Fractional Branching Processes

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Tesi di Dottorato XXIII Ciclo

Dipartimento di Scienze Statistiche

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Preface

Questa tesi è frutto di tre anni di vita. Voglio ringraziare chi mi è stato vicino e ne ha reso possibile la stesura. La mia famiglia, Vincenza e la sua famiglia, Enzo Orsingher, i miei amici del dottorato/università, i miei amici del CNR, Dexter Cahoy.

Voglio anche ricordare che i risultati presenti in questa tesi possono essere trovati anche nei seguenti articoli:

1. Orsingher, E., Polito, F. *Fractional Pure Birth Processes*, "Bernoulli" 16(3), 858-881, 2010.
2. Orsingher, E., Polito, F., Sakhno, L. *Fractional Non-Linear, Linear and Sublinear Death Processes*, "Journal of Statistical Physics" 141(1), 68-93, 2010.
3. Orsingher, E., Polito, F. *On a Fractional Linear Birth-Death Process*, In stampa su "Bernoulli".
4. Cahoy, D. O., Polito, F. *Simulation and estimation for the fractional Yule process*, In stampa su "Methodology and Computing in Applied Probability".
5. Orsingher, E., Polito, F. *Randomly Stopped Nonlinear Fractional Birth Processes*, Sottoposto.

Fractional Branching Processes

— Can the meaning of derivatives with integer order be generalised to derivatives with non integer order?
— What if the order will be 1/2?
— It will lead to a paradox. From this apparent paradox, one day useful consequences will be drawn.

Gottfried Wilhelm Von Leibniz
& Guillaume De l'Hôpital
Around A.D. 1695

Chapter 1

Introduction

Summary

In this first chapter, along with the fundamental motivations that lead us to the study of fractional branching processes, we review some basic classical properties of the point processes treated in the following parts. We describe the Poisson as well as the nonlinear pure birth and pure death processes and the linear birth-death process. Furthermore we highlight relations between Poisson subordination and compound Poisson processes. At the end of the chapter, a brief overview of the basic concepts of fractional calculus is given. The material presented in this chapter is necessary to understand the construction of the fractional growth processes described in the succeding chapters.

1.1 Motivations and basic facts on growth processes

The aim of the present work is to study a novel class of stochastic processes thought to modelling growth phenomena. The proposed models are constructed on top of some well-known classical point processes, namely those belonging to the class of birth-death processes, and then modified by means of fractional calculus.

The structure of this dissertation is as follows. In the first chapter, we review some basic notions about the Poisson, pure birth, pure death, and birth-death processes. A particular construction of compound Poisson processes by means of the principle of subordination is also derived and commented in section 1.1.1. The last section of the first chapter is devoted to a brief review of some fundamental properties and notions of fractional calculus.

Following some recent developements in the theory of anomalous diffusion and fractional phenomena in general, in the remaining chapters we construct *fractional growth processes*, thus obtaining a substantial generalisation of classical processes. References on fractional processes in general can be for example Wyss (1986), Schneider and Wyss (1988), Nigmatullin (1986), Zaslavsky (2002), Zaslavsky (2006). In particular, we develop and analyse in details the fractional nonlinear pure birth process (Chapter 2), the fractional nonlinear pure death process (Chapter 3) as well as several specific cases. Due to the complexity of the general birth-death case, in Chapter 4, we just treat its linear case, deriving, amongst other properties, and as in the previous chapters, the explicit form of the state probability distribution. Chapter 5 is devoted to studying the fractional linear pure birth process (named also fractional Yule process) in more details. Therein the inter-birth waiting time distribution is derived as well as some interesting representations involving time-changed Poisson processes. Simulations of the fractional Yule process are also performed and estimators based on the method of moments, defined and then tested on simulated data. The last chapter concerns the study of various generalisations of the proposed models. In particular, In order to furnish the process with further randomness, we construct subordinated birth processes, substituting the deterministic time t with random times represented as time-continuous stochastic processes.

The fractional growth processes herein analysed can be succesfully implemented in modelling real phenomena such as epidemics, anomalously expanding or contracting populations, branching in fractals and so forth. Some possible applications are depicted in the chapters' introductions.

1.1.1 Poisson process and related models

Recall that a continuous-time Markov chain is a stochastic process $X(t)$, $t > 0$, with continuous time and finite or countable state space satisfying the Markov property, that is, for a sequence of instants $0 \leq t_0 < \cdots < t_n$, we have that

$$\Pr\{X(t_n) = i_n \mid X(t_0) = i_0, \dots, X(t_{n-1}) = i_{n-1}\} = \Pr\{X(t_n) = i_n \mid X(t_{n-1}) = i_{n-1}\}. \quad (1.1)$$

In other words, the future behaviour of the process is determined by the process' state at the last known time.

The homogeneous Poisson process $\mathfrak{N}(t)$, $t > 0$, is a continuous-time Markov chain taking values in $\mathbb{N} \cup \{0\}$, defined by the following properties:

1. $\Pr\{\mathfrak{N}(0) = 0\} = 1$;

2. $\Pr\{\mathfrak{N}(t, t+dt] = 1\} = \lambda dt + o(dt)$, where $\lambda > 0$;

3. $\Pr\{\mathfrak{N}(t, t+dt] = 0\} = 1 - \lambda dt + o(dt)$;

4. $\Pr\{\mathfrak{N}(t, t+dt] > 0\} = o(dt)$;

5. If $(t_1, t_2) \cap (t_3, t_4) = \emptyset$, with $t_1 < t_2 < t_3 < t_4$, then $\mathfrak{N}(t_1, t_2)$ and $\mathfrak{N}(t_3, t_4)$ are independent random variables.

The Poisson process is a model for the counting process associated to the random occurrence of points events. Indeed, with $N(1, t]$ we indicate the number of events occurred in the time interval $(1, t] \subset \mathbb{R}^+$. Considering the above properties, it is immediate to note that, in a small time interval dt, the Poisson process $\mathfrak{N}(t)$, $t > 0$, can either remain in the current state, say j, with infinitesimal probability $1 - \lambda dt$, or move to the above state, $j+1$, with probability λdt. Here $\lambda > 0$ is the *rate of occurrence of events.* Transitions are admitted only upwards.

The Poisson process is undoubtedly the simplest and most studied of all time-continuous Markov chains. Over the years it has been considered as a model for completely different phenomena such as the arrival of customers at a desk (queueing systems), or to describe other point events: floods, eruptions, damages occurring in machining etc. Despite its simplicity, the Poisson process exhibits interesting properties. In the following we will review some of them.

Consider the state probabilities for the homogeneous Poisson process, i.e.

$$\mathfrak{p}_k(t) = \Pr\{\mathfrak{N}(t) = k \mid \mathfrak{N}(0) = 0\}, \qquad k \geq 0. \tag{1.2}$$

The state probabilities $\mathfrak{p}_k(t)$, $t > 0$, $k \geq 0$, satisfy the following difference-differential equations:

$$\frac{d}{dt}\mathfrak{p}_k(t) = -\lambda \mathfrak{p}_k(t) + \lambda \mathfrak{p}_{k-1}(t), \qquad k \geq 0, \tag{1.3}$$

subject to the initial condition

$$\mathfrak{p}_k(0) = \begin{cases} 0, & k = 0, \\ 1, & k > 0, \end{cases} \tag{1.4}$$

and considering that $\mathfrak{p}_{-1}(t) = 0$.

In order to obtain equations (1.3), we write

$$\begin{aligned} \mathfrak{p}_k(t+dt) &= \Pr\{\mathfrak{N}(t+dt) = k\} \\ &= \Pr\{(\mathfrak{N}(t) = k, \mathfrak{N}(t, t+dt) = 0) \cup (\mathfrak{N}(t) = k-1, \mathfrak{N}(t, t+dt) = 1) \\ &\quad \cup \bigcup_{j=2}^{k} (\mathfrak{N}(t) = k-j, \mathfrak{N}(t, t+dt) = j)\} \\ &= \mathfrak{p}_k(t)(1 - \lambda dt + o(dt)) + \mathfrak{p}_{k-1}(t)(\lambda dt + o(dt)) + \sum_{j=2}^{k} \mathfrak{p}_{k-j}(t) o(dt). \end{aligned} \tag{1.5}$$

By rearranging (1.5) and letting $dt \to 0$, we immediately retrieve formula (1.3).

From equation (1.5), we derive the differential equation satisfied by the probability generating function

$$\mathfrak{G}(u, t) = \sum_{k=0}^{\infty} u^k \Pr\{\mathfrak{N}(t) = k\}, \qquad t > 0,\ |u| \leq 1, \tag{1.6}$$

is

$$\begin{cases} \frac{\partial}{\partial t}\mathfrak{G}(u,t) = \lambda(u-1)\frac{\partial}{\partial u}\mathfrak{G}(u,t), \\ \mathfrak{G}(u,0) = 1. \end{cases} \tag{1.7}$$

In turn, the solution to (1.7) is

$$\mathfrak{G}(u,t) = e^{\lambda t(u-1)} = \sum_{k=0}^{\infty} u^k \frac{e^{-\lambda t}(\lambda t)^k}{k!}. \tag{1.8}$$

A comparison of (1.8) with (1.6) makes clear that the state probabilities have the following form:

$$\mathfrak{p}_k(t) = \Pr\{\mathfrak{N}(t) = k \mid \mathfrak{N}(0) = 0\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \qquad k \geq 0,\ t > 0 \tag{1.9}$$

(Poisson distribution with parameter λt). A peculiar characteristic of the Poisson distribution is that its mean value equals the variance:

$$\mathbb{E}\mathfrak{N}(t) = \mathbb{V}\mathrm{ar}\mathfrak{N}(t) = \lambda t. \tag{1.10}$$

Many other properties can be derived for the homogeneous Poisson process (for a complete treatment, the reader can consult e.g. Bailey (1964) or Gikhman and Skorokhod (1996)); here we just point out the following important fact. Let $T_1, \dots, T_k$, be the inter-arrival times (elapsed time between two successive events), that is T_j represents the time between the jth and $(j+1)$th event. It can be easily proved that, for every continuous-time Markov chain, the inter-arrival times are exponentially distributed with specific parameters depending on the structure of the chain. For the homogeneous Poisson process of rate λ, we have that $\Pr\{T_j \leq t\} = 1 - e^{-\lambda t}$.

A possible generalisation of the Poisson process consists of letting the arrival rate λ to be time-varying. The basic properties defining the process become:

1. $\Pr\{\mathfrak{N}(0) = 0\} = 1$;
2. $\Pr\{\mathfrak{N}(t, t+dt] = 1\} = \lambda(t)dt + o(dt)$, where $\lambda(t) > 0$, $t > 0$;
3. $\Pr\{\mathfrak{N}(t, t+dt] = 0\} = 1 - \lambda(t)dt + o(dt)$;
4. $\Pr\{\mathfrak{N}(t, t+dt] > 0\} = o(dt)$;
5. If $(t_1, t_2) \cap (t_3, t_4) = \emptyset$, with $t_1 < t_2 < t_3 < t_4$, then $\mathfrak{N}(t_1, t_2)$ and $\mathfrak{N}(t_3, t_4)$ are independent random variables.

The process $\mathfrak{N}(t)$, $t > 0$, satisfying the above properties is called *non-homogeneous* (or *inhomogeneous*) *Poisson process*, and the state probabilities $\mathfrak{p}_k(t) = \Pr\{\mathfrak{N}(t) = k \mid \mathfrak{N}(0) = 0\}$ solve the Cauchy problem

$$\begin{cases} \frac{d}{dt}\mathfrak{p}_k(t) = -\lambda(t)\mathfrak{p}_k(t) + \lambda(t)\mathfrak{p}_{k-1}(t), \\ \mathfrak{p}_k(0) = \begin{cases} 1, & k = 0, \\ 0, & k > 0, \end{cases} \end{cases} \tag{1.11}$$

taking into account that $\mathfrak{p}_{-1}(t) = 0$. From equation (1.11), we also obtain the partial differential equation satisfied by the probability generating function $\mathfrak{G}(u,t) = \sum_{k=0}^{\infty} \mathfrak{p}_k(t)$, $t > 0$, $|u| \leq 1$:

$$\begin{cases} \frac{\partial}{\partial t}\mathfrak{G}(u,t) = \lambda(t)(u-1)\mathfrak{G}(u,t), \quad \lambda(t) > 0, \\ \mathfrak{G}(u,t) = 1. \end{cases} \tag{1.12}$$

By means of standard methods, we arrive at the solution to (1.12):

$$\begin{aligned}\mathfrak{G}(u,t) &= e^{-(1-u)\int_0^t \lambda(s)ds} \qquad (1.13)\\ &= e^{-\int_0^t \lambda(s)ds} \sum_{k=0}^{\infty} u^k \frac{\left(\int_0^t \lambda(s)ds\right)^k}{k!}.\end{aligned}$$

Therefore, the probability distribution of the non-homogeneous Poisson process reads

$$\mathfrak{p}_k(t) = \Pr\{\mathfrak{N}(t) = k \mid \mathfrak{N}(0) = 0\} = e^{-\int_0^t \lambda(s)ds} \frac{\left(\int_0^t \lambda(s)ds\right)^k}{k!}, \qquad k \geq 0,\ t > 0. \qquad (1.14)$$

In the homogeneous or non-homogeneous Poisson process, the jump width is always 1. A possible generalisation leading to the *compound Poisson process* consists in letting the jumps to have arbitrary (random) size.

Definition 1.1.1 (Compound Poisson process). *Let $\mathfrak{N}(t)$, $t > 0$, be a homogeneous Poisson process with rate $\lambda > 0$. Furthermore, let ξ_j, $j = 1, 2, \dots$ be i.i.d. random variables, sharing the same mean $\mathbb{E}\xi_j = \eta$, and independent of $\mathfrak{N}(t)$. The compound Poisson process is defined as*

$$\mathfrak{X}(t) = \sum_{j=1}^{\mathfrak{N}(t)} \xi_j, \qquad t > 0. \qquad (1.15)$$

The behaviour of the compound Poisson process (which is a rather broad generalisation of the Poisson process) mainly depends on the structure of the i.i.d. random variable, for example if they are completely positively skewed, the process is increasing. Now, we present the explicit form of the mean and the moment generating function of the compound Poisson process $\mathfrak{X}(t)$, $t > 0$. The mean value reads

$$\begin{aligned}\mathbb{E}\mathfrak{X}(t) &= \sum_{k=0}^{\infty} \mathbb{E}\{\mathfrak{X}(t) \mid \mathfrak{N} = k\} \Pr\{\mathfrak{N}(t) = k\} \qquad (1.16)\\ &= \sum_{k=0}^{\infty} k\eta\, e^{-\lambda t} \frac{(\lambda t)^k}{k!} = \eta \lambda t e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} = \eta\lambda t,\end{aligned}$$

while the moment generating function is

$$\begin{aligned}\mathbb{E}\exp\left(u\mathfrak{X}(t)\right) &= \mathbb{E}\exp\left(u \sum_{j=1}^{\mathfrak{N}(t)} \xi_j\right) \qquad (1.17)\\ &= \Pr\{\mathfrak{N}(t) = 0\} + \sum_{k=0}^{\infty} \mathbb{E}\left[e^{u\sum_{j=1}^k \xi_j} \middle| \mathfrak{N}(t) = k \right] \Pr\{\mathfrak{N}(t) = k\}\\ &= e^{-\lambda t} + \sum_{k=1}^{\infty} \left[\mathbb{E}e^{u\xi}\right]^k e^{-\lambda t} \frac{(\lambda t)^k}{k!} = e^{-\lambda t}\left(1 + \sum_{k=1}^{\infty} \frac{[\lambda t \mathbb{E}e^{u\xi}]^k}{k!}\right)\\ &= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{[\lambda t \mathbb{E}e^{u\xi}]^k}{k!} = e^{\lambda t\left(\mathbb{E}[e^{u\xi}]-1\right)}.\end{aligned}$$

In the following subsection we present an alternative way of constructing compound Poisson processes; we give a simple example but the same technique can be exploited to obtain more general compound processes.

A compound Poisson process through subordination

Consider two independent homogeneous Poisson processes, say $\mathfrak{N}_1(t)$, $t > 0$, with rate $\lambda > 0$, and $\mathfrak{N}_2(t)$, $t > 0$, with rate β, $t > 0$. We are interested in the subordinated process $\hat{\mathfrak{N}}(t) = \mathfrak{N}_1(\mathfrak{N}_2(t))$, $t > 0$. Subordination technique (Bochner, 1955) permits to introduce in the system further randomness, thus permitting to model phenomena which exhibit either accelerated or slowed down behaviour. In particular, the type of subordination implemented in this subsection is a little different. Poisson processes have at most countable state space so that the inner Poisson process $\mathfrak{N}_2(t)$, $t > 0$, in addition to randomize the time, operates a sampling of the external process $\mathfrak{N}_1(t)$, $t > 0$. Note that this sampling allows the subordinated process to have jumps of arbitrary positive discrete size.

The state probabilities $\hat{\mathfrak{p}}_k(t) = \Pr\{\mathfrak{N}_1(\mathfrak{N}_2(t)) = k\}$, $k \geq 0$, $t > 0$, can be determined as follows.

$$\hat{\mathfrak{p}}_k(t) = \sum_{r=0}^{\infty} \frac{e^{-\lambda r}(\lambda r)^k}{k!} \frac{e^{-\beta t}(\beta t)^r}{r!} = \frac{\lambda^k}{k!} e^{-\beta t} \sum_{r=0}^{\infty} \frac{e^{-\lambda r} r^k (\beta t)^r}{r!}. \tag{1.18}$$

By considering that

$$e^{-x} \sum_{r=0}^{\infty} \frac{r^k x^r}{r!} = \mathfrak{B}_k(x), \tag{1.19}$$

is the nth order Bell polynomial (for a review of Bell polynomials, the reader can consult for example Boyadzhiev (2009)), we obtain that

$$\hat{\mathfrak{p}}_k(t) = \frac{\lambda^k}{k!} e^{-\beta t(1-e^{-\lambda})} \mathfrak{B}_k\left(\beta t e^{-\lambda}\right), \qquad k \geq 0,\ t > 0. \tag{1.20}$$

By recalling the exponential generating function for Bell polynomials

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} \mathfrak{B}_k(x) = e^{x(e^z - 1)}, \tag{1.21}$$

it is immediate to check that $\sum_{k=0}^{\infty} \hat{\mathfrak{p}}_k(t) = 1$. The mean value is directly calculated as follows.

$$\begin{aligned} \mathbb{E}[\mathfrak{N}_1(\mathfrak{N}_2(t))] &= \sum_{k=0}^{\infty} k \sum_{r=0}^{\infty} \frac{e^{-\lambda r}(\lambda r)^k}{k!} \frac{e^{-\beta t}(\beta t)^r}{r!} = \sum_{r=0}^{\infty} \frac{e^{-\beta t}(\beta t)^r}{r!} e^{-\lambda r} \sum_{k=0}^{\infty} k \frac{(\lambda r)^k}{k!} \\ &= \sum_{r=0}^{\infty} \frac{e^{-\beta t}(\beta t)^r}{r!} e^{-\lambda r} \lambda r \sum_{(\lambda r)^k} k! = e^{-\beta t} \sum_{r=0}^{\infty} \frac{\lambda r (\beta t)^r}{r!} \\ &= e^{-\beta t} \lambda \beta t \sum_{r=0}^{\infty} \frac{(\beta t)^r}{r!} = \lambda \beta t, \qquad t > 0. \end{aligned} \tag{1.22}$$

In order to determine the variance we first calculate the second order moment:

$$\mu_2 = \sum_{k=0}^{\infty} k^2 \sum_{r=0}^{\infty} \frac{e^{-\lambda r}(\lambda r)^k}{k!} \frac{e^{-\beta t}(\beta t)^r}{r!} = e^{-\beta t} \sum_{r=0}^{\infty} \frac{(\beta t)^r}{r!} e^{-\lambda r} \sum_{k=0}^{\infty} k^2 \frac{(\lambda r)^k}{k!}. \tag{1.23}$$

Considering that

$$\begin{aligned} \sum_{k=0}^{\infty} k^2 \frac{(\lambda r)^k}{k!} &= \sum_{k=1}^{\infty} k \frac{(\lambda r)^k}{(k-1)!} = \sum_{k=0}^{\infty} (k+1) \frac{(\lambda r)^{k+1}}{k!} \\ &= \sum_{k=1}^{\infty} k \frac{(\lambda r)^{k+1}}{k!} + \sum_{k=0}^{\infty} \frac{(\lambda r)^{k+1}}{k!} \end{aligned} \tag{1.24}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{(\lambda r)^{k+2}}{k!} + \lambda r e^{\lambda r} \\
&= \lambda r e^{\lambda r}(\lambda r + 1),
\end{aligned}
$$

we have

$$
\begin{aligned}
\mu_2 &= e^{-\beta t} \sum_{r=0}^{\infty} \frac{\lambda r(\lambda r+1)(\beta t)^r}{r!} = e^{-\beta t}\lambda \sum_{r=1}^{\infty} \frac{(\lambda r+1)(\beta t)^r}{(r-1)!} \\
&= e^{-\beta t}\lambda \sum_{r=0}^{\infty} \frac{(\lambda r+\lambda+1)(\beta t)^{r+1}}{r!} = e^{-\beta t}\lambda \left[(\lambda+1)\sum_{r=0}^{\infty} \frac{(\beta t)^{r+1}}{r!} + \lambda \sum_{r=1}^{\infty} r \frac{(\beta t)^{r+1}}{r!} \right] \\
&= \lambda(\lambda+1)\beta t + e^{-\beta t}\lambda^2 \sum_{r=0}^{\infty} \frac{(\beta t)^{r+2}}{r!} = \lambda(\lambda+1)\beta t + (\lambda \beta t)^2.
\end{aligned} \tag{1.25}
$$

Therefore, the variance of the process reads

$$
\mathbb{V}\mathrm{ar}\left[\mathfrak{N}_1(\mathfrak{N}_2(t))\right] = \lambda(\lambda+1)\beta t, \qquad t > 0. \tag{1.26}
$$

The probability generating function $\hat{\mathfrak{G}}(u,t) = \sum_{k=0}^{\infty} u^k \hat{\mathfrak{N}}(t)$ can be found in the following way:

$$
\begin{aligned}
\hat{\mathfrak{G}}(u,t) &= \sum_{k=0}^{\infty} u^k \sum_{r=0}^{\infty} \frac{e^{-\lambda r}(\lambda r)^k}{k!} \frac{e^{-\beta t}(\beta t)^r}{r!} = e^{-\beta t\left(1-e^{-\lambda}\right)} \sum_{k=0} u^k \frac{\lambda^k}{k!} \mathfrak{B}\left(\beta t e^{-\lambda}\right) \\
&= e^{-\beta t\left(1-e^{-\lambda}\right)} e^{\beta t e^{-\lambda}\left(e^{u\lambda}-1\right)} = e^{\beta t\left(e^{\lambda(u-1)}-1\right)}, \qquad t > 0,\ |u| \le 1.
\end{aligned} \tag{1.27}
$$

A comparison of the probability generating function (1.27) of the process $\hat{\mathfrak{N}}(t)$, $t > 0$, with (1.17) where $\mathbb{E}[e^{u\xi}] = e^{\eta(e^u-1)}$, makes clear that $\hat{\mathfrak{N}}(t)$, $t > 0$, is a compound Poisson process.

1.1.2 Pure birth process

In order to describe, for example, the evolution of a population, a useful model is that of pure birth. The behaviour of this model depends in practice on the size of the population through its *birth rates* $\lambda_k > 0$, $k = 1, 2, \cdots$. In the simplest particular case (so-called Yule–Furry process), the birth rates are linear, that is $\lambda_k = k\lambda$, $\lambda > 0$.

Let $\mathcal{N}(t)$, $t > 0$, the number of component in the population at time t. The evolution of the population is determined by the following properties.

1. $\Pr\{\mathcal{N}(0) = 1\}$;
2. $\Pr\{\mathcal{N}(t, t+dt] = 1 \mid \mathcal{N}(t) = k\} = \lambda_k dt + o(dt)$;
3. $\Pr\{\mathcal{N}(t, t+dt] = 0 \mid \mathcal{N}(t) = k\} = 1 - \lambda_k dt + o(dt)$;
4. $\Pr\{\mathcal{N}(t, t+dt] > 1 \mid \mathcal{N}(t) = k\} = o(dt)$.

It can be shown that the state probabilities $p_k(t) = \Pr\{\mathcal{N}(t) = k \mid \mathcal{N}(0) = 1\}$ solve the Cauchy problem

$$
\begin{cases}
\frac{d}{dt} p_k(t) = -\lambda_k p_k(t) + \lambda_{k-1} p_{k-1}(t), \\
p_k(0) = \begin{cases} 1, & k = 1, \\ 0, & k = 0. \end{cases}
\end{cases} \tag{1.28}
$$

The solution to (1.28), can be obtained by means of induction. A complete proof for the more general fractional case can be found in Chapter 2, and this can be adapted to the classical case setting $\nu = 1$. The state probabilities then turn out to be

$$\Pr\{\mathcal{N}(t) = k\} = \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{e^{-\lambda_m t}}{\prod_{l=1,l\neq m}^{k}(\lambda_l - \lambda_m)}, & k > 1, \\ e^{-\lambda_1 t}, & k = 1. \end{cases} \tag{1.29}$$

Remark 1.1.1. *The pure birth process can have different behaviour depending on the structure of the rates λ_k, $k \geq 1$. For example, if $\sum_{k=1}^{\infty}(1/\lambda_k) < \infty$ the process can* explode, *that is, an infinite number of events in a finite interval of time, can occur. This behaviour can be intuitively explained by noting that the series converges when the λ_k's (and thus the infinitesimal birth probabilities) are small. For a proof of this result see Grimmett and Stirzaker (2001), page 252.*

In the linear case (the so-called Yule–Furry process, indicated here as $N(t)$, $t > 0$), originally introduced by McKendrick (1914), the probability distribution (1.29), becomes much simpler. In this particular case the rates are $\lambda_k = k\lambda$, $k \geq 1$, $\lambda > 0$. By noticing that

$$\begin{aligned} \prod_{l=1,l\neq m}^{k} (\lambda_l - \lambda_m) &= \lambda^{k-1}\left[(1-m)(2-m)\dots(m-1-m)(m+1-m)\dots(k-m)\right] \\ &= \lambda^{k-1}(-1)^{m-1}(m-1)!(k-m)!, \end{aligned} \tag{1.30}$$

equation (1.29) can be written as

$$\begin{aligned} p_k(t) &= \lambda^{k-1}(k-1)! \sum_{m=1}^{k} \frac{e^{\lambda m t}}{\lambda^{k-1}(m-1)!(k-m)!}(-1)^{m-1} \\ &= \sum_{m=1}^{k} e^{-\lambda m t}(-1)^{m-1}\frac{(k-1)!}{(m-1)!(k-m)!} = \sum_{m=0}^{k-1} e^{-\lambda t(m+1)}(-1)^m \frac{(k-1)!}{m!(k-m-1)!} \\ &= e^{-\lambda t}\sum_{m=0}^{k-1}\binom{k-1}{m}(-1)^m e^{-\lambda m t} = e^{-\lambda t}(1-e^{-\lambda t})^{k-1}, \qquad k \geq 1. \end{aligned} \tag{1.31}$$

The state probabilities (1.31) can be, in a more classical way, directly derived by solving the governing difference-differential equations

$$\begin{cases} \frac{d}{dt}p_k(t) = -\lambda k p_k(t) + \lambda(k-1)p_{k-1}(t), \\ p_k(0) = \begin{cases} 1, & k = 1, \\ 0, & k = 0, \end{cases} \end{cases} \tag{1.32}$$

as follows.

Consider the probability generating function $G(u,t) = \sum_{k=1}^{\infty} u^k p_k(t)$, $|u| \leq 1$. From (1.32) we arrive at

$$\begin{cases} \frac{\partial}{\partial t}G(u,t) = \lambda u(u-1)\frac{\partial}{\partial u}G(u,t), \\ G(u,0) = u. \end{cases} \tag{1.33}$$

The solution to (1.33) is $G(u,t) = f\left(e^{-\lambda t}u/(1-u)\right)$. By means of the initial condition and letting $v = u/(1-u)$, we obtain immediately that $f(v) = v/(1-v)$. In conclusion, we arrive at the explicit form of the probability generating function

$$G(u,t) = \frac{e^{-\lambda t}\frac{u}{1-u}}{1+\frac{u}{1-u}e^{-\lambda t}} = \frac{ue^{-\lambda t}}{1-u(1-e^{-\lambda t})} \tag{1.34}$$

$$= ue^{-\lambda t}\sum_{k=0}^{\infty} u^k(1-e^{-\lambda t})^k = \sum_{k=1}^{\infty} u^k e^{-\lambda t}(1-e^{-\lambda t})^{k-1}, \tag{1.35}$$

and this confirms result (1.31).

From the above result, it is straightforward to obtain the mean value and the variance, which are, respectively

$$\mathbb{E}N(t) = e^{\lambda t}, \qquad t > 0; \tag{1.36}$$

$$\mathbb{V}\mathrm{ar}N(t) = e^{\lambda t}\left(e^{\lambda t}-1\right), \qquad t > 0. \tag{1.37}$$

1.1.3 Pure death process

Another stochastic process often used to modelling for example evolving populations and strictly related to the pure birth process analysed in the preceding section is the pure death process. Consider n_0 initial individuals subject to individual death according to the following rule:

$$\Pr\{\mathcal{M}(t,t+dt] = 1 \mid \mathcal{M}(t) = k\} = \mu_k dt + o(dt), \qquad 0 \le k \le n_0, \tag{1.38}$$

where $\mathcal{M}(t)$, $t > 0$, is the number of individuals in the population at time t, and $\mu_k > 0$, $0 \le k \le n_0$, are the death rates. The state probabilities $\rho_k(t) = \Pr\{\mathcal{M}(t) = k \mid \mathcal{M}(0) = n_0\}$, are thus subject to the difference-differential equations

$$\begin{cases} \frac{d}{dt}\rho_k(t) = \mu_{k+1}\rho_{k+1}(t) - \mu_k\rho_k(t), & 0 \le k \le n_0, \\ \rho_k(0) = \begin{cases} 1, & k = n_0, \\ 0, & 0 \le k < n_0, \end{cases} \end{cases} \tag{1.39}$$

and

$$\begin{cases} \frac{d}{dt}\rho_0(t) = \mu_1\rho_1(t), \\ \frac{d}{dt}\rho_{n_0}(t) = \mu_{n_0}\rho_{n_0}(t). \end{cases} \tag{1.40}$$

In this nonlinear case the solution to the Cauchy problem (1.40) reads

$$\rho_k(t) = \begin{cases} e^{-\mu_{n_0}t}, & k = n_0, \\ \prod_{j=k+1}^{n_0}\mu_j \sum_{m=k}^{n_0} \frac{e^{-\mu_m t}}{\prod_{\substack{h=k \\ h\ne m}}^{n_0}(\mu_h-\mu_m)}, & 0 < k < n_0, \\ 1 - \sum_{m=1}^{n_0}\prod_{\substack{h=1 \\ h\ne m}}^{n_0}\left(\frac{\mu_h}{\mu_h-\mu_m}\right)e^{-\mu_m t}, & k = 0,\ n_0 > 1. \end{cases} \tag{1.41}$$

Obviously, for $k = 0$, $n_0 = 1$,

$$\rho_0^{\nu}(t) = 1 - e^{-\mu_1 t}. \tag{1.42}$$

In the simple linear case, that is when $\mu_k = k\mu$, $\mu > 0$, $0 \le k \le n_0$, the state probabilities $\mathfrak{p}_k(t) = \Pr\{M(t) = k \mid M(0) = n_0\}$, where $M(t)$, $t > 0$ is the linear pure death process, take a particularly neat form:

$$\mathfrak{p}_k(t) = \binom{n_0}{k} e^{-k\mu t}(1 - e^{-\mu t})^{n_0 - k}, \qquad 0 \le k \le n_0. \tag{1.43}$$

For more results on the pure death process see Chapter 3, where a sublinear death process is also considered.

1.1.4 Birth-death process

Here, we summarise some basic results related to the linear birth-death process. Consider an evolving population; the initial number of individuals is fixed at 1 and the components of the population are subject to both birth and death, as specified by the following ruleset:

1. $\Pr\{\mathfrak{N}(0) = 1\} = 1$;
2. $\Pr\{\mathfrak{N}(t, t+dt] = 1 \mid \mathfrak{N}(t) = k\} = k\lambda dt + o(dt)$;
3. $\Pr\{\mathfrak{N}(t, t+dt] = -1 \mid \mathfrak{N}(t) = k\} = k\mu dt + o(dt)$;
4. $\Pr\{\mathfrak{N}(t, t+dt] = 0 \mid \mathfrak{N}(t) = k\} = 1 - k(\lambda + \mu)dt + o(dt)$;
5. $\Pr\{|\mathfrak{N}(t, t+dt]| > 1 \mid \mathfrak{N}(t) = k\} = o(dt)$,

where $\lambda, \mu > 0$ are respectively the birth and death rates, and $\mathfrak{N}(t)$, $t > 0$, is the number of individuals in the population at time t. The reader can immediately notice the linearity of this model (rates proportional to the population size).

The difference-differential equations satisfied by the state probabilities, in this case, read

$$\begin{cases} \frac{d}{dt}\mathfrak{p}_k(t) = \lambda(k-1)\mathfrak{p}_{k-1}(t) - (\lambda + \mu)k\mathfrak{p}_k(t) + \mu(k+1)\mathfrak{p}_{k+1}(t), & k \ge 1 \\ \frac{d}{dt}\mathfrak{p}_0(t) = \mu\mathfrak{p}_1(t), & k = 0. \end{cases} \tag{1.44}$$

From equation (1.44) one can directly comes upon the partial differential equation governing the moment generating function, thus obtaining

$$\begin{cases} \frac{\partial}{\partial t} h(s,t) = \left[\lambda\left(e^s - 1\right) + \mu\left(e^{-s} - 1\right)\right] \frac{\partial}{\partial s} h(s,t), \\ h(s,0) = e^s. \end{cases} \tag{1.45}$$

The above differential equation can be solved, for example, by recurring to Laplace transforms as in the preceding sections. We therefore arrive, in the case $\lambda \ne \mu$, at the explicit form of the moment generating function

$$\mathfrak{h}(s,t) = \frac{\frac{\mu(e^s - 1)e^{(\lambda-\mu)t}}{\lambda e^s - \mu} - 1}{\frac{\lambda(e^s - 1)e^{(\lambda-\mu)t}}{\lambda e^s - \mu} - 1}, \tag{1.46}$$

and consequently to that of the probability generating function:

$$\mathfrak{G}(u,t) = \frac{\frac{\mu(u - 1)e^{(\lambda-\mu)t}}{\lambda u - \mu} - 1}{\frac{\lambda(u - 1)e^{(\lambda-\mu)t}}{\lambda u - \mu} - 1}, \tag{1.47}$$

which, in turn, leads to the state probability distribution of the linear birth-death process:

$$\begin{cases} \mathfrak{p}_k(t) = \left[1 - \frac{\mu\left(e^{(\lambda-\mu)t}-1\right)}{\lambda e^{(\lambda-\mu)t}-\mu}\right]\left[1 - \frac{\lambda\left(e^{(\lambda-\mu)t}-1\right)}{\lambda e^{(\lambda-\mu)t}-\mu}\right]\left[\frac{\lambda\left(e^{(\lambda-\mu)t}-1\right)}{\lambda e^{(\lambda-\mu)t}-\mu}\right]^{k-1}, & k \geq 1 \\ \mathfrak{p}_0(t) = \left[\frac{\mu\left(e^{(\lambda-\mu)t}-1\right)}{\lambda e^{(\lambda-\mu)t}-\mu}\right], & k = 0. \end{cases} \tag{1.48}$$

When $\lambda = \mu$, we obtain that the moment generating function and the probability generating function, respectively read

$$\mathfrak{h}(s,t) = \frac{1-(\lambda t-1)(e^s-1)}{1-\lambda t(e^s-1)}, \tag{1.49}$$

$$\mathfrak{G}(u,t) = \frac{1-(\lambda t-1)(u-1)}{1-\lambda t(u-1)}, \tag{1.50}$$

thus obtaining the following state probabilities:

$$\begin{cases} \mathfrak{p}_k(t) = \frac{(\lambda t)^{k-1}}{(1+\lambda t)^{k+1}}, & k \geq 1 \\ \mathfrak{p}_0(t) = \frac{\lambda t}{1+\lambda t}, & k = 0. \end{cases} \tag{1.51}$$

The mean value $\mathbb{E}\mathfrak{N}(t)$, $t > 0$, from (1.47) and (1.49), can be written as

$$\mathbb{E}\mathfrak{N}(t) = \begin{cases} e^{(\lambda-\mu)t}, & \lambda \neq \mu \\ 1, & \lambda = \mu, \end{cases} \tag{1.52}$$

which tells us that, for $\lambda \neq \mu$, the mean value can be either increasing or decreasing depending on the sign of $(\lambda - \mu)$. Also, in the other case $(\lambda = \mu)$, the mean value turns out to be constant and equal to the initial number of progenitors. More insights on the evolution of the process, can be revealed by analysing the limiting behaviour of the extinction probability $\mathfrak{p}_0(t)$, $t > 0$. From (1.48) and (1.51), we obtain that

$$\lim \mathfrak{p}_0(t) = \begin{cases} 1, & \lambda \leq \mu, \\ \frac{\mu}{\lambda}, & \lambda > \mu. \end{cases} \tag{1.53}$$

Note that, in the case $\lambda = \mu$, although $\mathbb{E}\mathfrak{N}(t) = 1$, the extinction occurs with probability 1.

For more in-depth information about the linear birth-death process, the reader can consult the book by Bailey (1964).

1.2 A brief introduction to fractional calculus

Fractional calculus is meant as a direct generalisation of the usual integer-order calculus to arbitrary order. The whole term “fractional calculus” is actually a misnomer, as it should be correctly named instead “arbitrary-order calculus”. In the following, we present some basic definitions and properties that prove to be useful in defining and analysing fractional growth processes.

1.2.1 Fractional integral, derivatives, and their properties

Consider the well-known Cauchy formula for evaluating multiple integrals:

$$I^n f(x) = \int_0^x dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{n-1}} f(x_n) dx_n \tag{1.54}$$

$$= \frac{1}{\Gamma(n)} \int_0^x (x - x_n)^{n-1} f(x_n) dx_n,$$

where I^n, $n \in \mathbb{N}$, is the multiple integral operator, and $\Gamma(z)$ is the gamma function, defined as

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\dots(z+n)}, \qquad z \in \mathbb{C}. \tag{1.55}$$

When $z > 0$, the following integral expansion also holds.

$$\Gamma(z) = \int_0^\infty y^{z-1} e^{-y} dy, \qquad z > 0. \tag{1.56}$$

For details on gamma function, the reader can consult any good book on special functions, e.g. Lebedev (1972); here we present just few properties without proofs.

1. If $n \in \mathbb{N}$, then $\Gamma(n+1) = n!$.
2. It has poles in $-n$, $n \in \mathbb{N} \cup \{0\}$.
3. $\Gamma(z+1) = z\Gamma(z)$.
4. $\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$.

Formula (1.54) permits us to get rid of the multiple integral and retain just a single convolution-type integral. Formula (1.54) is also a perfect means to generalising the notion of *multiple integral* to that of *fractional integral*.

Let us consider a slight modification of formula (1.54) in which n is replaced by a new parameter $\alpha \in \mathbb{C}$, $\Re(\alpha) \in \mathbb{R}^+$.

Definition 1.2.1 (Riemann–Liouville fractional integral). *For an analytic function $f(x)$, $x \in \mathbb{R}^+$, we have that*

$${}_0I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y) dy. \tag{1.57}$$

The operator ${}_0I^\alpha$ is called Riemann–Liouville fractional integral.

In particular, formula (1.57) is the Riemann formula. A more general definition is the following:

$${}_cI^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-y)^{\alpha-1} f(y) dy. \tag{1.58}$$

When $c = -\infty$, we obtain the Liouville formula. Note that the fractional integral can be no longer interpreted as a repeated integral, exactly as a power with a real exponent cannot be interpreted as a repeated product.

After having defined the fractional integral, it is now necessary to construct a derivative operator. We have several possibilities and in the following we will describe the two major definitions. For a more in-depth explanation of possible construction of fractional derivatives operators, we refer to the classical books by Podlubny (1999) or Kilbas et al. (2006).

Definition 1.2.2 (Riemann–Liouville fractional derivative). *Let $0 < \nu < 1$, we have that*

$$\frac{\partial^\nu}{\partial x^\nu} f(x) = \frac{d}{dx} \left[{}_0I^{1-\nu} f(x) \right], \qquad x \in \mathbb{R}. \tag{1.59}$$

By exchanging the order of the derivative and integral operators in (1.59), we arrive at the second definition:

Definition 1.2.3 (Caputo fractional derivative). *Let $0 < \nu < 1$, we have that*

$$\frac{d^\nu}{dx^\nu} f(x) = {}_0I^{1-\nu} \left[\frac{d}{dx} f(x) \right], \qquad x \in \mathbb{R}. \tag{1.60}$$

For the fractional integral ${}_0I^\nu$, the following two properties hold:

1. Linearity:

$$ {}_0I^{1-\nu}(af(x) + bg(x)) = a {}_0I^{1-\nu} f(x) + b {}_0I^{1-\nu} g(x), \qquad a, b \text{ arbitrary constants.} \tag{1.61}$$

2. Semigroup property:

$$ {}_0I^{\nu_1}({}_0I^{\nu_2} f(x)) = {}_0I^{\nu_1+\nu_2} f(x) = {}_0I^{\nu_2}({}_0I^{\nu_1} f(x)), \qquad \nu_1, \nu_2 \in (0, 1). \tag{1.62}$$

For the Riemann–Liouville fractional derivative, the linearity property holds as well

$$\frac{\partial^\nu}{\partial x^\nu} [af(x) + bg(x)] = a \frac{\partial^\nu}{\partial x^\nu} f(x) + b \frac{\partial^\nu}{\partial x^\nu} g(x), \tag{1.63}$$

while the semigroup property does not hold:

$$\frac{\partial^{\nu_1}}{\partial x^{\nu_1}} \left[\frac{\partial^{\nu_2}}{\partial x^{\nu_2}} f(x) \right] \neq \frac{\partial^{\nu_1+\nu_2}}{\partial x^{\nu_1+\nu_2}} f(x), \qquad \nu_1, \nu_2 \in (0, 1). \tag{1.64}$$

Another important fact is that, differently from the classical case ($\nu = 1$),

$$ {}_0I^{1-\nu} \left[\frac{\partial^\nu}{\partial x^\nu} f(x) \right] \neq f(x), \tag{1.65}$$

but

$$\frac{\partial^\nu}{\partial x^\nu} \left[{}_0I^{1-\nu} f(x) \right] = f(x). \tag{1.66}$$

A fundamental tool for our next analyses is the Laplace transform. Therefore, we highlight the following relations involving Laplace transforms of the two main definitions of fractional derivative.

For the Riemann–Liouville fractional derivative we have:

$$\int_0^\infty e^{-\mu x} \frac{\partial^\nu}{\partial x^\nu} f(x) dx = \mathcal{L} \left\{ \frac{\partial^\nu}{\partial x^\nu} f(x) \right\} (\mu) = \mu^\nu \mathcal{L} \{f(x)\} (\mu) - \left. \frac{\partial^{\nu-1}}{\partial x^{\nu-1}} f(x) \right|_{x=0}, \tag{1.67}$$

while for the Caputo fractional derivative we have:

$$\int_0^\infty e^{-\mu x} \frac{d^\nu}{dx^\nu} f(x) dx = \mathcal{L} \left\{ \frac{d^\nu}{dx^\nu} f(x) \right\} (\mu) = \mu^\nu \mathcal{L} \{f(x)\} (\mu) - \left. \mu^{\nu-1} f(x) \right|_{x=0}. \tag{1.68}$$

Chapter 2

Fractional Pure Birth Processes

Summary

In this chapter we consider a fractional version of the classical nonlinear birth process of which the Yule–Furry model is a particular case. Fractionality is obtained by replacing the first-order time derivative in the difference-differential equations which govern the probability law of the process, with the Dzhrbashyan–Caputo fractional derivative. We derive the probability distribution of the number $\mathcal{N}^{\nu}(t)$ of individuals at an arbitrary time t. We also present an interesting representation for the number of individuals at time t, in the form of the subordination relationship $\mathcal{N}^{\nu}(t) = \mathcal{N}(T_{2\nu}(t))$, where $\mathcal{N}(t)$ is the classical generalised birth process and $T_{2\nu}(t)$ is a random time whose distribution is related to the fractional diffusion equation. The fractional linear birth process is examined in detail in section 2.2 and various forms of its distribution are given and discussed.

2.1 Introduction

We consider a birth process and denote by $\mathcal{N}(t)$, $t > 0$ the number of components in a stochastically developing population at time t. Possible examples are the number of particles produced in a radioactive disintegration and the number of particles in a cosmic ray shower where death is not permitted. The probabilities $p_k(t) = \Pr\{\mathcal{N}(t) = k\}$ satisfy the difference-differential equations

$$\frac{dp_k}{dt} = -\lambda_k p_k + \lambda_{k-1} p_{k-1}, \qquad k \geq 1, \tag{2.1}$$

where at time $t = 0$

$$p_k(0) = \begin{cases} 1, & k = 1, \\ 0, & k \geq 2. \end{cases} \tag{2.2}$$

This means that we have initially one progenitor igniting the branching process. For information on this process consult Gikhman and Skorokhod (1996) page 322.

We here examine a fractional version of the birth process where the probabilities are governed by

$$\frac{d^\nu p_k}{dt^\nu} = -\lambda_k p_k + \lambda_{k-1} p_{k-1}, \qquad k \geq 1, \tag{2.3}$$

and where the fractional derivative is understood in the Dzhrbashyan–Caputo sense, that is as

$$\frac{d^\nu p_k}{dt^\nu} = \frac{1}{\Gamma(1-\nu)} \int_0^t \frac{\frac{d}{ds} p_k(s)}{(t-s)^\nu} ds, \qquad \text{for } 0 < \nu < 1, \tag{2.4}$$

(see Podlubny (1999)). The use of Dzhrbashyan–Caputo derivative is preferred because in this case initial conditions can be expressed in terms of integer-order derivatives.

Extensions of continuous-time point processes like the homogeneous Poisson process to the fractional case have been considered in Jumarie (2001), Cahoy (2007), Laskin (2003), Wang and Wen (2003), Wang et al. (2006), Wang et al. (2007), Uchaikin and Sibatov (2008), Repin and Saichev (2000) and Beghin and Orsingher (2009b). A recently published paper (Uchaikin et al. (2008)) considers a fractional version of the Yule–Furry process where the mean value $\mathbb{E}N^\nu(t)$ is analysed.

By solving recursively the equation (2.3) (we write $p_k(t)$, $t > 0$ in equations (2.3) and $p_k^\nu(t)$ for the solutions) we obtain that

$$p_k^\nu(t) = \Pr\{\mathcal{N}_\nu(t) = k\} = \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \left\{ \frac{1}{\prod_{\substack{l=1 \\ l \neq m}}^{k} (\lambda_l - \lambda_m)} E_{\nu,1}(-\lambda_m t^\nu) \right\}, & k > 1, \\ E_{\nu,1}(-\lambda_1 t^\nu), & k = 1. \end{cases} \tag{2.5}$$

Result (2.5) generalises the classical distribution of the birth process (see Gikhman and Skorokhod (1996) page 322, or Bartlett (1978), page 59) where instead of the exponentials we have the Mittag–Leffler functions defined as

$$E_{\nu,1}(x) = \sum_{h=0}^{\infty} \frac{x^h}{\Gamma(\nu h + 1)}, \qquad x \in \mathbb{R}, \quad \nu > 0. \tag{2.6}$$

The fractional pure birth process has some specific features entailed by the fractional derivative appearing in (2.4) which is a non-local operator. The process governed by fractional equations (and therefore the related probabilities $p_k^\nu(t) = \Pr\{\mathcal{N}^\nu(t) = k\}$, $k \geq 1$) displays a slowly decreasing memory which seems a characteristic feature of all real systems (for example the hereditariety and the related aspects observed in phenomena such as the fatigue of metals, magnetic hysteresis and others). Fractional equations of various types have proved to be useful in representing different phenomena in optics (light propagation through random media), transport of charge carriers and also in economics (a survey of applications can be found in Podlubny (1999)). We show below that for the linear birth process $N^\nu(t)$, $t > 0$ the mean values $\mathbb{E}N^\nu(t)$, $\mathbb{V}\mathrm{ar}N^\nu(t)$ are increasing functions as the order of fractionality ν decreases. This shows that the fractional birth process is capable of representing explosively developing epidemics, accelerated cosmic showers and in general very rapidly expanding populations. This is a feature which the fractional pure birth process shares with its Poisson fractional counterpart whose practical applications are studied in recent works (see for example Laskin (2003) and Cahoy (2007)).

We are able to show that the fractional birth process $\mathcal{N}_\nu(t)$ can be represented as

$$\mathcal{N}_\nu(t) = \mathcal{N}(T_{2\nu}(t)), \qquad t > 0,\ 0 < \nu \leq 1, \tag{2.7}$$

where $T_{2\nu}(t)$, $t > 0$ is the random time process whose distribution at time t is obtained from the fundamental solution to the fractional diffusion equation (the fractional derivative is defined in (2.4))

$$\frac{\partial^{2\nu} u}{\partial t^{2\nu}} = \frac{\partial^2 u}{\partial s^2}, \qquad 0 < \nu \leq 1, \tag{2.8}$$

subject to the initial conditions $u(s,0) = \delta(s)$ for $0 < \nu \leq 1$ and also $u_t(s,0) = 0$ for $1/2 < \nu \leq 1$, as

$$\Pr\{T_{2\nu}(t) \in ds\} = \begin{cases} 2u_{2\nu}(s,t)\, ds & \text{for } s > 0, \\ 0 & \text{for } s < 0. \end{cases} \tag{2.9}$$

This means that the fractional birth process is a classical birth process with a random time $T_{2\nu}(t)$ which is the sole component of (2.7) affected by the fractional derivative. In equation (2.8) and throughout the whole chapter the fractional derivative must be understood in Dzhrbashyan–Caputo sense (2.3). The representation (2.7) leads to

$$\Pr\{\mathcal{N}_\nu(t) = k\} = \int_0^\infty \Pr\{\mathcal{N}(s) = k\} \Pr\{T_{2\nu}(t) \in ds\}, \tag{2.10}$$

where

$$\Pr\{\mathcal{N}(s) = k\} = \begin{cases} \displaystyle\prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{e^{-\lambda_m s}}{\prod_{\substack{l=1 \\ l \neq m}}^{k} (\lambda_l - \lambda_m)}, & k > 1,\ s > 0, \\ e^{-\lambda_1 s}, & k = 1,\ s > 0. \end{cases} \tag{2.11}$$

Formula (2.10) shows that $\sum_k \Pr\{\mathcal{N}_\nu(t) = k\} = 1$ iff $\sum_k \Pr\{\mathcal{N}(t) = k\} = 1$. It is well-known that the process $\mathcal{N}(t)$, $t > 0$ is such that $\Pr(\mathcal{N}(t) < \infty) = 1$ for all $t > 0$ (non-exploding) if $\sum_k \lambda_k^{-1} = \infty$ (consult Feller (1968), page 452).

