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Perfect simulation for the infinite random cluster model, Ising and Potts models at low or high temperature

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Abstract In this article we create a new algorithm for the perfect simulation of the infinite random cluster model for a sufficiently small or a sufficiently high value of the parameters. This implies the simulation of the Ising and Potts models with free boundary conditions.

Keywords Perfect simulation · Random-cluster model · Percolation

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1 Introduction

Given a finite or countable graph $G = (V, E)$, a positive real number q and parameters $p = \{p_e \in [0, 1] : e \in E\}$, the random cluster measure is defined on the measurable space $(\Omega, \mathcal{F})$, where $\Omega = \{0, 1\}^E$ and $\mathcal{F}$ is the σ-algebra generated by finite cylinders. This measure was introduced by Fortuin and Kasteleyn as a way to study the Ising and Potts models (see [6]). Notice that, in our paper, we do not require the parameters p_e to be all equal to the same constant; however we limit our attention only on the models with $q > 1$. This choice has been made both to maintain the article simpler and also because the case $q > 1$ has important connections with the statistical mechanics and in particular with the models of Ising and Potts.

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Our aim is to construct an algorithm which gives a perfect simulation of a random cluster measure on a finite region of an infinite graph. Notice that even if the perfect simulation is obtained only on a finite region, it takes into account the fact that the random field on this region is influenced by the value of the field on the whole infinite graph.

Now we briefly explain how this simulation is obtained. As will be recalled in Sects. 3 and 4 the random cluster measure is invariant under a Markovian dynamics. Introduce a countable number of copies of the graph G and think them as placed at level, 0, -1, -2, etc. Choose also an order of the edges of G: e_1, e_2, etc. For a configuration $\omega \in \Omega$ of the graph at level N create new configurations at level above N updating the value of ω_{e_k} one at the time, according to the conditional probabilities that depends on the geometry of the configuration. The details of this dynamics are given in Sect. 4. Let us just recall here that the law used to update the value of ω_{e_k} depends on the existence of a connected path of edges e, different from e_k, such that $\omega_e = 1$, joining the end vertices of e_k. The construction of this dynamics can be seen as a particular case of a Glauber dynamics. In the study of the random cluster measures similar dynamics where already considered, for example by Grimmett in [9].

We construct a coupling of all the dynamics for each possible initial configuration. We color with black, gray or white, in an independent way, the edges of all the copies of G. The law used to color the edge e at a certain level depends only on the parameter p_e. The coupling is constructed in such a way that for all possible initial configurations ω, the value of the configuration in e at level ℓ will be 1 whenever the edge e at level ℓ is black and 0 whenever it is white.

Now fix a finite set of edges F of G. In Theorem 9, in the case of high temperature (which corresponds to p_e close to zero), we prove that, for almost all coloring as above, there exists a finite region C_F^b of the union of all the copies of G that is "surrounded" by white edges and containing the region F at level 0. Finally, in Theorem 6 we prove that, given a coloring, if the region C_F^b is finite then we can determine a bigger finite region $\bar{H}$ such that the output of the dynamics described above at level 0 in the region F does not depend on the choice of ω and on the coloring outside the region $\bar{H}$.

Similar results are proved in Theorems 5 and 8 in the case of low temperature which corresponds to p_e close to one. However in this case the meaning of the word "surrounded" is different. To treat this case we have to make some further assumptions on the geometry of the graph G. For this reason in Sect. 5 we introduce the concept of a simplicial graph. These are the graphs that can be obtained as the vertices and edges of a tessellation of a Euclidean space. The first example we have in mind is the cubic lattice $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}_d)$. In Sect. 5 we prove also the results required in the proof of Theorem 5.

In Sect. 8 we summarize these results and we show how to obtain an algorithm for the perfect simulation of the random cluster measure. We give the details only in the case of low temperature which is the most difficult and interesting. The case of high temperature can be treated in a similar way. As a byproduct we also obtain the uniqueness of the random cluster measure at low or high temperature, which, at least in the case of $\mathbb{L}^d$ is well known (see [9,11]).

In Sect. 9 we describe the objects introduced in the paper in an explicit example. We also illustrate the simulation in one example regarding the random cluster measure with $q = 2$, the parameters p_e constant and free boundary conditions at low temperature.

This example is related to the Ising model. In the end of Sect. 9 we briefly recall how to deduce from the simulation of the random cluster measure the simulation of the Ising and Potts model with free boundary conditions.

We also refer to some literatures on perfect simulation. In [13] was introduced the perfect simulation algorithm for Markov chains. If the chain is ergodic with this algorithm it is possible to simulate the unique stationary measure associated with the chain. This paper has started some new research fields. One area of research concerns the Markov fields (see [3,12]); a second one concerns the processes with infinite memory (see [1,4,5,7]). Recently, these two areas of research have been in some sense unified by studying Gibbs measures with infinite interaction range (see [2,8]). Our paper is included in the latter context.

2 Some notations on graphs

In this section we recall some definitions on graphs that will be used in the sequel. In this paper a graph will be a collection of two sets, V called the set of vertices and E called the set of edges, and of a map from E to the set of unordered pairs of different elements of V. The pair associated to an edge e are called the end vertices of e and the two vertices are said to be *adjacent*. As it is common in the literature we will denote a graph by $G = (V, E)$. A *path* in G joining the vertices u and v is a sequence $e_1, \dots, e_m$ in E such that e_i and e_{i+1} have a common vertex, u is an end vertex of e_1 and v is an end vertex of e_m. The integer m is called the *length* of the path. Two vertices are said to be in the same *connected component* if there is a path joining them. The *graph-distance* of two vertices u and v is the length of a minimal path joining them, and it is infinite if the two vertices are in different connected components. We denote by $B_G(v, r)$ the ball of center v and radius r with respect to this distance.

3 The random cluster measure

In this section we define the random cluster measure introduced by Fortuin and Kasteleyn as explained in the book of Grimmett [11]. Since our setting will be slightly more general than the one exposed by Grimmett we give the construction of the measure. However all the arguments given in his book easily generalize to our setting, so we refer to [11], Chapter 4, for the details.

3.1 Construction as thermodynamic limit

For our constructions we fix a graph $G = (V, E)$. We further assume that it is countable of finite degree, meaning that V is countable and that every vertex is an end vertex of a finite number of edges.

Set $\Omega = \{0, 1\}^E$ and let $\mathcal{F}$ be the σ-algebra generated by finite cylinders. If $\omega \in \Omega$ we denote by $E(\omega)$ the set of the elements $e \in E$ such that $\omega_e = 1$.

The space Ω is also equipped , as usual, with the partial order given by: $\omega \leq \widehat{\omega}$ if $\omega(e) \leq \widehat{\omega}(e)$ for all $e \in E$.

An event $A \in \mathcal{F}$ is called increasing (respectively, decreasing) if $\omega \in A$ implies that $\widehat{\omega} \in A$ whenever $\omega \leq \widehat{\omega}$ (respectively, whenever $\widehat{\omega} \leq \omega$).

As in [11] Sect. 3.1 for two probability measures μ_1 and μ_2 on $(\Omega, \mathcal{F})$ we say that μ_1 is *stochastically smaller* than μ_2, writing $\mu_1 \leq_{st} \mu_2$, if $\mu_1(A) \leq \mu_2(A)$ for each increasing event $A \in \mathcal{F}$.

To define a random cluster measure we also fix parameters $p = (p_e \in [0, 1] : e \in E)$, and $q \in (0, \infty)$. For simplicity in this paper we will assume $q \geq 1$ which is the more significant for the application to statistical mechanics.

There are two ways of defining random cluster measure on G. The first method is as a limit on finite subgraphs and is called the thermodynamic limit. The second method is by giving the conditional probabilities on all finite subgraphs and it is called the Dobrushin-Lanford-Ruelle or DLR method. We now explain briefly the construction as thermodynamic limit.

Given $\xi \in \Omega$ and $F \subset E$ a finite set, let $\Omega_F^{\xi} = \{\omega \in \Omega : \omega_e = \xi_e \text{ for all } e \notin F\}$. We define the measure $\phi_{F,p,q}^{\xi}$ on Ω by:

$$\phi_{F,p,q}^{\xi}(\omega) = \begin{cases} \frac{1}{Z_{\xi,F}} \left[\prod_{e\in F} p_e^{\omega_e}(1-p_e)^{1-\omega_e}\right] q^{k(\omega,F)} & \text{if } \omega \in \Omega_F^{\xi}, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $k(\omega, F)$ is the number of connected components of the graph $(V, E(\omega))$ that intersects F and $Z_{\xi,F}$ is just the normalizing constant.

Following [11] Definition 4.15 we say that a probability ϕ on $(\Omega, \mathcal{F})$ is a *limit random cluster measure* if there is a sequence (ξ_n, F_n) such that ϕ is the weak limit of the measures $\phi_{F_n,p,q}^{\xi_n}$ and we denote by $\mathcal{W}_{p,q}$ the set of these measures.

If we fix ξ_n to be constantly equal to 1 (resp. to 0) the limit of the measures $\phi_{F_n,p,q}^{\xi_n}$ exists, and it does not depend on the choice of the sequence F_n (see Theorem 4.19 in [11]). This limit will be denoted by $\phi_{p,q}^1$ (resp. $\phi_{p,q}^0$). Moreover, for all $\phi \in \mathcal{W}_{p,q}$, we have

$$\phi_{p,q}^0 \leq_{st} \phi \leq_{st} \phi_{p,q}^1.$$

Another important property of these measures is the so called finite energy property. Let

$$\hat{p}_e = p_e/(p_e + q(1-p_e)), \tag{2}$$

then for all $\phi \in \mathcal{W}_{p,q}$ and $e \in E$, we have

$$\hat{p}_e \leq \phi(L_e|\mathcal{T}_e)(\omega) \leq p_e, \qquad a.s.$$

where $L_e = \{\omega : \omega_e = 1\}$, $\mathcal{T}_e$ is the σ-algebra generated by the finite cylinders with base contained in $E\backslash\{e\}$.

