Vector valued hermitian and quaternionic modular forms

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Abstract

Extending the method of the paper [FS3] and [Ib] we prove three structure theorems for vector valued modular forms, where two correspond to 4-dimensional cases (two hermitian modular groups, one belonging to the field of Eisenstein numbers, the other to the field of Gaussian numbers.) and one to a 6-dimensional case (a quaternionic modular group).

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Introduction

Extending the method of the papers [FS3] and [Ib] we prove three structure theorems for vector valued modular forms, where two correspond to 4-dimensional cases (two hermitian modular groups) and one to a 6-dimensional case (a quaternionic modular group). In the literature there have been treated several cases of dimension ≤ 3. We refer to [FS3] to some comments about what has been done in this area.

Vector valued modular forms belong to automorphy factors which are related to rational representations of the complexification of the maximal compact group of the underlying Lie group. In the case of the symplectic group $\mathrm{Sp}(n,\mathbb{R})$ this is $\mathrm{GL}(n,\mathbb{C})$. In the hermitian case (Lie group $\mathrm{U}(n,n)$) it is $\mathrm{GL}(n,\mathbb{C})\times\mathrm{GL}(n,\mathbb{C})$ and in the quaternary case (Lie group $\mathrm{Sp}(n,\mathbb{H})$ or, in the notation of Helgason, $\mathrm{SO}^*(4n)$) it is $\mathrm{GL}(2n,\mathbb{C})$. We will describe these general facts briefly in this paper. More details can be found in the Heidelberg diploma theses [Hey] (hermitian case) and [SH] (quaternionic case).

The basic automorphy factor is the Jacobian. In the hermitian case it corresponds to the external tensor product $\mathrm{St}\otimes\mathrm{St}$ of two standard representations of $\mathrm{GL}(n,\mathbb{C})$. In the quaternionic case it corresponds to the representation ϱ_{Jac} of $\mathrm{GL}(2n,\mathbb{C})$ with highest weight $(1,1,0,\dots,0)$.

We then restrict to $n=2$. First we treat two hermitian modular groups, one belonging to the field of Eisenstein numbers, the other to the field of Gaussian numbers. In the Eisenstein case we take the congruence group of level $\sqrt{-3}$ and in the Gauss case the congruence group of level $1+\mathrm{i}$. In both cases the groups have to be extended by the external involution $\tau(Z)=Z':={}^tZ$. Their rings of modular forms have been determined in [Ma] and [FS2]. In both cases the ring of modular forms is a polynomial ring in 5 generators of equal weight. This implies that the corresponding modular varieties are projective spaces $\mathbb{P}^4$. So the two groups are very distinguished. The main result in these cases states that in both cases a certain module of vector valued modular forms with respect to the representations $\det^r\mathrm{St}\otimes\mathrm{St}$ is generated by Rankin–Cohen brackets $\{f,g\}$ among the generators of the rings of modular forms (Theorems 3.3, 4.3).

In the quaternionic case we treat a very particular modular group of degree two which belongs to the ring of Hurwitz integers, more precisely to an extension of index 6 of the principal congruence subgroup of level $\mathfrak{p}=(1+\mathrm{i}_1)$. This group has been introduced in [FH]. Its ring of modular forms has been determined in [FS1]. It is a weighted polynomial ring where the generators have weights $3,1,1,1,1,1,1$. The direct sum of the spaces of vector valued modular forms with respect to $\det^r\varrho_{\mathrm{Jac}}$ is a module over this ring whose structure will be determined (Theorem 7.2). Again generators are given by Rankin–Cohen brackets.

1. The hermitian symplectic group

The hermitian symplectic group is the unitary group $\mathrm{U}(n,n)$. We use the matrix

$$J=\begin{pmatrix}0 & -E\\ E & 0\end{pmatrix}\qquad (E \text{ denotes the } n\times n\text{-unit matrix}).$$

The matrix $\mathrm{i}J$ is hermitian of signature (n,n). We denote its unitary group by $\mathrm{U}(n,n)$. So this is the subgroup of $\mathrm{GL}(2n,\mathbb{C})$ defined by $\bar{M}'JM=J$.

We denote by

$$\mathcal{H}_n:=\{Z\in\mathbb{C}^{n\times n};\quad \mathrm{i}(\bar{Z}'-Z)>0 \text{ (positive definite)}\}$$

the hermitian half plane. This is an open subset of the $\mathbb{C}$-vector space $\mathbb{C}^{n\times n}$. The group $\mathrm{U}(n,n)$ acts on $\mathcal{H}_n$ through the usual formula

$$MZ = (AZ+B)(CZ+D)^{-1}, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

There is an extra automorphism $Z \mapsto Z'$ and one has $M(Z')' = \bar{M}(Z)$.

The following description of vector valued automorphy factors is due to Heyen [Hey]. The maps

$$\mathrm{U}(n,n) \times \mathcal{H}_n \longrightarrow \mathrm{GL}(n,\mathbb{C}), \quad \begin{aligned} (M,Z) &\longmapsto CZ+D, \\ (M,Z) &\longmapsto \bar{C}Z'+\bar{D}, \end{aligned}$$

are both automorphy factors. Since they have equal rights, we introduce the group $\mathrm{GL}(n,\mathbb{C})\times\mathrm{GL}(n,\mathbb{C})$ (which is the complexification of the maximal compact subgroup $\mathrm{U}(n)\times\mathrm{U}(n)$ of $\mathrm{U}(n,n)$). If

$$\varrho : \mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C}) \longrightarrow \mathrm{GL}(\mathcal{Z})$$

is a rational representation on a finite dimensional complex vector space $\mathbb{C}$, we can consider the automorphic factor

$$\varrho(CZ+D, \bar{C}Z'+\bar{D}).$$

Usually we restrict to irreducible representations. They are tensor products of two irreducible representations of $\mathrm{GL}(n,\mathbb{C})$. An irreducible rational representation of $\mathrm{GL}(n,\mathbb{C})$ is called *reduced* if it is polynomial and if it does not vanish along the determinant hypersurface $\det(W)=0$. We can write every irreducible rational representation ϱ of $\mathrm{GL}(n,\mathbb{C})\times\mathrm{GL}(n,\mathbb{C})$ in the form

$$\varrho(A,B) = \det A^{r_1} \det B^{r_2} \varrho_1(A) \otimes \varrho_2(B)$$

with reduced representations ϱ_i. We denote by $\mathcal{Z}$ the representation space of ϱ.

