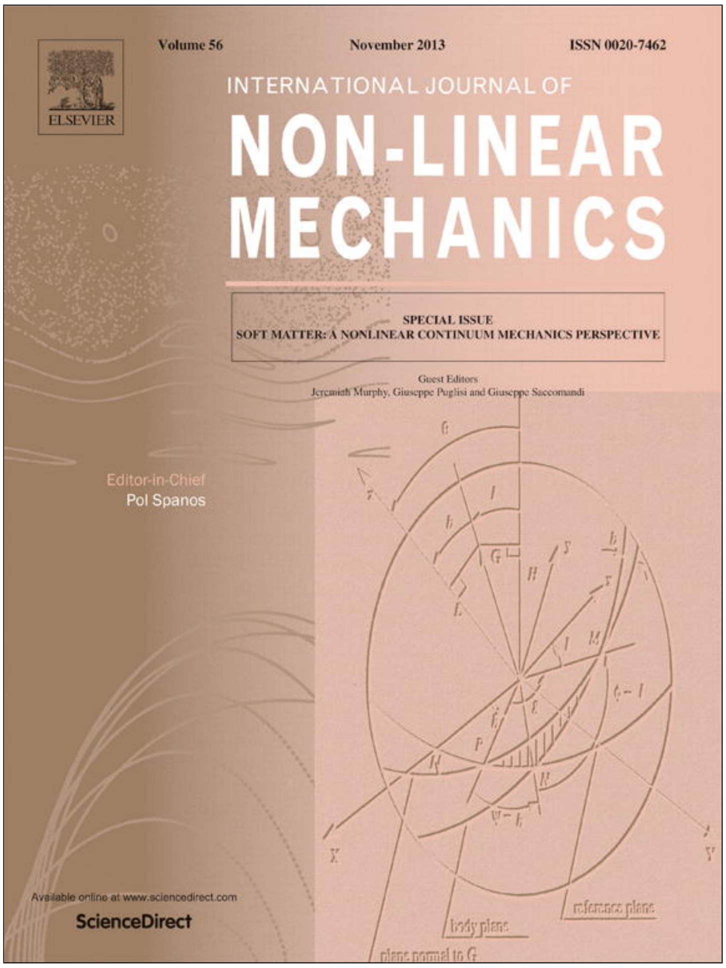
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The elastic metric: A review of elasticity with large distortions

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ABSTRACT

The mechanical behavior of soft matter is characterized by large shape changes, often accompanied by small changes of elastic energy; non-linear elasticity, with large, inhomogeneous, and anisotropic distortions, that may evolve in time, proved to be an effective modeling tool for many of such soft materials. Here, we deal with the definition of an appropriate strain measure, called the elastic metric, upon which the elastic energy density can be defined. Moreover, we discuss two key issues about distortions: one deals with the notion of compatible distortions, that is, distortion fields yielding a global configuration without any change of the elastic energy; the other concerns the symmetries of the material responses. We also present few selected examples of non-linear, anisotropic, elastic response where distortions play a key role.

1. Introduction

Both the modelling and the fabrication of three-dimensional elastic structures whose shapes can result from a given imposed pattern of growth or swelling are issues largely investigated in the recent years, and continue to represent challenges for physicists and experimentalists [1–7]. Another topic strictly related concerns the chances of getting the desired shapes without changing the elastic energy of the body; and it has received less attention thus far, even if its implications in the design of smart elastic structures are noteworthy.

Here, we have in mind the large, in general non-isotropic, shape changes which can be realized in elastic matter as a consequence of many different actuation mechanisms. The mechanical framework to describe these continuous large shape transformations of elastic materials is the theory of non-linear, inhomogeneous, anisotropic elasticity with distortions. The notion of distortion has been introduced more than 60 years ago to describe the emergence of plastic response in solids [8–11], and since then received very much attention in materials science, see [12] and the extensive references therein. The same notion has been proposed to model finite growth in soft matter [13], and more recently [14], the theory has been augmented with an additional balance law for the accretive forces, independent of the standard force balance, ruling the time evolution of distortions; applications of that can be found in [15,16].

