right)\delta}{576(A - C)^2}$$

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