

Exponential Dephasing of Oscillators in the Kinetic Kuramoto Model

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Abstract We study the kinetic Kuramoto model for coupled oscillators with coupling constant below the synchronization threshold. We manage to prove that, for any analytic initial datum, if the interaction is small enough, the order parameter of the model vanishes exponentially fast, and the solution is asymptotically described by a free flow. This behavior is similar to the phenomenon of Landau damping in plasma physics. In the proof we use a combination of techniques from Landau damping and from abstract Cauchy–Kowalewskaya theorem.

Keywords Kuramoto model · Dephasing · Landau damping · Abstract Cauchy–Kowalewskaya theorem

Mathematics Subject Classification 35A10 · 35Q92 · 74A25 · 76N10 · 92B25

1 Introduction

The Kuramoto model is a mean-field model of coupled oscillators proposed by Kuramoto to describe synchronization phenomena (see [1, 17, 22]). Any oscillator has a phase ϑ , that can be considered defined mod 2π , i.e. in the one-dimensional torus \mathcal{T} , and a “natural frequency” $\omega \in \mathbb{R}$. In the kinetic limit, the equation for the probability density $f(t, \vartheta, \omega)$ on $\mathcal{T} \times \mathbb{R}$ is

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$$\begin{cases} \partial_t f(t, \vartheta, \omega) + \partial_{\vartheta}(v(t, \vartheta, \omega)f(t, \vartheta, \omega)) = 0 \\ v(t, \vartheta, \omega) = \omega - \mu \int_{\mathcal{T} \times \mathbb{R}} \sin(\vartheta - \vartheta') f(t, \vartheta', \omega') d\vartheta' d\omega', \end{cases} \quad (1.1)$$

where $\int_{\mathcal{T} \times \mathbb{R}} \sin(\vartheta - \vartheta') f(t, \vartheta', \omega') d\vartheta' d\omega'$ is the mean field interaction term, and $\mu > 0$ is the coupling constant. The distribution of the natural frequencies is $g(\omega) = \int_{\mathcal{T}} f(t, \vartheta, \omega) d\vartheta$, which is a conserved quantity.

It can be useful to represent the system (1.1) in the unitary circle of the complex plane by considering the oscillators as particles with position $e^{i\vartheta}$. The center of mass is in the point

$$R(t)e^{i\varphi(t)} = \int_{\mathcal{T} \times \mathbb{R}} f(t, \vartheta, \omega)e^{i\vartheta} d\vartheta d\omega. \quad (1.2)$$

R and φ are the “order parameters” of the model. By this notation the coupling term can be rewritten as

$$\int_{\mathcal{T} \times \mathbb{R}} \sin(\vartheta - \vartheta') f(t, \vartheta', \omega') d\vartheta' d\omega' = R(t) \sin(\vartheta - \varphi(t)); \quad (1.3)$$

so that the mean field interaction between the particles can be read as an attraction towards the phase of the center of mass, modulated by $R(t)$.

Existence and uniqueness results for the system (1.1) are obtained in [18], where the (1.1) is rigorously derived by doing the kinetic limit of the particle model introduced by Kuramoto.

The model has been intensively studied in the case where g has compact support and the coupling constant μ is sufficiently large to observe the complete asymptotic synchronization phenomenon (see [8, 10, 15], and [5, 12] for the simpler case of g being a Dirac delta). If g has not compact support, the complete synchronization is impossible, and for large μ it is expected only a partial synchronization: some oscillators synchronize and a group of oscillators drifts along the circle (see [20, 22]).

The behavior at small values of μ was first suggested by Kuramoto in the case of even and unimodal distributions of natural frequencies: he formulated the conjecture that $R = 0$ is asymptotically stable if $\mu < \mu_c$, with $\mu_c = 2/(\pi g(0))$; while if $\mu > \mu_c$ synchronized states would emerge. This conjecture is related to the stability of the “fully incoherent state” $g(\omega)/(2\pi)$, which is the only stationary solution with $R = 0$. Note that in the sup norm this solution can not be asymptotically stable: in particular in the free flow case ($\mu = 0$) R vanishes for any initial datum f_{in} , but the density converges only weakly to the fully incoherent state. In [23] the linear instability of the fully incoherent state is proved for $\mu > \mu_c$.

