HÉNON TYPE EQUATIONS AND CONCENTRATION ON SPHERES

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Abstract. In this paper we study the concentration profile of various kind of symmetric solutions of some semilinear elliptic problems arising in astrophysics and in diffusion phenomena. Using a reduction method we prove that *doubly symmetric* positive solutions in a $2m$-dimensional ball must concentrate and blow up on $(m-1)$-spheres as the concentration parameter tends to infinity. We also consider axially symmetric positive solutions in a ball in $\mathbb{R}^N$, $N \geq 3$, and show that concentration and blow up occur on two antipodal points, as the concentration parameter tends to infinity.

1. Introduction

We consider semilinear elliptic problems of the type

$$\begin{cases} -\Delta u = h(x)|u|^{p-2}u \quad \text{in} \quad B_N(0,1), \\ u = 0 \quad \text{on} \quad \partial B_N(0,1), \end{cases} \tag{1.1}$$

where $B_N(0,1)$ is the unit open ball centered at the origin in $\mathbb{R}^N$, $N \geq 3$, and $p > 2$.

If $N = 2m$, $m > 1$ and $x = (y_1, y_2)$, $y_i \in \mathbb{R}^m$, $i = 1, 2$, we take

$$h(x) = |x|^\alpha = |(y_1, y_2)|^\alpha, \quad \alpha > 0 \tag{1.2}$$

or

$$h(x) = |y_2|^\alpha, \quad \alpha > 0. \tag{1.3}$$

If $N \geq 3$ and $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ we take

$$h(x) = |x_N|^\alpha, \quad \alpha > 0. \tag{1.4}$$

The first choice corresponds to the case of the well-known Hénon equation [11], while the other two are variants which have interest in applications as we will explain later.

From the mathematical point of view problem (1.1) has an interesting and rich structure and various results have been proved so far, some of which will be described in the sequel. Let us recall that, in the case of (1.2), it was first observed in [16] that the presence of the weight $|x|^\alpha$ modifies the consequences of the Pohožahev identity and produces a new critical exponent, namely $2(N + \alpha)/(N - 2)$, for the existence of classical solutions. In [7, 8, 23], symmetry breaking, asymptotics and single point concentration profile at the boundary of the least energy solutions, as $\alpha \to \infty$, are described. Moreover, in [17, 24], it is proved

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that any least energy solution is foliated Schwarz symmetric. More recently, the existence of infinitely many positive solutions have been proved in [26], in the case $p = 2N/(N-2)$.

The main purpose of this paper is to present a new feature of the Hénon equation and of its variant with $h(x)$ given by (1.3), namely the existence of positive solutions concentrating on spheres, as $\alpha \to \infty$.

In the study of semilinear elliptic equations with power-like nonlinearities there are few results of this type, unlike the case the concentration at a single point about which a large literature is available. For the case of (1.4) we will show instead concentration at antipodal points for some axially symmetric solutions, as $\alpha \to \infty$.

Before stating precisely our results let us outline the connections between our problems and some mathematical models arising in astrophysics and diffusion processes.

It was during the golden age of general relativity, from 1960 to 1975, that astrophysicists started to devote intensive attention to understand and to detect the existence of black holes in globular clusters. In 1972, Peebles [20, 21] published seminal works describing a stationary distribution of stars near a massive collapsed object, such as a black hole, located at the center of a globular cluster. In recent years, as in [15], the existence of single black holes in globular clusters have been perceived. More recently, it was reported in [25] the presence of two flat-spectrum radio sources in the Milky Way globular cluster M22.

It can be derived that the existence of stationary stellar dynamics models, cf. [3, 4, 13, 14, 5], is equivalent to the solvability of the equation

$$-\Delta U(x) = f(|x|, U(x)) \quad \text{in} \quad \mathbb{R}^3,$$

which, in the case $f(|x|, U) = |x|^\alpha |U|^{p-2}U$, $\alpha > 0$, $p > 2$, becomes the Hénon equation:

$$-\Delta U(x) = |x|^\alpha |U|^{p-2}U \quad \text{in} \quad \mathbb{R}^3,$$

where the weight $|x|^\alpha$ represents a black hole located at the center of the cluster, whose absorption strength increases as α increases. This corresponds to the case of problem (1.1) with $h(x)$ given by (1.2), while when we take $h(x)$ as in (1.3) or (1.4) it corresponds to a supermassive absorbing object represented by a higher dimensional body.

Besides its application to astrophysics, problem (1.1) also models steady-state distributions in other diffusion processes. For example, suppose $u(x)$ represents the density of some chemical solute, as a function of the position x, confined in a ball B. In this case $|u|^{p-2}u$ corresponds to the reaction term and the weight $h(x)$ is an intrinsic property of the medium B, which inhibits diffusion at the region A where $h = 0$ and hampers diffusion close to A. So, in case h is one of (1.2), (1.3) or (1.4), stronger obstructions correspond to larger values of α. This line of reasoning leads that, as $\alpha \to \infty$, concentration on parts of the domain far away from A should occurs; cf. [7, 8] in the case of (1.2).

Our results show that models having objects which inhibit diffusion, represented by a point as in (1.2) or by objects of higher dimension as in (1.3), can produce concentration on spheres, located as far as possible from the absorbing object as the parameter α tends to infinity.

We point out that our results, in the cases of (1.2) and (1.3) holds in balls contained in $\mathbb{R}^{2m}$, $m > 1$ and hence not in $\mathbb{R}^3$ which is the relevant case for the astrophysics models. We believe that the concentration phenomenon we describe should give insights also for the 3-dimensional case. However, for other diffusion processes it is meaningful to pose the problem in higher dimensional spaces.

The result we get regarding problem (1.1) with $h(x)$ given by (1.4) instead applies to any dimension $N \geq 3$, so, in particular, covers the case of the astrophysical model.

To state precisely our results we need to introduce some notations that we will keep throughout the paper. For each $r > 0$, we set

$$B_k(0,r) := \{y \in \mathbb{R}^k; |y| < r\}, \quad S_r^{k-1} := \{y \in \mathbb{R}^k; |y| = r\} \quad \text{and} \quad S^{k-1} := \{y \in \mathbb{R}^k; |y| = 1\}. \tag{1.5}$$

In addition, for sake of clarity, since we will perform changes of variables that will affect the space dimension, for any function $u : \Omega \subset \mathbb{R}^k \to \mathbb{R}$ we denote the usual Laplacian of u in $\mathbb{R}^k$ by

$$\Delta_k u(x) = \sum_{i=1}^{k} u_{x_i x_i}(x), \quad \forall\, x \in \Omega.$$

We will say that $u : B_{2m}(0,1) \to \mathbb{R}$, $m \geq 1$, is doubly symmetric if

$$u(y_1, y_2) = u(|y_1|, |y_2|), \quad \forall\, (y_1, y_2) \in B_{2m}(0,1).$$

Our first result concerns the case (1.2) and describes the concentration profile of doubly symmetric solutions of the Hénon equation, that is:

$$\begin{cases} -\Delta_{2m} u = |(y_1,y_2)|^\alpha |u|^{p-2} u, & (y_1,y_2) \in B_{2m}(0,1), \\ u = 0 \quad \text{on} \quad \partial B_{2m}(0,1). \end{cases} \tag{1.6}$$

Theorem 1.1. *Assume $m > 1$, $2 < p < \frac{2(m+1)}{m-1}$. Then there exists $\alpha_0 = \alpha_0(p,m) > 4$ such that for each $\alpha > \alpha_0$ the following holds. Let u_α be any least energy solution among the doubly symmetric solutions of (1.6). Then, up to replacing u_α by $-u_\alpha$ and up to commuting y_1 and y_2, one has $u_\alpha > 0$ in $B_{2m}(0,1)$, u_α is not radially symmetric and there exists $0 < r_\alpha < 1$ such that*

$$\mathcal{M}_\alpha := \max_{(y_1,y_2) \in B_{2m}(0,1)} u_\alpha(y_1,y_2) = u_\alpha(y_1, 0), \quad \forall\, y_1 \in S_{r_\alpha}^{m-1}.$$

Moreover, u_α concentrates and blows up on $S^{m-1} \times \{0\} \subset \mathbb{R}^{2m}$, i.e., $r_\alpha \to 1$, $\mathcal{M}_\alpha \approx \alpha^{2/(p-2)}$ and $\alpha(1-r_\alpha) \to \ell$ for some positive number l, as $\alpha \to \infty$.

Next we consider the case (1.3) and describe the concentration profile of doubly symmetric solutions of the equation

$$\begin{cases} -\Delta_{2m} u = |y_2|^\alpha |u|^{p-2} u, & (y_1,y_2) \in B_{2m}(0,1), \\ u = 0 \quad \text{on} \quad \partial B_{2m}(0,1). \end{cases} \tag{1.7}$$

Theorem 1.2. *Assume $m > 1$, $2 < p < \frac{2(m+1)}{m-1}$. Then there exists $\alpha_0 = \alpha_0(p,m) > 4$ such that for each $\alpha > \alpha_0$ the following holds. Let u_α be any least energy solution among the doubly symmetric solutions of (1.7). Then, up to replacing u_α by $-u_\alpha$, one has $u_\alpha > 0$ in $B_{2m}(0,1)$, u_α is not radially symmetric and there exists $0 < r_\alpha < 1$ such that*

$$\mathfrak{M}_\alpha := \max_{(y_1,y_2) \in B_{2m}(0,1)} u_\alpha(y_1,y_2) = u_\alpha(0, y_2), \quad y_2 \in S_{r_\alpha}^{m-1}.$$

Moreover, u_α concentrates and blows up on $\{0\} \times S^{m-1} \subset \mathbb{R}^{2m}$, i.e., $r_\alpha \to 1$, $\mathfrak{M}_\alpha \approx \alpha^{2/(p-2)}$ and $\alpha(1-r_\alpha) \to \ell$ for some positive number l, as $\alpha \to \infty$.

Let us stress that from the physical point of view, the concentration phenomenon described in Theorem 1.2 is indeed expected. Since the set

$$\mathfrak{D}_{\mathfrak{m}} = \{(y_1, 0) \in \mathbb{R}^m \times \mathbb{R}^m; |y_1| < 1\}$$

inhibits diffusion and its hindrance to diffusion increases with α, the maximum point of the density u is expected to be as far apart from $\mathfrak{D}_{\mathfrak{m}}$ as possible when α tends to infinity. However, the concentration profile described in Theorem 1.1 is a bit less evident. Indeed in this case, the set $\mathfrak{D}_{\mathfrak{o}}$ which inhibits diffusion is reduced to the origin $(0,0)$. Then there are three possible doubly symmetric sets, as far away as possible from $\mathfrak{D}_{\mathfrak{o}}$, where in principle, concentration could occurs, namely $S^{m-1} \times \{0\} \subset \mathbb{R}^{2m}$ (which up to rotation is the same as $\{0\} \times S^{m-1} \subset \mathbb{R}^{2m}$), $S^{m-1} \times S^{m-1} \subset \mathbb{R}^{2m}$ or $S^{2m-1} \subset \mathbb{R}^{2m}$. Nevertheless, thinking about reducing the energy, the second and third possible concentration profiles can be excluded since solutions concentrating on $S^{m-1} \times \{0\} \subset \mathbb{R}^{2m}$ have lower energy than those that concentrate on $S^{m-1} \times S^{m-1} \subset \mathbb{R}^{2m}$ or on $S^{2m-1} \subset \mathbb{R}^{2m}$. This is in fact the meaning of Theorem 1.1.

Note that in Theorem 1.1 and Theorem 1.2 the exponent $2 < p < 2(m+1)/(m-1)$ can be larger than the critical Sobolev exponent in dimension $2N$, namely $4N/(2N-2)$, and that $2(m+1)/(m-1)$ is the critical Sobolev exponent in dimension $m+1$.

Finally we turn to the case (1.4) where we are able to get results in any dimension $N \geq 3$. Let us point out that in the 3-dimensional case with $x = (x_1, x_2, x_3) \in B_3(0,1)$, the weight $|x_3|^\alpha$ represents a two dimensional absorbing object described by

$$\mathfrak{D}_2 = \{(x_1, x_2, 0) \in \mathbb{R}^2 \times \mathbb{R}; |(x_1, x_2)| < 1\},$$

whose absorption strength increases as α increases. In this case, reasoning as above, a concentration phenomenon on the antipodal points $(0,0,1)$ and $(0,0,-1)$ is expected as $\alpha \to \infty$. Indeed, we prove this result in any dimension.

