# Controlling the flow rate of a Bingham fluid undergoing stress-induced degradation in a pipe

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## 1 Introduction

As it is well-known, for some non-Newtonian fluids the internal structure can prevent flow unless the shear stress $\tau$ exceeds a yield value $\tau_0$. Among them, the name *Bingham plastic* is generally reserved to those fluids which, when $\tau > \tau_0$, show a Newtonian behaviour, viz.

$$(\tau - \tau_0)_+ = \eta_B \dot{\gamma}, \tag{1}$$

$(.)_+$ being the positive part of $(.)$, $\eta_B > 0$ the (constant) plastic viscosity, and $\dot{\gamma}$ the shear rate.

Although real Bingham plastic behaviour is encountered somewhat rarely, small enough departure from the ideal Bingham model is frequent in several materials of industrial interest (some slurries, paper pulp, chocolate mixtures, toothpaste, margarine, etc.). This circumstance justifies the widespread and frequent utilization of this model in technological design.

In a recent paper [1] an unusual mathematical model for the pipe flow of a Bingham fluid was introduced; the distinguishing feature is that $\tau_0$ is increasing with time and shearing because of internal power dissipation. This behaviour has been observed in some coal-water slurries (CWS's) and has been the object of a number of papers (see [2][3][4] and references therein). The model proposed in [1] fits very well the observed negative thixotropic behaviour (progressive increase of the apparent viscosity) of these CWS's. This very peculiar aspect of CWS's rheology has received a rather complex explanation at microscopic level [2].

The mathematical model presented in [1] is worked out under some particular but physically significant hypotheses (axisymmetry and *quasi*-steadyness). In particular it turns out that the boundary between the yielded region and the non-yielded one changes with time. Several situations may occur depending on the behavior with time of the driving pressure gradient . In the worst case the flow can stop in a finite time either because the boundary of the inner rigid core hits the pipe wall or because a second free boundary develops independently starting from the pipe wall and meets the first one.

From the commercial point of view, one of the major concerns is to maintain a constant flow rate by adjusting conveniently the pressure gradient. This problem is classical and already solved when $\tau_0$ does not depend on $t$ (see [8], pp. 71-75): given the desired flow rate, one needs to solve the (fourth order *algebraic*) Buckingham equation in the unknown (stationary) radius of the inner rigid core, this parameter being directly related to the driving pressure gradient. In this paper we solve the same problem for the model developed in [1]: however the equation corresponding to the Buckingham equation is now a nonlinear *functional* equation: we prove existence and uniqueness using fixed point arguments. The result is local in time and *cannot* be improved in the class of bounded pressure gradients (the one in which we prove existence and uniqueness). Indeed, as it was proved in [1], when the pressure gradient remains bounded the motion always comes to rest as time goes to infinity. At least in the easier case of an initially Newtonian fluid we can give an upper bound for the critical time for the existence of the solution: interestingly enough, in a typical case this turns out to be of order 10 *hours* at a mainstream speed of $1\, m \times s^{-1}$, which, for industrial design, is undoubtedly very significant.

## 2 Generalized Buckingham equation

Let us consider a laminar axisymmetric flow of a Bingham plastic in a infinitely long pipe of radius R under the assumption that the yield stress $\tau_o$ increases at a rate proportional to the internal power dissipation, according to the law

$$\frac{\partial}{\partial t}\tau_o = \alpha\tau\left|\frac{\partial}{\partial r}v\right| = \alpha\tau\frac{1}{\eta_B}(\tau - \tau_o)_+, \tag{2}$$

where $\alpha$ is a non-negative dimensionless constant.