A lot of physical applications of the finite elasticity with distortions may be found in the recent scientific literature on soft matter concerning isotropic–nematic transitions in gels [2], growth of elastic tissues [5], swelling or shrinkage of hydrogels [17], voltage-induced deformations in ionic polymer–metal composites [18], growth of plants [19], muscle activation [20]. Also, general investigations on stability issues related to volumetric growth in soft materials, particularly important when rapid changes of shape due to instability phenomena are involved, have been recently carried out in [21–24].

Here, we subsume the key ideas presented so far in the realm of both plasticity and growth, using a same unifying language. We review constitutive issues; we discuss the geometrical implications of finite elasticity with distortions, following [25–29,16], and tackle the problem of compatibility, discussing the existence and the possible representation of compatible distortions, realizable without any change of the elastic energy [30]. We present two examples of material response, one isotropic, the other one transversely isotropic, to highlight the important fact that distortions sharing the same metric can be distinguished by anisotropic material response. In the end, through a simple yet not trivial example, we discuss the role of the orthogonal component of the distortion fields, typically not largely considered.

2. Kinematics

Let us fix a few basic notions and notations; any further details are given in Appendix. For us $\mathcal{E}$ is the three-dimensional Euclidean ambient space, and the vector space $V\mathcal{E}$ is the corresponding translation space of $\mathcal{E}$. We denote with $\mathbb{L}\mathrm{in} = V\mathcal{E} \otimes V\mathcal{E} = \mathbb{S}\mathrm{ym} \oplus \mathbb{S}\mathrm{kw}$

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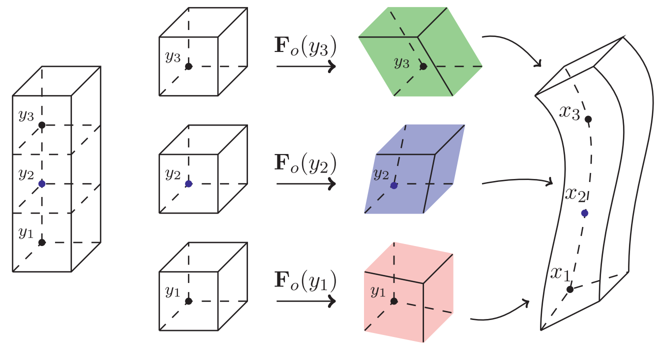

Fig. 1. To envision the notion of distortions, consider a stack of volume elements attached to three neighboring body points y_i, $i=1,2,3$ (left); then, cut and distort each body element by a smooth field $\mathbf{F}_o$. The new body elements are now in a relaxed state, but in general, they cannot be glued together. To realize an actual configuration (right), a further strain is necessary.

the space of double tensors on $\mathcal{V}\mathcal{E}$ (linear maps of $\mathcal{V}\mathcal{E}$ into itself) and with $\mathbb{LIN}=\mathbb{L}\text{in}\otimes\mathbb{L}\text{in}$ the space of fourth-order tensors on $\mathbb{L}\text{in}$.

The two-layer kinematics which is behind the theory of finite elasticity with distortions is basically reviewed from [14]; in particular, a detailed discussion on the elastic metric which naturally appears within that format is here given.

2.1. Placement

Given a body manifold $\mathcal{B}$, we define *placement* of the body a smooth embedding of $\mathcal{B}$ onto $\mathcal{E}$, described by the field

$$p:\mathcal{B}\to\mathcal{E} \tag{2.1}$$

$$y\mapsto x=p(y)=y+\mathbf{u}(y), \tag{2.2}$$

associating to any material point $y\in\mathcal{B}$, its position $x=p(y)\in\mathcal{E}$ in space. The vector-valued field $\mathbf{u}$ is the displacement field, and the set $p(\mathcal{B})$ describes an actual configuration of $\mathcal{B}$. The tangent space $T_y(\mathcal{B})$ to $\mathcal{B}$ at y is called *body element* at y; given a point y and a placement p, the corresponding *gradient* $\mathbf{F}$, *cofactor* $\mathbf{F}^*$, and *Jacobian determinant* J, defined as

$$\mathbf{F}:=\nabla p=\mathbf{I}+\nabla\mathbf{u},\quad \mathbf{F}^*:=J\mathbf{F}^{-\top},\quad J:=\det(\mathbf{F}), \tag{2.3}$$

perform key geometrical functions on the body element at y. Precisely, given $\mathbf{a},\mathbf{b},\mathbf{c}\in T_y(\mathcal{B})$ and built the hierarchy of (infinitesimally) small one-, two-, and three-dimensional parallelepipedal cells corresponding to