Theorem 1.3. *Let $N \geq 3$ and $2 < p < 2N/(N-2)$. Consider the problem*

$$\begin{cases} -\Delta_N u = |x_N|^\alpha |u|^{p-2} u, & x = (x_1, \dots, x_N) \in B_N(0,1), \\ u = 0 \quad on \quad \partial B_N(0,1). \end{cases} \tag{1.8}$$

Let u_α be any least energy solution of (1.8) among the solutions that are axially symmetric with respect to $\mathbb{R}e_N \subset \mathbb{R}^N$ and symmetric with respect to x_N. Then, up to replacing u_α by $-u_\alpha$, one has $u_\alpha > 0$ in $B_N(0,1)$ and there exists $0 \leq r_\alpha < 1$ such that

$$\mathbf{M}_\alpha := \max_{(x_1,\dots,x_N) \in B_N(0,1)} u_\alpha(x_1, \dots, x_N) = u_\alpha(0, \dots, 0, r_\alpha) = u_\alpha(0, \dots, 0, -r_\alpha).$$

Moreover, u_α concentrates and blows up, simultaneously, on the antipodal points $(0, \dots, 0, 1)$ and $(0, \dots, 0, -1)$, i.e., $r_\alpha \to 1$, $\mathbf{M}_\alpha \approx \alpha^{2/(p-2)}$ and $\alpha(1 - r_\alpha) \to \ell$ for some positive number l, as $\alpha \to \infty$.

Let us explain shortly the strategies of the proofs of the above theorems.

To prove Theorem 1.1 and Theorem 1.2 we perform a change of variables which allows us to reduce problems (1.6) and (1.7) to other semilinear problems in $\mathbb{R}^{m+1}$. The main feature of this transformation is that it sends in a bijective way doubly symmetric solutions of (1.6) or (1.7) to axially symmetric solutions of the reduced problems. This explain why the exponent p can be larger than the critical exponent in dimension $2N$. We mention that this approach was introduced in [18] to study concentration on spheres of some singularly perturbed problems in $\mathbb{R}^{2m}$ and relied on an idea of [22].

So, in the case of (1.6) and (1.7), we are led to study the $(m+1)$-dimensional problems (3.4) and (4.4) respectively, for which analyzing the behavior, as $\alpha \to \infty$, of the solutions we get point concentration results, which, going back to the $2m$-dimensional problems, imply concentration on $(m-1)$-spheres.

We stress that while the reduced problem (3.4) corresponding to (1.6) is still a Hénon problem for whose analysis we can use existing results, the reduced problem (4.4) corresponding to (1.7) needs a complete new analysis, in particular for what concerns the study of a *limit problem* in $\mathbb{R}^{m+1}_+$.

A similar analysis is also used in the proof of Theorem 1.3 to study problem (1.8) directly, without using any reduction argument.

The outline of this paper is the following. In Section 2 we explain the reduction method. In Section 3 we prove Theorem 1.1 while in Section 4, after several preliminary crucial estimates, we prove Theorem 1.2. Finally in Section 5 we prove Theorem 1.3.

2. A preliminary reduction lemma

Given an integer $m \geq 1$ we set $\mathcal{G}_m := \mathcal{O}(m) \times \mathcal{O}(m) \subset \mathcal{O}(2m)$. Then $g \in \mathcal{G}_m$ if, and only if, there exist $g_1, g_2 \in \mathcal{O}(m)$ such that

$$g(y_1, y_2) = (g_1 y_1, g_2 y_2), \quad \forall\, y_1, y_2 \in \mathbb{R}^m.$$

Definition 2.1. *We say that a set $\Omega \subset \mathbb{R}^{2m}$ is invariant by the action of $\mathcal{G}_m$ if $g\Omega = \Omega$ for all $g \in \mathcal{G}_m$. Given a function $u : \Omega \to \mathbb{R}$ defined on an invariant set Ω, we say that u is doubly symmetric if*

$$u(g(y_1, y_2)) = u(g_1 y_1, g_2 y_2) = u(y_1, y_2), \quad \forall\, (y_1, y_2) \in \Omega, \quad \forall\, g = (g_1, g_2) \in \mathcal{G}_m.$$

As above, a function $u : \Omega \to \mathbb{R}$, defined on a invariant set $\Omega \subset \mathbb{R}^{2m}$, is said to be doubly symmetric if

$$u(y_1, y_2) = u(|y_1|, |y_2|), \quad \forall\, (y_1, y_2) \in \Omega.$$

Now we perform a suitable change of variables as in [18]. Given any point $(y_1, y_2) \in B_{2m}(0,1)$ we write:

$$\begin{cases} |y_1| := r\cos\theta; \quad |y_2| := r\sin\theta; \quad r := \sqrt{|y_1|^2 + |y_2|^2}; \quad r \in [0,1),\ \theta \in \left[0, \dfrac{\pi}{2}\right]; \\ r = \sqrt{2\rho}; \quad \theta = \dfrac{\sigma}{2}; \quad \rho \in \left[0, \dfrac{1}{2}\right) \text{ and } \sigma \in [0, \pi]. \end{cases} \tag{2.1}$$

Given any doubly symmetric C^2-function $u : B_{2m}(0,1) \to \mathbb{R}$, we write

$$u(y_1, y_2) = u(|y_1|, |y_2|) = u(r\cos\theta, r\sin\theta) = u(r, \theta) \tag{2.2}$$

and we get

$$\Delta_{2m} u(y_1, y_2) = u_{rr}(r,\theta) + \frac{2m-1}{r} u_r(r,\theta) + \frac{m-1}{r^2}\left(\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}\right) u_\theta(r,\theta) + \frac{u_{\theta\theta}(r,\theta)}{r^2}.$$

Then we write $v(\rho, \sigma) := u\left(\sqrt{2\rho}, \dfrac{\sigma}{2}\right)$, with $r = \sqrt{2\rho}$ and $\theta = \dfrac{\sigma}{2}$ and we get

$$\Delta_{2m} u(y_1, y_2) = 2\rho\left(v_{\rho\rho}(\rho,\sigma) + \frac{m}{\rho} v_\rho(\rho,\sigma) + \frac{m-1}{\rho^2}\frac{\cos\sigma}{\sin\sigma} v_\sigma(\rho,\sigma) + \frac{v_{\sigma\sigma}(\rho,\sigma)}{\rho^2}\right).$$

Now we recall that, if $v : B_{m+1}(0, 1/2) \to \mathbb{R}$ is an axially symmetric C^2-function with respect to the axis $\mathbb{R}e_{m+1} \subset \mathbb{R}^{m+1}$ and if we set

$$z = (z_1, \ldots, z_m, z_{m+1}); \quad \rho = |z|; \quad z_{m+1} = \rho\cos\sigma \quad \rho \in \left[0, \frac{1}{2}\right) \text{ and } \sigma \in [0, \pi], \tag{2.3}$$

then

$$v(z_1, \ldots, z_m, z_{m+1}) = v(\rho, \sigma) \tag{2.4}$$

and

$$\Delta_{m+1} v(z_1, \ldots, z_m, z_{m+1}) = v_{\rho\rho}(\rho, \sigma) + \frac{m}{\rho} v_\rho(\rho, \sigma) + \frac{m-1}{\rho^2}\frac{\cos\sigma}{\sin\sigma} v_\sigma(\rho, \sigma) + \frac{v_{\sigma\sigma}(\rho, \sigma)}{\rho^2}.$$

At this point we have proved the following lemma; see also [18, Section 3].

Lemma 2.2. *There exists a one to one correspondence between doubly symmetric C^2-function $u : B_{2m}(0, 1)\backslash\{0\} \to \mathbb{R}$ and the axially symmetric, with respect to the axis $\mathbb{R}e_{m+1} \subset \mathbb{R}^{m+1}$, C^2-functions, $v : B_{m+1}(0, 1/2)\backslash\{0\} \to \mathbb{R}$. This correspondence is given by*

$$u(y_1, y_2) = u(|y_1|, |y_2|) = u(r\cos\theta, r\sin\theta) = u(r, \theta) = u\left(\sqrt{2\rho}, \frac{\sigma}{2}\right) = v(\rho, \sigma) = v(z_1, \ldots, z_m, z_{m+1}), \tag{2.5}$$

with the change of variables (2.1) and (2.3). Moreover,

$$\Delta_{2m} u(y_1, y_2) = 2|z|\Delta_{m+1} v(z_1, \ldots, z_m, z_{m+1}). \tag{2.6}$$

We also stress that the changes of variables (2.1) and (2.3) lead to

$$\left.\begin{aligned} |y_1| = r\cos\theta = \sqrt{2\rho}\cos\left(\frac{\sigma}{2}\right) = \sqrt{2|z|}\sqrt{\frac{1+\cos\sigma}{2}} = \sqrt{|z| + |z|\cos\sigma} = \sqrt{|z| + z_{m+1}}\,, \\ |y_2| = r\sin\theta = \sqrt{2\rho}\sin\left(\frac{\sigma}{2}\right) = \sqrt{2|z|}\sqrt{\frac{1-\cos\sigma}{2}} = \sqrt{|z| - |z|\cos\sigma} = \sqrt{|z| - z_{m+1}}\,, \\ |(y_1, y_2)| = \sqrt{|y_1|^2 + |y_2|^2} = \sqrt{2|z|}\,. \end{aligned}\right\} \tag{2.7}$$

Remark 2.3. *Due to the singularity of $|z|$ at $z = 0$, Lemma 2.2 does not hold between doubly symmetric C^2-function $u : B_{2m}(0, 1) \to \mathbb{R}$ and the axially symmetric with respect to the axis $\mathbb{R}e_{m+1} \subset \mathbb{R}^{m+1}$ C^2-functions, $v : B_{m+1}(0, 1/2) \to \mathbb{R}$. Indeed, for each $m \geq 2$ consider*

$$u(y_1, y_2) = |(y_1, y_2)|^2, \qquad (y_1, y_2) \in B_{2m}(0, 1).$$

Then $u \in C^\infty(B_{2m}(0, 1))$ and $\Delta_{2m} u(y_1, y_2) = 4m$. The function $v : B_{m+1}(0, 1/2) \to \mathbb{R}$ associated to u is given by

$$v(z_1, \ldots, z_m, z_{m+1}) = 2|z|, \qquad z \in B_{m+1}(0, 1/2),$$

which is singular at $z = 0$.

3. Radially invariant problems and proof of Theorem 1.1

In the search of doubly symmetric solutions of

$$\begin{cases} -\Delta_{2m} u = f(|(y_1, y_2)|, u), & (y_1, y_2) \in B_{2m}(0, 1), \\ u = 0 \quad \text{on} \quad \partial B_{2m}(0, 1), \end{cases} \tag{3.1}$$

we perform the change of variables from Section 2 and we are led to investigate the existence of axially symmetric, with respect to $\mathbb{R}e_{m+1} \subset \mathbb{R}^{m+1}$, solutions of

$$\begin{cases} -\Delta_{m+1} v = \dfrac{f(\sqrt{2|z|}, v)}{2|z|}, & z \in B_{m+1}(0, 1/2), \\ v = 0 \quad \text{on} \quad \partial B_{m+1}(0, 1/2). \end{cases} \tag{3.2}$$

Due to the singularity of $|z|$ at $z = 0$ as well as the possible singularity of $\dfrac{f(\sqrt{2|z|}, v)}{2|z|}$ at $z = 0$, the claim about the equivalence between problems (3.1) and (3.2) must be careful checked; see Remark 2.3 and also [9, Theorem 2.3] for a related regularity problem. Note that in [18] the domains considered are annuli, so the singularity at the origin does not appear. Nevertheless, for the Hénon equation, that is in the case $f(|(y_1, y_2)|, u) = |(y_1, y_2)|^\alpha |u|^{p-2}u$, we get it. Our arguments are based on some regularity results, namely equivalence between weak and classical solutions. In this direction we mention that the classical regularity results as in [1, 6] does not apply to our problems posed in $\mathbb{R}^{2m}$ since we are working with problems that may be supercritical in the sense that $2 < p < 2(m + 1)/(m - 1)$ allows $p > 4m/(2m - 2)$.