In [24] the Kuramoto conjecture for $\mu < \mu_c$ is interpreted as a Landau-damping behavior: as in the case of the Vlasov–Poisson equation, the damping (i.e. $R \rightarrow 0$) is due to phase mixing, modified by the self-consistent decaying order parameter R . This dephasing result is proved in the linearized case: the order parameter R of a perturbation of the fully incoherent state relaxes to zero as the time goes to infinity and the density is asymptotically transported by a free flow.

The phenomenon of nonlinear dephasing in the Kuramoto model has been studied in the recent work [9] and the contemporary [4, 11, 14]. In [9], by using spectral analysis applied to dynamical systems, for a special class of distributions g of natural frequencies, it is obtained the first result on the Kuramoto conjecture. By studying the bifurcation at μ_c of the fully incoherent state, it is shown that, if $\mu < \mu_c$, the order parameter decays exponentially.

In the most complete [11], the bifurcation analysis carried out in [9] is extended to more general distributions g and the nonlinear stability of $R = 0$ is proved for both Sobolev and analytical regularity of initial data. The dephasing result $R \rightarrow 0$ is achieved by showing the polynomial (in the Sobolev case) or exponential (in the analytical case) decay of all the

Fourier modes of the solutions. It is also proved a global stability result for $R = 0$: it exists a μ_{ec} , depending only on the L^1 norm of \hat{g} , such that if $\mu < \mu_{ec}$ then $\int_0^{+\infty} R^2(s) ds < +\infty$

On the other hand, in [14], by following the work [13] about the Landau-damping for the Vlasov-HMF model, it is shown the polynomial decaying of R for initial data of Sobolev regularity. Unlike the above mentioned works, it is studied the solution along the free flow $f(t, \vartheta + \omega, \omega)$ showing that it converges, so that the solution of the system (1.1) is asymptotically transported by the free flow.

A complementary result which shows the richness of possible asymptotic dephased states is obtained in [4] where, by following the works [7, 16] about the Vlasov–Poisson equation, it is proved the existence of regular dephasing solutions with any prescribed asymptotic behavior.

In this work, along the same line of the works [3, 13] on Landau-damping for the Vlasov equation [21], and of the work [2] for the Euler equation, we show a Landau-damping behavior for the Kuramoto model. For small values of μ , we prove the asymptotic dephasing in the case of analytic initial data; in particular, we show that $R(t)$ vanishes exponentially fast. Our results are weaker than those in [11] (and possible analytical analogues of [14]): giving up the benefits of the linear analysis, we lose the possibility of finding the critical value. On the other hand, we adapt abstract Cauchy–Kowalewskaya techniques to obtain a brief proof of the result. In particular we prove the dephasing via a globally-in-time existence result in analytic spaces, while the abstract Cauchy–Kowalewskaya Theorem gives only finite time existence.

2 Dephasing

In terms of the order parameters, the kinetic Kuramoto equation reads as

$$\begin{cases} \partial_t f(t, \vartheta, \omega) + \partial_{\vartheta}(v(t, \vartheta, \omega) f(t, \vartheta, \omega)) = 0 \\ v(t, \vartheta, \omega) = \omega - \mu R(t) \sin(\vartheta - \varphi(t)) \\ R(t) e^{i\varphi(t)} = \int_{\mathcal{T} \times \mathbb{R}} f(t, \vartheta, \omega) e^{i\vartheta} d\vartheta d\omega. \end{cases} \tag{2.1}$$

If $\mu = 0$, the solutions of Eq. (2.1) is

$$f(t, \vartheta, \omega) = f_{in}(\vartheta - \omega t, \omega)$$

where f_{in} is the initial datum. Our aim is to show that, if $f_{in}(\vartheta, \omega)$ is bounded in some analytic norm, and $\mu > 0$ is sufficiently small, there exists an asymptotic state h_{∞} such that

$$f(t, \vartheta + \omega t, \omega) \rightarrow h_{\infty}(\vartheta, \omega)$$

exponentially fast. In other words, the solutions $f(t, \vartheta, \omega)$ asymptotically approaches the incoherent state $h_{\infty}(\vartheta - \omega t, \omega)$, i.e. a function transported by the free flow. In this sense, the solutions show an asymptotic dephasing. The key ingredient of the proof is the exponential decay of the order parameter $R(t)$, which, as noted in [19], can only be obtained for analytic initial data.