1.1 Lemma. *Let be $M \in \mathrm{U}(n,n)$ and let be $Z \in \mathcal{H}_n$. Then*

$$\det(CZ+D) = \det M \det(\bar{C}Z'+\bar{D}).$$

The proof can be given by means of the usual generators of the group $\mathrm{U}(n,n)$ [Kr]. Details can be found in [Hey]. □

Taking into account this Lemma, we are led to Heyen's definition of automorphy factors

$$\det M^{\epsilon} \det(CZ+D)^r \varrho_1(CZ+D) \otimes \varrho_2(\bar{C}Z'+\bar{D})$$

where ϵ and r are integers and where ϱ_i are two reduced representations of $\mathrm{GL}(n,\mathbb{C})$. These are automorphy factors for the whole Lie group $\mathrm{U}(n,n)$.

2. Vector valued hermitian modular forms

The hermitian modular group with respect to an imaginary quadratic field F is

$$\Gamma_F = \{M \in \mathrm{U}(n,n); \quad M \text{ is integral in } F\}.$$

In the following $\Gamma \subset \Gamma_F$ denotes a subgroup of finite index. We introduce vector valued modular forms for automorphy factors as defined in the previous section. Since the discrete group Γ may have more characters than the power of the determinant, we should now consider instead of an integer $\epsilon \in \mathbb{Z}$ an arbitrary character $\chi : \Gamma \to \mathbb{C}^*$. Let $r \in \mathbb{Z}$ be a weight and ϱ_1, ϱ_2 two reduced representations of $\mathrm{GL}(n,\mathbb{C})$ that induce the representation $\varrho := \varrho_1 \otimes \varrho_2$ of $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$. We can consider the automorphy factor

$$J(M,Z) = \chi(M)\det(CZ+D)^r \varrho(CZ+D, \bar{C}Z' + \bar{D}).$$

A modular form with automorphy factor $J(M,Z)$ is a holomorphic function $f : \mathcal{H}_n \to \mathcal{Z}$ with the transformation property

$$f(MZ) = J(M,Z)f(Z) \quad \text{for} \quad M \in \Gamma.$$

In the case $n = 1$ the usual regularity condition at the cusps has to be added. Modular forms with the same automorphy factor form a finite dimensional vector space. We denote this space by

$$[\Gamma, r, \chi, \varrho].$$

More generally, one can consider meromorphic solutions f of the functional equation. As meromorphicity condition at the cusps, we require that there exists a scalar valued holomorphic modular form g such that gf is holomorphic. We denote the space of meromorphic forms by

$$\{\Gamma, r, \chi, \varrho\}.$$

The field of modular functions is

$$K(\Gamma) = \{\Gamma, 0, \mathrm{triv}, \mathrm{triv}\}.$$

We want to study the following example. Let St = id be the standard representation of $\mathrm{GL}(n,\mathbb{C})$. We want to take $\varrho_1 = \varrho_2 = \mathrm{St}$. The representation $\varrho = \mathrm{St} \otimes \mathrm{St}$ can be realized on the space $\mathcal{Z} = \mathbb{C}^{n\times n}$ of $n \times n$-matrices by

$$\varrho(A,B)(W) = AWB'.$$

Hence modular forms in this case are functions $f : \mathcal{H}_n \to \mathbb{C}^{n\times n}$ with the transformation property

$$f(MZ) = \chi(M)\det(CZ+D)^r(CZ+D)f(Z)(\bar{C}Z'+\bar{D})'.$$

We want to compute the derivative of a substitution $Z \mapsto MZ$. The Jacobian of a substitution $M \in \mathrm{U}(n,n)$ at a point $Z \in \mathcal{H}_n$ can be considered as a linear map

$$\mathrm{Jac}(M,Z) : \mathbb{C}^{n\times n} \longrightarrow \mathbb{C}^{n\times n}.$$

The following lemma is due to Pfrommer [Pf]. It can be proved by means of the standard generators of $\mathrm{U}(n,n)$.

2.1 Lemma. *The Jacobian* $\mathrm{Jac}(M,Z)$ *of a substitution* $M \in \mathrm{U}(n,n)$ *at a point* $Z \in \mathcal{H}$ *is given by the linear map*

$$\mathrm{Jac}(M,Z)(W) = (\bar{C}Z'+\bar{D})'^{-1}W(CZ+D)^{-1}.$$

Its determinant is

$$\det \mathrm{Jac}(M,Z) = \det M^n \det(CZ+D)^{-2n}.$$

We can consider the differentials dz_{ik} on $\mathcal{H}_n$ and collect them to a matrix $dZ = (dz_{ik})$. From Lemma 2.1 we obtain the formula

$$dZ|M = (\bar{C}Z'+\bar{D})'^{-1}dZ(CZ+D)^{-1}.$$

We want to consider meromorphic differentials on $\mathcal{H}_n$. They can be written in the form $\mathrm{tr}(fZ)$ where f is an $n\times n$-matrix of meromorphic functions on $\mathcal{H}_n$. This differential is invariant under Γ if and only if

$$f(MZ) = (CZ+D)f(Z)(\bar{C}Z'+\bar{D})',$$

a formula which can be found already in [Pf]. This is the transformation law of a form in $\{\Gamma, r, \chi, \varrho\}$ with $\chi(M) = 1$, $r = 0$ and $\varrho = \mathrm{St}\otimes\mathrm{St}$.

3. Eisenstein numbers

Let

$$F = \mathbb{Q}[\omega], \quad \omega := \frac{-1+\sqrt{-3}}{2},$$

be the field of Eisenstein numbers. In the paper [FS1] the hermitian modular group of degree two

$$\Gamma_F = \mathrm{U}(2,2) \cap \mathrm{GL}(4, \mathbb{Z}[\omega])$$

and its congruence subgroup of level $\sqrt{-3}$

$$\Gamma[\sqrt{-3}] = \Gamma_F[\sqrt{-3}] := \mathrm{kernel}(\Gamma_F \longrightarrow \mathrm{GL}(4, \mathbb{Z}[\omega]/\sqrt{-3}))$$

have been considered.