In order to proceed with (3.1) and (3.2) with $f(|(y_1, y_2)|, u) = |(y_1, y_2)|^\alpha |u|^{p-2}u$, that is,

$$\begin{cases} -\Delta_{2m} u = |(y_1, y_2)|^\alpha |u|^{p-2}u, & (y_1, y_2) \in B_{2m}(0, 1), \\ u = 0 \quad \text{on} \quad \partial B_{2m}(0, 1), \end{cases} \tag{3.3}$$

and

$$\begin{cases} -\Delta_{m+1} v = |2z|^{\frac{\alpha-2}{2}} |v|^{p-2}v, & z \in B_{m+1}(0, 1/2), \\ v = 0 \quad \text{on} \quad \partial B_{m+1}(0, 1/2). \end{cases} \tag{3.4}$$

we need to introduce some notation. Also observe that if we write

$$w(z) = \left(\frac{1}{4}\right)^{1/(p-2)} v\left(\frac{z}{2}\right), \tag{3.5}$$

then v is a solution of (3.4) if, and only if, w is a solution of

$$\begin{cases} -\Delta_{m+1} w = |z|^{\frac{\alpha-2}{2}} |w|^{p-2}w, & z \in B_{m+1}(0, 1), \\ w = 0 \quad \text{on} \quad \partial B_{m+1}(0, 1). \end{cases} \tag{3.6}$$

Definition 3.1. *Assume $m \geq 2$, $2 < p < \frac{2(m+1)}{m-1}$ and set*

$$\mathcal{H}_m := \{u \in H_0^1(B_{2m}(0, 1));\ u(g_1 y_1, g_2 y_2) = u(y_1, y_2),\ \forall\, (y_1, y_2) \in B_{2m}(0, 1),\ \forall\, g = (g_1, g_2) \in \mathcal{G}_m\},$$

with $\mathcal{G}_m$ as defined in Section 2. Then, cf. [2, Theorem 2.1 and Corollary 2.3]*, there exists $\alpha_0 = \alpha_0(p, m) > 4$ such that $\mathcal{H}_m$ is compactly imbedded in $L^p(B_{2m}(0, 1), |(y_1, y_2)|^\alpha)$ for every $\alpha > \alpha_0$. Assume $\alpha > \alpha_0$.*

1. *We say that U is a weak doubly symmetric solutions of (3.3) if U is a critical point of the $C^1(\mathcal{H}_m, \mathbb{R})$-functional*

$$I_m(u) = \frac{1}{2} \int_{B_{2m}(0,1)} |\nabla u|^2 d(y_1, y_2) - \frac{1}{p} \int_{B_{2m}(0,1)} |(y_1, y_2)|^\alpha |u|^p d(y_1, y_2), \quad u \in \mathcal{H}_m.$$

2. *We say that $u_\alpha \in \mathcal{H}_m$ is a least energy solution among the doubly symmetric solutions of (3.3) if u_α is a nontrivial doubly symmetric solution of (3.3) and*

$$I_m(u_\alpha) = \min\{I_m(u);\ u \text{ is a nontrivial doubly symmetric solution of } (3.3)\}.$$

3. *We say that W is a weak solution of* (3.6) *if W is a critical point of the $C^1(H_0^1(B_{m+1}(0,1),\mathbb{R})$-functional*
$$J(w) = \frac{1}{2}\int_{B_{m+1}(0,1)} |\nabla w|^2 dz - \frac{1}{p}\int_{B_{m+1}(0,1)} |z|^{\frac{\alpha-2}{2}}|w|^p dz, \quad w \in H_0^1(B_{m+1}(0,1)).$$
4. *We say that $w_\alpha \in H_0^1(B_{m+1}(0,1))$ is a least energy solution of* (3.6) *if w_α is a nontrivial solution of* (3.6) *and*
$$J(w_\alpha) = \min\{J(w);\ w \text{ is nontrivial solution of } (3.6)\}.$$

Lemma 3.2 ([9, Propositions 5.4 and 5.5])**.** *Assume $m \geq 2$ and $2 < p < \frac{2(m+1)}{m-1}$. There exists $\alpha_0 = \alpha_0(p,m) > 4$ such that for every $\alpha > \alpha_0$, u is a weak doubly symmetric solution of* (3.3) *if, and only if, u is a classical doubly symmetric solutions of* (3.3)*. In this case, $u \in C^{2,\gamma}(\overline{B}_{2m}(0,1))$ for all $0 < \gamma < 1$.*

Proposition 3.3. *Assume $m \geq 2$, $2 < p < \frac{2(m+1)}{m-1}$. Then there exists $\alpha_0 = \alpha_0(p,m) > 4$ such that for every $\alpha > \alpha_0$,* (2.5) *provides a bijective correspondence between*
$$X = \{u \in C^2(\overline{B}_{2m}(0,1));\ u \text{ is a doubly symmetric classical solution of } (3.3)\}$$
and
$$Y = \{v \in C^2(\overline{B}_{m+1}(0,1/2));\ v \text{ is an axially symmetric, w.r.t. } \mathbb{R}e_{m+1} \subset \mathbb{R}^{m+1}, \text{ classical solution of } (3.4)\}.$$
In addition, any $u \in X$ and any $v \in Y$ are such that $u \in C^{2,\gamma}(\overline{B}_{2m}(0,1))$, $v \in C^{2,\gamma}(\overline{B}_{m+1}(0,1/2))$ for all $0 < \gamma < 1$.

Proof. Let $u \in X$. Then u is a weak doubly symmetric solution of (3.3). So, after changing variables, we get that the function v, associated to u by (2.5), is a weak solution of (3.4) in the sense of $H_0^1(B_{m+1}(0,1/2))$; we have also used the classical result of Palais [19]. Hence, since we have subcritical growth for the problem posed in $B_{m+1}(0,1/2) \subset \mathbb{R}^{m+1}$, we apply [1] to get that v is a classical solution of (3.4).

On the other hand, let $v \in Y$. Then, after changing variables, we get that the function u, associated to v by (2.5), is a weak doubly symmetric solution of (3.3), hence classical by Lemma 3.2. □

We mention that is proved in [2] that the Hénon equation has doubly symmetric solutions, that are non radially symmetric, in case $2 < p < 2(m+1)/(m-1)$ and α is sufficiently large.

Now, from [17, 23, 24, 7, 8], we collect some results about the least energy solutions of (3.6).

Proposition 3.4. *Assume $m \geq 2$, $2 < p < \frac{2(m+1)}{m-1}$. Then there exists $\alpha_0 = \alpha_0(p,m) > 4$ such that for each $\alpha > \alpha_0$, any least energy solution w_α of* (3.6) *(up to rotation and up to replacing w_α by $-w_\alpha$) is such that:*

(i) $w_\alpha > 0$ in $B_{m+1}(0,1)$; w_α is not radially symmetric; w_α is Schwarz foliated w.r.t. the vector $e_{m+1} \in \mathbb{R}^{m+1}$, in particular w_α is axially symmetric w.r.t. $\mathbb{R}e_{m+1} \subset \mathbb{R}^{m+1}$.

(ii) w_α concentrates at the point $(0,\dots,0,1)$ as $\alpha \to \infty$. In addition, let $0 < \tau_\alpha < 1$ be such that
$$\mathcal{M}'_\alpha = \max_{z \in B_{m+1}(0,1)} w_\alpha(z) = w_\alpha((0,\dots,0,\tau_\alpha)).$$
Then $\alpha(1-\tau_\alpha) \to \ell$ for some positive number l and $\mathcal{M}'_\alpha \approx \alpha^{2/(p-2)}$ as $\alpha \to \infty$.

Proposition 3.5. *Assume $m \geq 2$, $2 < p < \frac{2(m+1)}{m-1}$. Then there exists $\alpha_0 = \alpha_0(p,m) > 4$ such that for each $\alpha > \alpha_0$, u_α is a least energy solution among the doubly symmetric solutions of* (3.3) *if, and only if, w_α is a least energy solution of* (3.6) *and u_α and w_α are related by* (2.5) *and* (3.5)*.*

Proof. It follows from the change of variables involving u_α and w_α by means of (2.5) and (3.5). □

Corollary 3.6. *Assume $m \geq 2$, $2 < p < \frac{2(m+1)}{m-1}$. Then there exists $\alpha_0 = \alpha_0(p, m) > 4$ such that for each $\alpha > \alpha_0$ the following holds. Let u_α be a least energy solution among the doubly symmetric solutions of (3.3). Then, up to replacing u_α by $-u_\alpha$, one has $u_\alpha > 0$ in $B_{2m}(0, 1)$, u_α is not radially symmetric, there exists $0 < r_\alpha < 1$ and $\theta_* \in \left\{0, \frac{\pi}{2}\right\}$ such that*

$$\max_{(y_1,y_2)\in B_{2m}(0,1)} u_\alpha(y_1, y_2) = u_\alpha(r_\alpha, \theta_*).$$

We stress that the above corollary guarantees that, up to replacing $u_\alpha(y_1, y_2)$ by $\overline{u}_\alpha(y_1, y_2) := u_\alpha(y_2, y_1)$, we have

$$\max_{(y_1,y_2)\in B_{2m}(0,1)} u_\alpha(y_1, y_2) = u_\alpha(y_1, 0), \quad |y_1| = r_\alpha.$$

Proof of Theorem 1.1. It is a straightforward consequence of Propositions 3.4, 3.5 and Corollary 3.6. □

4. Partially symmetric problems and proof of Theorem 1.2

In the search of doubly symmetric solutions of

$$\begin{cases} -\Delta_{2m} u = f(|y_1|, |y_2|, u), & (y_1, y_2) \in B_{2m}(0, 1), \\ u = 0 \quad \text{on} \quad \partial B_{2m}(0, 1), \end{cases} \tag{4.1}$$

we perform the change of variables from Section 2, see (2.1), (2.3), (2.7), and we are led to investigate the existence of axially symmetric, with respect to $\mathbb{R}e_{m+1} \subset \mathbb{R}^{m+1}$, solutions of

$$\begin{cases} -\Delta_{m+1} v = \dfrac{f(\sqrt{|z| + z_{m+1}}, \sqrt{|z| - z_{m+1}}, v)}{2|z|}, \ z \in B_{m+1}(0, 1/2), \\ \\ v = 0 \quad \text{on} \quad \partial B_{m+1}(0, 1/2). \end{cases} \tag{4.2}$$

In this part we consider the particular problem

$$\begin{cases} -\Delta_{2m} u = |y_2|^\alpha |u|^{p-2} u, & (y_1, y_2) \in B_{2m}(0, 1), \\ u = 0 \quad \text{on} \quad \partial B_{2m}(0, 1). \end{cases} \tag{4.3}$$

Applying the moving planes technique [10] we know that any positive classical solution of (4.3) is such that $u(y_1, y_2) = u(|y_1|, y_2)$ and, for each y_2, $u(y_1, y_2)$ is decreasing with respect to $|y_1|$. Therefore, if we look for positive doubly symmetric solutions of (4.3) we obtain that for any such solution, there exists $0 \leq r < 1$ such that

$$\max_{(y_1,y_2)\in B_{2m}(0,1)} u(y_1, y_2) = u(0, y_2), \quad \forall\, y_2 \in S_r^{m-1},$$

with S_r^{m-1} as defined in (1.5).

From now on in this section we will proceed to prove Theorem 1.2.

First, by arguing similarly to the proof of Proposition 3.3, we can prove following equivalence.