A special case of the above fractional birth process is the fractional linear birth process where $\lambda_k = \lambda k$. The distribution (2.5) reduces in this case to the simple form

$$p_k^\nu(t) = \sum_{j=1}^{k} \binom{k-1}{j-1} (-1)^{j-1} E_{\nu,1}(-\lambda j t^\nu), \qquad k \geq 1,\ t > 0. \tag{2.12}$$

For $\nu = 1$, we retrieve from (2.12) the classical geometric structure of the linear birth process with a single progenitor, that is

$$p_k^1(t) = \left(1 - e^{-\lambda t}\right)^{k-1} e^{-\lambda t}, \qquad k \geq 1,\ t > 0. \tag{2.13}$$

An interesting qualitative feature of the fractional linear birth process can be extracted from (2.12); it permits us to highlight the dependence of the branching speed on the order of fractionality ν. We show in section 2.3 that

$$\Pr\left\{N^{\nu}\left(dt\right)=n_0+1\mid N^{\nu}\left(0\right)=n_0\right\}\sim\frac{\lambda n_0\left(dt\right)^{\nu}}{\Gamma\left(\nu+1\right)},\tag{2.14}$$

and this proves that a decrease in the order of fractionality ν speeds up the reproduction of individuals. We are not able to generalise (2.14) to the case

$$\Pr\left\{N^{\nu}\left(t+dt\right)=n_0+1\mid N^{\nu}\left(t\right)=n_0\right\},\tag{2.15}$$

because the process we are investigating is not time-homogeneous. For the fractional linear birth process the representation (2.7) reduces to the form

$$N^{\nu}\left(t\right)=N\left(T_{2\nu}\left(t\right)\right),\qquad t>0,\ 0<\nu\leq 1,\tag{2.16}$$

and has an interesting special structure when $\nu=1/2^n$. For example for $n=2$ the random time appearing in (2.16) becomes a folded iterated Brownian motion. This means that

$$N^{\frac{1}{4}}\left(t\right)=N\left(\left|\mathcal{B}_1\left(\left|\mathcal{B}_2\left(t\right)\right|\right)\right|\right).\tag{2.17}$$

Clearly $|\mathcal{B}_2(t)|$ is a reflecting Brownian motion starting from zero and $|\mathcal{B}_1(|\mathcal{B}_2(t)|)|$ is a reflecting iterated Brownian motion. This permits us to write the distribution of (2.17) in the following form

$$\begin{aligned}&\Pr\left\{N^{\frac{1}{4}}\left(t\right)=k\mid N^{\frac{1}{4}}\left(0\right)=1\right\}\\&=\int_0^{\infty}\left(1-e^{-\lambda s}\right)^{k-1}e^{-\lambda s}\left\{2^2\int_0^{\infty}\frac{e^{-\frac{s^2}{4\omega}}}{\sqrt{2\pi 2\omega}}\frac{e^{-\frac{\omega^2}{4t}}}{\sqrt{2\pi 2t}}d\omega\right\}ds.\end{aligned}\tag{2.18}$$

The case $\nu=1/2^n$ involves the $(n-1)$-times iterated Brownian motion

$$\mathcal{I}_{n-1}\left(t\right)=\mathcal{B}_1\left(\left|\mathcal{B}_2\left(\cdots\left|\mathcal{B}_n\left(t\right)\right|\cdots\right)\ \right|\right),\tag{2.19}$$

with distribution

$$\begin{aligned}&\Pr\left\{\left|\mathcal{B}_1\left(\left|\mathcal{B}_2\left(\cdots\left|\mathcal{B}_n\left(t\right)\right|\cdots\right)\ \right)\right|\in ds\right\}\\&=ds\,2^n\int_0^{\infty}\frac{e^{-\frac{s^2}{4\omega_1}}}{\sqrt{4\pi\omega_1}}d\omega_1\int_0^{\infty}\frac{e^{-\frac{\omega_1^2}{4\omega_2}}}{\sqrt{4\pi\omega_2}}d\omega_2\cdots\int_0^{\infty}\frac{e^{-\frac{\omega_{n-1}^2}{4t}}}{\sqrt{4\pi t}}d\omega_{n-1}.\end{aligned}\tag{2.20}$$

For details on this point see Orsingher and Beghin (2009).

2.2 The distribution function for the generalised fractional birth process

We present now the explicit distribution

$$\Pr\left\{\mathcal{N}_{\nu}\left(t\right)\mid\mathcal{N}\left(0\right)=1\right\}=p_k^{\nu}\left(t\right),\qquad t>0,\ k\geq 1,\ 0<\nu\leq 1\tag{2.21}$$

of the number of individuals in the population expanding according to (2.3). Our technique is based on successive applications of the Laplace transform. Our first result is the next theorem below.

Theorem 2.2.1. *The solution to the fractional equations*

$$\begin{cases} \frac{d^\nu p_k}{dt^\nu} = -\lambda_k p_k + \lambda_{k-1} p_{k-1} & k \geq 1, 0 < \nu \leq 1, \\ p_k(0) = \begin{cases} 1 & k = 1, \\ 0 & k \geq 2, \end{cases} \end{cases} \tag{2.22}$$

is given by

$$p_k^\nu(t) = Pr\{\mathcal{N}_\nu(t) = k\} = \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \left\{ \frac{1}{\prod_{\substack{l=1 \\ l \neq m}}^{k} (\lambda_l - \lambda_m)} E_{\nu,1}(-\lambda_m t^\nu) \right\}, & k > 1, \\ E_{\nu,1}(-\lambda_1 t^\nu), & k = 1. \end{cases} \tag{2.23}$$

Proof. We prove result (2.23) by a recursive procedure.

For $k = 1$, the equation

$$\frac{d^\nu p_1}{dt^\nu} = -\lambda_1 p_1, \qquad p_1(0) = 1, \tag{2.24}$$

is immediately solved by

$$p_1^\nu(t) = E_{\nu,1}(-\lambda_1 t^\nu). \tag{2.25}$$

For $k = 2$ equation (2.3) becomes

$$\begin{cases} \frac{d^\nu p_2}{dt^\nu} = -\lambda_2 p_2 + \lambda_1 E_{\nu,1}(-\lambda_1 t^\nu), \\ p_2(0) = 0. \end{cases} \tag{2.26}$$

In view of the fact that

$$\int_0^\infty e^{-\mu t} E_{\nu,1}(-\lambda_1 t^\nu)\, dt = \frac{\mu^{\nu-1}}{\mu^\nu + \lambda_1}, \tag{2.27}$$

the Laplace transform of (2.26) yields

$$L_2(\mu) = \frac{\lambda_1 \mu^{\nu-1}}{\lambda_2 - \lambda_1} \left[\frac{1}{\mu^\nu + \lambda_1} - \frac{1}{\mu^\nu + \lambda_2} \right]. \tag{2.28}$$

In light of (2.27), from (2.28) we can extract the probability $p_2^\nu(t)$

$$p_2^\nu(t) = \left[E_{\nu,1}(-\lambda_1 t^\nu) - E_{\nu,1}(-\lambda_2 t^\nu) \right] \frac{\lambda_1}{\lambda_2 - \lambda_1}. \tag{2.29}$$

Now the Laplace transform of

$$\frac{d^\nu p_3}{dt^\nu} = -\lambda_3 p_3 + \frac{\lambda_2 \lambda_1}{\lambda_2 - \lambda_1} \left[E_{\nu,1}(-\lambda_1 t^\nu) - E_{\nu,1}(-\lambda_2 t^\nu) \right] \tag{2.30}$$

yields, after some computations

$$\begin{aligned} L_3(\mu) = \lambda_2 \lambda_1 \mu^{\nu-1} \bigg[& \frac{1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} \frac{1}{\mu^\nu + \lambda_1} \\ & + \frac{1}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} \frac{1}{\mu^\nu + \lambda_2} + \frac{1}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \frac{1}{\mu^\nu + \lambda_3} \bigg]. \end{aligned} \tag{2.31}$$

From this result it is clear that

$$p_3^\nu(t) = \lambda_2 \lambda_1 \left[\frac{1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} E_{\nu,1}(-\lambda_1 t^\nu) \right. \tag{2.32}$$

$$+\frac{1}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}E_{\nu,1}(-\lambda_2 t^\nu)+\frac{1}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}E_{\nu,1}(-\lambda_3 t^\nu)\Big].$$

The procedure for $k>3$ becomes more complicated. However the special case $k=4$ is instructive and we treat it first.

The Laplace transform of the equation

$$\begin{aligned}\frac{d^\nu p_4}{dt^\nu} =& -\lambda_4 p_4+\lambda_1\lambda_2\lambda_3\Big[\frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}E_{\nu,1}(-\lambda_1 t^\nu) \\ &+\frac{1}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}E_{\nu,1}(-\lambda_2 t^\nu)+\frac{1}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}E_{\nu,1}(-\lambda_2 t^\nu)\Big],\end{aligned} \tag{2.33}$$

subject to the initial condition $p_4(0)=0$ becomes

$$\begin{aligned}L_4(\mu)=&\lambda_1\lambda_2\lambda_3\mu^{\nu-1}\Big[\frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)}\left\{\frac{1}{\mu^\nu+\lambda_1}-\frac{1}{\mu^\nu+\lambda_4}\right\} \\ &+\frac{1}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)}\left\{\frac{1}{\mu^\nu+\lambda_2}-\frac{1}{\mu^\nu+\lambda_4}\right\} \\ &+\frac{1}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)}\left\{\frac{1}{\mu^\nu+\lambda_3}-\frac{1}{\mu^\nu+\lambda_4}\right\}\Big].\end{aligned} \tag{2.34}$$

The critical point of the proof is to show that

$$\begin{aligned}&-[(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)-(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)(\lambda_4-\lambda_3) \\ &\quad+(\lambda_2-\lambda_1)(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)]/[(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)(\lambda_3-\lambda_2) \\ &\quad\times(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)] \\ &=\frac{1}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}.\end{aligned} \tag{2.35}$$

We note that

$$\begin{aligned}0=&\det\begin{pmatrix}1&1&1&1\\1&1&1&1\\\lambda_1&\lambda_2&\lambda_3&\lambda_4\\\lambda_1^2&\lambda_2^2&\lambda_3^2&\lambda_4^2\end{pmatrix}=\det\begin{pmatrix}1&1&1\\\lambda_2&\lambda_3&\lambda_4\\\lambda_2^2&\lambda_3^2&\lambda_4^2\end{pmatrix} \\ &-\det\begin{pmatrix}1&1&1\\\lambda_1&\lambda_3&\lambda_4\\\lambda_1^2&\lambda_3^2&\lambda_4^2\end{pmatrix}+\det\begin{pmatrix}1&1&1\\\lambda_1&\lambda_2&\lambda_4\\\lambda_1^2&\lambda_2^2&\lambda_4^2\end{pmatrix}-\det\begin{pmatrix}1&1&1\\\lambda_1&\lambda_2&\lambda_3\\\lambda_1^2&\lambda_2^2&\lambda_3^2\end{pmatrix} \\ =&(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)-(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)(\lambda_4-\lambda_3) \\ &+(\lambda_2-\lambda_1)(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)-(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2),\end{aligned} \tag{2.36}$$

where in the last step the Vandermonde formula is applied.

By inserting (2.36) into (2.34) we now have that

$$\begin{aligned}L_4(\mu)=\lambda_1\lambda_2\lambda_3\mu^{\nu-1}&\Big[\frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)}\frac{1}{\mu^\nu+\lambda_1} \\ &+\frac{1}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)}\frac{1}{\mu^\nu+\lambda_2} \\ &+\frac{1}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)}\frac{1}{\mu^\nu+\lambda_3} \\ &+\frac{1}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}\frac{1}{\mu^\nu+\lambda_4}\Big],\end{aligned} \tag{2.37}$$

so that by inverting (2.37) we extract the following result

$$p_4^\nu(t) = \prod_{j=1}^{3} \lambda_j \left\{ \sum_{m=1}^{4} \frac{1}{\prod_{\substack{l=1\\ l\neq m}}^{4} (\lambda_l - \lambda_m)} E_{\nu,1}(-\lambda_m t^\nu) \right\}. \tag{2.38}$$

We now tackle the problem of showing that (2.23) solves the Cauchy problem (2.22) for all $k > 1$, by induction. This means that we must solve

$$\begin{cases} \frac{d^\nu p_k}{dt^\nu} = -\lambda_k p_k + \prod_{j=1}^{k-1} \lambda_j \left\{ \sum_{m=1}^{k-1} \frac{1}{\prod_{\substack{l=1\\ l\neq m}}^{k-1} (\lambda_l - \lambda_m)} E_{\nu,1}(-\lambda_m t^\nu) \right\}, \\ \\ p_k(0) = 0, \end{cases} \qquad k > 4\ . \tag{2.39}$$

The Laplace transform of (2.39) reads

$$L_k(\mu) = \prod_{j=1}^{k-1} \lambda_j \left[\sum_{m=1}^{k-1} \frac{\mu^{\nu-1}}{\prod_{\substack{l=1\\ l\neq m}}^{k} (\lambda_l - \lambda_m)} \frac{1}{\mu^\nu + \lambda_m} - \frac{\mu^{\nu-1}}{\mu^\nu + \lambda_k} \sum_{m=1}^{k-1} \frac{1}{\prod_{\substack{l=1\\ l\neq m}}^{k} (\lambda_l - \lambda_m)} \right]. \tag{2.40}$$

We must now prove that

$$-\sum_{m=1}^{k-1} \frac{1}{\prod_{\substack{l=1\\ l\neq m}}^{k} (\lambda_l - \lambda_m)} = \frac{1}{\prod_{\substack{l=1\\ l\neq k}}^{k} (\lambda_l - \lambda_k)}, \tag{2.41}$$

and this relationship is important also for the proof of (2.11).

In order to prove (2.41) we rewrite the left-hand side as

$$-\sum_{m=1}^{k-1} \frac{\prod_{h=1}^{k-1} \prod_{l>h}^{k} (\lambda_l - \lambda_h)}{\prod_{\substack{l=1\\ l\neq m}}^{k} (\lambda_l - \lambda_m)} \cdot \frac{1}{\prod_{h=1}^{k-1} \prod_{l>i}^{k} (\lambda_l - \lambda_h)}, \tag{2.42}$$

and concentrate our attention on the numerator of (2.42). In analogy with the calculations in (2.36) we have that

$$\begin{aligned} 0 = \det &\begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m & \cdots & \lambda_k \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{k-2} & \lambda_2^{k-2} & \cdots & \lambda_m^{k-2} & \cdots & \lambda_k^{k-2} \end{pmatrix} \\ = \sum_{m=1}^{k} (-1)^{m-1} \det &\begin{pmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_{m-1} & \lambda_{m+1} & \cdots & \lambda_k \\ \lambda_1^{k-2} & \cdots & \lambda_{m-1}^{k-2} & \lambda_{m+1}^{k-2} & \cdots & \lambda_k^{k-2} \end{pmatrix} \\ = \sum_{m=1}^{k} \frac{\prod_{h=1}^{k-1} \prod_{l>h}^{k} (\lambda_l - \lambda_h)}{\prod_{\substack{l=1\\ l\neq m}}^{k} (\lambda_l - \lambda_m)} &= \sum_{m=1}^{k-1} \frac{\prod_{h=1}^{k-1} \prod_{l>h}^{k} (\lambda_l - \lambda_h)}{\prod_{\substack{l=1\\ l\neq m}}^{k} (\lambda_l - \lambda_m)} + \frac{\prod_{h=1}^{k-1} \prod_{l>h}^{k} (\lambda_l - \lambda_h)}{\prod_{\substack{l=1\\ l\neq k}}^{k} (\lambda_l - \lambda_k)}. \end{aligned} \tag{2.43}$$

In the third step of (2.43) we applied the Vandermonde formula and considered that the n-th column is missing. It must also be taken into account that

$$\frac{\prod_{l>1}^{k}(\lambda_l-\lambda_1)}{(\lambda_m-\lambda_1)}\cdot\frac{\prod_{l>2}^{k}(\lambda_l-\lambda_2)}{(\lambda_m-\lambda_2)}\cdots\frac{\prod_{l>m-1}^{k}(\lambda_l-\lambda_{m-1})}{(\lambda_m-\lambda_{m-1})} \tag{2.44}$$
$$\cdot\frac{\prod_{l>m}^{k}(\lambda_l-\lambda_m)}{\prod_{l>m}^{k}(\lambda_l-\lambda_m)}\cdot\prod_{l>m+1}^{k}(\lambda_l-\lambda_{m+1})\cdots\prod_{l>k-1}^{k}(\lambda_l-\lambda_{k-1})=\frac{\prod_{h=1}^{k-1}\prod_{l>h}^{k}(\lambda_l-\lambda_h)}{(-1)^{m-1}\prod_{\substack{l=1\\l\neq m}}^{k}(\lambda_l-\lambda_m)}.$$

From (2.42) and (2.43) we have that

$$-\sum_{m=1}^{k-1}\frac{1}{\prod_{\substack{l=1\\l\neq m}}^{k}(\lambda_l-\lambda_m)}=-\sum_{m=1}^{k-1}\frac{\prod_{h=1}^{k-1}\prod_{l>h}^{k}(\lambda_l-\lambda_h)}{\prod_{\substack{l=1\\l\neq m}}^{k}(\lambda_l-\lambda_m)}\cdot\frac{1}{\prod_{i=1}^{k-1}\prod_{l>i}^{k}(\lambda_l-\lambda_i)}=\frac{1}{\prod_{\substack{l=1\\l\neq k}}^{k}(\lambda_l-\lambda_k)}. \tag{2.45}$$

In view of (2.45) we can write that

$$L_k(\mu)=\prod_{j=1}^{k-1}\lambda_j\sum_{m=1}^{k}\frac{\mu^{\nu-1}}{\prod_{\substack{l=1\\l\neq m}}^{k}(\lambda_l-\lambda_m)}\cdot\frac{1}{\mu^\nu+\lambda_m}, \tag{2.46}$$

because the k-th term of (2.46) coincides with the last term of (2.40) and therefore by inversion of the Laplace transform we get (2.23). □

Remark 2.2.1. *We now prove that for the generalised fractional birth process the representation*

$$\mathcal{N}^\nu(t)=\mathcal{N}(T_{2\nu}(t)),\qquad t>0,\ 0<\nu\le 1, \tag{2.47}$$

holds. This means that the process under investigation can be viewed as a generalised birth process at a random time $T_{2\nu}(t)$, $t>0$, whose distribution is the folded solution to the fractional diffusion equation (2.8).

$$\begin{aligned}
&\int_0^\infty e^{-\mu t}\mathcal{G}_\nu(u,t)\,dt \qquad (2.48)\\
&\overset{by\ (2.23)}{=}\int_0^\infty\left\{\sum_{k=2}^\infty u^k\prod_{j=1}^{k-1}\lambda_j\sum_{m=1}^{k}\frac{E_{\nu,1}(-\lambda_m t^\nu)}{\prod_{j\neq m}^{k}(\lambda_j-\lambda_m)}+uE_{\nu,1}(-\lambda_1 t^\nu)\right\}e^{-\mu t}dt\\
&=\sum_{k=2}^\infty u^k\prod_{j=1}^{k-1}\lambda_j\sum_{m=1}^{k}\frac{\mu^{\nu-1}}{\mu^\nu+\lambda_m}\frac{1}{\prod_{j\neq m}^{k}(\lambda_j-\lambda_m)}+\frac{u\mu^{\nu-1}}{\mu^\nu+\lambda_1}\\
&=\int_0^\infty\left\{\sum_{k=2}^\infty u^k\prod_{j=1}^{k-1}\lambda_j\sum_{m=1}^{k}\frac{\mu^{\nu-1}}{\prod_{j\neq m}^{k}(\lambda_j-\lambda_m)}e^{-s(\mu^\nu+\lambda_m)}+ue^{-s(\mu^\nu+\lambda_1)}\right\}ds\\
&=\int_0^\infty\mathcal{G}(u,s)\,\mu^{\nu-1}e^{-s\mu^\nu}ds=\int_0^\infty\mathcal{G}(u,s)\int_0^\infty e^{-\mu t}f_{T_{2\nu}}(s,t)\,dt\,ds\\
&=\int_0^\infty e^{-\mu t}\left\{\int_0^\infty\mathcal{G}(u,s)\,f_{T_{2\nu}}(s,t)\,ds\right\}dt,
\end{aligned}$$

where

$$\int_0^\infty e^{-\mu t} f_{T_{2\nu}}(s,t)\,dt = \mu^{\nu-1} e^{-s\mu^\nu}, \qquad s>0, \tag{2.49}$$

is the Laplace transform of the folded solution to (2.8). From (2.48) we infer that

$$\mathcal{G}_\nu(u,t) = \int_0^\infty \mathcal{G}(u,s)\, f_{T_{2\nu}}(s,t)\,ds, \tag{2.50}$$

and from this the representation (2.47) follows.

Remark 2.2.2. *The relationship (2.47) permits us to conclude that the functions (2.23) are non-negative because*

$$Pr\{\mathcal{N}^\nu(t)=k\} = \int_0^\infty Pr\{\mathcal{N}(s)=k\}\, Pr\{T_{2\nu}(t)\in ds\}, \tag{2.51}$$

and $Pr\{\mathcal{N}(s)=k\}>0$ and $\sum_k Pr\{\mathcal{N}(s)=k\}=1$ as shown, for example in Feller (1968), page 452. Furthermore the fractional birth process is non-exploding if and only if $\sum_k (1/\lambda_k)=\infty$, for all values of $0<\nu\le 1$.

2.3 The fractional linear birth process

In this section we examine in detail a special case of the previous fractional birth process, namely the fractional linear birth process which generalises the classical Yule–Furry model. The birth rates in this case have the form

$$\lambda_k = \lambda k, \qquad \lambda>0,\, k\ge 1, \tag{2.52}$$

and indicate that new births occur with a probability proportional to the size of the population. We denote by $N^\nu(t)$ the number of individuals in the population expanding according to the rates (2.52) and we have that the probabilities

$$p_k^\nu(t) = \Pr\left\{N^\nu(t)=k \middle| N^\nu(0)=1\right\}, \qquad k\ge 1, \tag{2.53}$$

satisfy the difference-differential equations

$$\begin{cases} \dfrac{d^\nu p_k}{dt^\nu} = -\lambda k p_k + \lambda(k-1)p_{k-1}, & 0<\nu\le 1,\, k\ge 1, \\ p_k(0) = \begin{cases} 1 & k=1, \\ 0 & k\ge 2. \end{cases} \end{cases} \tag{2.54}$$

The distribution (2.53) can be obtained as a particular case of (2.23) or directly by means of a completely different approach, as follows.

Theorem 2.3.1. *The distribution of the fractional linear birth process with a simple initial progenitor reads*

$$\begin{aligned} p_k^\nu(t) &= \Pr\{N^\nu(t)=k \mid N^\nu(0)=1\} \\ &= \sum_{j=1}^k \binom{k-1}{j-1} (-1)^{j-1} E_{\nu,1}(-\lambda j t^\nu), \qquad k\ge 1,\, 0<\nu\le 1, \end{aligned} \tag{2.55}$$

where $E_{\nu,1}(x)$ is the Mittag–Leffler function (2.6).

Proof. We can prove result (2.55) by solving equation (2.54) recursively. This means that $p_{k-1}^{\nu}(t)$ has the form (2.55) so $p_k^{\nu}(t)$ mantains the same structure. This is tantamount to solving the Cauchy problem

$$\begin{cases} \dfrac{d^{\nu} p_k(t)}{dt^{\nu}} = -\lambda k p_k(t) + \lambda(k-1)\displaystyle\sum_{j=1}^{k-1}\binom{k-2}{j-1}(-1)^{j-1} E_{\nu,1}(-\lambda j t^{\nu}), \\ p_k(0) = 0, \qquad k > 1. \end{cases} \tag{2.56}$$

By applying the Laplace transform $L_{k,\nu}(\mu) = \int_0^{\infty} e^{-\mu t} p_k(t)\, dt$ to (2.56) we have that

$$L_{k,\nu}(\mu) = \lambda(k-1)\left\{\sum_{j=1}^{k-1}\binom{k-2}{j-1}(-1)^{j-1}\frac{\mu^{k-1}}{\mu^{\nu}+\lambda j}\right\}\frac{1}{\mu^{\nu}+\lambda k}. \tag{2.57}$$

The Laplace transform (2.57) can conveniently be written as

$$\begin{aligned} L_{k,\nu}(\mu) &= \mu^{\nu-1}\left\{\left[\frac{1}{\mu^{\nu}+\lambda} - \frac{1}{\mu^{\nu}+\lambda k}\right] - (k-1)\left[\frac{1}{\mu^{\nu}+2\lambda} - \frac{1}{\mu^{\nu}+\lambda k}\right]\right. \\ &\quad + \frac{(k-1)(k-2)}{2}\left[\frac{1}{\mu^{\nu}+3\lambda} - \frac{1}{\mu^{\nu}+k\lambda}\right] + \cdots \\ &\quad \left. + (k-1)(-1)^{k-2}\left[\frac{1}{\mu^{\nu}+(k-1)\lambda} - \frac{1}{\mu^{\nu}+\lambda k}\right]\right\} \\ &= \mu^{\nu-1}\sum_{j=1}^{k-1}\binom{k-1}{j-1}(-1)^{j-1}\frac{1}{\mu^{\nu}+j\lambda} - \frac{\mu^{\nu-1}}{\mu^{\nu}+\lambda k}\sum_{j=1}^{k-1}\binom{k-1}{j-1}(-1)^{j-1}. \end{aligned} \tag{2.58}$$

This permits us to conclude that

$$L_{k,\nu}(\mu) = \mu^{\nu-1}\sum_{j=1}^{k}\binom{k-1}{j-1}(-1)^{j-1}\frac{1}{\mu^{\nu}+j\lambda}. \tag{2.59}$$

By inverting (2.59) we arrive immediately at result (2.55). □

For $\nu = 1$ (2.59) can be written as

$$\begin{aligned} \int_0^{\infty} e^{-\mu t} p_k^1(t)\, dt &= \int_0^{\infty} e^{-\lambda t} e^{-\mu t}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j e^{-\lambda j t} dt \\ &= \int_0^{\infty} e^{-\mu t}\left\{e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{k-1}\right\} dt, \end{aligned} \tag{2.60}$$

and this is an alternative derivation of the Yule–Furry linear birth process distribution.

Remark 2.3.1. *An alternative form of the distribution (2.55) can be derived by writing explicitly the Mittag–Leffler function and manipulating conveniently the double sums obtained. We have therefore*

$$\begin{aligned} p_k^{\nu}(t) &= \sum_{m=0}^{k-1}\frac{(-\lambda t^{\nu})^m}{\Gamma(\nu m+1)}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j (j+1)^m \\ &\quad + \sum_{m=k}^{\infty}\frac{(-\lambda t^{\nu})^m}{\Gamma(\nu m+1)}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j (j+1)^m \\ &= \frac{(\lambda t^{\nu})^{k-1}(k-1)!}{\Gamma(\nu(k-1)+1)} + \sum_{m=k}^{\infty}\frac{(-\lambda t^{\nu})^m}{\Gamma(\nu m+1)}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j (j+1)^m. \end{aligned} \tag{2.61}$$

*The last step of (2.61) is justified by the following formulae (*0.154(6) *and* 0.154(5) *Gradshteyn and Ryzhik (1980), page 4:)*

$$\sum_{k=0}^{N}(-1)^k\binom{N}{k}(\alpha+k)^{n-1}=0, \qquad \textit{valid for } N\geq n\geq 1, \tag{2.62}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\alpha+k)^{n}=(-1)^n n!\,. \tag{2.63}$$

What is remarkable about (2.63) is that the result is independent of α. This can be ascertained as follows

$$S_n^\alpha=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{r=0}^{n}\binom{n}{r}\alpha^r k^{n-r}=\sum_{r=0}^{n}\binom{n}{r}\alpha^r\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^{n-r+1-1}. \tag{2.64}$$

By formula 0.154(3) of Gradshteyn and Ryzhik (1980), page 4, the inner sum in (2.64) equals zero for $1\leq n-r+1\leq n$, that is for $1\leq r\leq n$. Therefore (see Gradshteyn and Ryzhik (1980) formula 0.154(4), page 4)

$$S_n^\alpha=\binom{n}{0}\alpha^0\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^n=(-1)^n n!\,. \tag{2.65}$$

We now provide a direct proof that the distribution (2.55) sums to unity. This is based on combinatorial arguments and will subsequently be validated by resorting to the representation of $N^\nu(t)$ as a composition of the Yule–Furry model with the random time $T_{2\nu}(t)$.

Theorem 2.3.2. *The distribution (2.55) is such that*

$$\sum_{k=1}^{\infty}p_k^\nu(t)=\sum_{k=1}^{\infty}\sum_{j=1}^{k}\binom{k-1}{j-1}(-1)^{j-1}E_{\nu,1}(-\lambda j t^\nu)=1. \tag{2.66}$$

Proof. We start by evaluating the Laplace transform $L_\nu(\mu)$ of (2.66) as follows

$$L_\nu(\mu)=\sum_{k=1}^{\infty}\sum_{j=1}^{k}\binom{k-1}{j-1}\frac{(-1)^{j-1}\mu^{\nu-1}}{\mu^\nu+\lambda j}=\frac{\mu^{\nu-1}}{\lambda}\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\frac{1}{\frac{\mu^\nu}{\lambda}+1+j}. \tag{2.67}$$

A crucial role is here played by the well-known formula (see Kirschenhofer (1996))

$$\sum_{k=0}^{N}\binom{N}{k}(-1)^k\frac{1}{x+k}=\frac{N!}{x(x+1)\cdots(x+N)}. \tag{2.68}$$

Therefore

$$\begin{aligned}L_\nu(\mu)&=\frac{\mu^{\nu-1}}{\lambda}\sum_{k=1}^{\infty}\frac{(k-1)!}{\left(\frac{\mu^\nu}{\lambda}+1\right)\left(\frac{\mu^\nu}{\lambda}+2\right)\cdots\left(\frac{\mu^\nu}{\lambda}+k\right)}=\frac{\mu^{\nu-1}}{\lambda}\sum_{l=0}^{\infty}\frac{\Gamma(l+1)\Gamma\left(\frac{\mu^\nu}{\lambda}+1\right)}{\Gamma\left(\frac{\mu^\nu}{\lambda}+1+(l+1)\right)}\\&=\frac{\mu^{\nu-1}}{\lambda}\sum_{l=0}^{\infty}\mathrm{B}\left(l+1,\frac{\mu^\nu}{\lambda}+1\right)=\frac{\mu^{\nu-1}}{\lambda}\int_0^1\sum_{l=0}^{\infty}x^l(1-x)^{\frac{\mu^\nu}{\lambda}}\,dx\\&=\frac{\mu^{\nu-1}}{\lambda}\int_0^1(1-x)^{\frac{\mu^\nu}{\lambda}-1}\,dx=\int_0^\infty e^{-\mu t}dt,\end{aligned} \tag{2.69}$$

where $\mathrm{B}(h,q)=\int_0^1 x^{h-1}(1-x)^{q-1}\,dx$, for $h,q>0$. This concludes the proof of (2.66). □

The presence of alternating sums in (2.55) imposes the check that $p_k^\nu(t) \geq 0$ for all k. This is the task of the next remark.

Remark 2.3.2. *In order to check the non-negativity of (2.55) we exploit the results of the proof of theorem 2.3.2, suitably adapted. The expression*

$$\sum_{k=1}^{\infty} \int_0^\infty e^{-\mu t} p_k^\nu(t)\, dt = \frac{\mu^{\nu-1}}{\lambda} \sum_{k=1}^{\infty} B\left(k, \frac{\mu^\nu}{\lambda} + 1\right), \tag{2.70}$$

which emerges from (2.69), permits us to write that

$$\int_0^\infty e^{-\mu t} p_k^\nu(t)\, dt = \int_0^1 x^{k-1} \frac{\mu^{\nu-1}}{\lambda} (1-x)^{\frac{\mu^\nu}{\lambda}}\, dx = \int_0^1 x^{k-1} \frac{\mu^{\nu-1}}{\lambda} e^{\frac{\mu^\nu}{\lambda}\ln(1-x)} dx \tag{2.71}$$
$$= \int_0^1 x^{k-1} \frac{\mu^{\nu-1}}{\lambda} e^{-\frac{\mu^\nu}{\lambda} \sum_{r=1}^{\infty} \frac{x^r}{r}} dx = \int_0^1 x^{k-1} \frac{\mu^{\nu-1}}{\lambda} e^{-\frac{\mu^\nu x}{\lambda}} \prod_{r=2}^{\infty} e^{-\frac{\mu^\nu x^r}{\lambda r}} dx.$$

The terms

$$e^{-\frac{\mu^\nu x^r}{\lambda r}} = \mathbb{E} e^{-\mu X_r} = \int_0^\infty e^{-\mu t} q_\nu^r(x,t)\, dt, \tag{2.72}$$

are the Laplace transforms of stable r.v's $X_r = S(\sigma_r, 1, 0)$ where $\sigma_r = \left(\frac{x^r}{\lambda r}\cos\frac{\pi\nu}{2}\right)^{\frac{1}{\nu}}$ (for details on this point see Samorodnitsky and Taqqu (1994), page 15). The term $\frac{\mu^{\nu-1}}{2\lambda}\exp(-\frac{\mu^\nu |x|}{\lambda})$, is the Laplace transform of the solution of the fractional diffusion equation

$$\begin{cases} \frac{\partial^{2\nu} u}{\partial t^{2\nu}} = \lambda^2 \frac{\partial^2 u}{\partial x^2}, & 0 < \nu \leq 1, \\ u(x,0) = \delta(x), \end{cases} \tag{2.73}$$

with the additional condition $u_t(x,0) = 0$ for $1/2 < \nu \leq 1$, and can be written as

$$u_{2\nu}(x,t) = \frac{1}{2\lambda\Gamma(1-\nu)} \int_0^t \frac{p_\nu(x,s)}{(t-s)^\nu} ds \tag{2.74}$$

(see formula (3.5) Orsingher and Beghin (2004)), where $p_\nu(x,1) = q_\nu^1(x,1)$ is the stable law with $\sigma_1 = \left(\frac{x}{\lambda}\cos\frac{\pi\nu}{2}\right)^{\frac{1}{\nu}}$. We can represent the product

$$\frac{\mu^{\nu-1}}{\lambda} e^{-\frac{x\mu^\nu}{\lambda}} \prod_{r=2}^{\infty} e^{-\frac{\mu^\nu x^r}{\lambda r}} = \int_0^\infty e^{-\mu t} \left\{ \int_0^t u_{2\nu}(x,s)\, q_\nu(x,t-s)\, ds \right\} dt, \tag{2.75}$$

where

$$\int_0^\infty e^{-\mu t} q_\nu(x,t)\, dt = \prod_{r=2}^{\infty} e^{-\frac{\mu^\nu x^r}{\lambda r}}. \tag{2.76}$$

Thus $q_\nu(x,t)$ appears as an infinite convolution of stable laws whose parameters depend on r and x. In light of (2.75) we have therefore that

$$\int_0^\infty e^{-\mu t} p_k^\nu(t)\, dt = 2\int_0^\infty e^{-\mu t} \int_0^1 x^{k-1} \int_0^t u_{2\nu}(x,s)\, q_\nu(x,t-s)\, ds\, dx\, dt. \tag{2.77}$$

Since $p_k^\nu(t)$ appears as the result of the integral of probability densities, we can conclude that $p_k^\nu(t) \geq 0$ for all $k \geq 1$ and $t > 0$.

We provide an alternative proof of the non-negativity of $p_k^\nu(t)$, $t > 0$, and of $\sum_k p_k^\nu(t) = 1$ based on the representation of the fractional linear birth process $N^\nu(t)$ as

$$N^\nu(t) = N(T_{2\nu}(t)), \qquad 0 < \nu \leq 1, \tag{2.78}$$

where $T_{2\nu}(t)$ possesses distribution coinciding with the folded solution of the fractional diffusion equation

$$\begin{cases} \dfrac{\partial^{2\nu} u}{\partial t^{2\nu}} = \dfrac{\partial^2 u}{\partial x^2}, & 0 < \nu \leq 1, \\ u(x,0) = \delta(x), \end{cases} \tag{2.79}$$

with the further condition $u_t(x,0) = 0$ for $1/2 < \nu \leq 1$.

Theorem 2.3.3. *The probability generating function $G_\nu(u,t) = \mathbb{E}u^{N^\nu(t)}$ of $N^\nu(t)$, $t > 0$ has the following Laplace transform*

$$\int_0^\infty e^{-\mu t} G_\nu(u,t)\, dt = \int_0^\infty \frac{ue^{-\lambda t}}{1 - u(1 - e^{-\lambda t})} \mu^{\nu-1} e^{-\mu^\nu t} dt. \tag{2.80}$$

Proof. We evaluate the Laplace transform (2.80) as follows

$$\begin{aligned}
&\int_0^\infty e^{-\mu t} G_\nu(u,t)\, dt \qquad (2.81)\\
&= \int_0^\infty e^{-\mu t} \sum_{k=1}^\infty u^k \sum_{j=1}^k \binom{k-1}{j-1} (-1)^{j-1} E_{\nu,1}(-\lambda j t^\nu)\, dt \\
&= \sum_{k=1}^\infty u^k \sum_{j=1}^k \binom{k-1}{j-1} (-1)^{j-1} \frac{\mu^{\nu-1}}{\mu^\nu + \lambda j} \\
&= \frac{\mu^{\nu-1}}{\lambda} \sum_{k=1}^\infty u^k \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \frac{1}{\frac{\mu^\nu}{\lambda} + 1 + j} \\
&\overset{\text{(by (2.68))}}{=} \frac{\mu^{\nu-1}}{\lambda} \sum_{k=1}^\infty u^k \frac{(k-1)!}{\left(\frac{\mu^\nu}{\lambda}+1\right)\left(\frac{\mu^\nu}{\lambda}+2\right)\cdots\left(\frac{\mu^\nu}{\lambda}+k\right)} \\
&= \frac{u\mu^{\nu-1}}{\lambda} \sum_{l=0}^\infty u^l \frac{l!}{\left(\frac{\mu^\nu}{\lambda}+1\right)\cdots\left(\frac{\mu^\nu}{\lambda}+1+l\right)} \\
&= \frac{u\mu^{\nu-1}}{\lambda} \sum_{l=0}^\infty u^l \mathrm{B}\left(l+1, \frac{\mu^\nu}{\lambda}+1\right) \\
&= \frac{u\mu^{\nu-1}}{\lambda} \int_0^1 \sum_{l=0}^\infty u^l x^l (1-x)^{\frac{\mu^\nu}{\lambda}}\, dx, \qquad \text{for } 0 < ux < 1 \\
&= \frac{u\mu^{\nu-1}}{\lambda} \int_0^1 \frac{(1-x)^{\frac{\mu^\nu}{\lambda}}}{(1-ux)} dx \\
&\overset{(1-x=e^{-\lambda t})}{=} \int_0^\infty \frac{ue^{-\lambda t}}{1-u(1-e^{-\lambda t})} e^{-t\mu^\nu} \mu^{\nu-1} dt.
\end{aligned}$$

□

Remark 2.3.3. *In order to extract from (2.80) the representation (2.78) we note that*

$$\begin{aligned}
&\int_0^\infty e^{-\mu t} \left\{ \sum_{k=0}^\infty u^k Pr\{N(T_{2\nu}(t)) = k\} \right\} dt \qquad (2.82)\\
&= \int_0^\infty e^{-\mu t} \left\{ \int_0^\infty \sum_{k=0}^\infty u^k Pr\{N(s) = k\} f_{T_{2\nu}}(s,t)\, ds \right\} dt \\
&= \int_0^\infty G(u,s)\, \mu^{\nu-1} e^{-\mu^\nu s} ds,
\end{aligned}$$

which coincides with (2.80). It can be shown that

$$\int_0^\infty e^{-\mu t} f_{T_{2\nu}}(s,t)\,dt = \mu^{\nu-1} e^{-s\mu^\nu}, \qquad s>0, \tag{2.83}$$

is the Laplace transform of the folded solution to

$$\frac{\partial^{2\nu} u}{\partial t^{2\nu}} = \frac{\partial^2 u}{\partial s^2}, \qquad 0<\nu\le 1, \tag{2.84}$$

with the initial condition $u(s,0)=\delta(s)$ *for* $0<\nu\le 1$ *and also* $u_t(s,0)=0$ *for* $1/2<\nu\le 1$.

In force of (2.78) the non-negativity of $p_k^\nu(t)$ *is immediate because*

$$Pr\{N^\nu(t)=k\} = \int_0^\infty Pr\{N(s)=k\}\, Pr\{T_{2\nu}(t)\in ds\}. \tag{2.85}$$

The relation (2.85) leads to the conclusion that $\sum_{k=1}^\infty Pr\{N^\nu(t)=k\}=1$.

Some explicit expressions for (2.85) can be given when the $\Pr\{T_{2\nu}(t)\in ds\}$ can be worked out in detail.

We know that for $\nu = 1/2^n$ we have that

$$\begin{aligned}\Pr\left\{T_{\frac{1}{2^{n-1}}}(t)\in ds\right\} &= \Pr\{|\mathcal{B}_1(|\mathcal{B}_2(\cdots|\mathcal{B}_n(t)|\cdots)|\,)\in ds\} \\ &= ds\, 2^n \int_0^\infty \frac{e^{-\frac{s^2}{4\omega_1}}}{\sqrt{4\pi\omega_1}} d\omega_1 \int_0^\infty \frac{e^{-\frac{\omega_1^2}{4\omega_2}}}{\sqrt{4\pi\omega_2}} d\omega_2 \cdots \int_0^\infty \frac{e^{-\frac{\omega_{n-1}^2}{4t}}}{\sqrt{4\pi t}} d\omega_{n-1}.\end{aligned} \tag{2.86}$$

For details about (2.86) see theorem 2.2 of Orsingher and Beghin (2009), where the differences of the constants depend on the fact that the diffusion coefficient in equation (2.84) equals 1 instead of $2^{(1/2^n)-2}$. The distribution (2.86) represents the density of the folded $(n-1)$-times iterated Brownian motion and therefore $\mathcal{B}_1,\cdots,\mathcal{B}_n$ are independent Brownian motions with volatility equal to 2.

For $\nu=1/3$ the process (2.78) has the form $N_{\frac{1}{3}}(t) = N(|\mathcal{A}(t)|)$, where $\mathcal{A}(t)$ is a process whose law is the solution of

$$\frac{\partial^{\frac{2}{3}} u}{\partial t^{\frac{2}{3}}} = \frac{\partial^2 u}{\partial x^2}, \qquad u(x,0)=\delta(x). \tag{2.87}$$

In Orsingher and Beghin (2009) it is shown that the solution to (2.87) is

$$u_{\frac{2}{3}}(x,t) = \frac{3}{2}\frac{1}{\sqrt[3]{3t}} \mathcal{A}_i\left(\frac{|x|}{\sqrt[3]{3t}}\right), \tag{2.88}$$

where

$$\mathcal{A}_i(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\alpha x + \frac{\alpha^3}{3}\right) d\alpha, \tag{2.89}$$

is the Airy function. Therefore the distribution (2.85) in this case reads

$$p_k^{\frac{1}{3}}(t) = \int_0^\infty e^{-\lambda s}\left(1-e^{-\lambda s}\right)^{k-1} \frac{3}{\sqrt[3]{3t}} \mathcal{A}_i\left(\frac{s}{\sqrt[3]{3t}}\right) ds, \qquad k\ge 1,\, t>0. \tag{2.90}$$

Remark 2.3.4. *From (2.54) it is straightforward to show that the probability generating function* $G_\nu(u,t) = \mathbb{E}u^{N^\nu(t)}$ *satisfies the partial differential equation*

$$\begin{cases} \frac{\partial^\nu}{\partial t^\nu} G(u,t) = \lambda u(u-1)\frac{\partial}{\partial u} G(u,t), \quad 0<\nu\le 1, \\ G(u,0)=u, \end{cases} \tag{2.91}$$

and thus $\mathbb{E}N^{\nu}(t) = \frac{\partial G}{\partial u}\big|_{u=1}$ *is the solution to*

$$\begin{cases} \dfrac{d^{\nu}}{dt^{\nu}}\mathbb{E}N^{\nu} = \lambda \mathbb{E}N^{\nu}, 0<\nu\leq 1, \\ \mathbb{E}N^{\nu}(0) = 1. \end{cases} \tag{2.92}$$

The solution of (2.92) is

$$\mathbb{E}N^{\nu}(t) = E_{\nu,1}(\lambda t^{\nu}), \qquad t>0. \tag{2.93}$$

Clearly, result (2.93) can be also derived by evaluating the following Laplace transform

$$\begin{aligned} \int_0^{\infty} e^{-\mu t}\mathbb{E}N^{\nu}(t)\,dt &= \int_0^{\infty} e^{-\mu t}\left\{\sum_{k=1}^{\infty} k\int_0^{\infty} Pr\{N(s)=k\}\,Pr\{T_{2\nu}(t)\in ds\}\right\}dt \\ &= \int_0^{\infty} e^{-\mu t}\int_0^{\infty} e^{\lambda s} Pr\{T_{2\nu}(t)\in ds\}\,dt \\ &= \int_0^{\infty} e^{\lambda s}\mu^{\nu-1}e^{-s\mu^{\nu}}ds \\ &= \frac{\mu^{\nu-1}}{\mu^{\nu}-\lambda} = \int_0^{\infty} e^{-\mu t}E_{\nu,1}(\lambda t^{\nu})\,dt, \end{aligned}$$

and this confirms (2.93). The mean-value (2.93) can be obtained in a third manner.

$$\begin{aligned} \int_0^{\infty} e^{-\mu t}\mathbb{E}N^{\nu}(t) &= \sum_{k=1}^{\infty} k\sum_{j=1}^{k}\binom{k-1}{j-1}(-1)^{j-1}\int_0^{\infty} E_{\nu,1}(-\lambda j t^{\nu})\,e^{-\lambda t}dt \\ &= \sum_{k=1}^{\infty} k\sum_{j=1}^{k}\binom{k-1}{j-1}(-1)^{j-1}\frac{\mu^{\nu-1}}{\mu^{\nu}+\lambda j} \\ &= \frac{\mu^{\nu-1}}{\lambda}\sum_{k=1}^{\infty} k\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^{j}\frac{1}{\frac{\mu^{\nu}}{\lambda}+1+j} \\ &= \frac{\mu^{\nu-1}}{\lambda}\sum_{k=1}^{\infty} k\frac{(k-1)!}{\left(\frac{\mu^{\nu}}{\lambda}+1\right)\cdots\left(\frac{\mu^{\nu}}{\lambda}+k\right)} \\ &= \frac{\mu^{\nu-1}}{\lambda}\sum_{k-1}^{\infty} k\frac{\Gamma(k)\,\Gamma\left(\frac{\mu^{\nu}}{\lambda}+1\right)}{\Gamma\left(\frac{\mu^{\nu}}{\lambda}+k+1\right)} \\ &= \frac{\mu^{\nu-1}}{\lambda}\int_0^{1}\sum_{k=1}^{\infty} kx^{k-1}(1-x)^{\frac{\mu^{\nu}}{\lambda}} \\ &= \frac{\mu^{\nu-1}}{\mu^{\nu}-\lambda} = \int_0^{\infty} e^{-\mu t}E_{\nu,1}(\lambda t^{\nu})\,dt. \end{aligned} \tag{2.94}$$

The result of remark 2.3.4, $\mathbb{E}N^{\nu}(t) = E_{\nu,1}(\lambda t^{\nu})$ should be compared with the results of Uchaikin et al. (2008).

An interesting representation of (2.93) following from (2.78) gives that

$$\mathbb{E}N^{\nu}(t) = \int_0^{\infty} e^{\lambda s}\Pr\{T_{2\nu}(t)\in ds\} = \int_0^{\infty} \mathbb{E}N(s)\Pr\{T_{2\nu}(t)\in ds\}. \tag{2.95}$$

Remark 2.3.5. *By twice deriving (2.91) w.r.t. u we obtain the fractional equation for the second-order factorial moment*

$$\mathbb{E}\{N^{\nu}(t)(N^{\nu}(t)-1)\} = g_{\nu}(t), \tag{2.96}$$

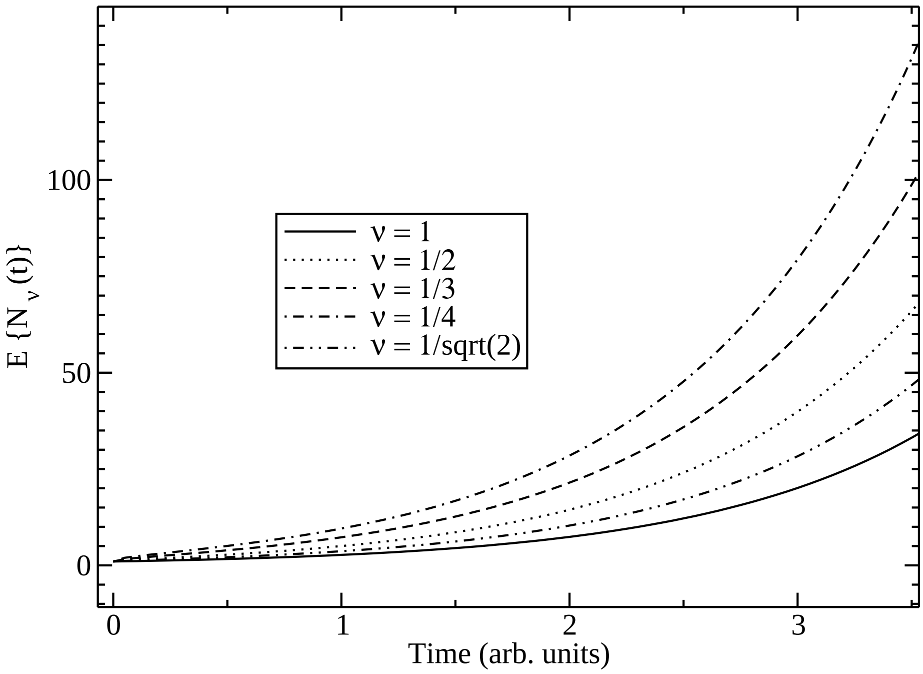

Figure 2.1: Mean number of individuals at time t for various values of ν.

that is

$$\begin{cases} \frac{\partial^\nu}{\partial t^\nu} g_\nu(t) = 2\lambda g_\nu(t) + 2\lambda \mathbb{E} N^\nu(t), \quad 0 < \nu \leq 1, \\ g_\nu(0) = 0. \end{cases} \tag{2.97}$$

The Laplace transform of the solution to (2.97) is

$$H_\nu(t) = \int_0^\infty e^{-\mu t} g_\nu(t)\, dt = \frac{2\lambda\mu^{\nu-1}}{(\mu^\nu - \lambda)(\mu^\nu - 2\lambda)} = 2\mu^{\nu-1}\left\{\frac{1}{\mu^\nu - 2\lambda} - \frac{1}{\mu^\nu - \lambda}\right\}. \tag{2.98}$$

The inverse Laplace transform of (2.98) is

$$\mathbb{E}\{N^\nu(t)(N^\nu(t) - 1)\} = 2E_{\nu,1}(2\lambda t^\nu) - 2E_{\nu,1}(\lambda t^\nu). \tag{2.99}$$

It is now straightforward to obtain the variance from (2.99)

$$\mathbb{V}ar N^\nu(t) = 2E_{\nu,1}(2\lambda t^\nu) - E_{\nu,1}(\lambda t^\nu) - E^2_{\nu,1}(\lambda t^\nu). \tag{2.100}$$

For $\nu = 1$ we retrieve from (2.100) the well-known expression of the variance of the linear birth process

$$\mathbb{V}ar N^1(t) = e^{\lambda t}\left(e^{\lambda t} - 1\right). \tag{2.101}$$

Remark 2.3.6. *If $X_1, \cdots, X_n$ are i.i.d. r.v's with common distribution $F(x) = Pr(X < x)$ then we can write the following probability*

$$\begin{aligned} & Pr\{\max(X_1, \cdots, X_{N^\nu(t)}) < x\} \\ & = \sum_{k=1}^\infty (Pr\{X < x\})^k \, Pr\{N^\nu(t) = k\} \\ & \overset{by\ (2.78)}{=} \int_0^\infty G(F(x), s)\, Pr\{T_{2\nu}(t) \in ds\} \end{aligned} \tag{2.102}$$

$$= \int_0^\infty \frac{F(x) e^{-\lambda s}}{1 - F(x)(1 - e^{-\lambda s})} Pr\{T_{2\nu}(t) \in ds\}.$$

Analogously we have that

$$\begin{aligned} &Pr\{\min(X_1, \cdots, X_{N^\nu(t)}) > x\} \\ &= \int_0^\infty \frac{(1 - F(x)) e^{-\lambda s}}{1 - (1 - F(x))(1 - e^{-\lambda s})} Pr\{T_{2\nu}(t) \in ds\}. \end{aligned} \tag{2.103}$$

Remark 2.3.7. *If the initial number of the components of the population is n_0 then the p.g.f. becomes*

$$\begin{aligned} &\mathbb{E}\left(u^{N^\nu(t)} \mid N_\nu(0) = n_0\right) \\ &= \sum_{k=0}^\infty u^{k+n_0} \int_0^\infty e^{-\lambda z n_0} \binom{n_0 + k - 1}{k} \left(1 - e^{-\lambda z}\right)^k Pr\{T_{2\nu}(t) \in dz\}. \end{aligned} \tag{2.104}$$

From (2.104) we can extract the distribution of the population size at time t as

$$\begin{aligned} &Pr\{N^\nu(t) = k + n_0 \mid N^\nu(0) = n_0\} \\ &= \binom{n_0 + k - 1}{k} \int_0^\infty e^{-\lambda z n_0} \left(1 - e^{-\lambda z}\right)^k Pr\{T_{2\nu}(t) \in dz\}, \qquad k \geq 0. \end{aligned} \tag{2.105}$$

If we write $k + n_0 = k'$ we can rewrite (2.105) as

$$\begin{aligned} &Pr\{N^\nu(t) = k' \mid N^\nu(0) = n_0\} \\ &= \binom{k' - 1}{k' - n_0} \int_0^\infty e^{-\lambda z n_0} \left(1 - e^{-\lambda z}\right)^{k' - n_0} Pr\{T_{2\nu}(t) \in dz\}, \qquad k' \geq n_0, \end{aligned} \tag{2.106}$$

where k' is the number of individuals in the population at time t. For $n_0 = 1$ formulae (2.105), (2.106) coincide with (2.55). The random time $T_{2\nu}(t)$, $t > 0$, appearing in (2.105) and (2.106) has a distribution which is related to the fractional equation

$$\frac{\partial^{2\nu} u}{\partial t^{2\nu}} = \frac{\partial^2 u}{\partial z^2}, \qquad 0 < \nu \leq 1. \tag{2.107}$$

It is possible to change a little bit the structure of formulae (2.105), (2.106) by means of the transformation $\lambda z = y$ so that the distribution of $T_{2\nu}(t)$ becomes related to equation

$$\frac{\partial^{2\nu} u}{\partial t^{2\nu}} = \lambda^2 \frac{\partial^2 u}{\partial y^2}, \qquad 0 < \nu \leq 1, \tag{2.108}$$

where (2.52) *shows the connection between the diffusion coefficient in* (2.108) *and the birth rate.*

Remark 2.3.8. *If we assume that the initial number of individuals in the population is $N^\nu(0) = n_0$ we can generalise the result (2.55) offering a representation of the distribution of $N^\nu(t)$ alternative to (2.106). If we take the Laplace transform of (2.106) we have that*

$$\begin{aligned} &\int_0^\infty e^{-\mu t} Pr\{N^\nu(t) = k + n_0 \mid N^\nu(0) = n_0\} \, dt \\ &= \int_0^\infty \binom{n_0 + k - 1}{k} \int_0^\infty e^{-\lambda z n_0} \left(1 - e^{-\lambda z}\right)^k Pr\{T_{2\nu}(t) \in dz\} \, dt \\ &\overset{by\ (2.83)}{=} \int_0^\infty \binom{n_0 + k - 1}{k} e^{-\lambda z n_0} \left(1 - e^{-\lambda z}\right)^k \mu^{\nu - 1} e^{-\mu^\nu z} dz \end{aligned} \tag{2.109}$$

$$= \binom{n_0+k-1}{k} \mu^{\nu-1} \int_0^\infty e^{-z(\lambda n_0+\mu^\nu)} \left(1-e^{-\lambda z}\right)^k dz$$
$$= \binom{n_0+k-1}{k} \mu^{\nu-1} \sum_{r=0}^k \binom{k}{r} (-1)^r \int_0^\infty e^{-z(\lambda n_0+\lambda r+\mu^\nu)} dz$$
$$= \binom{n_0+k-1}{k} \mu^{\nu-1} \sum_{r=0}^k \binom{k}{r} (-1)^r \frac{1}{\lambda (n_0+r)+\mu^\nu}.$$

By taking the inverse Laplace transform of (2.109) we have that

$$\begin{aligned} & Pr\{N^\nu(t) = k+n_0 \mid N^\nu(0) = n_0\} \\ &= \binom{n_0+k-1}{k} \sum_{r=0}^k \binom{k}{r} (-1)^r E_{\nu,1}\left(-(n_0+r)\lambda t^\nu\right). \end{aligned} \tag{2.110}$$

From (2.110) we can infer the following interesting information.

$$\begin{aligned} & Pr\{N^\nu(dt) = n_0+1 \mid N^\nu(0) = n_0\} \\ &= n_0 \sum_{r=0}^1 \binom{1}{r} (-1)^r E_{\nu,1}\left(-(n_0+r)\lambda (dt)^\nu\right) \\ &= n_0 \left[E_{\nu,1}\left(-n_0 \lambda (dt)^\nu\right) - E_{\nu,1}\left(-\lambda (n_0+1)(dt)^\nu\right)\right] \\ &\sim n_0 \frac{\lambda (dt)^\nu}{\Gamma(\nu+1)}, \end{aligned} \tag{2.111}$$

by writing only the lower order terms. This shows that the probability of a new offspring at the beginning of the process is proportional to $(dt)^\nu$ and to the initial number of progenitors. From our point of view this is the most important qualitative feature of our results, since it makes explicit the dependence on the order ν of the fractional birth process.

