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BLOW UP OF SOLUTIONS OF SEMILINEAR HEAT EQUATIONS IN GENERAL DOMAINS

VALERIA MARINO* and FILOMENA PACELLA†

Dipartimento di Matematica,
Università di Roma "La Sapienza"
P.le A. Moro 2, 00185 Roma, Italy
**valeria.marino.m@gmail.com*
†pacella@mat.uniroma1.it

BERARDINO SCIUNZI

Dipartimento di Matematica, UNICAL
Ponte Pietro Bucci 31B
87036 Arcavacata di Rende, Cosenza, Italy
sciunzi@mat.unical.it

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Consider the nonlinear heat equation $v_t - \Delta v = |v|^{p-1}v$ in a bounded smooth domain $\Omega \subset \mathbb{R}^n$ with $n > 2$ and Dirichlet boundary condition. Given u_p a sign-changing stationary classical solution fulfilling suitable assumptions, we prove that the solution with initial value ϑu_p blows up in finite time if $|\vartheta - 1| > 0$ is sufficiently small and if p is sufficiently close to the critical exponent $\frac{n+2}{n-2}$. Since for $\vartheta = 1$ the solution is global, this shows that, in general, the set of the initial data for which the solution is global is not star-shaped with respect to the origin. This phenomenon had been previously observed in the case when the domain is a ball and the stationary solution is radially symmetric.

Keywords: Semilinear heat equation; finite-time blow-up; sign-changing stationary solutions; linearized operator; asymptotic behavior.

Mathematics Subject Classification 2010: 35K91, 35B35, 35B44, 35J91

1. Introduction

We consider a nonlinear heat equation of the type

$$\begin{cases} v_t - \Delta v = |v|^{p-1}v & \text{in } \Omega \times (0,T), \\ v = 0 & \text{on } \partial\Omega \times (0,T), \\ v(0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a bounded smooth domain, $p > 1$, $T \in (0, +\infty]$ and $v_0 \in C_0(\Omega)$ where

$$C_0(\Omega) = \{v \in C(\overline{\Omega}), v(x) = 0 \text{ for } x \in \partial\Omega\}. \tag{1.2}$$

It is well known that the initial value problem (1.1) is locally well posed in $C_0(\Omega)$. Denoting with T_{v_0} the maximal existence time of the solution of (1.1) with initial datum v_0, we consider the set of the initial data for which the corresponding solution is global, namely:

$$\mathcal{G} = \{v_0 \in C_0(\Omega), T_{v_0} = \infty\}.$$

It is interesting to understand the geometrical properties of the set $\mathcal{G}$. If we consider $v_0 = \vartheta w$, with $w \in C_0(\Omega)$ and $\vartheta \in \mathbb{R}$, it is well known that, if $|\vartheta|$ is small enough, the solution of (1.1) with initial datum ϑw exists globally. Moreover, if $|\vartheta|$ is sufficiently large and $w \in C_0(\Omega) \cap H_0^1(\Omega)$, it is easy to see that the solution blows up in finite time as a consequence of the fact that it has negative energy (see [12, 1]). It is interesting to understand what happens for intermediate values of ϑ. The case when w is positive is completely clear, as a matter of fact from the maximum principle for the heat equation it follows that there exists $\widetilde{\vartheta} > 0$ such that if $0 < \vartheta < \widetilde{\vartheta}$ then the solution with initial value $\vartheta\, w$ is globally defined, while if $\vartheta > \widetilde{\vartheta}$ it blows up in finite time. In the borderline case both global existence or blow-up in finite time can occur.

Thus, if we define $\mathcal{G}^+ = \{v_0 \in \mathcal{G}, v_0 \geq 0\}$, we can assert that $\mathcal{G}^+$ is star-shaped with respect to 0 (indeed it is a convex set). When the initial value changes sign the situation is different and, in general, the set $\mathcal{G}$ may be not star-shaped. In fact, if we define by u_p a radial sign changing solution of the stationary problem

$$\begin{cases} -\Delta u_p = |u_p|^{p-1} u_p & \text{in } \Omega, \\ u_p = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is the unit ball in $\mathbb{R}^n$, with $n > 2$ and $p > 1$, it has been shown in [6] that there exists $p^* < p_S$, with $p_S = \frac{n+2}{n-2}$ and there exists $\epsilon > 0$ such that if $p^* < p < p_S$ and $0 < |1 - \vartheta| < \epsilon$ then $\vartheta u_p \notin \mathcal{G}$, i.e. the solution of (1.1), with initial datum ϑu_p, blows up in finite time both for ϑ slightly greater and slightly smaller than 1. Hence $\mathcal{G}$ is not star-shaped with respect to the origin since $u_p \in \mathcal{G}$. Recently a similar result has been proved in [7] in the case when the dimension is two and the exponent p is sufficiently large.

Such a result does not hold in the case $n = 1$ (always considering $p > 1$). As a matter of fact in the one-dimensional case we have that for $|\vartheta| < 1$, $v_{\vartheta,p}$ (the solution with initial value $\vartheta\, u_p$) is global and converges uniformly to zero, while it blows up in finite time if $|\vartheta| > 1$.

The proofs of the results of [6, 7] exploit strongly the radial symmetry of the stationary solutions. Hence it is natural to ask whether a similar result holds also in general domains and what kind of sign changing stationary solutions give rise to this phenomenon. Note that this cannot be true for any sign changing stationary solution as it is easy to see considering, for example, a nodal solution in the ball which is odd with respect to a symmetry hyperplane and has only two nodal domains. Here we show that, in the case when $n > 2$ and for exponents close to the critical one, the same blow-up phenomenon occurs in any bounded domain where a suitable class of sign changing solutions u_p of (1.3) exists. More precisely we deal with solutions u_p of (1.3) with the following properties:

(a) $\int_\Omega |\nabla u_p|^2 dx \to 2S^{\frac{n}{2}}$ as $p \to p_S$,

(b) $\frac{\max u_p}{\min u_p} \to -\infty$ as $p \to p_S$,

where S is the best Sobolev constant for the embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$. Note that in what stated before and in the sequel we consider classical solutions of (1.3). However we observe that all weak (i.e. $H_0^1(\Omega)$) solutions of (1.3) are indeed classical by standard elliptic regularity theory. As explained in Sec. 2 (see Proposition 2.2) there are domains Ω which are not a ball for which nodal solutions satisfying (a) and (b) exist. We show this in Proposition 2.2. Note also that in [3] it has been proved that condition (a) implies that $\Omega \backslash \{x \in \Omega \,|\, u_p(x) = 0\}$ has exactly two connected components while, when $n \geq 4$, (b) implies that the nodal surface of u_p does not intersect the boundary $\partial\Omega$ and the positive part u_p^+ and the negative part u_p^- concentrate at the same point. One could easily verify that (a) is equivalent to

$$E_p(u_p) = \frac{1}{2}\int_\Omega |\nabla u_p|^2 dx - \frac{1}{p+1}\int_\Omega |u_p|^{p+1} dx \to \frac{2}{n} S^{\frac{n}{2}} \quad \text{as } p \to p_S.$$

We refer to [3] for further properties of such solutions.

Our goal is to prove the following theorem.

Theorem 1.1. *Let $\{u_p\}$ be a family of sign changing solutions of problem* (1.3) *with $n > 2$ satisfying* (a) *and* (b). *Then there exists $1 < \underline{p^*} < p_S$ with the following property: if $p^* < p < p_S$ then there exists $0 < \underline{\vartheta} < 1 < \overline{\vartheta}$ such that for $\underline{\vartheta} < \vartheta < \overline{\vartheta}$ and $\vartheta \neq 1$ the solution $v_{\vartheta,p}$ of* (1.1), *with initial value ϑu_p, blows up in finite time.*

To prove Theorem 1.1 we use the following result which has been proved in [6] for general domains.

Proposition 1.1. *Let u_p be a sign changing solution of* (1.3) *and let $\varphi_{1,p}$ be a first eigenfunction of the linearized operator L_p at u_p. Assume that*

$$\int_\Omega u_p\, \varphi_{1,p} \neq 0.$$

Then there exists $\varepsilon > 0$ such that if $0 < |1 - \vartheta| < \varepsilon$, then the solution $v_{\vartheta,p}$, of (1.1), *with initial value ϑu_p, blows up in finite time.*

Thus Theorem 1.1 will be a consequence of the following.

Theorem 1.2. *Let $n > 2$, $1 < p < p_S$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and consider $\{u_p\}$ a family of sign changing solutions of* (1.3) *satisfying conditions* (a) *and* (b). *Then there exists $p^* < p_S$ such that for $p^* < p < p_S$*

$$\int_\Omega u_p \varphi_{1,p}\, dx > 0, \tag{1.4}$$

where $\varphi_{1,p}$ is the first positive eigenfunction of the linearized operator L_p at u_p.

Let us point out that for the proof of Theorem 1.2 the property (b) of our stationary solutions is crucial. Note that both properties (a) and (b) are actually satisfied in the special case of radial sign changing solutions of (1.3) (in the ball) with two nodal regions. So this clarifies that it is neither the symmetry nor the one-dimensional character of the solution which leads to the blow-up result obtained in [6] but rather these properties of the stationary solution that can hold in other bounded domains. Therefore we believe that also for other semilinear problems where such solutions exist, the same blow-up result should be true.

The proof of Theorem 1.2 is based on a rescaling argument about the maximum point of u_p. Indeed, analyzing the asymptotic behavior of the rescaled solutions and of the rescaled first eigenfunctions, we are able to prove (1.4) by using the properties of the solutions of the limit problem.

The same result of Theorem 1.2 can be easily extended to the case when the initial datum is a nodal solution $u_{p,\mathcal{K}}$ of (1.3) with a fixed number $\mathcal{K} > 2$ of nodal regions satisfying:

(a)$_\mathcal{K}$ $\int_\Omega |\nabla u_{p,\mathcal{K}}|^2 dx \le C$,

(b)$_\mathcal{K}$ there exists a nodal region Ω_p^1 such that, setting

$$u^1_{p,\mathcal{K}} := u_{p,\mathcal{K}} \cdot \chi_{\Omega^1_p} \quad \text{and} \quad \hat{u}_{p,\mathcal{K}} := u_{p,\mathcal{K}} \cdot \chi_{\Omega\setminus\Omega^1_p}$$

then

$$\int_{\Omega^1_p} |\nabla u^1_{p,\mathcal{K}}|^2 dx \to S^{\frac{n}{2}} \quad \text{as } p \to p_S$$

and

$$\frac{\|u^1_{p,\mathcal{K}}\|_\infty}{\|\hat{u}^1_{p,\mathcal{K}}\|_\infty} \to \infty \quad \text{as } p \to p_S.$$

Solutions of this type have been found in [16, 15] for some domains (see Sec. 2) but other kind of solutions with the same properties could be considered.

The outline of the proof is as follows. In Sec. 2 we prove some preliminary results, while in Sec. 3 we study the asymptotic behavior of the first eigenvalue and of the first eigenfunction of the linearized operator at u_p. Finally in Sec. 4 we prove Theorem 1.2.

2. Preliminaries

Let us consider the limit problem in $\mathbb{R}^n$, that is

$$\begin{cases} -\Delta u = |u|^{p_S - 1} u = |u|^{\frac{4}{n-2}} u & \text{in } \mathbb{R}^n, \\ \displaystyle\int_{\mathbb{R}^n} |\nabla u|^2 \, dx < +\infty. \end{cases} \tag{2.1}$$

It is well known that classical positive solutions to (2.1) are radial and radially decreasing. This was first proved in [9] via the moving plane method (see [17]) under suitable decay assumption on the solution and later, in [4], via the Kelvin transformation without any *a-priori* assumption. Consequently the solutions to (2.1) can be classified (see [18]) up to translations and scale invariance and they are given by

$$\delta_{(a,\mu)}(x) = \frac{\alpha_n \mu^{\frac{n-2}{2}}}{(1 + \mu^2 |x - a|^2)^{\frac{n-2}{2}}} \quad \text{for } x \in \mathbb{R}^n \tag{2.2}$$

with $\alpha_n := (n(n-2))^{\frac{n-2}{4}}$, $a \in \mathbb{R}^n$ and $\mu > 0$. Then if we assume $U(0) = \|U\|_{L^\infty(\mathbb{R}^n)} = 1$, it follows that the unique solution to (2.1) is given by:

$$U(x) = \left(\frac{n(n-2)}{n(n-2) + |x|^2} \right)^{\frac{n-2}{2}}.$$

Moreover

$$\int_{\mathbb{R}^n} |\nabla U|^2 \, dx = S^{\frac{n}{2}}.$$

We also observe that any sign changing solution of (2.1) has energy larger than $2S^{\frac{n}{2}}$.