Proposition 4.1. *Assume $m \geq 2$, $2 < p < \frac{2(m+1)}{m-1}$. Consider (4.3) and*

$$\begin{cases} -\Delta_{m+1} v = \dfrac{(|z| - z_{m+1})^{\frac{\alpha}{2}}}{2|z|} |v|^{p-2} v, & z \in B_{m+1}(0, 1/2), \\ v = 0 \quad on \quad \partial B_{m+1}(0, 1/2). \end{cases} \tag{4.4}$$

Then there exists $\alpha_0 = \alpha_0(p, m) > 4$ such that for every $\alpha > \alpha_0$ (2.5) provides a bijective correspondence between

$$X = \{u \in C^2(\overline{B}_{2m}(0,1));\ u \text{ is a doubly symmetric classical solution of } (4.3)\}$$

and

$$Y = \{v \in C^2(\overline{B}_{m+1}(0,1/2));\ v \text{ is an axially symmetric, w.r.t. } \mathbb{R}e_{m+1} \subset \mathbb{R}^{m+1}, \text{ classical solution of } (4.4)\}.$$

In addition, any $u \in X$ and any $v \in Y$ are such that $u \in C^{2,\gamma}(\overline{B}_{2m}(0,1))$, $v \in C^{2,\gamma}(\overline{B}_{m+1}(0,1/2))$ for all $0 < \gamma < 1$.

We recall that in the proof of Proposition 3.3 we used [9, Propositions 5.4 and 5.5], which assert about classical regularity of weak doubly symmetric solutions of the Hénon equation. The proof of Proposition 4.1 follows as the proof of Proposition 3.3, if we replace Lemma 3.2 by [12, Theorem 2.5], which in particular guarantees classical regularity of weak doubly symmetric solutions of (4.3).

We mention that, as in Proposition 3.5, we can show the correspondence between least energy solutions among the doubly symmetric solutions of (4.3) and least energy solutions among the axially symmetric, with respect to $\mathbb{R}e_{m+1} \subset \mathbb{R}^{m+1}$, solutions of (4.4). We then turn our attention to (4.4). Observe that for every $\alpha > 2$

$$\frac{(|z| - z_{m+1})^{\alpha/2}}{2|z|} \leq (|z| - z_{m+1})^{(\alpha-2)/2} \leq 1 \quad \forall\, z \in B_{m+1}(0,1/2)\backslash\{0\} \quad \text{and} \quad \lim_{z\to 0} \frac{(|z| - z_{m+1})^{\alpha/2}}{2|z|} = 0.$$

Let v be a positive and axially symmetric, with respect to $\mathbb{R}e_{m+1} \subset \mathbb{R}^{m+1}$, solution of (4.4). By Proposition 4.1, if u is associated to v by means of (2.5), then u is a positive doubly symmetric solution of (4.3). Then as observed before, by the moving planes technique, there exists $0 \leq r < 1$ such that

$$\max_{(y_1,y_2)\in B_{2m}(0,1)} u(y_1, y_2) = u(0, y_2), \quad \forall\, y_2 \in S_r^{m-1}.$$

Then, with $\rho = \dfrac{r^2}{2}$, we have that

$$\max_{z\in B_{m+1}(0,1/2)} v(z) = v(0, \dots, 0, -\rho).$$

Now let v_α be a least energy solution among the axially symmetric ones with respect to $\mathbb{R}e_{m+1} \subset \mathbb{R}^{m+1}$, solutions of (4.4). Then, up to a multiplicative constant, by the principle of symmetric criticality [19], we characterize such solution as a minimizer of a Rellich quotient among the functions in $H_0^1(B_{m+1}(0,1/2))$ invariant by the action of the group

$$\boldsymbol{G}_m = \{\sigma \in \mathcal{O}(m+1); \exists\, g \in \mathcal{O}(m) \text{ s.t. } \sigma(z_1, \dots z_m, z_{m+1}) = (g(z_1, \dots, z_m), z_{m+1})\}.$$

We can assume that $v_\alpha > 0$ in $B_{m+1}(0,1/2)$. So arguing as in the previous paragraph, there exists $0 \leq \rho_\alpha < \dfrac{1}{2}$ such that

$$\mathfrak{M}_\alpha := \max_{z\in B_{m+1}(0,1/2)} v_\alpha(z) = v_\alpha(0, \dots, 0, -\rho_\alpha). \tag{4.5}$$

Let

$$w_\alpha(z) = \left(\frac{1}{4}\right)^{1/(p-2)} v_\alpha\left(\frac{z}{2}\right). \tag{4.6}$$

Then $w_\alpha > 0$ in $B_{m+1}(0,1)$ and w_α is a least energy solution among the axially symmetric, with respect to $\mathbb{R}e_{m+1} \subset \mathbb{R}^{m+1}$, solutions of

$$\begin{cases} -\Delta_{m+1} w = h_\alpha(z)|w|^{p-2}w, & z \in B_{m+1}(0,1), \\ w = 0 \quad \text{on} \quad \partial B_{m+1}(0,1), \end{cases} \tag{4.7}$$

with

$$h_\alpha(z) := \frac{\left(\dfrac{|z| - z_{m+1}}{2}\right)^{\frac{\alpha}{2}}}{|z|}, \quad z \in B_{m+1}(0,1).$$

Also observe that for every $\alpha > 2$

$$h_\alpha(z) = \frac{\left(\dfrac{|z| - z_{m+1}}{2}\right)^{\frac{\alpha}{2}}}{|z|} < |z|^{\frac{\alpha-2}{2}} \quad \forall\, z \in B_{m+1}(0,1)\backslash\{0\}. \tag{4.8}$$

Now we compare (4.7) and

$$\begin{cases} -\Delta_{m+1} \psi = |z|^{\frac{\alpha-2}{2}}|\psi|^{p-2}\psi, & z \in B_{m+1}(0,1), \\ \psi = 0 \quad \text{on} \quad \partial B_{m+1}(0,1). \end{cases} \tag{4.9}$$

We set

$$H_m := \{w \in H_0^1(B_{m+1}(0,1)); gu = u \;\forall\, g \in \boldsymbol{G}_m\},$$

the space of functions in $H_0^1(B_{m+1}(0,1))$ that are axially symmetric with respect to $\mathbb{R}e_{m+1} \subset \mathbb{R}^{m+1}$. We also set

$$S_{\alpha,p} := \inf_{\psi \in H_0^1(B_{m+1}(0,1))\backslash\{0\}} \frac{\int |\nabla \psi|^2 dz}{\left(\int |z|^{\frac{\alpha-2}{2}}|\psi|^p dz\right)^{2/p}} \qquad \text{and} \qquad S'_{\alpha,p} := \inf_{w \in H_m\backslash\{0\}} \frac{\int |\nabla w|^2 dz}{\left(\int h_\alpha(z)|w|^p dz\right)^{2/p}}.$$

Then, from [24, 17], we have that any minimizer ψ of $S_{\alpha,p}$, up to rotation, is such that $\psi \in H_m$. Then, from (4.8) we conclude that

$$S'_{\alpha,p} > S_{\alpha,p} \quad \text{for every} \quad \alpha > 2. \tag{4.10}$$

We recall that

$$\frac{S_{\alpha,p}}{\alpha^{[2(m+1)-p(m-1)]/p}} = \frac{m_{1,p}}{2^{[2(m+1)-p(m-1)]/p}} + o(1) \quad \text{as} \quad \alpha \to \infty, \tag{4.11}$$

where

$$m_{\gamma,p} = \inf\left\{\int |\nabla w|^2 dz; w \in \mathcal{D}_0^{1,2}(\mathbb{R}_+^{m+1}),\; \int_{\mathbb{R}_+^{m+1}} e^{-\gamma z_{m+1}}|w|^p dz = 1\right\},$$

which is attained for every $\gamma > 0$ and $2 < p < \frac{2(m+1)}{m-1}$; see [8, Theorem 2.1 and Remark 4.8]. In particular, from (4.11), there exist $C_1, C_2 > 0$ such that

$$C_1 \alpha^{[2(m+1)-p(m-1)]/p} \le S_{\alpha,p} \le C_2 \alpha^{[2(m+1)-p(m-1)]/p} \quad \text{as} \quad \alpha \to \infty. \tag{4.12}$$

Moreover, the equation

$$-\Delta w = e^{-z_{m+1}}|w|^{p-2}w \quad \text{in} \quad \mathbb{R}_+^{m+1} \tag{4.13}$$

is called the limit problem associated to (4.9), since after suitable rescaling, as showed in [8], least energy solutions of (4.9) converge to least energy solutions of (4.13) as $\alpha \to \infty$.

Next we prove that $S'_{\alpha,p}$ may also be controlled as in (4.12). Indeed we show that the limit problem associated to (4.7) is a slight variation of (4.13).

Proposition 4.2. *There holds*

$$\frac{S'_{\alpha,p}}{\alpha^{[2(m+1)-p(m-1)]/p}} = m_{1/2,p} + o(1) \quad \text{as} \quad \alpha \to \infty. \tag{4.14}$$

We prove some preliminary lemmas in order to go through the proof of Proposition 4.2.

Lemma 4.3. *There exist C_1, C_2 positive constants such that*

$$C_1 \alpha^{[2(m+1)-p(m-1)]/p} \leq S'_{\alpha,p} \leq C_2 \alpha^{[2(m+1)-p(m-1)]/p} \quad \text{as} \quad \alpha \to \infty. \tag{4.15}$$

Proof. Given $\epsilon > 0$, choose $w_\epsilon \in C_c^\infty(\mathbb{R}^{m+1}_+)$ such that, $w_\epsilon \neq 0$, w_ϵ is axially symmetric with respect to $\mathbb{R}e_{m+1} \subset \mathbb{R}^{m+1}$ and

$$\frac{\int_{\mathbb{R}^{m+1}_+} |\nabla w_\epsilon(s)|^2 ds}{\left(\int_{\mathbb{R}^{m+1}_+} e^{-(s_{m+1}/2)} |w_\epsilon(s)|^p ds\right)^{2/p}} < m_{1/2,p} + \epsilon.$$

Set

$$\overline{w}_\epsilon(z) = w_\epsilon(\alpha z', \alpha[(1-|z'|^2)^{1/2} + z_{m+1}]), \quad z = (z', z_{m+1}) \in B_{m+1}(0,1).$$

Then, it is easy to see that $\overline{w}_\epsilon \in H_m$ for any large α.

We will perform the change of variables for $x = (x', x_{m+1}), s = (s', s_{m+1}) \in \mathbb{R}^m \times \mathbb{R}$:

$$x = \alpha e_{m+1} + \alpha z \quad \text{and} \quad s' = x',\ s_{m+1} = x_{m+1} + \alpha(-1 + (1-\alpha^{-2}|x'|^2)^{1/2}). \tag{4.16}$$

Then, since w_ϵ has compact support in $\mathbb{R}^{m+1}_+$ we get:

$$\int_{B_{m+1}(0,1)} |\nabla \overline{w}_\epsilon|^2 dz = \alpha^2 \int_{B_{m+1}(0,1)} \Bigg\{ \sum_{i=1}^m \Bigg[\Bigg| \partial_i w_\epsilon(\alpha z', \alpha[(1-|z'|^2)^{1/2} + z_{m+1}])$$
$$- \frac{z_i}{(1-|z'|^2)^{1/2}} \partial_{m+1} w_\epsilon(\alpha z', \alpha[(1-|z'|^2)^{1/2} + z_{m+1}]) \Bigg|^2 \Bigg] + \Big| \partial_{m+1} w_\epsilon(\alpha z', \alpha[(1-|z'|^2)^{1/2} + z_{m+1}]) \Big|^2 \Bigg\} dz$$
$$= \alpha^{1-m} \int_{B_{m+1}(\alpha e_{m+1}, \alpha)} \Bigg\{ \sum_{i=1}^m \Bigg[\Bigg| \partial_i w_\epsilon(x', (\alpha^2 - |x'|^2)^{1/2} + x_{m+1} - \alpha)$$
$$- \frac{\alpha^{-1} x_i}{(1-\alpha^{-2}|x'|^2)^{1/2}} \partial_{m+1} w_\epsilon(x', (\alpha^2 - |x'|^2)^{1/2} + x_{m+1} - \alpha) \Bigg|^2 \Bigg] + \Big| \partial_{m+1} w_\epsilon(x', (\alpha^2 - |x'|^2)^{1/2} + x_{m+1} - \alpha) \Big|^2 \Bigg\} dx$$
$$= \alpha^{1-m} \int_{\mathbb{R}^{m+1}_+} \left\{ \sum_{i=1}^m \left[\left| \partial_i w_\epsilon(s) - \frac{\alpha^{-1} s_i}{(1-\alpha^{-2}|s'|^2)^{1/2}} \partial_{m+1} w_\epsilon(s) \right|^2 \right] + |\partial_{m+1} w_\epsilon(s)|^2 \right\} ds$$
$$= \alpha^{1-m} \left[\int_{\mathbb{R}^{m+1}_+} |\nabla w_\epsilon(s)|^2 ds + O(\alpha^{-1}) \right].$$