To state precisely our result, we define $h(t, \vartheta, \omega) = f(t, \vartheta + \omega t, \omega)$, with initial datum $h(0, \vartheta, \omega) = f_{in}(\vartheta, \omega)$ and which verifies, from Eq. (2.1), the equation

$$\partial_t h(t, \vartheta, \omega) = \mu R(t) \partial_{\vartheta} (\sin(\vartheta + \omega t - \varphi(t)) h(t, \vartheta, \omega)). \tag{2.2}$$

Defining for $k \in \mathbb{Z}$ and $\eta \in \mathbb{R}$

$$\hat{h}_k(t, \eta) = \frac{1}{2\pi} \int_{\mathcal{T} \times \mathbb{R}} d\vartheta \, d\omega h(t, \vartheta, \omega) e^{-i\vartheta k - i\omega \eta}$$

Eq. (2.2) in Fourier space is

$$\partial_t \hat{h}_k(t, \eta) = \mu \widehat{\mathbf{L}_t h}(t, k, \eta) \tag{2.3}$$

where the operator L_t acts on a function $u(t, \vartheta, \omega)$ as

$$\widehat{\mathbf{L}_t u}(t, k, \eta) \doteq k \sum_{m=\pm 1} \frac{m}{2} z_m(t) \hat{u}_{k-m}(t, \eta - mt), \tag{2.4}$$

where the order parameters read as

$$z_{\pm 1}(t) = \hat{h}_{\pm 1}(t, \pm t) = R(t) e^{\mp i\varphi(t)}, \quad |z_{\pm 1}(t)| = R(t).$$

Integrating in time Eq. (2.3) we have

$$\hat{h}_k(t, \eta) = \hat{h}_k(0, \eta) + \mu k \sum_{m \in \pm 1} \frac{m}{2} \int_0^t z_m(s) \hat{h}_{k-m}(s, \eta - ms) \, ds. \tag{2.5}$$

In the sequel, we consider separately the evolution in time of the order parameter $z_{\pm 1}$, and of the function h ; in this sense \mathbf{L}_t can be considered a linear operator in h .

We use analytic norms for h so, to make the notation lighter, we give the following definitions:

$$\langle t \rangle = (1 + t^2)^{\frac{1}{2}}, \quad \langle k, \eta \rangle = (1 + k^2 + \eta^2)^{\frac{1}{2}}; \tag{2.6}$$

note that $\langle \cdot \rangle$ verifies the triangular inequality, as follows from easy calculation:

$$\langle k_1 + k_2, \eta_1 + \eta_2 \rangle \leq \langle k_1, \eta_1 \rangle + \langle k_2, \eta_2 \rangle. \tag{2.7}$$

For $\lambda, p \geq 0$, we define the weight

$$A_k^{\lambda, p}(\eta) = e^{\lambda \langle k, \eta \rangle} \langle k, \eta \rangle^p, \tag{2.8}$$

and the norms:

$$\|f\|_{\lambda, p} = \sup_{k \in \mathbb{Z}, \eta \in \mathbb{R}} A_k^{\lambda, p}(\eta) |\hat{f}_k(\eta)|. \tag{2.9}$$

We call $\mathcal{X}_{\lambda, p}$ the space of function f with finite $\|f\|_{\lambda, p}$ norm.

Using this norm, it is easy to show that if $\mu = 0$, we obtain the exponential decay of the order parameter $z_{\pm 1}(t)$. Let us first obtain an equation for $z_1 = R(t) e^{-i\varphi(t)}$ setting $k = 1$ and $\eta = t$ in (2.5)

$$z_1(t) = \hat{h}_1(0, t) + \mu \sum_{m=\pm 1} \frac{m}{2} \int_0^t z_m(s) \hat{h}_{1-m}(s, t - ms) \, ds, \tag{2.10}$$

where $h(0, \vartheta, \eta)$ is the initial datum $f_{in}(\vartheta, \eta)$. Choosing $\lambda, p \geq 0$, the first term, due to the free flow, is bounded by

$$\|f_{in}\|_{\lambda, p} e^{-\lambda \langle 0, t \rangle} \langle 0, t \rangle^{-p} \leq C e^{-\lambda t} \langle t \rangle^{-p} \|f_{in}\|_{\lambda, p}$$

where, here and in the following, C is a suitable time independent constant. This estimate suggests that we can control the quantities

$$r_{\lambda, p}(t) = R(t) e^{\lambda t} \langle t \rangle^p. \tag{2.11}$$

A uniform in time bound of $r_{\lambda,p}(t)$ is equivalent to an exponential decay of $R(t)$. The aim of this paper is to show that, if μ is sufficiently small, the other terms in (2.10) do not prevent a uniform estimate for $r_{\lambda,p}$.