(i) a *line element*, gauged by the vector $\mathbf{a}$,
(ii) an area element (*facet*) $(\mathbf{a},\mathbf{b})$, gauged by its Gibbs representative $\mathbf{a}\times\mathbf{b}$,
(iii) a *volume element* $(\mathbf{a},\mathbf{b},\mathbf{c})$, gauged by its (oriented) volume $\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}$,

the images of the line, area, and volume element under the action of p are gauged respectively by

(i) $\mathbf{F}(y)\mathbf{a}$;
(ii) $\mathbf{F}^*(y)(\mathbf{a}\times\mathbf{b})=(\mathbf{F}(y)\mathbf{a})\times(\mathbf{F}(y)\mathbf{b})$;
(iii) $J(y)(\mathbf{a}\times\mathbf{b}\cdot\mathbf{c})=(\mathbf{F}(y)\mathbf{a})\times(\mathbf{F}(y)\mathbf{b})\cdot(\mathbf{F}(y)\mathbf{c})$;

and are attached to $x=p(y)$.

2.2. Distortions

Distortions are described by a smooth tensor-valued field

$$\mathbf{F}_o:\mathcal{B}\to\mathbb{L}\text{in}, \tag{2.4}$$

with positive Jacobian determinant $J_o:=\det\mathbf{F}_o>0$; they act on the body element at y, as $\mathbf{F}$ and its products do. However, while the notion of deformation gradient only involves kinematics, distortions have a two-fold nature: a kinematical nature, as they add further information on the kinematics of $T_y(\mathcal{B})$ which is independent on the placement of y (precisely, we have 9 further degrees of freedom); a dynamical nature, as they describe a *relaxed state*, *i.e.* a zero-stress, of the body element. We quote verbatim from [14]: "$\mathbf{F}_o$ cannot even be conceived without the standard notion of stress and some constitutive information on it." Thus, the use of the term "relaxed state", that here anticipates the specification of any free energy, calls attention to the fact the distortions do not alter the value of the free-energy density of body elements.

It is of the essence to emphasize the fact that distortions are not required to be compatible, that is, they are not required to be the gradient of any diffeomorphism from $\mathcal{B}$; as a consequence, a zero-stress state may not be realizable, not even locally. If such is the case, any actual configuration of the body will be accompanied by a change of its free energy (see Fig. 1).

We discuss the transformations of a body element due to the pair $(\mathbf{F},\mathbf{F}_o)$ starting from a unit material fiber $(y,\mathbf{e})\in\mathcal{B}\times\mathcal{V}\mathcal{E}$ at y ($|\mathbf{e}|=\sqrt{\mathbf{e}\cdot\mathbf{e}}=1$), and considering (see Fig. 2)

$$\begin{aligned}(y,\mathbf{e}_o)&=(y,\mathbf{F}_o(y)\mathbf{e}), &&\text{the distorted fiber at } y,\\ (x,\mathbf{f})&=(x,\mathbf{F}(y)\mathbf{e}), &&\text{the actual fiber at } x=p(y).\end{aligned} \tag{2.5}$$

The last line prompts the introduction of the notion of *elastic deformation* $\mathbf{F}_e$ as the difference between the distortion $\mathbf{F}_o$ and the *visible* deformation $\mathbf{F}$ in the sense of the multiplicative composition:

$$\mathbf{F}_e=\mathbf{F}\mathbf{F}_o^{-1}, \tag{2.6}$$

whose introduction dates back to [9,10]. The elastic deformation $\mathbf{F}_e$ maps distorted fibers $\mathbf{e}_o$ onto their actual state $\mathbf{f}$; as $\mathbf{F}_o$, also $\mathbf{F}_e$ is not, in general, the gradient of any field. The notion of fiber leads naturally into the notion of stretch [31]. Firstly, we define the change in length of a fiber as the length of its image under a deformation, minus its original length, divided by its length, *i.e.* given the fiber $\mathbf{e}$ and the deformation $\mathbf{F}$, we get