For integral r we denote by $[\Gamma[\sqrt{-3}], r]$ the space of all scalar valued holomorphic modular forms of transformation type

$$f(MZ) = \det M^r \det(CZ+D)^r f(Z) \qquad (M \in \Gamma[\sqrt{-3}]).$$

So, in our vector valued notation, this is

$$[\Gamma[\sqrt{-3}], r] = [\Gamma[\sqrt{-3}], r, \det{}^r, \mathrm{triv}].$$

We are interested in the subspace of all symmetric forms

$$[\Gamma[\sqrt{-3}], r]^{\mathrm{sym}} := \{f \in [\Gamma[\sqrt{-3}], r];\ f(Z) = f(Z')\}.$$

We collect these spaces to the ring

$$A(\Gamma[\sqrt{-3}]) := \bigoplus_r [\Gamma[\sqrt{-3}], r]^{\mathrm{sym}}.$$

In [FS1] we proved that this ring is a polynomial ring in five theta constants which have been introduced by Dern and Krieg in [DK]:

$$\Theta_p(Z) := \sum_{g \in \mathfrak{o}_F^2} e^{2\pi \mathrm{i} \overline{(g+p)}' Z (g+p)},$$

where $\sqrt{-3}p$ runs through the five

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

These five series $\Theta_1, \dots, \Theta_5$ are algebraically independent modular forms of weight one. From [FS1] we know the following result.

3.1 Theorem. *One has*

$$A(\Gamma[\sqrt{-3}]) = \mathbb{C}[\Theta_1, \dots, \Theta_5].$$

Hence $A(\Gamma[\sqrt{-3}])$ *is a polynomial ring in five variables.*

We now consider the space of symmetric vector valued modular forms

$$\mathcal{M}(r) = [\Gamma[\sqrt{-3}], r, \det{}^r, \mathrm{St} \otimes \mathrm{St}]^{\mathrm{sym}}.$$

Recall that these are holomorphic functions $f : \mathcal{H}_n \to \mathbb{C}^{2\times 2}$ with the transformation property

$$f(MZ) = \det M^r \det(CZ+D)^r (CZ+D) f(Z)(\bar{C}Z' + \bar{D})' \text{ and } f(Z) = f(Z').$$

We collect them to

$$\mathcal{M} = \bigoplus_r \mathcal{M}(r).$$

This is a graded module over $A(\Gamma[\sqrt{-3}])$.

Pfrommer [Pf] constructed examples of vector valued hermitian modular forms by means of Rankin–Cohen brackets. Following his construction, we obtain elements from $\mathcal{M}$ as follows. Let $f, g \in [\Gamma[\sqrt{-3}], 1]$. Then

$$\{f, g\} = g^2 d(f/g)$$

can be considered as element of $\mathcal{M}(2)$. In particular, we are interested in the forms $\{\Theta_i, \Theta_j\}$.

3.2 Lemma (Dern–Krieg). *If we consider the* 4 *elements* $\{\Theta_1, \Theta_i\}$, $1 < i \le 5$, *as* 4×4*-matrix, then its determinant equals, up to a constant factor,* $\Theta_1^3 \phi_9$.

Here ϕ_9 *is the skew symmetric modular form of weight* 9 *with respect to the full modular group which has been introduced in [DK].*

This lemma is just a reformulation of Corollary 6 in [DK]. □

3.3 Theorem.

$$\mathcal{M} = \sum_{1 \le i < j \le 5} A(\Gamma[\sqrt{-3}])\{\Theta_i, \Theta_j\}.$$

Defining relations are

$$\Theta_k\{\Theta_i, \Theta_j\} = \Theta_j\{\Theta_i, \Theta_k\} + \Theta_i\{\Theta_k, \Theta_j\}.$$

Proof. The space

$$\{\Gamma[\sqrt{-3}], r, \det^r, \mathrm{St}\otimes\mathrm{St}\}^{\mathrm{sym}}$$

is a vector space over the field of symmetric modular functions of dimension ≤ 4 (which is the rank of the representation $\mathrm{St}\otimes\mathrm{St}$). We get a basis if we multiply $\{\Theta_1,\Theta_i\}$, $i>1$, by Θ_i^{r-1}. Hence an arbitrary $T\in\{\Gamma[\sqrt{-3}], r, \det^r, \mathrm{St}\otimes\mathrm{St}\}^{\mathrm{sym}}$ can be written in the form

$$T=\sum_{i=2}^{5} g_i\{\Theta_1,\Theta_i\}$$

where g_i are scalar valued meromorphic modular forms. From Lemma 3.2 follows that $\Theta_1^3\phi_9 g_i$ is holomorphic. Since this is a skew modular form, Corollary 7 of [DK] can be applied to show that that $\Theta_1^3 g_i$ is holomorphic. In other words

$$\mathcal{M}\subset\frac{1}{\Theta_1^3}\sum_{j=2}^{5} A(\Gamma[\sqrt{-3}])\{\Theta_1,\Theta_j\}.$$

Since we can interchange the variables, we get

$$\mathcal{M}\subset\bigcap_i\frac{1}{\Theta_i^3}\sum_j A(\Gamma[\sqrt{-3}])\{\Theta_i,\Theta_j\}.$$

Theorem 3.3 is now an easy consequence. The argument can be found in [Wi]. We just sketch it. The module $\mathcal{M}$ is a submodule of the free module $\mathcal{M}'=\sum A(\Gamma[\sqrt{-3}])\,d\Theta_i$. An element $\sum a_i d\Theta_i$ is contained in $\mathcal{M}$ if and only if the relation $\sum a_i\Theta_i=0$ holds. Since $\mathcal{M}'$ is a free module over a factorial domain A, it is the intersection of two localizations by two coprime elements of A. □

4. Gauss numbers

Now we consider the Gauss number field $K=\mathbb{Q}[\mathrm{i}]$, its hermitian modular group

$$\Gamma_K=\mathrm{U}(2,2)\cap\mathrm{GL}(4,\mathbb{Z}[i])$$

and its congruence group of level $1+\mathrm{i}$

$$\Gamma[1+\mathrm{i}]=\Gamma_K[1+\mathrm{i}]:=\mathrm{kernel}(\Gamma_K\longrightarrow\mathrm{GL}(4,\mathbb{Z}[\mathrm{i}]/(1+\mathrm{i}))).$$