On the other hand, by the change of variables (4.16), we have

$$h_\alpha(z) = \left(\frac{|x - \alpha e_{m+1}| - (x_{m+1} - \alpha)}{2\alpha} \right)^{\alpha/2} \Bigg/ \left| \frac{x}{\alpha} - e_{m+1} \right|$$
$$= \left(\frac{\sqrt{|s'|^2 + (s_{m+1} - (\alpha^2 - |s'|^2)^{1/2})^2} - (s_{m+1} - (\alpha^2 - |s'|^2)^{1/2})}{2\alpha} \right)^{\alpha/2} \Bigg/ \sqrt{\left|\frac{s'}{\alpha}\right|^2 + \left(\frac{s_{m+1}}{\alpha} - \left(1 - \left|\frac{s'}{\alpha}\right|^2\right)^{1/2} \right)^2}.$$

Now, if $s \in supp\, w_\epsilon$, then

$$\sqrt{\left|\frac{s'}{\alpha}\right|^2 + \left(\frac{s_{m+1}}{\alpha} - \left(1 - \left|\frac{s'}{\alpha}\right|^2\right)^{1/2}\right)^2} = 1 + O(\alpha^{-1})$$

and

$$\frac{\sqrt{|s'|^2 + (s_{m+1} - (\alpha^2 - |s'|^2)^{1/2})^2} - (s_{m+1} - (\alpha^2 - |s'|^2)^{1/2})}{2\alpha}$$
$$= -\frac{s_{m+1}}{\alpha} + 1 + \frac{\sqrt{|s'|^2 + (s_{m+1} - (\alpha^2 - |s'|^2)^{1/2})^2} + (s_{m+1} - (\alpha^2 - |s'|^2)^{1/2})}{2\alpha} + \frac{(\alpha^2 - |s'|^2)^{1/2} - \alpha}{\alpha}$$
$$= 1 - \frac{s_{m+1}/2}{\alpha/2} + O(\alpha^{-2}).$$

Then, if $s \in supp\, w_\epsilon$ we have,

$$h_\alpha(z) = e^{-\frac{s_{m+1}}{2} + O(\alpha^{-1})} + O(\alpha^{-1}) \tag{4.17}$$

and so

$$\int_{B_{m+1}(0,1)} h_\alpha(z)\overline{w}_\epsilon^p(z)dz = \alpha^{-(m+1)}\left[\int_{\mathbb{R}^{m+1}_+} e^{-\frac{s_{m+1}}{2} + O(\alpha^{-1})} w_\epsilon^p(s)ds + O(\alpha^{-1})\right]$$
$$= \alpha^{-(m+1)}\left[\int_{\mathbb{R}^{m+1}_+} e^{-\frac{s_{m+1}}{2}} w_\epsilon^p(s)ds + O(\alpha^{-1})\right].$$

Hence, by the definition of $S'_{\alpha,p}$, we have

$$S'_{\alpha,p} \le \alpha^{[2(m+1)-p(m-1)]/p} \frac{\int_{\mathbb{R}^{m+1}_+} |\nabla w_\epsilon|^2 ds + O(\alpha^{-1})}{\left(\int_{\mathbb{R}^{m+1}_+} e^{-\frac{s_{m+1}}{2}} w_\epsilon^p(s)ds + O(\alpha^{-1})\right)^{2/p}}$$
$$= \alpha^{[2(m+1)-p(m-1)]/p} \frac{\int_{\mathbb{R}^{m+1}_+} |\nabla w_\epsilon|^2 ds}{\left(\int_{\mathbb{R}^{m+1}_+} e^{-\frac{s_{m+1}}{2}} w_\epsilon^p(s)ds\right)^{2/p} + O(\alpha^{-1})} \le m_{1/2,p} + \epsilon + O(\alpha^{-1}).$$

From (4.10), (4.12) and the last inequality we have that there exist $C_1 > 0$ such that

$$C_1 \le \frac{S'_{\alpha,p}}{\alpha^{[2(m+1)-p(m-1)]/p}} \le m_{1/2,p} + o(1) \quad \text{as} \quad \alpha \to \infty. \tag{4.18}$$

□

Let $w_\alpha > 0$ be a least energy solution among the axially symmetric, with respect to $\mathbb{R}e_{m+1} \subset \mathbb{R}^{m+1}$, solutions of (4.7). Then

$$\int_{B_{m+1}(0,1)} |\nabla w_\alpha|^2 dz = \int_{B_{m+1}(0,1)} h_\alpha(z) w_\alpha^p dz = \left(S'_{\alpha,p}\right)^{p/p-2}$$

and from (4.15), there exist C_1, C_2 positive constants such that

$$C_1\alpha^{[2(m+1)-p(m-1)]/(p-2)} \le \int_{B_{m+1}(0,1)} |\nabla w_\alpha|^2 dz = \int_{B_{m+1}(0,1)} h_\alpha(z) w_\alpha^p dz \le C_2\alpha^{[2(m+1)-p(m-1)]/(p-2)} \tag{4.19}$$

as $\alpha \to \infty$.

Set

$$\overline{w}_\alpha(z) = \alpha^{-2/(p-2)} w_\alpha\left(\frac{z}{\alpha}\right), \quad z \in B_{m+1}(0,\alpha).$$

Then we have

$$-\Delta \overline{w}_\alpha = h_\alpha\left(\frac{z}{\alpha}\right)(\overline{w}_\alpha)^{p-1}, \quad z \in B_{m+1}(0,\alpha) \quad \text{with} \quad \overline{w}_\alpha = 0 \quad \text{on} \quad \partial B_{m+1}(0,\alpha)$$

and

$$\int_{B_{m+1}(0,\alpha)} |\nabla \overline{w}_\alpha|^2 dz = \alpha^{-[2(m+1)-p(m-1)]/(p-2)} \int_{B_{m+1}(0,1)} |\nabla w_\alpha|^2 dz$$

and hence

$$C_1 \leq \int_{B_{m+1}(0,\alpha)} |\nabla \overline{w}_\alpha|^2 dz \leq C_2 \quad \text{as} \quad \alpha \to \infty. \tag{4.20}$$

On the other hand, as proved in [8, pp. 473 and 474], there exists $C_3 > 0$ such that

$$\inf_{u \in H_0^1(B_{m+1}(0,1)),\ u \neq 0} \frac{\int |\nabla u|^2 dz}{\int |z|^{(\alpha-2)/2} u^2 dz} \geq C_3 \alpha^2 \quad \text{as} \quad \alpha \to \infty.$$

As a consequence

$$\frac{\int |\nabla w_\alpha|^2 dz}{\int h_\alpha(z) w_\alpha^2 dz} > \frac{\int |\nabla w_\alpha|^2 dz}{\int |z|^{(\alpha-2)/2} w_\alpha^2 dz} \geq C_3 \alpha^2 \quad \text{as} \quad \alpha \to \infty.$$

Then we combine (4.19) and the last inequality to get

$$\begin{aligned} \int_{B_{m+1}(0,\alpha)} h_\alpha\left(\frac{z}{\alpha}\right) \overline{w}_\alpha^2(z) dz &= \alpha^{m+1-\frac{4}{p-2}} \int_{B_{m+1}(0,1)} h_\alpha(z) w_\alpha^2(z) dz \\ &\leq C \alpha^{[p(m-1)-2(m+1)]/(p-2)} \int_{B_{m+1}(0,1)} |\nabla w_\alpha|^2 dz \leq C \quad \text{as} \quad \alpha \to \infty. \end{aligned} \tag{4.21}$$

Lemma 4.4. *There exist C_1, C_2 positive constants such that*

$$C_1 \leq \max_{z \in B_{m+1}(0,\alpha)} \overline{w}_\alpha(z) \leq C_2 \quad \textit{as} \quad \alpha \to \infty, \tag{4.22}$$

that is,

$$C_1 \alpha^{2/(p-2)} \leq \max_{z \in B_{m+1}(0,1)} w_\alpha(z) \leq C_2 \alpha^{2/(p-2)} \quad \textit{as} \quad \alpha \to \infty. \tag{4.23}$$

Proof. From (4.20) and (4.21), it follows that

$$\begin{aligned} 0 < C_1 \leq \int_{B_{m+1}(0,\alpha)} |\nabla \overline{w}_\alpha|^2 dx &= \int_{B_{m+1}(0,\alpha)} h_\alpha\left(\frac{z}{\alpha}\right)(\overline{w}_\alpha)^p dz \\ &\leq \max_{z \in B_{m+1}(0,\alpha)} (\overline{w}_\alpha)^{p-2} \int_{B_{m+1}(0,\alpha)} h_\alpha\left(\frac{z}{\alpha}\right)(\overline{w}_\alpha)^2 dz \leq C \max_{z \in B_{m+1}(0,\alpha)} (\overline{w}_\alpha)^{p-2} \quad \text{as} \quad \alpha \to \infty. \end{aligned}$$

Now we prove the reverse inequality. By contradiction, suppose that $(\|\overline{w}_\alpha\|_\infty)$ is not bounded from above as $\alpha \to \infty$. Then there exists a sequence (α_n) such that $\|\overline{w}_{\alpha_n}\|_\infty \to \infty$ and $\alpha_n \to \infty$ as $n \to \infty$. Let $z_{\alpha_n} \in B_{m+1}(0,\alpha_n)$ such that $\|\overline{w}_{\alpha_n}\|_\infty = \overline{w}_{\alpha_n}(z_{\alpha_n})$ and set

$$\overline{v}_{\alpha_n}(z) = \frac{1}{\|\overline{w}_{\alpha_n}\|_\infty} \overline{w}_{\alpha_n}\left(\|\overline{w}_{\alpha_n}\|_\infty^{-(p-2)/2} z + z_{\alpha_n}\right), \quad z \in B_{m+1}(-z_{\alpha_n}\|\overline{w}_{\alpha_n}\|_\infty^{(p-2)/2}, \alpha_n \|\overline{w}_{\alpha_n}\|_\infty^{(p-2)/2})$$

Then

$$-\Delta \overline{v}_{\alpha_n} = h_{\alpha_n}\left(\frac{\|\overline{w}_{\alpha_n}\|_\infty^{-(p-2)/2} z}{\alpha_n} + z_{\alpha_n}\right)(\overline{v}_{\alpha_n})^{p-1} \quad \text{in} \quad B_{m+1}(-z_{\alpha_n}\|\overline{w}_{\alpha_n}\|_\infty^{(p-2)/2}, \alpha_n \|\overline{w}_{\alpha_n}\|_\infty^{(p-2)/2})$$

with homogenous Dirichlet boundary condition. Observe that, from (4.20) and $2 < p < \frac{2(m+1)}{m-1}$, it follows that

$$\int |\nabla \overline{v}_{\alpha_n}|^2 dz = \|\overline{w}_{\alpha_n}\|_\infty^{[p(m-1)-2(m+1)]/2} \int |\nabla \overline{w}_{\alpha_n}|^2 dz \to 0 \quad \text{as} \quad \alpha \to \infty.$$

Then $B_{m+1}(-z_{\alpha_n}\|\overline{w}_{\alpha_n}\|_\infty^{(p-2)/2}, \alpha_n\|\overline{w}_{\alpha_n}\|_\infty^{(p-2)/2}) \to \Omega$ as $\alpha_n \to \infty$, where $\Omega = \mathbb{R}^{m+1}$ or Ω is a (possibly affine) half-space in $\mathbb{R}^{m+1}$ and we get the existence of v a solution of

$$-\Delta v = 0 \quad \text{in} \quad \Omega \quad \text{with} \quad \|v\|_\infty = 1, \quad v \in \mathcal{D}_0^{1,2}(\Omega),$$

which contradicts the classical Liouville's theorem. □

Let $0 \le \tau_\alpha < 1$, see (4.5) and (4.6), such that

$$\max_{z \in B_{m+1}(0,1)} w_\alpha(z) = w_\alpha(-\tau e_{m+1}).$$

Lemma 4.5. *The product*

$$\alpha(1-\tau_\alpha) \quad \textit{remains bounded as} \quad \alpha \to \infty.$$