3.2 DLR construction

In the case of a finite graph $G = (V, E)$ the definition given above furnishes a unique measure on $(\Omega, \mathcal{F})$ which is characterized by the conditional probability of $\omega_e = 1$ given the values of ω in $E\backslash\{e\}$. In this model these probabilities depend on the existence of a path in $E(\omega)\backslash\{e\}$ joining the two end vertices of the edge e. Let K_e be the set of configurations ω having this property.

In the case of infinite graph this property can be formalized as follows:

$$\phi(L_e|\mathcal{T}_e)(\omega) = \begin{cases} p_e & \text{if } \omega \in K_e, \\ \hat{p}_e & \text{if } \omega \notin K_e. \end{cases} \tag{3}$$

A probability ϕ on $(\Omega, \mathcal{F})$ is a *DLR random cluster measure* if it satisfies Eq. (3) for all $e \in E$. We denote by $\mathcal{R}_{p,q}$ the set of these measures.

In the infinite setting the two definitions of random cluster measure are not always equivalent. However (see [11], Chapter 4, Section 4) it is known that $\mathcal{R}_{p,q}$ is not empty, in particular $\phi^0_{p,q}$ and $\phi^1_{p,q}$ are elements of $\mathcal{R}_{p,q}$ and for all $\phi \in \mathcal{R}_{p,q}$ one has

$$\phi^0_{p,q} \leq_{st} \phi \leq_{st} \phi^1_{p,q}.$$

In particular notice that

$$\text{card}(\mathcal{W}_{p,q}) = \text{card}(\mathcal{R}_{p,q}) = 1 \quad \text{if and only if} \quad \phi^0_{p,q} = \phi^1_{p,q}.$$

4 Construction of the dynamics

In this paper, following the literature, to give a sample of a measure on $(\Omega, \mathcal{F})$ we introduce families of auxiliary random variables $(u_e)_{e\in E}$ that are independent and uniformly distributed on $[0, 1]$, which in the algorithm we present in Sect. 8 can be thought as the output of a pseudorandom function on a computer.

We define a stochastic dynamic such that the measures in $\mathcal{R}_{p,q}$ are invariant. In the study of the random cluster measure a similar dynamics was already considered by Grimmett in [9].

Let $G = (V, E)$ be a countable graph of finite degree and we choose an order for its edges so that $E = \{e_1, e_2, \ldots\}$.

For a negative number N we define

$$\mathcal{A}_N = \{(n, k) \in \mathbb{Z} \times \mathbb{Z} : -1 \geq n \geq N \text{ and } k \geq 1\}, \qquad \mathcal{A} = \bigcup_{N\leq -1} \mathcal{A}_N \tag{4}$$

and $\mathcal{U} = [0, 1]^{\mathcal{A}}$. On $\mathcal{U}$ we put the Lebesgue product measure so that the coordinates $u_{n,k}$ are i.i.d. random variables having uniform distribution on $[0, 1]$. We define also $\tilde{\mathcal{A}}_N = \mathcal{A}_N \cup \{(n, 0) : n = N, \ldots, 0\}$ and $\tilde{\mathcal{A}} = \bigcup_N \tilde{\mathcal{A}}_N$.

For a fixed $N < 0$ and for a fixed $X_{N,0} : \mathcal{U} \longrightarrow \Omega$ let ($X_{n,k} : \mathcal{U} \longrightarrow \Omega$, for $(n,k) \in \tilde{\mathcal{A}}_N$) be a process with values in Ω constructed in the following way: given $(n,k) \in \tilde{\mathcal{A}}_N$ define

$$(X_{n,k+1}(u))_e = \begin{cases} (X_{n,k}(u))_e & \text{if } e \neq e_{k+1}, \\ 1 & \text{if } u_{n,k+1} < \hat{p}_e,\ e = e_{k+1}, \\ 1 & \text{if } \hat{p}_e \leq u_{n,k+1} < p_e,\ e = e_{k+1},\ X_{n,k}(u) \in K_e, \\ 0 & \text{if } \hat{p}_e \leq u_{n,k+1} < p_e,\ e = e_{k+1},\ X_{n,k}(u) \notin K_e, \\ 0 & \text{if } p_e \leq u_{n,k+1},\ e = e_{k+1}. \end{cases} \tag{5}$$

Furthermore notice that for all n and all e there exists the limit $(X_{n,k}(u))_e$ for k going to infinity. We construct $X_{n+1,0}$ as this limit. We call such a process an $FK^N_{p,q}$-process.

All DLR random cluster measures are invariant under this process. Indeed if $\phi \in \mathcal{R}_{p,q}$ and $X_{n,k}$ has law ϕ then $X_{n,k+1}$ has the same law. Hence, for $h > k$ $X_{n,h}$ has also law ϕ. Moreover any probability measure on $(\Omega, \mathcal{F})$ is determined when it is given on finite cylinders. By construction given a finite cylinder A the probability that $X_{n+1,0}(u)$ belongs to A is the same of the probability that $X_{n,h}(u)$ belongs to A for h large enough, hence it is also equal to $\phi(A)$, proving that the law associated to $X_{n+1,0}$ is equal to ϕ. Hence the law associated to $X_{m,k}$ is equal to ϕ for $m > n$.

Finally, for any integer $N < 0$ and any $\omega \in \Omega$ we denote with $X^{(\omega,N)}_{n,k}(u)$ the $FK^N_{p,q}$-process constructed starting with $X_{N,0}(u) = \omega$.

In the main result of this paper, under suitable assumption on the parameters p, q and on the graph G, given a finite $F \subset E$ and for almost all $u \in \mathcal{U}$ we will show how to determine an integer N such that $(X^{(\omega,N)}_{0,0}(u))_e$ does not depend on $\omega \in \Omega$ for all $e \in F$. Moreover we show how to determine a finite region $\bar{H} \supset F$ such that $(X^{(\omega,N)}_{0,0}(u))_e$ does not depend on the values of $u_{n,k}$ for $e_k \notin \bar{H}$.

Since the $\mathcal{R}_{p,q}$-measures are invariant in this way we prove that they are all equal and we give a perfect simulation of them on any finite subset of E. In particular we prove $\mathcal{R}_{p,q} = \mathcal{W}_{p,q} = \{\phi^0_{p,q}\}$.

5 Simplicial graph

In this section we define the notion of simplicial graph and we prove some geometric properties of these graphs. It is possible that these results are already known, maybe with different notations, however we could not find any reference.

The prototypical graph we have in mind is the graph $\mathbb{L}^d$ whose vertices' are the elements of $\mathbb{Z}^d$ and whose edges are the segments of length one joining them. More in general a simplicial graph will be the graph obtained by considering vertices and segments of a polyhedral tessellation of $\mathbb{R}^d$.

Before giving the details and to introduce the necessary and somehow heavy notation, we explain roughly the problem we want to consider in the next sections. Let $G = (V, E)$ be a graph and color each edge of G white or black in a random way (more precisely we will color the graph black, white or grey but it is not important here). Consider a finite subset F of E. We want to determine a region $\bar{H}$ containing F such that for all $e \in F$ if there is a path of black edges joining the two end vertices of e then there is a path contained in $\bar{H}$ of black edges joining the two end vertices of e (see Theorem 5). Moreover we want $\bar{H}$ to be small as possible so that if the probability of an edge to be black is high, then, if F is finite, $\bar{H}$ is also almost surely finite (see Theorem 8). We construct first a set H by adding inductively to F white edges until its "boundary" is entirely composed by black edges and then we set $\bar{H}$ to be equal to the union of H with its boundary. The idea is that, in this way, if two points are in the boundary of $\bar{H}$, then they can be connected by a path contained in the boundary and in particular of black edges (for the precise statement see Proposition 3). In this way a path of black edges joining two end vertices x, y of an edge in F can be replaced by a path of black edges joining x and y contained in $\bar{H}$, by replacing the pieces outside $\bar{H}$ with paths along the boundary. A first try could be to construct H by adding edges to F until there are white edges that "have a vertex in common" with the set. In this way H would be the union of F with the connected components of the subgraph of white edges having non trivial intersection with F. However it is immediate to see that this H has not the required properties. For a simplicial graph we can construct H replacing the condition "to have a vertex in common" with a different condition. We construct H by adding to F white edges until there are white edges in the "boundary" of this set. The correct notion of "boundary" is defined in Sect. 5.3, now we explain it in the case of the simplicial graph $\mathbb{L}^d$. We say that an edge e' is in the "boundary" of an edge e if they are different and if there exists a d-dimensional hypercube of side 1 with vertices in $\mathbb{Z}^d$ containing both e and e'. In this section we prove that this notion of boundary has the required geometrical properties (see Proposition 3).

5.1 Definitions

Let A be a closed convex subset of $\mathbb{R}^\ell$ which is the intersection of a finite number of closed half-spaces. Such a set will be called a *convex cell* and will be the starting point of our constructions. For such a set we can identify the subset of vertices, edges and i-dimensional faces and we denote by A_i the set of i-dimensional faces of A.

In some constructions will be useful to have a more general notion of cell. If A is a convex cell of dimension m and $\varphi : A \longrightarrow \mathbb{R}^d$ is a piecewise affine continuous injective map we call the image σ of A a *cell* of $\mathbb{R}^d$ of dimension m. We define also the collections $\sigma_i = \{\varphi(B) : B \in A_i\}$ and the datum of $\sigma_0, \ldots, \sigma_m$ will be called a *polyhedron*.

The assumption piecewise affine on φ could be highly relaxed, however this assumption makes some of the arguments below more elementary and does not change the generality of our applications.

A *polytope* $\mathcal{P}$ in $\mathbb{R}^d$ is a collection of cells in $\mathbb{R}^d$ which intersect properly. More precisely $\mathcal{P}$ is the datum of sets $P_0, P_1, \ldots, P_m$ such that

(i) the elements of P_i are i-dimensional cells of $\mathbb{R}^d$;
(ii) P_i is locally finite: this means that for all bounded regions R of $\mathbb{R}^d$ $\sigma \cap R = \varnothing$ for all $\sigma \in P_i$ but a finite number;

(iii) if, for all $\sigma \in P_i$ and all $j \leq i$, we denote by $P_j(\sigma) = \{\tau \in P_j : \tau \subset \sigma\}$ then the datum

$$\mathcal{P}(\sigma) : P_0(\sigma), \ldots, P_i(\sigma)$$

is a polyhedron and the set $P_j(\sigma)$ will be called the set of j-faces of σ;

(iv) for all $\sigma \in P_i$ and $\tau \in P_j$ the intersection $\sigma \cap \tau$ is either empty or a union of faces of σ and τ.