Now, for $x, y \in \Omega$, let $H(x,y)$ be the regular part of the Green function of a bounded smooth domain Ω. Namely

$$-\Delta_x H(x,y) = 0 \quad \text{in } \Omega, \quad H(x,y) = \frac{1}{|x-y|^{n-2}} \quad \text{if } x \in \partial\Omega.$$

Let us define the so-called Robin function τ by

$$\tau(x) := H(x,x), \quad x \in \Omega. \tag{2.3}$$

Definition 2.1. We say that a bounded smooth domain Ω satisfies the condition $\mathcal{C}_\tau$ if the function τ admits at least a strict local minimum in Ω.

Example of domains of class $\mathcal{C}_\tau$ are domains convex and symmetric with respect to the hyperplanes $\{x_i = 0\}$, $i = 1, \ldots, n$ as proved in [11]. Moreover, in strictly convex domains, there is only one minimum point of τ which is strict since the function τ is strictly convex as shown in [5].

Now we show that in domains of class $\mathcal{C}_\tau$ there are nodal solutions of (1.3) satisfying the properties (a) and (b).

Proposition 2.2. *If Ω is of class $\mathcal{C}_\tau$, there exists at least a family of nodal solutions $\{u_p\}$ to* (1.3) *satisfying the properties* (a) *and* (b).

Proof. In [15] it is proved (see the beginning of the proof of Theorem 1.2 at p. 7) that in such domains, setting $p_\varepsilon = \frac{n+2}{n-2} - \varepsilon$, $0 < \varepsilon < \frac{4}{n-2}$, there exists a solution u_ε satisfying

$$\|u_\varepsilon - P\delta_{(a_{\varepsilon,1},\mu_{\varepsilon,1})} + P\delta_{(a_{\varepsilon,2},\mu_{\varepsilon,2})}\|_{H_0^1(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0, \tag{2.4}$$

where $\delta_{(a_{\varepsilon,j},\mu_{\varepsilon,j})}$, $j = 1, 2$, are defined in (2.2), $P\delta_{(a_{\varepsilon,j},\mu_{\varepsilon,j})}$, $j = 1, 2$, are the projections on $H_0^1(\Omega)$ and $a_{\varepsilon,1}, a_{\varepsilon,2}$ tend to a fixed strict local minimum point ξ of the function $\tau(x)$. Note that any local minimum point of τ is in the interior of Ω since $\tau(x) \to +\infty$ as $x \to x_0 \in \partial\Omega$. Moreover, as shown in [15, Theorem 1.2],

$$\mu_{\varepsilon,j} = \frac{C}{\varepsilon^{\frac{2j-1}{n-2}}}$$

hence

$$|\mu_{\varepsilon,j}|^\varepsilon \to 1 \quad \text{as } \varepsilon \to 0 \ \text{ for } j = 1, 2.$$

By (2.4)

$$\int_\Omega |\nabla u_\varepsilon|^2\, dx \to 2S^{\frac{n}{2}} \quad \text{as } \varepsilon \to 0,$$

thus condition (a) holds. To prove that u_ε satisfy (b) let us argue by contradiction assuming that

$$\exists\, \eta > 0 \text{ such that } \eta \le -\frac{\max u_\varepsilon}{\min u_\varepsilon} \le \frac{1}{\eta}.$$

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Then, since the family $\{u_\varepsilon\}$ satisfies the conclusions in [3, Theorem 1.1] we can apply Theorem 1.2 of [3] (see (1.7) there) and get that

$$|a_{\varepsilon,1} - a_{\varepsilon,2}| \ge \gamma > 0.$$

This is a contradiction since $a_{\varepsilon,1}$ and $a_{\varepsilon,2}$ tend to a same point. $\square$

Remark 2.3. Note that the statement of Theorem 1.1 and the statement of Theorem 1.2 (Theorem 1.2 derives from Theorem 1.1) in [15] are not precise since the concentration point ξ cannot be just a minimum but it must be a strict local minimum which in particular holds if ξ is a nondegenerate local minimum. This can be seen from the proof of [15, Theorem 1.1] (see Sec. 7). Therefore the result of [11] leads to the existence of domains for which Proposition 2.2 holds. Moreover we point out that the result of [15] holds also for nondegenerate critical points of $\tau(x)$ other than strict local minima by the result of [8] which extends easily to the sub-critical case. Note also that in [14] it is proved that generically (with respect to smooth perturbations of domains) the function τ has only nondegenerate critical points. This implies that, up to smooth perturbations of the domain, nodal solutions of (1.3) satisfying the properties (a) and (b) exist in all bounded smooth domains.

Let us now recall some properties of our solutions.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and let (u_p) be a family of sign-changing solutions of* (1.3) *satisfying* (a). *Then*

(i) $\int_\Omega |\nabla u_p^+|^2 dx \xrightarrow{p \to p_S} S^{\frac{n}{2}}$, $\int_\Omega |\nabla u_p^-|^2 dx \xrightarrow{p \to p_S} S^{\frac{n}{2}}$,
(ii) $\int_\Omega (u_p^+)^{\frac{2n}{n-2}} dx \xrightarrow{p \to p_S} S^{\frac{n}{2}}$, $\int_\Omega (u_p^-)^{\frac{2n}{n-2}} dx \xrightarrow{p \to p_S} S^{\frac{n}{2}}$,
(iii) $u_p \rightharpoonup 0$ *in* $H_0^1(\Omega)$ *as* $p \to p_S$,
(iv) $M_{p,+} := \max_\Omega u_p^+ \xrightarrow{p \to p_S} +\infty$, $M_{p,-} := \max_\Omega u_p^- \xrightarrow{p \to p_S} +\infty$,

with $u_p^+ = \max_\Omega(u_p, 0)$ *and* $u_p^- = \max_\Omega(-u_p, 0)$.