$$\text{change in length of } \mathbf{e} \text{ under } \mathbf{F}=\frac{|\mathbf{F}\mathbf{e}|-|\mathbf{e}|}{|\mathbf{e}|}=\frac{|\mathbf{F}\mathbf{e}|}{|\mathbf{e}|}-1;$$

then, denote the ratio $|\mathbf{F}\mathbf{e}|/|\mathbf{e}|$ as the *stretch* of the fiber $\mathbf{e}$ under $\mathbf{F}$. At the same point $y\in\mathcal{B}$, there are different stretches of different fibers, and we have

$$\lambda(\mathbf{e})=\frac{|\mathbf{f}|}{|\mathbf{e}|}=\left|\mathbf{F}\mathbf{e}\right|=(\mathbf{F}^\top\mathbf{F}\cdot\mathbf{e}\otimes\mathbf{e})^{1/2},\ \text{stretch of } \mathbf{e} \text{ under } \mathbf{F},$$

$$\lambda_o(\mathbf{e})=\frac{|\mathbf{e}_o|}{|\mathbf{e}|}=\left|\mathbf{F}_o\mathbf{e}\right|=(\mathbf{F}_o^\top\mathbf{F}_o\cdot\mathbf{e}\otimes\mathbf{e})^{1/2},\ \text{stretch of } \mathbf{e} \text{ under } \mathbf{F}_o,$$

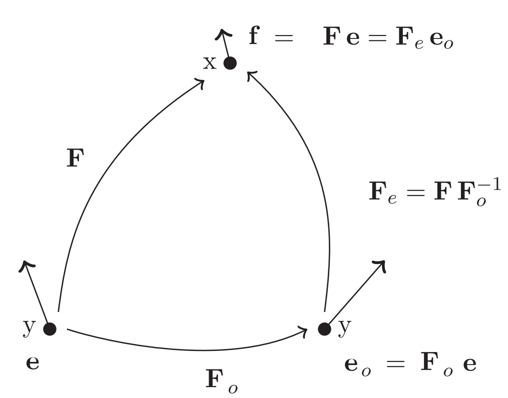

Fig. 2. This triangular diagram is based on the exploitation of the unit fiber $\mathbf{e}$ at y into its material $(y,\mathbf{e})$ and distorted image $(y,\mathbf{e}_o)$; the corresponding actual image $(x,\mathbf{f})$ of the same fiber realizes the third vertex of the triangle.

$$\lambda_e(\mathbf{e}_o) = \frac{|\mathbf{f}|}{|\mathbf{e}_o|} = \frac{|\mathbf{F}\mathbf{e}|}{|\mathbf{F}_o\mathbf{e}|} = \lambda(\mathbf{e})\lambda_o(\mathbf{e})^{-1}, \text{ elastic stretch of } \mathbf{e}_o. \tag{2.7}$$

The elastic stretch defined in $(2.7)_3$ measures the stretching of $\mathbf{f}$ with respect to $\mathbf{e}_o$; the formula $\lambda_e(\mathbf{e}_o) = \lambda(\mathbf{e})\lambda_o^{-1}(\mathbf{e})$ is a difference between the two stretches λ and λ_o in the sense of the multiplicative composition, exactly as (2.6) is.

2.3. Strain measures

As stated in [31], the use of $\mathbf{C} = \mathbf{F}^{\top}\mathbf{F}$ as strain measure in finite elasticity is a straightforward consequence of $(2.7)_1$; inspired by this, we rewrite equation $(2.7)_3$ as follows:

$$\lambda_e(\mathbf{e}_o) = \frac{|\mathbf{f}|}{|\mathbf{e}_o|} = \frac{|\mathbf{F}_e\mathbf{e}_o|}{|\mathbf{e}_o|} = \left(\mathbf{F}_e^{\top}\mathbf{F}_e \cdot \frac{\mathbf{e}_o}{|\mathbf{e}_o|} \otimes \frac{\mathbf{e}_o}{|\mathbf{e}_o|}\right)^{1/2} \tag{2.8}$$