The elements of P_i will be called the i-cells of $\mathcal{P}$ and in particular we will call P_0 (resp. P_1) the set of vertices (resp. of edges) of $\mathcal{P}$. The union of all the cells of $\mathcal{P}$ will be called the *support* of $\mathcal{P}$ and will be denoted by $supp\,(\mathcal{P})$.

The graph $G(\mathcal{P}) = (V, E)$ associated to $\mathcal{P}$ is defined as follows: $V = P_0$, $E = P_1$ and the end vertices of an edge $e \in E$ is the pair of vertices contained in e. Notice that if x and y are two vertices of $\mathcal{P}$ then they are in the same connected component of $supp\,(\mathcal{P})$ if and only if they are in the same connected component of $G(\mathcal{P})$.

The simplest possible convex cell are the simplexes. The ℓ dimensional standard simplex is the set $S = \{(x_0, \ldots, x_\ell) \in \mathbb{R}^{\ell+1} : x_i \geq 0$ for all i and $\sum_i x_i = 1\}$. A n-dimensional *convex simplex* (resp. an ℓ-dimensional *simplex*) is the image of S under an affine (resp. piecewise affine and continuous) injective map. We notice that every cell can be obtained as the support of a polytope whose cells are simplexes.

A refinement of a polytope $\mathcal{P}$ is a polytope $\mathcal{P}'$ such that $supp\,(\mathcal{P}) = supp\,(\mathcal{P}')$ and each cell of $\mathcal{P}'$ is contained in a cell of $\mathcal{P}$.

5.2 Internal and external part of codimension one smooth polytopes

We say that a polytope $\mathcal{C}$ in $\mathbb{R}^d$ is *smooth* if its support is smooth as a topological variety. In this case this means that for every $x \in supp\,(\mathcal{C})$ there exists a natural number j, a neighborhood W of x in $\mathbb{R}^d$ an open ball W' of $\mathbb{R}^d$ and a piecewise affine continuous map $\psi : W \longrightarrow W'$ which defines an homeomorphism between W and W', such that $\psi(x) = 0$ and $\psi(W \cap supp\,(\mathcal{C})) = \{(t_1, \ldots, t_n) \in W' : t_1 = \cdots = t_j = 0\}$. If, moreover all the connected components of $supp\,(\mathcal{C})$ have dimension $n-1$ we say that it is a smooth polytope of codimension one. Notice that if $\mathcal{C}$ is smooth of codimension one in $\mathbb{R}^d$ then for all x in $supp\,(\mathcal{C})$ there exists a neighborhood W of x in $\mathbb{R}^d$ which is divided by $supp\,(\mathcal{C})$ into two open connected components. Now we give a more global construction of these components.

Let $\mathcal{C}$ be a codimension one smooth polytope in $\mathbb{R}^d$ with a finite number of vertices and let U be the complement of $supp\,(\mathcal{C})$. For all $x \in U$ we consider the set $\mathsf{S}_x(\mathcal{C})$ of half-lines ℓ starting in x and whose intersection with $supp\,(\mathcal{C})$ is generic. More precisely we require that for all cells σ of $\mathcal{C}$ if $\ell \cap \sigma \neq \varnothing$ then $\sigma \in P_{d-1}$ and $\ell \cap \sigma$ is a finite set and moreover this intersection is contained in the set of points of σ that are linearly smooth: for all $y \in \ell \cap \sigma$ there exists a neighborhood U of y and an hyperplane such that $\sigma \cap U = U \cap H$. Then the parity of the cardinality of $\ell \cap supp\,(\mathcal{C})$ does not depend on $\ell \in \mathsf{S}_x(\mathcal{C})$. Moreover this parity is locally constant on $x \in U$.

Hence we define the *internal part* of $\mathcal{C}$, that we will denote by $\operatorname{Int}\mathcal{C}$ as the set of points $x \in U$ such that this cardinality is odd and the *external part*, that we will denote by $\mathrm{Est}\,\mathcal{C}$, as the set of points such that this cardinality is even. Notice that $\mathrm{Int}\,\mathcal{C}$ and $\mathrm{Est}\,\mathcal{C}$ are two open subsets of $\mathbb{R}^d$ with boundary equal to $supp\,(\mathcal{C})$. In particular for all path joining an element of $\mathrm{Int}\,\mathcal{C}$ with an element of $\mathrm{Est}\,\mathcal{C}$ the intersection of γ with $supp\,(\mathcal{C})$ is not empty. Finally notice that $\mathrm{Int}\,\mathcal{C}$ is bounded.

Notice that for all $x \in supp\,(\mathcal{C})$ if W is a neighborhood of x as in the beginning of this section, then the two connected components of $W \backslash supp\,(\mathcal{C})$ are the intersection of W with $\mathrm{Int}\,\mathcal{C}$ and $\mathrm{Est}\,\mathcal{C}$. We say that a path γ cross $\mathcal{C}$ in x if $\gamma(t_0) = x$ for some t_0 and there exist sequences $\{t_n\}$ and $\{s_n\}$ going to t_0 such that $\gamma(s_n) \in \mathrm{Est}\,\mathcal{C}$ and $\gamma(t_n) \in \mathrm{Int}\,\mathcal{C}$ for all n.

Lemma 1 *Let $\mathcal{C}$ be a codimension one smooth polytope in $\mathbb{R}^d$ with a finite number of vertices. Then for all $x, y \in supp\,(\mathcal{C})$ such that there exists a path in $\overline{\mathrm{Int}\,\mathcal{C}}$ joining x and y, and a path in $\overline{\mathrm{Est}\,\mathcal{C}}$ joining x and y, then x and y are in the same connected component of $supp\,(\mathcal{C})$. In particular if x, y are vertices then they are in the same connected component of $G(\mathcal{C})$.*

Proof Let α (resp. β) be a path joining x and y in $\overline{\mathrm{Int}\,\mathcal{C}}$ (resp. $\overline{\mathrm{Est}\,\mathcal{C}}$). Consider the connected component D of $supp\,(\mathcal{C})$ containing x and let $\mathcal{D}$ be the polytope whose cells are the cells of $\mathcal{C}$ contained in D, so that $supp\,(\mathcal{D}) = D$. The polytope $\mathcal{D}$ is a connected codimension one smooth polytope with a finite number of vertices. Notice that, by what noticed above, in a small neighborhood W of D we have that $W \backslash D$ has two connected components W_1 and W_2 and we have $W_1 = W \cap \mathrm{Int}\,\mathcal{C}$ and $W_2 = W \cap \mathrm{Est}\,\mathcal{C}$. Similarly W_1 and W_2 must be the intersection with $\mathrm{Int}\,\mathcal{D}$ and $\mathrm{Est}\,\mathcal{D}$, and both the possibilities

$$\begin{cases} W_1 = W \cap \mathrm{Int}\,\mathcal{C} \\ W_2 = W \cap \mathrm{Est}\,\mathcal{C} \end{cases} \quad \text{and} \quad \begin{cases} W_1 = W \cap \mathrm{Est}\,\mathcal{C} \\ W_2 = W \cap \mathrm{Int}\,\mathcal{C} \end{cases}$$

can occur. Since α and β do never cross $\mathcal{C}$ they also never cross $\mathcal{D}$ and we have that α is contained in $\overline{\mathrm{Int}\mathcal{D}}$ and β is contained in $\overline{\mathrm{Est}\mathcal{D}}$ or the opposite.

Hence the final point y of α and β belongs to $\overline{\mathrm{Int}\,\mathcal{D}} \cap \overline{\mathrm{Est}\,\mathcal{D}}$ hence it is in D. $\square$

5.3 Polyhedral tessellation and simplicial graph

We say that a polytope $\mathcal{P}$ is a *polyhedral tessellation* of $\mathbb{R}^d$ if $supp\,(\mathcal{P}) = \mathbb{R}^d$. In this case we say that $G = G(\mathcal{P})$ is a *simplicial graph*.

If $A \subset P_0$ we define the *boundary* $\Delta_{\mathcal{P}}(A)$ of A as the set of vertices $v \in P_0 \backslash A$ for which there exists, $w \in A$ and a cell σ of $\mathcal{P}$, such that $v, w \in \sigma$ and we define $G(A)$ as the graph whose set of vertices is equal to A and whose edges are given by the elements in P_1 joining two elements of A. Notice also that if $x \in A$ and $y \in P_0 \backslash A$ any path in $G(\mathcal{P})$ joining x and y intersects $\Delta_{\mathcal{P}}(A)$.

Proposition 2 *Let $\mathcal{P}$ be a polyhedral tessellation of $\mathbb{R}^d$ and let V be its set of vertices. Let A be a finite subset of V and set $B = V \backslash A$. Let $x, y \in A$ adjacent respectively to $x', y' \in B$. Assume now that x, y are connected in $G(A)$ and that x', y' are connected in $G(B)$. Then x', y' are connected in $G(\Delta_{\mathcal{P}}(A))$.*

Proof The proof follows exactly the same lines in the case of a tessellation using convex cells and in the case of a general tessellation. However in the first case all construction are more intuitive and direct. For this reason we give first the proof in the case of a tessellation made of convex cells and then we briefly explain how to change the proof in the general case.

In the first step of the proof we assume also that all cells are convex simplexes. We construct a smooth polytope $\mathcal{C}$ of codimension one that separate A and B in the following way.

We define the set of vertices C_0 of $\mathcal{C}$ as the set of middle points of edges joining an element of A and an element of B. For all i-dimensional simplexes σ of $\mathcal{C}$ which contain an element of C_0 the convex envelope of $\sigma \cap C_0$ is a $(i-1)$-cell. Let C_{i-1} be the collection of these cells and let $\mathcal{C}$ be the polytope whose i-dimensional cells are given by C_i for $i = 0, \dots, n-1$. By construction $\mathcal{C}$ is a smooth polytope of codimension one with a finite number of vertices. Notice also that a path in $G(A)$ or in $G(B)$ will never cross $supp\,(\mathcal{C})$.