For *even* r we denote by $[\Gamma[1+\mathrm{i}],r]$ the space of all scalar valued holomorphic modular forms of transformation type

$$f(MZ)=\det M^{r/2}\det(CZ+D)^r f(Z)\qquad(M\in\Gamma[1+\mathrm{i}]).$$

So, in our vector valued notation, this is

$$[\Gamma[1+\mathrm{i}],r]=[\Gamma[1+\mathrm{i}],r,\det{}^{r/2},\mathrm{triv}]\qquad(r\equiv 0 \bmod 2).$$

We are interested in the subspace of all symmetric forms

$$[\Gamma[1+\mathrm{i}],r]^{\mathrm{sym}}:=\{f\in[\Gamma[1+\mathrm{i}],r];\ f(Z)=f(Z')\}.$$

We collect these spaces to the ring

$$A(\Gamma[1+\mathrm{i}]):=\bigoplus_{r\equiv 0 \bmod 2}[\Gamma[1+\mathrm{i}],r]^{\mathrm{sym}}.$$

Basic elements of these are the squares of the ten theta series

$$\Theta[m]=\sum_{g\in\mathbb{Z}[\mathrm{i}]^2}e^{\pi\mathrm{i}(\overline{(g+a/2)}'Z(g+a/2)+b'(g+a/2))},\quad m=\begin{pmatrix}a\\b\end{pmatrix}.$$

Here a,b are two columns of the special form

$$a=(1+\mathrm{i})\alpha,\quad b=(1+\mathrm{i})\beta,\qquad \alpha\in\{0,1\}^2,\ \beta\in\{0,1\}^2,\ \alpha'\beta\equiv 0 \bmod 2.$$

These theta functions have been introduced in [Fr]. The following result is due to Matsumoto [Ma] and has been reproved by Hermann in [He].

4.1 Theorem (Matsumoto). *We have*

$$A(\Gamma[1+\mathrm{i}])=\mathbb{C}[\dots\Theta[m]^2\dots].$$

The ten theta squares generate a five dimensional space. Hence $A(\Gamma[1+\mathrm{i}])$ is a polynomial ring in five variables.

For sake of completeness we mention that the following five $\binom{\alpha}{\beta}$ give generators:

$$\begin{pmatrix}1&0&1&0&1\\1&0&0&0&1\\0&1&0&0&1\\0&1&0&0&1\end{pmatrix}$$

We denote the five corresponding thetas by $\Theta(1),\dots\Theta(5)$.

We consider (for even r) the space of symmetric vector valued modular forms

$$\mathcal{M}(r)=[\Gamma[1+\mathrm{i}],r,\det{}^{r/2},\mathrm{St}\otimes\mathrm{St}]^{\mathrm{sym}}.$$

Recall that these are holomorphic functions $f:\mathcal{H}_n\to\mathbb{C}^{2\times 2}$ with the transformation property

$$f(MZ)=\det M^{r/2}\det(CZ+D)^r(CZ+D)f(Z)(\bar{C}Z'+\bar{D})'\text{ and }f(Z)=f(Z').$$

We collect them to

$$\mathcal{M} = \bigoplus_{r\equiv 0 \bmod 2} \mathcal{M}(r).$$

This is a graded module over $A(\Gamma[1+\mathrm{i}])$. The forms

$$\{\Theta(i)^2, \Theta(j)^2\} = \Theta(j)^4 d(\Theta(i)^2/\Theta(j)^2)$$

can be considered as element of $\mathcal{M}(4)$.

We need two scalar valued modular forms ϕ_4 and ϕ_{10} which do not belong to the ring $A(\Gamma[1+\mathrm{i}])$. The form ϕ_{10} has been introduced in [Fr] as the product of the ten thetas $\Theta[m]$. It is a symmetric modular form of weight 10 with respect to the full modular group Γ_K and the trivial multiplier system. The form ϕ_4 is a skew symmetric modular form of weight 4 with respect to the full modular group and with trivial multiplier system too. It has been introduced in [Ma] (see Lemma 3.1.3) and occurs also in [He] and [DK].

The zero set of ϕ_4 is the $\Gamma[1+\mathrm{i}]$-orbit of the set defined by $Z = Z'$ (the Siegel space of genus 2). The zero orders are one. Hence every skew symmetric modular form with respect to $\Gamma[1+\mathrm{i}]$ is divisible by ϕ_4.

The zero sets of the single $\Theta[m]$ are also known. Since the full modular group permutes the 10 one-dimensional spaces, generated by them, transitively, it is sufficient to treat one case. In the case $m' = (1,1,1,1)$ the function $\Theta[m]$ vanishes along the fixed point set of the transformation

$$Z \longmapsto Z' \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{bmatrix} := \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix} Z' \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}$$

in first order and every zero is equivalent mod $\Gamma[1+\mathrm{i}]$ to a point of this set.

4.2 Lemma. *If we consider the* 4 *elements* $\{\Theta(1)^2, \Theta(i)^2\}$, $1 < i \le 5$, *as* 4×4*-matrix, then its determinant equals, up to constant factor,* $\Theta(1)^6\phi_4\phi_{10}$.