Theorem 2.3.4. *The Laplace transform of the probability generating function $G_\nu(t,u)$ of the fractional linear birth process has the form*

$$\begin{aligned} H_\nu(\mu,u) &= \int_0^\infty e^{-\mu t} G_\nu(t,u)\, dt \\ &= \frac{u\mu^{\nu-1}}{\lambda} \int_0^1 \frac{(1-x)^{\frac{\mu^\nu}{\lambda}}}{1-xu} dx, \qquad 0<u<1,\ \mu>0. \end{aligned} \tag{2.112}$$

Proof. We have seen above that the function G_ν solves the Cauchy problem

$$\begin{cases} \dfrac{\partial^\nu G_\nu}{\partial t^\nu} = \lambda u (u-1) \dfrac{\partial G_\nu}{\partial u}, & 0<\nu\le 1, \\ G_\nu(u,0) = u. \end{cases} \tag{2.113}$$

By taking the Laplace transform of (2.113) we have that

$$\mu^\nu H_\nu - \mu^{\nu-1} u = \lambda u (u-1) \frac{\partial H_\nu}{\partial u}. \tag{2.114}$$

By inserting (2.112) into (2.114) and performing some integration by parts we have that

$$\begin{aligned} & \frac{u\mu^{2\nu-1}}{\lambda} \int_0^1 \frac{(1-x)^{\frac{\mu^\nu}{\lambda}}}{1-xu} dx - u\mu^{\nu-1} \\ &= \lambda u (u-1) \left[\frac{\mu^{\nu-1}}{\lambda} \int_0^1 \frac{(1-x)^{\frac{\mu^\nu}{\lambda}}}{1-xu} dx + \frac{u\mu^{\nu-1}}{\lambda} \int_0^1 \frac{(1-x)^{\frac{\mu^\nu}{\lambda}} x}{(1-xu)^2} dx \right] \end{aligned} \tag{2.115}$$

$$= \lambda u (u-1) \left[\frac{\mu^{\nu-1}}{\lambda} \int_0^1 \frac{(1-x)^{\frac{\mu^\nu}{\lambda}}}{1-xu} dx + \frac{\mu^{\nu-1}}{\lambda} \frac{x(1-x)^{\frac{\mu^\nu}{\lambda}}}{1-xu} \Bigg|_{x=0}^{x=1} \right.$$

$$\left. - \frac{\mu^{\nu-1}}{\lambda} \int_0^1 \frac{(1-x)^{\frac{\mu^\nu}{\lambda}}}{(1-xu)} dx + \frac{\mu^{2\nu-1}}{\lambda^2} \int_0^1 \frac{x(1-x)^{\frac{\mu^\nu}{\lambda}-1}}{(1-xu)} dx \right]$$

$$= \frac{u(u-1)\mu^{2\nu-1}}{\lambda} \int_0^1 \frac{x(1-x)^{\frac{\mu^\nu}{\lambda}-1}}{(1-xu)} dx$$

$$= -u\mu^{\nu-1} + \frac{u\mu^{2\nu-1}}{\lambda} \int_0^1 \frac{(1-x)^{\frac{\mu^\nu}{\lambda}}}{(1-xu)} dx.$$

This concludes the proof of theorem 2.3.4. □

Remark 2.3.9. *We note that* $H_\nu(\mu, u)\big|_{u=1} = 1/\mu$ *because* $G_\nu(t,1) = 1$. *Furthermore*

$$\frac{\partial H_\nu(\mu, u)}{\partial u}\Bigg|_{u=1} = \frac{\mu^{\nu-1}}{\mu^\nu - \lambda} = \int_0^\infty e^{-\mu t} E_{\nu,1}(\lambda t^\nu)\, dt, \tag{2.116}$$

which accords well with (2.93).

Chapter 3

Fractional Pure Death Processes

Summary

This chapter is devoted to the study of a fractional version of nonlinear $\mathcal{M}^\nu(t)$, $t > 0$, linear $M^\nu(t)$, $t > 0$ and sublinear $\mathfrak{M}^\nu(t)$, $t > 0$, death processes. Fractionality is introduced by replacing the usual integer-order derivative in the difference-differential equations governing the state probabilities, with the fractional derivative understood in the sense of Dzhrbashyan–Caputo. We derive explicitly the state probabilities of the three death processes and examine the related probability generating functions and mean values. A useful subordination relation is also proved, allowing us to express the death processes as compositions of their classical counterparts with the random time process $T_{2\nu}(t)$, $t > 0$. This random time has one-dimensional distribution which is the folded solution to a Cauchy problem of the fractional diffusion equation.

3.1 Introduction

We assume that we have a population of n_0 individuals or objects. The components of this population might be the set of healthy people during an epidemic or the set of items being sold in a store, or even, say, melting ice pack blocks. However even a coalescence of particles can be treated in this same manner, leading to a large ensemble of physical analogues suited to the method. The main interest is to model the fading process of these objects and, in particular, to analyse how the size of the population decreases.

The classical death process is a model describing this type of phenomena and, its linear version is analysed in Bailey (1964), page 90. The most interesting feature of the extinguishing population is the probability distribution

$$\rho_k(t) = \Pr\{M(t) = k \mid M(0) = n_0\}, \qquad t > 0,\ 0 \le k \le n_0, \tag{3.1}$$

where $M(t)$, $t > 0$ is the point process representing the size of the population at time t. If the death rates are proportional to the population size, the process is called *linear* and the probabilities (3.1) are solutions to the initial-value problem

$$\begin{cases} \frac{d}{dt}\rho_k(t) = \mu(k+1)\rho_{k+1}(t) - \mu k \rho_k(t), & 0 \le k \le n_0, \\ \rho_k(0) = \begin{cases} 1, & k = n_0, \\ 0, & 0 \le k < n_0, \end{cases} \end{cases} \tag{3.2}$$

with $\rho_{n_0+1}(t) = 0$.

The distribution satisfying (3.2) is

$$\rho_k(t) = \binom{n_0}{k} e^{-\mu k t}\left(1 - e^{-\mu t}\right)^{n_0 - k}, \qquad 0 \le k \le n_0. \tag{3.3}$$

The equations (3.2) are based on the fact that the death rate of each component of the population is proportional to the number of existing individuals.

In the nonlinear case, where the death rates are μ_k, $0 \le k \le n_0$, equations (3.2) must be replaced by

$$\begin{cases} \frac{d}{dt}\rho_k(t) = \mu_{k+1}\rho_{k+1}(t) - \mu_k \rho_k(t), & 0 \le k \le n_0, \\ \rho_k(0) = \begin{cases} 1, & k = n_0, \\ 0, & 0 \le k < n_0. \end{cases} \end{cases} \tag{3.4}$$

In this chapter we consider fractional versions of the processes described above, where fractionality is obtained by substitution of the integer-order derivatives appearing in (3.2) and (3.4), with the fractional derivative called Caputo or Dzhrbashyan–Caputo derivative, defined as follows

$$\begin{cases} \frac{d^\nu f(t)}{dt^\nu} = \frac{1}{\Gamma(1-\nu)} \int_0^t \frac{f'(s)}{(t-s)^\nu}\, ds, & 0 < \nu < 1, \\ f'(t), & \nu = 1. \end{cases} \tag{3.5}$$

The main advantage of the Dzhrbashyan–Caputo fractional derivative over the usual Riemann–Liouville fractional derivatives is that the former requires only integer-order derivatives in the initial conditions.

The fractional derivative operator is vastly present in the physical and mathematical literature. It appears for example in generalisations of diffusion-type differential equations (see Wyss (1986), Schneider and Wyss (1988), Nigmatullin (1986) and Mainardi (1996)), hyperbolic equations such as telegraph equation (see Orsingher and Beghin (2004)), reaction-diffusion equations (see Saxena et al. (2006)), or in the study of continuous time random walks (CTRW) scaling limits (see Bening et al. (2007), Meerschaert et al. (2002)). Fractional calculus has also been considered by some authors to describe cahotic Hamiltonian dynamics in low dimensional systems (see e.g. Zaslavsky (1994), Saichev and Zaslavsky (1997), Saxena et al. (2002), Saxena et al. (2004a) and Saxena et al. (2004b)). For a complete review of fractional kinetics the reader can consult Zaslavsky (2002) or the book by Zaslavsky (2006). In the literature are also present fractional generalisations of point processes, such as the Poisson process (see Repin and Saichev (2000), Laskin (2003), Mainardi and Gorenflo (2004), Cahoy (2007), Uchaikin and Sibatov (2008) and Beghin and Orsingher (2009b)) and the birth and birth-death processes (see Uchaikin et al. (2008), Orsingher and Polito (2010), Orsingher and Polito (2011)). Fractional models are also used in other fields, for example finance (Meerschaert and Scalas (2006), Scalas et al. (2000)).

The population size is governed by

$$\begin{cases} \frac{d^\nu}{dt^\nu} \rho_k(t) = \mu_{k+1}\rho_{k+1}(t) - \mu_k \rho_k(t), \quad 0 \le k \le n_0, \\ \rho_k(0) = \begin{cases} 1, & k = n_0, \\ 0, & 0 \le k < n_0, \end{cases} \end{cases} \tag{3.6}$$

and is denoted by $\mathcal{M}^\nu(t)$, $t > 0$.

Let us assume that a crack has the form of a process $T_{2\nu}(t)$, $t > 0$. For $\nu = 1/2$, this coincides with a reflecting Brownian motion and has been described and derived in Chudnovsky and Kunin (1987). For $\nu \neq 1/2$, the process $T_{2\nu}(t)$, $t > 0$, can be identified with a stable process (see for details on this point Orsingher and Beghin (2009)). The ensemble of n_0 particles moves on the fracture and, at the same time, undergoes a decaying process which respects the same probabilistic rules of the usual death process. For the number of existing particles, we have therefore

$$\rho_k^\nu(t) = \int_0^\infty \rho_k(s) \Pr\{T_{2\nu}(t) \in ds\}. \tag{3.7}$$

We observe that

$$\Pr\{T_{2\nu}(t) \in ds\} = q(s,t)ds, \tag{3.8}$$

is a solution to

$$\frac{\partial^{2\nu}}{\partial t^{2\nu}} q(s,t) = \frac{\partial^2}{\partial s^2} q(s,t), \qquad s > 0,\ t > 0, \tag{3.9}$$

with the necessary initial conditions. Furthermore we recall that

$$\int_0^\infty e^{-zt} q(s,t)dt = z^{\nu-1} e^{-z^\nu s}, \qquad z > 0,\ s > 0. \tag{3.10}$$

The distribution $q(s,t)$ is also a solution to

$$\frac{\partial^\nu}{\partial t^\nu} q(s,t) = -\frac{\partial}{\partial s} q(s,t), \qquad s > 0, \tag{3.11}$$

as can be ascertained directly. If we take the fractional derivative in (3.7) we get

$$\begin{aligned} \frac{d^\nu}{dt^\nu} \rho_k^\nu(t) &= \int_0^\infty \rho_k(s) \frac{\partial^\nu}{\partial t^\nu} \Pr\{T_{2\nu}(t) \in ds\} \\ &= -\int_0^\infty \rho_k(s) \frac{\partial}{\partial s} q(s,t) ds \end{aligned} \tag{3.12}$$

$$
\begin{aligned}
&= -q(s,t)\rho_k(s)\Big|_0^\infty + \int_0^\infty \frac{d\rho_k(s)}{ds} q(s,t) ds \\
&= \int_0^\infty \left[-\mu_k \rho_k(s) + \mu_{k+1}\rho_{k+1}(s)\right] q(s,t) ds \\
&= -\mu_k \rho_k^\nu(t) + \mu_{k+1}\rho_{k+1}^\nu(t).
\end{aligned}
$$

This shows that replacing the time derivative with the fractional derivative corresponds to considering a death process (annihilating process) on particles displacing on a crack.

We now give some details about (3.11). By taking the Laplace transform of both members of (3.11) we have that

$$
\begin{aligned}
\int_0^\infty e^{-zt} \frac{\partial^\nu}{\partial t^\nu} q(s,t) dt &= -\frac{\partial}{\partial s}\left(\int_0^\infty e^{-zt} q(s,t) dt\right) \\
&= -\frac{\partial}{\partial s}\left(z^{\nu-1} e^{-sz^\nu}\right) = z^{2\nu-1} e^{-sz^\nu}.
\end{aligned} \tag{3.13}
$$

Furthermore,

$$
\begin{aligned}
\int_0^\infty e^{-zt} \frac{\partial^\nu}{\partial t^\nu} q(s,t) dt &= z^\nu \int_0^\infty e^{-zt} q(s,t) dt - z^{\nu-1} q(s,0) \\
&= z^\nu \left(z^{\nu-1} e^{-sz^\nu}\right) - z^{\nu-1}\delta(s),
\end{aligned} \tag{3.14}
$$

and therefore, for $s > 0$, this establishes that $q(s,t)$ solves equation (3.11). We note that a gas particle moving on a fracture has inspired to different authors the iterated Brownian motion (see DeBlassie (2004)).

The distribution

$$
\rho_k^\nu(t) = \Pr\left\{\mathcal{M}^\nu(t) = k \mid \mathcal{M}^\nu(0) = n_0\right\}, \qquad 0 \le k \le n_0, \tag{3.15}
$$

is obtained explicitly and reads

$$
\rho_k^\nu(t) = \begin{cases}
E_{\nu,1}(-\mu_{n_0} t^\nu), & k = n_0, \\
\prod_{j=k+1}^{n_0} \mu_j \sum_{m=k}^{n_0} \frac{E_{\nu,1}(-\mu_m t^\nu)}{\prod_{\substack{h=k \\ h\neq m}}^{n_0} (\mu_h - \mu_m)}, & 0 < k < n_0, \\
1 - \sum_{m=1}^{n_0} \prod_{\substack{h=1 \\ h\neq m}}^{n_0} \left(\frac{\mu_h}{\mu_h - \mu_m}\right) E_{\nu,1}(-\mu_m t^\nu), & k = 0,\ n_0 > 1.
\end{cases} \tag{3.16}
$$

Obviously, for $k = 0$, $n_0 = 1$,

$$
\rho_0^\nu(t) = 1 - E_{\nu,1}(-\mu_1 t^\nu). \tag{3.17}
$$

The Mittag–Leffler functions appearing in (3.16) are defined as

$$
E_{\nu,\gamma}(x) = \sum_{h=0}^\infty \frac{x^h}{\Gamma(\nu h + \gamma)}, \qquad x \in \mathbb{R}, \quad \nu, \gamma > 0. \tag{3.18}
$$

For $\nu = \gamma = 1$, $E_{1,1}(x) = e^x$ and formulae (3.16) provide the explicit distribution of the classical nonlinear death process.

For $\mu_k = k\mu$ the distribution of the fractional linear death process can be obtained either directly by solving the Cauchy problem (3.6) with $\mu_k = k \cdot \mu$ and $\rho_{n_0+1}(t) = 0$, or by specialising

(3.16) resulting in the following form

$$\rho_k^\nu(t) = \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r E_{\nu,1}(-(k+r)\mu t^\nu). \tag{3.19}$$

A technical tool necessary for our manipulations is the Laplace transform of Mittag–Leffler functions which we write here for the sake of completeness:

$$\int_0^\infty e^{-zt} t^{\gamma-1} E_{\nu,\gamma}(\pm\vartheta t^\nu)dt = \frac{z^{\nu-\gamma}}{z^\nu \mp \vartheta}, \qquad \Re(z) > |\vartheta|^{\frac{1}{\nu}}. \tag{3.20}$$

Another special case is the so-called fractional sublinear death process (for sublinear birth processes consult Donnelly et al. (1993)) where the death rates have the form $\mu_k = \mu(n_0 + 1 - k)$. In the sublinear process, the annihilation of particles or individuals accelerates with decreasing population size.

The distribution $\wp_k^\nu(t)$, $0 \le k \le n_0$ of the fractional sublinear death process $\mathfrak{M}^\nu(t)$, $t > 0$, is strictly related to that of the fractional linear birth process $N^\nu(t)$, $t > 0$ (see, for details on this point, Orsingher and Polito (2010)):

$$\Pr\{\mathfrak{M}^\nu(t) = 0 \mid \mathfrak{M}^\nu(0) = n_0\} = \Pr\{N^\nu(t) > n_0 \mid N^\nu(0) = 1\}. \tag{3.21}$$

In general, the connection between the fractional sublinear death process and the fractional linear birth process is expressed by the relation

$$\begin{aligned} &\Pr\{\mathfrak{M}^\nu(t) = n_0 - (k-1) \mid \mathfrak{M}^\nu(0) = n_0\} \\ &= \Pr\{N^\nu(t) = k \mid N^\nu(0) = 1\}, \qquad 1 \le k \le n_0. \end{aligned} \tag{3.22}$$

This shows a sort of symmetry in the evolution of fractional linear birth and fractional sublinear death processes.

For all fractional processes considered in this chapter, a subordination relationship holds. In particular, for the fractional linear death process we can write that

$$M^\nu(t) = M(T_{2\nu}(t)), \qquad 0 < \nu < 1,\ t > 0, \tag{3.23}$$

where $T_{2\nu}(t)$ is a process for which

$$\Pr\{T_{2\nu}(t) \in ds\} = q(s,t)ds, \tag{3.24}$$

is a solution to the following Cauchy problem (see Beghin and Orsingher (2009a))

$$\begin{cases} \frac{\partial^{2\nu}}{\partial t^{2\nu}} q(s,t) = \frac{\partial^2}{\partial s^2} q(s,t), & t > 0,\ s > 0, \\ \frac{\partial}{\partial t} q(s,t)\big|_{s=0} = 0, & \\ q(s,0) = \delta(s), & 0 < \nu \le 1, \end{cases} \tag{3.25}$$

with the additional initial condition

$$q_t(s,0) = 0, \qquad 1/2 < \nu \le 1. \tag{3.26}$$

In equation (3.23), $M(t)$, $t > 0$, represents the classical linear death process. Subordination relations of this type are extensively treated in Orsingher and Beghin (2009) and Kolokoltsov (2009).

We also show that all the fractional death processes considered below can be viewed as classical death processes with rate $\mu \cdot \Xi$, where Ξ is a Wright-distributed random variable.

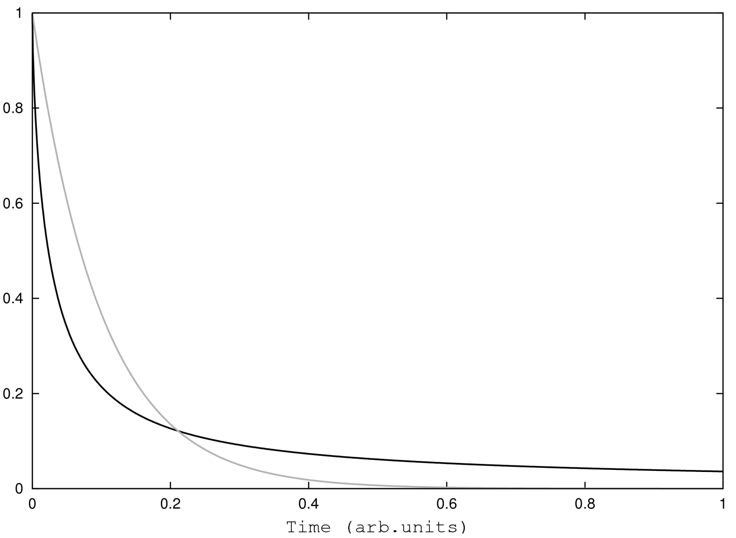

Figure 3.1: Plot of $\rho^{0.7}_{n_0}(t)$ (in black) and $\rho^{1}_{n_0}(t)$ (in grey), both with $n_0 = 10$.

3.2 The fractional linear death process and its properties

In this section we derive the distribution of the fractional linear death process as well as some interesting related properties and interpretations.

Theorem 3.2.1. *The distribution of the fractional linear death process* $M^\nu(t)$, $t > 0$ *with* n_0 *initial individuals and death rates* $\mu_k = \mu \cdot k$, *is given by*

$$\begin{aligned}\rho^\nu_k(t) &= Pr\{M^\nu(t) = k \mid M^\nu(0) = n_0\} \qquad (3.27)\\ &= \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r E_{\nu,1}(-(k+r)\mu t^\nu),\end{aligned}$$

where $0 \le k \le n_0$, $t > 0$ *and* $\nu \in (0,1]$. *The function* $E_{\nu,1}(x)$ *is the Mittag–Leffler function previously defined in* (3.18).

Proof. The state probability $\rho^\nu_{n_0}(t)$, $t > 0$ is readily obtained by applying the Laplace transform to equation (3.6), with $\mu_k = \mu \cdot k$, and then transforming back the results, thus yielding

$$\rho^\nu_{n_0}(t) = E_{\nu,1}(-n_0\mu t^\nu), \qquad t > 0,\ \nu \in (0,1]. \tag{3.28}$$

When $k = n_0 - 1$, in order to solve the related differential equation, we can write

$$\begin{aligned} z^\nu \mathcal{L}\{\rho_{n_0-1}\}(z) &= \mu n_0 \frac{z^{\nu-1}}{z^\nu + n_0\mu} - \mu(n_0-1)\mathcal{L}\{\rho_{n_0-1}\}(z) \qquad (3.29)\\ \Leftrightarrow \quad \mathcal{L}\{\rho_{n_0-1}\}(z) &= \mu n_0 z^{\nu-1} \frac{1}{z^\nu + n_0\mu} \cdot \frac{1}{z^\nu + (n_0-1)\mu}\\ \Leftrightarrow \quad \mathcal{L}\{\rho_{n_0-1}\}(z) &= n_0 z^{\nu-1} \left(\frac{1}{z^\nu + (n_0+1)\mu} - \frac{1}{z^\nu + n_0\mu} \right).\end{aligned}$$

By inverting equation (3.29), we readily obtain that

$$\rho^\nu_{n_0-1}(t) = n_0 \left(E_{\nu,1}(-(n_0-1)\mu t^\nu) - E_{\nu,1}(-n_0\mu t^\nu) \right). \tag{3.30}$$

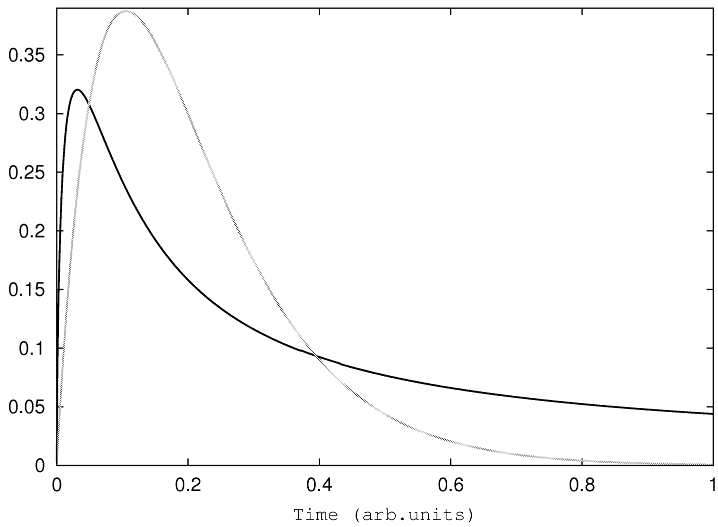

Figure 3.2: Plot of $\rho^{0.7}_{n_0-1}(t)$ (in black) and $\rho^{1}_{n_0-1}(t)$ (in grey). Here $n_0 = 10$.

For general values of k, with $0 \leq k < n_0$, we must solve the following Cauchy problem:

$$\begin{aligned} \frac{d^\nu}{dt^\nu}\rho_k(t) &= \mu(k+1)\binom{n_0}{k+1} \\ &\times \sum_{r=0}^{n_0-k-1}\binom{n_0-k-1}{r}(-1)^r E_{\nu,1}(-(k+1+r)\mu t^\nu) - \mu k \rho_k(t), \end{aligned} \tag{3.31}$$

subject to the initial condition $\rho_k(0) = 0$ and with $\nu \in (0,1]$. The solution can be found by resorting to the Laplace transform, as we see in the following.

$$\begin{aligned} z^\nu \mathcal{L}\{\rho_k\}(z) &= \mu(k+1)\binom{n_0}{k+1} \\ &\times \sum_{r=0}^{n_0-k-1}\binom{n_0-k-1}{r}(-1)^r \frac{z^{\nu-1}}{z^\nu+(k+1+r)\mu} - \mu k \mathcal{L}\{\rho_k\}(z). \end{aligned} \tag{3.32}$$

The Laplace transform $\mathcal{L}\{\rho_k\}(z)$ can thus be written as

$$\begin{aligned} &\mathcal{L}\{\rho_k\}(z) \\ &= \mu(k+1)\binom{n_0}{k+1}\sum_{r=0}^{n_0-k-1}\binom{n_0-k-1}{r}(-1)^r\frac{z^{\nu-1}}{z^\nu+(k+1+r)\mu}\cdot\frac{1}{z^\nu+k\mu} \\ &= \binom{n_0}{k}\sum_{r=0}^{n_0-k-1}\binom{n_0-k}{r+1}(-1)^r z^{\nu-1}\left(\frac{1}{z^\nu+k\mu}-\frac{1}{z^\nu+(k+1+r)\mu}\right) \\ &= \binom{n_0}{k}\sum_{j=1}^{n_0-k}\binom{n_0-k}{j}(-1)^{j-1} z^{\nu-1}\left(\frac{1}{z^\nu+k\mu}-\frac{1}{z^\nu+(k+j)\mu}\right) \\ &= \binom{n_0}{k}\sum_{j=1}^{n_0-k}\binom{n_0-k}{j}(-1)^j\frac{z^{\nu-1}}{z^\nu+(k+j)\mu} \\ &\quad -\binom{n_0}{k}\frac{z^{\nu-1}}{z^\nu+k\mu}\sum_{j=1}^{n_0-k}\binom{n_0-k}{j}(-1)^j \end{aligned} \tag{3.33}$$

$$= \binom{n_0}{k} \sum_{j=1}^{n_0-k} \binom{n_0-k}{j} (-1)^j \frac{z^{\nu-1}}{z^\nu + (k+j)\mu} + \binom{n_0}{k} \frac{z^{\nu-1}}{z^\nu + k\mu}$$

$$= \binom{n_0}{k} \sum_{j=0}^{n_0-k} \binom{n_0-k}{j} (-1)^j \frac{z^{\nu-1}}{z^\nu + (k+j)\mu}.$$

□

By taking now the inverse Laplace transform of (3.33)*, we obtain the claimed result* (3.19)*.*

Remark 3.2.1. *When* $\nu = 1$*, equation* (3.19) *easily reduces to the distribution of the classical linear death process, i.e.*

$$\rho_k(t) = \binom{n_0}{k} e^{-k\mu t} \left(1 - e^{-\mu t}\right)^{n_0-k}, \qquad t > 0,\ 0 \le k \le n_0. \tag{3.34}$$

In the following theorem we give a proof of an interesting subordination relation.

Theorem 3.2.2. *The fractional linear death process* $M^\nu(t)$*,* $t > 0$ *can be represented as*

$$M^\nu(t) \overset{i.d.}{=} M(T_{2\nu}(t)), \qquad t > 0,\ \nu \in (0,1], \tag{3.35}$$

where $M(t)$*,* $t > 0$ *is the classical linear death process (see e.g. Bailey (1964), page 90) and* $T_{2\nu}(t)$*,* $t > 0$*, is a random time process whose one-dimensional distribution coincides with the folded solution to the following fractional diffusion equation*

$$\begin{cases} \frac{\partial^{2\nu}}{\partial t^{2\nu}} q(s,t) = \frac{\partial^2}{\partial s^2} q(s,t), & t > 0,\ \nu \in (0,1], \\ q(s,0) = \delta(s), \end{cases} \tag{3.36}$$

with the additional condition $q_t(s,0) = 0$ *if* $\nu \in (1/2, 1]$ *(see Beghin and Orsingher (2009a)).*

Proof. By evaluating the Laplace transform of the generating function of the fractional linear death process $M^\nu(t)$, $t > 0$, we obtain that

$$\int_0^\infty e^{-zt} G^\nu(u,t) dt \tag{3.37}$$

$$= \int_0^\infty e^{-zt} \sum_{k=0}^{n_0} u^k \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r E_{\nu,1}(-(k+r)\mu t^\nu) dt$$

$$= \sum_0^{n_0} u^k \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r \frac{z^{\nu-1}}{z^\nu + (k+r)\mu}$$

$$= \int_0^\infty \sum_{k=0}^{n_0} u^k \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r z^{\nu-1} e^{-s(z^\nu + (k+r)\mu)} ds$$

$$= \int_0^\infty e^{-sz^\nu} z^{\nu-1} \left\{ \sum_{k=0}^{n_0} u^k \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r e^{-s(k+r)\mu} \right\} ds$$

$$= \int_0^\infty e^{-sz^\nu} z^{\nu-1} \left\{ \sum_{k=0}^{n_0} u^k \binom{n_0}{k} e^{-\mu s k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r e^{-sr\mu} \right\} ds$$

$$= \int_0^\infty e^{-sz^\nu} z^{\nu-1} \left\{ \sum_{k=0}^{n_0} u^k \binom{n_0}{k} e^{-\mu s k} (1 - e^{-\mu s})^{n_0-k} \right\} ds$$

$$= \int_0^\infty e^{-sz^\nu} z^{\nu-1} G(u,s) ds$$

$$= \int_0^\infty e^{-zt} \int_0^\infty \sum_{k=0}^{n_0} u^k \Pr\{M(s)=k\} f_{T_{2\nu}}(s,t)ds\, dt$$
$$= \int_0^\infty e^{-zt} \left\{ \sum_{k=0}^{\infty} u^k \Pr\{M(T_{2\nu}(t))=k\} \right\} dt,$$

and this is sufficient to prove that (3.35) holds. Note that we used two facts. The first one is that

$$\int_0^\infty e^{-zt} f_{T_{2\nu}}(s,t)dt = z^{\nu-1}e^{-sz^\nu}, \qquad s>0,\ z>0, \tag{3.38}$$

is the Laplace transform of the solution to (3.36). The second fact is that the Laplace transform of the Mittag–Leffler function is

$$\int_0^\infty e^{-zt} E_{\nu,1}(-\vartheta t^\nu)dt = \frac{z^{\nu-1}}{z^\nu+\vartheta}. \tag{3.39}$$

□

In figures 3.1 and 3.2, we compare the behaviour of the fractional probabilities $\rho^{0.7}_{n_0}(t)$ and $\rho^{0.7}_{n_0-1}(t)$ with their classical counterparts $\rho^1_{n_0}(t)$ and $\rho^1_{n_0-1}(t)$, $t>0$. What emerges from the inspection of both figures is that, for large values of t, the probabilities, in the fractional case, decrease more slowly than $\rho^1_{n_0}(t)$ and $\rho^1_{n_0-1}(t)$. The probability $\rho^{0.7}_{n_0-1}(t)$, increases initially faster than $\rho^1_{n_0-1}(t)$, but after a certain time lapse, $\rho^1_{n_0-1}(t)$ dominates $\rho^{0.7}_{n_0-1}(t)$.

Remark 3.2.2. *For $\nu=1/2$, in view of the integral representation*

$$E_{\frac{1}{2},1}(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-w^2+2xw}dw, \qquad x\in\mathbb{R}, \tag{3.40}$$

we extract from (3.19) *that*

$$\begin{aligned} \rho^{\frac{1}{2}}_k(t) &= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-w^2} \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r}(-1)^r e^{-2w(k+r)\mu t^{\frac{1}{2}}} \\ &= \int_0^\infty \frac{e^{-\frac{y^2}{4t}}}{\sqrt{\pi t}} \rho^1_k(y)dy = Pr\{M(|B(t)|)=k\}, \end{aligned} \tag{3.41}$$

where $B(t)$, $t>0$ is a Brownian motion with volatility equal to 2.

Remark 3.2.3. *We can interpret formula* (3.19) *in an alternative way, as follows. For each integer $k\in[0,n_0]$ we have that*

$$\begin{aligned} \rho^\nu_k(t) &= Pr\{M^\nu(t)=k \mid M^\nu(0)=n_0\} \\ &= \int_0^\infty \rho_k(s) Pr\{T_{2\nu}(t)\in ds\} \\ &= \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r}(-1)^r \int_0^\infty e^{-\mu(k+r)s} Pr\{T_{2\nu}(t)\in ds\} \\ &= \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r}(-1)^r \int_0^\infty e^{-\mu(k+r)s} t^{-\nu} W_{-\nu,1-\nu}(-st^{-\nu})ds \\ &= \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r}(-1)^r \int_0^\infty e^{-\xi\mu(k+r)t^\nu} W_{-\nu,1-\nu}(-\xi)d\xi \\ &= \int_0^\infty W_{-\nu,1-\nu}(-\xi) Pr\{M_\xi(t^\nu)=k \mid M_\xi(0)=n_0\}\, d\xi, \end{aligned} \tag{3.42}$$

where $W_{-\nu,1-\nu}(-\xi)$ is the Wright function defined as

$$W_{-\nu,1-\nu}(-\xi) = \sum_{r=0}^{\infty} \frac{(-\xi)^r}{r!\Gamma(1-\nu(r+1))}, \qquad 0 < \nu \le 1. \tag{3.43}$$

We therefore obtain an interpretation in terms of a classical linear death process $M_{\Xi}(t)$, $t > 0$ evaluated on a new time scale and with random rate $\mu \cdot \Xi$, where Ξ is a random variable, $\xi \in \mathbb{R}^+$, with Wright density

$$f_{\Xi}(\xi) = W_{-\nu,1-\nu}(-\xi), \qquad \xi \in \mathbb{R}^+. \tag{3.44}$$

From equation (3.6) with $\mu_k = k \cdot \mu$, the related fractional differential equation governing the probability generating function, can be easily obtained, leading to

$$\begin{cases} \frac{\partial^\nu}{\partial t^\nu} G^\nu(u,t) = -\mu u(u-1)\frac{\partial}{\partial u} G^\nu(u,t), & \nu \in (0,1], \\ G^\nu(u,0) = u^{n_0}. \end{cases} \tag{3.45}$$

From this, and by considering that $\mathbb{E}M^\nu(t) = \frac{\partial}{\partial u} G^\nu(u,t)\big|_{u=1}$, we obtain that

$$\begin{cases} \frac{d^\nu}{dt^\nu} \mathbb{E}M^\nu(t) = -\mu \mathbb{E}M^\nu(t), & \nu \in (0,1], \\ \mathbb{E}M^\nu(t) = n_0. \end{cases} \tag{3.46}$$

Equation (3.46) is easily solved by means of the Laplace transforms, yielding

$$\mathbb{E}M^\nu(t) = n_0 E_{\nu,1}(-\mu t^\nu), \qquad t > 0,\ \nu \in (0,1]. \tag{3.47}$$

Remark 3.2.4. *The mean value $\mathbb{E}M^\nu(t)$ can also be directly calculated.*

$$\begin{aligned} \mathbb{E}M^\nu(t) &= \sum_{k=0}^{n_0} k \rho_k^\nu(t) \\ &= \sum_{k=0}^{n_0} k \binom{n_0}{k} \sum_{r=k}^{n_0} \binom{n_0-k}{r-k} (-1)^{r-k} E_{\nu,1}(-r\mu t^\nu) \\ &= \sum_{r=0}^{n_0} E_{\nu,1}(-r\mu t^\nu)(-1)^r \sum_{k=1}^{r} k \binom{n_0}{k} \binom{n_0-k}{r-k} (-1)^k \\ &= \sum_{r=1}^{n_0} E_{\nu,1}(-r\mu t^\nu)(-1)^r n_0 \binom{n_0-1}{r-1} \sum_{k=1}^{r} \binom{r-1}{k-1} (-1)^k \\ &= n_0 E_{\nu,1}(-\mu t^\nu). \end{aligned} \tag{3.48}$$

This last step in (3.48) holds because

$$\sum_{k=1}^{r} \binom{r-1}{k-1} (-1)^k = \sum_{k=0}^{r-1} \binom{r-1}{k} (-1)^{k+1} = \begin{cases} -1, & r = 1, \\ 0, & r > 1. \end{cases} \tag{3.49}$$

3.3 Related models

In this section we present two models which are related to the fractional linear death process. The first one is its natural generalisation to the nonlinear case i.e. we consider death rates in the form $\mu_k > 0$, $0 \le k \le n_0$. The second one is a sublinear process (see Donnelly et al. (1993)), namely with death rates in the form $\mu_k = \mu(n_0 + 1 - k)$; the death rates are thus an increasing sequence as the number of individuals in the population decreases towards zero.

3.3.1 Generalisation to the nonlinear case

Let us denote by $\mathcal{M}^\nu(t)$, $t > 0$ the random number of components of a nonlinear fractional death process with death rates $\mu_k > 0$, $0 \le k \le n_0$.

The state probabilities $\rho_k^\nu(t) = \Pr\{\mathcal{M}^\nu(t) = k \mid \mathcal{M}^\nu(0) = n_0\}$, $t > 0$, $0 \le k \le n_0$, $\nu \in (0,1]$ are governed by the following difference-differential equations

$$\begin{cases} \frac{d^\nu}{dt^\nu}\rho_k(t) = \mu_{k+1}\rho_{k+1}(t) - \mu_k\rho_k(t), & 0 < k < n_0, \\ \frac{d^\nu}{dt^\nu}\rho_0(t) = \mu_1\rho_1(t), & k = 0, \\ \frac{d^\nu}{dt^\nu}\rho_{n_0}(t) = -\mu_{n_0}\rho_{n_0}(t), & k = n_0, \\ \rho_k(0) = \begin{cases} 0, & 0 \le k < n_0, \\ 1, & k = n_0. \end{cases} \end{cases} \tag{3.50}$$

The fractional derivatives appearing in (3.50) provide the system with a global memory; i.e. the evolution of the state probabilities $\rho_k^\nu(t)$, $t > 0$, is influenced by the past, as definition (3.5) shows. This is a major difference with the classical nonlinear (and, of course, linear and sublinear) death processes, and reverberates in the slowly decaying structure of probabilities extracted from (3.50).

In the nonlinear process, the dependence of death rates from the size of the population is arbitrary, and this explains the complicated structure of the probabilities obtained. Further generalisation can be considered by assuming that the death rates depend on t (non-homogeneous, nonlinear death process).

We outline here the evaluation of the probabilities $\rho_k^\nu(t)$, $t > 0$, $0 \le k \le n_0$, which can be obtained, as in the linear case, by means of a recursive procedure (similar to that implemented in Orsingher and Polito (2010) for the fractional linear birth process).

Let $k = n_0$. By means of the Laplace transform applied to equation (3.50) we immediately obtain that

$$\rho_{n_0}^\nu(t) = E_{\nu,1}(-\mu_{n_0}t^\nu). \tag{3.51}$$

When $k = n_0 - 1$ we get

$$\begin{aligned} & z^\nu \mathcal{L}\left\{\rho_{n_0-1}^\nu\right\}(z) = -\mu_{n_0-1}\mathcal{L}\left\{\rho_{n_0-1}^\nu\right\}(z) + \mu_{n_0}\frac{z^{\nu-1}}{z^\nu + \mu_{n_0}} \\ & \Leftrightarrow \mathcal{L}\left\{\rho_{n_0-1}^\nu\right\}(z) = \mu_{n_0}\frac{z^{\nu-1}}{z^\nu+\mu_{n_0}} \cdot \frac{1}{z^\nu + \mu_{n_0-1}} \\ & \Leftrightarrow \mathcal{L}\left\{\rho_{n_0-1}^\nu\right\}(z) = \mu_{n_0}z^{\nu-1}\left[\frac{1}{z^\nu+\mu_{n_0}} - \frac{1}{z^\nu+\mu_{n_0-1}}\right]\frac{1}{\mu_{n_0-1}-\mu_{n_0}} \\ & \Leftrightarrow \rho_{n_0-1}^\nu(t) = \frac{\mu_{n_0}}{\mu_{n_0-1}-\mu_{n_0}}\left\{E_{\nu,1}(-\mu_{n_0}t^\nu) - E_{\nu,1}(-\mu_{n_0-1}t^\nu)\right\}. \end{aligned} \tag{3.52}$$

For $k = n_0 - 2$ we obtain in the same way that

$$\begin{aligned} & z^\nu\mathcal{L}\left\{\rho_{n_0-2}^\nu\right\}(z) \\ & = -\mu_{n_0-2}\mathcal{L}\left\{\rho_{n_0-2}^\nu\right\}(z) + \frac{\mu_{n_0}\mu_{n_0-1}}{\mu_{n_0-1}-\mu_{n_0}}\left[\frac{z^{\nu-1}}{z^\nu+\mu_{n_0}} - \frac{z^{\nu-1}}{z^\nu+\mu_{n_0-1}}\right], \end{aligned} \tag{3.53}$$

so that

$$\mathcal{L}\left\{\rho_{n_0-2}^\nu\right\}(z) \tag{3.54}$$

$$
\begin{aligned}
&= \frac{\mu_{n_0}\mu_{n_0-1}}{\mu_{n_0-1}-\mu_{n_0}} z^{\nu-1}\left[\frac{1}{z^\nu+\mu_{n_0}} - \frac{1}{z^\nu+\mu_{n_0-1}}\right]\frac{1}{z^\nu+\mu_{n_0-2}} \\
&= \frac{\mu_{n_0}\mu_{n_0-1}}{\mu_{n_0-1}-\mu_{n_0}} z^{\nu-1}\left[\left(\frac{1}{z^\nu+\mu_{n_0}} - \frac{1}{z^\nu+\mu_{n_0-2}}\right)\frac{1}{\mu_{n_0-2}-\mu_{n_0}}\right. \\
&\quad \left. - \left(\frac{1}{z^\nu+\mu_{n_0-1}} - \frac{1}{z^\nu+\mu_{n_0-2}}\right)\frac{1}{\mu_{n_0-2}-\mu_{n_0-1}}\right].
\end{aligned}
$$

By inverting the Laplace transform we readily arrive at the following result

$$
\begin{aligned}
\rho^\nu_{n_0-2}(t) &= \mu_{n_0}\mu_{n_0-1}\left[\frac{E_{\nu,1}(-\mu_{n_0}t^\nu)}{(\mu_{n_0-1}-\mu_{n_0})(\mu_{n_0-2}-\mu_{n_0})}\right. \\
&\quad - \frac{E_{\nu,1}(-\mu_{n_0-2}t^\nu)}{(\mu_{n_0-1}-\mu_{n_0})(\mu_{n_0-2}-\mu_{n_0})} - \frac{E_{\nu,1}(-\mu_{n_0-1}t^\nu)}{(\mu_{n_0-1}-\mu_{n_0})(\mu_{n_0-2}-\mu_{n_0-1})} \\
&\quad \left. + \frac{E_{\nu,1}(-\mu_{n_0-2}t^\nu)}{(\mu_{n_0-1}-\mu_{n_0})(\mu_{n_0-2}-\mu_{n_0-1})}\right] \\
&= \mu_{n_0}\mu_{n_0-1}\left[\frac{E_{\nu,1}(-\mu_{n_0}t^\nu)}{(\mu_{n_0-1}-\mu_{n_0})(\mu_{n_0-2}-\mu_{n_0})}\right. \\
&\quad + \frac{E_{\nu,1}(-\mu_{n_0-2}t^\nu)}{(\mu_{n_0-1}-\mu_{n_0})}\left(\frac{1}{\mu_{n_0-2}-\mu_{n_0-1}} - \frac{1}{\mu_{n_0-2}-\mu_{n_0}}\right) \\
&\quad \left. - \frac{E_{\nu,1}(-\mu_{n_0-1}t^\nu)}{(\mu_{n_0-1}-\mu_{n_0})(\mu_{n_0-2}-\mu_{n_0-1})}\right] \\
&= \mu_{n_0}\mu_{n_0-1}\left[\frac{E_{\nu,1}(-\mu_{n_0}t^\nu)}{(\mu_{n_0-1}-\mu_{n_0})(\mu_{n_0-2}-\mu_{n_0})}\right. \\
&\quad \left. + \frac{E_{\nu,1}(-\mu_{n_0-2}t^\nu)}{(\mu_{n_0-2}-\mu_{n_0-1})(\mu_{n_0-2}-\mu_{n_0})} - \frac{E_{\nu,1}(-\mu_{n_0-1}t^\nu)}{(\mu_{n_0-1}-\mu_{n_0})(\mu_{n_0-2}-\mu_{n_0-1})}\right].
\end{aligned}
\tag{3.55}
$$

The structure of the state probabilities for arbitrary values of $k = n_0 - l$, $0 \le l < n_0$, can now be easily obtained. The proof follows the lines of the derivation of the state probabilities for the fractional nonlinear pure birth process adopted in Theorem 2.1 in Orsingher and Polito (2010). We have that

$$
\rho^\nu_{n_0-l}(t) = \begin{cases} \prod_{j=0}^{l-1}\mu_{n_0-j}\sum_{m=0}^{l}\frac{E_{\nu,1}(-\mu_{n_0-m}t^\nu)}{\prod_{h=0, h\neq m}^{l}(\mu_{n_0-h}-\mu_{n_0-m})}, & 1 \le l < n_0, \\ E_{\nu,1}(-\mu_{n_0}t^\nu), & l = 0. \end{cases}
\tag{3.56}
$$

By means of some changes of indices, formula (3.56) can also be written as

$$
\rho^\nu_k(t) = \begin{cases} \prod_{j=k+1}^{n_0}\mu_j\sum_{m=k}^{n_0}\frac{E_{\nu,1}(-\mu_m t^\nu)}{\prod_{h=k, h\neq m}^{n_0}(\mu_h-\mu_m)}, & 0 < k < n_0, \\ E_{\nu,1}(-\mu_{n_0}t^\nu), & k = n_0. \end{cases}
\tag{3.57}
$$

For the extinction probability, we have to solve the following initial value problem:

$$
\begin{cases} \frac{d^\nu}{dt^\nu}\rho_0(t) = \mu_1\prod_{j=2}^{n_0}\mu_j\sum_{m=1}^{n_0}\frac{E_{\nu,1}(-\mu_m t^\nu)}{\prod_{h=1, h\neq m}^{n_0}(\mu_h-\mu_m)}, & n_0 > 1, \\ \frac{d^\nu}{dt^\nu}\rho_0(t) = \mu_1 E_{\nu,1}(-\mu_1 t^\nu), & n_0 = 1, \\ \rho_0(0) = 0, & n_0 \ge 1. \end{cases}
\tag{3.58}
$$

When $n_0 > 1$, starting from (3.58) and by resorting to the Laplace transform once again, we have that

$$\mathcal{L}\left\{\rho_0^\nu\right\}(z) = \prod_{j=1}^{n_0} \mu_j \sum_{m=1}^{n_0} \frac{1}{\prod\limits_{\substack{h=1 \\ h\neq m}}^{n_0} (\mu_h - \mu_m)} \cdot \frac{z^{-1}}{z^\nu + \mu_m}. \tag{3.59}$$

The inverse Laplace transform of (3.59) leads to

$$\begin{aligned}
\rho_0^\nu(t) &= \prod_{j=1}^{n_0} \mu_j \sum_{m=1}^{n_0} \frac{1}{\prod\limits_{\substack{h=1 \\ h\neq m}}^{n_0} (\mu_h - \mu_m)} t^\nu E_{\nu,\nu+1}(-\mu_m t^\nu) \\
&= \prod_{j=1}^{n_0} \mu_j \sum_{m=1}^{n_0} \frac{1}{\prod\limits_{\substack{h=1 \\ h\neq m}}^{n_0} (\mu_h - \mu_m)} \cdot \frac{1}{\mu_m} \left[1 - E_{\nu,1}(-\mu_m t^\nu)\right] \\
&= \sum_{m=1}^{n_0} \prod_{\substack{h=1 \\ h\neq m}}^{n_0} \left(\frac{\mu_h}{\mu_h - \mu_m}\right) - \sum_{m=1}^{n_0} \prod_{\substack{h=1 \\ h\neq m}}^{n_0} \left(\frac{\mu_h}{\mu_h - \mu_m}\right) E_{\nu,1}(-\mu_m t^\nu) \\
&= 1 - \sum_{m=1}^{n_0} \prod_{\substack{h=1 \\ h\neq m}}^{n_0} \left(\frac{\mu_h}{\mu_h - \mu_m}\right) E_{\nu,1}(-\mu_m t^\nu).
\end{aligned} \tag{3.60}$$

Note that, in the last step, we used the following fact:

$$\sum_{m=1}^{n_0} \prod_{\substack{h=1 \\ h\neq m}}^{n_0} \left(\frac{\mu_h}{\mu_h - \mu_m}\right) = 1. \tag{3.61}$$

This can be ascertained by observing that

$$\prod_{1\le h<l\le n_0} (\mu_h - \mu_l) = \det \boldsymbol{A} = \sum_{j=1}^{n_0} a_{1,j} (-1)^{j+1} \mathrm{Min}_{1,j} \tag{3.62}$$

where

$$\boldsymbol{A} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \mu_1 & \mu_2 & \cdots & \mu_{n_0} \\ \mu_1^2 & \mu_2^2 & \cdots & \mu_{n_0}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \mu_1^{n_0-1} & \mu_2^{n_0-1} & \dots & \mu_{n_0}^{n_0-1} \end{vmatrix}, \tag{3.63}$$

is a Vandermonde matrix and $\mathrm{Min}_{1,j}$ is the determinant of the matrix resulting from $\boldsymbol{A}$ by removing the first row and the j-th column.