Proof. The assertion is exactly the same as in [3, Lemma 2.1] whose proof is similar to that of [2, Lemma 2.1] for the Brezis–Nirenberg problem. We provide some details for the reader's convenience.

Let us consider $\tilde{\Omega}$ a connected component of $\Omega \backslash \{u_p = 0\}$. Note that, considering $u_p \cdot \chi_{\tilde{\Omega}}$ as test function in the weak formulation of (1.3) we get:

$$\int_{\tilde{\Omega}} |\nabla u_p|^2 = \int_{\tilde{\Omega}} |u_p|^{p+1}. \tag{2.5}$$

Exploiting the fact that $p \to p_S$, Hölder's inequality and the Sobolev embedding, it is easy to deduce from (2.5) that

$$\int_{\tilde{\Omega}} |\nabla u_p|^2 \geq S^{\frac{N}{2}}(1 + o(1)), \tag{2.6}$$

for each connected component $\tilde{\Omega}$. From (2.6) and property (a) we deduce that $\Omega \backslash \{u_p = 0\}$ has exactly two connected components, at least for p close to p_S. Let therefore $\Omega^+ = \{u > 0\}$ and $\Omega^- = \{u < 0\}$. As $p \to p_S$ (i) follows now by (2.6) and property (a). By (2.5) with $\tilde{\Omega} = \Omega^\pm$ we deduce that

$$\int_\Omega (u_p^+)^{p+1} dx \xrightarrow{p \to p_S} S^{\frac{n}{2}} \quad \text{and} \quad \int_\Omega (u_p^-)^{p+1} dx \xrightarrow{p \to p_S} S^{\frac{n}{2}}.$$

Since

$$\left(\int_\Omega (u^\pm)^{p+1} \right)^{\frac{2n}{(n-2)(p+1)}} \cdot |\Omega|^{-\frac{(n+2)-p(n-2)}{(n-2)(p+1)}}$$
$$\leq \int_\Omega (u^\pm)^{\frac{2n}{n-2}} \leq S^{-\frac{n}{n-2}} \left(\int_\Omega |\nabla u_p^\pm|^2 \right)^{\frac{n}{n-2}},$$

from (i) it follows (ii).

Moreover (iii) follows from (i) and (ii) since the best Sobolev constant is never achieved in bounded domains.

To prove (iv) we observe that if it did not hold then, up to a subsequence,

$$\int_\Omega (u_p^\pm)^{\frac{2n}{n-2}} dx \xrightarrow{p\to p_S} 0,$$

by dominated convergence theorem. This would contradict (ii). $\square$

We now describe the rescaled problem. Let us define

$$\widetilde{u}_p(x) := \frac{1}{M_p} u_p\left(a_p + \frac{x}{M_p^{\frac{p-1}{2}}}\right), \quad \text{for } x \in \widetilde{\Omega}_p := M_p^{\frac{p-1}{2}}(\Omega - a_p). \tag{2.7}$$

where a_p and M_p are such that $|u_p(a_p)| = \|u_p\|_{L^\infty(\Omega)} =: M_p$. Without loss of generality, we can assume that $u_p(a_p) > 0$.

We have the following lemma.

Lemma 2.2. *For $p \to p_S$*

$$\widetilde{u}_p \to U \text{ in } C^2_{\text{loc}}(\mathbb{R}^n).$$

Proof. It is contained in the proof of [3, Theorem 1.1, pp. 353–354]. $\square$

Now we study the linearization of the limit problem (2.1), so we define the operator

$$L^*(v) := -\Delta v - p_S|U|^{p_S-1}v, \quad v \in H^2(\mathbb{R}^n),$$

where U is the solution of (2.1). The Rayleigh functional associated to L^* is

$$\mathcal{R}(v) = \int_{\mathbb{R}^n} |\nabla v|^2 - p_S|U|^{p_S-1}v^2 dx$$

and we define

$$\lambda_1^* := \inf_{\substack{v\in H^1(\mathbb{R}^n),\\ \|v\|_{L^2(\mathbb{R}^n)}=1}} \mathcal{R}(v). \tag{2.8}$$

We observe that $\lambda_1^* > -\infty$, since U is bounded.

Remark 2.4. It can be shown, with standard arguments, that there exists a unique positive minimizer φ_1^* to (2.8) which is radial and radially nonincreasing; moreover λ_1^* is an eigenvalue of L^* and φ_1^* is an eigenvector associated to λ_1^*. For further details see [13].

Proposition 2.5. *We have the following*:

(i) $\lambda_1^* < 0$,

(ii) *every minimizing sequence of* (2.8) *has a subsequence which strongly converges in* $L^2(\mathbb{R}^n)$.

Proof. This is a known result so we refer the reader to [6, Proposition 3.4] for a detailed proof. $\square$

3. Asymptotic Spectral Analysis

We consider the linearized operator at u_p, that is,

$$L_p = -\Delta - p|u_p|^{p-1} I.$$

We denote by $\lambda_{1,p}$ the first eigenvalue of L_p in Ω and by $\varphi_{1,p}$ the corresponding positive eigenfunction such that $\varphi_{1,p} > 0$ and $\|\varphi_{1,p}\|_{L^2(\Omega)} = 1$. We have

$$-\Delta\varphi_{1,p} - p|u_p|^{p-1}\varphi_{1,p} = \lambda_{1,p}\varphi_{1,p} \quad \text{in } \Omega. \tag{3.1}$$