Proof. We denote the determinant by

$$D = \det(\{\Theta(1)^2, \Theta(2)^2\}, \dots, \{\Theta(1)^2, \Theta(5)^2\}).$$

One can check that D, up to a factor $\Theta(1)^6$, is equal to the Jacobian determinant of the the five $\Theta(i)^2$ that is a modular form with respect to the full modular group. Expanding the determinant D one shows easily that $\Theta(1)^{-7}D$ is holomorphic. From this it follows that $\Theta(1)^{-6}D$ is a modular form with respect to the full modular group. It is divisible by $\Theta(1)$ and hence by all $\Theta(i)$. So $D/(\Theta(1)^6\phi_{10})$ is a holomorphic modular form of weight 4 with the same character of ϕ_4 and hence $D/(\Theta(1)^6\phi_4\phi_{10})$ is constant. □

4.3 Theorem.

$$\mathcal{M} = \sum_{1\le i<j\le 5} A(\Gamma[1+\mathrm{i}])\{\Theta(i)^2,\Theta(j)^2\}.$$

Defining relations are

$$\Theta(k)^2\{\Theta(i)^2,\Theta(j)^2\} = \Theta(j)^2\{\Theta(i)^2,\Theta(k)^2\} + \Theta(i)^2\{\Theta(k)^2,\Theta(j)^2\}.$$

Proof. Similar to the Eisenstein case an arbitrary

$$T \in \{\Gamma[1+\mathrm{i}], r, \det{}^{r/2}, \mathrm{St}\otimes\mathrm{St}\}^{\mathrm{sym}}$$

can be written in the form

$$T = \sum_{i=2}^{5} g_i\{\Theta(1)^2,\Theta(i)^2\}$$

where g_i are scalar valued meromorphic modular forms. We assume that T is holomorphic. Then from Lemma 4.2 follows that $\Theta(1)^6\phi_4\phi_{10}g_i$ is holomorphic. Since it is skew symmetric, it is divisible by ϕ_4 and we obtain that $\Theta(1)^6\phi_{10}g_i$ is holomorphic. We want to show that this form is divisible by ϕ_{10}. It is sufficient to show that it is divisible by each $\Theta[m]$ and, using the action of the full modular group, it is sufficient to restrict to $m' = (1,1,1,1)$. The forms $\Theta(1)^2$ and g_i are invariant under

$$Z \longmapsto Z'\begin{bmatrix}1 & 0\\ 0 & \mathrm{i}\end{bmatrix}$$

but ϕ_{10} changes its sign. Hence we can divide by $\Theta[m]$ and we obtain that $\Theta(1)^6 g_i$ is holomorphic.

In other words

$$\mathcal{M} \subset \frac{1}{\Theta(1)^6}\sum_{j=2}^{5} A(\Gamma[1+\mathrm{i}])\{\Theta(1)^2,\Theta(j)^2\}.$$

Since we can interchange the variables we get

$$\mathcal{M} \subset \bigcap_i \frac{1}{\Theta(i)^6}\sum_j A(\Gamma[1+\mathrm{i}])\{\Theta(i)^2,\Theta(j)^2\}.$$

Theorem 4.3 is now an easy consequence (compare Theorem 3.3). □

5. The quaternionic symplectic group

Let $\mathbb{H} = \mathbb{R} + \mathbb{R}\mathrm{i}_1 + \mathbb{R}\mathrm{i}_2 + \mathbb{R}\mathrm{i}_3$ be the field of quaternions. The defining relations are

$$\mathrm{i}_1^2 = \mathrm{i}_2^2 = \mathrm{i}_3^2 = -1,$$
$$\mathrm{i}_1\mathrm{i}_2 = -\mathrm{i}_2\mathrm{i}_1 = \mathrm{i}_3,\ \mathrm{i}_2\mathrm{i}_3 = -\mathrm{i}_3\mathrm{i}_2 = \mathrm{i}_1,\ \mathrm{i}_3\mathrm{i}_1 = -\mathrm{i}_1\mathrm{i}_3 = \mathrm{i}_2.$$

The conjugate $\bar{x}$ of a quaternion $x = x_0 + x_1\mathrm{i}_1 + x_2\mathrm{i}_2 + x_3\mathrm{i}_3$ is

$$\bar{x} := x_0 - x_1\mathrm{i}_1 - x_2\mathrm{i}_2 - x_3\mathrm{i}_3.$$

We use the embedding

$$\mathbb{H} \longrightarrow \mathbb{C}^{2\times 2}, \quad x \longmapsto \check{x},$$
$$x_0 + x_1\mathrm{i}_1 + x_2\mathrm{i}_2 + x_3\mathrm{i}_3 \longmapsto \begin{pmatrix} x_0 + \mathrm{i}x_1 & x_2 + \mathrm{i}x_3 \\ -x_2 + \mathrm{i}x_3 & x_0 - \mathrm{i}x_1 \end{pmatrix}.$$

It induces an isomorphism of algebras

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \ \xrightarrow{\sim}\ \mathbb{C}^{2\times 2}.$$

We extend this to an isomorphism of matrix algebras

$$(\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C})^{n\times n} \ \xrightarrow{\sim}\ \mathbb{C}^{2n\times 2n}$$

by applying the check operator componentwise. In particular, we obtain an embedding

$$\mathrm{GL}(n,\mathbb{H}) \longrightarrow \mathrm{GL}(2n,\mathbb{C}), \quad A \longmapsto \check{A}.$$

This is a complexification in the sense that the Lie algebra of $\mathrm{GL}(2n,\mathbb{C})$ arises as complexification of the Lie algebra of $\mathrm{GL}(n,\mathbb{H})$.

The quaternionic symplectic group $\mathrm{Sp}(n,\mathbb{H})$ consists of all $2n \times 2n$-matrices M with entries in $\mathbb{H}$ such that

$$\bar{M}' J M = J, \qquad J = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \quad (E \text{ unit matrix}).$$

(This is the group $\mathrm{SO}^*(4n)$ in the notation of Helgason.) Let $\mathcal{X}_n \subset \mathbb{H}^{n\times n}$ be the set of quaternionic hermitian $n \times n$ matrices and $\mathcal{Y}_n$ the cone of all positive definit ones. The quaternionic half plane is

$$\mathcal{H}_n = \mathcal{X}_n + \mathrm{i}\mathcal{Y}_n \subset (\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C})^n.$$

This is an open subset of the complex vector space

$$\mathcal{Z}_n = \mathcal{X}_n \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{X}_n + \mathrm{i}\mathcal{X}_n.$$

Its dimension is $2 + 2n(n-1)$.

The group $\mathrm{Sp}(n,\mathbb{H})$ acts on $\mathcal{H}_n$ through the usual formula

$$Z \longmapsto MZ = (AZ+B)(CZ+D)^{-1}, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The standard generators of $\mathrm{Sp}(n,\mathbb{H})$ and their actions are

a) $\begin{pmatrix} E & H \\ 0 & E \end{pmatrix}(Z) = Z + H \qquad (\bar{H}' = H \in \mathcal{X}_n).$

b) $\begin{pmatrix} U & 0 \\ 0 & \bar{U}'^{-1} \end{pmatrix}(Z) = UZ\bar{U}' \qquad (U \in \mathrm{GL}(n,\mathbb{H})).$

c) $J(Z) = -Z^{-1}$.