For a sufficiently large pressure gradient $-G(t)$, we have an inner rigid core $0 < r < s(t)$ while for $s(t) < r < R$ and $t > 0$ the velocity $v(r,t)$ satisfies the equation

$$\rho\frac{\partial}{\partial t}v = G - \frac{1}{2}\frac{\partial}{\partial r}(r\tau), \tag{3}$$

$\rho$ being the fluid density; we assume the no-slip condition at $r = R$, i.e. $v(R,t) = 0$, and the free-boundary conditions

$$\left(\frac{\partial}{\partial r}v\right)\Big|_{r\,=\,s(t)} = 0, \tag{4}$$

$$\left(\rho\frac{\partial}{\partial t}v\right)\Big|_{r\,=\,s(t)} = G(t) - \frac{2\tau_o}{s(t)}, \tag{5}$$

which express the absence of the strain rate at the boundary and the momentum balance of a unit length portion of the rigid core respectively (see [6],[7]).

The quasi-steady approximation used in [1] is justified by the very small order of magnitude of the multiplying coefficient of the $\frac{\partial}{\partial t}v$ in the dimensionless form of momentum equation (see [1] for the details). This in turn depends on choosing the time scale to be that of the slow phenomenon (degradation). The *quasi*-steady free-boundary problem in dimensionless form, after rescaling, is

$$
(6)\qquad \begin{cases}
(\tau-\tau_o)_+ = \dfrac{\zeta}{4}\left|\dfrac{\partial}{\partial r}v\right|, & \text{for } 0<r<1, \text{ and } t>0,\\[2mm]
G(t) = \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\tau), & \text{for } 0<r<1, \text{ and } t>0,\\[2mm]
\left(\dfrac{\partial}{\partial r}v\right)\Big|_{r=s(t)} = 0, & \text{for } t>0,\\[2mm]
s(t) = 2G^{-1}(t)\tau_o(s(t),t), & \text{for } t>0,\\[2mm]
\left(\dfrac{\partial}{\partial t}\tau_o\right) = \tau\left|\dfrac{\partial}{\partial r}v\right|, & \text{for } 0<r<1, \text{ and } t>0,
\end{cases}
$$

with the boundary condition

$$
(7)\qquad v(1,t) = 0,
$$

and the initial conditions

$$
(8)\qquad \begin{cases}
\tau_o(r,0) = 1, & 0\le r\le 1,\\
v(r,0) = v_o(r), & 0\le r\le 1,\\
s(0) = s_o = 2/\zeta,
\end{cases}
$$

where $\zeta = G(0) > 2$ and

$$
(9)\qquad \begin{cases}
v_o(r) = 1 - r^2 - 2s_o(1-r), & s_o < r \le 1,\\
v_o(r) = (1-s_o)^2, & 0\le r\le s_o.
\end{cases}
$$

If we set $Y(r,t) = \tau(r,t) - \tau_o(r,t)$, the fluid velocity is given by

$$
(10)\qquad v(r,t) = \frac{4}{\zeta}\int_r^1 [Y(r',t)]_+ dr'.
$$

Equivalent forms of the volumetric flow rate $Q(t)$ in dimensionless units are listed below:

$$
(11)\qquad Q(t) = 2\pi\int_0^1 rv(r,t)\,dr = \frac{4\pi}{\zeta}\int_0^1 r^2 Y_+(r,t)\,dr = -\pi\int_{s(t)}^1 r^2\frac{\partial}{\partial r}v\,dr,
$$

(the second expression follows after an integration by parts).

In [1] it is proved, in particular, that if $G(t)$ is not decreasing then the inner core is not expanding, no other free boundary can develop and a unique solution exists for all times. The unique free boundary is given by $s(t) = 2/G(t)$. Moreover, if $G(t)$ is bounded, it is impossible to maintain the initial flow rate $Q(t)$ for ever. Indeed in this case $v$ goes to zero uniformly in $(0,1)$ as $t$ goes to infinity.