Let us define $\widetilde{\varphi}_{1,p}$ by

$$\widetilde{\varphi}_{1,p}(x) = \left(\frac{1}{M_p^{\frac{p-1}{2}}}\right)^{\frac{n}{2}} \varphi_{1,p}\left(a_p + \frac{x}{M_p^{\frac{p-1}{2}}}\right) \quad \text{in } \widetilde{\Omega}_p$$

and $\widetilde{\varphi}_{1,p} = 0$ outside $\widetilde{\Omega}_p$. It is easy to see that $\|\widetilde{\varphi}_{1,p}\|_{L^2(\mathbb{R}^n)} = 1$ and $\widetilde{\varphi}_{1,p}$ satisfies

$$-\Delta\widetilde{\varphi}_{1,p} - V_p\widetilde{\varphi}_{1,p} = \widetilde{\lambda}_{1,p}\widetilde{\varphi}_{1,p} \quad \text{in } \widetilde{\Omega}_p,$$

where

$$V_p(x) = p\frac{1}{M_p^{p-1}}\left|u_p\left(a_p + \frac{x}{M_p^{\frac{p-1}{2}}}\right)\right|^{p-1} = p|\widetilde{u}_p(x)|^{p-1}$$

and

$$\widetilde{\lambda}_{1,p} = \frac{\lambda_{1,p}}{M_p^{p-1}}.$$

This means that $\widetilde{\varphi}_{1,p}$ is a first eigenfunction of the operator

$$\widetilde{L}_p = -\Delta - p|\widetilde{u}_p|^{p-1} I \quad \text{in } \widetilde{\Omega}_p$$

with zero Dirichlet boundary condition and $\widetilde{\lambda}_{1,p}$ is the corresponding first eigenvalue.

Lemma 3.1. *The set* $\{\widetilde{\varphi}_{1,p}, 1 < p < p_S\}$ *is bounded in* $H^1(\mathbb{R}^n)$.

Proof. As we have already remarked $\|\widetilde{\varphi}_{1,p}\|_{L^2(\mathbb{R}^n)} = 1$. Moreover, since $\lambda_{1,p} < 0$ and $p < p_S$, we get

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla\widetilde{\varphi}_{1,p}(x)|^2 dx &= \frac{1}{M_p^{p-1}} \int_{\widetilde{\Omega}_p} \left(\frac{1}{M_p^{\frac{p-1}{2}}}\right)^n \left|\nabla\varphi_{1,p}\left(a_p + \frac{x}{M_p^{\frac{p-1}{2}}}\right)\right|^2 dx \\
&= \frac{1}{M_p^{p-1}} \int_\Omega |\nabla\varphi_{1,p}(x)|^2 dx \\
&= \frac{1}{M_p^{p-1}} \int_\Omega p|u_p|^{p-1}\varphi_{1,p}^2 dx + \frac{\lambda_{1,p}}{M_p^{p-1}} \int_\Omega \varphi_{1,p}^2 dx
\end{aligned}$$

$$\leq \int_\Omega p \left(\frac{|u_p|}{M_p}\right)^{p-1} \varphi_{1,p}^2 dx$$

$$\leq p \int_\Omega \varphi_{1,p}^2 dx < p_S,$$

i.e. the assertion. □

Theorem 3.2. *We have*

$$\widetilde{\lambda}_{1,p} \to \lambda_1^* \quad as\ p \to p_S. \tag{3.2}$$

Proof. We divide the proof in two steps.

Step 1. We show that for $\epsilon > 0$ we have

$$\lambda_1^* \leq \widetilde{\lambda}_{1,p} + \epsilon \quad \text{for } p \text{ sufficiently close to } p_S. \tag{3.3}$$

By (2.8), we have $\lambda_1^* \leq \mathcal{R}(\widetilde{\varphi}_{1,p})$. Thus

$$\begin{aligned}
\lambda_1^* &\leq \int_{\mathbb{R}^n} |\nabla \widetilde{\varphi}_{1,p}|^2 - p_S |U|^{p_S-1} \widetilde{\varphi}_{1,p}^2 dx \\
&= \int_{\widetilde{\Omega}_p} |\nabla \widetilde{\varphi}_{1,p}|^2 - p|\widetilde{u}_p|^{p-1} \widetilde{\varphi}_{1,p}^2 dx - \int_{\widetilde{\Omega}_p} (p_S|U|^{p_S-1} - p|\widetilde{u}_p|^{p-1}) \widetilde{\varphi}_{1,p}^2 dx \\
&= \widetilde{\lambda}_{1,p} - \int_{\widetilde{\Omega}_p} (p_S|U|^{p_S-1} - p|\widetilde{u}_p|^{p-1}) \widetilde{\varphi}_{1,p}^2 dx \\
&= \widetilde{\lambda}_{1,p} - \int_{\widetilde{\Omega}_p \cap |x| \leq R} (p_S|U|^{p_S-1} - p|\widetilde{u}_p|^{p-1}) \widetilde{\varphi}_{1,p}^2 dx \\
&\quad - \int_{\widetilde{\Omega}_p \cap |x| > R} (p_S|U|^{p_S-1} - p|\widetilde{u}_p|^{p-1}) \widetilde{\varphi}_{1,p}^2 dx
\end{aligned}$$

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where $R > 0$. Let us first consider the last integral. We want to show that it can be made arbitrarily small. We have

$$\begin{aligned}
\left| \int_{\widetilde{\Omega}_p \cap |x| > R} p_S |U|^{p_S-1} \widetilde{\varphi}_{1,p}^2 dx \right| &\leq p_S \int_{\widetilde{\Omega}_p \cap |x| > R} |U|^{p_S-1} \widetilde{\varphi}_{1,p}^2 dx \\
&\leq \frac{C_1}{R^4} \int_{\mathbb{R}^n} \widetilde{\varphi}_{1,p}^2 dx \leq \frac{C_1}{R^4}
\end{aligned} \tag{3.4}$$

for some constant $C_1 > 0$. Therefore we can choose R so large that

$$\left| \int_{\widetilde{\Omega}_p \cap |x| > R} p_S |U|^{p_S-1} \widetilde{\varphi}_{1,p}^2 dx \right| \leq \epsilon.$$

To estimate the term

$$\left| \int_{\widetilde{\Omega}_p \cap |x| > R} p |\widetilde{u}_p|^{p-1} \widetilde{\varphi}_{1,p}^2 dx \right|$$

note that we can split the integral on $\widetilde{\Omega}_p$ in the integral on

$$\widetilde{\Omega}_p^+ = \{x \in \widetilde{\Omega}_p : \widetilde{u}_p(x) \geq 0\} \tag{3.5}$$

and the one on

$$\widetilde{\Omega}_p^- = \{x \in \widetilde{\Omega}_p : \widetilde{u}_p(x) < 0\}. \tag{3.6}$$