3 A-priori Estimates

In the case $\mu > 0$, $r_{\lambda,p}(t)$ can be estimated as in the following proposition.

Proposition 1 For $\lambda, p \geq 0$

$$\begin{aligned}
 r_{\lambda,p}(t) &\leq C \|f_{in}\|_{\lambda,p} + \mu C \|f_{in}\|_{\lambda,p} \int_0^t r_{\lambda,p}(s) \left(\frac{1}{\langle s \rangle^p} + \frac{1}{\langle t-s \rangle^p} \right) ds \\
 &\quad + \mu C \int_0^t r_{\lambda,p}(s) \frac{\|h(s)\|_{\lambda,p}}{\langle s \rangle^p} ds.
 \end{aligned}
 \tag{3.1}$$

Proof The first term of the r.h.s. of (2.10) is simply bounded by

$$C e^{-\lambda t} \langle t \rangle^{-p} \|f_{in}\|_{\lambda,p}.$$

Since $\hat{f}_0(t, \eta) = \hat{f}_0(0, \eta)$, the term with $m = 1$ is

$$\int_0^t z_1(s) \hat{f}_0(s, t-s) ds = \int_0^t z_1(s) \hat{f}_0(0, t-s) ds,$$

which is bounded by

$$\begin{aligned}
 &C \|f_{in}\|_{\lambda,p} \int_0^t e^{-\lambda(0,t-s)} \langle t-s \rangle^{-p} |z_1(s)| ds \\
 &\leq C \|f_{in}\|_{\lambda,p} \int_0^t r_{\lambda,p}(s) e^{-\lambda s} \langle s \rangle^{-p} e^{-\lambda(t-s)} \langle t-s \rangle^{-p} ds,
 \end{aligned}
 \tag{3.2}$$

where we used that $\langle 0, t-s \rangle \geq \langle t-s \rangle$. Multiplying by $e^{\lambda t} \langle t \rangle^p$, we have the estimate

$$\int_0^t r_{\lambda,p}(s) \frac{\langle t \rangle^p}{\langle t-s \rangle^p \langle s \rangle^p} ds \leq \int_0^t r_{\lambda,p}(s) \left(\frac{1}{\langle s \rangle^p} + \frac{1}{\langle t-s \rangle^p} \right) ds,$$

because $\langle t \rangle^p \leq C(\langle s \rangle^p + \langle t-s \rangle^p)$.

The term with $m = -1$ is bounded by

$$\int_0^t R(s) |h_2(s, t+s)| ds \leq \int_0^t r_{\lambda,p}(s) \|h(s)\|_{\lambda,p} e^{-\lambda s - \lambda(t+s)} \langle s \rangle^{-p} \langle t+s \rangle^{-p} ds. \tag{3.3}$$

We conclude the estimate (3.1) by multiplying by $e^{\lambda t} \langle t \rangle^p$ and noting that $\langle t \rangle^p \leq \langle t+s \rangle^p$. \square

In order to estimate $\|h\|_{\lambda,p}$, we need to control the time derivative of h .

Proposition 2 Given $z_{\pm 1}(t)$, for $\lambda, p \geq 0$, \mathbf{L}_t is a continuous operator from $\mathcal{X}_{\lambda,p+1}$ to $\mathcal{X}_{\lambda,p}$:

$$\|\mathbf{L}_t h(t)\|_{\lambda,p} \leq C (r_{\lambda,0}(t) \|h(t)\|_{\lambda,p+1} + r_{\lambda,p}(t) \|h(t)\|_{\lambda,1}). \tag{3.4}$$

\mathbf{L}_t is also continuous from $\mathcal{X}_{\lambda',p}$ to $\mathcal{X}_{\lambda,p}$, when $\lambda' > \lambda$, in fact

$$\|f\|_{\lambda,p+1} \leq \frac{1}{\lambda' - \lambda} \|f\|_{\lambda',p}. \tag{3.5}$$