In the case b) we can apply the transformation to all $Z \in \mathcal{Z}_n$. This gives a representation

$$\varrho : \mathrm{GL}(n,\mathbb{H}) \longrightarrow \mathrm{GL}(\mathcal{Z}_n), \quad U \longmapsto (W \mapsto UW\bar{U}').$$

5.1 Remark. *There exists a unique rational (holomorphic) representation* $\varrho_{\mathrm{Jac}} : \mathrm{GL}(2n,\mathbb{C}) \longrightarrow \mathrm{GL}(\mathcal{Z}_n)$ *with the property*

$$\varrho_{\mathrm{Jac}}(\check{U}) = \varrho(U) \quad \text{for} \quad U \in \mathrm{GL}(n,\mathbb{H}).$$

Proof. The derived representation of ϱ extends to a (complex) representation of the Lie algebra of $\mathrm{GL}(2n,\mathbb{C})$. Each representation of the Lie algebra of $\mathrm{GL}(m,\mathbb{C})$ on a finite dimensional complex vector space is induced by a rational representation of $\mathrm{GL}(m,\mathbb{C})$. □

We can consider the Jacobian

$$\mathrm{Jac} : \mathrm{Sp}(n,\mathbb{H}) \times \mathcal{H}_n \longrightarrow \mathrm{GL}(\mathcal{Z}_n).$$

5.2 Lemma. *We have*

$$\mathrm{Jac}(M,Z) = \varrho_{\mathrm{Jac}}(\check{C}\check{Z} + \check{D})^{-1}.$$

Corollary.

$$\det \mathrm{Jac}(M,Z) = \det(\check{C}\check{Z} + \check{D})^{-3}.$$

Proof. Since both sides are automorphy factors, it is sufficient to prove this for generators. For the translations $Z \mapsto Z + H$ and unimodular transformations $Z \mapsto \bar{U}'ZU$ the statement is trivial. Hence it is sufficient to treat the case $M = J$. The Jacobian can be computed easily.

The Jacobian of the transformation $Z \mapsto -Z^{-1}$ is the linear map $W \mapsto Z^{-1}WZ^{-1}$.

Since the claimed formula in Lemma 5.2 is an identity between rational functions in $Z \in \mathcal{Z}_n$, it is sufficient to prove it for a *real* invertible matrix $Z \in \mathcal{X}_n$. For real Z we have $\varrho_{\text{Jac}}(\hat{Z})^{-1} = \varrho(Z)^{-1}$. This is the linear map $W \mapsto Z^{-1}WZ^{-1}$. □

From Lemma 5.2 we get a different description of the representation ϱ_{Jac}. The stabilizer $\mathrm{Sp}(n, \mathbb{H})_{\mathrm{i}E}$ of the point $\mathrm{i}E$ consists of matrices of the form

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

We attach to such a matrix $\check{A} + \mathrm{i}\check{B}$. This defines an ismorphism

$$\mathrm{Sp}(n, \mathbb{H})_{\mathrm{i}E} \longrightarrow \mathrm{U}(2n).$$

Hence the map

$$\mathrm{Sp}(n, \mathbb{H})_{\mathrm{i}E} \longrightarrow \mathrm{GL}(2n, \mathbb{C})$$

is a complexification map. The assignment $M \mapsto \mathrm{Jac}(M, \mathrm{i}E)$ is a representation of $\mathrm{Sp}(n, \mathbb{H})_{\mathrm{i}E}$. Its complexification is ϱ_{Jac}.

5.3 Lemma. *The representation*

$$\varrho_{\text{Jac}} : \mathrm{GL}(2n, \mathbb{C}) \longrightarrow \mathrm{GL}(\mathcal{Z}_n) \qquad (\dim_{\mathbb{C}} \mathcal{Z}_n = 2 + 2n(n-1))$$

is irreducible. Its highest weight is $(1, 1, 0, \ldots, 0)$.

Proof. Recall that a highest weight vector is a vector in $\mathcal{Z}_n$ that is invariant under all unipotent upper-triangular matrices. If $A \in \mathrm{GL}(n, \mathbb{H})$ is a unipotent upper-triangular matrix, then $\check{A}$ has the same property. Hence highest weight vectors have the property $\varrho(A)a = a$. It follows that the matrix $E_{11} \in \mathcal{Z}_n$ whose $(1, 1)$-entry is 1 and all other entries are 0 is up to a constant factor the only highest weight vector. Hence the representation is irreducible.

The statement about the highest weight says that a diagonal matrix $D \in \mathrm{GL}(2n, \mathbb{C})$ acts on the highest weight vector by multiplication by $d_{11}d_{22}$. Since the representation is holomorphic, it is sufficient to proof this for diagonal matrices of special shape. We can assume that the diagonal elements are of the form $d_1, \bar{d}_1, \ldots, d_n, \bar{d}_n$. Then the matrix has to act by multiplication by $|d_1|^2$. Write $d_i = \alpha_i + \mathrm{i}\beta_i$. Consider in $\mathrm{GL}(n, \mathbb{H})$ the diagonal matrix D_1 with diagonal entries $\alpha_i + \mathrm{i}_1\beta_i$. Then $\check{D}_1 = D$. Now the stated formula can be verified. □

6. A ring of quaternionic modular forms of degree two

We denote by Ω_n the image of $\mathrm{Sp}(n,\mathbb{H})$ in the group of biholomorphic self maps of $\mathcal{H}_n$. Since two symplectic matrices define the same transformation if and only if they agree up to the sign, we have

$$\Omega_n = \mathrm{Sp}(n,\mathbb{H})/\{\pm E\}.$$

The map

$$j : \mathrm{Sp}(n,\mathbb{H}) \times \mathcal{H}_2 \longrightarrow \mathbb{C}^*, \quad (M,Z) \longmapsto \det(\check{C}\check{Z}+\check{D}),$$

is a scalar valued factor of automorphy. Since $j(M,Z)=j(-M,Z)$ it factors through an automorphy factor of Ω_n,

$$j : \Omega_n \times \mathcal{H}_n \longrightarrow \mathbb{C}^*.$$

In the case $n>2$ the group Ω_n is the full group of biholomorphic self maps of $\mathcal{H}_n$. But in the case $n=2$ the full group is an extension of index two

$$\hat{\Omega}_2 = \Omega \cup \tau\Omega, \quad \tau(Z)=Z'.$$

The automorphy factor j can be extended to an automorphy factor

$$j : \hat{\Omega}_2 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_2, \quad j(\tau,Z)=1.$$