Proof. By contradiction assume that there exists a sequence (α_n) such that

$$\alpha_n \to \infty, \quad \alpha_n(1-\tau_{\alpha_n}) \to \infty \quad \text{as} \quad n \to \infty.$$

Set

$$\widetilde{w}_{\alpha_n}(z) = \alpha^{-2/(p-2)} w_{\alpha_n}\left(\frac{z}{\alpha_n} - \tau_{\alpha_n} e_{m+1}\right), \quad z \in \Omega_n := B_{m+1}(\alpha_n \tau_{\alpha_n} e_{m+1}, \alpha_n).$$

Then

$$\begin{cases} -\Delta \widetilde{w}_{\alpha_n} = h_{\alpha_n}\left(\frac{z}{\alpha_n} - \tau_{\alpha_n} e_{m+1}\right)(\widetilde{w}_{\alpha_n})^{p-1}, \quad z \in \Omega_n \quad \text{with} \quad \widetilde{w}_{\alpha_n} = 0 \quad \text{on} \quad \partial\Omega_n, \\ 0 < C_1 \le \widetilde{w}_{\alpha_n}(0) = \max_{\Omega_n} \widetilde{w}_{\alpha_n} \le C_2, \quad \text{as} \quad n \to \infty \text{ by (4.22)}, \\ (\widetilde{w}_{\alpha_n}) \quad \text{is bounded in } \mathcal{D}^{1,2}(\mathbb{R}^{m+1}) \text{ by (4.20)}, \\ \Omega_n \to \mathbb{R}^{m+1} \quad \text{as} \quad n \to \infty, \\ h_{\alpha_n}\left(\frac{z}{\alpha_n} - \tau_{\alpha_n} e_{m+1}\right) \to 0 \quad L^\infty_{loc}(\mathbb{R}^{m+1}) \quad \text{as} \quad n \to \infty. \end{cases}$$

As a consequence, we obtain $w \in \mathcal{D}^{1,2}(\mathbb{R}^{m+1})$ a bounded positive solution of

$$-\Delta w = 0 \quad \text{in} \quad \mathbb{R}^{m+1},$$

which contradicts the classical Liouville's theorem. □

Proposition 4.6. *We have the convergence*

$$\int_{\mathbb{R}^{m+1}_+} |\nabla \widehat{w}_\alpha - \nabla w|^2 dz \to 0 \quad \textit{as} \quad \alpha \to \infty, \tag{4.24}$$

where

$$\widehat{w}_\alpha(z) = \alpha^{-2/(p-2)} w_\alpha\left(\frac{z}{\alpha} - e_{m+1}\right), \quad z \in \Omega_\alpha := B_{m+1}(\alpha e_{m+1}, \alpha)$$

and, up to normalizing, w minimizes $m_{1/2,p}$.

Proof. It follows from (4.20) that $(\widehat{w}_\alpha)$ remains bounded in $\mathcal{D}_0^{1,2}(\mathbb{R}_+^{m+1})$ as $\alpha \to \infty$. Then there exist $w \in \mathcal{D}_0^{1,2}(\mathbb{R}_+^{m+1})$ such that $\widehat{w}_\alpha \rightharpoonup w$ in $\mathcal{D}_0^{1,2}(\mathbb{R}_0^{m+1})$ as $\alpha \to \infty$. Observe that

$$-\Delta \widehat{w}_\alpha(z) = h_\alpha\left(\frac{z}{\alpha} - e_{m+1}\right)(\widehat{w}_\alpha)^{p-1}(z) \quad \text{in} \quad \Omega_\alpha \quad \text{and} \quad \widehat{w}_\alpha = 0 \quad \text{on} \quad \partial\Omega_\alpha.$$

Also observe that

$$0 \leq h_\alpha\left(\frac{z}{\alpha} - e_{m+1}\right) \leq 1 \quad \forall\, z \in \Omega_\alpha$$

and from (4.23), there exist $C_1, C_2 > 0$ such that

$$C_1 \leq \widehat{w}_\alpha(z) \leq C_2 \quad \forall\, z \in \Omega_\alpha.$$

Then, from classical regularity results for second order elliptic equations as in [1] and classical Sobolev imbeddings, we obtain that

$$\widehat{w}_\alpha \to w \quad \text{in} \quad C^1_{loc}(\mathbb{R}_+^{m+1}). \tag{4.25}$$

Now observe that

$$h_\alpha\left(\frac{z}{\alpha} - e_{m+1}\right) = \left(\frac{|z - \alpha e_{m+1}| - (z_{m+1} - \alpha)}{2\alpha}\right)^{\alpha/2} \Bigg/ \left|\frac{z}{\alpha} - e_{m+1}\right| \forall\, z \in \Omega_\alpha.$$

Then, as we did at (4.17), we conclude that w solves

$$\begin{cases} -\Delta w = e^{-z_{m+1}/2} w^{p-1} \quad \text{in} \quad \mathbb{R}_+^{m+1}, \\ w > 0 \quad \text{in} \quad \mathbb{R}_+^{m+1} \quad \text{and} \quad w \in \mathcal{D}_0^{1,2}(\mathbb{R}_+^{m+1}). \end{cases} \tag{4.26}$$

Then, from (4.26) and from the definition of $m_{1/2,p}$ we conclude that

$$\int_{\mathbb{R}_+^{m+1}} |\nabla w|^2 dz = \int_{\mathbb{R}_+^{m+1}} e^{-z_{m+1}/2} w^p dz \geq m_{1/2,p}^{p/(p-2)}. \tag{4.27}$$

With $R > 0$ large with $R < \alpha$ we define

$$B_{R,\alpha} = \left\{z; \frac{z}{\alpha} - e_{m+1} \in B_{m+1}(-e_{m+1}, R/\alpha) \cap B_{m+1}(0,1)\right\}.$$

From (4.18) and with the change of variables $x = \frac{z}{\alpha} - e_{m+1}$ we have

$$\begin{aligned} \left(m_{1/2,p}^{p/p-2} + o(1)\right) \alpha^{[2(m+1)-p(m-1)]/(p-2)} &\geq (S'_{\alpha,p})^{p/(p-2)} = \int_{B_{m+1}(0,1)} |\nabla w_\alpha|^2 dx \\ &= \int_{B_{m+1}(0,1)\cap B_{m+1}(-e_{m+1},R/\alpha)} |\nabla w_\alpha|^2 dx + \int_{B_{m+1}(0,1)\setminus B_{m+1}(-e_{m+1},R/\alpha)} |\nabla w_\alpha|^2 dx \\ &= \alpha^{[2(m+1)-p(m-1)]/(p-2)} \left[\int_{B_{R,\alpha}} |\nabla \widehat{w}_\alpha|^2 dz + \int_{B_{m+1}(\alpha e_{m+1},\alpha)\setminus B_{R,\alpha}} |\nabla \widehat{w}_\alpha|^2 dz\right] \\ &\geq \alpha^{[2(m+1)-p(m-1)]/(p-2)} \int_{B_{R,\alpha}} |\nabla \widehat{w}_\alpha|^2 dz. \end{aligned}$$

Then from (4.25) and (4.27) it follows that

$$\begin{aligned} \left(m_{1/2,p}^{p/p-2} + o(1)\right) &\geq \int_{B_{R,\alpha}} |\nabla \widehat{w}_\alpha|^2 dz = \int_{\mathbb{R}_+^{m+1}\cap B_{m+1}(0,R)} |\nabla \widehat{w}_\alpha|^2 dz + o(1) \\ &= \int_{\mathbb{R}_+^{m+1}\cap B_{m+1}(0,R)} |\nabla w|^2 dz + o(1) \geq m_{1/2,p}^{p/(p-2)} + o_R(1) + o(1). \end{aligned}$$

Hence we obtain

$$\int_{B_{m+1}(\alpha e_{m+1},\alpha)\setminus B_{R,\alpha}} |\nabla \widehat{w}_\alpha|^2 dz = o(1) + o_R(1) \quad \text{when} \quad R < \alpha,\, R \to \infty,$$

$$\int_{B_{R,\alpha}} |\nabla \widehat{w}_\alpha|^2 dz = \int_{\mathbb{R}^{m+1}_+ \cap B_{m+1}(0,R)} |\nabla w|^2 dz + o(1) = m_{1/2,p}^{p/(p-2)} + o_R(1) + o(1) \quad \text{when} \quad R < \alpha,\, R \to \infty.$$

Then, from (4.25) and since $(\widehat{w}_\alpha)$ is bounded in $\mathcal{D}^{1,2}_0(\mathbb{R}^{m+1}_+)$, it follows that

$$\begin{aligned}\int_{\mathbb{R}^{m+1}_+} |\nabla \widehat{w}_\alpha - \nabla w|^2 dz &= \int_{\mathbb{R}^{m+1}_+ \cap B_{m+1}(0,R)} |\nabla \widehat{w}_\alpha - \nabla w|^2 dz + \int_{\mathbb{R}^{m+1}_+ \setminus B_{m+1}(0,R)} |\nabla \widehat{w}_\alpha - \nabla w|^2 dz \\ &\le o(1) + \int_{B_{m+1}(\alpha e_{m+1},\alpha)\setminus B_{R,\alpha}} |\nabla \widehat{w}_\alpha - \nabla w|^2 dz + \int_{\mathbb{R}^{m+1}_+ \setminus B_{m+1}(0,R)} |\nabla w|^2 dz \\ = o(1) + o_R(1) - 2\int_{B_{m+1}(\alpha e_{m+1},\alpha)\setminus B_{R,\alpha}} \nabla \widehat{w}_\alpha \nabla w dz &\le o(1) + o_R(1) + 2C\left(\int_{B_{m+1}(\alpha e_{m+1},\alpha)\setminus B_{R,\alpha}} |\nabla w|^2 dz\right)^{1/2} \\ &= o(1) + o_R(1).\end{aligned}$$

Hence we conclude that

$$\int_{\mathbb{R}^{m+1}_+} |\nabla \widehat{w}_\alpha - \nabla w|^2 dz \to 0 \quad \text{as} \quad \alpha \to \infty,$$

and that $\int_{\mathbb{R}^{m+1}_+} |\nabla w|^2 dz = m_{1/2,p}^{p/(p-2)}$ and so w minimizes $m_{1/2,p}$. □

Proposition 4.7. *There exists $l > 0$ such that*

$$\alpha(1 - \tau_\alpha) \to l \quad \textit{as} \quad \alpha \to \infty. \tag{4.28}$$

Proof. We have proved that $\widehat{w}_\alpha \to w$ in $C^1_{loc}(\mathbb{R}^{m+1}_+)$ as $\alpha \to \infty$ and from Lemma 4.5 we know that $\alpha(1-\tau_\alpha)$ remains bounded as $\alpha \to \infty$.