The following proposition can be immediately proved:

**Theorem 1** *Let $\dot{Q} = 0$; then $\dot{G} > 0$.*

Consequently $G$ has an inverse $\tilde{G}$; from (6) one deduces an equation satisfied by $Y(r,t)$ in the region $Y \geq 0$:

$$\frac{\partial Y}{\partial t} + \frac{2r}{\zeta}G(t)Y = \frac{r}{2}\dot{G}(t) \tag{12}$$

which, integrated, gives (see eq.'s (3.6) and (3.6') in [1]):

$$\begin{cases} Y(r,t) = \dfrac{r\displaystyle\int_0^t e^{rF(\theta)}\dot{G}(\theta)d\theta + Y_o(r)}{2e^{rF(t)}}, & \text{for } s_o \leq r \leq 1,\ t > 0, \\[2ex] Y(r,t) = \dfrac{r\displaystyle\int_{t_o(r)}^t e^{rF(\theta)}\dot{G}(\theta)d\theta}{2e^{rF(t-t_o(r))}}, & \text{for } s(t) \leq r \leq s_o,\ t > t_o(r), \end{cases} \tag{13}$$

with

$$Y_o(r) = Y|_{t=0} = \zeta\frac{r}{2} - 1,\quad t_o(r) = \delta(s_o - r)\tilde{G}(2/r),\quad F(t) = (2/\zeta)\int_0^t G(\xi)d\xi. \tag{14}$$

and $\delta(x)$ is the Heaviside function.

A particular case, also met with some CWS's, is that of an *initially Newtonian fluid* $(\tau_o(r,0) = 0)$. With a slightly different rescaling the problem remains formally the same. In particular $G$ is rescaled with half of its initial value, which implies now to fix $\zeta = 2$. For $\dot{G} > 0$ the fluid is everywhere sheared for all $t \geq 0$ (see [1], **Proposition 5.1**), and (13) simplifies as

$$Y(r,t) = \frac{r}{2}e^{[-rF_N(t)]}\{\int_0^t e^{rF_N(\theta)}\dot{G}(\theta)d\theta + 2\}, \tag{15}$$

where now

$$F_N(t) = \int_0^t G(\xi)d\xi. \tag{16}$$

Equation (15) holds everywhere in $[0,1] \times [0,\infty)$, since no free boundary $\widetilde{s}(t)$ can ever develop in this case. By using (6) it is not difficult to express $\dot{Q}$ in terms of $G$, $\dot{G}$ and $Y$ alone. We then put $\dot{Q} = 0$ and integrate with respect to time; after inserting (13) into the resulting equation we get a functional equation $G = \mathcal{F}(G, \widetilde{G})$ which is the required generalization to our case of the Buckingham equation: to be precise we have

$$(17) \qquad \begin{cases} G & = \widetilde{\mathcal{M}}\Big(\int_0^t [\mathcal{N}_1(G,\widetilde{G}) + \mathcal{N}_2(G)]\, d\tau\Big) \\ \widetilde{G} \circ G & = I \ \ (I \text{ is the identity map in } R) \end{cases}$$

where $\widetilde{\mathcal{M}}$ is the inverse function of

$$(18) \qquad \mathcal{M}(x) = \frac{\zeta}{16}[ln(\frac{x}{\zeta}) + 4(x^{-4} - \zeta^{-4})]$$

in $[\zeta, \infty)$ and

$$(19) \qquad \begin{cases} \mathcal{N}_1[G,\widetilde{G}](\tau) & = \frac{1}{2}\int_{s(\tau)}^{s_o} r^4 e^{-rF(\tau - t_o(r))}\{G(\tau)e^{rF(\tau)} \\ & - \frac{2}{r}e^{rF(t_o(r))} - \frac{2}{\zeta}\int_{t_o(r)}^{\tau} rG^2(s)e^{rF(s)}\ ds\}\, dr, \\ \mathcal{N}_2[G](\tau) & = G(\tau)\frac{(1-s_o^5)}{10} - \int_{s_0}^1 r^3 e^{-rF(\tau)}\ dr \\ & - \frac{1}{\zeta}\int_0^{\tau}\int_{s_o}^1 r^5 G^2(s) e^{-r[F(\tau)-F(s)]}\ dr\, ds. \end{cases}$$

The complexity of (17)-(19) ( which is natural to define as the *generalized Buckingham equation*) in the unknown function $G$ should be compared with the classical case.