Therefore we get

$$\begin{aligned}
&\left| \int_{\widetilde{\Omega}_p \cap |x|>R} p|\widetilde{u}_p|^{p-1} \widetilde{\varphi}_{1,p}^2 dx \right| \\
&\quad \leq \int_{\widetilde{\Omega}_p^+ \cap |x|>R} p|\widetilde{u}_p|^{p-1} \widetilde{\varphi}_{1,p}^2 dx + \int_{\widetilde{\Omega}_p^- \cap |x|>R} p|\widetilde{u}_p|^{p-1} \widetilde{\varphi}_{1,p}^2 dx.
\end{aligned} \tag{3.7}$$

As for the first term of (3.7) we have

$$\begin{aligned}
&\int_{\widetilde{\Omega}_p^+ \cap |x|>R} p|\widetilde{u}_p|^{p-1} \widetilde{\varphi}_{1,p}^2 dx \\
&\quad \leq p \left(\int_{\widetilde{\Omega}_p^+ \cap |x|>R} |\widetilde{u}_p|^{\frac{n(p-1)}{2}} dx \right)^{\frac{2}{n}} \left(\int_{\widetilde{\Omega}_p^+ \cap |x|>R} \widetilde{\varphi}_{1,p}^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \\
&\quad \leq p \left(\int_{\widetilde{\Omega}_p^+ \cap |x|>R} |\widetilde{u}_p|^{\frac{n(p-1)}{2}} dx \right)^{\frac{2}{n}} \|\widetilde{\varphi}_{1,p}\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\
&\quad \leq C_2 \left(\int_{\widetilde{\Omega}_p^+ \cap |x|>R} |\widetilde{u}_p|^{\frac{n(p-1)}{2}} dx \right)^{\frac{2}{n}},
\end{aligned} \tag{3.8}$$

where we have used Hölder's inequality (with exponents $\frac{n}{2}$ and $\frac{n}{n-2}$) for the first estimate and the fact that, as a consequence of Lemma 3.1, $\widetilde{\varphi}_{1,p}$ is bounded in $L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ to obtain the last inequality.

In order to estimate the last term in (3.8), we use (ii) of Lemma 2.1 to get

$$\begin{aligned}
&\int_{\widetilde{\Omega}_p^+ \cap |x|\leq R} |\widetilde{u}_p|^{\frac{n(p-1)}{2}} dx + \int_{\widetilde{\Omega}_p^+ \cap |x|>R} |\widetilde{u}_p|^{\frac{n(p-1)}{2}} dx \\
&\quad = \int_{\widetilde{\Omega}_p^+} |\widetilde{u}_p|^{\frac{n(p-1)}{2}} dx \xrightarrow{p \to p_S} S^{\frac{n}{2}} = \int_{\mathbb{R}^n} |U|^{\frac{2n}{n-2}} dx \\
&\quad = \int_{|x|\leq R} |U|^{\frac{2n}{n-2}} dx + \int_{|x|>R} |U|^{\frac{2n}{n-2}} dx.
\end{aligned}$$

As $\widetilde{u}_p \xrightarrow{p \to p_S} U$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$, we have

$$\int_{\widetilde{\Omega}_p^+ \cap |x|\leq R} |\widetilde{u}_p|^{\frac{n(p-1)}{2}} dx \xrightarrow{p \to p_S} \int_{|x|\leq R} |U|^{\frac{2n}{n-2}} dx$$

and so

$$\int_{\widetilde{\Omega}_p^+\cap|x|>R} |\widetilde{u}_p|^{\frac{n(p-1)}{2}} dx \xrightarrow{p\to p_S} \int_{|x|>R} |U|^{\frac{2n}{n-2}} dx \tag{3.9}$$

but, as $U \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)$, the term on the right-hand side of (3.9) can be made as small as we like, choosing R sufficiently large. Thus we have that, chosen R large enough, we can take p sufficiently close to p_S so that

$$\int_{\widetilde{\Omega}_p^+\cap|x|>R} |\widetilde{u}_p|^{\frac{n(p-1)}{2}} dx \le \epsilon. \tag{3.10}$$

Let us now estimate the second term of (3.7)

$$\begin{aligned}\int_{\widetilde{\Omega}_p^-\cap|x|>R} p|\widetilde{u}_p|^{p-1}\widetilde{\varphi}_{1,p}^2 dx &\le p\left(\frac{\|u_p^-\|_{L^\infty(\Omega)}}{\|u_p^+\|_{L^\infty(\Omega)}}\right)^{p-1}\left(\int_{\widetilde{\Omega}_p^-\cap|x|>R}\widetilde{\varphi}_{1,p}^2 dx\right)\\ &\le p\left(\frac{\|u_p^-\|_{L^\infty(\Omega)}}{\|u_p^+\|_{L^\infty(\Omega)}}\right)^{p-1} \xrightarrow{p\to p_S} 0,\end{aligned}$$

where we used the fact that $\|\widetilde{\varphi}_{1,p}\|_{L^2(\mathbb{R}^n)} = 1$ and condition (b) satisfied by our solutions.

Recalling that $\widetilde{u}_p \xrightarrow{p\to p_S} U$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$, for R fixed as above and p sufficiently close to p_S, we have

$$\int_{\widetilde{\Omega}_p\cap|x|\le R} (p_S|U|^{p_S-1} - p|\widetilde{u}_p|^{p-1})\widetilde{\varphi}_{1,p}^2 dx \le \epsilon. \tag{3.11}$$

Thus (3.3) follows from (3.4)–(3.11).