Let now $u, v \in C_0$ be the middle points of the edges joining x, x' and y and y' respectively. By Proposition 1, u and v are in the same connected component of $G(\mathcal{C})$. Hence there exists a sequence of vertices $w_0 = u, w_1, \dots, w_m = v$ in C_0 determining the path connecting u and v in $G(\mathcal{C})$. Furthermore let $t_i \in A$ and $t_i' \in B$ be such that w_i is the middle point of the edge joining t_i and t_i'. Then (here we use the cells are simplexes) $t_0' = x', t_1', \dots, t_m' = y'$ determine a path in $\Delta_{\mathcal{P}}(A)$ joining x' and y'.

Let now $\mathcal{P}$ be any tessellation with convex cells. We construct a sequence $\mathcal{P}^{(i)}$ of refinements of $\mathcal{P}$ and of finite subsets $A^{(i)}$ of the vertices of $\mathcal{P}^{(i)}$.

- $\mathcal{P}^{(1)}$ is $\mathcal{P}$ and $A^{(1)} = A$.
- $\mathcal{P}^{(2)}$ is the tessellation obtained by adding a vertex v_σ in the barycentre of all 2-dimensional faces $\sigma \in P_2$ which are not a simplex and adding the edges joining v_σ with the vertices of σ. Finally we set $A^{(2)} = A \cup \{v_\sigma : \sigma \cap A \neq \varnothing\}$.
- more generally given $\mathcal{P}^{(i-1)}$, $\mathcal{P}^{(i)}$ will be the tessellation obtained adding a vertex v_σ in the barycentre of every i-dimensional cell σ of $\mathcal{C}^{(i-1)}$ which is not a simplex and adding all the j-dimensional cells obtained by joining this vertex with the $j-1$ cells contained in σ. Finally we set $A^{(i)} = A^{(i-1)} \cup \{v_\sigma : \sigma \cap A^{(i-1)} \neq \varnothing\}$.

Set $\mathcal{P}' = \mathcal{P}^{(d)}$ and $A' = A^{(d)}$, and B' is the complement of A' in the set of vertices of $\mathcal{P}'$. Notice that all cells of $\mathcal{C}'$ are convex simplexes and that $\Delta_{\mathcal{P}}(A) = \Delta_{\mathcal{P}'}(A')$. Hence we can apply to this situation what we have already proved.

In the case of a tessellation whose cells are not convex some of the constructions we have described have to be modified. Indeed it does not make sense to consider the middle point of an edge or the convex envelope of the middle points in the first part of the proof or the barycentre in the second part of the proof. As an example we show which changes are necessary in the construction of the sequence $\mathcal{P}^{(i)}$.

Suppose we have already constructed $\mathcal{P}^{(i-1)}$ and that all j-dimensional cells of $\mathcal{P}^{(i-1)}$ are simplexes for $j \leq i-1$. Now consider a i-dimensional cell σ of $\mathcal{P}^{(i-1)}$ which is not a simplex. Let $\varphi : A \longrightarrow \sigma$ be the piecewise affine map which parametrise σ where A is a convex cells. Notice that by induction all faces of A are convex simplexes. Now let v be the barycentre of A and that we can divide A into simplexes

A_τ joining v with the faces of A. We obtain the new tessellation by considering the restriction of φ to the simplexes A_τ. □

The result we will use in the next section is a variation of Proposition 2 where the set of vertices A is replaced by a set of edges.

If $H \subset P_1$ we define the *boundary* $\Delta_{\mathcal{P}}(H)$ of H as the set of edges $e \in P_1 \backslash H$ for which there exist $f \in H$ and $\sigma \in P_d$ such that $e, f \subset \sigma$. Let also $V_H = \{v \in P_0 : v \in e \text{ for some } e \in H\}$ and let $G(H)$ be the subgraph (V_H, H) of $G(\mathcal{P})$.

Proposition 3 *Let $\mathcal{P}$ be a polyhedral tessellation of $\mathbb{R}^d$ and $H \subset P_1$ be a finite set. Let $x, y \in V_H$ be connected in $G(H)$ and in $G(P_1 \backslash H)$. Then x and y are connected in $G(\Delta_{\mathcal{P}}(H))$.*

Proof Let $e_1, \ldots, e_m$ be a path in H which join x and y and let $f_1, \ldots, f_n$ be a path in $P_1 \backslash H$ which joins x and y. Let $x = u_0, \ldots, u_m = y$ and $x = v_0, \ldots, v_n = y$ be the sequences of adjacent vertices determined respectively by the path in H and in $P_1 \backslash H$. Dividing the path into smaller pieces we can assume that $u_i \neq v_j$ for $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$.

We construct a refinement $\mathcal{Q}$ of $\mathcal{P}$ replacing each vertex w with a small convex cell so that we will be able to apply Proposition 2 to this situation.

More precisely for each vertex w we consider a "small" ball B_w with center w. Here small means that two balls do not intersect and that the geometry of the polytope inside D_w is the geometry of a cone with vertex in w. We set also D_w to be the intersection of the boundary of B_w with the edges in P_1 containing w and A_w to be the intersection of the boundary of B_w with the edges in H containing w. We set

$$Q_0 = P_0 \cup \bigcup_{w \in P_0} D_w \quad \text{and} \quad A = \{u_i : i = 0, \ldots, m\} \cup \bigcup_{w \in P_0} A_w.$$

We now describe the higher dimensional cells of $\mathcal{Q}$. Let F_w be convex envelope of the set D_w and let M_w be its boundary and N_w its open part. If B_w is small enough for every i-dimensional cell σ of $\mathcal{P}$ containing w the intersections $\sigma_w = \sigma \cap F_w$ and $\sigma_w^\partial = \sigma \cap M_w$ are respectively an i-dimensional cell and an $(i-1)$-dimensional cell. Finally for all i-dimensional cells σ of $\mathcal{P}$ the set $\tilde{\sigma} = \sigma \backslash \bigcup_w N_w$ is also an i-dimensional cell. The tessellation $\mathcal{Q}$ is the collection of the cells $\tilde{\sigma}$, σ_w and σ_w^∂ for σ a cell of $\mathcal{P}$ and $w \in P_0$.

Notice that if $\tilde{\sigma}$ is an edge in $G(\Delta_{\mathcal{Q}}(A))$ then σ is an edge in $G(\Delta_{\mathcal{P}}(H))$.

We now apply Proposition 2. The sequence of edges

$$(e_1)_{u_0}, \tilde{e}_1, (e_1)_{u_1}, (e_2)_{u_1}, \tilde{e}_2, \ldots, \tilde{e}_m, (e_m)_{u_m}$$

is a path in $G(A)$ joining x and y. Let $x' = (f_1)_x^\partial$ and $y' = (f_n)_y^\partial$ then they are two vertices adjacent respectively to x and y and the sequence of edges

$$\tilde{f}_1, (f_1)_{v_1}, (f_2)_{v_1}, \tilde{f}_2, \ldots, (f_n)_{v_{n-1}}, \tilde{f}_n$$

is a path in $G(Q_0 \backslash A)$ joining x' and y'. Hence there exists a path

$$\tau_1, \ldots, \tau_{r_1}, \tilde{\sigma}_1, \tau_{r_1+1}, \ldots, \tau_{r_2}, \tilde{\sigma}_2, \ldots, \tau_{r_d}$$

in $G(\Delta_{\mathcal{Q}}(A))$ joining x' and y' where we assume that τ_i is of the form η_w or η_w^{∂} for all i. Then $\sigma_1, \ldots, \sigma_{d-1}$ is a path in $G(\Delta_{\mathcal{P}}(H))$ joining x and y. □

For the application of our results to the Potts Model with constant boundary condition we need also a dual version of the result above. To state it we need to introduce some further notations. For a subset H of E we denote by $\Gamma(H)$ the set of edges which are not in H and which have a vertex in common with an edge in H. Given a tessellation $\mathcal{P}$ with associated simplicial graph $G(\mathcal{P}) = (V, E)$, we introduce also the graph $\mathcal{G} = \mathcal{G}(H, \mathcal{P}) = (\mathcal{V}, \mathcal{E})$ defined as follows: $\mathcal{V} = H$ and two elements $e, e' \in \mathcal{V}$ are joined by an edge in $\mathcal{E}$ if and only if there exists a cell in $\mathcal{P}$ which contains e and e'.

Proposition 4 *Let H be a finite subset of E. Let $e, e' \in \Gamma(H)$ with end vertices respectively x, y and x', y' with $x, x' \in V_H$ and $y, y' \in V \backslash V_H$. Assume that x, x' are connected in $G(H)$ and y, y' are connected in $G(E \backslash (H \cup \Gamma(H))$. Then e, e' are in the same connected component of $\mathcal{G}(\Gamma(H), \mathcal{P})$.*

Proof We indicate only the main steps of the proof which is completely analogous to the proof of Proposition 2. First we analyze the case where the cells are convex simplexes producing, as in the proof of Proposition 2, a smooth polytope of codimension one separating $A = V_H$ and $B = V \backslash A$, and apply Proposition 1. Then to analyze the general case, one produce a refinement of $\mathcal{P}$ as in the proof of Proposition 2 and adding to H the new edges joining a new vertex with a vertex in A. □

6 Coupling of the random cluster measure at low or high temperatures

In this section we fix a polyhedral tessellation $\mathcal{P}$ in $\mathbb{R}^d$ and we denote by $G = (V, E)$ the underlying simplicial graph. We introduce a new graph $G^* = (V^*, E^*)$ where $V^* = \mathbb{Z} \times V$ and $E^* = \mathbb{Z} \times E \cup \mathbb{Z} \times V$ where if x, y are the end vertices of $e \in E$ then $(n, x), (n, y)$ are the end vertices of $(n, e) \in E^*$ and if $v \in V$ and $n \in \mathbb{Z}$ then we denote with $e_{n,v}$ the corresponding element in E^* and its end vertices are $(n, v), (n + 1, v) \in V^*$. Notice that G^* is the simplicial graph of a polyhedral tessellation $\mathcal{P}^*$ of $\mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d$ whose i-dimensional cells are the collection of the cells of the form $\{n\} \times \sigma$ where σ is an i-dimensional cell of $\mathcal{P}$ and $n \in \mathbb{Z}$ and of the cells of the form $[n, n + 1] \times \tau$ and τ is an $(i - 1)$-dimensional cell of $\mathcal{P}$ and $n \in \mathbb{Z}$. We consider G as the subgraph $G \times \{-1\}$ of G^*.