When $n_0 = 1$ we obtain

$$\mathcal{L}\left\{\rho_0^\nu\right\}(z) = \mu_1 \frac{z^{-1}}{z^\nu + \mu_1}, \tag{3.64}$$

so that the inverse Laplace transform can be written as

$$\begin{aligned}
\rho_0^\nu(t) &= \mu_1 t^\nu E_{\nu,\nu+1}(-\mu_1 t^\nu) \\
&= 1 - E_{\nu,1}(-\mu_1 t^\nu).
\end{aligned} \tag{3.65}$$

We can therefore summarise the results obtained as follows:

$$\rho_k^\nu(t) = \begin{cases} \prod_{j=k+1}^{n_0} \mu_j \sum_{m=k}^{n_0} \frac{E_{\nu,1}(-\mu_m t^\nu)}{\prod_{\substack{h=k\\h\neq m}}^{n_0} (\mu_h - \mu_m)}, & 0 < k < n_0,\ n_0 > 1, \\ E_{\nu,1}(-\mu_{n_0} t^\nu), & k = n_0,\ n_0 \geq 1, \end{cases} \tag{3.66}$$

and

$$\rho_0^\nu(t) = \begin{cases} 1 - \sum_{m=1}^{n_0} \prod_{\substack{h=1\\h\neq m}}^{n_0} \left(\frac{\mu_h}{\mu_h - \mu_m} \right) E_{\nu,1}(-\mu_m t^\nu), & n_0 > 1, \\ 1 - E_{\nu,1}(-\mu_1 t^\nu), & n_0 = 1. \end{cases} \tag{3.67}$$

3.3.2 A fractional sublinear death process

We consider in this section the process where the infinitesimal death probabilities have the form

$$\Pr\{\mathfrak{M}(t, t+dt] = -1 \mid \mathfrak{M}(t) = k\} = \mu(n_0 + 1 - k)dt + o(dt), \tag{3.68}$$

where n_0 is the initial number of individuals in the population. The state probabilities

$$\wp_k(t) = \Pr\{\mathfrak{M}(t) = k \mid \mathfrak{M}(0) = n_0\}, \qquad 0 \leq k \leq n_0, \tag{3.69}$$

satisfy the equations

$$\begin{cases} \frac{d}{dt}\wp_k(t) = -\mu(n_0 + 1 - k)\wp_k(t) + \mu(n_0 - k)\wp_{k+1}(t), & 1 \leq k \leq n_0, \\ \frac{d}{dt}\wp_0(t) = \mu n_0 \wp_1(t), & k = 0, \\ \wp_k(0) = \begin{cases} 1, & k = n_0, \\ 0, & 0 \leq k < n_0. \end{cases} \end{cases} \tag{3.70}$$

In this model the death rate increases with decreasing population size.

The probabilities $\wp_k^\nu(t) = \Pr\{\mathfrak{M}^\nu(t) = k \mid \mathfrak{M}^\nu(0) = n_0\}$ of the fractional version of this process are governed by the equations

$$\begin{cases} \frac{d^\nu}{dt^\nu}\wp_k(t) = -\mu(n_0 + 1 - k)\wp_k(t) + \mu(n_0 - k)\wp_{k+1}(t), & 1 \leq k \leq n_0, \\ \frac{d^\nu}{dt^\nu}\wp_0(t) = \mu n_0 \wp_1(t), & k = 0, \\ \wp_k(0) = \begin{cases} 1, & k = n_0, \\ 0, & 0 \leq k < n_0. \end{cases} \end{cases} \tag{3.71}$$

We first observe that the solution to the Cauchy problem

$$\begin{cases} \frac{d^\nu}{dt^\nu}\wp_{n_0}(t) = -\mu\wp_{n_0}(t), \\ \wp_{n_0}(0) = 1, \end{cases} \tag{3.72}$$

is $\wp_{n_0}^\nu(t) = E_{\nu,1}(-\mu t^\nu)$, $t > 0$.

In order to solve the equation

$$\begin{cases} \frac{d^\nu}{dt^\nu}\wp_{n_0-1}(t) = -2\mu\wp_{n_0-1}(t) + \mu E_{\nu,1}(-\mu t^\nu), \\ \wp_{n_0-1}(0) = 0, \end{cases} \tag{3.73}$$

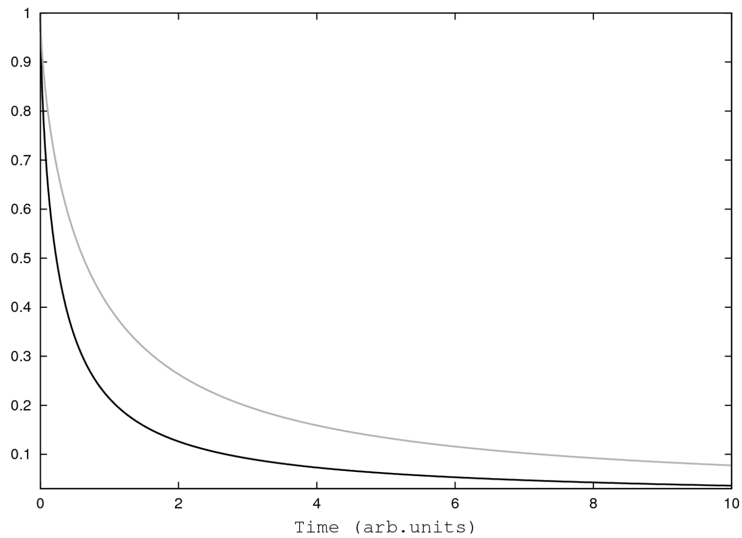

Figure 3.3: Plot of $\rho_{n_0}^{0.7}(t)$ (in black) and $\wp_{n_0}^{0.7}(t)$ (in grey), with $n_0 = 2$.

we resort to the Laplace transform and obtain that

$$\mathcal{L}\left\{\wp_{n_0-1}^{\nu}\right\}(z) = \mu z^{\nu-1}\frac{1}{z^\nu+\mu}\cdot\frac{1}{z^\nu+2\mu} = z^{\nu-1}\left(\frac{1}{z^\nu+\mu}-\frac{1}{z^\nu+2\mu}\right). \tag{3.74}$$

By inverting (3.74) we extract the following result

$$\wp_{n_0-1}^{\nu}(t) = E_{\nu,1}(-\mu t^\nu) - E_{\nu,1}(-2\mu t^\nu). \tag{3.75}$$

By the same technique we solve

$$\begin{cases} \frac{d^\nu}{dt^\nu}\wp_{n_0-2}(t) = -3\mu\wp_{n_0-2}(t) + 2\mu\left[E_{\nu,1}(-\mu t^\nu) - E_{\nu,1}(-2\mu t^\nu)\right], \\ \wp_{n_0-2}(0) = 0, \end{cases} \tag{3.76}$$

thus obtaining

$$\begin{aligned} \mathcal{L}\left\{\wp_{n_0-2}^{\nu}\right\}(z) &= 2\mu z^{\nu-1}\left[\frac{1}{z^\nu+\mu}-\frac{1}{z^\nu+2\mu}\right]\frac{1}{z^\nu+3\mu} \\ &= 2\mu z^{\nu-1}\left[\left(\frac{1}{z^\nu+\mu}-\frac{1}{z^\nu+3\mu}\right)\frac{1}{2\mu} - \left(\frac{1}{z^\nu+2\mu}-\frac{1}{z^\nu+3\mu}\right)\frac{1}{\mu}\right] \\ &= \frac{z^{\nu-1}}{z^\nu+\mu} - 2\frac{z^{\nu-1}}{z^\nu+2\mu} + \frac{z^{\nu-1}}{z^\nu+3\mu}. \end{aligned} \tag{3.77}$$

In light of (3.77), we infer that

$$\wp_{n_0-2}^{\nu}(t) = E_{\nu,1}(-\mu t^\nu) - 2E_{\nu,1}(-2\mu t^\nu) + E_{\nu,1}(-3\mu t^\nu). \tag{3.78}$$

For all $1 \le n_0 - m \le n_0$, by similar calculations, we arrive at the general result

$$\wp_{n_0-m}^{\nu} = \sum_{l=0}^{m}\binom{m}{l}(-1)^l E_{\nu,1}\left(-\left(l+1\right)\mu t^\nu\right), \qquad 1 \le n_0 - m \le n_0. \tag{3.79}$$

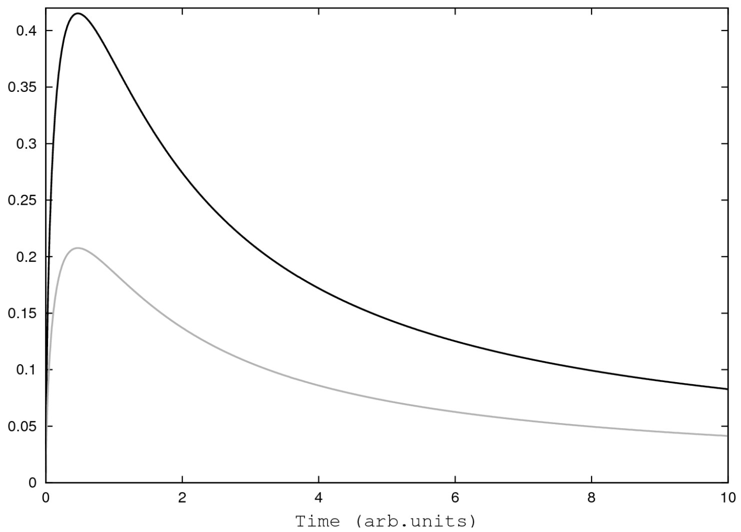

Figure 3.4: Plot of $\rho^{0.7}_{n_0-1}(t)$ (in black) and $\wp^{0.7}_{n_0-1}(t)$ (in grey), with $n_0 = 2$.

Introducing the notation $n_0 - m = k$, we rewrite the state probabilities (3.79) in the following manner

$$\wp^\nu_k = \sum_{l=0}^{n_0-k} \binom{n_0-k}{l} (-1)^l E_{\nu,1}\left(-(l+1)\mu t^\nu\right), \qquad 1 \le k \le n_0. \tag{3.80}$$

For the extinction probability we must solve the following Cauchy problem

$$\begin{cases} \frac{d^\nu}{dt^\nu}\wp_0(t) = \mu n_0 \sum_{l=0}^{n_0-1} \binom{n_0-1}{l}(-1)^l E_{\nu,1}\left(-(l+1)\mu t^\nu\right), \\ \wp_0(0) = 0. \end{cases} \tag{3.81}$$

The Laplace transform of (3.81) yields

$$z^\nu \mathcal{L}\left\{\wp^\nu_0\right\}(z) = \mu n_0 \sum_{l=0}^{n_0-1} \binom{n_0-1}{l} (-1)^l \frac{z^{\nu-1}}{z^\nu + \mu(l+1)}. \tag{3.82}$$

The inverse Laplace transform can be written down as

$$\wp^\nu_0(t) = \mu n_0 \sum_{l=0}^{n_0-1} \binom{n_0-1}{l} (-1)^l \frac{1}{\Gamma(\nu)} \int_0^t E_{\nu,1}\left(-(l+1)\mu s^\nu\right)(t-s)^{\nu-1} ds. \tag{3.83}$$

The integral appearing in (3.83) can be suitably evaluated as follows

$$\begin{aligned} &\int_0^t E_{\nu,1}\left(-(l+1)\mu s^\nu\right)(t-s)^{\nu-1} \\ &= \sum_{m=0}^\infty \frac{(-(l+1)\mu)^m}{\Gamma(\nu m+1)} \int_0^t s^{\nu m}(t-s)^{\nu-1} ds \\ &= \sum_{m=0}^\infty \frac{(-(l+1)\mu)^m}{\Gamma(\nu m+1)} \frac{t^{\nu(m+1)}\Gamma(\nu)\Gamma(\nu m+1)}{\Gamma(\nu m+\nu+1)} \\ &= \frac{\Gamma(\nu)}{(-\mu(l+1))} \sum_{m=0}^\infty \frac{(-(l+1)\mu t^\nu)^{m+1}}{\Gamma(\nu(m+1)+1)} \\ &= \frac{\Gamma(\nu)}{(-\mu(l+1))} \left[E_{\nu,1}(-(l+1)\mu t^\nu) - 1\right]. \end{aligned} \tag{3.84}$$

By inserting result (3.84) into (3.83), we obtain

$$\begin{aligned}\wp_0^\nu(t) &= n_0 \sum_{l=0}^{n_0-1} \binom{n_0-1}{l} \frac{(-1)^{l+1}}{l+1} \left[E_{\nu,1}(-(l+1)\mu t^\nu) - 1\right] \\ &= \sum_{l=0}^{n_0-1} \binom{n_0}{l+1} (-1)^{l+1} \left[E_{\nu,1}(-(l+1)\mu t^\nu) - 1\right] \\ &= \sum_{l=1}^{n_0} \binom{n_0}{l} (-1)^l E_{\nu,1}(-l\mu t^\nu) - \sum_{l=1}^{n_0} \binom{n_0}{l} (-1)^l \\ &= 1 + \sum_{l=1}^{n_0} \binom{n_0}{l} (-1)^l E_{\nu,1}(-l\mu t^\nu) \\ &= \sum_{l=0}^{n_0} \binom{n_0}{l} (-1)^l E_{\nu,1}(-l\mu t^\nu).\end{aligned} \tag{3.85}$$

Remark 3.3.1. *We check that the probabilities* (3.80) *and* (3.85) *sum up to unity. We start by analysing the following sum:*

$$\sum_{k=1}^{n_0} \wp_k^\nu(t) = \sum_{k=1}^{n_0} \sum_{l=0}^{n_0-k} \binom{n_0-k}{l} (-1)^l E_{\nu,1}(-(l+1)\mu t^\nu). \tag{3.86}$$

In order to evaluate (3.86)*, we resort to the Laplace transform*

$$\sum_{k=1}^{n_0} \mathcal{L}\left\{\wp_k^\nu\right\}(z) = \frac{z^{\nu-1}}{\mu} \sum_{k=1}^{n_0} \sum_{l=0}^{n_0-k} \binom{n_0-k}{l} (-1)^l \frac{1}{\frac{z^\nu}{\mu} + 1 + l}. \tag{3.87}$$

By using formula (6) of Kirschenhofer (1996) (see also Graham et al. (1994), formula (5.41), page 188), we obtain that

$$\begin{aligned}\sum_{k=1}^{n_0} \mathcal{L}\left\{\wp_k^\nu\right\}(z) &= \frac{z^{\nu-1}}{\mu} \sum_{k=1}^{n_0} \frac{\Gamma(n_0-k+1)}{\left(\frac{z^\nu}{\mu}+1\right)\left(\frac{z^\nu}{\mu}+2\right)\dots\left(\frac{z^\nu}{\mu}+1+n_0-k\right)} \\ &= \frac{z^{\nu-1}}{\mu} \sum_{k=1}^{n_0} \frac{\Gamma\left(\frac{z^\nu}{\mu}+1\right)\Gamma\left(n_0-k+1\right)}{\Gamma\left(\frac{z^\nu}{\mu}+1+n_0-k\right)} \\ &= \frac{z^{\nu-1}}{\mu} \sum_{k=1}^{n_0} \int_0^1 x^{\frac{z^\nu}{\mu}} (1-x)^{n_0-k} dx \\ &= \frac{z^{\nu-1}}{\mu} \int_0^1 x^{\frac{z^\nu}{\mu}-1} \left[1-(1-x)^{n_0}\right] dx \\ &= \frac{1}{z} - \frac{z^{\nu-1}}{\mu} \int_0^1 x^{\frac{z^\nu}{\mu}-1} (1-x)^{n_0} dx \\ &\overset{(-\ln x = y)}{=} \frac{1}{z} - \frac{z^{\nu-1}}{\mu} \int_0^\infty e^{-y\frac{z^\nu}{\mu}} \left(1-e^{-y}\right)^{n_0} dy \\ &\overset{(y/\mu = w)}{=} \frac{1}{z} z^{\nu-1} \int_0^\infty e^{-wz^\nu} \left(1-e^{-\mu w}\right)^{n_0} dw \\ &= \frac{1}{z} - z^{\nu-1} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k \int_0^\infty e^{-z^\nu w - \mu w k} dw \\ &= \frac{1}{z} - z^{\nu-1} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k \frac{1}{z^\nu + \mu k}.\end{aligned} \tag{3.88}$$

The inverse Laplace transform of (3.88) *is therefore*

$$\sum_{k=1}^{n_0} \wp_k^\nu(t) = 1 - \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k E_{\nu,1}(-\mu k t^\nu) \tag{3.89}$$
$$= -\sum_{k=1}^{n_0} \binom{n_0}{k} (-1)^k E_{\nu,1}(-\mu k t^\nu).$$

By putting (3.85) *and* (3.89) *together, we conclude that*

$$\sum_{k=0}^{n_0} \wp_k^\nu(t) = 1, \tag{3.90}$$

as it should be.

Remark 3.3.2. *We observe that, in the linear and sublinear death processes, the extinction probabilities coincide. This implies that although the state probabilities* $\rho_k^\nu(t)$ *and* $\wp_k^\nu(t)$ *differ for all* $1 \le k \le n_0$*, we have that*

$$\sum_{k=1}^{n_0} \rho_k^\nu(t) = \sum_{k=1}^{n_0} \wp_k^\nu(t). \tag{3.91}$$

This can be checked by performing the following sum

$$\begin{aligned}
&\sum_{k=1}^{n_0} \mathcal{L}\left\{\rho_k^\nu(t)\right\}(z) \qquad (3.92)\\
&= \sum_{k=1}^{n_0} \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r \frac{z^{\nu-1}}{z^\nu + \mu(k+r)} \\
&= \frac{z^{\nu-1}}{\mu} \sum_{k=1}^{n_0} \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r \frac{1}{\frac{z^\nu}{\mu} + k + r} \\
&= \frac{z^{\nu-1}}{\mu} \sum_{k=1}^{n_0} \binom{n_0}{k} \frac{(n_0-k)!}{\left(\frac{z^\nu}{\mu}+k\right)\left(\frac{z^\nu}{\mu}+k+1\right)\dots\left(\frac{z^\nu}{\mu}+n_0\right)} \\
&= \frac{z^{\nu-1}}{\mu} \sum_{k=1}^{n_0} \binom{n_0}{k} \frac{\Gamma(n_0-k+1)\,\Gamma\left(\frac{z^\nu}{\mu}+k\right)}{\Gamma\left(\frac{z^\nu}{\mu}+n_0+1\right)} \\
&= \frac{z^{\nu-1}}{\mu} \int_0^1 (1-x)^{\frac{z^\nu}{\mu}-1} \sum_{k=1}^{n_0} \binom{n_0}{k} x^{n_0-k}(1-x)^k dx \\
&= \frac{z^{\nu-1}}{\mu} \int_0^1 (1-x)^{\frac{z^\nu}{\mu}-1} \left(1-x^{n_0}\right) dx \\
&= \frac{1}{z} - \frac{z^{\nu-1}}{\mu} \int_0^1 x^{n_0}(1-x)^{\frac{z^\nu}{\mu}-1} dx.
\end{aligned}$$

This coincides with the fourth-to-last step of (3.88) *and therefore we can conclude that*

$$\sum_{k=1}^{n_0} \rho_k^\nu(t) = -\sum_{k=1}^{n_0} \binom{n_0}{k} (-1)^k E_{\nu,1}(-\mu k t^\nu) = \sum_{k=1}^{n_0} \wp_k^\nu(t). \tag{3.93}$$

Mean value

Theorem 3.3.1. *Consider the fractional sublinear death process* $\mathfrak{M}^\nu(t)$*,* $t > 0$ *defined above. The probability generating function* $\mathfrak{G}^\nu(u,t) = \sum_{k=0}^{n_0} u^k \wp_k^\nu(t)$*,* $t > 0$*,* $|u| \le 1$*, satisfies the* *following partial differential equation:*

$$\frac{\partial^\nu}{\partial t^\nu}\mathfrak{G}^\nu(u,t) = \mu(n_0+1)\left(\frac{1}{u}-1\right)\left[\mathfrak{G}^\nu(u,t) - \wp_0^\nu(t)\right] + \mu(u-1)\frac{\partial}{\partial u}\mathfrak{G}^\nu(u,t). \tag{3.94}$$

subject to the initial condition $\mathfrak{G}^\nu(u,0) = u^{n_0}$*, for* $|u| \le 1$*,* $t > 0$*.*

Proof. Starting from (3.71), we obtain that

$$\begin{aligned} &\frac{d^\nu}{dt^\nu}\sum_{k=0}^{n_0} u^k \wp_k^\nu(t) \\ &= -\mu\sum_{k=1}^{n_0} u^k(n_0+1-k)\wp_k^\nu(t) + \mu\sum_{k=0}^{n_0-1} u^k(n_0-k)\wp_{k+1}^\nu(t), \end{aligned} \tag{3.95}$$

so that

$$\begin{aligned} \frac{\partial^\nu}{\partial t^\nu}\mathfrak{G}^\nu(u,t) = & -\mu(n_0+1)\left[\mathfrak{G}^\nu(u,t) - \wp_0^\nu(t)\right] + \mu u\frac{\partial}{\partial u}\mathfrak{G}^\nu(u,t) \\ & + \frac{\mu(n_0+1)}{u}\left[\mathfrak{G}^\nu(u,t) - \wp_0^\nu(t)\right] - \mu\frac{\partial}{\partial u}\mathfrak{G}^\nu(u,t) \\ = & \ \mu(n_0+1)\left(\frac{1}{u}-1\right)\left[\mathfrak{G}^\nu(u,t) - \wp_0^\nu(t)\right] + \mu(u-1)\frac{\partial}{\partial u}\mathfrak{G}^\nu(u,t). \end{aligned} \tag{3.96}$$

□

Theorem 3.3.2. *The mean number of individuals* $\mathbb{E}\mathfrak{M}^\nu(t)$*,* $t>0$ *in the fractional sublinear death process, reads*

$$\mathbb{E}\mathfrak{M}^\nu(t) = \sum_{k=1}^{n_0}\binom{n_0+1}{k+1}(-1)^{k+1}E_{\nu,1}(-\mu k t^\nu), \qquad t>0,\ \nu\in(0,1]. \tag{3.97}$$

Proof. From (3.94) and by considering that $\mathbb{E}\mathfrak{M}^\nu(t) = \frac{\partial}{\partial u}\mathfrak{G}^\nu(u,t)\big|_{u=1}$, we directly arrive at the following initial value problem:

$$\begin{cases} \frac{d^\nu}{dt^\nu}\mathbb{E}\mathfrak{M}^\nu(t) = -\mu(n_0+1)\left[1-\wp_0^\nu(t)\right] + \mu\mathbb{E}\mathfrak{M}^\nu(t), \\ \mathbb{E}\mathfrak{M}^\nu(0) = n_0, \end{cases} \tag{3.98}$$

which can be solved by resorting to the Laplace transform, as follows:

$$\begin{aligned} \mathcal{L}\left\{\mathbb{E}\mathfrak{M}^\nu(t)\right\}(z) &= n_0\frac{z^{\nu-1}}{z^\nu-\mu} + \mu(n_0+1)\sum_{k=1}^{n_0}\binom{n_0}{k}(-1)^k\frac{z^{\nu-1}}{z^\nu+\mu k}\cdot\frac{1}{z^\nu-\mu} \\ &= n_0\frac{z^{\nu-1}}{z^\nu-\mu} + \sum_{k=1}^{n_0}\binom{n_0+1}{k+1}(-1)^k\left[\frac{z^{\nu-1}}{z^\nu-\mu} - \frac{z^{\nu-1}}{z^\nu+\mu k}\right]. \end{aligned} \tag{3.99}$$

In (3.99), formula (3.85) must be considered. By inverting the Laplace transform we obtain that

$$\begin{aligned} &\mathbb{E}\mathfrak{M}^\nu(t) \\ &= n_0E_{\nu,1}(\mu t^\nu) + \sum_{k=1}^{n_0}\binom{n_0+1}{k+1}(-1)^k\left[E_{\nu,1}(\mu t^\nu) - E_{\nu,1}(-\mu k t^\nu)\right] \\ &= n_0E_{\nu,1}(\mu t^\nu) + E_{\nu,1}(\mu t^\nu)\sum_{k=1}^{n_0}\binom{n_0+1}{k+1}(-1)^k \end{aligned} \tag{3.100}$$

$$
\begin{aligned}
&-\sum_{k=1}^{n_0}\binom{n_0+1}{k+1}(-1)^k E_{\nu,1}(-\mu k t^\nu) \\
&=\sum_{k=1}^{n_0}\binom{n_0+1}{k+1}(-1)^{k+1} E_{\nu,1}(-\mu k t^\nu),
\end{aligned}
$$

as desired. □

Remark 3.3.3. *The mean value* (3.97) *can also be directly derived as follows.*

$$
\begin{aligned}
\mathbb{E}\mathfrak{M}^\nu(t) &= \sum_{k=0}^{n_0} k \wp_k^\nu(t) \qquad (3.101)\\
&= \sum_{k=1}^{n_0} k \sum_{l=0}^{n_0-k}\binom{n_0-k}{l}(-1)^l E_{\nu,1}(-(l+1)\mu t^\nu) \\
&= \sum_{k=1}^{n_0} k \sum_{l=1}^{n_0+1-k}\binom{n_0-k}{l-1}(-1)^{l-1} E_{\nu,1}(-\mu l t^\nu) \\
&= \sum_{l=1}^{n_0}(-1)^{l-1} E_{\nu,1}(-\mu l t^\nu) \sum_{k=1}^{n_0+1-l} k\binom{n_0-k}{l-1}.
\end{aligned}
$$

It is now sufficient to show that

$$
\sum_{k=1}^{n_0+1-l} k\binom{n_0-k}{l-1} = \binom{n_0+1}{l+1}. \qquad (3.102)
$$

Indeed,

$$
\begin{aligned}
\sum_{k=1}^{n_0+1-l} k\binom{n_0-k}{l-1} &= \sum_{k=l-1}^{n_0-1}(n_0-k)\binom{k}{l-1} \qquad (3.103)\\
&= \sum_{k=l-1}^{n_0-1}(n_0+1-k-1)\binom{k}{l-1} \\
&= (n_0+1)\sum_{k=l-1}^{n_0-1}\binom{k}{l-1} - l\sum_{k=l-1}^{n_0-1}\binom{k+1}{l} \\
&= (n_0+1)\sum_{k=l}^{n_0}\binom{k-1}{l-1} - l\sum_{k=l+1}^{n_0+1}\binom{k-1}{l} \\
&= (n_0+1)\binom{n_0}{l} - l\binom{n_0+1}{l+1} \\
&= \binom{n_0+1}{l+1}.
\end{aligned}
$$

The crucial step of (3.103) *is justified by the following formula*

$$
\sum_{k=j}^{n_0}\binom{k-1}{j-1} = 1 + \binom{j}{j-1} + \cdots + \binom{n_0-1}{j-1} = \binom{n_0}{j}. \qquad (3.104)
$$

Figure 3.5 shows that in the sublinear case, the mean number of individuals in the population, decays more slowly than in the linear case, as expected.

Note that (3.97) satisfies the initial condition $\mathbb{E}\mathfrak{M}^\nu(0) = n_0$. In order to check this, it is sufficient to show that

$$
\sum_{k=1}^{n_0}\binom{n_0+1}{k+1}(-1)^{k+1} = \sum_{r=2}^{n_0+1}\binom{n_0+1}{r}(-1)^r \qquad (3.105)
$$

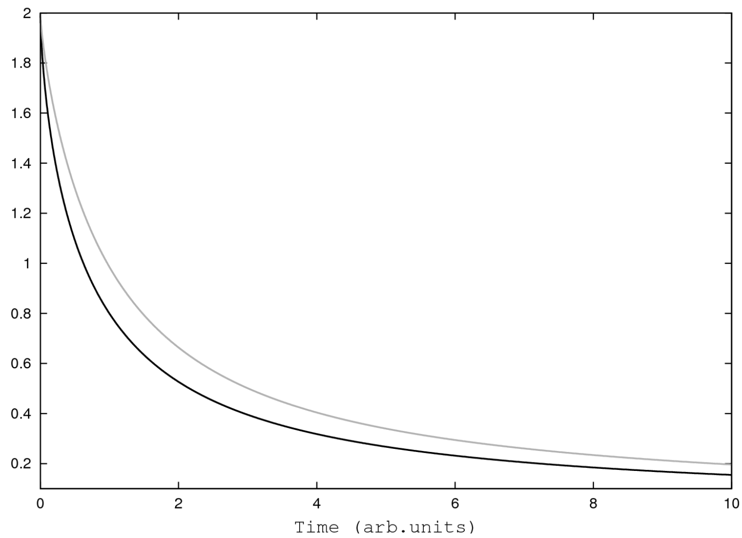

Figure 3.5: Plot of $\mathbb{E}M^{0.7}(t)$ (in black) and $\mathbb{E}\mathfrak{M}^{0.7}(t)$ (in grey), $n_0 = 2$.

$$= \left[\sum_{r=0}^{n_0+1} \binom{n_0+1}{r} (-1)^r \right] - 1 + \binom{n_0+1}{1} = n_0.$$

The details in (3.105) explain also the last step of (3.100).

Comparison of $\mathfrak{M}^\nu(t)$ with the fractional linear death process $M^\nu(t)$ and the fractional linear birth process $N^\nu(t)$

The distributions of the fractional linear and sublinear processes examined above display a behaviour which is illustrated in Table 3.1.

Table 3.1: State probabilities $\rho_k^\nu(t)$ for the fractional linear death process $M^\nu(t)$, $t > 0$, and $\wp_k^\nu(t)$ for the fractional sublinear death process $\mathfrak{M}^\nu(t)$.

State Probabilities
$\rho_{n_0}^\nu(t) = E_{\nu,1}(-\mu n_0 t^\nu)$
$\wp_{n_0}^\nu(t) = E_{\nu,1}(-\mu t^\nu)$
$\rho_{n_0-1}^\nu(t) = n_0 \left[E_{\nu,1}(-(n_0-1)\mu t^\nu) - E_{\nu,1}(-n_0\mu t^\nu)\right]$
$\wp_{n_0-1}^\nu(t) = E_{\nu,1}(-\mu t^\nu) - E_{\nu,1}(-2\mu t^\nu)$
$\vdots$
$\rho_k^\nu(t) = \binom{n_0}{k} \sum_{l=0}^{n_0-k} \binom{n_0-k}{l} (-1)^l E_{\nu,1}(-(k+l)\mu t^\nu)$
$\wp_k^\nu(t) = \sum_{l=0}^{n_0-k} \binom{n_0-k}{l} (-1)^l E_{\nu,1}\left(-(l+1)\mu t^\nu\right)$
$\vdots$
$\rho_1^\nu(t) = n_0 \sum_{l=0}^{n_0-1} \binom{n_0-1}{l} (-1)^l E_{\nu,1}(-(1+l)\mu t^\nu)$
$\wp_1^\nu(t) = \sum_{l=0}^{n_0-1} \binom{n_0-1}{l} (-1)^l E_{\nu,1}\left(-(l+1)\mu t^\nu\right)$
$\rho_0^\nu(t) = \sum_{l=0}^{n_0} \binom{n_0}{l} (-1)^l E_{\nu,1}(-l\mu t^\nu)$
$\wp_0^\nu(t) = \sum_{l=0}^{n_0} \binom{n_0}{l} (-1)^l E_{\nu,1}(-l\mu t^\nu)$

Table 3.2: Mean values for the fractional linear birth $N^\nu(t)$, fractional linear death $M^\nu(t)$ and fractional sublinear death $\mathfrak{M}^\nu(t)$ processes.

$\mathbb{E}N^\nu(t) = E_{\nu,1}(\lambda t^\nu)$
$\mathbb{E}M^\nu(t) = n_0 E_{\nu,1}(-\mu t^\nu)$
$\mathbb{E}\mathfrak{M}^\nu(t) = \sum_{k=1}^{n_0} \binom{n_0+1}{k+1} (-1)^{k+1} E_{\nu,1}(-\mu k t^\nu)$

The most striking fact about the models dealt with above, is that the linear probabilities decay faster than the corresponding sublinear ones, for small values of k; whereas, for large values of k, the sublinear probabilities take over and the extinction probabilities in both cases coincide. The reader should also compare the state probabilities of the death models examined here with those of the fractional linear pure birth process (with birth rate λ and one progenitor). These read

$$p_k^\nu(t) = \sum_{j=1}^{k} \binom{k-1}{j-1} (-1)^{j-1} E_{\nu,1}(-\lambda j t^\nu), \qquad k \geq 1. \tag{3.106}$$

Note that $p_1^\nu(t) = E_{\nu,1}(-\lambda t^\nu)$ is of the same form as $\wp_{n_0}^\nu(t) = E_{\nu,1}(-\mu t^\nu)$. We now show that

$$\begin{aligned} \sum_{k=n_0+1}^{\infty} p_k^\nu(t) &= 1 - \sum_{k=1}^{n_0} p_k^\nu(t) \\ &= 1 - \sum_{k=1}^{n_0} \sum_{j=1}^{k} \binom{k-1}{j-1} (-1)^{j-1} E_{\nu,1}(-\lambda j t^\nu) \\ &= 1 - \sum_{j=1}^{n_0} (-1)^{j-1} E_{\nu,1}(-\lambda j t^\nu) \sum_{k=j}^{n_0} \binom{k-1}{j-1} \\ &= 1 - \sum_{j=1}^{n_0} (-1)^{j-1} \binom{n_0}{j} E_{\nu,1}(-\lambda j t^\nu) \\ &= (3.85) \text{ with } \lambda \text{ replacing } \mu. \end{aligned} \tag{3.107}$$

Note that in the above step we used formula (3.104).

By comparing formulae (3.4) of Orsingher and Polito (2010) and (3.80) above, we arrive at the conclusion that (for $\lambda = \mu$)

$$\begin{aligned} &\Pr\{N^\nu(t) = k \mid N^\nu(0) = 1\} \\ &= \sum_{j=1}^{k} \binom{k-1}{j-1} (-1)^{j-1} E_{\nu,1}(-\lambda_j t^\nu) \\ &= \Pr\{\mathfrak{M}^\nu(t) = n_0 + 1 - k \mid \mathfrak{M}(0) = n_0\}, \qquad 1 \leq k \leq n_0. \end{aligned} \tag{3.108}$$

For $k = 0$ the probability of extinction corresponds to the probability of the event $\{N^\nu(t) > n_0\}$ for the fractional linear birth process.

Chapter 4

Fractional Linear Birth-Death Processes

Summary

In this chapter we introduce and examine a fractional linear birth-death process $\mathfrak{N}^{\nu}(t)$, $t > 0$, whose fractionality is obtained by replacing the time-derivative with a fractional derivative in the system of difference-differential equations governing the state probabilities $\mathfrak{p}_k^{\nu}(t)$, $t > 0$, $k \geq 0$. We present a subordination relation connecting $\mathfrak{N}^{\nu}(t)$, $t > 0$, with the classical birth-death process $\mathfrak{N}(t)$, $t > 0$, by means of the time process $T_{2\nu}(t)$, $t > 0$, whose distribution is related to a time-fractional diffusion equation. We obtain explicit formulae for the extinction probability $\mathfrak{p}_0^{\nu}(t)$, and the state probabilities $\mathfrak{p}_k^{\nu}(t)$, $t > 0$, $k \geq 1$, in the three relevant cases $\lambda > \mu$, $\lambda < \mu$, $\lambda = \mu$ (where λ and μ are respectively the birth and the death rates) and discuss their behaviour in specific situations. We highlight the connection of the fractional linear birth-death process with the fractional pure birth process. Finally the mean values $\mathbb{E}\mathfrak{N}^{\nu}(t)$ and $\mathbb{V}\mathrm{ar}\mathfrak{N}^{\nu}(t)$ are derived and analysed.

4.1 Introduction

In Chapter 2 (see also Orsingher and Polito (2010)) we constructed a fractional version of the pure birth process $\mathcal{N}^{\nu}(t), t>0$ (both in the general and in the linear case), by considering the fractional equations governing their distributions. In this chapter we examine the linear birth-death process $\mathfrak{N}^{\nu}(t)$, $t>0$, where the state probabilities

$$\mathfrak{p}_k^{\nu}(t) = \Pr\left\{\mathfrak{N}^{\nu}(t) = k \mid \mathfrak{N}^{\nu}(0) = 1\right\}, \tag{4.1}$$

are assumed to satisfy the fractional difference-differential equations

$$\frac{d^{\nu}\mathfrak{p}_k(t)}{dt^{\nu}} = -(\lambda+\mu)k\mathfrak{p}_k(t) + \lambda(k-1)\mathfrak{p}_{k-1}(t) + \mu(k+1)\mathfrak{p}_{k+1}(t), \tag{4.2}$$

with $k \geq 1$, $0<\nu\leq 1$.

The fractional operator appearing in (4.2) is defined as

$$\begin{cases} \dfrac{d^{\nu}f(t)}{dt^{\nu}} = \dfrac{1}{\Gamma(1-\nu)}\displaystyle\int_0^t \frac{\frac{d}{ds}f(s)}{(t-s)^{\nu}}ds, & 0<\nu<1, \\ f'(t), & \nu=1. \end{cases} \tag{4.3}$$

The derivative (4.3) is usually called Caputo or Dzhrbashyan–Caputo fractional derivative and differs from the classical Riemann–Liouville derivative by exchanging the integral and derivative operators (see Podlubny (1999)). An advantage of Caputo over Riemann–Liouville is that Caputo does not require fractional-order derivatives in the initial conditions which is good for practical purposes. The positive parameters λ and μ are respectively the birth and death rates.

The exact distribution of the linear birth-death process reads (see Bailey (1964) page 91, Feller (1968) page 454)

$$\mathfrak{p}_k^1(t) = (\lambda-\mu)^2 e^{-(\lambda-\mu)t}\frac{\left(\lambda\left(1-e^{-(\lambda-\mu)t}\right)\right)^{k-1}}{\left(\lambda-\mu e^{-(\lambda-\mu)t}\right)^{k+1}}, \qquad k\geq 1,\ t>0,\ \mu\neq\lambda. \tag{4.4}$$

When $\lambda=\mu$ the distribution (4.4) is much simpler and takes the form

$$\mathfrak{p}_k^1(t) = \frac{(\lambda t)^{k-1}}{(1+\lambda t)^{k+1}}, \qquad t>0,\ k\geq 1. \tag{4.5}$$

The exact expressions for the extinction probabilities are

$$\mathfrak{p}_0^1(t) = \begin{cases} \dfrac{\lambda t}{1+\lambda t} & \lambda=\mu, \\ \dfrac{\mu-\mu e^{-t(\lambda-\mu)}}{\lambda-\mu e^{-t(\lambda-\mu)}} & \lambda\neq\mu. \end{cases} \tag{4.6}$$

From (4.2) we can infer that the probability generating function of $\mathfrak{N}^{\nu}(t)$, $t>0$,

$$G_{\nu}(u,t) = \mathbb{E}u^{\mathfrak{N}^{\nu}(t)}, \qquad |u|\leq 1,\ 0<\nu\leq 1,\ t>0, \tag{4.7}$$

satisfies the Cauchy problem

$$\begin{cases} \dfrac{\partial^{\nu}}{\partial t^{\nu}}G_{\nu}(u,t) = (\lambda u-\mu)(u-1)\dfrac{\partial}{\partial u}G_{\nu}(u,t), & \nu\in(0,1],\ |u|\leq 1, \\ G_{\nu}(u,0) = u. \end{cases} \tag{4.8}$$

We will show below that from (4.8) one can arrive at the subordination relation

$$\mathfrak{N}^{\nu}(t) \overset{\text{i.d.}}{=} \mathfrak{N}(T_{2\nu}(t)), \qquad t>0, \tag{4.9}$$

where $T_{2\nu}(t)$, $t>0$, is the random time process whose distribution is obtained by folding the solution of the following fractional diffusion equation

$$\begin{cases} \dfrac{\partial^{2\nu} q}{\partial t^{2\nu}} = \dfrac{\partial^2 q}{\partial x^2}, & 0<\nu\le 1,\ x\in\mathbb{R},\ t>0, \\ q(x,0)=\delta(x). \end{cases} \tag{4.10}$$

The process $\mathfrak{N}(t), t>0$, found in (4.9), is the classical linear birth-death process whose distribution is given in (4.4), (4.5) and (4.6). A relation similar to (4.9) holds also for the fractional pure birth process (Orsingher and Polito (2010)) and the fractional Poisson process (Beghin and Orsingher (2009b)). In this context it represents the main tool of our analysis and leads to a number of interesting explicit distributions. We consider the subordinator related to (4.10) because the p.g.f. of the distribution of (4.9) satisfies the simplest fractional equation generalising the classical one.

For the extinction probabilities of the fractional linear birth-death process we have the following attractive formulas

$$\mathfrak{p}_0^{\nu}(t) = \begin{cases} \dfrac{\mu}{\lambda} - \dfrac{\lambda-\mu}{\lambda}\displaystyle\sum_{m=1}^{+\infty}\left(\frac{\mu}{\lambda}\right)^m E_{\nu,1}\left(-t^{\nu}(\lambda-\mu)m\right), & \lambda>\mu, \\ 1 - \dfrac{\mu-\lambda}{\lambda}\displaystyle\sum_{m=1}^{+\infty}\left(\frac{\lambda}{\mu}\right)^m E_{\nu,1}\left(-t^{\nu}(\mu-\lambda)m\right), & \lambda<\mu, \\ 1-\displaystyle\int_0^{+\infty} e^{-w}E_{\nu,1}\left(-\lambda t^{\nu}w\right)dw & \lambda=\mu. \end{cases} \tag{4.11}$$

for $t>0,\ 0<\nu\le 1$.

The function $E_{\alpha,\beta}(x)$, appearing in (4.11) is the generalised Mittag–Leffler function, defined as

$$E_{\alpha,\beta}(x) = \sum_{m=0}^{+\infty}\frac{x^m}{\Gamma(\alpha m+\beta)}, \qquad x\in\mathbb{R},\ \alpha>0,\ \beta>0. \tag{4.12}$$

From (4.11) we can easily retrieve the classical extinction probabilities (4.6) for $\nu=1$ by holding in mind that $E_{1,1}(x)=e^x$.

For the state distributions $\mathfrak{p}_k^{\nu}(t)$, $t>0$, $k\ge 1$ we have formulas similar to (4.11) but with a more complicated structure.

$$\mathfrak{p}_k^{\nu}(t) = \begin{cases} \left(\dfrac{\lambda-\mu}{\lambda}\right)^2\displaystyle\sum_{l=0}^{\infty}\binom{l+k}{l}\left(\frac{\mu}{\lambda}\right)^l \\ \qquad\times\displaystyle\sum_{r=0}^{k-1}(-1)^r\binom{k-1}{r}E_{\nu,1}\left(-(l+r+1)(\lambda-\mu)t^{\nu}\right), & \lambda>\mu, \\ \left(\dfrac{\lambda}{\mu}\right)^{k-1}\left(\dfrac{\mu-\lambda}{\mu}\right)^2\displaystyle\sum_{l=0}^{\infty}\binom{l+k}{l}\left(\frac{\lambda}{\mu}\right)^l \\ \qquad\times\displaystyle\sum_{r=0}^{k-1}(-1)^r\binom{k-1}{r}E_{\nu,1}\left(-(l+r+1)(\mu-\lambda)t^{\nu}\right), & \lambda<\mu, \\ \dfrac{(-1)^{k-1}\lambda^{k-1}}{k!}\dfrac{d^k}{d\lambda^k}\left[\lambda\left(1-\mathfrak{p}_0^{\nu}(t)\right)\right], & \lambda=\mu. \end{cases} \tag{4.13}$$

Also from (4.13), for $\nu = 1$ one can reobtain the distributions (4.4), (4.5).

We will show below that the probabilities $\mathfrak{p}_k^\nu(t)$, $t > 0$, $k \geq 1$, appearing in (4.13) are strictly related to the distributions of the fractional linear pure birth process $N^\nu(t)$, $t > 0$, with an arbitrary number of progenitors and a birth rate equal to $\lambda - \mu$ with $\lambda > \mu$. In particular, we can extract from the first of (4.13) that

$$\begin{aligned} &\Pr\{\mathfrak{N}^\nu(t) = k \mid \mathfrak{N}^\nu(0) = 1\} \\ &= \frac{\lambda - \mu}{\lambda} \sum_{l=0}^{\infty} \left[\left(1 + \frac{\mu}{k(\lambda - \mu)}\right) \Pr\{\mathcal{G} = l\} + \frac{\mu}{k} \frac{d}{d\mu} \Pr\{\mathcal{G} = l\} \right] \\ &\quad \times \Pr\{N^\nu(t) = k + l \mid N^\nu(0) = l + 1\}, \end{aligned} \tag{4.14}$$

where

$$\Pr\{\mathcal{G} = l\} = \left(1 - \frac{\mu}{\lambda}\right)\left(\frac{\mu}{\lambda}\right)^l, \qquad l \geq 0, \tag{4.15}$$

is a geometric law for the number of progenitors. We also note that the distribution (4.13), for $\lambda = \mu$ can be expressed in terms of the extinction probability (4.11):

$$\Pr\{\mathfrak{N}^\nu(t) = k\} = \frac{(-1)^{k-1}\lambda^{k-1}}{k!} \frac{d^k}{d\lambda^k} \left[\lambda\left(1 - \Pr\{\mathfrak{N}^\nu(t) = 0\}\right)\right], \qquad k \geq 1,\ t > 0. \tag{4.16}$$

The extinction probability (4.11) can be viewed as being a suitable weighted mean of the waiting times of the fractional Poisson process $\mathcal{P}_\lambda^\nu(t)$, $t > 0$, for which it is well-known that (Beghin and Orsingher (2009b))

$$\Pr\{\mathcal{P}_\lambda^\nu(t) = 0\} = E_{\nu,1}(-\lambda t^\nu), \qquad t > 0. \tag{4.17}$$

The fractional linear birth-death process dealt with in this chapter, provides a generalisation of the classical linear birth-death process and may well prove to be capable of modelling queues in service systems, epidemics and the evolution populations under accelerating conditions. The introduction of the fractional derivative furnishes the system with a global memory. Furthermore, the qualitative features illustrated in the last section show that the fractional counterpart of the linear birth-death process has a faster mean evolution (and variance expansion) as was pointed out in similar fractional generalisations, e.g. for the Poisson process (see Cahoy (2007), Uchaikin and Sibatov (2008), Laskin (2003) and Beghin and Orsingher (2009b)), for fractional branching processes (Uchaikin et al. (2008)) and for pure birth processes (Orsingher and Polito (2010)).

4.2 The extinction probabilities

We begin this section by proving the subordination relation (4.9) which is interlaced with all the distributional results of this chapter.

Theorem 4.2.1. *The fractional linear birth-death process $\mathfrak{N}^\nu(t)$, $t > 0$, can be represented as*

$$\mathfrak{N}^\nu(t) = \mathfrak{N}(T_{2\nu}(t)), \qquad t > 0,\ 0 < \nu \leq 1, \tag{4.18}$$

where $\mathfrak{N}(t)$, $t > 0$, is the classical linear birth-death process and $T_{2\nu}(t)$, $t > 0$, is a random process whose one-dimensional distribution coincides with the folded solution to the fractional diffusion equation

$$\frac{\partial^{2\nu} q}{\partial t^{2\nu}} = \frac{\partial^2 q}{\partial x^2}, \qquad 0 < \nu \leq 1,\ x \in \mathbb{R},\ t > 0, \tag{4.19}$$

subject to the initial conditions $q(x, 0) = \delta(x)$ for $0 < \nu \leq 1$ and $q_t(x, 0) = 0$ for $1/2 < \nu \leq 1$.

Proof. The Laplace transform $\tilde{G}_\nu(u,z) = \int_0^\infty e^{-zt} G_\nu(u,t)\,dt$, applied to the fractional p.d.e.

$$\begin{cases} \frac{\partial^\nu}{\partial t^\nu} G_\nu(u,t) = (\lambda u - \mu)(u-1)\frac{\partial}{\partial u} G_\nu(u,t), \quad 0 < \nu \le 1, \\ G_\nu(u,0) = u. \end{cases} \tag{4.20}$$

yields

$$z^\nu \tilde{G}_\nu(u,z) - z^{\nu-1}u = (\lambda u - \mu)(u-1)\frac{\partial}{\partial u}\tilde{G}_\nu(u,z), \qquad 0<\nu\le 1,\ z>0,\ |u|\le 1. \tag{4.21}$$

We now observe that

$$\tilde{G}_\nu(u,z) = \int_0^\infty e^{-zt}\left[\sum_{k=0}^\infty u^k \Pr\{\mathfrak{N}^\nu(t)=k\}\right]dt. \tag{4.22}$$

If (4.18) holds then

$$\begin{aligned} \tilde{G}_\nu(u,z) &= \int_0^\infty e^{-zt}\left[\sum_{k=0}^\infty u^k \int_0^\infty \Pr\{\mathfrak{N}(s)=k\}\Pr\{T_{2\nu}(t)\in ds\}\right]dt \\ &= \int_0^\infty e^{-zt}\left[\int_0^\infty G(u,s)\Pr\{T_{2\nu}(t)\in ds\}\right]dt \\ &= \int_0^\infty G(u,s)\, z^{\nu-1}e^{-sz^\nu}ds. \end{aligned} \tag{4.23}$$

In the last step we applied the folded version of equation (3.3) in Orsingher and Beghin (2004) for $c=1$, that being therefore

$$\int_0^\infty e^{-zt}\Pr\{T_{2\nu}(t)\in ds\} = e^{-sz^\nu}z^{\nu-1}ds. \tag{4.24}$$

We now show that (4.23) satisfies equation (4.21); by inserting the Laplace transform into (4.21) we obtain

$$z^\nu z^{\nu-1}\int_0^\infty G(u,s)\,e^{-sz^\nu}ds - z^{\nu-1}u = (\lambda u-\mu)(u-1)\,z^{\nu-1}\int_0^\infty \frac{\partial}{\partial u}G(u,s)\,e^{-sz^\nu}ds. \tag{4.25}$$

The inversion of the integral with $\partial/\partial u$ is justified because

$$\left|\frac{\partial}{\partial u}G(u,s)\right| = \left|\sum_{k=1}^\infty k u^{k-1}\Pr\{\mathfrak{N}(s)=k\}\right| \le \sum_{k=1}^\infty k\Pr\{\mathfrak{N}(s)=k\} = \mathbb{E}\mathfrak{N}(s) < \infty. \tag{4.26}$$

Taking into account that $G(u,t)$ satisfies the first-order p.d.e.

$$\frac{\partial G}{\partial s} = (\lambda u-\mu)(u-1)\frac{\partial G}{\partial u}, \tag{4.27}$$

from (4.25), we have that

$$\begin{aligned} z^\nu\int_0^\infty G(u,s)\,e^{-sz^\nu}ds - u &= \int_0^\infty \frac{\partial}{\partial s}G(u,s)\,e^{-sz^\nu}ds \\ &= G(u,s)\,e^{-sz^\nu}\Big|_{s=0}^{s=\infty} + z^\nu\int_0^\infty G(u,s)\,e^{-sz^\nu}ds \\ &= -u + z^\nu\int_0^\infty G(u,s)\,e^{-sz^\nu}ds. \end{aligned} \tag{4.28}$$

This shows that (4.18) holds for the one-dimensional distributions. This concludes the proof of theorem 4.2.1. □

Remark 4.2.1. *For $\nu = 1/2^n$, $n \in \mathbb{N}$, the density $f_{T_{2\nu}}$ of the random time $T_{2\nu}$ appearing in* (4.18) *becomes the probability density of an $(n-1)$-iterated Brownian motion, i.e.*

$$Pr\left\{T_{\frac{1}{2^{n-1}}}(t) \in ds\right\} = Pr\{|\mathcal{B}_1(|\mathcal{B}_2(\cdots|\mathcal{B}_n(t)|\cdots)|)| \in ds\} \tag{4.29}$$
$$= ds\, 2^n \int_0^\infty \frac{e^{-\frac{s^2}{4\omega_1}}}{\sqrt{4\pi\omega_1}} d\omega_1 \int_0^\infty \frac{e^{-\frac{\omega_1^2}{4\omega_2}}}{\sqrt{4\pi\omega_2}} d\omega_2 \cdots \int_0^\infty \frac{e^{-\frac{\omega_{n-1}^2}{4t}}}{\sqrt{4\pi t}} d\omega_{n-1},$$

as can easily be inferred from Theorem 2.1 in Orsingher and Beghin (2009). The differences between (4.29) *and its corresponding formula in the cited paper, is that here the diffusion coefficient is equal to 1.*

In the following theorems we derive separately the three different expressions of the probability of extinction in the cases $\lambda > \mu$, $\lambda < \mu$ and $\lambda = \mu$. We prefer to treat them separately because their proofs are somewhat different.

Theorem 4.2.2. *For a fractional linear birth-death process with rates $\lambda > \mu$ the probability of extinction has the following form*

$$\mathfrak{p}_0^\nu(t) = Pr\{\mathfrak{N}^\nu(t) = 0\} = \frac{\mu}{\lambda} - \frac{\lambda-\mu}{\lambda} \sum_{m=1}^\infty \left(\frac{\mu}{\lambda}\right)^m E_{\nu,1}\left(-t^\nu(\lambda-\mu)m\right), \tag{4.30}$$

for $t > 0$, $0 < \nu \leq 1$, and where $E_{\nu,1}(x)$ is the Mittag–Leffler function (4.12).