Step 2. Now we show that for $\epsilon > 0$ we have

$$\widetilde{\lambda}_{1,p} \le \lambda_1^* + \epsilon \quad \text{for } p \text{ sufficiently close to } p_S. \tag{3.12}$$

Let us consider a regular cut-off function $\psi_R(x) = \psi_R(r)$, for $R > 0$, such that

- $0 \le \psi_R \le 1$ and $\psi_R(r) = 1$ for $r \le R$, $\psi_R(r) = 0$ for $r \ge 2R$,
- $|\nabla\psi_R| \le \frac{2}{R}$

and let us set

$$w_R := \frac{\psi_R\varphi_1^*}{\|\psi_R\varphi_1^*\|_{L^2(\mathbb{R}^n)}}.$$

Thus

$$\begin{aligned}\widetilde{\lambda}_{1,p} &\le \int_{\mathbb{R}^n} |\nabla w_R|^2 - p|\widetilde{u}_p|^{p-1} w_R^2 dx\\ &= \int_{\mathbb{R}^n} |\nabla w_R|^2 - p_S|U|^{p_S-1} w_R^2 dx\\ &\quad + \int_{\mathbb{R}^n} (p_S|U|^{p_S-1} - p|\widetilde{u}_p|^{p-1}) w_R^2 dx.\end{aligned} \tag{3.13}$$

It is easy to see that $w_R \to \varphi_1^*$ in $H^1(\mathbb{R}^n)$ as $R \to \infty$. Therefore, by (2.8), we have that given $\epsilon > 0$ we can fix $R > 0$ such that

$$\int_{\mathbb{R}^n} |\nabla w_R|^2 - p_S|U|^{p_S-1} w_R^2 dx \leq \lambda_1^* + \epsilon.$$

For such a fixed value of R, arguing as in Step 1, we obtain that

$$\int_{\mathbb{R}^n} (p_S|U|^{p_S-1} - p|\widetilde{u}_p|^{p-1}) w_R^2 dx \leq \epsilon \tag{3.14}$$

for p close enough to p_S. Then (3.12) follows from (3.13) and (3.14). By (3.3) and (3.12) we deduce (3.2). □

Corollary 3.1. $\widetilde{\varphi}_{1,p}$ *strongly converges to* φ_1^* *in* $L^2(\mathbb{R}^n)$.

Proof. By the definition of $\widetilde{\lambda}_{1,p}$, and what is stated in Theorem 3.2, we have

$$\int_{\widetilde{\Omega}_p} |\nabla \widetilde{\varphi}_{1,p}|^2 - p|U|^{p-1} \widetilde{\varphi}_{1,p}^2 dx = \widetilde{\lambda}_{1,p} \to \lambda_1^* \quad \text{as } p \to p_S.$$

This implies that $\widetilde{\varphi}_{1,p}$ is a minimizing sequence for (2.8), and so the assertion follows by Proposition 2.5 (see also Remark 2.4). □

4. Proof of Theorem 1.2

We now proceed proving Theorem 1.2.

Proof of Theorem 1.2. Using $\varphi_{1,p} \in H_0^1(\Omega)$ as a test function in (1.3) we have

$$\int_\Omega \nabla u_p \cdot \nabla \varphi_{1,p} dx = \int_\Omega |u_p|^{p-1} u_p \varphi_{1,p} dx, \tag{4.1}$$

while using u_p as a test function in (3.1) we obtain

$$\int_\Omega \nabla u_p \cdot \nabla \varphi_{1,p} dx = \int_\Omega p|u_p|^{p-1} u_p \varphi_{1,p} dx + \lambda_{1,p} \int_\Omega u_p \varphi_{1,p} dx. \tag{4.2}$$

Subtracting (4.1) from (4.2) we get

$$-\frac{p-1}{\lambda_{1,p}} \int_\Omega |u_p|^{p-1} u_p \varphi_{1,p} dx = \int_\Omega u_p \varphi_{1,p} dx.$$

Taking into account that $\lambda_{1,p}$ is negative, we have that, to determine the sign of $\int_\Omega u_p \varphi_{1,p} dx$, we can study the sign of

$$\int_\Omega |u_p|^{p-1} u_p \varphi_{1,p} dx. \tag{4.3}$$

For convenience we consider

$$M_p^{(\frac{p-1}{2})\frac{n}{2}-p}\int_\Omega |u_p|^{p-1}u_p\varphi_{1,p}dx,$$

which has the same sign of (4.3). Now we prove that

$$M_p^{(\frac{p-1}{2})\frac{n}{2}-p}\int_\Omega |u_p|^{p-1}u_p\varphi_{1,p}dx \xrightarrow{p\to p_S} \int_{\mathbb{R}^n}|U|^{p_S-1}U\varphi_1^*dx = \int_{\mathbb{R}^n}U^{p_S}\varphi_1^*dx. \quad (4.4)$$

Since the term on the right-hand side of (4.4) is positive, this will lead to the assertion of Theorem 1.2. By a simple change of variables it follows that

$$\begin{aligned}&\left|M_p^{(\frac{p-1}{2})\frac{n}{2}-p}\int_\Omega |u_p|^{p-1}u_p\varphi_{1,p}dx - \int_{\mathbb{R}^n}|U|^{p_S-1}U\varphi_1^*dx\right|\\ &\quad = \left|\int_{\widetilde{\Omega}_p}|\widetilde{u}_p|^{p-1}\widetilde{u}_p\widetilde{\varphi}_{1,p}dx - \int_{\mathbb{R}^n}|U|^{p_S-1}U\varphi_1^*dx\right|. \end{aligned} \quad (4.5)$$

We take $\epsilon > 0$ and choose $R > 0$ such that

$$\int_{|x|>R}|U|^{p_S-1}U\varphi_1^*dx = \int_{|x|>R}U^{p_S}\varphi_1^*dx \le \epsilon,$$

this is possible arguing as we did in the proof of (3.4). We rewrite (4.5) in the following way:

$$\begin{aligned}&\left|\int_{\widetilde{\Omega}_p\cap|x|>R}|\widetilde{u}_p|^{p-1}\widetilde{u}_p\widetilde{\varphi}_{1,p}dx + \int_{\widetilde{\Omega}_p\cap|x|\le R}|\widetilde{u}_p|^{p-1}\widetilde{u}_p\widetilde{\varphi}_{1,p}dx\right.\\ &\qquad\left. - \int_{|x|\le R}|U|^{p_S-1}U\varphi_1^*dx - \int_{|x|>R}|U|^{p_S-1}U\varphi_1^*dx\right|\\ &\quad\le \left|\int_{\widetilde{\Omega}_p\cap|x|>R}|\widetilde{u}_p|^{p-1}\widetilde{u}_p\widetilde{\varphi}_{1,p}dx\right| + \left|\int_{|x|>R}|U|^{p_S-1}U\varphi_1^*dx\right|\\ &\qquad + \left|\int_{\widetilde{\Omega}_p\cap|x|\le R}|\widetilde{u}_p|^{p-1}\widetilde{u}_p\widetilde{\varphi}_{1,p}dx - \int_{|x|\le R}|U|^{p_S-1}U\varphi_1^*dx\right|.\end{aligned}$$