Fix $u \in \mathcal{U}$. We define the following coloring of the edges E^* of G^*:

$$\begin{aligned} W &= W(u) = \{(n, e_k) \in \mathbb{Z} \times E : n \leq -1 \quad \text{and} \quad p_{e_k} \leq u_{n,k} \leq 1\}, \\ M &= M(u) = \{(n, e_k) \in \mathbb{Z} \times E : n \leq -1 \quad \text{and} \quad \hat{p}_{e_k} \leq u_{n,k} < p_{e_k}\}, \\ B &= B(u) = E^* \backslash (M(u) \cup W(u)). \end{aligned}$$

We define also $B_n = \{e \in E : (n, e) \in B\}$. We say that the elements of B are black, the elements of M are gray and the elements of W are white.

Given a subset F of $M(u)$ we define the *cluster of white or gray edges* $C_F^w = C_F^w(u)$ as the minimum subset H of E^* containing F such that if $e \in E^*$ and $e \in \Delta_{\mathcal{P}^*}(H)$

then $e \in B(u)$. More in general if $F \subset E$ is not necessarily a subset of M we define $C_F^w = F \cup C_{F\cap M}^w$. Equivalently if $F \subset M$ we can construct C_F^w inductively by adding the white or gray edges "near" to F as follows. Let $D_0 = F \cap M$ and $D_{i+1} = D_i \cup (\Delta_{\mathcal{P}^*}(D_i)\backslash B)$ then $C_F^w = F \cup \bigcup_i D_i$.

Notice that as the numbers p_e grow the probability that C_F^w is finite increases. In the Ising model the p_e's are related to a parameter called temperature and when this parameter is small the p_e are closer to 1. For this reason we refer to the case in which C_F^w is almost surely finite as the situation at low temperature.

Assume now that $C_F^w(u)$ is finite for a given $u \in \mathcal{U}$. In this case we define $N^w(u, F)$ as the biggest negative integer N such that $C_F^w \subset E \times [N+1, -1]$. We define

$$H_n^w = H_n^w(u, F) = \left\{e \in E : (n, e) \in C_F^w \text{ or } (n-1, e) \in C_F^w\right\} \tag{6}$$

and we set $\bar{H}_n^w(u, F) = H_n^w \cup \Delta_{\mathcal{P}}(H_n^w)$ and finally $\bar{H}^w(u, F) = \bigcup_n \{n\} \times \bar{H}_n^w$.

Theorem 5 *Fix $u \in \mathcal{U}$ such that $C_F^w(u)$ is finite and an integer $N \leq N^w(u, F)$. Let $u' \in \mathcal{U}_N$ such that $u'_{n,k} = u_{n,k}$ for all $(n, e_k) \in \bar{H}^w(u, F)$. Then for all $\omega, \omega' \in \Omega$ we have*

$$\left(X_{0,0}^{(\omega,N)}(u)\right)_e = \left(X_{0,0}^{(\omega',N)}(u')\right)_e$$

for all $e \in \bar{H}_{-1}^w(u, F)$.

Proof Let $\eta_{n,k} = X_{n,k}^{(\omega,N)}(u)$ and $\eta'_{n,k} = X_{n,k}^{(\omega',N)}(u')$. Let also $C = C_F^w$ and $C_n = \{e \in E : (n, e) \in C\}$, $H_n = C_n \cup C_{n-1}$ and $\bar{H}_n = H_n \cup \Delta_{\mathcal{P}}(H_n)$. We notice first that $\Delta_{\mathcal{P}}(H_n) \subset B_n \cap B_{n-1}$. Indeed let $e \in \Delta_{\mathcal{P}}(H_n)$, then there exists $e' \in H_n$ and a cell σ in $\mathcal{P}$ such that e and e' are contained in σ. If $e \notin B_n$ then by the definition of C and of the cells of $\mathcal{P}^*$ we have that $(n, e) \in C$, hence $e \in C_n$ which is in contradiction with $e \in \Delta_{\mathcal{P}}(H_n)$. Similarly we get an absurd if $e \notin B_{n-1}$. By the definition of a $FK_{p,q}^N$-process this implies that

$$\Delta_{\mathcal{P}}(H_m) \subset E(\eta_{m,h}) \cap E(\eta'_{m,h}) \tag{7}$$

for all $m > N$ and $h \geq 0$.

We will prove that

$$\left(\eta_{m,h}\right)_e = \left(\eta'_{m,h}\right)_e \tag{8}$$

for all $(m, h) \in \tilde{\mathcal{A}}_N$ and for all $e \in \bar{H}_m$ (for the definition of $\mathcal{A}$ see Eq. (4)). We prove this by induction starting with $m = N$ and $h = 0$. For $m \leq N^w(u, F)$ and for all h the equality (8) is trivially satisfied since $C_m = \varnothing$.

Now we prove that if Eq. (8) holds for $m = N, \ldots, n-1$ and for all h and for $m = n$ and $h = 0, 1, \ldots, k$ then it holds also for $m = n$ and $h = k+1$. Let $e \in \bar{H}_n$. If $e \neq e_{k+1}$ then, by induction, and definition of $FK_{p,q}^N$-process we get

$$\left(\eta_{n,k+1}\right)_e = \left(\eta_{n,k}\right)_e = \left(\eta'_{n,k}\right)_e = \left(\eta'_{n,k+1}\right)_e$$

proving the claim. If $e = e_{k+1}$ and $e \in \bar{H}_n$ we compute $\left(\eta_{n,k+1}\right)_{e_{k+1}}$. If $u_{n,k+1} \geq p_{e_{k+1}}$ then we have $\left(\eta_{n,k+1}\right)_{e_{k+1}} = 0$ and similarly for η'. If $u_{n,k+1} < \hat{p}_{e_{k+1}}$ then we have $\left(\eta_{n,k+1}\right)_{e_{k+1}} = 1$ and similarly for η'. If $\hat{p}_{e_{k+1}} < u_{n,k+1} \leq p_{e_{k+1}}$ we need to prove that

$$\eta_{n,k} \in K_{e_{k+1}} \quad \text{iff} \quad \eta'_{n,k} \in K_{e_{k+1}}. \tag{9}$$

Let x and y be the end vertices of e_{k+1} and assume that there is a path $\gamma : \varepsilon_1, \ldots, \varepsilon_m$ in $E(\eta_{n,k})\backslash\{e_{k+1}\}$ joining x and y.

If the path is contained in $\bar{H}_n = C_n \cup C_{n-1} \cup \Delta_{\mathcal{P}}(H_n)$ then we prove that the same path is contained in $E(\eta'_{n,k})\backslash\{e_{k+1}\}$. Let e_r be an edge of the path. By induction $(\eta_{n,k})_{e_r} = (\eta'_{n,k})_{e_r}$ hence $e_r \in E(\eta'_{n,k})$.

If γ is not contained in $\bar{H}_n$ we show there is another path joining x and y contained in $\bar{H}_n$. By (7) and the fact that $\hat{p}_{e_{k+1}} < u_{n,k+1} \leq p_{e_{k+1}}$ we have $e_{k+1} \in H_n$. Let D be the connected component of H_n containing e_{k+1}. Let $\varepsilon_1, \ldots, \varepsilon_{i-1} \in D$, $\varepsilon_i, \ldots, \varepsilon_j \notin D$ and $\varepsilon_{j+1} \in D$. Let x' be the vertex common to ε_{i-1} and ε_i and y' the vertex common to ε_j and ε_{j+1}. We can apply Proposition 3 and we construct a path β in $\Delta_{\mathcal{P}}(D) \subset \Delta_{\mathcal{P}}(H_n)$ joining x' and y'. Since $\Delta_{\mathcal{P}}(H_n) \subset E(\eta_{n,k}) \cap E(\eta'_{n,k})$ we can replace γ with the path $\gamma' : \varepsilon_1, \ldots, \varepsilon_{i-1}, \beta, \varepsilon_{j+1}, \ldots, \varepsilon_m$. Repeating this process we see that we can substitute the path γ with a path entirely contained in $D \cup \Delta_{\mathcal{P}}(D) \subset \bar{H}_n$ as claimed. Hence we are reduced to the previous case.

Finally we prove that if (8) holds for a fixed m and all $h \geq 1$ then it holds also for $m + 1, 0$. Let $e \in C_{m+1}$. If $e \in \bar{H}_m$ this follows by definition of the $FK^N_{p,q}$-process. If $e = e_r \in \bar{H}_{m+1}\backslash C_m$ then $e \in B_m$ otherwise e would be an element of C_m). Then

$$(\eta_{m+1,0})_{e_r} = (\eta_{m,r})_{e_r} = 1 = (\eta'_{m,r})_{e_r} = (\eta_{m+1,0})_{e_r}$$

proving the claim. □

6.1 Coupling at high temperatures

We give now a similar result corresponding, in the Ising model, to high temperatures.

Let $G = (V, E)$ be a countable graph (in this case we do not assume simplicial). Define $\bar{G}$ as the graph with set of vertices $\bar{V} = \mathbb{Z}_{<0} \times V$ and edges $\bar{E} = \mathbb{Z}_{<0} \times E$ where if $e \in E$ has end vertices x, y then the edge (n, e) has end vertices (n, x) and (n, y). We consider G as the subgraph $G \times \{-1\}$ of $\bar{G}$. For all $u \in \mathcal{U}$ define $M(u)$, $W(u)$ as in the previous section.

Recall from Sect. 5.3 that for a subset H of E we denote by $\Gamma(H)$ the set of edges which are not in H and which have a vertex in common with an edge in H.