Proof. In light of the subordination relation (4.18) of Theorem 4.2.1 and by taking into account the extinction probability of the classical linear birth-death process

$$\Pr\{\mathfrak{N}(t) = 0\} = \frac{\mu - \mu e^{-t(\lambda-\mu)}}{\lambda - \mu e^{-t(\lambda-\mu)}}, \qquad t > 0, \tag{4.31}$$

we can write that

$$\Pr\{\mathfrak{N}^\nu(t) = 0\} = \int_0^{+\infty} \frac{\mu - \mu e^{-s(\lambda-\mu)}}{\lambda - \mu e^{-s(\lambda-\mu)}} \Pr\{T_{2\nu}(t) \in ds\}, \tag{4.32}$$

for all $t > 0$ and $0 < \nu \leq 1$. By taking the Laplace transform of (4.32) we obtain that

$$\int_0^\infty e^{-zt}\Pr\{\mathfrak{N}^\nu(t) = 0\}\, dt \tag{4.33}$$
$$= \int_0^\infty \frac{\mu - \mu e^{-s(\lambda-\mu)}}{\lambda - \mu e^{-s(\lambda-\mu)}} z^{\nu-1} e^{-sz^\nu} ds$$
$$= \frac{\mu}{\lambda}\int_0^\infty \left(1 - e^{-s(\lambda-\mu)}\right) \sum_{m=0}^\infty \left(\frac{\mu}{\lambda}e^{-s(\lambda-\mu)}\right)^m z^{\nu-1}e^{-sz^\nu} ds$$
$$= \frac{\mu}{\lambda}\sum_{m=0}^\infty \left(\frac{\mu}{\lambda}\right)^m z^{\nu-1}\left[\int_0^\infty \left(e^{-s(\lambda-\mu)m - sz^\nu} - e^{-s(\lambda-\mu)(1+m)-sz^\nu}\right) ds\right]$$
$$= \frac{\mu}{\lambda}z^{\nu-1}\left\{\sum_{m=0}^\infty \left(\frac{\mu}{\lambda}\right)^m \frac{1}{(\lambda-\mu)m + z^\nu} - \sum_{m=1}^\infty \left(\frac{\mu}{\lambda}\right)^{m-1}\frac{1}{(\lambda-\mu)m+z^\nu}\right\}$$
$$= \frac{\mu}{\lambda}z^{\nu-1}\sum_{m=1}^\infty \frac{1}{(\lambda-\mu)m+z^\nu}\left(\frac{\mu}{\lambda}\right)^m\left(1-\frac{\lambda}{\mu}\right) + \frac{\mu}{\lambda}z^{\nu-1}\frac{1}{z^\nu}.$$

The above steps are valid because $0 < (\mu/\lambda)e^{-s(\lambda-\mu)} < 1$ for $\lambda > \mu$. By keeping in mind the Laplace transform of the Mittag–Leffler function $E_{\nu,1}(-xt^\nu)$

$$\int_0^\infty e^{-st}E_{\nu,1}(-xt^\nu)dt = \frac{s^{\nu-1}}{s^\nu + x}, \tag{4.34}$$

we readily arrive at the claimed result. □

Remark 4.2.2. *When $\nu = 1$, from (4.30) we obtain the form of the extinction probability (4.31) for the classical birth-death model.*

$$Pr\{\mathfrak{N}(t) = 0\} = \frac{\mu-\lambda}{\lambda}\left[\sum_{m=1}^{+\infty}\left(\frac{\mu}{\lambda}\right)^m e^{-(\lambda-\mu)mt}\right] + \frac{\mu}{\lambda} \tag{4.35}$$

$$= \frac{\mu-\lambda}{\lambda}\left[\frac{1}{1-\frac{\mu}{\lambda}e^{-(\lambda-\mu)t}} - 1\right] + \frac{\mu}{\lambda} = \frac{\mu-\lambda}{\lambda}\left[\frac{\frac{\mu}{\lambda}e^{-t(\lambda-\mu)}}{1-\frac{\mu}{\lambda}e^{-t(\lambda-\mu)}}\right] + \frac{\mu}{\lambda} = \frac{\mu-\mu e^{-t(\lambda-\mu)}}{\lambda-\mu e^{-t(\lambda-\mu)}}.$$

From (4.32) for $\nu = 1$, $Pr\{T_2(t) \in ds\} = \delta(s-t)$ and again we retrieve result (4.31).

Remark 4.2.3. *From (4.30) we note that*

$$Pr\{\mathfrak{N}^\nu(t) = 0\} \overset{t\to+\infty}{\longrightarrow} \frac{\mu}{\lambda}, \qquad \forall\, \nu \in (0,1] \tag{4.36}$$

which is the asymptotic extinction probability irrespective of the value of ν.

Let us now deal with the case $\lambda < \mu$, i.e. when the rate of birth is strictly lower than the rate of death.

Theorem 4.2.3. *For $\mu > \lambda$ the probability $\mathfrak{p}_0^\nu(t) = Pr\{\mathfrak{N}^\nu(t) = 0\}$ of complete extinction of the population is*

$$\mathfrak{p}_0^\nu(t) = 1 - \frac{\mu-\lambda}{\lambda}\sum_{m=1}^{+\infty}\left(\frac{\lambda}{\mu}\right)^m E_{\nu,1}\left(-t^\nu(\mu-\lambda)m\right), \tag{4.37}$$

where $t > 0$, $0 < \nu \leq 1$ and $E_{\nu,1}(x)$ is the Mittag–Leffler function (4.12).

Proof. We start by rewriting (4.31) as

$$\mathfrak{p}_0(t) = \frac{\mu e^{-t(\mu-\lambda)} - \mu}{\lambda e^{-t(\mu-\lambda)} - \mu}. \tag{4.38}$$

Using (4.18) we are able to write

$$\mathfrak{p}_0^\nu(t) = \int_0^{+\infty} \frac{\mu e^{-s(\mu-\lambda)} - \mu}{\lambda e^{-s(\mu-\lambda)} - \mu}\Pr\{T_{2\nu}(t) \in ds\}. \tag{4.39}$$

By applying the Laplace transform to (4.39) we obtain that

$$L_0^\nu(z) = \int_0^{+\infty} \frac{\mu e^{-s(\mu-\lambda)} - \mu}{\lambda e^{-s(\mu-\lambda)} - \mu} z^{\nu-1}e^{-sz^\nu}ds = \int_0^{+\infty} \frac{e^{-s(\mu-\lambda)} - 1}{\frac{\lambda}{\mu}e^{-s(\mu-\lambda)} - 1} z^{\nu-1}e^{-sz^\nu}ds \tag{4.40}$$

$$= z^{\nu-1}\int_0^{+\infty}\left(1-e^{-s(\mu-\lambda)}\right)e^{-sz^\nu}\sum_{m=0}^{+\infty}\left[\frac{\lambda}{\mu}e^{-s(\mu-\lambda)}\right]^m$$

$$= z^{\nu-1}\sum_{m=0}^{+\infty}\left(\frac{\lambda}{\mu}\right)^m\int_0^{+\infty}\left(1-e^{-s(\mu-\lambda)}\right)e^{-sz^\nu}e^{-s(\mu-\lambda)m}ds$$

$$= z^{\nu-1}\sum_{m=0}^{+\infty}\left(\frac{\lambda}{\mu}\right)^m\int_0^{+\infty} e^{-s(\mu-\lambda)m-sz^\nu} - e^{-s(\mu-\lambda)(m+1)-sz^\nu}ds$$

$$= z^{\nu-1}\sum_{m=0}^{+\infty}\left(\frac{\lambda}{\mu}\right)^m\left\{\frac{1}{(\mu-\lambda)m+z^\nu} - \frac{1}{(\mu-\lambda)(m+1)+z^\nu}\right\}$$

$$
\begin{aligned}
&= z^{\nu-1}\left\{\sum_{m=0}^{+\infty}\left(\frac{\lambda}{\mu}\right)^{m}\frac{1}{(\mu-\lambda)m+z^{\nu}}-\sum_{m=1}^{+\infty}\left(\frac{\lambda}{\mu}\right)^{m-1}\frac{1}{(\mu-\lambda)m+z^{\nu}}\right\}\\
&= z^{\nu-1}\left\{\frac{1}{z^{\nu}}+\sum_{m=1}^{+\infty}\left(\frac{\lambda}{\mu}\right)^{m}\frac{1}{(\mu-\lambda)m+z^{\nu}}-\frac{\mu}{\lambda}\sum_{m=1}^{+\infty}\left(\frac{\lambda}{\mu}\right)^{m}\frac{1}{(\mu-\lambda)m+z^{\nu}}\right\}\\
&= z^{\nu-1}\left\{\frac{1}{z^{\nu}}+\left[1-\frac{\mu}{\lambda}\right]\sum_{m=1}^{+\infty}\left(\frac{\lambda}{\mu}\right)^{m}\frac{1}{(\mu-\lambda)m+z^{\nu}}\right\}\\
&= \frac{1}{z}+\left[1-\frac{\mu}{\lambda}\right]\sum_{m=1}^{+\infty}\left(\frac{\lambda}{\mu}\right)^{m}\frac{z^{\nu-1}}{(\mu-\lambda)m+z^{\nu}}.
\end{aligned}
$$

By inverting (4.40) by means of (4.34) we retrieve formula (4.37). □

Remark 4.2.4. *When $\nu=1$ from* (4.37) *we reobtain the extinction probability of the classical birth-death process* (4.38).

$$
\begin{aligned}
\mathfrak{p}_0^1(t) &= 1-\left[\frac{\mu-\lambda}{\lambda}\right]\sum_{m=1}^{+\infty}\left(\frac{\lambda}{\mu}\right)^{m}e^{-(\mu-\lambda)mt} \qquad (4.41)\\
&= 1-\left(\frac{\mu-\lambda}{\lambda}\right)\left(\frac{1}{1-\frac{\lambda}{\mu}e^{-(\mu-\lambda)t}}-1\right)\\
&= 1-\left(\frac{\mu-\lambda}{\lambda}\right)\left[\frac{\frac{\lambda}{\mu}e^{-\lambda(\mu-\lambda)t}}{1-\frac{\lambda}{\mu}e^{-(\mu-\lambda)t}}\right]\\
&= 1+\frac{\frac{\lambda^2}{\mu}e^{-(\mu-\lambda)t}-\lambda e^{-(\mu-\lambda)t}}{\lambda-\frac{\lambda^2}{\mu}e^{-(\mu-\lambda)t}}=\frac{\lambda-\lambda e^{-(\mu-\lambda)t}}{\lambda-\frac{\lambda^2}{\mu}e^{-(\mu-\lambda)t}}\\
&= \frac{1-e^{-(\mu-\lambda)t}}{1-\frac{\lambda}{\mu}e^{-(\mu-\lambda)t}}=\frac{\mu e^{-t(\mu-\lambda)}-\mu}{\lambda e^{-t(\mu-\lambda)}-\mu}.
\end{aligned}
$$

Remark 4.2.5. *Population extinction in the long run is evident from* (4.37) *as*

$$
Pr\{\mathfrak{N}^{\nu}(t)=0\}\overset{t\to+\infty}{\longrightarrow}1 \qquad (4.42)
$$

due to the death rate exceeding the birth rate for all $0<\nu\le 1$.

In the next theorem we treat the remaining case i.e. when $\mu=\lambda$.

Theorem 4.2.4. *For the fractional linear birth process, when the rate of birth equals the rate of death (i.e. when $\lambda=\mu$), the extinction probability $\mathfrak{p}_0^{\nu}(t)$ reads*

$$
\mathfrak{p}_0^{\nu}(t)=\frac{\lambda t^{\nu}}{\nu}\int_0^{+\infty}e^{-w}E_{\nu,\nu}\left(-w\lambda t^{\nu}\right)dw=1-\int_0^{+\infty}e^{-w}E_{\nu,1}\left(-\lambda t^{\nu}w\right)dw \qquad (4.43)
$$

with $t>0$, $0<\nu\le 1$ and $E_{\nu,1}(x)$ is the Mittag–Leffler function (4.12).

Proof. Resorting again to (4.18) we write

$$
\mathfrak{p}_0^{\nu}(t)=\int_0^{+\infty}\frac{\lambda s}{1+\lambda s}\Pr\{T_{2\nu}(t)\in ds\}. \qquad (4.44)
$$

We now apply the Laplace transform once again thus obtaining

$$
L_0^{\nu}(z)=\int_0^{+\infty}\frac{\lambda s z^{\nu-1}e^{-z^{\nu}s}}{\lambda s+1}ds \qquad (4.45)
$$

$$
\begin{aligned}
&= \lambda z^{\nu-1} \int_0^{+\infty} s e^{-z^\nu s} \int_0^{+\infty} e^{-w(\lambda s+1)} dw\, ds \\
&= \lambda z^{\nu-1} \int_0^{+\infty} e^{-w} \int_0^{+\infty} s e^{-z^\nu s - w\lambda s} ds\, dw \\
&\overset{(y=s(z^\nu+\lambda w))}{=} \lambda z^{\nu-1} \int_0^{+\infty} e^{-w} \int_0^{+\infty} \frac{y}{z^\nu+\lambda w} e^{-y} \frac{dy}{z^\nu+\lambda w} dw \\
&= \lambda \int_0^{+\infty} e^{-w} \frac{1}{z^\nu+\lambda w} \cdot \frac{z^{\nu-1}}{z^\nu+\lambda w} dw.
\end{aligned}
$$

By inverting the Laplace transform we obtain the integral form

$$
\mathfrak{p}_0^\nu(t) = \lambda \int_0^{+\infty} e^{-w} \int_0^t u^{\nu-1} E_{\nu,\nu}(-w\lambda u^\nu) E_{\nu,1}\left(-w\lambda (t-u)^\nu\right) du\, dw, \tag{4.46}
$$

which involves convolutions of generalised Mittag–Leffler functions $E_{\alpha,\beta}(t)$, defined, for example in Podlubny (1999), page 17, equation (1.56). The inner integral in (4.46) can be worked out explicitly as follows

$$
\begin{aligned}
&\int_0^t u^{\nu-1} E_{\nu,\nu}(-w\lambda u^\nu) E_{\nu,1}\left(-w\lambda(t-u)^\nu\right) du \qquad (4.47)\\
&= \sum_{m=0}^\infty \sum_{r=0}^\infty \frac{(-w\lambda)^m}{\Gamma(\nu m+\nu)} \frac{(-w\lambda)^r}{\Gamma(\nu r+1)} \int_0^t u^{\nu-1} u^{\nu m} (t-u)^{\nu r} du \\
&= \sum_{m=0}^\infty \sum_{r=0}^\infty \frac{(-w\lambda)^{m+r}}{\Gamma(\nu m+\nu)\Gamma(\nu r+1)} t^{\nu+\nu(m+r)} \frac{\Gamma(\nu m+\nu)\Gamma(\nu r+1)}{\Gamma(\nu(m+r)+\nu+1)} \\
&\overset{(m+r=n)}{=} \sum_{m=0}^\infty \sum_{n=m}^\infty \frac{(-w\lambda)^n}{\Gamma(\nu n+\nu+1)} t^{\nu+\nu n} = \sum_{n=0}^\infty \frac{(-w\lambda)^n}{\Gamma(\nu n+\nu+1)} t^{\nu+\nu n}(n+1) \\
&= \frac{t^\nu}{\nu} \sum_{n=0}^\infty \frac{(-w\lambda t^\nu)^n}{\Gamma(\nu(n+1))} = \frac{t^\nu}{\nu} E_{\nu,\nu}(-w\lambda t^\nu).
\end{aligned}
$$

The extinction probability now reads

$$
\mathfrak{p}_0^\nu(t) = \frac{\lambda t^\nu}{\nu} \int_0^\infty e^{-w} E_{\nu,\nu}(-w\lambda t^\nu) dw. \tag{4.48}
$$

Using the following relation

$$
\frac{d}{dx} E_{\nu,1}(x) = \frac{1}{\nu} E_{\nu,\nu}(x), \tag{4.49}
$$

the extinction probability (4.48) takes the alternative form (4.43), because

$$
\begin{aligned}
&\mathfrak{p}_0^\nu(t) \overset{(-w\lambda t^\nu=y)}{=} -\frac{\lambda t^\nu}{\nu} \int_0^{-\infty} E_{\nu,\nu}(y) e^{\frac{y}{\lambda t^\nu}} dy \qquad (4.50)\\
&= \frac{1}{\nu} \int_{-\infty}^0 E_{\nu,\nu}(y) e^{\frac{y}{\lambda t^\nu}} dy \overset{(4.49)}{=} \int_{-\infty}^0 e^{\frac{y}{\lambda t^\nu}} \frac{d}{dy} E_{\nu,1}(y) dy \\
&= 1 - \frac{1}{\lambda t^\nu} \int_{-\infty}^0 e^{\frac{y}{\lambda t^\nu}} E_{\nu,1}(y) dy \overset{(w=-\frac{y}{\lambda t^\nu})}{=} 1 - \int_0^\infty e^{-w} E_{\nu,1}(-\lambda t^\nu w) dw.
\end{aligned}
$$

This concludes the proof of (4.43). □

Remark 4.2.6. *From* (4.43)*, when* $\nu = 1$*, again we retrieve the classical form*

$$
\mathfrak{p}_0(t) = \frac{\lambda t}{\lambda t+1}, \tag{4.51}
$$

as expected.

Remark 4.2.7. *The limiting extinction probability when $\mu = \lambda$ is*

$$Pr\{\mathfrak{N}^{\nu}(t) = 0\} \overset{t \to +\infty}{\longrightarrow} 1 \tag{4.52}$$

for all values of $0 < \nu \leq 1$.

Remark 4.2.8. *The last expression in* (4.43) *is in some way similar to the Riemann limit for* $\mu \to \lambda$ *of* (4.30) *and* (4.37).

Remark 4.2.9. *We can rewrite the probabilities* (4.11) *in an alternative form which permits us to give an interesting interpretation to their structure.*

For the case $\lambda > \mu$ *we can write*

$$\begin{aligned} \mathfrak{p}_0^{\nu}(t) &= \frac{\mu}{\lambda}\left[1 - \frac{\lambda}{\mu}\frac{\lambda-\mu}{\lambda}\sum_{m=1}^{\infty}\left(\frac{\mu}{\lambda}\right)^m E_{\nu,1}\left(-t^{\nu}(\lambda-\mu)m\right)\right] \qquad (4.53) \\ &= \frac{\mu}{\lambda}\left[1 - \sum_{m=1}^{\infty} Pr\{\mathcal{G} = m \mid \mathcal{G} \geq 1\} E_{\nu,1}\left(-t^{\nu}(\lambda-\mu)m\right)\right], \end{aligned}$$

where $\mathcal{G}$ *is a geometric r.v. with distribution*

$$Pr(\mathcal{G} = m \mid \mathcal{G} \geq 1) = \frac{Pr(\mathcal{G} = m)}{Pr(\mathcal{G} \geq 1)} = \frac{\lambda-\mu}{\lambda}\left(\frac{\mu}{\lambda}\right)^m \frac{\lambda}{\mu}, \qquad m \geq 1. \tag{4.54}$$

The treatment of the opposite case $\lambda < \mu$ *is similar except that a different conditional geometric r.v.* $\mathcal{G}'$ *must be introduced, defined as*

$$Pr(\mathcal{G}' = m \mid \mathcal{G}' \geq 1) = \frac{\mu}{\lambda}\left(\frac{\lambda}{\mu}\right)^m \frac{\mu-\lambda}{\mu}, \qquad m \geq 1, \tag{4.55}$$

and thus

$$\mathfrak{p}_0^{\nu}(t) = 1 - \sum_{m=1}^{\infty} Pr(\mathcal{G}' = m \mid \mathcal{G}' \geq 1)\, E_{\nu,1}\left(-t^{\nu}(\mu-\lambda)m\right)). \tag{4.56}$$

A well-known property for a fractional Poisson process $\mathfrak{n}_{\lambda}^{\nu}(t), t > 0$, *of degree* $0 < \nu \leq 1$ *and parameter* $\lambda > 0$ *is that (Beghin and Orsingher (2009b))*

$$Pr\{\mathfrak{N}_{\lambda}^{\nu}(t) = 0\} = E_{\nu,1}(-t^{\nu}\lambda) = Pr(\mathcal{T}_{\nu} \geq t) \tag{4.57}$$

where $\mathcal{T}_{\nu} = \inf(s\colon \mathcal{P}_{\lambda}^{\nu}(s) = 1)$. *This permits us to rewrite the extinction probabilities also in terms of waiting times of a fractional Poisson process with a random rate* $\lambda\mathcal{G}$.

For the case $\lambda = \mu$ *the interpretation is straightforward because the waiting time of the related fractional Poisson process has a rate* $\lambda\mathcal{E}$ *where* $\mathcal{E}$ *is an exponentially distributed r.v. with parameter equal to 1.*

Remark 4.2.10. *In the case* $\mu = \lambda$, *it is well-known that the extinction probability in the classical birth-death process,* $\mathfrak{p}_0(s)$, $s > 0$ *satisfies the nonlinear Riccati differential equation*

$$\mathfrak{p}_0'(s) + 2\lambda\mathfrak{p}_0(s) = \lambda + \lambda\left[\mathfrak{p}_0(s)\right]^2. \tag{4.58}$$

By using (4.58) *we can provide an alternative proof for the subordination relation* (4.18)

$$\int_0^{\infty} \mathfrak{p}_0'(s)\, Pr(T_{2\nu}(t) \in ds) + 2\lambda\mathfrak{p}_0^{\nu}(t) = \lambda + \lambda\int_0^{\infty}\left[\mathfrak{p}_0(s)\right]^2 Pr(T_{2\nu}(t) \in ds) \tag{4.59}$$

$$\Leftrightarrow \int_0^\infty \frac{\lambda Pr(T_{2\nu}(t) \in ds)}{(1+\lambda s)^2} + 2\lambda \mathfrak{p}_0^\nu(t) = \lambda + \lambda \int_0^\infty \frac{\lambda^2 s^2}{(1+\lambda s)^2} Pr(T_{2\nu}(t) \in ds)$$
$$\Leftrightarrow \int_0^\infty \lambda \frac{\left(1-\lambda^2 s^2\right)}{(1+\lambda s)^2} Pr(T_{2\nu}(t) \in ds) = \lambda - 2\lambda \mathfrak{p}_0^\nu(t)$$
$$\Leftrightarrow \int_0^\infty \frac{1-\lambda s}{1+\lambda s} Pr(T_{2\nu}(t) \in ds) = 1 - 2\mathfrak{p}_0^\nu(t)$$
$$\Leftrightarrow 2\mathfrak{p}_0^\nu(t) = 2\int_0^\infty \frac{\lambda s}{1+\lambda s} Pr(T_{2\nu}(t) \in ds)$$
$$\Leftrightarrow \mathfrak{p}_0^\nu(t) = \int_0^\infty \frac{\lambda s}{1+\lambda s} Pr(T_{2\nu}(t) \in ds). \tag{4.60}$$

Remark 4.2.11. *By exploiting the subordination relation* (4.18) *and the fact that the extinction probability in the classical case satisfies the following integral equation*

$$\mathfrak{p}_0(t) = \int_0^t e^{-(\lambda+\mu)u} \mu du + \int_0^t \lambda e^{-(\lambda+\mu)u} \left[\mathfrak{p}_0(t-u)\right]^2 du \tag{4.61}$$

we can offer an integral form for $\mathfrak{p}_0^\nu(t)$ *as*

$$\mathfrak{p}_0^\nu(t) = \int_0^{+\infty} \left\{ \int_0^s e^{-(\lambda+\mu)u} \mu du + \int_0^s \lambda e^{-(\lambda+\mu)u} \left[\mathfrak{p}_0(s-u)\right]^2 du \right\} Pr\{T_{2\nu}(t) \in ds\}. \tag{4.62}$$

We note that the first integral of (4.62) *can be worked out explicitly as follows*

$$\mu \int_0^\infty e^{-zt} \left[\int_0^\infty \int_0^s e^{-(\lambda+\mu)u} Pr\{T_{2\nu}(t) \in ds\} du \right] dt = \frac{\mu}{z} \frac{1}{\lambda+\mu+z^\nu}. \tag{4.63}$$

This can be directly inverted so as to obtain

$$\int_0^\infty \int_0^s e^{-(\lambda+\mu)u} \mu \, Pr\{T_{2\nu}(t) \in ds\} du = \mu \int_0^t w^{\nu-1} E_{\nu,\nu}\left(-(\lambda+\mu) w^\nu\right) dw \tag{4.64}$$
$$= \frac{\mu t^\nu}{\nu} \sum_{m=0}^\infty \frac{\left(-(\lambda+\mu) t^\nu\right)^m}{(m+1)\Gamma(\nu m+\nu)} = \frac{\mu}{\lambda+\mu}\left[1 - E_{\nu,1}(-(\lambda+\mu)t^\nu)\right].$$

4.3 The state probabilities

Here we present three theorems concerning the structure of the state probabilities $\Pr\{\mathfrak{N}^\nu(t) = k\}$, $t > 0$, with $0 < \nu \le 1$. Three cases must be distinguished and treated separately as in section 4.2, namely $\lambda > \mu$, $\lambda < \mu$ and $\lambda = \mu$.

Theorem 4.3.1. *For the case* $\lambda > \mu$, *the state probabilities* $\mathfrak{p}_k^\nu(t)$, $k \ge 1$, $t > 0$, $0 < \nu \le 1$ *in the fractional linear birth-death process* $\mathfrak{N}^\nu(t)$, $t > 0$, *have the following form*

$$\mathfrak{p}_k^\nu(t) \tag{4.65}$$
$$= \left(\frac{\lambda-\mu}{\lambda}\right)^2 \sum_{l=0}^\infty \binom{l+k}{l} \left(\frac{\mu}{\lambda}\right)^l \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} E_{\nu,1}\left(-(l+r+1)(\lambda-\mu) t^\nu\right).$$

Proof. By exploiting the subordination relation (4.18) and by conveniently rewriting the well-known form of the state probabilities of the classical linear birth-death process, we have that

$$\mathfrak{p}_k^\nu(t) = (\lambda-\mu)^2 \lambda^{k-1} \int_0^\infty e^{-(\lambda-\mu)s} \frac{\left(1-e^{-(\lambda-\mu)s}\right)^{k-1}}{\left(\lambda-\mu e^{-(\lambda-\mu)s}\right)^{k+1}} \Pr\left(T_{2\nu}(t) \in ds\right). \tag{4.66}$$

By applying the Laplace transform we obtain

$$\begin{aligned} L_k^\nu(z) &= (\lambda-\mu)^2 \lambda^{k-1} \int_0^\infty e^{-(\lambda-\mu)s} \frac{\left(1-e^{-(\lambda-\mu)s}\right)^{k-1}}{\left(\lambda-\mu e^{-(\lambda-\mu)s}\right)^{k+1}} z^{\nu-1} e^{-sz^\nu} ds \qquad (4.67) \\ &= (\lambda-\mu)^2 \lambda^{k-1} \sum_{l=0}^\infty \sum_{r=0}^{k-1} \binom{-(k+1)}{l} (-1)^l \left(\frac{\mu}{\lambda}\right)^l \lambda^{-(k+1)} \cdot \\ &\quad \cdot \binom{k-1}{r} (-1)^r z^{\nu-1} \int_0^\infty e^{-sz^\nu} e^{-(\lambda-\mu)sl} e^{-(\lambda-\mu)sr} e^{-(\lambda-\mu)s} ds \\ &= \left(\frac{\lambda-\mu}{\lambda}\right)^2 \sum_{l=0}^\infty \sum_{r=0}^{k-1} \binom{l+k}{l} \binom{k-1}{r} (-1)^r \left(\frac{\mu}{\lambda}\right)^l z^{\nu-1} \int_0^\infty e^{-s(z^\nu+(\lambda-\mu)(l+r+1))} ds \\ &= \left(\frac{\lambda-\mu}{\lambda}\right)^2 \sum_{l=0}^\infty \sum_{r=0}^{k-1} \binom{l+k}{l} \binom{k-1}{r} (-1)^r \left(\frac{\mu}{\lambda}\right)^l \frac{z^{\nu-1}}{z^\nu+(\lambda-\mu)(l+r+1)}, \end{aligned}$$

which can be easily inverted by using (4.34), thus obtaining (4.65). □

Remark 4.3.1. *We check that, for $\nu = 1$, formula* (4.65) *converts into the well-known distribution of the linear birth-death process, thus being its fractional extension. For $\nu = 1$ we get from* (4.65) *that*

$$\mathfrak{p}_k^1(t) = \left(\frac{\lambda-\mu}{\lambda}\right)^2 \sum_{l=0}^\infty \binom{l+k}{l} \left(\frac{\mu}{\lambda}\right)^l \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} e^{-(\lambda-\mu)t(l+r+1)}. \qquad (4.68)$$

We now observe that

$$\sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} e^{-t(\lambda-\mu)r} = \left(1-e^{-(\lambda-\mu)t}\right)^{k-1}, \qquad (4.69)$$

$$\sum_{l=0}^\infty \binom{l+k}{l} \left(\frac{\mu}{\lambda}\right)^l e^{-(\lambda-\mu)tl} = \left(1-\frac{\mu}{\lambda} e^{-(\lambda-\mu)t}\right)^{-(k+1)}, \qquad (4.70)$$

where in (4.70) *we applied the binomial expression*

$$\sum_{l=0}^\infty \binom{a+l}{l} b^l = \sum_{l=0}^\infty \binom{-(a+1)}{l} (-b)^l = (1-b)^{-(a+1)}. \qquad (4.71)$$

This permits us to write

$$\mathfrak{p}_k^1(t) = \left(\frac{\lambda-\mu}{\lambda}\right)^2 e^{-(\lambda-\mu)t} \frac{\left(1-e^{-(\lambda-\mu)t}\right)^{k-1}}{\left(1-\frac{\mu}{\lambda} e^{-(\lambda-\mu)t}\right)^{k+1}}, \qquad \mu < \lambda, \qquad (4.72)$$

which coincides with (4.4).

Remark 4.3.2. *In order to prove that $\sum_{k=0}^\infty \mathfrak{p}_k^\nu(t) = 1$ for $\lambda > \mu$ (formula* (4.65)*), we can resort again to the Laplace transform and prove that $\sum_{k=0}^\infty \int_0^\infty e^{-zt} \mathfrak{p}_k^\nu(t) dt = 1/z$. We first calculate*

$$\begin{aligned} &\sum_{k=1}^\infty \int_0^\infty e^{-zt} \mathfrak{p}_k^\nu(t) dt \qquad (4.73) \\ &= \sum_{k=1}^\infty \left(\frac{\lambda-\mu}{\lambda}\right)^2 \sum_{l=0}^\infty \sum_{r=0}^{k-1} \binom{l+k}{l} \binom{k-1}{r} (-1)^r \left(\frac{\mu}{\lambda}\right)^l \frac{z^{\nu-1}}{z^\nu+(\lambda-\mu)(l+r+1)} \end{aligned}$$

$$= \sum_{k=1}^{\infty} \left(\frac{\lambda-\mu}{\lambda}\right)^2 \sum_{l=0}^{\infty}\sum_{r=0}^{k-1} \binom{-(k+1)}{l} (-1)^l \binom{k-1}{r} (-1)^r \left(\frac{\mu}{\lambda}\right)^l z^{\nu-1}$$
$$\times \int_0^\infty e^{-sz^\nu} e^{-ls(\lambda-\mu)} e^{-sr(\lambda-\mu)} e^{-s(\lambda-\mu)} ds.$$

By keeping in mind formulae (4.69) *and* (4.70), *we have that*

$$\sum_{k=1}^{\infty} \int_0^\infty e^{-zt} \mathfrak{p}_k^\nu(t) dt = \sum_{k=1}^{\infty} (\lambda-\mu)^2 \lambda^{k-1} \int_0^\infty e^{-s(\lambda-\mu)} \frac{\left(1-e^{-s(\lambda-\mu)}\right)^{k-1}}{\left(\lambda-\mu e^{-s(\lambda-\mu)}\right)^{k+1}} z^{\nu-1} e^{-sz^\nu} ds \tag{4.74}$$
$$= (\lambda-\mu)\, z^{\nu-1} \int_0^\infty \frac{e^{-sz^\nu}}{\lambda-\mu e^{-s(\lambda-\mu)}}.$$

By using the Laplace transform of the extinction probability (second line of formula (4.33)*) we finally obtain*

$$\sum_{k=0}^{\infty} \int_0^\infty e^{-zt} p_k^\nu(t) dt \tag{4.75}$$
$$= (\lambda-\mu)\, z^{\nu-1} \int_0^\infty \frac{e^{-sz^\nu}}{\lambda-\mu e^{-s(\lambda-\mu)}} + \int_0^\infty \frac{\mu-\mu e^{-s(\lambda-\mu)}}{\lambda-\mu e^{-s(\lambda-\mu)}} z^{\nu-1} e^{-sz^\nu} ds$$
$$= \int_0^\infty z^{\nu-1} e^{-sz^\nu} ds = \frac{1}{z},$$

as desired.

Remark 4.3.3. *The distribution* (4.65) *can be expressed in terms of the probability law of a fractional linear birth process with rate* $\lambda-\mu$ *which reads*

$$p_k^\nu(t) = Pr\{N^\nu(t) = k+l \mid N^\nu(0) = l+1\} \tag{4.76}$$
$$= \binom{k+l-1}{k-1} \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} E_{\nu,1}\left(-(r+1+l)(\lambda-\mu)t^\nu\right)$$

where $l+1$ *initial progenitors are assumed (see Orsingher and Polito (2010) formula (3.59)). If we write*

$$Pr\{G=l\} = \left(1-\frac{\mu}{\lambda}\right)\left(\frac{\mu}{\lambda}\right)^l, \qquad l \geq 0, \tag{4.77}$$

then formula (4.65) *can be rewritten as*

$$\mathfrak{p}_k^\nu(t) = \left(\frac{\lambda-\mu}{\lambda}\right)^2 \sum_{l=0}^{\infty} \frac{l+k}{k} \left(\frac{\mu}{\lambda}\right)^l Pr\{N^\nu(t) = k+l \mid N^\nu(0) = l+1\} \tag{4.78}$$
$$= \frac{\lambda-\mu}{\lambda} \sum_{l=0}^{\infty} \left[\left(1+\frac{\mu}{k(\lambda-\mu)}\right) Pr(G=l) + \frac{\mu}{k}\frac{d}{d\mu} Pr(G=l)\right]$$
$$\times Pr\{N^\nu(t) = k+l \mid N^\nu(0) = l+1\},$$

because

$$\frac{\mu}{k}\frac{d}{d\mu} Pr(G=l) = \frac{l}{k}\left(1-\frac{\mu}{\lambda}\right)\left(\frac{\mu}{\lambda}\right)^l - \frac{\mu}{k(\lambda-\mu)} Pr(G=l). \tag{4.79}$$

Result (4.78) *shows that for large values of k we have the following interesting approximation*

$$\mathfrak{p}_k^\nu(t) \sim \frac{\lambda-\mu}{\lambda} \sum_{l=0}^{\infty} Pr(G=l)\, Pr\{N^\nu(t) = k+l \mid N^\nu(0) = l+1\}.$$

Theorem 4.3.2. *For a fractional linear birth-death process $\mathfrak{N}^{\nu}(t)$, $t>0$, with $\mu>\lambda$, the probabilities $\mathfrak{p}_k^{\nu}(t)=Pr\{\mathfrak{N}^{\nu}(t)=k\}$, $k\geq 1$, have the following form*

$$\begin{aligned}\mathfrak{p}_k^{\nu}(t) = {} & \left(\frac{\mu-\lambda}{\mu}\right)^2\left(\frac{\lambda}{\mu}\right)^{k-1}\sum_{l=0}^{\infty}\binom{l+k}{l}\left(\frac{\lambda}{\mu}\right)^l \\ & \times\sum_{r=0}^{k-1}(-1)^r\binom{k-1}{r}E_{\nu,1}\left(-(l+r+1)(\mu-\lambda)t^{\nu}\right).\end{aligned} \tag{4.80}$$

Proof. By resorting again to relation (4.18), in virtue of formula (4.4) suitably rearranged, we can write

$$\mathfrak{p}_k^{\nu}(t)=\int_0^{\infty}(\mu-\lambda)^2e^{-(\mu-\lambda)s}\lambda^{k-1}\frac{\left(e^{-(\mu-\lambda)s}-1\right)^{k-1}}{\left(\lambda e^{-(\mu-\lambda)s}-\mu\right)^{k+1}}\Pr\left(T_{2\nu}(t)\in ds\right). \tag{4.81}$$

By applying the Laplace transform we have (we skip here some steps similar to those of the proof of the previous theorem)

$$\begin{aligned}L_k^{\nu}(z) = {} & \int_0^{\infty}(\mu-\lambda)^2e^{-(\mu-\lambda)s}(-\lambda)^{k-1}\frac{\left(1-e^{-(\mu-\lambda)s}\right)^{k-1}}{(-\mu)^{k+1}\left(1-\frac{\lambda}{\mu}e^{-(\mu-\lambda)s}\right)^{k+1}}z^{\nu-1}e^{-sz^{\nu}}ds \\ = {} & \left(\frac{\mu-\lambda}{\mu}\right)^2\left(\frac{\lambda}{\mu}\right)^{k-1}\sum_{l=0}^{\infty}\binom{l+k}{l}\left(\frac{\lambda}{\mu}\right)^l \\ & \times\sum_{r=0}^{k-1}(-1)^r\binom{k-1}{r}z^{\nu-1}\int_0^{\infty}e^{-s(z^{\nu}+(\mu-\lambda)(l+r+1))}ds \\ = {} & \left(\frac{\mu-\lambda}{\mu}\right)^2\left(\frac{\lambda}{\mu}\right)^{k-1}\sum_{l=0}^{\infty}\binom{l+k}{l}\left(\frac{\lambda}{\mu}\right)^l \\ & \times\sum_{r=0}^{k-1}(-1)^r\binom{k-1}{r}\frac{z^{\nu-1}}{z^{\nu}+(\mu-\lambda)(l+r+1)}.\end{aligned} \tag{4.82}$$

By transforming equation (4.82) we easily arrive at the result (4.80). □

Remark 4.3.4. *When $k=1$ equation (4.65) takes a simple form*

$$\begin{aligned}\mathfrak{p}_1^{\nu}(t) &= \left(\frac{\lambda-\mu}{\lambda}\right)^2\sum_{l=0}^{\infty}(l+1)\left(\frac{\mu}{\lambda}\right)^l E_{\nu,1}\left(-(l+1)(\lambda-\mu)t^{\nu}\right) \\ &= \left(\frac{\lambda-\mu}{\lambda}\right)^2\sum_{l=1}^{\infty}l\left(\frac{\mu}{\lambda}\right)^{l-1}E_{\nu,1}\left(-l(\lambda-\mu)t^{\nu}\right).\end{aligned} \tag{4.83}$$

where $\lambda>\mu$. For the case $\lambda<\mu$ we obtain essentially the same expression with λ and μ exchanged.

An interpretation similar to that in (4.78) is valid for the case $\mu>\lambda$ as well. The following theorem describes the structure of the state probabilities $\mathfrak{p}_k^{\nu}(t)$, $k\geq 1$ in the case where $\mu=\lambda$, i.e. when the birth rate equals the death rate.

Theorem 4.3.3. *In the case $\mu=\lambda$ the probabilities $\mathfrak{p}_k^{\nu}(t)=Pr\{\mathfrak{N}^{\nu}(t)=k\}$ of the fractional linear birth-death process read*

$$Pr\{\mathfrak{N}^{\nu}(t)=k\}=\frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\left[\lambda\left(1-\mathfrak{p}_0^{\nu}(t)\right)\right], \tag{4.84}$$

with $k\geq 1$ and $t>0$.

Proof. The explicit form of the distribution $\Pr\{\mathfrak{N}^\nu(t)=k\}, k\geq 1$, for the fractional linear birth-death process, in the case $\lambda=\mu$, can be evaluated in the following manner. In light of (4.9) we have that

$$\Pr\{\mathfrak{N}^\nu(t)=k\}=\int_0^\infty \Pr\{\mathfrak{N}(s)=k\}\Pr\{T_{2\nu}(t)\in ds\} \tag{4.85}$$

so that

$$L_k^\nu(z)=\int_0^\infty e^{-zt}\Pr\{\mathfrak{N}^\nu(t)=k\}\,dt=\int_0^\infty \frac{(\lambda s)^{k-1}}{(1+\lambda s)^{k+1}}z^{\nu-1}e^{-sz^\nu}ds. \tag{4.86}$$

This is because for the $\lambda=\mu$ case of the classical birth-death process, we have that (see Bailey (1964), page 95, formula (8.53))

$$\Pr\{\mathfrak{N}(t)=k\}=\frac{(\lambda t)^{k-1}}{(1+\lambda t)^{k+1}}, \qquad k\geq 1. \tag{4.87}$$

We note that the extinction probability cannot be extracted from the above formula since it reads

$$\Pr\{\mathfrak{N}(t)=0\}=\frac{\lambda t}{1+\lambda t}. \tag{4.88}$$

This implies that we have a different expression for $k\geq 1$ and $k=0$ for the fractional linear birth-death process as well.

Formula (4.86) can be worked out as

$$\begin{aligned} L_k^\nu(z) &= \frac{(-1)^k\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\int_0^\infty \frac{1}{s(1+\lambda s)}z^{\nu-1}e^{-sz^\nu}ds \qquad (4.89)\\ &= \frac{(-1)^k\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\int_0^\infty \left(\frac{1}{s}-\frac{\lambda}{1+\lambda s}\right)z^{\nu-1}e^{-sz^\nu}ds\\ &= \frac{(-1)^k\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\left[\int_0^\infty\int_0^\infty \left(e^{-ws}-\lambda e^{-w(1+\lambda s)}\right)z^{\nu-1}e^{-sz^\nu}ds\,dw\right]\\ &= \frac{(-1)^k\lambda^{k-1}}{k!}z^{\nu-1}\frac{d^k}{d\lambda^k}\left[-\int_0^\infty \frac{\lambda e^{-w}}{w\lambda+z^\nu}dw+\int_0^\infty\frac{dw}{w+z^\nu}\right]. \end{aligned}$$

By inverting the Laplace transform we have that

$$\begin{aligned} \Pr\{\mathfrak{N}^\nu(t)=k\} &= \frac{(-1)^k\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\left[\int_0^\infty \left(E_{\nu,1}(-wt^\nu)-\lambda e^{-w}E_{\nu,1}(-\lambda wt^\nu)\right)dw\right] \qquad (4.90)\\ &= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\left[\lambda\int_0^\infty e^{-w}E_{\nu,1}(-\lambda wt^\nu)\,dw\right]\\ &= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\left[\lambda\left(1-p_0^\nu(t)\right)\right]. \end{aligned}$$

Formula (4.84) is thus proved. $\square$

It is important to note how all the state probabilities $\mathfrak{p}_k^\nu(t)$ depend on the extinction probability $\mathfrak{p}_0^\nu(t)$.

Remark 4.3.5. *For $\nu=1$ we can extract from (4.84) the classical formula (4.87) because*

$$\mathfrak{p}_k^1(t)=Pr\{\mathfrak{N}(t)=k\}=\frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\left[\frac{\lambda}{1+\lambda t}\right], \tag{4.91}$$

and by considering that

$$\frac{d^k}{d\lambda^k}\left[\frac{\lambda}{1+\lambda t}\right]=\sum_{j=0}^k\binom{k}{j}\frac{d^j}{d\lambda^j}\lambda\frac{d^{k-j}}{d\lambda^{k-j}}\left(\frac{1}{1+\lambda t}\right) \tag{4.92}$$

$$
\begin{aligned}
&= \lambda \frac{d^k}{d\lambda^k}\left(\frac{1}{1+\lambda t}\right) + k\frac{d^{k-1}}{d\lambda^{k-1}}\left(\frac{1}{1+\lambda t}\right) \\
&= \lambda \frac{(-1)^k k! t^k}{(1+\lambda t)^{k+1}} + k\frac{(-1)^{k-1}(k-1)! t^{k-1}}{(1+\lambda t)^k} \\
&= \frac{(k-1)! t^{k-1}}{(1+\lambda t)^{k+1}}(-1)^{k-1}\left[-\lambda k t + (1+\lambda t)k\right] \\
&= \frac{k! t^{k-1}}{(1+\lambda t)^{k+1}}(-1)^{k-1}.
\end{aligned}
$$

Remark 4.3.6. *From the representation on the last line of* (4.90) *it is possible to give an alternative proof of the subordination relation* (4.18) *when* $k \geq 1$ *as follows*

$$
\begin{aligned}
\mathfrak{p}_k^\nu(t) &= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\left[\lambda\left(1-\mathfrak{p}_0^\nu(t)\right)\right] && (4.93)\\
&= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\left[\lambda - \int_0^\infty \frac{\lambda^2 s}{1+\lambda s} Pr(T_{2\nu}(t)\in ds)\right] \\
&= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\left[\int_0^\infty \frac{d^k}{d\lambda^k}\left[\frac{\lambda}{1+\lambda s}\right] Pr(T_{2\nu}(t)\in ds)\right]. && (4.94)
\end{aligned}
$$

Exploiting (4.92) *we readily obtain*

$$
\mathfrak{p}_k^\nu(t) = \int_0^\infty \frac{(\lambda s)^{k-1}}{(1+\lambda s)^{k+1}} Pr(T_{2\nu}(t)\in ds). \tag{4.95}
$$

Remark 4.3.7. *Here we present two other interesting relations. The first one is simply a particular case of formula* (4.84) *when* $k=1$, *i.e. the probability of having one individual in the process at time* t *is*

$$
Pr\{\mathfrak{N}^\nu(t)=1\} = \frac{d}{d\lambda}\left[\lambda\left(1-\mathfrak{p}_0^\nu(t)\right)\right]. \tag{4.96}
$$

The second relation is again a particular case of formula (4.84) *with* $\nu = 1/2$. *In that case, recalling that*

$$
E_{\frac{1}{2},1}(x) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-y^2+2yx}dy \tag{4.97}
$$

we obtain

$$
\begin{aligned}
Pr\left\{\mathfrak{N}^{\frac{1}{2}}(t)=k\right\} &= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\left[\lambda\left(1-\mathfrak{p}_0^{\frac{1}{2}}(t)\right)\right] && (4.98)\\
&= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\left[\lambda\int_0^\infty e^{-w}E_{\frac{1}{2},1}\left(-\lambda t^{\frac{1}{2}}w\right)dw\right] \\
&= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\left[\frac{2\lambda}{\sqrt{\pi}}\int_0^\infty e^{-w}\int_0^\infty e^{-y^2-2y\lambda t^{\frac{1}{2}}w}dw\,dy\right] \\
&= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\left[\frac{2\lambda}{\sqrt{\pi}}\int_0^\infty \frac{e^{-y^2}}{1+2\lambda y\sqrt{t}}dy\right] \\
&= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\left[2\lambda\int_0^\infty \frac{e^{-\frac{w^2}{2t}}}{1+\lambda\sqrt{2}w}\frac{1}{\sqrt{2\pi t}}dw\right] \\
&= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\mathbb{E}\left[\frac{2\lambda}{1+\lambda\sqrt{2}B(t)}\right],
\end{aligned}
$$

where $B(t)$, $t>0$, *is a standard Brownian motion.*

4.4 Some further properties

The analysis of the moments of the fractional linear birth-death process gives us useful information on the behaviour of the system. Starting from (4.8) we easily see that

$$\mathbb{E}\mathfrak{N}^{\nu}(t) = \left.\frac{\partial G}{\partial u}\right|_{u=1}, \tag{4.99}$$

is the solution to

$$\begin{cases} \dfrac{d^{\nu}}{dt^{\nu}}\mathbb{E}\mathfrak{N}^{\nu} = (\lambda-\mu)\,\mathbb{E}\mathfrak{N}^{\nu}, & 0<\nu\leq 1, \\ \mathbb{E}\mathfrak{N}^{\nu}(0) = 1. \end{cases} \tag{4.100}$$

By resorting again to the Laplace transform we have that the solution to (4.100) is

$$\mathbb{E}\mathfrak{N}^{\nu}(t) = E_{\nu,1}\left((\lambda-\mu)\,t^{\nu}\right), \qquad t>0. \tag{4.101}$$

In the case $\lambda > \mu$, result (4.101) shows that the mean size of the population coincides with that of a fractional linear pure birth process with rate $(\lambda-\mu)>0$ (see Orsingher and Polito (2010)). Result (4.101) can also be derived by means of the subordination relation: (4.18)

$$\begin{aligned} \mathbb{E}\mathfrak{N}^{\nu}(t) &= \sum_{k=0}^{\infty} k \Pr\left\{\mathfrak{N}^{\nu}(t)=k\right\} \\ &= \sum_{k=0}^{\infty} k \int_0^{\infty} \Pr\left\{\mathfrak{N}(s)=k\right\} \Pr\left\{T_{2\nu}(t)\in ds\right\} \\ &= \int_0^{\infty} e^{(\lambda-\mu)s} \Pr\left\{T_{2\nu}(t)\in ds\right\}. \end{aligned} \tag{4.102}$$

The Laplace transform of (4.102) yields

$$\begin{aligned} \int_0^{\infty} e^{-zt}\mathbb{E}\mathfrak{N}^{\nu}(t)\,dt &= \int_0^{\infty} e^{(\lambda-\mu)s} z^{\nu-1} e^{-sz^{\nu}}\,ds \\ &= \frac{z^{\nu-1}}{z^{\nu}-(\lambda-\mu)} = \int_0^{\infty} e^{-zt} E_{\nu,1}\left((\lambda-\mu)\,t^{\nu}\right) dt, \end{aligned} \tag{4.103}$$

and this confirms (4.101).

By again applying (4.8) it is also possible to derive the variance $\mathbb{V}\mathrm{ar}\mathfrak{N}^{\nu}(t)$, $t>0$, of the number of individuals in the population at time t. We start by evaluating the second order factorial moment $\mu_{(2)}(t) = \mathbb{E}\left[\mathfrak{N}^{\nu}(t)\left(\mathfrak{N}^{\nu}(t)-1\right)\right]$, $t>0$. From (4.8), after some straightforward steps, we see that

$$\mu_{(2)}(t) = \mathbb{E}\left[\mathfrak{N}^{\nu}(t)\left(\mathfrak{N}^{\nu}(t)-1\right)\right] = \left.\frac{\partial^2 G}{\partial u^2}\right|_{u=1} \tag{4.104}$$

is the solution to the following differential equation

$$\begin{cases} \dfrac{d^{\nu}}{dt^{\nu}}\mu_{(2)}(t) = 2\lambda\mathbb{E}\mathfrak{N}^{\nu}(t) + 2(\lambda-\mu)\,\mu_{(2)}(t), & 0<\nu\leq 1, \\ \mu_{(2)}(0) = 0. \end{cases} \tag{4.105}$$

In order to solve (4.105) we apply the Laplace transform obtaining, in the case $\lambda\neq\mu$,

$$\begin{aligned} \int_0^{\infty} e^{-zt}\mu_{(2)}(t)\,dt &= 2\lambda\frac{z^{\nu-1}}{z^{\nu}-(\lambda-\mu)}\cdot\frac{1}{z^{\nu}-2(\lambda-\mu)} \\ &= \frac{2\lambda z^{\nu-1}}{\lambda-\mu}\left[\frac{1}{z^{\nu}-2(\lambda-\mu)} - \frac{1}{z^{\nu}-(\lambda-\mu)}\right]. \end{aligned} \tag{4.106}$$

The Laplace transform (4.106) can be inverted thus leading to the explicit expression of the second order factorial moment as

$$\mu_{(2)}(t) = \frac{2\lambda}{\lambda-\mu}\left[E_{\nu,1}\left(2(\lambda-\mu)t^{\nu}\right) - E_{\nu,1}\left((\lambda-\mu)t^{\nu}\right)\right]. \tag{4.107}$$

From the first expression of the Laplace transform in (4.106) we also have that

$$\mu_{(2)}(t) = 2\lambda\int_0^t s^{\nu-1}E_{\nu,\nu}\left(2(\lambda-\mu)s^{\nu}\right)E_{\nu,1}\left((\lambda-\mu)(t-s)^{\nu}\right)ds. \tag{4.108}$$

By applying similar calculations to those of (4.47) we prove result (4.107).

From (4.107) we can easily write that

$$\begin{aligned}\mathbb{V}\mathrm{ar}\mathfrak{N}^{\nu}(t) =& \frac{2\lambda}{\lambda-\mu}\left[E_{\nu,1}\left(2(\lambda-\mu)t^{\nu}\right) - E_{\nu,1}\left((\lambda-\mu)t^{\nu}\right)\right] + \\ &+ E_{\nu,1}\left((\lambda-\mu)t^{\nu}\right) - E^2_{\nu,1}\left((\lambda-\mu)t^{\nu}\right) \\ =& \frac{2\lambda}{\lambda-\mu}E_{\nu,1}\left(2(\lambda-\mu)t^{\nu}\right) - \frac{\lambda+\mu}{\lambda-\mu}E_{\nu,1}\left((\lambda-\mu)t^{\nu}\right) - E^2_{\nu,1}\left((\lambda-\mu)t^{\nu}\right).\end{aligned} \tag{4.109}$$

Remark 4.4.1. *When $\nu = 1$, from (4.109), we obtain the expression for the variance of the classical linear birth-death process as follows*

$$\begin{aligned}\mathbb{V}ar\mathfrak{N}(t) &= \frac{2\lambda}{\lambda-\mu}e^{2t(\lambda-\mu)} - \frac{\lambda+\mu}{\lambda-\mu}e^{t(\lambda-\mu)} - e^{2t(\lambda-\mu)} \\ &= \frac{\lambda+\mu}{\lambda-\mu}\left(e^{2t(\lambda-\mu)} - e^{t(\lambda-\mu)}\right) = \frac{\lambda+\mu}{\lambda-\mu}e^{t(\lambda-\mu)}\left(e^{t(\lambda-\mu)}-1\right).\end{aligned} \tag{4.110}$$

Remark 4.4.2. *When $\mu = 0$, that is in the case of pure linear birth, we obtain from (4.110) the expression of the variance of the classical linear pure birth process and from (4.109) that of the fractional linear birth process (see Orsingher and Polito (2010)).*

In the case $\lambda = \mu$, from (4.105), it is easy to show that

$$\mathbb{V}\mathrm{ar}\mathfrak{N}^{\nu}(t) = \frac{2\lambda t^{\nu}}{\Gamma(\nu+1)} \tag{4.111}$$

in accordance with the well-known result of the classical linear birth-death process for $\lambda = \mu$, which reads $\mathbb{V}\mathrm{ar}\mathfrak{N}(t) = 2\lambda t$.