and say that $\mathbf{C}_e = \mathbf{F}_e^{\top}\mathbf{F}_e$ is the local and exact strain measure to be used in the theory of finite elasticity with distortions. Indeed, the components of $\mathbf{C}_e$ in the (unit) direction $\mathbf{e}_o/|\mathbf{e}_o|$ yields the (square of the) actual length of a fiber which, once distorted, has that direction. In the end, we have the following different metric tensors to be used as (left Cauchy–Green) strain measures:

$$\begin{aligned} &\mathbf{C}_o = \mathbf{F}_o^{\top}\mathbf{F}_o, \quad \text{metric induced by the distortion } \mathbf{F}_o,\\ &\mathbf{C} = \mathbf{F}^{\top}\mathbf{F}, \quad \text{metric induced by the placement } \mathbf{F},\\ &\mathbf{C}_e = \mathbf{F}_e^{\top}\mathbf{F}_e \quad \text{elastic metric.} \end{aligned} \tag{2.9}$$

Due to the requirements made upon p and $\mathbf{F}_o$, all the three tensor fields in (2.9) are positive definite; it means that the body manifold $\mathcal{B}$ is naturally equipped with four different metric tensors. The trivial metric $\mathbf{I}$ on $\mathcal{B}$ corresponding to the trivial embedding $p(y) = y$ of $\mathcal{B}$ into $\mathcal{E}$, which implies that $\mathbf{F} = \mathbf{I}$, and the actual metric $\mathbf{C}$ induced by the placement p, are standard in the finite elasticity theory; $\mathbf{C}_o$ is the *intrinsic metric*[1] of $\mathcal{B}$ induced by the distortions $\mathbf{F}_o$. In general, $\mathbf{C}_o$ is not Euclidean, that is, flat Riemannian, but Riemannian in the general sense, a feature strictly related to the fact that the field $\mathbf{F}_o$ is not required to be compatible; as a consequence, the body manifold $\mathcal{B}$, endowed with such a metric, cannot be embedded in the 3D Euclidean space.

In the end, the elastic metric $\mathbf{C}_e$ gauges the elastic strain from the strain $\mathbf{C}$ and the distortion $\mathbf{F}_o$ of body elements; in particular, we can rewrite $(2.9)_3$ as follows:

$$\mathbf{C}_e = \mathbf{F}_o^{-\top}\mathbf{C}\mathbf{F}_o^{-1}, \tag{2.10}$$

we can define additional strain measures, as the Green–Saint Venant strain $\mathbf{E}_e$

$$\mathbf{E}_e = \tfrac{1}{2}(\mathbf{C}_e - \mathbf{I}). \tag{2.11}$$

It is worth noting that the strain $\mathbf{E}_e$ can be rewritten as

$$\mathbf{E}_e = \mathbf{F}_o^{-\top}(\mathbf{E}-\mathbf{E}_o)\mathbf{F}_o^{-1} = \mathbf{F}_o^{-\top}\tfrac{1}{2}(\mathbf{C}-\mathbf{C}_o)\mathbf{F}_o^{-1}, \tag{2.12}$$

being $\mathbf{E} = \frac{1}{2}(\mathbf{C}-\mathbf{I})$, $\mathbf{E}_o = \frac{1}{2}(\mathbf{C}_o-\mathbf{I})$. Different proposal may be made to measure the elastic strain, as example $\frac{1}{2}(\mathbf{C}-\mathbf{C}_o)$, see [32]. Therein, $\mathbf{C}$ is called the actual metric, and $\mathbf{C}_o$ the reference, or target, metric (see also [3]); their difference, which is not a metric tensor, not being positive-definite, is called the deviation of the actual metric from the reference metric. It is easy to verify that the Green–Saint Venant strain $\mathbf{E}_e$ and this last strain measure $\frac{1}{2}(\mathbf{C}-\mathbf{C}_o)$ satisfy (see Appendix)

$$\mathbf{E}_e\mathbf{e}_o \cdot \mathbf{e}_o = \tfrac{1}{2}(\mathbf{C}-\mathbf{C}_o)\mathbf{e} \cdot \mathbf{e}. \tag{2.13}$$

[1] We borrow this name from [27]; therein, the notion of distortions, dubbed *local configurations*, is described with $\mathbf{K} = \mathbf{F}_o^{-1}$, while the intrinsic metric is given in terms of the inverse of the left Cauchy–Green strain $\mathbf{B}^{-1} = \mathbf{K}^{-\top}\mathbf{K}^{-1}$, to be identified with our $\mathbf{C}_o$.