Now we analyze each term in the previous inequality. Splitting the integral on $\widetilde{\Omega}_p^+$ and on $\widetilde{\Omega}_p^-$ (see (3.5) and (3.6) for the definitions of such sets) we have

$$\left|\int_{\widetilde{\Omega}_p\cap|x|>R}|\widetilde{u}_p|^{p-1}\widetilde{u}_p\widetilde{\varphi}_{1,p}dx\right| \le \int_{\widetilde{\Omega}_p^+\cap|x|>R}|\widetilde{u}_p|^{p}\widetilde{\varphi}_{1,p}dx + \int_{\widetilde{\Omega}_p^-\cap|x|>R}|\widetilde{u}_p|^{p}\widetilde{\varphi}_{1,p}dx. \quad (4.6)$$

As for the first term of (4.6) we have

$$\begin{aligned}\int_{\widetilde{\Omega}_p^+\cap|x|>R}|\widetilde{u}_p|^p\widetilde{\varphi}_{1,p}dx &\le \left(\int_{\widetilde{\Omega}_p^+\cap|x|>R}|\widetilde{u}_p|^{\frac{2np}{n+2}}dx\right)^{\frac{n+2}{2n}}\left(\int_{\widetilde{\Omega}_p^+\cap|x|>R}\widetilde{\varphi}_{1,p}^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{2n}}\\ &\le \left(\int_{\widetilde{\Omega}_p^+\cap|x|>R}|\widetilde{u}_p|^{\frac{2np}{n+2}}dx\right)^{\frac{n+2}{2n}}\|\widetilde{\varphi}_{1,p}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}\\ &\le C_4\left(\int_{\widetilde{\Omega}_p^+\cap|x|>R}|\widetilde{u}_p|^{\frac{2np}{n+2}}dx\right)^{\frac{n+2}{2n}},\end{aligned}$$

where we have used Hölder's inequality (with exponents $\frac{2n}{n+2}$ and $\frac{2n}{n-2}$) for the first estimate and the fact that, as a consequence of Lemma 3.1, $\widetilde{\varphi}_{1,p}$ is bounded in $L^{\frac{2n}{n-2}}(\mathbb{R}^n)$. Thus, with the same argument used to obtain (3.10), we can state that, for every $\epsilon > 0$, having chosen R large enough and taking p close enough to p_S, we have

$$C_4\left(\int_{\widetilde{\Omega}_p^+\cap|x|>R}|\widetilde{u}_p|^{\frac{2np}{n+2}}dx\right)^{\frac{n+2}{2n}} < \epsilon.$$

Next we estimate the second term of (4.6). We have

$$\begin{aligned}\int_{\widetilde{\Omega}_p^-\cap|x|>R}|\widetilde{u}_p|^p\widetilde{\varphi}_{1,p}dx &\le \left(\int_{\widetilde{\Omega}_p^-\cap|x|>R}|\widetilde{u}_p|^{2p}dx\right)^{\frac{1}{2}}\left(\int_{\widetilde{\Omega}_p^-\cap|x|>R}\widetilde{\varphi}_{1,p}^2dx\right)^{\frac{1}{2}}\\ &= \left(\int_{\widetilde{\Omega}_p^-\cap|x|>R}|\widetilde{u}_p|^{2p-\frac{2n}{n-2}}|\widetilde{u}_p|^{\frac{2n}{n-2}}dx\right)^{\frac{1}{2}}\left(\int_{\widetilde{\Omega}_p^-\cap|x|>R}\widetilde{\varphi}_{1,p}^2dx\right)^{\frac{1}{2}}\\ &\le \left(\frac{\|u_p^-\|_{L^\infty(\Omega)}}{\|u_p^+\|_{L^\infty(\Omega)}}\right)^{p-\frac{n}{n-2}}\left(\int_{\widetilde{\Omega}_p^-\cap|x|>R}|\widetilde{u}_p|^{\frac{2n}{n-2}}dx\right)^{\frac{1}{2}}\\ &\le C_5\left(\frac{\|u_p^-\|_{L^\infty(\Omega)}}{\|u_p^+\|_{L^\infty(\Omega)}}\right)^{p-\frac{n}{n-2}}\xrightarrow{p\to p_S}0,\end{aligned}$$

where we have used Hölder's inequality (with exponent 2) for the first estimate, the fact that $\|\widetilde{\varphi}_{1,p}\|_{L^2(\mathbb{R}^n)} = 1$ for the second and condition (b) satisfied by our solution. Note in particular that, for p close to p_S, we may and do assume that $p > \frac{n}{n-2}$. Moreover, recalling once again that $\widetilde{u}_p \xrightarrow{p\to p_S} U$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$, we deduce that

$$\left|\int_{\widetilde{\Omega}_p\cap|x|\le R}|\widetilde{u}_p|^{p-1}\widetilde{u}_p\widetilde{\varphi}_{1,p}dx - \int_{|x|\le R}|U|^{p_S-1}U\varphi_1^*dx\right| < \epsilon, \qquad (4.7)$$

for R fixed as above and p sufficiently close to p_S. Finally, for R sufficiently large, the term

$$\int_{|x|>R} |U|^{p_S-1} U \varphi_1^* dx$$

can be made arbitrary small since $U \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ and φ_1^* is bounded.

Thus (4.5)–(4.7) and the arbitrary choice of ϵ imply (4.4) concluding the proof. □

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