Remark 4.4.3. *We can directly evaluate the mean value $\mathbb{E}\mathfrak{N}(t)$ for $\lambda = \mu$ in the following way:*

$$\begin{aligned}\mathbb{E}\mathfrak{N}(t) &= \sum_{k=1}^{\infty}k\left(\frac{(\lambda t)^{k-1}}{(1+\lambda t)^{k+1}}\right) = \frac{1}{(1+\lambda t)^2}\sum_{k=1}^{\infty}k\left(\frac{\lambda t}{1+\lambda t}\right)^{k-1} \\ &= \frac{1}{(1+\lambda t)^2}\frac{d}{dz}\sum_{k=1}^{\infty}z^k\Big|_{z=\frac{\lambda t}{1+\lambda t}} = \frac{1}{(1+\lambda t)^2}\frac{d}{dz}\frac{z}{1-z}\Big|_{z=\frac{\lambda t}{1+\lambda t}} \\ &= \frac{1}{(1+\lambda t)^2}\frac{1}{(1-z)^2}\Big|_{z=\frac{\lambda t}{1+\lambda t}} = 1.\end{aligned} \tag{4.112}$$

The assumption that $\lambda = \mu$ implies that the mean size of the population $\mathbb{E}\mathfrak{N}^{\nu}(t)$, $t > 0$, is equal to one (number of original progenitors) for all $t > 0$, and for all $0 < \nu \le 1$ (this is confirmed also by (4.101) for $\lambda = \mu$).

Chapter 5

Simulation and Estimation for the Fractional Linear Birth Process

Summary

In this chapter, we propose some representations of the generalised linear birth process called fractional Yule process (fYp). We also derive the probability distributions of the random birth and sojourn times. The inter-birth time distribution and the representations then yield algorithms on how to simulate sample paths of a fYp. We also attempt to estimate the model parameters. The estimation procedure is then tested using simulated data. We also illustrate some major characteristics of fYp which will be helpful in practice.

5.1 Introduction

The pure birth process is undoubtedly considered as one of the simplest branching processes. It has a Markovian structure and has already been extensively studied in the past. When the birth rate is linear, it is then usually called the pure linear birth or classical Yule or Yule–Furry process (Yp). The pure linear birth process has been introduced by McKendrick (1914), and has been widely used to model various stochastic dynamical systems such as cosmic showers, epidemics, and population growth to name a few. In finance, a birth process was recently used by Ding et al. (2009) to study the valuation and risk analysis of multiname credit derivatives. In particular, a time-changed birth process was introduced to model correlated event timing in portfolios of credit-sensitive securities such as bonds and loans.

For the sake of completeness, we review some known properties of the classical Yule process which will be used in the succeeding discussion. Let $N(t)$ be the number of individuals in a Yule process with a single initial progenitor and birth intensity $\lambda > 0$. The kth state probability or the probability of having exactly k individuals $p_k(t) = \Pr\{N(t) = k \mid N(0) = 1\}$ in a growing population at time $t > 0$ solves the following Cauchy problem:

$$\begin{cases} \frac{d}{dt} p_k(t) = -\lambda k p_k(t) + \lambda(k-1) p_{k-1}(t), & k \geq 1, \\ p_k(0) = \begin{cases} 1, & k = 1, \\ 0, & k \geq 1, \end{cases} \end{cases} \tag{5.1}$$

where $p_0(t) = 0$. The explicit solution to (5.1) is

$$p_k(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{k-1}, \qquad t > 0,\ k \geq 1, \tag{5.2}$$

with mean $\mathbb{E}N(t) = e^{\lambda t}$. To make the Yule process more flexible in taking into account more complex non-Markovian behaviour, some authors (Cahoy (2007), Orsingher and Polito (2010)) proposed a more general model called the fractional Yule process (fYp). A similar generalisation of other point processes such as the Poisson process has previously been carried out by Repin and Saichev (2000), Jumarie (2001), Laskin (2003), Wang and Wen (2003), Mainardi (1996), Wang et al. (2006), Wang et al. (2007), Mainardi et al. (2005), Cahoy (2007), Uchaikin and Sibatov (2008), Uchaikin et al. (2008) and Orsingher and Beghin (2004).

The aim of this chapter is twofold: We want to derive related representations of fYp in terms of some classical or standard processes, and we want to construct algorithms on how to simulate a fYp and estimate the parameters.

We organize the rest of the chapter as follows: Section 2 shows the fractional generalisation of the pure linear birth process. Section 3 illustrates that a pure linear birth process can also be viewed as a classical linear pure birth process with Wright-distributed random rates evaluated on a stretched time scale, i.e.,

$$N^\nu(t) \stackrel{d}{=} N_\Xi(t^\nu), \qquad \nu \in (0, 1], \tag{5.3}$$

where Ξ is a random variable having the Wright probability density function

$$W_{-\nu,1-\nu}(-\xi) = \sum_{r=0}^{\infty} \frac{(-\xi)^r}{r!(1 - \nu(r+1))}. \tag{5.4}$$

Furthermore, a representation in terms of a mixed non-homogeneous Poisson process with intensity function $\lambda(t) = \Omega\lambda\Xi e^{\lambda\Xi t}$ is derived, where Ω is a negative-exponential distributed random variable with mean 1. The following equalities in distribution:

$$N^{\nu}(t) \stackrel{d}{=} M\left[\Omega\left(e^{\lambda T_{2\nu}(t)}-1\right)\right], \qquad t>0,\ \nu\in(0,1], \tag{5.5a}$$

$$M(t) \stackrel{d}{=} N^{\nu}\left[\mathfrak{T}^{\nu}\left(\frac{1}{\lambda}\log\left(\frac{t}{\Omega}+1\right)\right)\right], \qquad t>0,\ \nu\in(0,1], \tag{5.5b}$$

are also proved, where $M(t)$ is a homogeneous Poisson process with rate 1, and $\mathfrak{T}^{\nu}(t)$ is a process with the following one-dimensional distribution (given in terms of Fox functions):

$$\Pr\left\{\mathfrak{T}^{\nu}(t)\in ds\right\} = t^{-\frac{1}{\nu}} H_{1,1}^{1,0}\left[t^{-\frac{1}{\nu}}s \,\middle|\, \begin{matrix}(1-1/\nu,1/\nu)\\(0,1)\end{matrix}\right] ds. \tag{5.6}$$

Section 4 derives the birth and inter-birth time distributions. The structural representation, fractional moments of the sojourn and birth times are also shown. The algorithms for generating sample paths of a fYp are in Section 5. Section 6 proposes an estimation procedure using the moments of the log-transformed data, and shows some empirical results. Section 7 concludes the chapter with a discussion on the key points and possible extensions of this study.

5.2 Generalisation of the Yule process

The fractional generalisation of the pure linear birth process was first carried out in Cahoy (2007), Section 8, and is described as follows: The authors defined the following difference-differential equations governing the state probabilities $\mathrm{p}_k^{\nu}(t)=\Pr\left\{\mathrm{N}^{\nu}(t)=k \mid \mathrm{N}^{\nu}(0)=1\right\}$:

$$\frac{\partial^{\nu}}{\partial t^{\nu}}\mathrm{p}_k(t) = \lambda\left[\sum_{l=1}^{k-1}\mathrm{p}_l(t)\mathrm{p}_{k-l}(t)-\mathrm{p}_k(t)\right]+\frac{t^{-\nu}}{\Gamma(1-\nu)}\delta_{k,1}, \qquad \nu\in(0,1],\ k\geq 1, \tag{5.7}$$

where the initial condition

$$\mathrm{p}_k(0)=\begin{cases}1, & k=1,\\ 0, & k>1,\end{cases} \tag{5.8}$$

is incorporated into equation (5.7) through the Kronecker delta $\delta_{k,1}$. The fractional derivative appearing in (5.7) is the so-called Riemann–Liouville operator, which is defined as

$$\begin{cases}\frac{\partial^{\nu}}{\partial t^{\nu}}\mathrm{p}_k(t) = \frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_0^t\frac{\mathrm{p}_k(s)}{(t-s)^{\nu}}ds, & \nu\in(0,1),\\ \mathrm{p}_k'(t), & \nu=1.\end{cases} \tag{5.9}$$

Furthermore, the mean number of individuals in the system was found to be

$$\mathbb{E}\left[\mathrm{N}^{\nu}(t)\right]=E_{\nu,1}(\lambda t^{\nu}), \qquad t>0,\ \nu\in(0,1], \tag{5.10}$$

where

$$E_{\alpha,\beta}(-\tau)=\sum_{r=0}^{\infty}\frac{(-\tau)^r}{\Gamma(\alpha r+\beta)}, \qquad \alpha,\beta,\tau\in\mathbb{R}^+, \tag{5.11}$$

is the generalised Mittag–Leffler function.

Let $N^{\nu}(t)$ be the number of individuals in a fractional linear birth process or fractional Yule (fYp) up to the time $t>0$. The state probabilities $p_k^{\nu}(t)=\Pr\left\{N^{\nu}(t)=k \mid N^{\nu}(0)=1\right\}$ solve the following Cauchy problem:

$$\begin{cases}\frac{d^{\nu}}{dt^{\nu}}p_k(t) = -\lambda k p_k(t)+\lambda(k-1)p_{k-1}(t), & k\geq 1,\\ p_k(0)=\begin{cases}1, & k=1,\\ 0, & k\geq 1,\end{cases}\end{cases} \tag{5.12}$$

which is a fractional generalisation of (5.1). The fractional derivative involved in (5.12) is the Caputo operator, which is defined as

$$\begin{cases} \frac{d^\nu}{dt^\nu} f(t) = \frac{1}{\Gamma(1-\nu)} \int_0^t \frac{f'(s)}{(t-s)^\nu} ds, & \nu \in (0,1), \\ f'(t), & \nu = 1. \end{cases} \tag{5.13}$$

Moreover, the Riemann–Liouville (5.9) and the Caputo (5.13) fractional derivatives are linked together by the following relation (see Kilbas et al. (2006), page 91):

$$\frac{d^\nu}{dt^\nu} f(t) = \frac{\partial^\nu}{\partial t^\nu} f(t) - \frac{f(0)}{\Gamma(1-\nu)} t^{-\nu}, \qquad \nu \in (0,1). \tag{5.14}$$

From (5.14), it is easy to see that both fractional derivatives coincide when $f(0) = 0$ for each $k > 1$. The solution to the Cauchy problem (5.12) is

$$p_k^\nu(t) = \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu), \qquad k \geq 1,\ \nu \in (0,1]. \tag{5.15}$$

Note that the mean number of individuals $\mathbb{E}\big[N^\nu(t)\big]$ in the fractional Yule process is the same as (5.10). From here on, we emphasize that the fractional derivative operation is performed in Caputo's sense.

5.3 Stretched Yule process with random rates and related representations

In this section, we present some relevant and interesting representations of the fractional Yule process (fYp). We start by proving a subordination relation that links the fractional Yule process with its classical counterpart.

Theorem 5.3.1. *Let $N^\nu(t)$ be the number of individuals in a fractional Yule process at time $t > 0$. Then the following equality in distribution holds:*

$$N^\nu(t) \overset{d}{=} N(T_{2\nu}(t)), \tag{5.16}$$

where $N(t)$ is a classical Yule process, $\nu \in (0,1]$, and $T_{2\nu}(t)$ is a random time whose distribution coincides with the solution of the following fractional diffusion equation:

$$\begin{cases} \frac{\partial^{2\nu}}{\partial t^{2\nu}} g(x,t) = \frac{\partial^2}{\partial x^2} g(x,t), & x > 0, \\ \frac{\partial}{\partial x} g(x,t)\big|_{x=0} = 0, \\ g(x,0) = \delta(x), \end{cases} \tag{5.17}$$

with the initial condition $g_t(x,0) = 0$, when $1/2 < \nu \leq 1$.

Proof. Let $G^\nu(u,t)$, $t > 0$, $|u| < 1$, be the probability generating function of the fractional Yule process. To prove (5.16), it is sufficient to observe that

$$\int_0^\infty e^{-zt} G^\nu(u,t) dt \tag{5.18}$$

$$
\begin{aligned}
&= \int_0^\infty e^{-zt} \sum_{k=1}^\infty u^k p_k^\nu(t) dt \\
&= \int_0^\infty e^{-zt} \sum_{k=1}^\infty u^k \sum_{l=1}^k \binom{k-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu) dt \\
&= \sum_{k=1}^\infty u^k \sum_{l=1}^k \binom{k-1}{l-1} (-1)^{l-1} \frac{z^{\nu-1}}{z^\nu + \lambda l} \\
&= \sum_{k=1}^\infty u^k \sum_{l=1}^k \binom{k-1}{l-1} (-1)^{l-1} z^{\nu-1} \int_0^\infty e^{-s(\lambda l + z^\nu)} ds \\
&= \int_0^\infty \sum_{k=1}^\infty u^k \sum_{l=1}^k \binom{k-1}{l-1} (-1)^{l-1} e^{-s\lambda l} z^{\nu-1} e^{-sz^\nu} ds \\
&= \int_0^\infty u^k \sum_{l=1}^k \binom{k-1}{l-1} (-1)^{l-1} e^{-\lambda l s} \int_0^\infty e^{-zt} \Pr\{T_{2\nu}(t) \in ds\} dt \\
&= \int_0^\infty e^{-zt} \left[\int_0^\infty \sum_{k=1}^\infty u^k \sum_{l=1}^k \binom{k-1}{l-1} (-1)^{l-1} e^{-\lambda l s} \Pr\{T_{2\nu}(t) \in ds\} \right] dt \\
&= \int_0^\infty e^{-zt} \left[\sum_{k=1}^\infty u^k \int_0^\infty \Pr\{N(s) = k\} \Pr\{T_{2\nu}(t) \in ds\} \right] dt \\
&= \int_0^\infty e^{-zt} \left[\sum_{k=1}^\infty u^k \Pr\{N(T_{2\nu}(t)) = k\} \right] dt.
\end{aligned}
$$

□

Remark 5.3.1. *In the proof of Theorem 5.3.1, we have taken into account the Laplace transform of $Pr\{T_{2\nu}(t) \in ds\}$ which is*

$$
\int_0^\infty e^{-zt} Pr\{T_{2\nu}(t) \in ds\} = z^{\nu-1} e^{-sz^\nu} ds, \qquad s > 0. \tag{5.19}
$$

In the next Theorem, we derive a random-rate representation of the fractional Yule process using the preceding subordination relation.

Theorem 5.3.2. *(Representation A) Let $t > 0$ and $\nu \in (0, 1]$. Then the following equality in distribution holds:*

$$
N^\nu(t) \overset{d}{=} N_\Xi(t^\nu), \tag{5.20}
$$

where $N_\Xi(t^\nu)$ is a classical linear birth process with random rate $\lambda\Xi$ evaluated at t^ν, Ξ is a Wright-distributed random variable with probability density function $W_{-\nu,1-\nu}(-\xi)$ in (5.4).

Proof. To prove equation (5.20), we use the subordination relation (5.16) as follows:

$$
\begin{aligned}
&\Pr\{N^\nu(t) = k \mid N^\nu(0) = 1\} \qquad (5.21) \\
&= \int_0^\infty \Pr\{N(s) = k \mid N(0) = 1\} \Pr\{T_{2\nu}(t) \in ds\} \\
&= \int_0^\infty \sum_{l=1}^k \binom{k-1}{l-1} (-1)^{l-1} e^{-\lambda l s} t^{-\nu} W_{-\nu,1-\nu}(-t^{-\nu} s) ds \\
&= \int_0^\infty \sum_{l=1}^k \binom{k-1}{l-1} (-1)^{l-1} e^{-\lambda l \xi t^\nu} W_{-\nu,1-\nu}(-\xi) d\xi
\end{aligned}
$$

$$= \int_0^\infty \Pr\{N_\xi(t^\nu) = k \mid N_\xi(0) = 1\} W_{-\nu,1-\nu}(-\xi) d\xi,$$

and this leads to (5.20). □

Note that in the second step of formula (5.21), we used the explicit form of the solution to the fractional diffusion equation (5.17) which is (see Podlubny (1999))

$$\Pr\{T_{2\nu}(t) \in ds\} = t^{-\nu} W_{-\nu,1-\nu}(-t^{-\nu}s) ds, \qquad s > 0. \tag{5.22}$$

Remark 5.3.2. *As noted above, representation* (5.20) *holds for the one-dimensional state probability distribution $p_k^\nu(t)$, $t > 0$, $k \geq 1$. This, however is sufficient in the sense that the process $N_\Xi(t^\nu)$ has distribution that solves* (5.1).

We now prove a further interesting representation of the fractional Yule process in terms of a specific mixed non-homogeneous Poisson process.

Starting from the second-to-last step of formula (5.21), we obtain

$$\begin{aligned} p_k^\nu(t) &= \int_0^\infty \sum_{l=1}^k \binom{k-1}{l-1} (-1)^{l-1} e^{-\lambda l \xi t^\nu} W_{-\nu,1-\nu}(-\xi) d\xi \qquad (5.23) \\ &= \int_0^\infty e^{-\lambda\xi t^\nu} \left[1 - e^{-\lambda\xi t^\nu}\right]^{k-1} W_{-\nu,1-\nu}(-\xi) d\xi \\ &= \int_0^\infty \frac{1}{\left[e^{\lambda\xi t^\nu}\right]^k} \left[e^{\lambda\xi t^\nu} - 1\right]^{k-1} W_{-\nu,1-\nu}(-\xi) d\xi. \end{aligned}$$

Recalling the identity

$$\int_0^\infty e^{-ax} x^r dx = a^{-(r+1)} r!, \qquad r \in \mathbb{N},\ \Re(a) > 0, \tag{5.24}$$

we get

$$\begin{aligned} p_k^\nu(t) &= \int_0^\infty \int_0^\infty e^{-\omega e^{\lambda\xi t^\nu}} \omega^{k-1} \frac{\left[e^{\lambda\xi t^\nu} - 1\right]^{k-1}}{(k-1)!} W_{-\nu,1-\nu}(-\xi) d\omega\, d\xi \qquad (5.25) \\ &= \int_0^\infty \int_0^\infty \frac{e^{-\omega\left[e^{\lambda\xi t^\nu} - 1\right]} \omega^{k-1} \left[e^{\lambda\xi t^\nu} - 1\right]^{k-1}}{(k-1)!} e^{-\omega} W_{-\nu,1-\nu}(-\xi) d\omega\, d\xi \\ &= \int_0^\infty \int_0^\infty \frac{e^{-\int_0^{t^\nu} \omega\lambda\xi e^{\lambda\xi s} ds} \left[\int_0^{t^\nu} \omega\lambda\xi e^{\lambda\xi s} ds\right]^{k-1}}{(k-1)!} e^{-\omega} W_{-\nu,1-\nu}(-\xi) d\omega\, d\xi. \end{aligned}$$

Thus, we have obtained a representation in terms of a mixed non-homogeneous Poisson process with intensity function

$$\lambda(t) = \Omega\lambda\Xi e^{\lambda\Xi t}, \qquad t > 0, \tag{5.26}$$

where the distribution of Ω is negative-exponential with mean equal to 1, and Ξ has probability density function (5.4). Note that the random variable Ω, conditional on $\Xi = \xi$, is such that

$$\frac{N_\xi(t^\nu)}{\mathbb{E}N_\xi(t^\nu)} \xrightarrow{\text{a.s.}} \Omega, \tag{5.27}$$

as $t \to \infty$ (see e.g. Keiding (1974), Waugh (1970), Harris (2002)).

Remark 5.3.3. *A simple change of variable also allows us to obtain a representation in terms of a mixed non-homogeneous Poisson process evaluated at the random time* $T_{2\nu}(t)$, $t > 0$. *From the second step of formula* (5.25), *we have*

$$\begin{aligned} p_k^\nu(t) &= \int_0^\infty \int_0^\infty \frac{e^{-\omega\left[e^{\lambda\xi t^\nu}-1\right]}\omega^{k-1}\left[e^{\lambda\xi t^\nu}-1\right]^{k-1}}{(k-1)!} e^{-\omega} W_{-\nu,1-\nu}(-\xi)d\omega\, d\xi \\ &= \int_0^\infty \int_0^\infty \frac{e^{-\omega\left[e^{\lambda s}-1\right]}\omega^{k-1}\left[e^{\lambda s}-1\right]^{k-1}}{(k-1)!} e^{-\omega} \frac{1}{t^\nu} W_{-\nu,1-\nu}\left(-\frac{s}{t^\nu}\right) ds\, d\omega. \end{aligned} \tag{5.28}$$

Consider a non-homogeneous Poisson process $\mathfrak{N}(t)$ *with intensity function* $\lambda(t) = \Omega\lambda e^{\lambda t}$. *Then the state probabilities of the fractional Yule process can be written as*

$$\begin{aligned} p_k^\nu(t) &= \int_0^\infty e^{-\omega} \int_0^\infty Pr\{\mathfrak{N}(s) = k-1\}\, Pr\{T_{2\nu}(t) \in ds\}\, d\omega \\ &= \mathbb{E}_\Omega \mathfrak{N}(T_{2\nu}(t)). \end{aligned} \tag{5.29}$$

In addition, the subordinated non-homogeneous Poisson process $\mathfrak{N}(T_{2\nu}(t))$ conditioned on $\Omega = \omega$ could be interesting as the fractional homogeneous Poisson process admits a similar representation (Orsingher and Beghin, 2004).

Let $\mathfrak{p}_k^\nu(t)$ be the state probabilities of $\mathfrak{N}(T_{2\nu}(t))$, i.e.,

$$\mathfrak{p}_k^\nu(t) = \Pr\left\{\mathfrak{N}(T_{2\nu}(t)) = k-1\right\}, \qquad t > 0,\ k \geq 1. \tag{5.30}$$

Then

$$\mathfrak{p}_k^\nu(t) = \int_0^\infty \frac{e^{-\omega\left[e^{\lambda s}-1\right]}\omega^{k-1}\left[e^{\lambda s}-1\right]^{k-1}}{(k-1)!} \Pr\left\{T_{2\nu}(t) \in ds\right\}. \tag{5.31}$$

Applying the Laplace transform to (5.31), we have

$$\begin{aligned} \int_0^\infty e^{-zt}\mathfrak{p}_k^\nu(t)dt &= \int_0^\infty \frac{e^{-\omega\left[e^{\lambda s}-1\right]}\omega^{k-1}\left[e^{\lambda s}-1\right]^{k-1}}{(k-1)!} z^{\nu-1}e^{-sz^\nu}ds \\ &= \int_0^\infty e^{\omega}\frac{e^{-\omega e^{\lambda s}}\omega^{k-1}\left[1-e^{\lambda s}\right]^{k-1}}{(k-1)!} (-1)^{k-1} z^{\nu-1}e^{-sz^\nu}ds, \end{aligned} \tag{5.32}$$

and by taking into account the relations

$$e^{-\omega e^{\lambda s}} = \sum_{l=0}^\infty \frac{(-\omega)^l e^{\lambda s l}}{l!}, \tag{5.33}$$

$$\left[1-e^{\lambda s}\right]^{k-1} = \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j e^{\lambda s j}, \tag{5.34}$$

we arrive at the equality

$$\begin{aligned} &\int_0^\infty e^{-zt}\mathfrak{p}_k^\nu(t)dt \\ &= \int_0^\infty \frac{e^\omega}{(k-1)!}(-1)^{k-1}\omega^{k-1}\sum_{l=0}^\infty\sum_{j=0}^{k-1}\frac{(-\omega)^l}{l!}e^{\lambda l s}\binom{k-1}{j}(-1)^j e^{\lambda s j} z^{\nu-1}e^{-sz^\nu}ds \\ &= \frac{e^\omega}{(k-1)!}(-1)^{k-1}\omega^{k-1}\sum_{l=0}^\infty \frac{(-\omega)^l}{l!}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j z^{\nu-1}\int_0^\infty e^{-s[z^\nu-\lambda(l+j)]}ds \end{aligned} \tag{5.35}$$

$$= \frac{e^{\omega}}{(k-1)!}(-\omega)^{k-1}\sum_{l=0}^{\infty}\frac{(-\omega)^l}{l!}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\frac{z^{\nu-1}}{z^{\nu}-\lambda(l+j)}.$$

Applying the inverse Laplace transform to equation (5.35), we obtain the explicit expression of the state probabilities as

$$\mathfrak{p}_k^{\nu}(t) = \frac{e^{\omega}(-\omega)^{k-1}}{(k-1)!}\sum_{l=0}^{\infty}\frac{(-\omega)^l}{l!}\sum_{j=1}^{k}\binom{k-1}{j-1}(-1)^{j-1}E_{\nu,1}\left[\lambda(l+j)t^{\nu}\right], \qquad k \geq 1. \tag{5.36}$$

Remark 5.3.4. *From equation* (5.36)*, it is straightforward to obtain the classical form of the state probabilities of the (conditional) non-homogeneous Poisson process* ($\nu = 1$) *with intensity function* $\lambda(t) = \omega\lambda e^{\lambda t}, t > 0$.

Theorem 5.3.3. *(Representation B) Let* $\mathfrak{N}(t)$ *be a homogeneous Poisson process of rate 1. The time-changed process can be defined as*

$$N^{\nu}(t) \overset{d}{=} \mathfrak{N}\left(\Omega\left[e^{\lambda T_{2\nu}(t)} - 1\right]\right), \tag{5.37}$$

where $t > 0$, $\nu \in (0,1]$, Ω *is a negative-exponential distributed random variable with parameter equal to unity, and* $T_{2\nu}(t)$ *is a random time whose one-dimensional distribution is a solution to the fractional diffusion equation* (5.17).

Proof. The proof directly follows from (5.29) and Theorem 1 of Kendall (1966). □

We introduce a definition and a lemma below which will be helpful in transforming fYp into a non-homogeneous Poisson process with rate 1.

Definition 5.3.1. *Let* $\mathfrak{T}^{\nu}(t)$ *be a random time process whose one-dimensional distribution is given by*

$$Pr\{\mathfrak{T}^{\nu}(t) \in ds\} = h(t,s)ds = t^{-\frac{1}{\nu}}H_{1,1}^{1,0}\left[t^{-\frac{1}{\nu}}s \,\middle|\, \begin{matrix}(1-1/\nu, 1/\nu)\\(0,1)\end{matrix}\right]ds, \tag{5.38}$$

where $t > 0$, $s > 0$, $\nu \in (0,1]$, *and the function*

$$H_{1,1}^{1,0}\left[t^{-\frac{1}{\nu}}s \,\middle|\, \begin{matrix}(1-1/\nu, 1/\nu)\\(0,1)\end{matrix}\right], \tag{5.39}$$

is a Fox function. Furthermore, $h(t,s)$ *has Mellin transform*

$$\int_0^{\infty} s^{\eta-1}h(t,s)ds = \frac{\Gamma(\eta)}{\Gamma\left(1-\frac{1}{\nu}+\frac{1}{\nu}\eta\right)}t^{\frac{\eta-1}{\nu}} \tag{5.40}$$

(see Mathai et al. (2010)).

Lemma 5.3.1. *Let* $N^{\nu}(t)$ *be a fractional Yule process with rate* $\lambda > 0$ *and* $t > 0$*. Then the process* $N^{\nu}(\mathfrak{T}^{\nu}(t))$ *is a classical Yule process with rate* λ.

Proof. Define $G^{\nu}(u,t)$ and $G(u,t)$, $t > 0$, $|u| \leq 1$ as the probability generating functions of fYp and the classical Yule process, respectively. Then

$$\int_0^{\infty} G^{\nu}(u,s)h(t,s)ds = \int_0^{\infty}\sum_{k=1}^{\infty}u^k\sum_{j=1}^{k}\binom{k-1}{j-1}(-1)^{j-1}E_{\nu,1}(-\lambda j s^{\nu})h(t,s)ds. \tag{5.41}$$

Using the Mellin–Barnes representation of the Mittag–Leffler function

$$E_{\nu,1}(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(z)\Gamma(1-z)}{\Gamma(1-\nu z)} x^{-z}\, dz, \tag{5.42}$$

we obtain

$$\begin{aligned} &\int_0^\infty G^\nu(u,s)h(t,s)ds \\ &= \sum_{k=1}^\infty u^k \sum_{j=1}^k \binom{k-1}{j-1} \frac{(-1)^{j-1}}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(z)\Gamma(1-z)}{\Gamma(1-\nu z)} (-\lambda j)^{-z} \int_0^\infty \frac{h(t,s)}{s^{\nu z}} ds\, dz. \end{aligned} \tag{5.43}$$

By applying formula (5.40), we can write

$$\begin{aligned} &\int_0^\infty G^\nu(u,s)h(t,s)ds \\ &= \sum_{k=1}^\infty u^k \sum_{j=1}^k \binom{k-1}{j-1} \frac{(-1)^{j-1}}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(z)\Gamma(1-z)}{\Gamma(1-\nu z)} (-\lambda j)^{-z} \frac{\Gamma(1-\nu z)}{\Gamma(1-z)} t^{-z} dz \\ &= \sum_{k=1}^\infty u^k \sum_{j=1}^k \binom{k-1}{j-1} \frac{(-1)^{j-1}}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(z)(-\lambda j t)^{-z} dz \\ &= \sum_{k=1}^\infty u^k \sum_{j=1}^k \binom{k-1}{j-1} (-1)^{j-1} e^{-\lambda j t} \\ &= \sum_{k=1}^\infty u^k e^{-\lambda t} \left[1-e^{-\lambda t}\right]^{k-1} \\ &= G(u,t). \end{aligned} \tag{5.44}$$

□

Remark 5.3.5. *Note that it is straightforward to generalize Lemma 5.3.1 to the more general (non-linear) case.*

Remark 5.3.6. *Letting $u = 1$ in (5.44), we have*

$$\begin{aligned} \sum_{k=1}^\infty \int_0^\infty p_k^\nu(s)h(t,s)ds &= \sum_{k=1}^\infty p_k(t) \\ \Leftrightarrow \int_0^\infty h(t,s)ds &= 1. \end{aligned} \tag{5.45}$$

Theorem 5.3.4. *Consider a fractional Yule process $N^\nu(t)$ with birth rate $\lambda > 0$, $t > 0$, and $\nu \in (0,1]$. Then the random time-changed process*

$$N^\nu\left[\mathfrak{T}^\nu\left(\frac{1}{\lambda}\log\left(\frac{t}{\Omega}+1\right)\right)\right] \tag{5.46}$$

has a one-dimensional distribution which coincides with that of a non-homogeneous Poisson process $\mathfrak{N}(t)$ with rate 1.

Proof. It readily follows from (5.29), Lemma 5.3.1 and Theorem 1 of Kendall (1966). □

5.4 Wait and sojourn time distributions

We now show that the sojourn or inter-birth time for fYp follows the Mittag–Leffler distribution. Let T_i^ν, $i \geq 1$, denote the time between the $(i-1)$th and ith birth. This means that T_i^ν is the time it takes for the population size to grow from i to $i+1$. More specifically, we will show that the sojourn times T_i^ν's are independent and T_i^ν is distributed

$$f_{T_i^\nu}(t) = i\lambda t^{\nu-1} E_{\nu,\nu}(-i\lambda t^\nu), \quad i \geq 1. \tag{5.47}$$

Recall that when $\nu = 1$, the inter-birth times T_i's of the Yp are independent and T_i is exponentially distributed with rate $i\lambda$, $i \geq 1$. Moreover, the waiting or birth time distribution for the pure linear birth process ($\nu = 1$) satisfies the following two equalities:

$$\Pr(\mathfrak{W}_j = T_1 + \cdots + T_j \leq t) = \Pr(N(t) \geq j+1 \big| N(0) = 1) \tag{5.48a}$$

and

$$p_j(t) = \Pr(\mathfrak{W}_{j-1} \leq t) - \Pr(\mathfrak{W}_j \leq t). \tag{5.48b}$$

Let $\mathfrak{W}_j^\nu = T_1^\nu + T_2^\nu + \cdots + T_j^\nu$ be the waiting time of the jth birth of the fYp. We now show that the preceding two equations hold true as well for the fractional or general case $(0 < \nu \leq 1)$, i.e.,

$$\Pr(\mathfrak{W}_j^\nu \leq t) = \Pr(N^\nu(t) \geq j+1 \big| N^\nu(0) = 1), \quad j \geq 1, \tag{5.49a}$$

and

$$p_j^\nu(t) = \Pr(\mathfrak{W}_{j-1}^\nu \leq t) - \Pr(\mathfrak{W}_j^\nu \leq t). \tag{5.49b}$$

Using (5.15), we obtain

$$\begin{aligned}
\Pr(N^\nu(t) \geq j+1 \big| N^\nu(0) = 1) &= \sum_{k=j+1}^{\infty} \Pr(N^\nu(t) = k \big| N^\nu(0) = 1) \\
&= 1 - \sum_{k=1}^{j} \Pr(N^\nu(t) = k \big| N^\nu(0) = 1) \\
&= 1 - \sum_{k=1}^{j} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu).
\end{aligned} \tag{5.50}$$

This implies that the jth waiting time $\mathfrak{W}_j^\nu$ has distribution

$$f_{\mathfrak{W}_j^\nu}(t) = \sum_{k=1}^{j} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} (\lambda l) t^{\nu-1} E_{\nu,\nu}(-\lambda l t^\nu), \quad t > 0,\ \nu \in (0,1]. \tag{5.51}$$

Integrating the preceding equation, we get

$$\begin{aligned}
\int_0^\infty f_{\mathfrak{W}_j^\nu}(t)dt &= \sum_{k=1}^{j} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \\
&= \sum_{k=1}^{j} \sum_{l=0}^{k-1} \binom{k-1}{l} (-1)^{l} = \sum_{k=1}^{j} (1-1)^{k-1} = 1.
\end{aligned} \tag{5.52}$$

The non-negativity of $f_{\mathfrak{W}_j^\nu}(t)$ follows from the non-negativity of $p_k^\nu(t)$ (see Orsingher and Polito (2010)), and the last line of (5.50) is a monotone increasing function of t. To see this, we can write

$$1 - \sum_{k=1}^{j} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu) \tag{5.53}$$

$$
\begin{aligned}
&= 1 - \sum_{k=1}^{j} p_k^\nu(t) \\
&= 1 - \sum_{k=1}^{j}\sum_{l=1}^{k} \binom{k-1}{l-1}(-1)^{l-1}\Pr(T_l > t) \\
&= \sum_{k=1}^{j}\sum_{l=1}^{k} \binom{k-1}{l-1}(-1)^{l-1}\Pr(T_l < t).
\end{aligned}
$$

Indeed, $f_{\mathfrak{W}_j^\nu}(t)$ is a probability density function. Note also that $f_{\mathfrak{W}_j^\nu}(t)$ has the following integral representation:

$$
f_{\mathfrak{W}_j^\nu}(t) = \frac{1}{t}\int_0^\infty e^{-\xi}\sum_{k=1}^{j}\sum_{l=1}^{k}\binom{k-1}{l-1}(-1)^{l-1} g(l\lambda t/\xi)d\xi, \tag{5.54}
$$

where $g(\eta) = \sin(\alpha\pi)/[\pi(\eta^\alpha + \eta^{-\alpha} + 2\cos(\alpha\pi))]$ (see Repin and Saichev (2000)). We now show that if the sojourn times are distributed as in (5.47), the cumulative distribution function $\Pr(\mathfrak{W}_j^\nu \le t)$ of the waiting or birth time equals the right-hand side of (5.49a). When $j = 1$, we get

$$
\Pr(\mathfrak{W}_1^\nu \le t) = \Pr(T_1^\nu \le t) = 1 - E_{\nu,1}(-\lambda t^\nu) = 1 - p_1^\nu(t). \tag{5.55}
$$

In the succeeding calculations, we use the following identities (see page 26 of Podlubny (1999)):

$$
\int_0^t E_{\nu,1}(-j\lambda(t-u)^\nu)u^{\nu-1}E_{\nu,\nu}(-\lambda l u^\nu)du = \frac{jE_{\nu,\nu+1}(-j\lambda t^\nu) - lE_{\nu,\nu+1}(-l\lambda t^\nu)}{j-l}t^\nu \tag{5.56}
$$

and

$$
E_{\nu,\nu+1}(\xi) = \frac{E_{\nu,1}(\xi) - 1}{\xi}, \qquad l \le j. \tag{5.57}
$$

Now,

$$
\begin{aligned}
\Pr(\mathfrak{W}_2^\nu \le t) &= \int_0^t \Pr\{T_1^\nu + T_2^\nu \le t \big| T_1^\nu = u\}dF_{T_1^\nu}(u) \qquad (5.58)\\
&= \int_0^t [1 - E_{\nu,1}(-2\lambda(t-u)^\nu)]\lambda u^{\nu-1}E_{\nu,\nu}(-\lambda u^\nu)du \\
&= 1 - E_{\nu,1}(-\lambda t^\nu) - [2\lambda t^\nu E_{\nu,\nu+1}(-2\lambda t^\nu) - t^\nu E_{\nu,\nu+1}(-\lambda t^\nu)] \\
&= 1 - E_{\nu,1}(-\lambda t^\nu) - [E_{\nu,1}(-\lambda t^\nu) - E_{\nu,1}(-2\lambda t^\nu)] \\
&= 1 - 2E_{\nu,1}(-\lambda t^\nu) + E_{\nu,1}(-2\lambda t^\nu) \\
&= 1 - \sum_{k=1}^{2} p_k^\nu(t),
\end{aligned}
$$

$$
\begin{aligned}
&\Pr(\mathfrak{W}_3^\nu \le t) \qquad (5.59)\\
&= \int_0^t \Pr\{T_1^\nu + T_2^\nu + T_3^\nu \le t \big| T_1^\nu + T_2^\nu = u\}dF_{T_1^\nu + T_2^\nu}(u) \\
&= \int_0^t [1 - E_{\nu,1}(-3\lambda(t-u)^\nu)][2\lambda u^{\nu-1}\{E_{\nu,\nu}(-\lambda u^\nu) - E_{\nu,\nu}(-2\lambda u^\nu)\}]du \\
&= 2[1 - E_{\nu,1}(-\lambda t^\nu))] - [1 - E_{\nu,1}(-2\lambda t^\nu)] \\
&\quad - \int_0^t E_{\nu,1}(-3\lambda(t-u)^\nu)2\lambda u^{\nu-1}E_{\nu,\nu}(-\lambda u^\nu)du \\
&\quad + \int_0^t E_{\nu,1}(-3\lambda(t-u)^\nu)2\lambda u^{\nu-1}E_{\nu,\nu}(-2\lambda u^\nu)du
\end{aligned}
$$

$$
\begin{aligned}
&= 2[1 - E_{\nu,1}(-\lambda t^{\nu})] - [1 - E_{\nu,1}(-2\lambda t^{\nu})] \\
&\quad - [3\lambda t^{\nu} E_{\nu,\nu+1}(-3\lambda t^{\nu}) - \lambda t^{\nu} E_{\nu,\nu+1}(-\lambda t^{\nu})] \\
&\quad + [3\lambda t^{\nu} E_{\nu,\nu+1}(-3\lambda t^{\nu}) - 2\lambda t^{\nu} E_{\nu,\nu+1}(-2\lambda t^{\nu})] \\
&= 2[1 - E_{\nu,1}(-\lambda t^{\nu})] - [1 - E_{\nu,1}(-2\lambda t^{\nu})] - [E_{\nu,1}(-\lambda t^{\nu}) - E_{\nu,1}(-3\lambda t^{\nu})] \\
&\quad + 2[E_{\nu,1}(-2\lambda t^{\nu}) - E_{\nu,1}(-3\lambda t^{\nu})] \\
&= 1 - 3E_{\nu,1}(-\lambda t^{\nu}) + 3E_{\nu,1}(-2\lambda t^{\nu}) - E_{\nu,1}(-3\lambda t^{\nu}) \\
&= 1 - \sum_{k=1}^{3} p_k^{\nu}(t),
\end{aligned}
$$

and in general, we can show by induction that

$$
\begin{aligned}
&\Pr(\mathfrak{W}_j^{\nu} \le t) \qquad (5.60) \\
&= \int_0^t \Pr\{\mathfrak{W}_j^{\nu} \le t \big| \mathfrak{W}_{j-1}^{\nu} = u\} dF_{\mathfrak{W}_{j-1}^{\nu}}(u) \\
&= \int_0^t [1 - E_{\nu,1}(-j\lambda(t-u)^{\nu})] f_{\mathfrak{W}_{j-1}^{\nu}}(u) du \\
&= \int_0^t [1 - E_{\nu,1}(-j\lambda(t-u)^{\nu})] \sum_{k=1}^{j-1}\sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} (\lambda l) u^{\nu-1} E_{\nu,\nu}(-\lambda l u^{\nu}) du \\
&= \sum_{k=1}^{j-1}\sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} [1 - E_{\nu,1}(-\lambda l t^{\nu})] \\
&\quad - \sum_{k=1}^{j-1}\sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \lambda l \int_0^t E_{\nu,1}(-j\lambda(t-u)^{\nu}) u^{\nu-1} E_{\nu,\nu}(-\lambda l u^{\nu}) du \\
&= \sum_{k=1}^{j-1}\sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} [1 - E_{\nu,1}(-\lambda l t^{\nu})] \\
&\quad - \sum_{k=1}^{j-1}\sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{l}{j-l} [E_{\nu,1}(-\lambda l t^{\nu}) - E_{\nu,1}(-\lambda j t^{\nu})] \\
&= 1 - \sum_{k=1}^{j-1}\sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \left(\frac{j}{j-l} E_{\nu,1}(-\lambda l t^{\nu}) - \frac{l}{j-l} E_{\nu,1}(-\lambda j t^{\nu}) \right) \\
&= 1 - \sum_{k=1}^{j-1}\sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{j}{j-l} E_{\nu,1}(-\lambda l t^{\nu}) \\
&\quad + \sum_{k=1}^{j-1}\sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{l}{j-l} E_{\nu,1}(-\lambda j t^{\nu}).
\end{aligned}
$$

Using the formulas on page 3 of Gradshteyn and Ryzhik (1980), we have

$$
\begin{aligned}
\sum_{k=1}^{j-1}\sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{l}{j-l} &= \sum_{l=1}^{j-1} (-1)^{l-1} \frac{l}{j-l} \sum_{k=l}^{j-1} \binom{k-1}{l-1} \qquad (5.61) \\
&= \sum_{l=1}^{j-1} (-1)^{l-1} \frac{l}{j-l} \sum_{k=0}^{j-1-l} \binom{k+l-1}{l-1} \\
&= \sum_{l=1}^{j-1} (-1)^{l-1} \frac{l}{j-l} \binom{j-1}{l}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{l=1}^{j-1} (-1)^{l-1} \frac{l}{j-l} \frac{(j-1)!}{l!(j-l-1)!} \\
&= \sum_{l=1}^{j-1} \frac{(j-1)!}{(l-1)!(j-l)!} \\
&= \sum_{l=0}^{j-2} (-1)^l \binom{j-1}{l} = (-1)^{j-2} \binom{j-2}{j-2} = (-1)^{j-2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\Pr(\mathfrak{W}_j^\nu \le t) \qquad (5.62)\\
&= 1 - \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{j}{j-l} E_{\nu,1}(-\lambda l t^\nu) - (-1)^{j-1} E_{\nu,1}(-\lambda j t^\nu) \\
&= 1 - \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{j}{j-l} E_{\nu,1}(-\lambda l t^\nu) \\
&\quad - \left(\sum_{l=1}^{j} \binom{j-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu) - \sum_{l=1}^{j-1} \binom{j-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu) \right) \\
&= 1 - \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{j}{j-l} E_{\nu,1}(-\lambda l t^\nu) \\
&\quad - \left(\sum_{l=1}^{j} \binom{j-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu) - \sum_{l=1}^{j-1} \binom{j-1}{l} (-1)^{l-1} \frac{l}{j-l} E_{\nu,1}(-\lambda l t^\nu) \right) \\
&= 1 - \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{j}{j-l} E_{\nu,1}(-\lambda l t^\nu) \\
&\quad - \left(\sum_{l=1}^{j} \binom{j-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu) - \sum_{l=1}^{j-1} (-1)^{l-1} \frac{l}{j-l} E_{\nu,1}(-\lambda l t^\nu) \sum_{k=l}^{j-1} \binom{k-1}{l-1} \right) \\
&= 1 - \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{j}{j-l} E_{\nu,1}(-\lambda l t^\nu) \\
&\quad - \left(\sum_{l=1}^{j} \binom{j-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu) - \sum_{l=1}^{j-1} \sum_{k=l}^{j-1} (-1)^{l-1} \frac{l}{j-l} \binom{k-1}{l-1} E_{\nu,1}(-\lambda l t^\nu) \right) \\
&= 1 - \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{j}{j-l} E_{\nu,1}(-\lambda l t^\nu) \\
&\quad - \left(\sum_{l=1}^{j} \binom{j-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu) - \sum_{l=1}^{j-1} \sum_{k=l}^{j-1} (-1)^{l-1} \frac{l}{j-l} \binom{k-1}{l-1} E_{\nu,1}(-\lambda l t^\nu) \right) \\
&= 1 - \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{j}{j-l} E_{\nu,1}(-\lambda l t^\nu) \\
&\quad - \left(\sum_{l=1}^{j} \binom{j-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu) - \sum_{l=k}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{l}{j-l} E_{\nu,1}(-\lambda l t^\nu) \right) \\
&= 1 - \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu) - \sum_{l=1}^{j} \binom{j-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu)
\end{aligned}
$$

$$= 1 - \sum_{k=1}^{j} p_k^\nu(t), \quad 1 \le k < j.$$

That is, equality (5.49a) is attained. Notice that when $\nu = 1$, we get $\Pr(\mathfrak{W}_j \le t) = (1 - e^{-\lambda t})^j$ which corresponds to the birth time distribution of the classical Yule process. Moreover, equality (5.49b) can be straightforwardly evaluated as

$$\begin{aligned} \Pr(\mathfrak{W}_{j-1}^\nu \le t) - \Pr(\mathfrak{W}_j^\nu \le t) &= \left(1 - \sum_{k=1}^{j-1}\sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu)\right) \\ &\quad - \left(1 - \sum_{k=1}^{j}\sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu)\right) \\ &= \sum_{l=1}^{j} \binom{j-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu) \\ &= p_j^\nu(t). \end{aligned} \tag{5.63}$$

In addition, the Laplace transform of the probability density $f_{T_i^\nu}(t)$ is

$$\int_0^\infty e^{-zt} f_{T_i^\nu}(t)dt = \frac{i\lambda}{i\lambda + z^\nu}. \tag{5.64}$$

This suggests that the distribution (eqn (5.47)) leads to the following known mixture or structural representation (see Cahoy et al. (2010)) of the inter-birth times as

$$T_i^\nu \stackrel{d}{=} V_i^{1/\nu} S_\nu, \tag{5.65}$$

where V_i has the exponential distribution with parameter $i\lambda$, i.e.,

$$f_{V_i}(v) = i\lambda e^{-i\lambda v}, \quad v > 0, \tag{5.66}$$

and is independent of the positive *Lévy* or ν-stable distributed random variable S_ν having the Laplace transform of the density function e^{-z^ν}. This also suggests that the qth fractional moment of the ith inter-birth time is given by

$$\mathbb{E}\left[T_i^\nu\right]^q = \frac{\pi\Gamma(1+q)}{(i\lambda)^q \Gamma(q/\nu)\sin(\pi q/\nu)\Gamma(1-q)}, \quad 0 < q < \nu, \tag{5.67}$$

which further implies that the qth fractional moment of the jth wait or birth time is

$$\mathbb{E}\left[\mathfrak{W}_j^\nu\right]^q = \frac{\pi\Gamma(1+q)}{\lambda^q \Gamma(q/\nu)\sin(\pi q/\nu)\Gamma(1-q)} \sum_{k=1}^{j}\sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \left(\frac{1}{l^q}\right), \tag{5.68}$$

where $0 < q < \nu$.

5.5 Sample paths of fYp

From Sections 2 and 3, it is now straightforward to simulate a trajectory of a fYp. However, we only propose the two simplest algorithms on how to generate a sample path of the fYp as the others follow. In particular, the random-rate representation (Representation A, Theorem 5.3.2) yields the algorithm below.

ALGORITHM 1:

i) Generate Ξ from the Wright distribution $W_{-\nu,1-\nu}(-\xi)$, and obtain ξ.

ii) Simulate a classical Yule process with birth rate $\lambda\xi$.

iii) Stretch the time scale to t^{ν}.

A simpler way to generate a realisation of fYp with n births is to directly exploit the known birth and/or sojourn time distributions as follows: Generate V_i from the exponential distribution in (5.66) with parameter $i\lambda$, and S_ν from the strictly positive stable distribution with parameter ν.

ALGORITHM 2:

i) Let $i = 1$ and $N^{\nu}(0) = 1$.

ii) Simulate $T_i^{\nu} = V_i^{1/\nu} S_\nu$, and let $\mathfrak{W}_i^{\nu} = T_1^{\nu} + T_2^{\nu} + \cdots + T_i^{\nu}$.

iii) $N^{\nu}(\mathfrak{W}_i^{\nu}) = i + 1$, and $i = i + 1$.

iv) Repeat ii-iii for $i = 2, \ldots, n-1$.

We now use the algorithms above to highlight some unique properties of the fractional Yule process that are related to its true mean given in (5.10). Figure 5.1 below shows both Yp and fYp as jump processes of size 1 in the time interval $(0, 5)$ with $\nu = 0.5$, and $\lambda = 1$. Using the same set of parameters, Figure 5.2 displays sample trajectories of a different/independent fYp and Yp which model a binary-split growth process. An important attribute that can be directly observed from these two graphs is that on the average, fYp grows more rapidly than the classical Yp shortly after it starts.

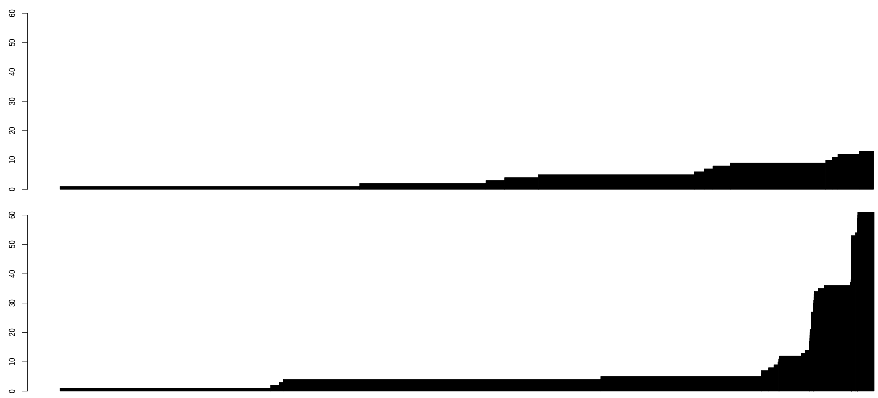

Figure 5.1: *Sample trajectories of the standard Yule process (top) and the fractional Yule process (bottom) in the interval* $(0, 5)$ *with parameters* $(\nu, \lambda) = (0.5, 1)$.

In addition, another characteristic of fYp is illustrated in Figure 5.3. The particular realisation of fYp below used the parameter values $\nu = 0.25$, $\lambda = 1$, and is observed in the time interval $(0, 5)$. It clearly suggests that fYp is more explosive than Yp when $\nu \to 0$. In general, the plots strongly validate the plausibility of fYp to model exploding and strictly growing processes. Note also that Representation A implies that the interaction between the random rate and time stretching of the classical Yule process can rapidly speed up or slow down fYp at any given time instance.

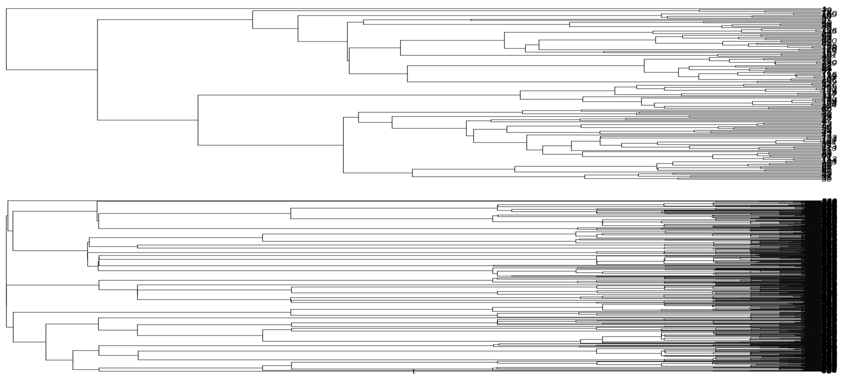

Figure 5.2: *Sample paths of the pure linear birth process (top) and the fractional Yule process (bottom) in the interval* $(0,5)$ *with parameters* $(\nu, \lambda) = (0.5, 1)$.