We denote by $\mathfrak{o}$ the ring of Hurwitz integers. It contains

$$\mathfrak{o}_0 := \mathbb{Z} + \mathbb{Z}\mathrm{i}_1 + \mathbb{Z}\mathrm{i}_2 + \mathbb{Z}\mathrm{i}_3$$

as subgroup of index two which is extended by the element

$$\omega := \frac{1+\mathrm{i}_1+\mathrm{i}_2+\mathrm{i}_3}{2}.$$

We consider the two sided ideal $\mathfrak{p}$ which can be generated by any element of norm 2 as for example $1+\mathrm{i}_1$.

In the paper [FH] a certain subgroup

$$\Gamma(\mathfrak{o})[\mathfrak{p}] \subset \hat{\Omega}_2$$

has been defined. It is related to the principal congruence subgroup

$$\mathrm{Sp}(2,\mathfrak{o})[\mathfrak{p}] = \mathrm{kernel}(\mathrm{Sp}(2,\mathfrak{o}) \longrightarrow \mathrm{Sp}(2,\mathfrak{o}/\mathfrak{p})).$$

We need a certain extension of this group. For this we need the following type of transformations. Let $\sigma : \mathbb{H} \to \mathbb{H}$ be an $\mathbb{R}$-linear transformation which is orthogonal in the sense that it preserves the standard scalar product $\bar{x}x$. We extend σ to hermitian matrices in $\mathcal{X}_2$ by

$$\sigma\begin{pmatrix} x_0 & x_1 \\ * & x_2 \end{pmatrix} = \begin{pmatrix} x_0 & \sigma(x_1) \\ * & x_2 \end{pmatrix}$$

Since positive definite matrices are mapped to positive definite ones, we can extend σ to a biholomorphic transformation of $\mathcal{H}_2$. Hence the orthogonal group $\mathrm{O}(4,\mathbb{R})$ acts on $\mathcal{H}_2$. This defines a certain subgroup of $\hat{\Omega}_2$. The group of diagonal matrices in $\mathrm{Sp}(2,\mathbb{H})$ produces the subgroup $\mathrm{SO}(4,\mathbb{R})$ and the extension by τ produces $\mathrm{O}(4,\mathbb{R})$.

6.1 Definition. *The group* $\mathrm{O}'(4,\mathbb{Z})$ *consists of all transformations*

$$(x_1,x_2,x_3,x_4)\longmapsto(\pm x_{\sigma(1)},\pm x_{\sigma(2)},\pm x_{\sigma(3)},\pm x_{\sigma(4)}),$$

where σ *is an arbitrary permutation and where the number of minus-signs is even.*

This is a subgroup of index 2 in $\mathrm{O}(4,\mathbb{Z})$.

6.2 Definition. *The group* $\Gamma(\mathfrak{o})[\mathfrak{p}]$ *is defined as*

$$\Gamma(\mathfrak{o})[\mathfrak{p}]=\langle \mathrm{Sp}(2,\mathfrak{o})[\mathfrak{p}]/\{\pm E\},\mathrm{O}'(4,\mathbb{Z})\rangle.$$

This group has been introduced in [FH]. Using the exceptional isogeny between $\mathrm{O}(2,6)$ and $\mathrm{Sp}(2,\mathbb{H})$, in this paper also a very natural orthogonal description of this group has been given.

6.3 Lemma. *The group* $\Gamma(\mathfrak{o})[\mathfrak{p}]$ *contains* $\mathrm{Sp}(2,\mathfrak{o})[\mathfrak{p}]/\{\pm E\}$ *as normal subgroup of index* 6*. The quotient group is isomorphic to* S_3*. It can be generated by the images of*

$$x\longmapsto \omega x\bar\omega,\qquad x\longmapsto \alpha\bar x\bar\alpha\quad \textit{where}\quad \alpha=\frac{1+\mathrm{i}_1}{\sqrt{2}}.$$

We consider modular forms for this group. The space $[\Gamma(\mathfrak{o})[\mathfrak{p}],r]$ for an integral r consists of all holomorphic functions $f:\mathcal{H}_2\to\mathbb{C}$ with the transformation property

$$f(\gamma(Z))=j(\gamma,Z)^r f(Z)\quad\text{for all}\quad \gamma\in\Gamma(\mathfrak{o})[\mathfrak{p}].$$

We collect them to the graded algebra

$$A(\Gamma(\mathfrak{o})[\mathfrak{p}])=\bigoplus_{r\in\mathbb{Z}}[\Gamma(\mathfrak{o})[\mathfrak{p}],r].$$

The structure of this algebra has been investigated in [FH] and, correcting an error in [FH], it has finally been determined in [FS2]. In [FS2] the orthogonal language has been used. The translation between the orthogonal and the symplectic picture can be found in detail in [FH].

6.4 Theorem. *The algebra* $A(\Gamma(\mathfrak{o})[\mathfrak{p}])$ *is a polynomial ring of* 7 *variables,* $G,\vartheta_1,\dots,\vartheta_6$ *where* G *is of weight* 3 *and the other six of weight* 1*.*

We mention that the weights of the symplectic picture are doubled in the orthogonal world. Hence in the paper [FS2] the weights are $6,2,\dots,2$.

7. Vector valued quaternionic modular forms of degree 2

In section 5 we studied the vector valued automorphy factor Jac. It factors through Ω_n and, in the case $n=2$, extends to $\hat\Omega_2$.

$$\mathrm{Jac}:\hat\Omega_2\times\mathcal{H}_2\longmapsto \mathrm{GL}(\mathcal{Z}_2).$$

We denote by $\mathcal{M}(r)$ the space of all holomorphic functions

$$f:\mathcal{H}_2\longrightarrow\mathcal{Z}_2$$

with the transformation property

$$f(\gamma(Z))=j(\gamma,Z)^r\mathrm{Jac}(\gamma,Z)f(Z)\quad\text{for all}\quad\gamma\in\Gamma(\mathfrak{o})[\mathfrak{p}].$$

The direct sum

$$\mathcal{M}=\bigoplus_{r\in\mathbb{Z}}\mathcal{M}(r)$$

is a graded module over $A=A(\Gamma(\mathfrak{o})[\mathfrak{p}])$.