3. Constitutive issue

There are two main constitutive assumptions behind the theory of finite elasticity with distortions, which are discussed in the next two subsections. The first deals with the frame-indifference requirements, which have to be re-formulated to account for the presence of the additional kinematical descriptor $\mathbf{F}_o$; the second involves the representation of the free-energy density when distortions are present.

3.1. Frame indifference

The requirement that the material response be invariant under a change of observer stands among the main axioms of mechanics. Following [14], it is assumed that a change of observer defined by $\mathbf{Q} \in \mathbb{R}\mathrm{ot}$, being $\mathbb{R}\mathrm{ot}$ the group of proper orthogonal transformations of $\mathcal{V}\mathcal{E}$, transforms the pair $(\mathbf{F}, \mathbf{F}_o)$ as follows:

$$(\mathbf{F}, \mathbf{F}_o) \mapsto (\mathbf{Q}\mathbf{F}, \mathbf{F}_o). \tag{3.1}$$

It follows from (3.1), that a change of observer transforms the elastic deformation $\mathbf{F}_e$ as $\mathbf{F}_e \mapsto \mathbf{Q}\mathbf{F}_e$ and the metrics as

$$\begin{aligned} &\mathbf{C} = \mathbf{F}^{\top}\mathbf{F} \mapsto \mathbf{F}^{\top}\mathbf{Q}^{\top}\mathbf{Q}\mathbf{F} = \mathbf{C},\\ &\mathbf{C}_o = \mathbf{F}_o^{\top}\mathbf{F}_o \mapsto \mathbf{F}_o^{\top}\mathbf{F}_o = \mathbf{C}_o,\\ &\mathbf{C}_e = \mathbf{F}_o^{-\top}\mathbf{C}\mathbf{F}_o^{-1} \mapsto \mathbf{C}_e. \end{aligned} \tag{3.2}$$

Likewise, given (2.11), $\mathbf{E}_e \mapsto \mathbf{E}_e$. It follows that any free-energy density having as argument the elastic metric $\mathbf{C}_e$, or the strain measure $\mathbf{E}_e$, is frame invariant, that is, attains the same value under any change of observer; invariance of the energy, in turns, implies invariance and symmetry of the (elastic part of the) actual stress [33].

3.2. Hyperelastic response

It is assumed that the free-energy density ψ_o per unit relaxed volume $dV_o = J_o\, dV$, being dV the volume element on $\mathcal{B}$, is a single-valued scalar function of the elastic deformation $\mathbf{F}_e$

$$\psi_o(y) = \psi_o(\mathbf{F}_e(y)). \tag{3.3}$$

Given ψ_o, and granted for (2.6), the strain energy density per unit reference volume ψ can be represented as a function of $\mathbf{F} = \mathbf{F}_e\mathbf{F}_o$, defined by

$$\psi(y) = J_o(y)\psi_o(\mathbf{F}_e(y)). \tag{3.4}$$

The standard tools of continuum mechanics deliver the following stress measures (see Fig. 3):

$$\begin{aligned} &\mathbf{S}_{oe} = \frac{\partial \psi_o}{\partial \mathbf{F}_e}, \quad \text{energetic stress per unit relaxed volume,}\\ &\mathbf{S}_e = \mathbf{S}_{oe}\mathbf{F}_o^{*}, \quad \text{reference stress (a.k.a. Piola–Kirchhoff stress) i.e. the pull back of } \mathbf{S}_{oe},\\ &\mathbf{T}_e = \mathbf{S}_e(\mathbf{F}^{*})^{-1}, \quad \text{actual stress (a.k.a. Cauchy stress) i.e. the push forward of } \mathbf{S}_e. \end{aligned} \tag{3.5}$$

The subscripts "e" point to the fact that we are only dealing with the energetic component of the stress; possible internal constraint would involve extra stress-components.