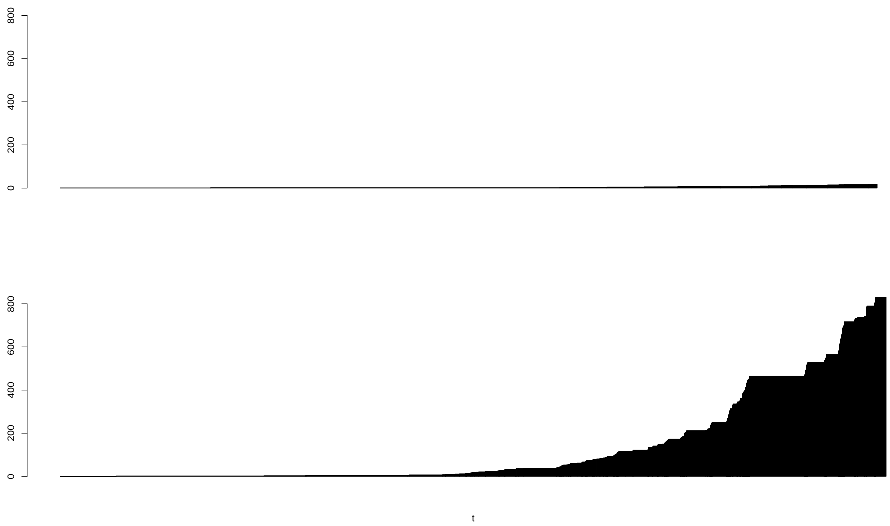

Figure 5.3: *Sample paths of the classical Yule process (top) and the fractional Yule process (bottom) in the interval* $(0,5)$ *with parameters* $(\nu, \lambda) = (0.25, 1)$.

5.6 Method-of-Moments (MoM) estimation

We now propose a method-of-moments estimation procedure for the parameters ν and λ to make fYp usable in practice. In this procedure, we assume that a particular realisation or complete history of the process is observed until the population is n, i.e., there are n births. We then attempt to use all the available data from the observed sample path of the fractional Yule process.

In particular, we use all the available inter-birth or sojourn times of the observed sample trajectory of the fractional Yule process. A direct way of estimating the parameters is to use the fractional moment estimators as follows: Choose constants $q_m < \nu, m = 1, 2$, and solve for the estimates $\hat{\lambda}$ and $\hat{\nu}$ using the equations

$$\frac{\sum_{i=1}^{n} [T_i^{\nu}]^{q_m}}{n} = \frac{\pi \Gamma(1+q_m)}{\hat{\lambda}^{q_m} \Gamma(q_m/\hat{\nu}) \sin(\pi q_m/\hat{\nu}) \Gamma(1-q_m)} \frac{\sum_{i=1}^{n} 1/i^{q_m}}{n}, \quad m = 1, 2. \tag{5.69}$$

Another approach is to use the first two integer-order moments of the log-transformed sojourn times (see Cahoy et al. (2010)) which are

$$\mathbb{E} \ln [T_i^{\nu}] = \frac{-\ln(i\lambda)}{\nu} - \mathbb{C}, \tag{5.70}$$

and

$$\mathbb{E} \ln [T_i^{\nu}]^2 = \pi^2 \left(\frac{1}{3\nu^2} - \frac{1}{6} \right) + \left(\frac{\ln(i\lambda)}{\nu} + \mathbb{C} \right)^2. \tag{5.71}$$

This further suggests that the parameter estimates can be computed using the two equations:

$$\frac{\sum_{i=1}^{n} \ln [T_i^{\nu}]}{n} = \frac{-\sum_{i=1}^{n} \ln(i\lambda)}{\nu n} - \mathbb{C}, \tag{5.72}$$

and

$$\frac{\sum_{i=1}^{n} (\ln [T_i^{\nu}])^2}{n} = \pi^2 \left(\frac{1}{3\nu^2} - \frac{1}{6} \right) + \frac{1}{n} \sum_{i=1}^{n} \left(\frac{\ln(i\lambda)}{\nu} + \mathbb{C} \right)^2, \tag{5.73}$$

where $\mathbb{C} = 0.577215664901532$ is the Euler–Mascheroni constant. A major advantage of this procedure over the fractional moment technique is that it does not require selection of constants a priori which will be dangerous in practice. Note also that the maximum likelihood estimators are more challenging to compute due to the required evaluation of the Mittag–Leffler function.

In addition, we tested our parameter estimation procedure. In doing so, we generated 10 random samples of inter-birth times of size 10000 each for $\nu = 0.1 + 0.1m$, $m = 0, \ldots, 9$ and $\lambda = 0.2, 10$. For each simulated data set, we computed the estimates using the first n observations in the set with $n = 100, 1000$, and 10000. The tables below show the simulation results for a single run, which further indicate that the proposed procedure performs relatively well. Note that these estimates could also serve as good starting values of an iterative estimation procedure.

5.7 Concluding remarks

We have derived one-dimensional representations of the fractional Yule process, which led to algorithms for simulating its sample paths. These representations are also deemed necessary in understanding the properties of fYp further. We have derived the birth and inter-birth or sojourn time distributions, which are of Mittag–Leffler type. The structural representation of the random sojourn time also led to an algorithm for simulating sample trajectories of fYp. We have also

Table 5.1: *Parameter estimates* $(\widehat{\nu}, \widehat{\lambda})$ *for* fYp *with* $\nu = 0.1(0.1)1$ *and* $\lambda = 0.2$.

	$n = 100$	$n = 1000$	$n = 10000$
$(\nu = 0.1, \lambda = 0.2)$	(0.095, 0.198)	(0.096, 0.185)	(0.100, 0.205)
$(\nu = 0.2, \lambda = 0.2)$	(0.228, 0.249)	(0.193, 0.189)	(0.199, 0.193)
$(\nu = 0.3, \lambda = 0.2)$	(0.283, 0.185)	(0.292, 0.193)	(0.303, 0.228)
$(\nu = 0.4, \lambda = 0.2)$	(0.381, 0.178)	(0.407, 0.218)	(0.402, 0.209)
$(\nu = 0.5, \lambda = 0.2)$	(0.481, 0.212)	(0.501, 0.197)	(0.500, 0.197)
$(\nu = 0.6, \lambda = 0.2)$	(0.599, 0.211)	(0.602, 0.186)	(0.595, 0.186)
$(\nu = 0.7, \lambda = 0.2)$	(0.759, 0.257)	(0.728, 0.250)	(0.700, 0.198)
$(\nu = 0.8, \lambda = 0.2)$	(0.818, 0.220)	(0.819, 0.229)	(0.803, 0.204)
$(\nu = 0.9, \lambda = 0.2)$	(0.850, 0.193)	(0.899, 0.211)	(0.907, 0.215)
$(\nu = 1.0, \lambda = 0.2)$	(0.977, 0.183)	(0.991, 0.199)	(0.999, 0.202)

Table 5.2: *Parameter estimates* $(\widehat{\nu}, \widehat{\lambda})$ *for* fYp *with* $\nu = 0.1(0.1)1$ *and* $\lambda = 10$.

	$n = 100$	$n = 1000$	$n = 10000$
$(\nu = 0.1, \lambda = 10)$	(0.107, 13.067)	(0.101, 10.599)	(0.101, 10.730)
$(\nu = 0.2, \lambda = 10)$	(0.203, 10.737)	(0.206, 12.384)	(0.201, 10.555)
$(\nu = 0.3, \lambda = 10)$	(0.299, 11.027)	(0.297, 9.359)	(0.295, 8.593)
$(\nu = 0.4, \lambda = 10)$	(0.391, 7.598)	(0.396, 8.899)	(0.397, 9.086)
$(\nu = 0.5, \lambda = 10)$	(0.517, 10.939)	(0.509, 11.428)	(0.501, 10.269)
$(\nu = 0.6, \lambda = 10)$	(0.630, 11.379)	(0.586, 8.308)	(0.597, 9.162)
$(\nu = 0.7, \lambda = 10)$	(0.716, 12.413)	(0.699, 10.634)	(0.710, 11.679)
$(\nu = 0.8, \lambda = 10)$	(0.782, 8.713)	(0.786, 8.186)	(0.804, 10.498)
$(\nu = 0.9, \lambda = 10)$	(0.919, 11.429)	(0.899, 9.043)	(0.897, 9.684)
$(\nu = 1.0, \lambda = 10)$	(0.969, 8.712)	(1.000, 10.427)	(1.001, 10.434)

proposed an estimation procedure using the moments of the log-transformed inter-birth times, which performed quite satisfactorily especially for larger populations.

Although some properties of fYp have already been studied, there are still a lot of open problems that need to be figured out. For instance, understanding fYp in more depth and the construction of more efficient estimators like the maximum likelihood would be worth pursuing in the future. Also, the application of fYp in finance is still in progress.

Chapter 6

Randomly Stopped Nonlinear Fractional Birth Processes

Summary

We present and analyse the nonlinear classical pure birth process $\mathcal{N}(t)$, $t > 0$, and the fractional pure birth process $\mathcal{N}^\nu(t)$, $t > 0$, subordinated to various random times, namely the first-passage time T_t of the standard Brownian motion $B(t)$, $t > 0$, the α-stable subordinator $\mathcal{S}^\alpha(t)$, $\alpha \in (0, 1)$, and others. For all of them we derive the state probability distribution $\hat{p}_k(t)$, $k \geq 1$ and, in some cases, we also present the corresponding governing differential equation.

We also highlight interesting interpretations for both the subordinated classical birth process $\hat{\mathcal{N}}(t)$, $t > 0$, and its fractional counterpart $\hat{\mathcal{N}}^\nu(t)$, $t > 0$ in terms of classical birth processes with random rates evaluated on a stretched or squashed time scale.

Various types of compositions of the fractional pure birth process $\mathcal{N}^\nu(t)$ have been examined in the last part of the chapter. In particular, the processes $\mathcal{N}^\nu(T_t)$, $\mathcal{N}^\nu(\mathcal{S}^\alpha(t))$, $\mathcal{N}^\nu(T_{2\nu}(t))$, have been analysed, where $T_{2\nu}(t)$, $t > 0$, is a process related to fractional diffusion equations. Also the related process $\mathcal{N}(\mathcal{S}^\alpha(T_{2\nu}(t)))$ is investigated and compared with $\mathcal{N}(T_{2\nu}(\mathcal{S}^\alpha(t))) = \mathcal{N}^\nu(\mathcal{S}^\alpha(t))$. As a byproduct of our analysis, some formulae relating Mittag–Leffler functions are obtained

6.1 Introduction

We here consider the pure birth process $\mathcal{N}(t)$, $t > 0$, (linear and nonlinear) composed with different processes like the first-passage time of Brownian motion T_t (possibly iterated n-times), the sojourn time of Brownian motion Γ_t and bridge $\mathfrak{G}_t$, and α-stable processes $\mathcal{S}^\alpha(t)$.

The subordination of processes (first introduced by Bochner (1955)) has been studied by several authors, over the years, in connection, for example, to modelling the wear of instruments during the real working time, or security trading which takes into account fluctuations of the economic activity during the time elapse t (see Lee and Whitmore (1993))).

The second part of the chapter concerns the subordination of the fractional pure birth process $\mathcal{N}^\nu(t)$, $t > 0$, $0 < \nu \leq 1$, with the processes $\mathcal{S}^\alpha(t)$ and $T_{2\alpha}(t)$, establishes that $\mathcal{N}^\nu(\mathcal{S}^\alpha(t)) = \mathcal{N}(T_{2\nu}(\mathcal{S}^\alpha(t)))$, and discuss its connection with $\mathcal{N}(\mathcal{S}^\alpha(T_{2\nu}(t)))$.

Subordinated processes connected with fractional and higher order partial differential equations are treated in numerous recent papers. Most of them concern compositions of time-continuous processes (see for example Baeumer et al. (2009)), but also point processes (Laskin (2003), Mainardi and Gorenflo (2004), Uchaikin et al. (2008), Beghin and Orsingher (2009b), Meerschaert et al. (2010)).

Birth processes stopped at different random times can be useful to model branching processes under laboratory conditions. For diseases started off artificially, the spread of the infected population can be stopped when the experiment leads to convincing conclusions. The cost of the investigation can play a certain role in stopping the artificially constructed experiment. The fluctuations of the temperature during the effective time t can influence the growth rapidity of cells or of bacteria and thus the population size can be thought as a function of the temperature modelled as a random time process. The same reasoning underlies experiments in physical studies on chain reactions. In the case of $\mathcal{N}(\Gamma_t)$, where Γ_t is the sojourn time of a Brownian motion on the positive half-line, the experiment can be interrupted immediately (if it proves useless), or at the end of the time interval $[0, t]$ (in the case that no evidence can be attained in a short time).

We recall that the distribution of the nonlinear fractional birth process (with one progenitor) reads

$$\Pr\{\mathcal{N}^\nu(t) = k \mid \mathcal{N}^\nu(0) = 1\} = \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{E_{\nu,1}(-\lambda_m t^\nu)}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)}, & k > 1, \\ E_{\nu,1}(-\lambda_1 t^\nu), & k = 1, \end{cases} \tag{6.1}$$

where

$$E_{\nu,\gamma}(x) = \sum_{h=0}^{\infty} \frac{x^h}{\Gamma(\nu h + \gamma)}, \tag{6.2}$$

is the Mittag–Leffler function and λ_k, $k \geq 1$, are the birth rates (see Orsingher and Polito (2010)).

For $\lambda_k = \lambda \cdot k$ (fractional linear birth process), formula (6.1) takes the simple form

$$\Pr\{N^\nu(t) = k \mid N^\nu(0) = 1\} = \sum_{m=1}^{k} \binom{k-1}{m-1} (-1)^{m-1} E_{\nu,1}(-\lambda m t^\nu), \qquad k \geq 1,\ t > 0. \tag{6.3}$$

For $\nu = 1$, we retrieve from (6.1) and (6.3) the classical distributions of nonlinear and linear pure birth process, by taking into account that $E_{1,1}(x) = e^x$.

The simplest subordinator considered is the first-passage time

$$T_t = \inf\{s \colon B(s) = t\}, \tag{6.4}$$

where B is a standard Brownian motion, independent of the birth process considered. For us it is relevant that the probability density of (6.4)

$$q(t,s)ds = \Pr\{T_t \in ds\}, \tag{6.5}$$

satisfies the following equation

$$\frac{\partial^2}{\partial t^2} q(t,s) = 2\frac{\partial}{\partial s} q(t,s), \qquad t>0,\ s>0, \tag{6.6}$$

as a direct check shows.

In view of (6.6) we can establish the following relation between the state probabilities

$$\hat{p}_k^\nu(t) = \Pr\{\mathcal{N}^\nu(T_t) = k\} \tag{6.7}$$

and (6.1):

$$\frac{d^2}{dt^2}\hat{p}_k^\nu(t) = -2\int_0^\infty q(t,s)\frac{d}{ds}\Pr\{\mathcal{N}^\nu(t)=k\}\,ds. \tag{6.8}$$

For $\nu = 1$, equation (6.8) becomes the second-order difference-differential equation

$$\frac{d^2}{dt^2}\hat{p}_k^\nu(t) = 2\left[\lambda_k \hat{p}_k^\nu(t) - \lambda_{k-1}\hat{p}_{k-1}^\nu(t)\right], \qquad k \geq 1. \tag{6.9}$$

Furthermore, for $\nu = 1$, the probability distribution (6.7) can be worked out explicitely and becomes

$$\hat{p}_k(t) = \begin{cases} \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^k \frac{e^{-t\sqrt{2\lambda_m}}}{\Pi_{l=1,l\neq m}^k(\lambda_l-\lambda_m)}, & k>1,\ t>0, \\ e^{-t\sqrt{2\lambda_1}}, & k=1,\ t>0. \end{cases} \tag{6.10}$$

For $0<\nu<1$, in light of the well-known integral representation of the Mittag–Leffler function

$$E_{\nu,1}(-\lambda t^\nu) = \frac{\sin\nu\pi}{\pi}\int_0^\infty \frac{r^{\nu-1}e^{-r\lambda^{\frac{1}{\nu}}t}}{r^{2\nu}+2r^\nu\cos\nu\pi+1}dr, \qquad \nu\in(0,1), \tag{6.11}$$

we obtain several different representations of the distributions of the subordinated processes.

For $\nu = 1/2$, we have the following result

$$\hat{p}_k^{\frac{1}{2}}(t) = \frac{\sqrt{2}}{\pi}\int_0^\infty \frac{1}{\left(\frac{w^2}{2}+1\right)}\Pr\{\mathcal{N}_w(t)=k \mid \mathcal{N}_w(0)=1\}\,dw, \tag{6.12}$$

which shows that $\mathcal{N}^{1/2}(T_t)$ is equivalent in distribution to a fractional pure birth process (denoted by $\mathcal{N}_W(t)$) with rates $\lambda_k \cdot W$, where W is a folded Cauchy distribution with scale parameter equal to $\sqrt{2}$.

We have also that

$$\hat{p}_k^{\frac{1}{2}}(t) = \int_0^\infty \Pr\{\mathcal{N}(s)=k\}\Pr\left\{\left|C(\sqrt{2}t)\right|\in ds\right\}. \tag{6.13}$$

In other words, $\mathcal{N}^{1/2}(T_t)$ is also equivalent in distribution to $\mathcal{N}(|C(\sqrt{2}t)|)$, C being a Cauchy process.

We generalise the previous framework by considering the iterated process

$$\tilde{\mathcal{N}}^\nu(t) = \mathcal{N}^\nu\left[T^1_{T^2_{\cdot_{\cdot_{T^n_t}}}}\right], \qquad t>0, \tag{6.14}$$

where $T_t^1, \dots, T_t^n$, are independent first-passage times and

$$T^j_{T^{j+1}_{\cdot_{\cdot_{T^n_t}}}} = \inf\left\{ s\colon B^j(s) = T^{j+1}_{T^{j+2}_{\cdot_{\cdot_{T^n_t}}}} \right\}, \qquad j = 1, \dots, (n-1), \tag{6.15}$$

where $B^j(t)$, $t > 0$, $1 \le j \le n$, are independent Brownian motions. In particular, for $\nu = 1$ we show that the state probabilities

$$\Pr\left\{ \tilde{\mathcal{N}}^1(t) = k \right\} = \tilde{p}^1_k(t), \tag{6.16}$$

satisfy the 2^nth order equations

$$\frac{d^{2^n}}{dt^{2^n}} \tilde{p}^1_k(t) = 2^{2^n-1} \left\{ \lambda_k \tilde{p}^1_k(t) - \lambda_{k-1} \tilde{p}^1_{k-1}(t) \right\}. \tag{6.17}$$

The distribution $\tilde{\mathcal{N}}^1(t) = \tilde{\mathcal{N}}(t)$ (for short) is directly derived and reads

$$\tilde{p}_k(t) = \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^k \frac{e^{-t\lambda_m^{\frac{1}{2^n}} 2^{\left(1-\frac{1}{2^n}\right)}}}{\prod_{l=1, l\neq m}^k (\lambda_l - \lambda_m)}, & k > 1, \\ e^{-t\lambda_1^{\frac{1}{2^n}} 2^{\left(1-\frac{1}{2^n}\right)}}, & k = 1. \end{cases} \tag{6.18}$$

For $n \to \infty$, we obtain from (6.18) that

$$\lim_{n\to\infty} \tilde{p}_k(t) = \begin{cases} e^{-2t}, & k = 1, \\ 0, & k > 1. \end{cases} \tag{6.19}$$

In the last part of the chapter we examine different types of compositions of the fractional pure birth process with positively skewed stable processes $\mathcal{S}^\alpha(t)$, $t > 0$, $0 < \alpha \le 1$. For $\alpha = \nu$, we show that

$$\mathcal{N}^\nu(\mathcal{S}^\nu(t)) \overset{\text{i.d.}}{=} \mathcal{N}(t\mathcal{W}_\nu), \qquad 0 < \nu < 1. \tag{6.20}$$

For the stable random variables $\mathcal{S}_1^\nu$, $\mathcal{S}_2^\nu$, it is well-known that the ratio

$$\mathcal{W}_\alpha = \left(\frac{\mathcal{S}_1^\nu}{\mathcal{S}_2^\nu} \right)^\alpha \tag{6.21}$$

(sometimes called Lamperti law), has probability density equal to

$$f_{\mathcal{W}_\alpha}(r) = \frac{\sin \nu\pi}{\alpha\pi} \frac{r^{\frac{\nu}{\alpha}-1}}{r^{2\frac{\nu}{\alpha}} + 2r^{\frac{\nu}{\alpha}} \cos \nu\pi + 1}, \qquad r > 0. \tag{6.22}$$

Furthermore, we show that $\mathcal{N}^\nu(\mathcal{S}^\nu(t)) = \mathcal{N}(T_{2\nu}(\mathcal{S}^\nu(t))) \overset{\text{i.d.}}{\neq} \mathcal{N}(\mathcal{S}^\nu(T_{2\nu}(t)))$. We are also able to prove that

$$\mathcal{N}^\nu(T_{2\alpha}(t)) = \mathcal{N}(T_{2\nu}(T_{2\alpha}(t))) \overset{\text{i.d.}}{=} \mathcal{N}(T_{2\nu\alpha}(t)) = \mathcal{N}^{\nu\alpha}(t). \tag{6.23}$$

As a byproduct of our analysis we obtain the following integral relation between Mittag–Leffler functions of different indices:

$$\begin{aligned} E_{\nu\alpha,1}(-\lambda_m t^{\nu\alpha}) &= \frac{\sin\nu\pi}{\pi} \int_0^\infty \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos\nu\pi + 1} E_{\alpha,1}\left(-r\lambda_m^{\frac{1}{\nu}} t^\alpha\right) dr \\ &= \frac{\sin\alpha\pi}{\pi} \int_0^\infty \frac{r^{\alpha-1}}{r^{2\alpha} + 2r^\alpha \cos\alpha\pi + 1} E_{\nu,1}\left(-r\lambda_m^{\frac{1}{\alpha}} t^\nu\right) dr, \qquad 0 < \alpha, \nu \le 1. \end{aligned} \tag{6.24}$$

6.2 Subordinated nonlinear birth processes

In this section we study in detail the nonlinear pure birth process stopped at T_t and we derive the state probabilities $\hat{p} = \Pr\{\mathcal{N}(T_t) = k \mid \mathcal{N}(0) = 1\}$, $k \geq 1$, and the corresponding governing differential equations.

We give some information about the process $\mathcal{N}(t)$, $t > 0$, evaluate explicitly its mean value $\mathbb{E}\mathcal{N}(t)$, and discuss also the linear birth process (sometimes referred as Yule–Furry process).

6.2.1 Preliminaries

The state probabilities $p_k(t) = \Pr\{\mathcal{N}(t) = k \mid \mathcal{N}(0) = 1\}$ read (see e.g. Gikhman and Skorokhod (1996), page 322)

$$p_k(t) = \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{e^{-\lambda_m t}}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)}, & k > 1,\ t > 0, \\ e^{-\lambda_1 t}, & k = 1,\ t > 0. \end{cases} \tag{6.25}$$

For the case of n_0 progenitors (see Chiang (1968), page 51), formula (6.25) must be replaced by

$$p_k(t) = \begin{cases} \prod_{j=n_0}^{k-1} \lambda_j \sum_{m=n_0}^{k} \frac{e^{-\lambda_m t}}{\prod_{l=n_0, l\neq m}^{k} (\lambda_l - \lambda_m)}, & k > n_0,\ t > 0, \\ e^{-\lambda_{n_0} t}, & k = n_0,\ t > 0. \end{cases} \tag{6.26}$$

We assume that $\sum_k 1/\lambda_k = \infty$ in such a way that the process is non-exploding (see Feller (1968), page 452). For a discussion on this point, consult Grimmett and Stirzaker (2001), page 252. The probabilities (6.25) satisfy the following difference-differential equations:

$$\frac{d}{dt} p_k(t) = -\lambda_k p_k(t) + \lambda_{k-1} p_{k-1}(t), \qquad k \geq 1. \tag{6.27}$$

We have our first result in the next theorem.

Theorem 6.2.1. *The mean value of the nonlinear birth process is*

$$\mathbb{E}\mathcal{N}(t) = 1 + \sum_{k=1}^{\infty} \left\{ 1 - \sum_{m=1}^{k} \prod_{l=1, l\neq m}^{k} \frac{\lambda_l}{\lambda_l - \lambda_m} e^{-\lambda_m t} \right\}. \tag{6.28}$$

Proof. From equation (6.27), we have that

$$\begin{aligned} \sum_{k=1}^{\infty} k \frac{d}{dt} p_k(t) &= -\sum_{k=1}^{\infty} k\lambda_k p_k(t) + \sum_{k=2}^{\infty} k \lambda_{k-1} p_{k-1} \\ &= \sum_{k=1}^{\infty} \lambda_k p_k(t). \end{aligned} \tag{6.29}$$

By integrating both members in $(0, t)$, we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} k p_k(t) - 1 &= \sum_{k=1}^{\infty} \lambda_k \int_0^t p_k(s) ds \\ &= \lambda_1 \int_0^t p_1(s) ds + \sum_{k=2}^{\infty} \lambda_k \left\{ \int_0^t \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{e^{-\lambda_m s}}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} ds \right\} \end{aligned} \tag{6.30}$$

$$
\begin{aligned}
&= 1 - e^{-\lambda_1 t} + \sum_{k=2}^{\infty} \left(\prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{\left(1 - e^{-\lambda_m t}\right)}{\lambda_m \prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} \right) \\
&= 1 - e^{-\lambda_1 t} + \sum_{k=2}^{\infty} \sum_{m=1}^{k} \prod_{l=1, l\neq m}^{k} \frac{\lambda_l}{\lambda_l - \lambda_m} \left(1 - e^{-\lambda_m t}\right) \\
&= 1 - e^{-\lambda_1 t} + \sum_{k=2}^{\infty} \left\{ 1 - \sum_{m=1}^{k} \prod_{l=1, l\neq m}^{k} \frac{\lambda_l}{\lambda_l - \lambda_m} e^{-\lambda_m t} \right\} \\
&= \sum_{k=1}^{\infty} \left\{ 1 - \sum_{m=1}^{k} \prod_{l=1, l\neq m}^{k} \frac{\lambda_l}{\lambda_l - \lambda_m} e^{-\lambda_m t} \right\},
\end{aligned}
$$

so that formula (6.28) emerges. In the second-to-last step of (6.30), we applied formula (3.12) of Orsingher et al. (2010) and, in the last step, we considered that, for $k = 1$, the set of numbers $\{1 \leq l \leq 1,\, l \neq m = 1\}$, is empty and the

$$
\prod_{l=1, l\neq m}^{k} \frac{\lambda_l}{\lambda_l - \lambda_m}, \tag{6.31}
$$

is taken equal to 1 by convention. □

Remark 6.2.1. *As a check we can extract, from* (6.28)*, the mean value in the linear case* $\lambda_m = m \cdot \lambda$. *Since*

$$
\begin{aligned}
&\sum_{m=1}^{k} \prod_{l=1, l\neq m}^{k} \frac{\lambda l}{\lambda l - \lambda m} e^{-\lambda m t} \qquad (6.32)\\
&= \sum_{m=1}^{k} \frac{1 \dots (m-1)(m+1) \dots k}{(m-1)(m-2) \dots 1 \cdot (-1)^{m-1} \cdot 1 \dots (k-m)} e^{-\lambda m t} \\
&= -\sum_{m=1}^{k} \frac{k!}{m!(k-m)!} (-1)^m e^{-\lambda m t} \\
&= -\sum_{m=1}^{k} \binom{k}{m} (-1)^m e^{-\lambda m t} \\
&= -\left[(1 - e^{-\lambda t})^k - 1 \right],
\end{aligned}
$$

we have that

$$
1 - \sum_{m=1}^{k} \prod_{l=1, l\neq m}^{k} \frac{\lambda l}{\lambda l - \lambda m} e^{-\lambda m t} = (1 - e^{-\lambda t})^k. \tag{6.33}
$$

From this we readily have that

$$
\begin{aligned}
\mathbb{E} N(t) &= 1 + \sum_{k=1}^{\infty} (1 - e^{-\lambda t})^k \qquad (6.34)\\
&= 1 + \frac{1}{1 - (1 - e^{-\lambda t})} - 1 = e^{\lambda t}.
\end{aligned}
$$

The aim of this section is to compose the growth process $\mathcal{N}(t)$ with the first-passage time $T_t = \inf(s\colon B(s) = t)$, where B is a Brownian motion independent of $\mathcal{N}(t)$.

Remark 6.2.2. *The probability density of* $T_t = \inf\{s\colon B(s) = t\}$, $t > 0$, *where* $B(t)$ *is a standard Brownian motion, namely*

$$
Pr\{T_t \in ds\}/ds = q(t,s) = t \frac{e^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}}, \tag{6.35}
$$

is the solution to the Cauchy problem

$$\begin{cases} \frac{\partial^2}{\partial t^2} q(t,s) = 2\frac{\partial}{\partial s} q(t,s), & t > 0,\ s > 0, \\ q(0,s) = \delta(s), \end{cases} \tag{6.36}$$

as a simple check shows.

6.2.2 Pure birth process stopped at T_t

Theorem 6.2.2. *Let $\mathcal{N}(t)$, $t > 0$ be a classical nonlinear pure birth process and let $q(t,s)$, $s > 0$, $t > 0$, the law of T_t. The process $\hat{\mathcal{N}}(t) = \mathcal{N}(T_t)$, $t > 0$, has the following distribution*

$$\hat{p}_k(t) = \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{e^{-t\sqrt{2\lambda_m}}}{\prod_{l=1,l\neq m}^{k} (\lambda_l - \lambda_m)}, & k > 1,\ t > 0, \\ e^{-t\sqrt{2\lambda_1}}, & k = 1,\ t > 0, \end{cases} \tag{6.37}$$

and mean value equal to

$$\mathbb{E}\mathcal{N}(T_t) = 1 + \sum_{k=1}^{\infty} \left(1 - \sum_{m=1}^{k} \prod_{l=1,l\neq m}^{k} \frac{\lambda_l}{\lambda_l - \lambda_m} e^{-t\sqrt{2\lambda_m}} \right). \tag{6.38}$$

The distribution (6.37) *is non-exploding under the condition that $\sum_k 1/\lambda_k = \infty$.*

Proof. The state probabilities are derived by straight calculations and by resorting to the Laplace transform of $q(t,s)$ which reads

$$\int_0^\infty e^{-\gamma s} q(t,s) ds = \int_0^\infty e^{-\gamma s} \frac{t e^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds = e^{-t\sqrt{2\gamma}}. \tag{6.39}$$

We treat the case $k > 1$ as follows. The case $k = 1$ is analogous.

$$\begin{aligned} \hat{p}_k(t) &= \Pr\{\mathcal{N}(T_t) = k \mid \mathcal{N}(0) = 1\} = \int_0^\infty p_k(s) q(t,s) ds \qquad (6.40) \\ &= \int_0^\infty \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{e^{-\lambda_m s}}{\prod_{l=1,l\neq m}^{k} (\lambda_l - \lambda_m)} \frac{t e^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds \\ &= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k} (\lambda_l - \lambda_m)} e^{-t\sqrt{2\lambda_m}}. \end{aligned}$$

In view of Theorem 6.2.1, we can evaluate the mean value

$$\begin{aligned} \mathbb{E}\mathcal{N}(T_t) &= \int_0^\infty \mathbb{E}\mathcal{N}(s) \frac{t e^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds \qquad (6.41) \\ &= 1 + \sum_{k=1}^{\infty} \left(1 - \sum_{m=1}^{k} \prod_{l=1,l\neq m}^{k} \frac{\lambda_l}{\lambda_l - \lambda_m} e^{-t\sqrt{2\lambda_m}} \right). \end{aligned}$$

□

In the linear case (6.38) can be written as

$$\mathbb{E}\mathcal{N}(T_t) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m e^{-t\sqrt{2\lambda m}}. \tag{6.42}$$

On the other side, this sum diverges because

$$\mathbb{E}\mathcal{N}(T_t) = \int_0^\infty e^{\lambda s} \Pr\{T_t \in ds\} = \infty. \tag{6.43}$$

Remark 6.2.3. *Note that $\forall t$, $\hat{p}_k(t)$, $k \geq 1$ is a proper probability distribution because of the composition $\hat{\mathcal{N}}(t) = \mathcal{N}(T_t)$. The process can be appropriately interpreted by rewriting* (6.40) *as follows*

$$\begin{aligned}\hat{p}_k(t) &= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k} (\lambda_l - \lambda_m)} \int_0^\infty e^{-\lambda_m s} \frac{t e^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds \\ &= \int_0^\infty \prod_{j=1}^{k-1} \vartheta\lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k} (\vartheta\lambda_l - \vartheta\lambda_m)} e^{-\lambda_m \vartheta t^2} \frac{e^{-\frac{1}{2\vartheta}}}{\sqrt{2\pi\vartheta^3}} d\vartheta.\end{aligned} \tag{6.44}$$

The process $\hat{\mathcal{N}}(t)$, $t > 0$ can be viewed as a classical nonlinear pure birth process evaluated at time t^2 with random birth rates $\Theta\lambda_k$, $k \geq 1$, where Θ is an inverse Gaussian random variable with p.d.f.

$$f_\Theta(\vartheta) = \frac{e^{-\frac{1}{2\vartheta}}}{\sqrt{2\pi\vartheta^3}}, \qquad \vartheta \in \mathbb{R}^+. \tag{6.45}$$

The composition of $\mathcal{N}(t)$, $t > 0$, with T_t leads to a second-order time derivative in the governing equations, as shown in the next theorem.

Theorem 6.2.3. *Let $\hat{p}_k(t)$, $t > 0$, $k \geq 1$, be the distribution of the process $\hat{\mathcal{N}}(t) = \mathcal{N}(T_t)$, $t > 0$, where T_t is the first-passage time process of the standard Brownian motion, having transition density $q(t,s)$, $s > 0$, $t > 0$. The state probabilities $\hat{p}_k(t)$, $t > 0$, $k \geq 1$, satisfy the following difference-differential equations*

$$\frac{d^2}{dt^2}\hat{p}_k(t) = 2\left[\lambda_k \hat{p}_k(t) - \lambda_{k-1}\hat{p}_{k-1}(t)\right], \qquad k \geq 1, \tag{6.46}$$

where λ_k, $k \geq 1$ are the birth rates of the nonlinear classical birth process $\mathcal{N}(t)$, $t > 0$.

Proof. Since

$$\hat{p}_k(t) = \int_0^\infty p_k(s) q(t,s) ds, \tag{6.47}$$

by taking the second-order derivative w.r.t. t, in view of Remark 6.2.2, we have that

$$\begin{aligned}\frac{d^2}{dt^2}\hat{p}_k(t) &= 2\int_0^\infty p_k(s) \frac{\partial}{\partial s} q(t,s) ds \\ &= 2q(t,s)p_k(s)\big|_{s=0}^{s=\infty} - 2\int_0^\infty \frac{d}{ds}p_k(s) q(t,s) ds \\ &= -2\int_0^\infty q(t,s)\left[-\lambda_k p_k(s) + \lambda_{k-1}p_{k-1}(s)\right] ds \\ &= 2\left[\lambda_k \hat{p}_k(t) - \lambda_{k-1}\hat{p}_{k-1}(t)\right] ds.\end{aligned} \tag{6.48}$$

In (6.48), we considered that $p_k(0) = 0$, for $k > 1$. □

Remark 6.2.4. *In the linear case, some calculations suffice to show that*

$$\hat{p}_k(t) = Pr\{N(T_t) = k\} = \sum_{m=1}^{k} \binom{k-1}{m-1} (-1)^{m-1} e^{-t\sqrt{2\lambda m}}, \qquad k \geq 1,\ t > 0, \tag{6.49}$$

and the state probabilities satisfy the equation

$$\frac{d^2}{dt^2}\hat{p}_k(t) = 2\lambda\hat{p}_k(t) - 2\lambda(k-1)\hat{p}_{k-1}(t), \qquad k \geq 1. \tag{6.50}$$

Iterated compositions

Theorem 6.2.4. *Let $\mathcal{N}(t)$, $t>0$, be a classical nonlinear birth process. Let $T_t^1, T_t^2, \dots, T_t^n$, $n \in \mathbb{N}$, be first-passage times of n independent standard Brownian motions. The process*

$$\tilde{\mathcal{N}}(t) = \mathcal{N}\left[T^1_{T^2_{\cdot_{\cdot_{T^n_t}}}}\right], \qquad t>0, \tag{6.51}$$

has the following distribution

$$\tilde{p}_k(t) = \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^k \frac{e^{-t\lambda_m^{\frac{1}{2^n}} 2^{\left(1-\frac{1}{2^n}\right)}}}{\prod_{l=1, l\neq m}^k (\lambda_l-\lambda_m)}, & k>1,\ t>0, \\ e^{-t\lambda_1^{\frac{1}{2^n}} 2^{\left(1-\frac{1}{2^n}\right)}}, & k=1,\ t>0. \end{cases} \tag{6.52}$$

Proof. We start by proving the case $n=2$ since the case $n=1$ is already proved in Theorem 6.2.2. We omit the details for the case $k=1$ and directly treat the case $k \geq 2$. We have that

$$\begin{aligned} \int_0^\infty \hat{p}_k(t) q(t,s) ds &= \int_0^\infty \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^k \frac{e^{-s\sqrt{2\lambda_m}}}{\prod_{l=1, l\neq m}^k (\lambda_l-\lambda_m)} \frac{te^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds \\ &= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^k \frac{1}{\prod_{l=1, l\neq m}^k (\lambda_l-\lambda_m)} \int_0^\infty e^{-s\sqrt{2\lambda_m}} \frac{te^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds \\ &= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^k \frac{1}{\prod_{l=1, l\neq m}^k (\lambda_l-\lambda_m)} e^{-t\sqrt{2\sqrt{2\lambda_m}}}. \end{aligned} \tag{6.53}$$

It is now straightforward to generalise formula (6.53) for n compositions, as follows

$$\begin{aligned} \tilde{p}_k(t) &= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^k \frac{1}{\prod_{l=1, l\neq m}^k (\lambda_l-\lambda_m)} e^{-t\lambda_m^{\frac{1}{2^n}} 2^{\sum_{i=1}^n \frac{1}{2^i}}} \\ &= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^k \frac{1}{\prod_{l=1, l\neq m}^k (\lambda_l-\lambda_m)} e^{-t\lambda_m^{\frac{1}{2^n}} 2^{\left(1-\frac{1}{2^n}\right)}}. \end{aligned} \tag{6.54}$$

□

When $n \to \infty$, equation (6.52) becomes

$$\lim_{n\to\infty} \tilde{p}_k(t) = \begin{cases} e^{-2t} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^k \frac{1}{\prod_{l=1, l\neq m}^k (\lambda_l-\lambda_m)} = 0, & k>1, \\ e^{-2t}, & k=1, \end{cases} \tag{6.55}$$

because of formula (3.4), page 51 of Chiang (1968). Therefore, the process (6.51) can either assume the state $k=1$ with probability e^{-2t}, or explode with probability $1-e^{-2t}$.

Theorem 6.2.5. *Let $\tilde{p}_k(t)$, $t>0$, $k \geq 1$, be the distribution of the process*

$$\tilde{\mathcal{N}}(t) = \mathcal{N}\left[T^1_{T^2_{\cdot_{\cdot_{T^n_t}}}}\right], \qquad t>0. \tag{6.56}$$

The state probabilities $\tilde{p}_k(t)$, $t>0$, $k\geq 1$, satisfy the following difference-differential equations

$$\frac{d^{2^n}}{dt^{2^n}} \tilde{p}_k(t) = 2^{2^n-1} \left\{ \lambda_k \tilde{p}_k(t) - \lambda_{k-1}\tilde{p}_{k-1}(t) \right\}, \tag{6.57}$$

where λ_k, $k \geq 1$, are the birth rates of the nonlinear classical birth process $\mathcal{N}(t)$, $t>0$.

Proof. For $n = 1$, equations (6.57) reduce to equations (6.46). For $n = 2$ we have that

$$\begin{aligned}
\frac{d^4}{dt^4}\tilde{p}_k(t) &= \int_0^\infty \int_0^\infty \hat{p}_k(w_1) q(w_2, w_1) \frac{\partial^4}{\partial t^4} q(t, w_2) dw_1\, dw_2 \qquad (6.58)\\
&= 2^2 \int_0^\infty \int_0^\infty \hat{p}_k(w_1) \frac{\partial^2}{\partial w_2^2} q(w_2, w_1) q(t, w_2) dw_1\, dw_2\\
&= 2^2 \int_0^\infty \int_0^\infty \hat{p}_k(w_1) \frac{\partial^2}{\partial w_2^2} q(w_2, w_1) q(t, w_2) dw_1\, dw_2\\
&= 2^3 \int_0^\infty \int_0^\infty \hat{p}_k(w_1) \frac{\partial}{\partial w_1} q(w_2, w_1) q(t, w_2) dw_1\, dw_2\\
&= -2^3 \int_0^\infty \int_0^\infty \frac{d}{dw_1} \hat{p}_k(w_1) q(w_2, w_1) q(t, w_2) dw_1\, dw_2\\
&= 2^3 \left\{ \lambda_k \tilde{p}_k(t) - \lambda_{k-1} \tilde{p}_{k-1}(t) \right\}.
\end{aligned}$$

The above reasoning can be generalised, thus arriving at equation (6.57). □

6.2.3 Other compositions

In this part we present the distributions of the classical nonlinear birth process $\mathcal{N}(t)$, $t > 0$, stopped at various random time processes, namely the sojourn time Γ_t of a standard Brownian motion, the sojourn time $\mathfrak{G}_t$ of a standard Brownian bridge and the stable subordinator $\mathcal{S}^\alpha(t)$ of order $\alpha \in (0, 1]$.

We start first by considering the nonlinear birth process at time $\Gamma_t = \int_0^t I_{[0,\infty)}(B(s))ds = \text{meas}\{s < t \colon B(s) > 0\}$. The process $\mathcal{N}(\Gamma_t)$, is a slowed down birth process. In the next theorem we provide its distribution.

Theorem 6.2.6. *We have that*

$$Pr\{\mathcal{N}(\Gamma_t) = k\} = \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{e^{-\frac{t}{2}\lambda_m} I_0\left(\frac{t}{2}\lambda_m\right)}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)}, & k > 1,\ t > 0,\\ e^{-\frac{t}{2}\lambda_1} I_0\left(\frac{t}{2}\lambda_1\right), & k = 1,\ t > 0, \end{cases} \tag{6.59}$$

where

$$I_0(z) = \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^{2k} \frac{1}{(k!)^2}, \tag{6.60}$$

is the zero-order Bessel function with imaginary argument.

Proof. The derivation of (6.59) is based on the evaluation of the following integral:

$$\int_0^t e^{-s\lambda_m} \frac{ds}{\pi\sqrt{s\,(t-s)}} = e^{-\frac{t}{2}\lambda_m} I_0\left(\frac{t}{2}\lambda_m\right). \tag{6.61}$$

□

Remark 6.2.5. *In view of the integral representation of the Bessel function*

$$I_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{z\cos\vartheta} d\vartheta, \tag{6.62}$$

we can give the following alternative, interesting representation of (6.59).

$$Pr\{\mathcal{N}(\Gamma_t) = k\} = \frac{1}{2\pi} \int_0^{2\pi} Pr\left\{ \mathcal{N}\left(t \sin^2 \frac{\vartheta}{2}\right) = k \right\} d\vartheta. \tag{6.63}$$

In other words,

$$\mathcal{N}(\Gamma_t) \overset{i.d.}{=} \mathcal{N}\left(t \sin^2 \frac{\Theta}{2}\right), \tag{6.64}$$

where Θ is a random variable uniform in $[0, 2\pi]$.

Theorem 6.2.7. *For the nonlinear birth process stopped at*

$$\mathfrak{G}_t = \int_0^t I_{[0,\infty)}(\bar{B}(s))ds, \tag{6.65}$$

$\bar{B}(s)$, $s > 0$, being a Brownian bridge, we have that

$$Pr\{\mathcal{N}(\mathfrak{G}_t) = k\} = \begin{cases} \frac{1}{\lambda_k t}\left\{1 - \sum_{m=1}^{k}\prod_{l=1, l\neq m}^{k}\left(\frac{\lambda_l}{\lambda_l - \lambda_m}\right)e^{-\lambda_m t}\right\}, & k > 1, \\ \frac{1-e^{-\lambda_1 t}}{\lambda_1 t}, & k = 1. \end{cases} \tag{6.66}$$

Proof. The calculation

$$\Pr\{\mathcal{N}(\mathfrak{G}_t) = k\} = \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k}\frac{1}{\prod_{l=1,l\neq m}^{k}(\lambda_l - \lambda_m)}\int_0^t e^{-\lambda_m s}\Pr\{\mathfrak{G}_t \in ds\}, \tag{6.67}$$

is sufficient to arrive at result (6.66), once the well-known fact that (6.65) is uniformly distributed in $[0, t]$ is considered. □

Remark 6.2.6. *For the the linear birth process, the distribution (6.66) takes a very simple form as the calculations below show. Since for $\lambda_k = \lambda \cdot k$, $\lambda > 0$, we have that*

$$\prod_{l=1,l\neq m}^{k}\frac{\lambda_l}{\lambda_l - \lambda_m} = \binom{k}{m}(-1)^{m-1}, \tag{6.68}$$

we can write that

$$\begin{aligned} Pr\{N(\mathfrak{G}_t) = k\} &= \frac{1}{\lambda k t}\left(1 - \sum_{m=1}^{k}\binom{k}{m}(-1)^{m-1}e^{-\lambda m t}\right) \\ &= \frac{1}{\lambda k t}\sum_{m=0}^{k}\binom{k}{m}(-1)^m e^{-\lambda m t} \\ &= \frac{\left(1 - e^{-\lambda t}\right)^k}{\lambda k t}, \qquad k \geq 1. \end{aligned} \tag{6.69}$$

The distribution (6.69) is logarithmic with parameter $1 - e^{-\lambda t}$. In the logarithmic distribution with parameter $0 < q < 1$, we have that

$$\mathbb{E}L = -\frac{q}{(1-q)\log(1-q)}, \tag{6.70}$$

$$\mathbb{V}arL = -\frac{q}{(1-q)^2\log(1-q)}\left[1 + \frac{q}{\log(1-q)}\right]. \tag{6.71}$$

In our case $q = 1 - e^{-\lambda t}$ so that

$$\mathbb{E}N(\mathfrak{G}_t) = \frac{e^{\lambda t} - 1}{\lambda t}, \tag{6.72}$$

$$\mathbb{V}arN(\mathfrak{G}_t) = \frac{e^{\lambda t}(e^{\lambda t} - 1)}{\lambda t}\left[1 - \frac{1 - e^{-\lambda t}}{\lambda t}\right]. \tag{6.73}$$

For large values of t we have that

$$\mathbb{E}N(\mathfrak{G}_t) \sim \frac{e^{\lambda t}}{\lambda t} = \frac{\mathbb{E}N(t)}{\lambda t}, \tag{6.74}$$

$$\mathbb{V}arN(\mathfrak{G}_t) \sim \frac{e^{\lambda t}(e^{\lambda t}-1)}{\lambda t} = \frac{\mathbb{V}arN(t)}{\lambda t}. \tag{6.75}$$

Theorem 6.2.8. *For the nonlinear birth process stopped at an α-stable time $\mathcal{S}^\alpha(t)$ with distribution $q_\alpha(t,s)$ and Laplace transform $\int_0^\infty e^{-\mu s} q_\alpha(t,s)ds = e^{-t\mu^\alpha}$, we have that*

$$Pr\{\mathcal{N}(\mathcal{S}^\alpha(t)) = k\} = \begin{cases} \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k} \frac{e^{-t\lambda_m^\alpha}}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)}, & k>1, \\ e^{-t\lambda_1^\alpha}, & k=1. \end{cases} \tag{6.76}$$

Proof. The following calculation is sufficient to prove result (6.76):

$$\Pr\{\mathcal{N}(\mathcal{S}^\alpha(t)) = k\} = \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)} \int_0^\infty e^{-\lambda_m s} q_\alpha(t,s)ds. \tag{6.77}$$

□

Remark 6.2.7. *Formula* (6.76) *can be further worked out as follows.*

$$\begin{aligned} Pr\{\mathcal{N}(\mathcal{S}^\alpha(t)) = k\} &= \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k} \frac{e^{-t\lambda_m^\alpha}}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)} \\ &\text{by exploiting the self-similarity of } \mathcal{S}^\alpha(t) \\ &= \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)} \int_0^\infty e^{-\lambda_m s} t^{-\frac{1}{\alpha}} q_\alpha(1, t^{-\frac{1}{\alpha}}s)ds \\ &= \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)} \int_0^\infty e^{-\lambda_m t^{\frac{1}{\alpha}}\zeta} q_\alpha(1,\zeta)d\zeta. \end{aligned} \tag{6.78}$$

The last result implies the following representation:

$$\mathcal{N}(\mathcal{S}^\alpha(t)) \overset{i.d.}{=} \mathcal{N}\left(t^{\frac{1}{\alpha}}Z\right), \tag{6.79}$$

where Z has distribution $q_\alpha(1,\zeta)$, $\zeta>0$.

Remark 6.2.8. *If we assume $\alpha = 1/2^n$ in the first line of* (6.78)*, and $s = t2^{1-\frac{1}{2^n}}$ in* (6.52)*, the distribution* (6.78) *suggests the following unexpected relation:*

$$Pr\left\{\mathcal{N}\left(\mathcal{S}^{\frac{1}{2^n}}(t)\right) = k\right\} = Pr\left\{\mathcal{N}\left[T^1_{T^2_{\cdot^{\cdot^{T^n_s}}}}\right] = k\right\}, \qquad k\geq 1. \tag{6.80}$$

Remark 6.2.9. *Many other compositions can be envisaged and in some cases they provide curious results. For example, we consider the standard Cauchy process $C(t)$, with law $h(t,s)$, $t>0$, $s\in\mathbb{R}$, satisfying the Laplace equation*

$$\frac{\partial^2 h}{\partial t^2} + \frac{\partial^2 h}{\partial s^2} = 0. \tag{6.81}$$

We can show that $\mathcal{N}(|C(t)|)$, $t>0$, is a birth process whose state probabilities $p_k^*(t)$, $t>0$, satisfy the difference-differential equations

$$\frac{d^2}{dt^2}p_k^*(t) = -\lambda_k^2 p_k^*(t) + \lambda_{k-1}(\lambda_k+\lambda_{k-1})p_{k-1}^*(t) - \lambda_{k-1}\lambda_{k-2}p_{k-2}^*(t). \tag{6.82}$$

6.3 Subordinated fractional birth processes

In a previous work of us (see Orsingher and Polito (2010)) we constructed and analysed a fractional (possibly nonlinear) pure birth process $\mathcal{N}^\nu(t)$, $t>0$, $\nu \in (0,1]$ by exchanging the integer-order time derivative with the Dzhrbashyan–Caputo fractional derivative in the difference-differential equation (6.27) governing the state probabilities. We recall that the Dzhrbashyan–Caputo derivative has the form, for $0<\nu\leq 1$

$$\frac{d^\nu}{dt^\nu} f(t) = \begin{cases} \frac{1}{\Gamma(1-\nu)} \int_0^t \frac{f'(s)}{(t-s)^\nu} ds, & 0<\nu<1, \\ f'(t), & \nu = 1. \end{cases} \tag{6.83}$$

In this section we examine properties of the subordinated processes $\mathcal{N}^\nu(T_t)$, $t>0$, $\mathcal{N}^\nu(T_{2\beta}(t))$, and $\mathcal{N}^\nu(\mathcal{S}^\alpha(t))$, $t>0$, $\nu,\alpha,\beta \in (0,1]$, bringing to the fore some interesting relations and discussing the interpretation for the results obtained.

6.3.1 Preliminaries

The state probabilities $p_k^\nu(t) = \Pr\{\mathcal{N}^\nu(t) = k\}$, $k \geq 1$ of the fractional pure birth process have the following form

$$p_k^\nu(t) = \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^k \frac{E_{\nu,1}(-\lambda_m t^\nu)}{\Pi_{l=1,l\neq m}^k (\lambda_l - \lambda_m)}, & k>1,\ t>0, \\ E_{\nu,1}(-\lambda_1 t^\nu), & k=1,\ t>0, \end{cases} \tag{6.84}$$

where $E_{\nu,1}(-\zeta t^\nu)$ is the Mittag–Leffler function defined as

$$E_{\nu,1}\left(-\zeta t^\nu\right) = \sum_{h=0}^\infty \frac{\left(-\zeta t^\nu\right)^h}{\Gamma\left(\nu h+1\right)}, \qquad \zeta \in \mathbb{R},\ \nu>0, \tag{6.85}$$

and with Laplace transform

$$\int_0^\infty e^{-zt} E_{\nu,1}\left(-\zeta t^\nu\right) dt = \frac{z^{\nu-1}}{z^\nu + \zeta}, \qquad \nu>0. \tag{6.86}$$

A useful integral representation for $E_{\nu,1}(-\zeta t^\nu)$ reads

$$E_{\nu,1}(-\zeta t^\nu) = \frac{\sin \nu\pi}{\pi} \int_0^\infty \frac{r^{\nu-1} e^{-r\zeta^{\frac{1}{\nu}} t}}{r^{2\nu} + 2r^\nu \cos\nu\pi + 1} dr, \qquad \nu \in (0,1). \tag{6.87}$$

In a previous work (see Orsingher and Polito (2010)) we proved a useful subordination representation for the fractional pure birth process (6.84). This can be viewed as a classical birth process stopped at a random time $T_{2\nu}(t)$ possessing density function coinciding with the folded solution to the fractional diffusion equation

$$\begin{cases} \frac{\partial^{2\nu} g}{\partial t^{2\nu}} = \frac{\partial^2 g}{\partial x^2}, & 0<\nu\leq 1, \\ g\left(x,0\right) = \delta\left(x\right), \end{cases} \tag{6.88}$$

with the additional condition $g_t\left(x,0\right) = 0$ for $1/2<\nu\leq 1$. In other words $\mathcal{N}^\nu(t) = \mathcal{N}(T_{2\nu}(t))$, $t>0$. It can be shown that $f_{T_{2\nu}}(s,t) = \Pr\left\{T_{2\nu}(t) \in ds\right\}$ is also a solution to

$$\frac{\partial^\nu f}{\partial t^\nu} = -\frac{\partial f}{\partial s} \tag{6.89}$$

(see Orsingher et al. (2010)).