For two non-zero homogenous elements of positive degree $f,g\in A$, we define

$$\{f,g\}:=\mathrm{wt}(g)g\,df-\mathrm{wt}(f)f\,dg.$$

Here $\mathrm{wt}(f)$ denotes the weight of f. Another way to write this is

$$\{f,g\}=\frac{g^{\mathrm{wt}(f)+1}}{f^{\mathrm{wt}(g)-1}}\,d\Big(\frac{f^{\mathrm{wt}(g)}}{g^{\mathrm{wt}(f)}}\Big).$$

This shows

$$\{f,g\}\in\mathcal{M}(\mathrm{wt}(f)+\mathrm{wt}(g)).$$

This is a skew-symmetric $\mathbb{C}$-bilinear pairing and it satisfies the following rule:

$$\mathrm{wt}(h)h\{f,g\}=\mathrm{wt}(g)g\{f,h\}+\mathrm{wt}(f)f\{h,g\}.$$

We identify the vector space $\mathcal{Z}_2$ with $\mathbb{C}^6$ using the coordinates z_0, z_2, z_{10}, z_{11}, z_{12}, z_{13} where

$$Z=\begin{pmatrix}z_0&z_1\\ *&z_2\end{pmatrix},\qquad z_1=z_{10}+\mathrm{i}_1z_{11}+\mathrm{i}_2z_{12}+\mathrm{i}_3z_{13}.$$

Then we can consider the brackets $\{f,g\}$ as columns with 6 entries.

7.1 Lemma.

$$\det(\{G,\vartheta_1\},\{\vartheta_2,\vartheta_1\},\dots,\{\vartheta_6,\vartheta_1\}) = \vartheta_1^5 D.$$

Here D is a modular form of weight 12 *with respect to a non-trivial character (more precisely: $D^2\in[\Gamma(\mathfrak{o})[\mathfrak{p}],24]$). The zero divisor of D is the $\Gamma(\mathfrak{o})[\mathfrak{p}]$-orbit of the set $z_{10}=z_{11}$. The vanishing order is one.*

Proof. We denote by ∇g the column of the partial derivatives of a holomorphic function g on $\mathcal{H}_2$ with respect to the variables $z_0,\dots,z_{13}$ In [FS2] it has been proved that the determinant

$$D = \det\begin{pmatrix} \vartheta_1 & \dots & \vartheta_6 & 3G \\ \nabla\vartheta_1 & \dots & \nabla\vartheta_6 & \nabla G \end{pmatrix}$$

is a modular form of weight 12 with respect to a nontrivial character. Now

$$\vartheta_1^6 D = \det\begin{pmatrix} \vartheta_1 & \vartheta_1\vartheta_2 & \dots & \vartheta_1\vartheta_6 & \vartheta_1 3G \\ \nabla\vartheta_1 & \vartheta_1\nabla\vartheta_2 & \dots & \vartheta_1\nabla\vartheta_6 & \vartheta_1\nabla G \end{pmatrix}.$$

Hence

$$\vartheta_1^5 D = \det\left(\{\vartheta_1,\vartheta_2\}\quad\dots\quad\{\vartheta_1,\vartheta_6\}\quad\{\vartheta_1,G\}\right).$$

In [FH] (Lemma 2.1 in connection with Proposition 8.4) has been proved that the zero divisor is the $\Gamma(\mathfrak{o})[\mathfrak{p}]$-orbit of the set $z_{10}=z_{11}$. The vanishing order is one. Lemma 7.1 is an immediate consequence. □

7.2 Theorem. *The $A(\Gamma(\mathfrak{o})[\mathfrak{p}])$-module $\mathcal{M}$ is generated by the brackets $\{f,g\}$ between the* 7 *generators. Defining relations are*

$$\{f,g\} = -\{g,f\} \quad \textit{and} \quad \operatorname{wt}(h)h\{f,g\} = \operatorname{wt}(g)g\{f,h\} + \operatorname{wt}(f)f\{h,g\}$$

for any f,g,h of the 7 *generators.*

Proof. Since the determinant of the matrix $(\{G,\vartheta_1\},\{\vartheta_2,\vartheta_1\},\dots,\{\vartheta_6,\vartheta_1\})$ does not vanish, every $T\in\mathcal{M}(r)$ can be written in the form

$$T = g_1\{G,\vartheta_1\} + g_2\{\vartheta_2,\vartheta_1\} + \dots + g_6\{\vartheta_6,\vartheta_1\}$$

with meromorphic functions g_i. They transform as modular forms in the ring $A(\Gamma(\mathfrak{o})[\mathfrak{p}])$. From Lemma 7.1 we see that $g_i\vartheta_1^5 D$ is holomorphic. Since D vanishes along $z_{10}=z_{11}$ in first order and since the transformation $z_{10}\leftrightarrow z_{11}$ is in the group $\Gamma(\mathfrak{o})[\mathfrak{p}]$, the forms $g_i\vartheta_1^5 D$ are skew symmetric with respect to this transformation. The forms g_i, ϑ_1 are symmetric. Hence $g_i\vartheta_1^5 D$ vanishes along $z_{10}=z_{11}$ and must be divisible by D. So we see

$$T \in \frac{1}{\vartheta_1^5}\big(A\{G,\vartheta_1\} + A\{\vartheta_2,\vartheta_1\} + \dots + A\{\vartheta_6,\vartheta_1\}\big).$$

One can take any ϑ_i instead of ϑ_1 and obtains

$$\mathcal{M} \subset \bigcap_{i=1}^{6} \frac{1}{\vartheta_i^5}\Big(A\{G,\vartheta_i\} + \sum_{j\neq i} A\{\vartheta_j,\vartheta_i\}\Big).$$

Theorem 7.2 is an easy consequence (compare Theorem 3.3). □

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