Fix $u \in \mathcal{U}$ and a subset F of $\bar{E}$. If $F \subset M(u)$ we define *the cluster of black or gray edges* $C^b_F = C^b_F(u)$ as the smallest set C of $\bar{E}$ containing F and such that for all $(n, e) \in \bar{E}\backslash C$ if either $(n-1, e)$ or (n, e) or $(n+1, e)$ have a vertex in common with an edge in C then $(n, e) \in W(u)$. If F is not necessarily contained in $M(u)$ we define $C^b_F(u) = C^b_{F \cap M(u)}(u) \cup F$. For all $n < 0$ we define

$$H^b_n = H^b_n(u, F) = \left\{e \in E : (n, e) \in C^b_F(u) \text{ or } (n-1, e) \in C^b_F(u)\right\}$$

and we set $\bar{H}_n^b(u, F) = H_n^b \cup \Gamma(H_n^b)$ and $\bar{H}^b(u, F) = \bigcup_n \{n\} \times \bar{H}_n^b$.

Finally if $C_F^b(u)$ is finite we define $N^b(u, F)$ as the biggest negative integer N such that $C_F^b \subset E \times [N+1, -1]$.

Theorem 6 *Fix $u \in \mathcal{U}$ such that $C_F^b(u)$ is finite and an integer $N \leq N^b(u, F)$. If $\omega, \omega' \in \Omega$ and $u' \in \mathcal{U}_N$ is such that $u'_{n,k} = u_{n,k}$ for all $(e_k, n) \in \bar{H}^b(u, F)$ then*

$$\left(X_{0,0}^{(\omega,N)}(u)\right)_e = \left(X_{0,0}^{(\omega',N)}(u')\right)_e$$

for all $e \in \bar{H}_{-1}^b(u, F)$.

Proof The proof follows exactly the same strategy of the proof of Theorem 5. However in this case it is simpler since we do not have to use the result of Sect. 5. We give here only the main lines. Indeed an argument analogous to proof of equation (7) gives

$$\Gamma(H^b(u, F)) \subset W(u). \tag{10}$$

Then we prove the equality (8) by induction as in the proof of Theorem 5. Also in this case we are reduced easily to prove the equivalence (9). This equivalence is easier in this case, since, by (10), we have that $E(\eta_{n,h}) \subset H_n^b$ for all $n > N$ and similarly for η' so we can assume that the path joining the extremal point of e_{k+1} is contained in H_n^b without using any further remark, while in the proof of Theorem 5 we need Proposition 3. The remaining part of the proof is completely similar to the proof of Theorem 5. □

7 Assumptions for the finiteness of clusters

In this section, given a countable graph G, we present some conditions on it and on the parameters p such that the cluster C_F^w of Theorem 5 is almost surely finite or such that the cluster C_F^b of Theorem 6 is almost surely finite.

We start by recalling a general lemma. Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be a graph and let $\pi = (\pi_v)_{v \in \mathbb{V}}$ be an element of $[0, 1]^{\mathbb{V}}$. Consider the product measure on the space $\Omega_{\mathbb{G}} = \{0, 1\}^{\mathbb{V}}$ such that $P(\omega_v = 0) = \pi_v$. For each $\omega \in \Omega_{\mathbb{G}}$ let $\mathbb{G}[\omega]$ be the subgraph of G with set of vertices $\mathbb{V}[\omega] = \{v \in \mathbb{V} : \omega_v = 0\}$ and with set of edges $\mathbb{E}[\omega]$ of the elements of $\mathbb{E}$ joining two vertices in $\mathbb{V}[\omega]$. Moreover for each $\omega \in \Omega_{\mathbb{G}}$ and for any $v \in \mathbb{V}$ set $\mathbb{G}_v[\omega]$ the connected component of $\mathbb{G}[\omega]$ containing v (possibly empty if $v \notin \mathbb{V}[\omega]$) and if n is a natural number let $\mathbb{V}_{v,n}[\omega]$ be the set of vertices in $\mathbb{G}_v[\omega]$ whose graph-distance from v in the graph $\mathbb{G}_v[\omega]$ is equal to n.

Lemma 7 *Let $\mathbb{G}$ and π be as above. For each vertex $v \in \mathbb{V}$ let A_v be the set of vertices adjacent to v and set $g_v = \sum_{w \in A_v} \pi_w$. If $g = \sup\{g_v : v \in \mathbb{V}\} < 1$ then $P(\{\omega \in \Omega_{\mathbb{G}} : \mathbb{V}_{v,n}[\omega] \neq \varnothing\}) \leq g^n$.*

Proof Define the random variables $Z_{v,n}[\omega] = \text{card}\left(\mathbb{V}_{v,n}[\omega]\right)$. The conditional mean value $E(Z_{v,n+1}|Z_{v,n})$ verifies $E(Z_{v,n+1}|Z_{v,n}) \leq g\, Z_{v,n}$, in particular the sequence $\{Z_{v,n}\}_n$ is a supermartingale. Hence the mean value $E(Z_{v,n})$ is less or equal to g^n. By Markov's inequality the claim follows. □

7.1 Assumptions for the finiteness of clusters at low temperatures

Now we give conditions on G and p such that the cluster $C_F^w(u)$, defined in Sect. 6, is finite for almost all $u \in \mathcal{U}$. We fix a polyhedral tessellation $\mathcal{P}$ of $\mathbb{R}^d$. Let $G = (V, E)$ be the associated simplicial graph and let $\mathcal{P}^*$ and $G^* = (V^*, E^*)$ be defined as in Sect. 6.

For the proof of our next theorem we set also $\mathbb{G} = \mathcal{G}(E^*, \mathcal{P}^*) = (\mathbb{V}, \mathbb{E})$. Recall from Sect. 5.3 that $\mathbb{V} = E^*$ and two elements $e, e' \in \mathbb{V}$ are joined by an edge in $\mathbb{E}$ if and only if there exist a cell in $\mathcal{P}^*$ which contains e and e'.

If $H \subset E^*$ and $u \in \mathcal{U}$ we define also the subgraph $\mathbb{G}(H, u) = (\mathbb{V}(H, u), \mathbb{E}(H, u))$ of $\mathbb{G}$ whose set of vertices $\mathbb{V}(H, u)$ is equal to $H \backslash B(u)$ and whose edges $\mathbb{E}(H, u)$ are all the edges of $\mathbb{E}$ joining two vertices in $\mathbb{V}(H, u)$.

For $e \in E$ define

$$\hat{g}_e = 2(1 - \hat{p}_e) + 3 \sum_{e' \in \Delta_{\mathcal{P}}(\{e\})} (1 - \hat{p}_{e'}), \tag{11}$$

where $\hat{p}_e$ is defined in (2).

Theorem 8 *Let $G = (V, E)$ be a simplicial graph in $\mathbb{R}^d$. Assume that $\hat{p}_e > 0$ for all $e \in E$, and that*

$$\lim_{\Lambda \uparrow E} \sup_{e \notin \Lambda} \hat{g}_e < 1.$$

Then for all $e \in E$ the cluster $C_e^w(u)$ is finite for almost all $u \in \mathcal{U}$.

Proof First we do a preliminary remark: the event $I = \cup_{e \in E} \{u \in \mathcal{U} : \text{card}\left(C_e^w(u)\right) = \infty\}$, is in the tail σ-algebra. Therefore, by Kolmogorov 0-1 law, this event has probability zero or one, in particular to prove our claim it is enough to prove that $P(I) < 1$.

Define $F_n = \{e \in E : \hat{g}_e > 1 - \frac{1}{n}\}$. By assumption there exists n_0 such that $\text{card}(F_{n_0}) < \infty$ and set $F = F_{n_0}$ and $g = 1 - \frac{1}{n_0}$. Set also $\tilde{F} = F \cup \Delta_{\mathcal{P}}(F)$ and define $\hat{F} = \mathbb{Z} \times F \subset \mathbb{Z} \times E \subset E^*$. Fix an edge $\hat{e} \in E$ and, for $\ell \geq 1$ set

$$\Lambda_\ell = [-\ell, -1] \times B_{\mathbb{G}}(\hat{e}, \ell) \subset \mathbb{Z} \times E$$

and set also $S_\ell = \text{card}(\Lambda_\ell)$.

Choose ℓ_0 such that $\frac{3\,\text{card}(\tilde{F})}{1-g} g^{\ell_0} < \frac{1}{2}$ and, for $\ell \geq \ell_0 + 1$, define the events

$$\begin{aligned} \mathcal{W}_\ell &= \{u \in \mathcal{U} : u_{n,k} < \hat{p}_{e_k} \text{ for any } (n, e_k) \in [-\ell - S_\ell, -\ell - S_\ell + \ell_0] \times F\}, \\ \mathcal{X}_\ell &= \{u \in \mathcal{U} : \text{card}\left(C_{\Lambda_\ell}^w(u)\right) = \infty\}. \end{aligned}$$

Notice that the probability $P_0 = P(\mathcal{W}_\ell)$ does not depend on ℓ and it is a positive constant being $\hat{p}_e > 0$ for any $e \in E$. We also notice that $\mathcal{X}_\ell \subset \mathcal{X}_{\ell+1}$ and that their union is equal to I. Therefore

$$P(I) = P\left(\bigcup_{\ell} \mathcal{X}_\ell\right) = \lim_{\ell\to\infty} P(\mathcal{X}_\ell).$$

Define the events

$$\begin{aligned}\mathcal{Y}_\ell = \{u \in \mathcal{U} :\ &\text{there exists a sequence } e_1, \ldots, e_m \in W(u) \cup M(u)\backslash \hat{F}\\ &\text{such that } e_i \text{ is adjacent to } e_{i+1} \text{ in the graph } \mathbb{G},\ e_1 \in \Lambda_\ell \text{ and } e_m \notin \Lambda_{\ell+S_\ell}\},\\ \tilde{\mathcal{Z}}_{\ell,i} = \{u \in \mathcal{U} :\ &\text{there exists a sequence } e_1, \ldots, e_m \in W(u) \cup M(u) \text{ such that}\\ &e_j \text{ is adjacent to } e_{j+1} \text{ in the graph } \mathbb{G},\\ &e_1 \in \{-\ell - S_\ell + i\} \times F,\ e_m \notin \Lambda_{\ell+S_\ell} \text{ and } e_j \notin \hat{F} \text{ for } 1 < j < m\}.\end{aligned}$$

Finally define $\mathcal{Z}_{\ell,1} = \tilde{\mathcal{Z}}_{\ell,1}$ and $\mathcal{Z}_{\ell,i} = \tilde{\mathcal{Z}}_{\ell,i}\backslash\tilde{\mathcal{Z}}_{\ell,i-1}$ for $i > 1$. It is clear that

$$\mathcal{X}_\ell \subset \mathcal{Y}_\ell \cup \left(\bigcup_{i=1}^{\ell} \mathcal{Z}_{\ell,i}\right),$$

in particular $P(\mathcal{X}_\ell|\mathcal{W}_\ell) \leq P(\mathcal{Y}_\ell|\mathcal{W}_\ell) + \sum_{i=1}^{\ell} P(\mathcal{Z}_{\ell,i}|\mathcal{W}_\ell)$.