Proof Recalling the definition of $A_k^{\lambda,p}$ in (2.8), we write

$$A_k^{\lambda,p}(\eta)|\widehat{\mathbf{L}_t h}(t, k, \eta)| \leq \frac{1}{2}|z_1(t)| |k| \sum_{m=\pm 1} e^{\lambda\langle k, \eta \rangle} \langle k, \eta \rangle^p |\widehat{h}_{k-m}(t, \eta - mt)|;$$

then, by the triangular inequality,

$$\langle k, \eta \rangle \leq \langle k - m, \eta - mt \rangle + \langle m, mt \rangle,$$

and that, when $|m| = 1$, $\langle m, mt \rangle \leq C + t$, we have

$$e^{\lambda\langle k, \eta \rangle} \leq C e^{\lambda t} e^{\lambda\langle k-m, \eta-mt \rangle}.$$

Since $\langle m, mt \rangle$ also verifies $\langle m, mt \rangle \leq C(t)$, it is true that

$$\langle k, \eta \rangle^p \leq C \left(\langle k - m, \eta - mt \rangle^p + \langle t \rangle^p \right),$$

which implies

$$\begin{aligned} & A_k^{\lambda,p}(\eta)|\widehat{\mathbf{L}_t h}(t, k, \eta)| \\ & \leq C e^{\lambda t} R(t) \sum_{m=\pm 1} e^{\lambda\langle k-m, \eta-mt \rangle} |k| \langle k - m, \eta - mt \rangle^p |\widehat{h}_{k-m}(\eta - mt)| \\ & \quad + C e^{\lambda t} R(t) \sum_{m=\pm 1} e^{\lambda\langle k-m, \eta-mt \rangle} |k| \langle t \rangle^p |\widehat{h}_{k-m}(\eta - mt)|. \end{aligned}$$

Using that $|k| \leq \langle k - m, \eta - mt \rangle$, we conclude the proof by estimating the first term with

$$C r_{\lambda,0}(t) \|h(t)\|_{\lambda,p+1}$$

and the second with

$$C r_{\lambda,p}(t) \|h(t)\|_{\lambda,1}.$$

□

Let us discuss how to choose the norms that will allow us to obtain closed estimates for h and $z_{\pm 1}$. In the Landau Damping type results in the case of Sobolev regularity of order γ , the choice of a suitable Hilbert space \mathcal{H}_γ , with norm $\|h\|_{\mathcal{H}_\gamma}$, guarantees that L_t is a continuous map from \mathcal{H}_γ in the same \mathcal{H}_γ (see [14]). Then the results are achieved estimating, globally in time, a term of the type $\|h(t)\|_{\mathcal{H}_\gamma}/\langle t \rangle$, for suitable values of γ , and, correspondingly, $\langle t \rangle^\gamma R(t)$. In the case of analytical regularity we can not obtain this behavior and we have to take into account that in (3.4) we can only estimate $\mathbf{L}_t h$ in a norm that is weaker than the one of h . We give closed estimates by mixing the typical norms used in Landau-Damping type results with the norms needed for the proof of the abstract Cauchy–Kowalewskaya theorem, following in particular [6].

Given $\lambda_0 > 0$ and a such that $0 < a < 2\lambda_0/\pi$, for $t \geq 0$ and $\lambda < \lambda_0$, we define the weight

$$\beta(t, \lambda) = \beta_a(t, \lambda) = \lambda_0 - \lambda - a \int_0^t \frac{ds}{\langle s \rangle^2} = \lambda_0 - \lambda - a \arctan t. \tag{3.6}$$

This function is positive for decreasing in time values of λ , and, as in abstract Cauchy–Kowalewskaya theorems, we use it to taking into account the loss of analytical regularity due to the spatial derivative in ϑ in the operator \mathbf{L}_t . In [6] and in other proofs of the abstract Cauchy–Kowalewskaya Theorem, the time dependence of the weight is linear and the solutions exists only for finite time. Here the Landau-Damping type estimates allow us to choose a weight convergent in time, which, if $a < 2\lambda_0/\pi$ give the analyticity also for $t \rightarrow +\infty$.