Let z_α be the maximum point of $\widehat{w}_\alpha$. Then

$$-\tau_\alpha e_{m+1} = \frac{z_\alpha}{\alpha} - e_{m+1} \quad \text{which implies} \quad z_\alpha = \alpha(1-\tau_\alpha) e_{m+1}. \tag{4.29}$$

and we obtain that the maximum point of $\widehat{w}_\alpha$ converges to the maximum point of w, which is precisely $l e_{m+1}$ for some $l > 0$, which follows from (4.26) and the moving planes technique as in [10]. Indeed w is axially symmetric with respect to $\mathbb{R}e_{m+1} \subset \mathbb{R}^{m+1}$ and decreasing with respect to $|z'|$. Therefore, from (4.29) we conclude that

$$\alpha(1-\tau_\alpha) \to l \quad \text{as} \quad \alpha \to \infty.$$

□

***Proof of Proposition* 4.2.** From (4.24) we obtain that

$$\begin{aligned} m_{1/2,p}^{p/(p-2)} &= \int_{\mathbb{R}^{m+1}_+} |\nabla w|^2 dz = \int_{B_{m+1}(\alpha e_{m+1},\alpha)} |\nabla \widehat{w}_\alpha|^2 dz + o(1) \\ &= \alpha^{[p(m-1)-2(m+1)]/(p-2)} \int_{B_{m+1}(0,1)} |\nabla w_\alpha|^2 dx + o(1) = \alpha^{[p(m-1)-2(m+1)]/(p-2)} (S'_{\alpha,p})^{p/(p-2)} + o(1). \end{aligned}$$

Therefore

$$\frac{S'_{\alpha,p}}{\alpha^{[2(m+1)-p(m-1)]/p}} = m_{1/2,p} + o(1) \quad \text{as} \quad \alpha \to \infty. \qquad \square$$

***Proof of Theorem* 1.2.** It follows from Lemma 4.4, Propositions 4.6 and 4.7. □

5. Hyperplanes preventing diffusion and proof of Theorem 1.3

In this section we consider the problem

$$\begin{cases} -\Delta_N u = |z_N|^\alpha |u|^{p-2}u, & z = (z_1, \ldots, z_N) \in B_N(0,1), \\ u = 0 \quad \text{on} \quad \partial B_N(0,1). \end{cases} \tag{5.1}$$

where p and N satisfy the conditions from Theorem 1.3. The procedure to study (5.1) is quite similar to that from Section 4, but due to its technicality we also include some details here.

By the moving planes technique [10] we know that any classical positive solution of (5.1) is such that $u(z_1, \ldots, z_{N-1}, z_N) = u(|(z_1, \ldots, z_{N-1})|, z_N)$ and that $u(\cdot, z_N)$ decreases with respect to $|(z_1, \ldots, z_{N-1})|$. Therefore, if we look for positive solutions of (5.1) such that $u(z_1, \ldots, z_{N-1}, z_N) = u(|(z_1, \ldots, z_{N-1})|, |z_N|)$ we obtain that for any such solution, there exists $0 \leq r < 1$ such that

$$\max_{(z_1, \ldots, z_N) \in B_N(0,1)} = u(re_N) = u(-re_N).$$

Now let u_α be a least energy solution among the solutions of (5.1) that depend only on $|(z_1, \ldots, z_{N-1})|$ and $|z_N|$. Then, by the principle of symmetric criticality [19], we characterize such solution as a minimizer of a Rellich quotient among the functions in $H_0^1(B_N(0,1))$ invariant by the action of the group

$$\mathfrak{G}_N = \mathcal{O}(N-1) \times \mathbb{Z}_2.$$

We can assume that $u_\alpha > 0$ in $B_N(0,1)$. So arguing as in the previous paragraph, there exists $0 \leq r_\alpha < 1$ such that

$$\mathbf{M}_\alpha := \max_{(z_1, \ldots, z_N) \in B_N(0,1)} u_\alpha(z_1, \ldots, z_N) = u_\alpha(r_\alpha e_n) = u_\alpha(-r_\alpha e_n).$$

We set

$$\mathcal{H}_{D,N} := \{u \in H_0^1(B_N(0,1)); gu = u \; \forall \; g \in \mathfrak{G}_N\},$$

the space of functions in $H_0^1(B_N(0,1))$ that are axially symmetric with respect to $\mathbb{R}e_N \subset \mathbb{R}^N$ and symmetric with respect to x_N. We also set

$$K_{\alpha,p} := \inf_{\psi \in H_0^1(B_N(0,1)) \setminus \{0\}} \frac{\int |\nabla \psi|^2 dz}{\left(\int |z|^\alpha |\psi|^p dz\right)^{2/p}} \qquad \text{and} \qquad K'_{\alpha,p} := \inf_{w \in \mathcal{H}_{D,N} \setminus \{0\}} \frac{\int |\nabla w|^2 dz}{\left(\int |z_N|^\alpha |w|^p dz\right)^{2/p}}.$$

Then, from [24, 17], we have that any minimizer ψ of $K_{\alpha,p}$, up to rotation, is such that ψ is axially symmetric with respect to $\mathbb{R}e_N$. Then, since $|z_N|^\alpha \leq |z|^\alpha$, we conclude that

$$K'_{\alpha,p} > K_{\alpha,p} \quad \text{for every} \quad \alpha > 0. \tag{5.2}$$

We recall that

$$\frac{K_{\alpha,p}}{\alpha^{[2N-p(N-2)]/p}} = m_{1,p} + o(1) \quad \text{as} \quad \alpha \to \infty, \tag{5.3}$$

where

$$m_{\gamma,p} = \inf\left\{\int |\nabla w|^2 dz; w \in \mathcal{D}_0^{1,2}(\mathbb{R}^N_+), \; \int_{\mathbb{R}^N_+} e^{-\gamma z_N} |w|^p dz = 1\right\},$$

which is attained for every $\gamma > 0$ and $2 < p < \frac{2N}{N-2}$; see [8, Remark 4.8 and Theorem 2.1]. In particular, from (5.3), there exist $C_1, C_2 > 0$ such that

$$C_1\alpha^{[2N-p(N-2)]/p} \leq K_{\alpha,p} \leq C_2\alpha^{[2N-p(N-2)]/p} \quad \text{as} \quad \alpha \to \infty. \tag{5.4}$$

Moreover, the equation

$$-\Delta w = e^{-z_N}|w|^{p-2}w \quad \text{in} \quad \mathbb{R}^N_+ \tag{5.5}$$

is called the limit problem associated to

$$-\Delta u = |z|^\alpha |u|^{p-2}u \quad \text{in} \quad B_N(0,1), \quad u = 0 \quad \text{on} \quad \partial B_N(0,1), \tag{5.6}$$

since after suitable rescaling, we can show that least energy solutions of (5.6) converges to least energy solutions of (5.5) as $\alpha \to \infty$.

Next we prove that $K'_{\alpha,p}$ may also be controlled as in (5.4). Indeed we show that the limit problem associated to (5.1), for solutions that are axially symmetric with respect to $\mathbb{R}e_N \subset \mathbb{R}^N$ and symmetric with respect to x_N, is also (5.5).

Proposition 5.1. *There holds*

$$\frac{K'_{\alpha,p}}{\alpha^{[2N-p(N-2)]/p}} = 2^{1-2/p}m_{1,p} + o(1) \quad \textit{as} \quad \alpha \to \infty. \tag{5.7}$$

We prove some preliminary lemmas in order to go through the proof of Proposition 5.1.

Lemma 5.2. *There exist C_1, C_2 positive constants such that*

$$C_1\alpha^{[2N-p(N-2)]/p} \leq K'_{\alpha,p} \leq C_2\alpha^{[2N-p(N-2)]/p} \quad \textit{as} \quad \alpha \to \infty. \tag{5.8}$$

Proof. Given $\epsilon > 0$, choose $u_\epsilon \in C_c^\infty(\mathbb{R}^N_+)$ such that, $u_\epsilon \neq 0$, u_ϵ is axially symmetric with respect to $\mathbb{R}e_N \subset \mathbb{R}^N$ and

$$\frac{\int_{\mathbb{R}^N_+} |\nabla u_\epsilon(s)|^2 ds}{\left(\int_{\mathbb{R}^N_+} e^{-s_N}|u_\epsilon(s)|^p ds\right)^{2/p}} < m_{1,p} + \epsilon.$$

Set

$$\overline{u}_\epsilon(z) = u_\epsilon(\alpha z', \alpha[(1-|z'|^2)^{1/2} - |z_N|]), \quad z = (z', z_N) \in B_N(0,1).$$

Then, it is easy to see that $\overline{u}_\epsilon \in \mathcal{H}_{D,N}$ for any $\alpha > 0$.

We will perform the change of variables

$$x = \alpha e_N + \alpha z \quad \text{and} \quad s' = x',\ s_N = x_N + \alpha(-1 + (1-\alpha^{-2}|x'|^2)^{1/2}). \tag{5.9}$$

Then, since u_ϵ has compact support in $\mathbb{R}^N_+$, for any α large we get:

$$\int_{B_N(0,1)} |\nabla \overline{u}_\epsilon|^2 dz = 2\alpha^2 \int_{B_N(0,1),\ z_N<0} \left\{ \sum_{i=1}^{N-1} \left[\Big| \partial_i u_\epsilon(\alpha z', \alpha[(1-|z'|^2)^{1/2} + z_N]) \right.\right.$$
$$\left.\left. - \frac{z_i}{(1-|z'|^2)^{1/2}} \partial_N u_\epsilon(\alpha z', \alpha[(1-|z'|^2)^{1/2} + z_N]) \Big|^2 \right] + \Big| \partial_N u_\epsilon(\alpha z', \alpha[(1-|z'|^2)^{1/2} + z_N]) \Big|^2 \right\} dz$$
$$= 2\alpha^{2-N} \int_{B_N(\alpha e_N,\alpha),\ x_N<\alpha} \left\{ \sum_{i=1}^{N-1} \left[\Big| \partial_i u_\epsilon(x', (\alpha^2-|x'|^2)^{1/2} + x_N - \alpha) \right.\right.$$
$$\left.\left. - \frac{\alpha^{-1} x_i}{(1-\alpha^{-2}|x'|^2)^{1/2}} \partial_N u_\epsilon(x', (\alpha^2-|x'|^2)^{1/2} + x_N - \alpha) \Big|^2 \right] + \Big| \partial_N u_\epsilon(x', (\alpha^2-|x'|^2)^{1/2} + x_N - \alpha) \Big|^2 \right\} dx$$
$$= 2\alpha^{2-N} \int_{\mathbb{R}^N_+} \left\{ \sum_{i=1}^{N-1} \left[\Big| \partial_i u_\epsilon(s) - \frac{\alpha^{-1} s_i}{(1-\alpha^{-2}|s'|^2)^{1/2}} \partial_N u_\epsilon(s) \Big|^2 \right] + |\partial_N u_\epsilon(s)|^2 \right\} ds$$
$$= 2\alpha^{2-N} \left[\int_{\mathbb{R}^N_+} |\nabla u_\epsilon(s)|^2 ds + O(\alpha^{-1}) \right].$$

On the other hand, by the change of variables (5.9), we have that for $z \in B_N(0,1)$ with $z_N < 0$ that $0 < x_N < \alpha$ and

$$|z_N|^\alpha = \left| 1 - \frac{x_N}{\alpha} \right|^\alpha = \left| \frac{s_N}{\alpha} - (1-\alpha^{-2}|s'|^2)^{1/2} \right|^\alpha.$$

Now, if $s \in supp\, u_\epsilon$, then

$$\left| \frac{s_N}{\alpha} - (1-\alpha^{-2}|s'|^2)^{1/2} \right| = 1 - \frac{s_N}{\alpha} + O(\alpha^{-2})$$

and

$$|z_N|^\alpha = e^{-s_N + O(\alpha^{-1})} + O(\alpha^{-1}). \tag{5.10}$$

Hence

$$\int_{B_N(0,1)} |z_N|^\alpha \overline{u}_\epsilon^p(z) dz = 2\int_{B_N(0,1),\ z_N<0} |z_N|^\alpha \overline{u}_\epsilon^p(z) dz = 2\alpha^{-N} \left[\int_{\mathbb{R}^N_+} e^{-s_N + O(\alpha^{-1})} u_\epsilon^p(s) ds + O(\alpha^{-1}) \right]$$
$$= 2\alpha^{-N} \left[\int_{\mathbb{R}^N_+} e^{-s_N} u_\epsilon^p(s) ds + O(\alpha^{-1}) \right].$$

By the definition of $K'_{\alpha,p}$, we have

$$K'_{\alpha,p} \leq 2^{1-2/p} \alpha^{[2N-p(N-2)]/p} \frac{\int_{\mathbb{R}^N_+} |\nabla u_\epsilon|^2 ds + O(\alpha^{-1})}{\left(\int_{\mathbb{R}^N_+} e^{-s_N} u_\epsilon^p(s) ds + O(\alpha^{-1}) \right)^{2/p}}$$
$$= 2^{1-2/p} \alpha^{[2N-p(N-2)]/p} \frac{\int_{\mathbb{R}^N_+} |\nabla u_\epsilon|^2 ds}{\left(\int_{\mathbb{R}^N_+} e^{-s_N} u_\epsilon^p(s) ds \right)^{2/p} + O(\alpha^{-1})} \leq 2^{1-2/p} \alpha^{[2N-p(N-2)]/p} (m_{1,p} + \epsilon) + O(\alpha^{-1}).$$

From (5.2), (5.4) and the last inequality we have that there exist $C_1 > 0$ such that

$$C_1 \leq \frac{K'_{\alpha,p}}{\alpha^{[2N-p(N-2)]/p}} \leq 2^{1-2/p} m_{1,p} + o(1) \quad \text{as} \quad \alpha \to \infty. \tag{5.11}$$

□

Let $u_\alpha > 0$ be a least energy solution among those which are axially symmetric with respect to $\mathbb{R}e_N \subset \mathbb{R}^N$ and symmetric with respect to x_N solutions of (5.1). Then

$$\int_{B_N(0,1)} |\nabla u_\alpha|^2 dz = \int_{B_N(0,1)} |z_N|^\alpha u_\alpha^p dz = \left(K'_{\alpha,p}\right)^{p/p-2}$$

and from (5.8), there exist C_1, C_2 positive constants such that

$$C_1 \alpha^{[2N-p(N-2)]/(p-2)} \leq \int_{B_N(0,1)} |\nabla u_\alpha|^2 dz = \int_{B_N(0,1)} |z_N|^\alpha u_\alpha^p dz \leq C_2 \alpha^{[2N-p(N-2)]/(p-2)} \tag{5.12}$$

as $\alpha \to \infty$.