Now we notice that the event $\mathcal{W}_\ell$ is decreasing meaning that if $u \in \mathcal{X}_\ell$ and $u' \in \mathcal{U}$ is such that $u'_{n,h} \leq u_{n,h}$ for all n, h then $u' \in \mathcal{W}_\ell$. With a similar definition the events $\mathcal{Y}_\ell$ and $\mathcal{Z}_{\ell,i}$ are increasing. Hence, by the FKG inequality we obtain $P(\mathcal{Y}_\ell|\mathcal{W}_\ell) \leq P(\mathcal{Y}_\ell)$ and $P(\mathcal{Z}_{\ell,i}|\mathcal{W}_\ell) \leq P(\mathcal{Z}_{\ell,i})$ (see [10], Chapter 2). Hence, noticing that $P(\mathcal{Z}_{\ell,i}|\mathcal{W}_\ell) = 0$ for $i < \ell_0$, we get $P(\mathcal{X}_\ell|\mathcal{W}_\ell) \leq P(\mathcal{Y}_\ell) + \sum_{i\geq\ell_0} P(\mathcal{Z}_{\ell,i})$.

Now we estimate $P(\mathcal{Y}_\ell)$ and $P(\mathcal{Z}_{\ell,i})$ using Lemma 7. We start with $\mathcal{Y}_\ell$. Consider the random graph $\mathbb{G}(\hat{F}^c, u)$ and set $\pi_e = 1 - \hat{p}_e$. Notice that $\mathcal{Y}_\ell \subset \bigcup_{e\in\Lambda_\ell}\{u \in \mathcal{U} : \mathbb{V}(\hat{F}^c, u)_{e,S_\ell} \neq \varnothing\}$ hence using Lemma 7 we get

$$P(\mathcal{Y}_\ell) \leq \text{card}(\Lambda_\ell)\, g^{S_\ell} = S_\ell\, g^{S_\ell},$$

for ℓ large enough. For $\mathcal{Z}_{\ell.i}$ we proceed in a similar way. Consider again the random graph $\mathbb{G}(\hat{F}^c, u)$. Notice that in the sequence $e_1, \ldots, e_m$ which appears in the definition of $\tilde{\mathcal{Z}}_{\ell,i}$ the subsequence $e_2 \ldots, e_{m-1}$ is in $\hat{F}^c$ and $e_2 \in \tilde{F}_{\ell,i} := \{-\ell - S_\ell + i - 1, -\ell - S_\ell + i, -\ell - S_\ell + i + 1\} \times \tilde{F}$. Hence $\mathcal{Z}_{\ell,i} \subset \bigcup_{e\in\tilde{F}_{\ell,i}}\{u \in \mathcal{U} : \mathbb{V}(\hat{F}^c, u)_{e,i} \neq \varnothing\}$ and using Lemma 7 we get

$$P(\mathcal{Z}_{\ell,i}) \leq \text{card}(\tilde{F}_{\ell,i})\, g^i = 3\ \text{card}(\tilde{F})\, g^i.$$

Recall that $P_0 = P(\mathcal{W}_\ell)$ does not depend on ℓ, hence we have

$$\begin{aligned}P(\mathcal{X}_\ell) &= P(\mathcal{X}_\ell|\mathcal{W}_\ell)P(\mathcal{W}_\ell) + P(\mathcal{X}_\ell|\mathcal{W}_\ell^c)P(\mathcal{W}_\ell^c) \leq P(\mathcal{W}_\ell^c) + P(\mathcal{X}_\ell|\mathcal{W}_\ell)P(\mathcal{W}_\ell)\\ &\leq 1 - P_0 + P_0\left(P(\mathcal{Y}_\ell) + \sum_{i=\ell_0}^{\infty} P(\mathcal{Z}_{\ell,i})\right)\end{aligned}$$

$$\leq 1 - P_0 + P_0 \left(S_\ell \, g^{S_\ell} + \frac{3 \ \text{card}(\tilde{F})}{1-g} g^{\ell_0} \right)$$
$$\leq 1 - \frac{P_0}{2} + P_0 S_\ell \, g^{S_\ell}.$$

Finally notice that $\lim_{\ell \to \infty} S_\ell \, g^{S_\ell} = 0$. Hence

$$\lim_{\ell \to \infty} P(\mathcal{X}_\ell) \leq 1 - \frac{P_0}{2} < 1$$

as claimed. □

7.2 Assumptions for the finiteness of clusters at high temperature

Let $G = (V, E)$ be a countable graph of finite degree. We remark that in this case we do not need to assume that G is a simplicial graph. For all $e \in E$ define

$$g_e = 2\, p_e + 3 \sum_{e' \in \Gamma(\{e\})} p_{e'}.$$

Theorem 9 *Let G be a countable graph of finite degree. If $p_e < 1$ for all $e \in E$ and*

$$\lim_{\Lambda \uparrow E} \sup_{e \notin \Lambda} g_e < 1$$

then for all $e \in E$ the set $C_e^b(u)$ is finite for almost all $u \in \mathcal{U}$.

The proof follows exactly the same line of the proof of Theorem 8, however we do not need any result from Sect. 5.

8 Perfect simulation of the random cluster measure at low or high temperature

As an application of the previous results we now explain how to prove uniqueness of the random cluster measure and how to obtain a perfect simulation of the random cluster measure. We consider the case of low temperature, the case of high temperatures can be obtained in a similar way. In this section, from now on we assume that G is a simplicial graph, that $p_e > 0$ for all $e \in E$ and that $\lim_{\Lambda \uparrow E} \sup_{e \notin \Lambda} \hat{g}_e < 1$. The uniqueness proved in the following Corollary is well known at least in the case of $\mathbb{L}^d$.

Corollary 10 *Assuming the hypotheses above the random cluster measure on G is unique.*

Proof Let ϕ, ϕ' be two DLR random cluster measures.

To prove that ϕ and ϕ' are equal we prove that for each finite subset F of E the projections ϕ_F and ϕ'_F of ϕ and ϕ' on $\{0,1\}^F$ are equal. We denote also by $X_F^{(N,\omega)} \in \{0,1\}^F$ the projection of $X_{0,0}^{(N,\omega)}$.

Let ω be a random variable with law ϕ and ω' a random variable with law ϕ'. By Theorem 5 we have that

$$\|X_F^{(N,\omega)} - X_F^{(N,\omega')}\|_{TV} \leq P\big(u \in \mathcal{U} : N^w(u,F) \leq N\big)$$

where the left hand side is the total variation distance between the law of $X_F^{(N,\omega)}$ and $X_F^{(N,\omega')}$. Recall now that as noticed in Sect. 4 the DLR random cluster measures are invariant under a $FK_{p,q}^N$-process. Hence, since ω has law ϕ, the random variable $X_F^{(N,\omega)}$ has law ϕ_F and $X_F^{(N,\omega')}$ has law ϕ'_F. Hence we get $\|\phi_F - \phi'_F\|_{TV} \leq P\big(u \in \mathcal{U} : N^w(u,F) \leq N\big)$. Finally by Theorem 8, $C_F^w(u)$ is finite for almost all $u \in \mathcal{U}$. Hence $P(u \in \mathcal{U} : N^w(u,F) \leq N)$ goes to zero as N goes to infinity. Hence ϕ and ϕ' are equal. □

8.1 Description of the algorithm

Under the same assumptions, given a finite subset F of E we briefly describe an algorithm which furnishes a sampling of the random cluster measure on $\bar{H}_{-1}$ and in particular on F.

The algorithm follows exactly the strategy of the proof of Theorem 5. We assume that the tessellation $\mathcal{P}$ of $\mathbb{R}^d$ and the graph $G=(V,E)$ is already implemented in an algorithmic way. This means that there is an algorithmic way to describe the graph G and given two edges is possible to say if they belongs to the same cell (in actual examples this implementation is very easy to do, see also Sect. 9). This allow, given a finite subset E' of E to compute the set $\Delta_{\mathcal{P}}(E')$ which we recall is the set of edges $f \notin E'$ for which there exists an edge $e \in E'$ such that e and f belong to the same cell. Notice that given a finite subset C of $\mathbb{Z}_{<0} \times E$ we can determine the set $\tilde{\Delta}(C) = \{(n,e_k) \in \mathbb{Z}_{<0} \times E$: there exists $(m,e_h) \in C$ with $|m-n| \leq 1$ and the edges e_h and e_k belonging to the same cell$\}$. The definition of this $\tilde{\Delta}$ is equivalent, for the purposes of this pseudocode, to the definition of $C \cup \Delta_{\mathcal{P}^*}(C)$ of Sect. 6.

Let also be given a generator of independent random numbers $u_{n,k}$. The algorithm takes a finite subset F of E as an input and gives as output subsets C, $\bar{H}$ of $\mathbb{Z}_{<0} \times E$, a subset $\bar{H}_{-1}$ of E and a configuration $Y \in \{0,1\}^{\bar{H}_{-1}}$. It uses also local variables $B, B', D, D', F', H, L, M$ and generates $u_{n,k}$ for $(n,e_k) \in \bar{H}$. We describe the algorithm with the following pseudocode where we have added some comments in brackets.