More precisely, we define the Banach space $\tilde{\mathcal{B}}_{a,p}$ as the space of the functions $h(t)$ such that, if $\beta(t, \lambda) > 0$, $h(t) \in \mathcal{X}_{\lambda,p}$. The norm in $\tilde{\mathcal{B}}_{a,p}$ is

$$\|h\|_{a,p} = \sup_{\lambda,t:\beta(t,\lambda)>0} \beta^{1/2}(t, \lambda) \|h\|_{\lambda,p}. \tag{3.7}$$

Finally, fixing $\gamma \geq 3$, we define the norm

$$\|h\|_a = \|h\|_{a,1} + \|h(\cdot)/\langle \cdot \rangle\|_{a,\gamma}, \tag{3.8}$$

and the corresponding Banach space \mathcal{B}_a , of the function h with $\|h\|_a$ bounded. With little abuse of notation, we write:

$$\|R\|_a = \sup_{\lambda,t:\beta(t,\lambda)>0} r_{\lambda,\gamma}(t) = \sup_{\lambda,t:\beta(t,\lambda)>0} R(t)e^{\lambda t} \langle t \rangle^\gamma \tag{3.9}$$

Now, we prove the a-priori estimates in \mathcal{B}_a which allow us to construct the solutions.

Proposition 3 *Given $z_{\pm 1}(t)$ with $\|R\|_a < +\infty$, if $h = h(t, \vartheta, \eta)$ solves Eq. (2.3) then it satisfies*

$$\|h\|_a \leq C \|f_{in}\|_{\lambda_0,\gamma} + C\mu \|R\|_a \|h\|_a. \tag{3.10}$$

Proof First we estimate $\|h(t)\|_{\lambda,1}$, for λ such that $\beta(t, \lambda) > 0$. Using (2.5) and the estimate (3.4) with $p = 1$, we have

$$\|h(t)\|_{\lambda,1} \leq C \|f_{in}\|_{\lambda_0} + C\mu \int_0^t r_{\lambda,\gamma}(s) \left(\frac{1}{\langle s \rangle^{\gamma-1}} \frac{\|h(s)\|_{\lambda,\gamma}}{\langle s \rangle} + \frac{1}{\langle s \rangle^{\gamma-1}} \|h(s)\|_{\lambda,1} \right) ds. \tag{3.11}$$

Multiplying by $\beta^{1/2}(t, \lambda)$:

$$\beta^{1/2}(t, \lambda) \|h(t)\|_{\lambda,1} \leq C \|f_{in}\|_{\lambda_0} + C\mu \|h\|_a \|R\|_a \int_0^t \frac{1}{\langle s \rangle^2} \frac{\beta^{1/2}(t, \lambda)}{\beta^{1/2}(s, \lambda)} ds, \tag{3.12}$$

where we have used that for $\gamma \geq 3$, $\langle s \rangle^{\gamma-1} \geq \langle s \rangle^2$. Using that $\beta(t) \leq \beta(s)$ we estimate the time integral with a constant, then

$$\|h\|_{a,1} \leq C \|f_{in}\|_{\lambda_0,\gamma} + C\mu \|R\|_a \|h\|_a.$$

Now we estimate $\|h\|_{\lambda,\gamma}$: using Eq. (2.5) and the estimates (3.4), (3.5) with $p = \gamma$

$$\|h(t)\|_{\lambda,\gamma} \leq C \|f_{in}\|_{\lambda_0,\gamma} + C\mu \int_0^t r_{\lambda,\gamma}(s) \left(\frac{1}{\langle s \rangle^\gamma} \frac{\|h(s)\|_{\lambda'(s),\gamma}}{\lambda'(s) - \lambda} + \|h(s)\|_{\lambda,1} \right) ds, \tag{3.13}$$

for any $\lambda'(s) > \lambda$ such that $\lambda_0 - \lambda'(s) - a \arctan(s) > 0$. Dividing by $\langle t \rangle$ and multiplying by $\beta^{1/2}(t, \lambda)$, we obtain

$$\frac{\beta^{1/2}(t, \lambda)}{\langle t \rangle} \|h\|_{\lambda,\gamma} \leq C \|f_{in}\|_{\lambda_0,\gamma} + C\mu \|h\|_a \|R\|_a (I_1 + I_2), \tag{3.14}$$

where

$$I_1 = \frac{\beta^{1/2}(t, \lambda)}{\langle t \rangle} \int_0^t \frac{ds}{\langle s \rangle^2 \beta^{1/2}(s, \lambda) (\lambda'(s) - \lambda)},$$

$$I_2 = \frac{1}{\langle t \rangle} \int_0^t \frac{\beta^{1/2}(t, \lambda)}{\beta^{1/2}(s, \lambda)} ds.$$

I_2 is less than a constant because $\beta(t, \lambda) \leq \beta(s, \lambda)$, for $s \leq t$.