Set

$$\overline{u}_\alpha(z) = \alpha^{-2/(p-2)} u_\alpha\left(\frac{z}{\alpha}\right), \quad z \in B_N(0,\alpha).$$

Then we have

$$-\Delta \overline{u}_\alpha = \left|\frac{z_N}{\alpha}\right|^\alpha (\overline{u}_\alpha)^{p-1}, \quad z \in B_N(0,\alpha) \quad \text{with} \quad \overline{w}_\alpha = 0 \quad \text{on} \quad \partial B_N(0,\alpha)$$

and

$$\int_{B_N(0,\alpha)} |\nabla \overline{u}_\alpha|^2 dz = \alpha^{-[2N-p(N-2)]/(p-2)} \int_{B_N(0,1)} |\nabla u_\alpha|^2 dz$$

and hence

$$C_1 \leq \int_{B_N(0,\alpha)} |\nabla \overline{u}_\alpha|^2 dz \leq C_2 \quad \text{as} \quad \alpha \to \infty. \tag{5.13}$$

Then we can proceed as in Section 4 to prove the estimate below.

Lemma 5.3. *There exist C_1, C_2 positive constants such that*

$$C_1 \leq \max_{z \in B_N(0,\alpha)} \overline{u}_\alpha(z) \leq C_2 \quad \text{as} \quad \alpha \to \infty, \tag{5.14}$$

that is,

$$C_1 \alpha^{2/(p-2)} \leq \max_{z \in B_N(0,1)} u_\alpha(z) \leq C_2 \alpha^{2/(p-2)} \quad \text{as} \quad \alpha \to \infty. \tag{5.15}$$

Proof. It follows exactly as in the proof of Lemma 4.4. □

Let $0 \leq r_\alpha < 1$ such that

$$\max_{z \in B_N(0,1)} u_\alpha(z) = u_\alpha(-r_\alpha e_N) = u_\alpha(r_\alpha e_N).$$

We can follow the proof of Lemma 4.5 to get the estimate below.

Lemma 5.4. *The product*

$$\alpha(1 - r_\alpha) \quad \text{remains bounded as} \quad \alpha \to \infty.$$

Proposition 5.5. *We have the convergence*

$$\int_{\mathbb{R}^N_+} |\nabla \widehat{u}_\alpha - \nabla u|^2 dz \to 0 \quad \text{as} \quad \alpha \to \infty, \tag{5.16}$$

where

$$\widehat{u}_\alpha(z) = \alpha^{-2/(p-2)} u_\alpha\left(\frac{z}{\alpha} - e_N\right), \quad z \in \Omega_\alpha := \{z \in B_N(\alpha e_N, \alpha); z_N < \alpha\}$$

and, up to normalization, u minimizes $m_{1,p}$.

Proof. It follows from (5.13) that $(\nabla \widehat{u}_\alpha)$ remains bounded in $L^2(\mathbb{R}^{m+1}_+)$ as $\alpha \to \infty$. Observe that

$$\begin{cases} -\Delta \widehat{u}_\alpha(z) = \left|\frac{z_N}{\alpha} - e_N\right|^\alpha (\widehat{u}_\alpha)^{p-1}(z) \quad \text{in} \quad \Omega_\alpha, \\ \widehat{w}_\alpha(z) = 0 \quad \text{on} \quad z \in \partial\Omega_\alpha \quad \text{s.t.} \quad 0 \le z_N < \alpha, \\ \frac{\partial \widehat{w}}{\partial \nu}(z) = 0 \quad \text{on} \quad z \in \partial\Omega_\alpha \quad \text{s.t.} \quad z_N = \alpha. \end{cases}$$

Also observe that

$$0 \le \left|\frac{z_N}{\alpha} - e_N\right|^\alpha \le 1 \quad \forall\, z \in \Omega_\alpha$$

and from (5.15), there exist $C_1, C_2 > 0$ such that

$$C_1 \le \widehat{u}_\alpha(z) \le C_2 \quad \forall\, z \in \Omega_\alpha.$$

Then, from classical regularity results for second order elliptic equations as in [1] and classical Sobolev imbeddings, we obtain that there exists $w \in \mathcal{D}^{1,2}_0(\mathbb{R}^N_+)$

$$\widehat{u}_\alpha \to u \quad \text{in} \quad C^1_{loc}(\mathbb{R}^N_+). \tag{5.17}$$

Then we conclude that w solves

$$\begin{cases} -\Delta u = e^{-z_N} u^{p-1} \quad \text{in} \quad \mathbb{R}^N_+, \\ u > 0 \quad \text{in} \quad \mathbb{R}^N_+ \quad \text{and} \quad u \in \mathcal{D}^{1,2}_0(\mathbb{R}^N_+). \end{cases} \tag{5.18}$$

Then, from (5.18) and from the definition of $m_{1,p}$ we conclude that

$$\int_{\mathbb{R}^N_+} |\nabla u|^2 dz = \int_{\mathbb{R}^N_+} e^{-z_N} u^p dz \ge m_{1,p}^{p/(p-2)}. \tag{5.19}$$

With $R > 0$ large with $R < \alpha$ we define

$$B_{R,\alpha} = \left\{ z; \frac{z}{\alpha} - e_N \in B_N(-e_N, R/\alpha) \cap B_N(0,1) \right\}.$$

From (5.11) and with the change of variables $x = \frac{z}{\alpha} - e_N$ we have

$$\begin{aligned} \left(2\, m_{1,p}^{p/p-2} + o(1)\right) \alpha^{[2(m+1)-p(m-1)]/(p-2)} &\ge (K'_{\alpha,p})^{p/(p-2)} = 2 \int_{B_N(0,1),\ z_N<0} |\nabla u_\alpha|^2 dx \\ &= 2 \int_{B_N(0,1)\cap B_N(-e_N, R/\alpha)} |\nabla u_\alpha|^2 dx + 2 \int_{(B_N(0,1),\ z_N<0)\setminus B_{m+1}(-e_{m+1}, R/\alpha)} |\nabla u_\alpha|^2 dx \\ &= 2\alpha^{[2N-p(N-2)]/(p-2)} \left[\int_{B_{R,\alpha}} |\nabla \widehat{u}_\alpha|^2 dz + \int_{\Omega_\alpha \setminus B_{R,\alpha}} |\nabla \widehat{u}_\alpha|^2 dz \right] \\ &\ge 2\alpha^{[2N-p(N-2)]/(p-2)} \int_{B_{R,\alpha}} |\nabla \widehat{u}_\alpha|^2 dz. \end{aligned}$$

Then from (5.17) and (5.19) it follows that

$$\begin{aligned} \left(2\, m_{1,p}^{p/p-2} + o(1)\right) &\ge 2 \int_{B_{R,\alpha}} |\nabla \widehat{u}_\alpha|^2 dz = 2 \int_{\mathbb{R}^N_+ \cap B_N(0,R)} |\nabla \widehat{u}_\alpha|^2 dz + o(1) \\ &= 2 \int_{\mathbb{R}^N_+ \cap B_N(0,R)} |\nabla u|^2 dz + o(1) \ge 2\, m_{1,p}^{p/(p-2)} + o_R(1) + o(1). \end{aligned}$$

Hence we obtain

$$\int_{\Omega_\alpha \setminus B_{R,\alpha}} |\nabla \widehat{u}_\alpha|^2 dz = o(1) + o_R(1) \quad \text{when} \quad R < \alpha,\ R \to \infty,$$

$$\int_{B_{R,\alpha}} |\nabla \widehat{u}_\alpha|^2 dz = \int_{\mathbb{R}^N_+ \cap B_N(0,R)} |\nabla u|^2 dz + o(1) = m_{1,p}^{p/(p-2)} + o_R(1) + o(1) \quad \text{when} \quad R < \alpha,\ R \to \infty.$$

Then, from (5.17) and since $(\nabla \widehat{u}_\alpha)$ is bounded in $L^2(\mathbb{R}^N_+)$, it follows that

$$\begin{aligned}\int_{\mathbb{R}^N_+} |\nabla \widehat{u}_\alpha - \nabla u|^2 dz &= \int_{\mathbb{R}^N_+ \cap B_N(0,R)} |\nabla \widehat{u}_\alpha - \nabla u|^2 dz + \int_{\mathbb{R}^N_+ \setminus B_N(0,R)} |\nabla \widehat{u}_\alpha - \nabla u|^2 dz \\ &\le o(1) + \int_{\Omega_\alpha \setminus B_{R,\alpha}} |\nabla \widehat{u}_\alpha - \nabla u|^2 dz + \int_{\mathbb{R}^N_+ \setminus B_N(0,R)} |\nabla u|^2 dz \\ = o(1) + o_R(1) - 2\int_{\Omega_\alpha \setminus B_{R,\alpha}} \nabla \widehat{u}_\alpha \nabla u dz &\le o(1) + o_R(1) + 2C\left(\int_{\Omega_\alpha \setminus B_{R,\alpha}} |\nabla u|^2 dz\right)^{1/2} = o(1) + o_R(1).\end{aligned}$$

Hence we conclude that

$$\int_{\mathbb{R}^N_+} |\nabla \widehat{u}_\alpha - \nabla u|^2 dz \to 0 \quad \text{as} \quad \alpha \to \infty,$$

and that $\int_{\mathbb{R}^N_+} |\nabla u|^2 dz = m_{1,p}^{p/(p-2)}$ and so w minimizes $m_{1,p}$. □

Proposition 5.6. *There exists $l > 0$ such that*

$$\alpha(1 - \tau_\alpha) \to l \quad \textit{as} \quad \alpha \to \infty.$$

Proof. Exactly as the proof of Proposition 4.7. □

***Proof of Proposition* 5.1.** From (5.16) we obtain that

$$\begin{aligned} m_{1,p}^{p/(p-2)} &= \int_{\mathbb{R}^N_+} |\nabla u|^2 dz = \int_{\Omega_\alpha} |\nabla \widehat{u}_\alpha|^2 dz + o(1) \\ &= \alpha^{[p(N-2)-2N]/(p-2)} \int_{B_N(0,1),\ z_N<0} |\nabla u_\alpha|^2 dx + o(1) = \frac{1}{2}\alpha^{[p(N-2)-2N]/(p-2)} (K'_{\alpha,p})^{p/(p-2)} + o(1).\end{aligned}$$

Therefore

$$\frac{K'_{\alpha,p}}{\alpha^{[2N-p(N-2)]/p}} = 2^{1-2/p} m_{1,p} + o(1) \quad \text{as} \quad \alpha \to \infty.$$ □

***Proof of Theorem* 1.3.** It follows from Lemma 5.3, Propositions 5.5 and 5.6. □

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