Step 1: generate the random numbers $u_{n,k}$ for $n=-1$ and $e_k \in F$ and set $F' = \{-1\} \times F$ [color randomly the starting set];
Step 2: set $M = \{(-1,e_k) \in F' : \hat{p}_{e_k} \leq u_{-1,e_k} < p_{e_k}\}$ and $B = \{(-1,e_k) \in F' : p_{e_k} \leq u_{-1,e_k}\}$ [compute black and gray edges];
Step 3: set $D = M$;
Step 4: set $L = \tilde{\Delta}(D)$ [enlarge the set D in "all possible directions"];

Step 5: generate the random numbers $u_{n,k}$ for $(n, e_k) \in L\backslash(B \cup D \cup F')$;
Step 6: set $B' = B \cup \{(n, e_k) \in L\backslash(B \cup D \cup F') : p_{e_k} \leq u_{n,e_k}\}$ [compute the new black edges];
Step 7: set $D' = D \cup (L\backslash B')$ [add the new white and gray edges to the cluster];
Step 8: if $D' \neq D$ assign to D the value given by D' and to B the value given by B' and goes to step 4 [if there is something new iterate the procedure];
Step 9: if $D' = D$ then $C = D \cup F'$ and set $N = \min\{n$: there exists k and $(n, e_k) \in C\} - 1$ [if there is nothing new start to compute the output];
Step 10: use the formula (6) to compute the sets H_n^w and define $\bar{H} = \bigcup_{n=N}^{-1}\{n\} \times \bar{H}_n$ where $\bar{H}_n = H_n^w \cup \Delta_{\mathcal{P}}(H_n^w)$ as in Sect. 6 [compute the set $\bar{H}$ and $\bar{H}_{-1}$];
Step 11: generate the random numbers $u_{n,k}$ for $(n, e_k) \in \bar{H}\backslash C$ [generate the missing random numbers];
Step 12: if $e \in \bar{H}_{-1}$ use the process described in Sect. 4 to compute the value $Y_e = X_{0,0}^{(\omega,N)}$ where $\omega_e = 0$ for all $e \in \bar{H}_N$ [as explained in the proof of Theorem 5, by Proposition 3, for this computation it is enough to know the value of u only inside the region $\bar{H}$].

Notice that under the assumption of Theorem 8 this algorithm will end in a finite number of steps almost surely. Notice also that the output C is the set $C_F^w(u, F)$, $\bar{H}$ is the set $\bar{H}^w(u, F)$. Finally, as explained in proof of uniqueness above, Y has law ϕ_F where ϕ_F is the projection onto $\{0, 1\}^F$ of the unique random cluster measure on $\{0, 1\}^E$.

Moreover if further assume that $\sup_{e\in E} \hat{g}_e < 1$ it can be easily proved that the average complexity of this algorithm goes linearly with the cardinality of F.

9 Examples

We illustrate the objects introduced in the paper in one example. To fix the ideas we consider the random cluster measure with $q = 2$. The associated graph is equal to $\mathbb{L}^2 = (\mathbb{Z}^2, \mathbb{E}_2)$ and can also be described as the graph obtained from the tessellation $\mathcal{P}$ of $\mathbb{R}^2$ by squares with vertices in $\mathbb{Z}^2$. For simplicity we assume also that the parameters p_e are all equal to a number $p = 1 - e^{-\beta}$, where $\beta \in (0, \infty)$ is the parameter called the inverse of the temperature and the interactions are fixed equal to 1.

In this case $\hat{p}_e = \frac{p}{2-p} =$, $\hat{g}_e = 40\frac{1-p}{2-p}$ and $g_e = 8p$, are independent of the edge e. The results of Sect. 7 can be stated in the following way. If $p > 38/39$ or equivalently the temperature $1/\beta < 1/\ln 39 \simeq 0.27$, then the cluster C_F^w is almost surely finite for any finite $F \subset \mathbb{E}_2$. Similarly, if $p < 1/8$ or equivalently the temperature $1/\beta > 1/(\ln 8 - \ln 7) \simeq 7.49$, then the cluster C_F^b is almost surely finite for any finite $F \subset \mathbb{E}_2$.

We now explain more concretely the construction of the geometric objects. If $A \subset \mathbb{E}_2$ then $\Delta_{\mathcal{P}}(A)$ is the set of edges not in A which are a side of a square containing an element of A (see Fig. 1), while $\Gamma(A)$ is the set of edges not in A which have a vertex in common with an edge in A (see Fig. 1).

The tessellation $\mathcal{P}^*$ is the tessellation of $\mathbb{R}^3$ by cubes with vertices in $\mathbb{Z}^3$ and the associated graph is $\mathbb{L}^3 = (\mathbb{Z}^3, \mathbb{E}_3)$. The tessellation $\mathcal{P}^*$ is introduced so that we can apply the results of Sect. 5. To understand the application to the original problem what it is really important are the horizontal sections of $\mathbb{L}^3$ corresponding to negative levels

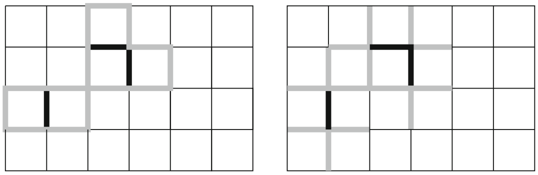

Fig. 1 On the *left* the description of $\Delta_{\mathcal{P}}(A)$ and on the *right* the description of $\Gamma(A)$. In both cases in *bold black* the set A and in *bold gray* the sets $\Delta_{\mathcal{P}}(A)$ and $\Gamma(A)$

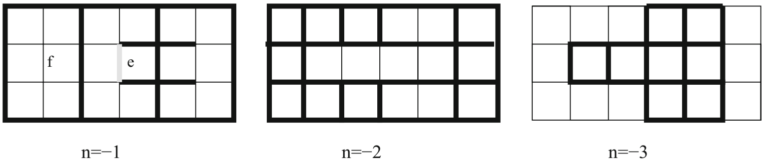

Fig. 2 Coloring of a region around e at levels $-1, -2, -3$. B is the set of edges *colored black* and *bold*, M is the set of edges *colored gray* and *bold* and W is the set of remaining edges

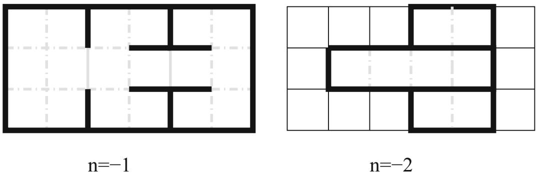

Fig. 3 Description of C_F^w (*bold dashed gray edges*), H_F^w (*dashed* or not *dashed bold gray edges*) and ΔH_F^w (*bold black edges*) at levels $n = -1, -2$

of the height. These graphs are all isomorphic to the original graph and their edges are parameterized by $\mathbb{Z}_{<0} \times \mathbb{E}_2$. For two different edges (m, e) and (n, f) of this type we have that $(m, e) \in \Delta_{\mathcal{P}^*}((n, f))$ if and only if they are sides of the same cube. In particular we consider the original graph placed in level -1.

We explain now the construction of the cluster of white or gray edges C_F^w. Assume F is just one single edge e initially colored with gray. In Fig. 2 we describe a coloring of a region around F at levels $n = -1, -2, -3$. For our purposes this is equivalent to give u on this set.

Recall that C_F^w is constructed by adding iteratively all the white or gray edges which belongs to its boundary. For example at the first step of the iteration we add the 2 edges colored white on the square on the left of e at level $n = -1$, and the 3 white edges at level $n = -2$ in the central row. The sets C_F^w and $\bar{H}_F^w$ resulting from this iteration are given in Fig. 3. In particular notice that $N = -3$ in this case.

Our definition does not depend on the ordering of the edges. For this reason we have to consider edges in the boundary looking also at levels in higher position. For example the element f in Fig. 2 is an element of C_F^w.

However to apply the dynamics described in Sect. 4 we have to fix an ordering of the edges. In this example this order is described in Fig. 4 where we have specified this order only for the edges which are relevant in our computations (in particular $e = e_{17}$).

We can now apply the dynamics and compute the process $(X_{n,k}^{(\omega,-3)})_e$ for $n = -1, -2$ and $(n, e_k) \in \bar{H}_F^w\}$ and for an arbitrary initial configuration ω using formula (5). In the final output we will have $(X_{0,0}^{(\omega,-3)})_e = 0$ (notice the if we exchange the

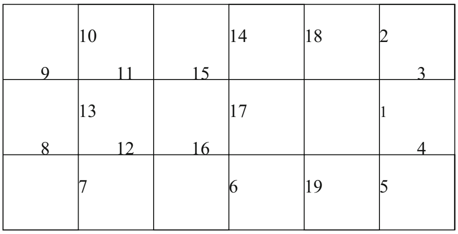

Fig. 4 Ordering of the edges

numbering of the edges now numbered 1 and 17 in the final output we would have $(X_{0,0}^{(\omega,-3)})_e = 1)$.

The example we have considered above is related with the two dimensional Ising model. We conclude our paper recalling how to apply the results on the random cluster measure to the Ising and Potts model with free boundary conditions. Following the original paper by Fortuin and Kasteleyn [6], the book of Grimmett [11], Chapter 1 it is clear the connection between the random cluster measure and the Ising or Potts model. Starting from the random cluster measure to construct the Ising or Potts model just need to know if two given vertices $v, w \in V$ are in the same component or not, then one color randomly any component independently from each other.

Under the hypotheses of Theorems 5 and 8, using our algorithm to simulate the random cluster measure on a finite $F \subset E$ one obtain a perfect simulated configuration Y on a larger finite set $\bar{H} \supset F$. Moreover the knowledge of the configuration Y on $\bar{H}$ is sufficient to establish if two vertices v, w belonging to some edges of F are in the same component or not (as a consequence of results in Sect. 5). Therefore one can color randomly and independently each component obtaining a perfect simulation for the Ising or Potts model. It is easy to translate this description in an actual pseudocode producing a configuration of the Ising or Potts model with free boundary conditions on a finite set $V' \subset V$. More care is required in dealing with the Ising or Potts model in the case of constant boundary conditions. Indeed, using the same idea, it is not possible to give a perfect simulation but, with a new hypothesis on the graph, for a prescribed error ε one can construct a simulation of a random field that has total variation distance from the Ising (resp. Potts) model lesser or equal than ε.

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