In I_1 , we chose $\lambda' = \lambda'(s) > \lambda$ as

$$\lambda'(s) = \frac{1}{2}(\lambda_0 - a \arctan s) + \frac{\lambda}{2},$$

which verifies

$$\beta(s, \lambda'(s)) = \frac{1}{2}\beta(s, \lambda) > 0,$$

and

$$\lambda'(s) - \lambda = \frac{1}{2}\beta(s) \geq \frac{1}{2}\beta(t) > 0.$$

Then I_1 is bounded by

$$I_1 \leq 2 \frac{\beta^{1/2}(t, \lambda)}{\langle t \rangle} \int_0^t \frac{ds}{\langle s \rangle^2 \beta(s, \lambda)^{3/2}}.$$

Since $d\beta/ds = -a/\langle s \rangle^2$, the time integral can be explicitly computed and gives:

$$\int_0^t \frac{ds}{\langle s \rangle^2 \beta(s)^{3/2}} = \frac{2}{a} \left(\frac{1}{\beta^{1/2}(t, \lambda)} - \frac{1}{\beta^{1/2}(0, \lambda)} \right),$$

then also I_1 is less than a constant. □

Now we estimate $\|R\|_a$.

Proposition 4 *Fixed h such that $\|h\|_a < +\infty$, if $z_{\pm 1}(t) = R(t)e^{\mp i\varphi(t)}$ solves (2.10), then*

$$\|R\|_a \leq C \|f_{in}\|_{\lambda_0, \gamma} (1 + \mu \|R\|_a) + C\mu \|R\|_a \|h\|_a. \tag{3.15}$$

Proof We use (3.1) with $p = \gamma$: the estimate of the first two terms are obvious; the last one is bounded by $\mu C \|R\|_a \|h\|_a$ times the integral

$$\int_0^t \frac{ds}{\langle s \rangle^2 \beta^{1/2}(s, \lambda)} = \frac{2}{a} (\beta^{1/2}(0, \lambda) - \beta^{1/2}(t, \lambda)) \leq \frac{2}{a} \lambda_0^{1/2}$$

4 The Main Theorem

Theorem 1 *For $\lambda_0 > 0$ and $\gamma \geq 3$, if $\|f_{in}\|_{\lambda_0, \gamma}$ is bounded, for μ sufficiently small, the unique solution $h(t, \vartheta, \omega)$ of (2.2) with initial datum $f_{in}(\vartheta, \omega)$ verifies $\|h\|_a < C$ and $\|R\|_a < C$.*

As a consequence, $R(t) \rightarrow 0$ exponentially fast and there exists $h_\infty(\vartheta, \omega)$ with $\|h_\infty\|_{\bar{\lambda}, \gamma} < +\infty$ for some $\bar{\lambda} > 0$, such that

$$f(t, \vartheta + \omega t, \omega) = h(t, \vartheta, \omega) \rightarrow h_\infty(\vartheta, \omega)$$

exponentially fast.

Proof We construct the solution with an iterative procedure. For $n \geq 0$

$$\partial_t \widehat{h}_k^{n+1}(t, \eta) = \mu \widehat{\mathbf{L}}_t^n h^{n+1}(t, k, \eta), \tag{4.1}$$

where the operators L_t^n act on a generic function $u(t, \vartheta, \omega)$ as

$$\widehat{\mathbf{L}}_t^n u(t, k, \eta) \doteq k \sum_{m=\pm 1}^n \frac{m}{2} z_m^n(t) \widehat{u}_{k-m}(t, \eta - mt), \tag{4.2}$$

with $z_m^n(t)$ solution of

$$z_1^n(t) = \hat{h}_1^n(0, t) + \mu \sum_{m=\pm 1} \frac{m}{2} \int_0^t z_m^n(s) \hat{h}_{1-m}^n(s, t - ms) ds, \quad z_{-1}^n(t) = \overline{z_1^n(t)}. \tag{4.3}$$

The procedure starts with

$$h^0(t, \vartheta, \omega) = f_{in}(\vartheta, \omega). \tag{4.4}$$

The linear problems in Eq. (4.3) and in Eq. (4.1) are easily solvable, and the solutions verify the analogous of the a-priori estimate provided by Propositions 3 and 4:

$$\|R^n\|_a \leq C \|f_{in}\|_{\lambda_0, \gamma} (1 + \mu \|R^n\|_a) + C\mu \|R^n\|_a \|h^n\|_a$$

and

$$\|h^{n+1}\|_a \leq C \|f_{in}\|_{\lambda_0, \gamma} + C\mu \|R^n\|_a \|h^{n+1}\|_a.$$