Theorem 6.3.1. *The fractional nonlinear pure birth process is a renewal process with intermediate waiting times T_k^ν with law*

$$Pr\{T_k^\nu \in ds\} = \lambda_k s^{\nu-1} E_{\nu,\nu}(-\lambda_k s^\nu) ds, \qquad k \geq 1,\ s > 0, \tag{6.90}$$

where T_k^ν is the random time separating the kth and $(k+1)$th birth.

Proof. We prove this result by induction. Denoting $Z_k^\nu = T_1^\nu + \cdots + T_k^\nu$, we can certainly write that

$$\Pr\{T_1^\nu + \cdots + T_k^\nu \in dt\} = \int_0^t \Pr\{T_k^\nu \in d(t-s)\} \Pr\left\{T_1^\nu + \cdots + T_{k-1}^\nu \in ds\right\}, \tag{6.91}$$

where $\Pr\left\{T_1^\nu + \cdots + T_{k-1}^\nu \in ds\right\}/ds = \frac{d}{ds}\Pr\{\mathcal{N}^\nu(t) \geq k\}$. By resorting to Laplace transforms, from (6.91), we obtain that

$$\begin{aligned}
&\int_0^\infty e^{-\mu t} \Pr\{T_1^\nu + \cdots + T_k^\nu \in dt\} \\
&= \int_0^\infty e^{-\mu t} dt \int_0^t \Pr\{T_k^\nu \in d(t-s)\} \Pr\left\{T_1^\nu + \cdots + T_{k-1}^\nu \in ds\right\} \\
&= \int_0^\infty \Pr\left\{T_1^\nu + \cdots + T_{k-1}^\nu \in ds\right\} \int_s^\infty e^{-\mu t} \Pr\{T_k^\nu \in d(t-s)\} \\
&= \int_0^\infty e^{-\mu s} \Pr\left\{T_1^\nu + \cdots + T_{k-1}^\nu \in ds\right\} \int_0^\infty e^{-\mu y} \Pr\{T_k^\nu \in dy\} \\
&= \prod_{j=1}^k \int_0^\infty e^{-\mu s} \Pr\left\{T_j^\nu \in ds\right\} = \prod_{j=1}^k \frac{\lambda_j}{\mu^\nu + \lambda_j}.
\end{aligned} \tag{6.92}$$

We observe that

$$\begin{aligned}
\Pr\{T_1 \in ds\}/ds &= \frac{d}{ds} \Pr\{\mathcal{N}^\nu(s) \geq 2\} \\
&= -\frac{d}{ds} E_{\nu,1}(-\lambda_1 s^\nu) \\
&= \lambda_1 s^{\nu-1} E_{\nu,\nu}(-\lambda_1 s^\nu),
\end{aligned} \tag{6.93}$$

and that

$$\begin{aligned}
&\Pr\{T_1^\nu + T_2^\nu \in ds\} \\
&= \frac{d}{ds}\left[1 - \Pr\{\mathcal{N}^\nu(s) = 1\} - \Pr\{\mathcal{N}^\nu(s) = 2\}\right] \\
&= \lambda_1 s^{\nu-1} E_{\nu,\nu}(-\lambda_1 s^\nu) + \lambda_1 \left[\lambda_1 \frac{E_{\nu,\nu}(-\lambda_1 s^\nu)}{\lambda_2 - \lambda_1} + \lambda_2 \frac{E_{\nu,\nu}(-\lambda_2 s^\nu)}{\lambda_1 - \lambda_2}\right] s^{\nu-1} \\
&= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} s^{\nu-1} \left[E_{\nu,\nu}(-\lambda_1 s^\nu) - E_{\nu,\nu(-\lambda_2 s^\nu)}\right].
\end{aligned} \tag{6.94}$$

For $k = 2$, relation (6.92) simplifies to

$$\int_0^\infty e^{-\mu t} \Pr\{T_1^\nu + T_2^\nu \in dt\} = \frac{\lambda_1}{\mu^\nu + \lambda_1} \frac{\lambda_2}{\mu^\nu + \lambda_2}, \tag{6.95}$$

and this coincides with the Laplace transform of (6.94). □

Theorem 6.3.2. *The mean value $\mathbb{E}\mathcal{N}^\nu(t)$, for the fractional nonlinear pure birth process has the form:*

$$\mathbb{E}\mathcal{N}^\nu(t) = 1 + \sum_{k=1}^\infty \left\{1 - \sum_{m=1}^k \prod_{l=1, l\neq m}^k \frac{\lambda_l}{\lambda_l - \lambda_m} E_{\nu,1}(-\lambda_m t^\nu)\right\}. \tag{6.96}$$

Proof. In light of the subordination relation $\mathcal{N}^\nu(t) = \mathcal{N}(T_{2\nu}(t))$, and of result (6.28), we can write that

$$\begin{aligned}
\mathbb{E}\mathcal{N}^\nu(t) &= \int_0^\infty \mathbb{E}\mathcal{N}(s)\Pr\{T_{2\nu}(t) \in ds\} &(6.97)\\
&= 1 + \sum_{k=1}^\infty \left\{ 1 - \sum_{m=1}^k \prod_{l=1, l\neq m}^k \frac{\lambda_l}{\lambda_l - \lambda_m} \int_0^\infty e^{-\lambda_m s} \Pr\{T_{2\nu}(t) \in ds\} \right\}\\
&= 1 + \sum_{k=1}^\infty \left\{ 1 - \sum_{m-1}^k \prod_{l=1, l\neq m}^k \frac{\lambda_l}{\lambda_l - \lambda_m} E_{\nu,1}(-\lambda_m t^\nu) \right\}.
\end{aligned}$$

□

In the previous steps we assumed that

$$\int_0^\infty e^{-\lambda_m s} f_{T_{2\nu}}(t,s) ds = \int_0^\infty e^{-\lambda_m s} \Pr\{T_{2\nu}(t) \in ds\} = E_{\nu,1}(-\lambda_m t^\nu). \tag{6.98}$$

We give here some details of this result. The density $f_{T_{2\nu}}(z,s)$, $z > 0$, $s > 0$, is obtained by folding the solution of the fractional diffusion equation

$$\frac{\partial^{2\nu} u}{\partial z^{2\nu}} = \frac{\partial^2 u}{\partial s^2}, \tag{6.99}$$

which reads

$$u(z,s) = \frac{1}{z^\nu} W_{-\nu, 1-\nu}\left(-\frac{s}{z^\nu}\right), \qquad s > 0,\ z > 0. \tag{6.100}$$

Therefore

$$\begin{aligned}
\frac{1}{z^\nu}\int_0^\infty e^{-\lambda_m s} W_{-\nu,1-\nu}\left(-\frac{s}{z^\nu}\right) ds &= \frac{1}{z^\nu} \int_0^\infty e^{-\lambda_m s} \sum_{k=0}^\infty \left(-\frac{s}{z^\nu}\right)^k \frac{1}{k!\Gamma(-\nu k + 1 - \nu)} ds &(6.101)\\
&= \frac{1}{z^\nu} \sum_{k=0}^\infty \frac{(-1)^k}{\lambda_m^{k+1}\Gamma(-\nu k + 1 - \nu)} \frac{1}{(z^\nu)^k}\\
&= \frac{1}{\lambda_m z^\nu} E_{-\nu,1-\nu}\left(-\frac{1}{\lambda_m z^\nu}\right)\\
&\text{(by formula (5.1) page 1825, Beghin and Orsingher (2009b))}\\
&= E_{\nu,1}(-\lambda_m z^\nu).
\end{aligned}$$

Remark 6.3.1. *We can extract, from* (6.96), *the mean value of the fractional linear birth process obtained in Orsingher and Polito (2010), formula (3.42), as follows. By considering that* $\lambda_m = \lambda \cdot m$, *formula* (6.96) *becomes*

$$\begin{aligned}
\mathbb{E}N^\nu &= 1 + \sum_{k=1}^\infty \left\{ 1 - \sum_{m=1}^k (-1)^{m-1} \frac{k!}{m!(k-m)!} E_{\nu,1}(-\lambda m t^\nu) \right\} &(6.102)\\
&= 1 + \sum_{k=1}^\infty \left\{ 1 + \sum_{m=1}^k (-1)^m \binom{k}{m} E_{\nu,1}(-\lambda m t^\nu) \right\}\\
&= 1 + \sum_{k=1}^\infty \sum_{m=0}^k \binom{k}{m} (-1)^m E_{\nu,1}(-\lambda m t^\nu)\\
&= \sum_{k=0}^\infty \sum_{m=0}^k \binom{k}{m} (-1)^m E_{\nu,1}(-\lambda m t^\nu).
\end{aligned}$$

In order to obtain the desired result we pass to Laplace transforms and extract from (6.102) *that*

$$\begin{aligned}\int_0^\infty e^{-\mu t}\mathbb{E}N^\nu(t)dt &= \sum_{k=0}^\infty \sum_{m=0}^k \binom{k}{m}(-1)^k \int_0^\infty e^{-\mu t} E_{\nu,1}(-\lambda m t^\nu)dt \\ &= \sum_{k=0}^\infty \sum_{m=0}^k \binom{k}{m}(-1)^k \frac{\mu^{\nu-1}}{\mu^\nu+\lambda m} \\ &= \mu^{\nu-1}\int_0^\infty \sum_{k=0}^\infty \sum_{m=0}^k \binom{k}{m}(-1)^k e^{-w\mu^\nu}e^{-w\lambda m}dw \\ &= \mu^{\nu-1}\int_0^\infty \sum_{k=0}^\infty e^{-w\mu^\nu}\left(1-e^{-w\lambda}\right)^k dw \\ &= \mu^{\nu-1}\int_0^\infty \frac{e^{-w\mu^\nu}}{1-(1-e^{-w\lambda})}dw \\ &= \mu^{\nu-1}\int_0^\infty e^{-w\mu^\nu+w\lambda}dw \\ &= \frac{\mu^{\nu-1}}{\mu^\nu-\lambda}.\end{aligned} \tag{6.103}$$

By inverting the Laplace transform above, we can conclude that

$$\mathbb{E}N^\nu(t) = E_{\nu,1}(\lambda t^\nu), \tag{6.104}$$

thus confirming our previous result.

Here we remark that another interpretation in terms of random birth rates can be highlighted. If we write

$$\begin{aligned}p_k^\nu(t) = \Pr\{\mathcal{N}^\nu(t)=k \mid \mathcal{N}^\nu(0)=1\} &= \int_0^\infty p_k(s)\Pr\{T_{2\nu}(t)\in ds\} \\ &= \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^k \frac{1}{\prod_{l=1,l\neq m}^k(\lambda_l-\lambda_m)}\int_0^\infty e^{-\lambda_m s}\Pr\{T_{2\nu}(t)\in ds\} \\ &= \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^k \frac{1}{\prod_{l=1,l\neq m}^k(\lambda_l-\lambda_m)}\int_0^\infty e^{-\lambda_m s}\frac{1}{t^\nu}W_{-\nu,1-\nu}\left(-\frac{s}{t^\nu}\right)ds \\ &= \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^k \frac{1}{\prod_{l=1,l\neq m}^k(\lambda_l-\lambda_m)}\int_0^\infty e^{-\lambda_m \xi t^\nu}W_{-\nu,1-\nu}(-\xi)d\xi \\ &= \int_0^\infty W_{-\nu,1-\nu}(-\xi)\Pr\{\mathcal{N}_\xi(t^\nu)=k \mid \mathcal{N}_\xi(0)=1\}d\xi,\end{aligned} \tag{6.105}$$

we have that a fractional nonlinear pure birth process can be considered as a classical nonlinear pure birth process evaluated at a rescaled time t^ν and with random rates $\lambda_k\Xi$, $k\geq 1$, where Ξ is a random variable with density function

$$f_\Xi(\xi) = W_{-\nu,1-\nu}(-\xi), \qquad \xi\in\mathbb{R}^+ \tag{6.106}$$

and where $W_{-\nu,1-\nu}(-\xi)$ is a Wright function defined as

$$W_{-\nu,1-\nu}(-\xi) = \sum_{r=0}^\infty \frac{(-\xi)^r}{r!\Gamma(1-\nu(r+1))}. \tag{6.107}$$

From (6.105), the following interpretation also holds:

$$\mathcal{N}^\nu(t) \overset{\text{i.d.}}{=} \mathcal{N}(\Xi t^\nu). \tag{6.108}$$

Note also that, from (6.87) and (6.105), we have that

$$E_{\nu,1}(-zt^\nu) = \int_0^\infty e^{-rzt^\nu} W_{-\nu,1-\nu}(-r)dr = \frac{\sin \nu\pi}{\pi} \int_0^\infty e^{-rz^{\frac{1}{\nu}}t} \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} dr, \tag{6.109}$$

which illustrates an interesting relation between the Wright function and the law of $\mathcal{W}_1$ (see (6.21)). Equation (6.109) can be derived directly as follows.

$$\begin{aligned}
&\int_0^\infty e^{-\gamma x} \frac{1}{\lambda t^\nu} W_{-\nu,1-\nu}\left(-\frac{x}{\lambda t^\nu}\right) dx \qquad (6.110)\\
&= \frac{1}{\lambda t^\nu} \sum_{m=0}^\infty \frac{(-1)^m}{m!\Gamma\left(-\frac{\nu}{m} + 1 - \nu\right)} \frac{1}{(\lambda t^\nu)^m} \int_0^\infty e^{-\gamma x} x^m dx \\
&= \frac{1}{\lambda t^\nu} \sum_{m=0}^\infty \frac{(-1)^m}{\Gamma\left(-\frac{\nu}{m} + 1 - \nu\right)} \frac{1}{\gamma(\gamma\lambda t^\nu)^m} \\
&= \frac{1}{\gamma\lambda t^\nu} E_{-\nu,1-\nu}\left(-\frac{1}{\gamma\lambda t^\nu}\right) \\
&\text{(by formula (5.1), page 1825, Beghin and Orsingher (2009b))} \\
&= E_{\nu,1}(-\gamma\lambda t^\nu) \\
&= \frac{\sin \nu\pi}{\pi} \int_0^\infty \frac{e^{-(\gamma\lambda)^{\frac{1}{\nu}} tr} r^{\nu-1}}{(r^\nu + \cos \nu\pi)^2 + \sin^2 \nu\pi} dr.
\end{aligned}$$

This yields (6.109) for $z = \gamma\lambda$.

For further details on fractional pure birth process the reader can refer to Orsingher and Polito (2010) while Mittag–Leffler functions are extensively analysed in Kilbas et al. (2006).

6.3.2 Fractional pure birth process stopped at T_t

We consider here the composition of a fractional nonlinear pure birth process, denoted as $\mathcal{N}^\nu(t)$, $t > 0$, $\nu \in (0,1]$ with the first-passage time T_t of a standard Brownian motion. In the following theorem we derive an interesting integral representation for the state probabilities $\hat{p}_k^\nu(t)$, $t > 0$, $k \geq 1$, of $\hat{\mathcal{N}}^\nu(t) = \mathcal{N}^\nu(T_t)$, $t > 0$, $\nu \in (0,1)$.

Theorem 6.3.3. *Let $\mathcal{N}^\nu(t)$, $t > 0$, $\nu \in (0,1)$, be a fractional nonlinear pure birth process and T_t be the first-passage time process of the standard Brownian motion with distribution $q(t,s)$. The state probabilities $\hat{p}_k^\nu(t) = Pr\{\mathcal{N}^\nu(T_t) = k \mid \mathcal{N}^\nu(0) = 1\}$ possess the following integral form*

$$\hat{p}_k^\nu(t) = \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^k \frac{1}{\prod_{l=1,l\neq m}^k (\lambda_l - \lambda_m)} \frac{1}{i\pi} \int_0^\infty \frac{E_{2\nu,1}\left(-x^{2\nu} e^{i\pi\nu}\right) - E_{2\nu,1}\left(-x^{2\nu} e^{-i\pi\nu}\right)}{x + t\sqrt{2\lambda_m^{\frac{1}{\nu}}}} dx, & k > 1, \\ \frac{1}{i\pi} \int_0^\infty \frac{E_{2\nu,1}\left(-x^{2\nu} e^{i\pi\nu}\right) - E_{2\nu,1}\left(-x^{2\nu} e^{-i\pi\nu}\right)}{x + t\sqrt{2\lambda_1^{\frac{1}{\nu}}}} dx, & k = 1. \end{cases} \tag{6.111}$$

Proof. It is sufficient to prove (6.111) in the case $k > 1$, since the case $k = 1$ is analogous. We have

$$\hat{p}_k^\nu(t) = \int_0^\infty p_k^\nu(s) q(t,s) ds \tag{6.112}$$

$$= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k} (\lambda_l - \lambda_m)} \int_0^\infty E_{\nu,1}(-\lambda_m s^\nu) \frac{te^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds.$$

In order to prove (6.111), by taking into consideration formula (6.87), we do the following calculations

$$\begin{aligned}
\int_0^\infty E_{\nu,1}(-\lambda_m s^\nu) \frac{te^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds &= \int_0^\infty \frac{te^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} \int_0^\infty \frac{\sin \nu\pi}{\pi} \frac{r^{\nu-1} e^{-r\lambda_m^{\frac{1}{\nu}} s}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} dr\, ds \qquad (6.113)\\
&= \frac{\sin \nu\pi}{\pi} \int_0^\infty \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} \int_0^\infty e^{-r\lambda_m^{\frac{1}{\nu}} s} \frac{te^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds\, dr \\
&= \frac{\sin \nu\pi}{\pi} \int_0^\infty \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} e^{-t\sqrt{2r\lambda_m^{\frac{1}{\nu}}}} dr \\
&= \frac{\sin \nu\pi}{\pi} \int_0^\infty \frac{r^{\nu-1} e^{-t\sqrt{2r\lambda_m^{\frac{1}{\nu}}}}}{(r^\nu + e^{i\pi\nu})(r^\nu + e^{-i\pi\nu})} dr \\
&= \frac{1}{2i\pi} \int_0^\infty \left(\frac{1}{r^\nu + e^{i\pi\nu}} - \frac{1}{r^\nu + e^{-i\pi\nu}} \right) r^{\nu-1} e^{-t\sqrt{2r\lambda_m^{\frac{1}{\nu}}}} dr \\
&= \frac{1}{i\pi} \int_0^\infty \left(\frac{1}{w^{2\nu} + e^{i\pi\nu}} - \frac{1}{w^{2\nu} + e^{-i\pi\nu}} \right) w^{2\nu-1} e^{-tw\sqrt{2\lambda_m^{\frac{1}{\nu}}}} dw \\
&= \frac{1}{i\pi} \int_0^\infty \left[\frac{w^{2\nu-1}}{w^{2\nu} + e^{i\pi\nu}} - \frac{w^{2\nu-1}}{w^{2\nu} + e^{-i\pi\nu}} \right] e^{-tw\sqrt{2\lambda_m^{\frac{1}{\nu}}}} dw.
\end{aligned}$$

By using the Laplace transform (6.86) we obtain

$$\begin{aligned}
&\int_0^\infty E_{\nu,1}(-\lambda_m s^\nu) \frac{te^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds \qquad (6.114)\\
&= \frac{1}{i\pi} \int_0^\infty \left[\int_0^\infty e^{-wx} E_{2\nu,1}(-x^{2\nu} e^{i\pi\nu}) dx - \int_0^\infty e^{-wx} E_{2\nu,1}(-x^{2\nu} e^{-i\pi\nu}) dx \right] e^{-tw\sqrt{2\lambda_m^{\frac{1}{\nu}}}} dw \\
&= \frac{1}{i\pi} \int_0^\infty \frac{E_{2\nu,1}\left(-x^{2\nu} e^{i\pi\nu}\right) - E_{2\nu,1}\left(x^{2\nu} e^{-i\pi\nu}\right)}{x + t\sqrt{2\lambda_m^{\frac{1}{\nu}}}} dx.
\end{aligned}$$

Formula (6.111) is then proved by combining (6.112) and (6.114). □

Remark 6.3.2. *If $\lambda_k = k\lambda$, $k \geq 1$, the state probabilities $\hat{p}_k^\nu(t) = Pr\{N^\nu(T_t) = k \mid N^\nu(0) = 1\}$ of a fractional linear pure birth process stopped at T_t read*

$$\hat{p}_k^\nu(t) = \begin{cases} \sum_{m=0}^{k-1} \binom{k-1}{m-1} (-1)^{m-1} \frac{1}{i\pi} \int_0^\infty \frac{E_{2\nu,1}\left(-x^{2\nu} e^{i\pi\nu}\right) - E_{2\nu,1}\left(-x^{2\nu} e^{-i\pi\nu}\right)}{x + t\sqrt{2\lambda^{\frac{1}{\nu}} m}} dx, & k > 1, \\ \frac{1}{i\pi} \int_0^\infty \frac{E_{2\nu,1}\left(-x^{2\nu} e^{i\pi\nu}\right) - E_{2\nu,1}\left(-x^{2\nu} e^{-i\pi\nu}\right)}{x + t\sqrt{2\lambda^{\frac{1}{\nu}}}} dx, & k = 1. \end{cases} \qquad (6.115)$$

This result can be obtained by means of methods similar to those of Theorem 6.3.3.

Remark 6.3.3. *By considering formula (6.112) and the representation (6.87) we can give an interesting interpretation of the process $\hat{\mathcal{N}}^{1/2}(t) = \mathcal{N}^{1/2}(T_t)$, $t > 0$, as follows (again, we treat the case $k \geq 1$ since the case $k = 1$ is analogous)*

$$\begin{aligned}
\hat{p}_k^\nu(t) &= \int_0^\infty p_k^\nu(s) q(t,s) ds \qquad (6.116)\\
&= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k} (\lambda_l - \lambda_m)} \int_0^\infty E_{\nu,1}(-\lambda_m s^\nu) \frac{te^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds \\
&= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k} (\lambda_l - \lambda_m)} \frac{\sin \nu\pi}{\pi} \int_0^\infty \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} e^{-t\sqrt{2r\lambda_m^{\frac{1}{\nu}}}} dr
\end{aligned}$$

$$= \frac{\sin \nu\pi}{\pi} \int_0^\infty \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k} (\lambda_l - \lambda_m)} e^{-t\sqrt{2r\lambda_m^{\frac{1}{\nu}}}} dr.$$

If $\nu = 1/2$ we obtain the following expression

$$\hat{p}_k^{\frac{1}{2}}(t) = \frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{r}\,(r+1)} \prod_{j=1}^{k-1} (\sqrt{2r}\lambda_j) \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k} \left(\sqrt{2r}\lambda_l - \sqrt{2r}\lambda_m\right)} e^{-t\sqrt{2r}\lambda_m} dr \tag{6.117}$$
$$= \frac{\sqrt{2}}{\pi} \int_0^\infty \frac{1}{\left(\frac{w^2}{2}+1\right)} Pr\{\mathcal{N}_w(t) = k \mid \mathcal{N}_w(0) = 1\}\, dw$$

where $\mathcal{N}_W(t)$, $t > 0$, is a classical nonlinear birth process (6.25) *with random birth rates* $(W\lambda_k)$, $k \geq 1$ *where W is a folded Cauchy r.v. with p.d.f.*

$$f_W(w) = \frac{\sqrt{2}}{\pi\left(\frac{w^2}{2}+1\right)}, \qquad w \in \mathbb{R}^+. \tag{6.118}$$

It is possible to highlight a further interpretation by rewriting formula (6.117) *in the following way*

$$\hat{p}_k^{\frac{1}{2}}(t) = \frac{\sqrt{2}}{\pi} \int_0^\infty \frac{1}{t\left(\frac{s^2}{2t^2}+1\right)} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k} (\lambda_l - \lambda_m)} e^{-s\lambda_m} ds \tag{6.119}$$
$$= \int_0^\infty Pr\{\mathcal{N}(s) = k\}\, Pr\left\{\left|C(\sqrt{2}t)\right| \in ds\right\}$$

where $C(\sqrt{2}t)$, $t > 0$ is a Cauchy process with rescaled time, possessing transition density

$$f_C(t,s) = \frac{1}{\pi} \frac{\sqrt{2}t}{s^2 + \left(\sqrt{2}t\right)^2}, \qquad t > 0,\ s \in \mathbb{R}. \tag{6.120}$$

The process $\hat{\mathcal{N}}^{1/2}(t) = \mathcal{N}^{1/2}(T_t)$ can thus be written as $\hat{\mathcal{N}}^{1/2}(t) = \mathcal{N}(\left|C(\sqrt{2}t)\right|)$.

Iterated compositions

In the next theorem we present the explicit form of the state probabilities $\tilde{p}_k^\nu(t)$, $t > 0$, $k \geq 1$, for the process

$$\tilde{\mathcal{N}}^\nu(t) = \mathcal{N}^\nu\left[T^1_{T^2_{\cdot_{\cdot_{T^n_t}}}}\right], \qquad t > 0,\ \nu \in (0,1]. \tag{6.121}$$

and in the following remark an interesting interpretation for that process when $\nu = 1/2^n$, $n \in \mathbb{N}$, is given.

Theorem 6.3.4. *Let $\mathcal{N}^\nu(t)$, $t > 0$, $\nu \in (0,1]$, be a fractional nonlinear pure birth process and let T_t^1, T_t^2, $\ldots T^n$, $n \in \mathbb{N}$, be n independent first-passage time processes at t of the standard Brownian motion. The process*

$$\tilde{\mathcal{N}}^\nu(t) = \mathcal{N}^\nu\left[T^1_{T^2_{\cdot_{\cdot_{T^n_t}}}}\right], \qquad t > 0,\ \nu \in (0,1]. \tag{6.122}$$

has the following distribution

$$\tilde{p}_k^\nu(t) = \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k} (\lambda_l - \lambda_m)} \frac{\sin \nu\pi}{\pi} \int_0^\infty \frac{r^{\nu-1} e^{-t r^{\frac{1}{2^n}} \lambda_m^{\frac{1}{\nu 2^n}} 2^{\left(1-\frac{1}{2^n}\right)}}}{r^{2\nu}+2r^\nu \cos \nu\pi+1} dr, & k > 1,\ t > 0, \\ \frac{\sin \nu\pi}{\pi} \int_0^\infty \frac{r^{\nu-1} e^{-t \lambda_1^{\frac{1}{2^n}} 2^{\left(1-\frac{1}{2^n}\right)}}}{r^{2\nu}+2r^\nu \cos \nu\pi+1} dr, & k = 1,\ t > 0. \end{cases} \tag{6.123}$$

Proof. We start by proving the case $n = 2$ since the case $n = 1$ is already proved in Theorem 6.3.3. We omit the details for the case $k = 1$ and directly treat the case $k \geq 2$. We have

$$\begin{aligned}
&\int_0^\infty \hat{p}_k^\nu(t) q(t,s) ds \qquad (6.124)\\
&= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} \frac{\sin \nu\pi}{\pi} \int_0^\infty \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} \int_0^\infty e^{-s(2r)^{\frac{1}{2}} \lambda_m^{\frac{1}{2\nu}}} \frac{t e^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds\, dr \\
&= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} \frac{\sin \nu\pi}{\pi} \int_0^\infty \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} e^{-t\sqrt{2\sqrt{2r\lambda_m^{\frac{1}{\nu}}}}} dr.
\end{aligned}$$

It is now straightforward to generalise formula (6.124) for n compositions, as follows

$$\begin{aligned}
\tilde{p}_k^\nu(t) &= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} \frac{\sin \nu\pi}{\pi} \int_0^\infty \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} e^{-tr^{\frac{1}{2^n}} \lambda_m^{\frac{1}{\nu 2^n}} 2^{\sum_{i=1}^{n} \frac{1}{2^i}}} dr \qquad (6.125)\\
&= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} \frac{\sin \nu\pi}{\pi} \int_0^\infty \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} e^{-tr^{\frac{1}{2^n}} \lambda_m^{\frac{1}{\nu 2^n}} 2^{\left(1-\frac{1}{2^n}\right)}} dr.
\end{aligned}$$

□

Remark 6.3.4. *Analogously to Remark 6.3.3, for $\nu = 1/2^n$, $n \in \mathbb{N}$, it is possible to interpret formula* (6.123) *as follows*

$$\begin{aligned}
&\tilde{p}_k^{\frac{1}{2^n}}(t) \qquad (6.126)\\
&= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} \frac{\sin \frac{\pi}{2^n}}{\pi} \int_0^\infty \frac{r^{\frac{1}{2^n}-1}}{r^{\frac{1}{2^{n-1}}} + 2r^{\frac{1}{2^n}} \cos \frac{\pi}{2^n} + 1} e^{-tr^{\frac{1}{2^n}} 2^{\left(1-\frac{1}{2^n}\right)} \lambda_m} dr \\
&\left(r = y^{2^n}\right) \\
&= \frac{\sin \frac{\pi}{2^n}}{\frac{\pi}{2^n}} \int_0^\infty \frac{dr}{r^2 + 2r \cos \frac{\pi}{2^n} + 1} Pr\left\{ \mathcal{N}\left(tr 2^{1-\frac{1}{2^n}} \right) = k \right\}.
\end{aligned}$$

Therefore, the following representation holds

$$\mathcal{N}^{\frac{1}{2^n}} \left[T^1_{T^2_{\cdot_{\cdot_{T^n_t}}}} \right] = \mathcal{N}\left(t\, \Omega\, 2^{1-\frac{1}{2^n}} \right), \qquad (6.127)$$

where Ω is a random variable with density

$$f_\Omega(r) = \frac{\sin \frac{\pi}{2^n}}{\frac{\pi}{2^n}} \frac{1}{r^2 + 2r \cos \frac{\pi}{2^n} + 1}, \qquad r \in \mathbb{R}^+. \qquad (6.128)$$

The density is a unimodal law which, for $n \to \infty$, becomes

$$f(r) = \frac{1}{(1+r)^2}, \qquad r \in \mathbb{R}^+. \qquad (6.129)$$

6.3.3 Fractional pure birth process stopped at $\mathcal{S}^\alpha(t)$

We consider the fractional nonlinear pure birth process stopped at a stable time $\mathcal{S}^\alpha(t)$ of order $0 < \alpha \leq 1$ with Laplace transform

$$\mathbb{E} e^{-z\mathcal{S}^\alpha(t)} = \int_0^\infty e^{-zs} q_\alpha(t,s) ds = e^{-tz^\alpha}, \qquad (6.130)$$

where $q_\alpha(t,s)$, $s>0$, is the density of the stable process $\mathcal{S}^\alpha(t)$, $t>0$.

We have that the probabilities

$$\begin{aligned}
\breve{p}_k^\nu(t) &= \Pr\{\mathcal{N}^\nu(\mathcal{S}^\alpha(t))=k \mid \mathcal{N}^\nu(0)=1\} \qquad (6.131)\\
&= \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)} \int_0^\infty E_{\nu,1}(-\lambda_m s^\nu) q_\alpha(t,s)ds \\
&= \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)} \frac{\sin\nu\pi}{\pi} \int_0^\infty \frac{r^{\nu-1}}{r^{2\nu}+2r^\nu\cos\nu\pi+1} \int_0^\infty e^{-r\lambda_m^{\frac{1}{\nu}}s} q_\alpha(t,s)ds\,dr \\
&= \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)} \frac{\sin\nu\pi}{\pi} \int_0^\infty \frac{r^{\nu-1}}{r^{2\nu}+2r^\nu\cos\nu\pi+1} e^{-t\lambda_m^{\frac{\alpha}{\nu}}r^\alpha}dr \\
&= \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)} \frac{\sin\nu\pi}{\alpha\pi} \int_0^\infty \frac{w^{\frac{\nu}{\alpha}-1}}{w^{2\frac{\nu}{\alpha}}+2w^{\frac{\nu}{\alpha}}\cos\nu\pi+1} e^{-t\lambda_m^{\frac{\alpha}{\nu}}w}dw \\
&= \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)} \mathbb{E}e^{-t\lambda_m^{\frac{\alpha}{\nu}}\mathcal{W}_\alpha},
\end{aligned}$$

where $\mathcal{W}_\alpha$ is a random variable with density

$$f_{\mathcal{W}_\alpha}(w) = \frac{\sin\nu\pi}{\alpha\pi} \frac{w^{\frac{\nu}{\alpha}-1}}{w^{2\frac{\nu}{\alpha}}+2w^{\frac{\nu}{\alpha}}\cos\nu\pi+1}, \qquad w>0,\ 0<\nu<1, \tag{6.132}$$

(first obtained by Lamperti (1958)). The density (6.132) coincides with the probability distribution of

$$\mathcal{W}_\alpha = \left(\frac{S_1^\nu}{S_2^\nu}\right)^\alpha, \tag{6.133}$$

where S_1^ν, S_2^ν, are independent stable random variables with Laplace transform

$$\mathbb{E}e^{-zS^\nu} = e^{-z^\nu}, \qquad z>0,\ 0<\nu<1. \tag{6.134}$$

If $\alpha=1$, $\mathbb{E}e^{-t\lambda_m^{\frac{1}{\nu}}\mathcal{W}_\alpha} = E_{\nu,1}(-\lambda_m t^\nu)$ and (6.131) are the state probabilities of a fractional pure birth process, while for $\alpha=\nu$, $\mathbb{E}e^{-t\lambda_m\mathcal{W}_\nu}$ are the state probabilities of a pure birth process at time $t\mathcal{W}_\nu$ or, equivalently, a pure birth process at time t with rates $\lambda_k\mathcal{W}_\nu$.

If we compare (6.131) with (6.78), we can conclude that the process

$$\mathcal{N}^\nu(\mathcal{S}^\alpha(t)) = \mathcal{N}(T_{2\nu}(\mathcal{S}^\alpha(t))), \tag{6.135}$$

can be represented as

$$\mathcal{N}\left(\mathcal{S}^{\alpha/\nu}\left(t\mathcal{W}_\alpha\right)\right), \tag{6.136}$$

if $0<\alpha<\nu<1$.

Remark 6.3.5. *From formula* (6.131), *when α takes the form $\alpha=\nu/2^n$, $n\in\mathbb{N}$, we have*

$$\breve{p}_k^\nu(t) = \frac{\sin\nu\pi}{\pi}\int_0^\infty \frac{r^{\nu-1}}{r^{2\nu}+2r^\nu\cos\nu\pi+1} \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k} \frac{e^{-t\lambda_m^{\frac{1}{2^n}} r^{\frac{\nu}{2^n}}}}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)} dr. \tag{6.137}$$

For $n\to\infty$, we obtain that $\breve{p}_k^\nu(t)\to 0$, $k>1$, and $\breve{p}_1^\nu(t)\to e^{-t}$. This shows that for $n\to\infty$, either the population istantaneously explodes or does not produce offsprings with exponential probability.

An alternative way of presenting the state probabilities (6.131) is based on the Mellin–Barnes representation of the Mittag–Leffler function

$$E_{\nu,\mu}(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(z)\Gamma(1-z)}{\Gamma(\mu-\nu z)} x^{-z}\,dz, \tag{6.138}$$

with $\nu > 0$, $x \in \mathbb{C}$, $|\arg(-x)| < \pi$ (see Kilbas et al. (2006), page 44, formula (1.8.32)).

In view of (6.138), we can write (6.131) as follows

$$\begin{aligned}\breve{p}_k^{\nu}(t) &= \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)} \int_0^\infty \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(z)\Gamma(1-z)}{\Gamma(1-\nu z)} (\lambda_m s^{\nu})^{-z} dz\, q_\alpha(t,s)ds \\ &= \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)} \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(z)\Gamma(1-z)}{\Gamma(1-\nu z)}\lambda_m^{-z}\int_0^\infty s^{-\nu z} q_\alpha(t,s)ds\,dz,\end{aligned} \tag{6.139}$$

where in the last member of (6.139) the Mellin transform of $q_\alpha(t,s)$ appears.

The Mellin transform of the stable subordinator $\mathcal{S}^\alpha(t)$, with Laplace transform

$$\mathbb{E}e^{-z\mathcal{S}^\alpha(t)} = e^{-tz^\alpha}, \tag{6.140}$$

reads

$$\begin{aligned}\mathbb{E}(\mathcal{S}^\alpha(t))^{\eta-1} &= \int_0^\infty s^{\eta-1} q_\alpha(t,s)ds \\ &= \frac{1}{\alpha}\Gamma\left(\frac{1-\eta}{\alpha}\right)\frac{1}{\Gamma(1-\eta)} t^{\frac{\eta-1}{\alpha}}.\end{aligned} \tag{6.141}$$

By inserting (6.141) into (6.139), we arrive at

$$\begin{aligned}\breve{p}_k^{\nu}(t) &= \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)} \frac{\alpha^{-1}}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(z)\Gamma(1-z)\Gamma\left(\frac{\nu}{\alpha}z\right)}{\Gamma(\nu z)\Gamma(1-\nu z)} \left(\lambda_m t^{\frac{\nu}{\alpha}}\right)^{-z} dz \\ &= \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)} \alpha^{-1} H_{2,3}^{2,1}\left[\lambda_m t^{\frac{\nu}{\alpha}} \middle| \begin{matrix}(0,1),(0,\nu)\\(0,1),(0,\nu/\alpha),(0,\nu)\end{matrix}\right].\end{aligned} \tag{6.142}$$

We examine now in detail the case $\alpha = \nu$ in the next theorem.

Theorem 6.3.5. *We have the following distributions:*

1. $Pr\{\mathcal{N}^\nu(\mathcal{S}^\nu(t)) = k\} = Pr\{\mathcal{N}(T_{2\nu}(\mathcal{S}^\nu(t))) = k\} = Pr\{\mathcal{N}(t\mathcal{W}_\nu) = k\}$,
2. $Pr\{\mathcal{N}(\mathcal{S}^\nu(T_{2\nu}(t))) = k\} = Pr\{\mathcal{N}(t\mathcal{W}_1) = k\}$,

for $k \geq 1$, $t > 0$, *where*

$$\mathcal{W}_\alpha = \left(\frac{S_1^\nu}{S_2^\nu}\right)^\alpha, \tag{6.143}$$

and has distribution (6.132).

Proof. For $k > 1$ we can write that

$$\Pr\{\mathcal{N}(T_{2\nu}(\mathcal{S}^\nu(t))) = k\} \tag{6.144}$$

$$
\begin{aligned}
&= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} \int_0^\infty e^{-\lambda_m s} \Pr\{T_{2\nu}(\mathcal{S}^\nu(t)) \in ds\} \\
&= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} \int_0^\infty e^{-\lambda_m s} \int_0^\infty f_{T_{2\nu}}(z,s) q_\nu(t,z) dz\, ds \\
&= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} \int_0^\infty E_{\nu,1}(-\lambda_m z^\nu) q_\nu(t,z) dz \\
&= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} \frac{\sin \nu\pi}{\pi} \int_0^\infty dr \int_0^\infty \frac{r^{\nu-1} e^{-\lambda_m^{1/\nu} zr}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} q_\nu(t,z) dz \\
&= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} \frac{\sin \nu\pi}{\pi} \int_0^\infty \frac{r^{\nu-1} e^{-t(r\lambda_m^{1/\nu})^\nu}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} dr \\
&= \int_0^\infty \Pr\{\mathcal{N}(tr^\nu) = k\} \frac{\sin \nu\pi}{\pi} \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} dr \\
&= \Pr\{\mathcal{N}(t\mathcal{W}_1^\nu) = k\} = \Pr\{\mathcal{N}(t\mathcal{W}_\nu) = k\}.
\end{aligned}
$$

This concludes the proof of the first result. In order to prove the second result we write

$$
\begin{aligned}
&\Pr\{\mathcal{N}(\mathcal{S}^\nu(T_{2\nu}(t))) = k\} \qquad (6.145)\\
&= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} \int_0^\infty \int_0^\infty e^{-\lambda_m s} q_\nu(z,s) f_{T_{2\nu}}(z,t) dz\, ds \\
&= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} \int_0^\infty e^{-\lambda_m^\nu z} f_{T_{2\nu}}(z,t) dz \\
&\overset{\text{by}(6.98)}{=} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} E_{\nu,1}(-\lambda_m^\nu t^\nu) \\
&= \int_0^\infty \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} \frac{\sin \nu\pi}{\pi} \frac{r^{\nu-1} e^{-\lambda_m tr}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} dr \\
&= \int_0^\infty \Pr\{\mathcal{N}(tr) = k\} \frac{\sin \nu\pi}{\pi} \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} dr \\
&= \Pr\{\mathcal{N}(t\mathcal{W}_1) = k\}.
\end{aligned}
$$

□

Remark 6.3.6. *By slightly changing the above calculations, we arrive at the following result (compare with* (6.131)*):*

$$
Pr\{\mathcal{N}(\mathcal{S}^\alpha(T_{2\nu}(t))) = k\} = \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} \mathbb{E} e^{-t\lambda_m^{\frac{\alpha}{\nu}} \mathcal{W}_1}. \qquad (6.146)
$$

Remark 6.3.7. *An alternative form of the distribution* (6.131)*, for* $\alpha = \nu$*, can be given as follows.*

$$
\begin{aligned}
&Pr\{\mathcal{N}^\nu(\mathcal{S}^\nu(t)) = k\} \qquad (6.147)\\
&= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} \frac{1}{2i\nu\pi} \int_0^\infty \left(\frac{1}{w + e^{-i\pi\nu}} - \frac{1}{w + e^{i\pi\nu}} \right) e^{-t\lambda_m w} dw \\
&= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} \frac{1}{2i\nu\pi} \left[e^{t\lambda_m e^{-i\pi\nu}} \mathrm{E}_1\left(t\lambda_m e^{-i\pi\nu}\right) - e^{t\lambda_m e^{i\pi\nu}} \mathrm{E}_1\left(t\lambda_m e^{i\pi\nu}\right) \right]
\end{aligned}
$$

$$= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k} (\lambda_l - \lambda_m)} \frac{e^{\lambda_m t \cos \nu\pi}}{2i\nu\pi} \left[e^{t\lambda_m \sin \nu\pi} \mathrm{E}_1 \left(t\lambda_m e^{-i\pi\nu} \right) - e^{t\lambda_m \sin \nu\pi} \mathrm{E}_1 \left(t\lambda_m e^{i\pi\nu} \right) \right],$$

where the function $\mathrm{E}_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt$, $|\arg z < \pi|$, *is the exponential integral.*

6.3.4 Fractional pure birth process stopped at $T_{2\alpha}(t)$

In this section we consider the process $\mathcal{N}^\nu(T_{2\alpha}(t))$, $t > 0$ (see the discussion related to formula (6.88). As we did before, here we treat the case $k \geq 2$. The state probabilities can be written as follows.

$$\begin{aligned} p_k^{\nu,\alpha}(t) &= \Pr\{\mathcal{N}^\nu(T_{2\alpha}(t)) = k \mid \mathcal{N}^\nu(0) = 1\} \\ &= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k} (\lambda_l - \lambda_m)} \int_0^\infty E_{\nu,1}(-\lambda_m s^\nu) \Pr\{T_{2\alpha}(t) \in ds\}. \end{aligned} \tag{6.148}$$

The integral in (6.148) can be further worked out by means of the Laplace transform:

$$\begin{aligned} &\int_0^\infty E_{\nu,1}(-\lambda_m s^\nu) \int_0^\infty e^{-zt} \Pr\{T_{2\alpha}(t) \in ds\} dt \\ &= \int_0^\infty E_{\nu,1}(-\lambda_m s^\nu) z^{\alpha-1} e^{-sz^\alpha} ds \\ &= z^{\alpha-1} \frac{(z^\alpha)^{\nu-1}}{z^{\alpha\nu} + \lambda_m} = \frac{z^{\alpha\nu-1}}{z^{\alpha\nu} + \lambda_m}. \end{aligned} \tag{6.149}$$

By taking the inverse Laplace transform of the above formula, we immediately obtain that

$$\begin{aligned} &\int_0^\infty E_{\nu,1}(-\lambda_m s^\nu) \Pr\{T_{2\alpha}(t) \in ds\} \\ &= \int_0^\infty E_{\nu,1}(-\lambda_m s^\nu) t^{-\alpha} W_{-\alpha,1-\alpha}(-t^{-\alpha} s) ds \\ &= E_{\nu\alpha,1}(-\lambda_m t^{\nu\alpha}). \end{aligned} \tag{6.150}$$

Therefore, the state probabilities for the process $\mathcal{N}^\nu(T_{2\alpha}(t))$, $t > 0$, result in the following form:

$$p_k^{\nu,\alpha}(t) = \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k} (\lambda_l - \lambda_m)} E_{\nu\alpha,1}(-\lambda_m t^{\nu\alpha}) = p_k^{\nu\alpha}(t), \qquad k \geq 2. \tag{6.151}$$

Note that the case $k = 1$ can be treated in the same manner. We thus obtain the following equalities in distribution:

$$\mathcal{N}^\nu(T_{2\alpha}(t)) = \mathcal{N}\{T_{2\nu}(T_{2\alpha}(t))\} = \mathcal{N}(T_{2\nu\alpha}(t)) = \mathcal{N}^{\nu\alpha}(t), \qquad t > 0. \tag{6.152}$$

Let now $\eta_n = \prod_{i=1}^n \nu_i$, where $n \in \mathbb{N}$, and ν_i are n indices such that $\nu_i \in (0,1]$ for $1 \leq i \leq n$. Formula (6.152) can be generalised as

$$\mathcal{N}\{T_{2\nu_1}(T_{2\nu_2}(\dots T_{2\nu_n}(t)\dots))\} = \mathcal{N}(T_{2\eta_n}(t)) = \mathcal{N}^{\eta_n}(t), \qquad t > 0 \tag{6.153}$$

where $\mathcal{N}^{\eta_n}(t)$ is a nonlinear fractional birth process.

Formula (6.148) can also be worked out in an alternative way. In the following calculations we will make use of the integral representation (6.87).

$$p_k^{\nu,\alpha}(t) = \Pr\{\mathcal{N}^\nu(T_{2\alpha}(t)) = k \mid \mathcal{N}^\nu(0) = 1\} \tag{6.154}$$

$$
\begin{aligned}
&= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k} (\lambda_l - \lambda_m)} \int_0^\infty E_{\nu,1}\left(-\lambda_m s^\nu\right) \Pr\left\{T_{2\alpha}(t) \in ds\right\} \\
&= \int_0^\infty \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k} (\lambda_l - \lambda_m)} \\
&\quad \times \int_0^\infty \frac{\sin \nu\pi}{\pi} \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} e^{-r\lambda_m^{\frac{1}{\nu}} s} \Pr\left\{T_{2\alpha}(t) \in ds\right\} dr \\
&= \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k} (\lambda_l - \lambda_m)} \int_0^\infty \frac{\sin \nu\pi}{\pi} \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} E_{\alpha,1}\left(-r\lambda_m^{\frac{1}{\nu}} t^\alpha\right).
\end{aligned}
$$

Remark 6.3.8. *By comparing formulae* (6.151) *and* (6.154)*, it is clear that the following expansion holds:*

$$
E_{\nu\alpha,1}(-\lambda_m t^{\nu\alpha}) = \frac{\sin \nu\pi}{\pi} \int_0^\infty \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} E_{\alpha,1}\left(-r\lambda_m^{\frac{1}{\nu}} t^\alpha\right) dr, \qquad \nu \in (0,1],\ \alpha \in (0,1]. \tag{6.155}
$$

We give a direct proof of (6.155) *by applying the Laplace transform to both members. Of course*

$$
\int_0^\infty e^{-\mu t} E_{\nu\alpha,1}(-\lambda_m t^{\nu\alpha}) dt = \frac{\mu^{\nu\alpha-1}}{\mu^{\nu\alpha} + \lambda_m}. \tag{6.156}
$$

Then, we must calculate the twofold integral

$$
\begin{aligned}
&\frac{\sin \nu\pi}{\pi} \int_0^\infty e^{-\mu t} \int_0^\infty \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} E_{\alpha,1}\left(-r\lambda_m^{\frac{1}{\nu}} t^\alpha\right) dt\, dr \qquad (6.157)\\
&= \frac{\sin \nu\pi}{\pi} \int_0^\infty \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} \frac{\mu^{\alpha-1}}{\mu^\alpha + r\lambda_m^{\frac{1}{\nu}}} dr \\
&= \frac{\sin \nu\pi}{\pi} \int_0^\infty \int_0^\infty \frac{\mu^{\alpha-1} r^{\nu-1} e^{-w\left(\mu^\alpha + r\lambda_m^{\frac{1}{\nu}}\right)}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1} dr\, dw \\
&\overset{by\ (6.87)}{=} \int_0^\infty \mu^{\alpha-1} e^{-\mu^\alpha w} E_{\nu,1}(-\lambda_m w^\nu) dw \\
&= \frac{(\mu^\alpha)^{\nu-1} \mu^{\alpha-1}}{(\mu^\alpha)^\nu + \lambda_m} = \frac{\mu^{\nu\alpha-1}}{\mu^{\nu\alpha} + \lambda_m}.
\end{aligned}
$$

Remark 6.3.9. *A number of interesting relations follow from formula* (6.155)*.*

The following integral relation holds:

$$
\begin{aligned}
&\frac{\sin \nu\pi}{\nu} \int_0^\infty \frac{1}{r^2 + 2r \cos \nu\pi + 1} E_{\alpha,1}\left(-r^{\frac{1}{\nu}} \lambda_m^{\frac{1}{\nu}} t^\alpha\right) dr \qquad (6.158)\\
&= \frac{\sin \nu\alpha\pi}{\nu\alpha} \int_0^\infty \frac{1}{r^2 + 2r \cos \nu\alpha\pi + 1} e^{-r^{\frac{1}{\nu\alpha}} \lambda_m^{\frac{1}{\nu\alpha}} t} dr.
\end{aligned}
$$

A sort of commutativity is valid for (6.155)*:*

$$
E_{\alpha\nu,1}(-\lambda_m t^{\alpha\nu}) = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \frac{r^{\alpha-1}}{r^{2\alpha} + 2r^\alpha \cos \alpha\pi + 1} E_{\nu,1}\left(-r\lambda_m^{\frac{1}{\alpha}} t^\nu\right), \qquad \nu \in (0,1],\ \alpha \in (0,1]. \tag{6.159}
$$

For $\alpha = 1$ *we recover, from* (6.155)*, the integral representation of Mittag-Leffler functions. By considering that*

$$
f_\nu(r) = \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \nu\pi + 1}, \tag{6.160}
$$

for $\nu = 1$*, becomes a delta function with pole at* $r = 1$*, we extract, from* (6.155)*, an identity.*

Furthermore, it is worth noticing that formulae similar to (6.155) *can be derived by repeated applications of the same formula. For example we have:*

$$\begin{aligned} E_{\nu\alpha\beta,1}\left(-\lambda_m t^{\nu\alpha\beta}\right) &= \frac{\sin\nu\pi}{\pi}\int_0^\infty \frac{r^{\nu-1}}{r^{2\nu}+2r^\nu\cos\nu\pi+1} E_{\alpha\beta,1}\left(-r\lambda_m^{\frac{1}{\nu}}t^{\alpha\beta}\right)dr \qquad (6.161)\\ &= \frac{\sin\nu\pi\sin\alpha\pi}{\pi^2}\int_0^\infty\int_0^\infty \frac{r^{\nu-1}w^{\alpha-1}E_{\beta,1}(-wr^{\frac{1}{\alpha}}\lambda_m^{\frac{1}{\nu\alpha}}t^\beta)}{(r^{2\nu}+2r^\nu\cos\nu\pi+1)(w^{2\alpha}+2w^\alpha\cos\alpha\pi+1)}dw\,dr. \end{aligned}$$

Let ν_i*,* $1 \le i \le n$ *be* n *indices such that for all* $1 \le i \le n$*,* $\nu_i \in (0,1]$*, and let us denote* $\eta_n = \prod_{i=1}^n \nu_i$*. In general, for* $n \ge 2$*, we obtain that*

$$\begin{aligned} & E_{\eta_n,1}\left(-\lambda_m t^{\eta_n}\right) \qquad (6.162)\\ &= \int_0^\infty \!\!\dots\int_0^\infty \prod_{j=1}^{n-1}\left(\frac{r_j^{\nu_j-1}}{r_j^{2\nu_j}+2r_j^{\nu_j}\cos\nu_j+1}\right)E_{\nu_n,1}\left(-r_1 r_2^{\frac{1}{\nu_1}} r_3^{\frac{1}{\nu_1\nu_2}}\dots r_{n-1}^{\frac{1}{\eta_{n-2}}}\lambda_m^{\frac{1}{\eta_{n-1}}}t^{\nu_n}\right)\prod_{j=1}^{n-1}dr_j\\ &= \mathbb{E}\left[E_{\nu_n,1}\left(-{}^{(1)}\mathcal{W}_1\,{}^{(2)}\mathcal{W}_1^{\frac{1}{\nu_1}}\,{}^{(3)}\mathcal{W}_1^{\frac{1}{\nu_1\nu_2}}\dots{}^{(n-1)}\mathcal{W}_1^{\frac{1}{\eta_{n-2}}}\lambda_m^{\frac{1}{\eta_{n-1}}}t^{\nu_n}\right)\right], \end{aligned}$$

where ${}^{(j)}\mathcal{W}_1$*,* $1 \le j \le n-1$*, are independent random variables, each with distribution* (6.132)*, with* $\alpha = 1$ *and* $\nu = \nu_j$*.*

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