## 3 Existence and uniqueness for the generalized Buckingham equation

Let $C^k([0,T])$ denote the Banach space of functions continuous up to their k-th derivative over $[0,T]$, for a fixed finite $T > 0$, equipped with the usual norm $||.||_k$. For given $M_1, M_2 > 0$ and $K > 0$ such that $K \leq \dot{G}(0)$ [1], let us define

$$\mathcal{X} = \{f \in C^1([0,T]) / \\ ||f||_0 \leq M_1, ||\dot{f}||_0 \leq M_2, \dot{f} \geq K, f(0) = G(0)(=\zeta), \dot{f}(0) = \dot{G}(0)\}$$

It is evident that $\mathcal{X}$ is a closed convex subset of $C^1([0,T])$.

[1] Using (12) it is easy to check that $\dot{G}(0) = (4/5)(2\zeta(1-(2/\zeta)^5) - 5(1-(2/\zeta)^4)) > 0$ for all $\zeta > 2$

**Theorem 2** *For a sufficiently small $T(M_1, M_2, K)$, there exists one and only one solution $G(t)$ of the generalized Buckingham equation in $\mathcal{X}$.*

The proof of the above result, based on fixed point arguments in $\mathcal{X}$, is rather long and delicate; for lack of space it will be presented elsewhere. However in the particular case of an initially Newtonian fluid, the functional equation is simpler and we can show some details. Instead of (17)-(19) we have now the following

$$G(t) = 2 + \frac{4}{5}\int_0^t G^2(s)\,ds \tag{20}$$

$$-4\int_0^t G(s)\int_0^1 r^5\{\int_0^s G^2(u)e^{r[F_N(u)-F_N(s)]}\,du\}\,dr\,ds$$

We can write (20) formally as $G = \mathcal{F}(G)$ and search for a solution in

$$\mathcal{S} = \{f \in C^0([0,T])/||f||_0 \leq M, f(0) = G(0)(=2)\}.$$

We first show that if $T \leq \frac{5}{32}$ than $\mathcal{F}$ maps $\mathcal{S}$ into itself: indeed

$$|\mathcal{F}(G)| \leq 2 + \frac{4}{5}\int_0^t G^2(s)\,ds \leq 2 + \frac{4}{5}M^2T, \forall t \in [0,T] \tag{21}$$

being the last addendum in (20) less than zero; then $2 + \frac{4}{5}M^2T \leq M$ iff $T \leq \max_M \frac{5(M-2)}{4M^2} = \frac{5}{32} \approx .156$

We prove now that $\mathcal{F}$ is a contraction mapping in $\mathcal{S}$. Let us put

$$\begin{aligned} E(r,u,s) &:= \exp\{r[F_N(u) - F_N(s)]\}, \\ I(s) &:= \int_0^1\int_0^s r^5G^2(u)E(r,u,s)\,du\,dr, \\ H(u,s) &:= -\int_u^s G(z)dz. \end{aligned}$$

Evidently $E \leq 1$ and

$$\begin{aligned} |E_2 - E_1| &= \exp\{r[H_2(u,s) + H_1(u,s)]\} \\ &\leq r|H_2 - H_1| \leq ||G_2 - G_1||_0(s-u); \end{aligned} \tag{22}$$

moreover

$$\begin{aligned} |\mathcal{F}(G_2) - \mathcal{F}(G_1)| &\leq (\tfrac{8}{5}MT + 4T\sup_{[0,T]} I_2(s))||G_2 - G_1||_0 \\ &+ 4\int_0^T G_1(s)|I_2 - I_1|\,ds \end{aligned}$$