Using that $\|h^0\|_a \leq C \|f_{in}\|_{\lambda_0, \gamma}$ we can inductively prove that, if $\mu \|f_{in}\|_{\lambda_0, \gamma}$ is sufficiently small, then

$$\|h^n\|_a \leq C \|f_{in}\|_{\lambda_0, \gamma}, \quad \text{and} \quad \|R^n\|_a \leq C \|f_{in}\|_{\lambda_0, \gamma},$$

uniformly in n . Choosing $a' > a$ with $a' < 2\lambda_0/\pi$, and estimating the operator L_t^n as in Proposition 2, we have that all the first derivative of h^n are uniformly bounded in the region defined by $\beta_{a'}(t, \lambda) > 0$; then, for subsequences, h^n converges to some $h \in \tilde{B}_{a', \gamma}$. Correspondingly, $z_{\pm 1}^n$ converges to $z_{\pm 1}$ with $\|z_{\pm 1}\|_a$ bounded. The function h and $z_{\pm 1}$ solve the coupled equations (2.3) and (2.10). Moreover, putting $k = 1$ and $\eta = t$ in (2.5)

$$\hat{h}_1(t, t) = \hat{h}_1(0, t) + \mu \sum_{m=\pm 1} \int_0^t \frac{m}{2} z_m(s) \hat{h}_{k-m}(s, \eta - ms) ds = z_1(t)$$

as follows from (2.10). Then h solves the non linear equation (2.3), and its uniqueness is guaranteed by the uniqueness of regular solutions (see [18]) (note that the uniqueness implies the convergence to h and z for the full sequences h_n and z_n).

Finally, let $\bar{\lambda} > 0$, with $\bar{\lambda} < \lambda_0 - a'\pi/2$. Then

$$\|L_t h\|_{0, \gamma} \leq \frac{C}{\bar{\lambda}} \|R\|_{a'} \|h\|_{a'} e^{-\bar{\lambda}t}.$$

This inequality implies the existence of

$$\lim_{t \rightarrow +\infty} h(t) = h_\infty,$$

with $h_\infty \in \mathcal{X}_{\bar{\lambda}, \gamma}^*$ because $h \in B_{a', \gamma}$. Being $\gamma \geq 3$, the norm $\|h\|_{0, \gamma}$ dominates the sup norm in ϑ and ω , then $h(t)$ converges exponentially fast to h_∞ in the sup norm. \square

Remark 1 In the analysis carried out in this work the term $\hat{h}_0(t, \eta) = \frac{\hat{g}(\eta)}{\sqrt{2\pi}}$ can be separated from the other Fourier modes: in the Proposition 1 and its following, we can bound separately the zero and nonzero modes, so that, being more careful in the estimates, it is true that

$$\begin{aligned} \|R\|_a &\leq C \left\| f_{in} - \frac{g}{2\pi} \right\|_{\lambda_0, \gamma} + C\mu \|g\|_a \|R\|_a + C\mu \|R\|_a \left\| h - \frac{g}{2\pi} \right\|_a \\ \left\| h - \frac{g}{2\pi} \right\|_a &\leq C \left\| f_{in} - \frac{g}{2\pi} \right\|_{\lambda_0, \gamma} + C\mu \|g\|_a \|R\|_a + C\mu \|R\|_a \left\| h - \frac{g}{2\pi} \right\|_a. \end{aligned}$$

Using these estimates, we can prove the thesis of the main theorem assuming $\mu \|g\|_a$ small, and f_{in} close to the fully incoherent state $\frac{g}{2\pi}$.

The same results have been proved in the Sobolev case in [14]. Note that in [11, 14] it is assumed a “stability condition” on μ of the type

$$1 - \frac{\mu}{2} \int_0^{+\infty} \hat{g}(\eta) e^{-\eta z} d\eta \neq 0$$

for $z \in \mathbb{C}$ with $\Re z > -b$, $b > 0$ (exponential decay) or $\Re z \geq 0$ (polynomial decay), and a smallness condition for $f_{in} - \frac{g}{2\pi}$. These conditions can be verified also for large values of μ , but are obviously satisfied if μ is small, as in our case.

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