Because of (22) we get

$$\begin{aligned}\int_0^T G_1(s)|I_2 - I_1|\,ds &\leq (M/2)\int_0^T\int_0^1 r^5\int_0^s\{|G_2^2(u) - G_1^2(u)|E_2(r,u,s)\\ &+ G_1^2(u)|E_2 - E_1|\}\,du\,dr\,dt\\ &\leq (M^2/12)T^2||G_2 - G_1||_0\\ &+ (M^3/2)\int_0^T\int_0^1 r^5\int_0^s |E_2 - E_1|\,du\,dr\,ds\\ &\leq (M^2/12)T^2||G_2 - G_1||_0\\ &+ (M^3/2)||G_2 - G_1||_0\int_0^T\int_0^1 r^6\int_0^s (u-s)\,du\,dr\,ds\\ &\leq (M^2/12)T^2||G_2 - G_1||_0 + ((MT)^3/84)||G_2 - G_1||_0.\end{aligned}$$

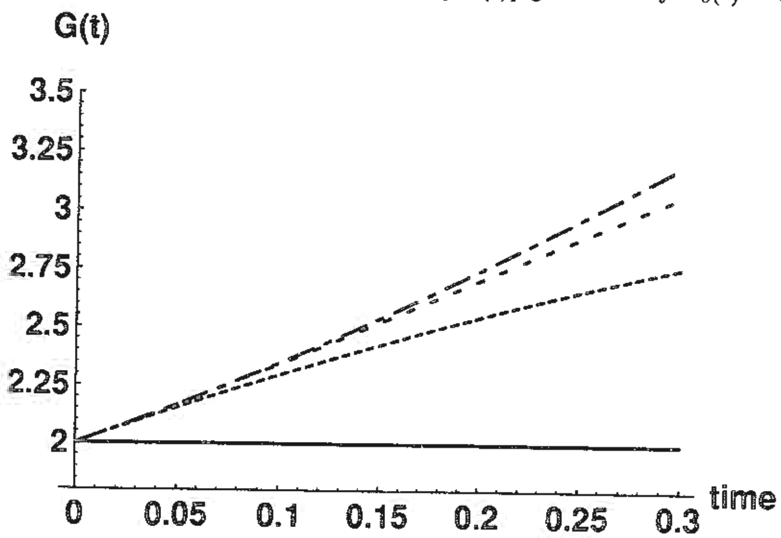
Therefore, since $\sup_{[0,T]} I_2(s)) = M^2T/6$, we finally obtain

$$|\mathcal{F}(G_2) - \mathcal{F}(G_1)| \leq C(M,T)||G_2 - G_1||_0 \tag{23}$$

where $C(M,T) := (\frac{8}{5}MT + M^2T^2 + \frac{M^3T^3}{21})$. By solving the inequality $C(M,T) < 1$ with respect to $T$ and maximizing the range of admissible $T's$ with respect to $M$ one easily gets $\approx .24$ . Thus the contraction mapping principle applies for $T \leq 5/32$ and $\mathcal{F}$ has a unique fixed point $G^* \in \mathcal{S}$.

## 4 Conclusions

The solution $G^*(t)$ can be calculated by successive approximations. The figure below shows the first elements of the sequence $\{G_n(t)\}$ generated by $G_0(t) = 2$.



For large values of $t$ the elements of $\{G_n(t)\}$ become very irregular and blow up in a finite time. It is interesting to evaluate the above upper bound in dimensional units: the time scale factor used is $t_0 = R/V\alpha$ ($V$ is the centerline velocity, while $R$ and $\alpha$ have been introduced at the very beginning of Section 2). Typical field values in CGS units could be

$$R \approx 30\,cm, \quad V \approx 100\,cm \times s^{-1}, \quad \alpha \approx 10^{-6}$$

which imply $t_0 \approx 3 \times 10^5\,s$. Then the value $T = 5/32$ means about $13\,hours$, which, at the commercial speed of $100\,cm \times s^{-1}$ corresponds to $\approx 46\,Kilometers$: this is a quite reasonable distance between two pumping stations of an industrial pipeline.

## References

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