Granted for (3.3), we introduce the frame invariant free-energy density ϕ_o as

$$\phi_o(y) = \phi_o(\mathbf{C}_e(y)) = \psi_o(\mathbf{F}_e(y)). \tag{3.6}$$

In terms of ϕ_o, the energetic stress $\mathbf{S}_{oe}$ per unit relaxed volume admits the following representation:

$$\mathbf{S}_{oe} = 2\mathbf{F}_e\frac{\partial \phi_o}{\partial \mathbf{C}_e} = \mathbf{F}_e\frac{\partial \phi_o}{\partial \mathbf{E}_e}. \tag{3.7}$$

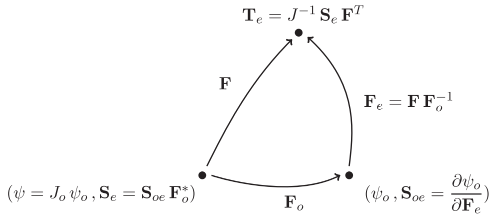

Fig. 3. Each energy density and stress measure are placed in the triangular diagram; hence, the stress measures related through pull back and push forward operations are highlighted.

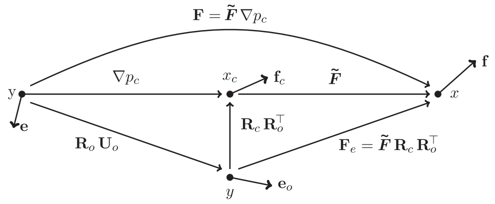

Fig. 4. Given a compatible distortion $\mathbf{F}_o = \mathbf{R}_o\mathbf{U}_o$, a unique relaxed configuration $p_c(\mathcal{B})$ is realized. The fiber $\mathbf{f}_c$ at $x_c = p_c(y)$ is the image of $\mathbf{e}$ at y under ∇p_c; however, it can be also viewed as the image of $\mathbf{e}_o$ at y under the rotation $\mathbf{R}_c(y)\mathbf{R}_o^\top(y)$. A further deformation $\tilde{\mathbf{F}}$ from $p_c(\mathcal{B})$ yields an elastic deformation $\mathbf{F}_e = \mathbf{F}\mathbf{F}_o^{-1}$ which depends on $\mathbf{R}_o$, too: $\mathbf{F}_e = \tilde{\mathbf{F}}\ \mathbf{R}_c\ \mathbf{R}_o^\top$.

The derivative of the energy with respect to $\mathbf{E}_e$ is usually called second Piola–Kirchhoff stress

$$\frac{\partial\phi_o}{\partial\mathbf{E}_e} = \mathbf{F}_e^{-1}\mathbf{S}_{oe}. \tag{3.8}$$

The stress $\mathbf{T}_e$ is then given by

$$\mathbf{T}_e = \frac{1}{J}\mathbf{S}_e\mathbf{F}^\top = \frac{1}{J}\mathbf{S}_{oe}\mathbf{F}_o^*\mathbf{F}^\top = \frac{1}{J}\mathbf{F}_e\frac{\partial\phi_o}{\partial\mathbf{E}_e}\mathbf{F}_o^*\mathbf{F}^\top = \frac{1}{J_e}\mathbf{F}_e\frac{\partial\phi_o}{\partial\mathbf{E}_e}\mathbf{F}_e^\top. \tag{3.9}$$

4. Compatibility

A key issue in the theory of elastic materials with large distortions is the existence of relaxed configurations corresponding to an assigned field $\mathbf{F}_o$. These are special solutions of the elastic problem as they correspond to a minimum of energy and to zero stress [25,26].

4.1. Compatible distortions

Given a smooth, positive-definite symmetric tensor field $\mathbf{G}$ on a simply connected domain $\mathcal{B}$, the necessary and sufficient condition for $\mathbf{G}$ to be a *metric tensor* of a realizable configuration, *i.e.* for the existence of a placement $p:\mathcal{B}\rightarrow\mathcal{E}$ such that

$$\nabla p^\top\nabla p = \mathbf{G} \tag{4.1}$$

is that the associated Riemann curvature tensor $\mathbb{R} = \mathbb{R}(\mathbf{G})$ be null. If it is the case, $\mathbf{G}$ is an Euclidean metric tensor, and the placement p satisfying (4.1) is *unique*, up to a global isometry [28]. Based on these results, we define a smooth distortion field $\mathbf{F}_o$ as *compatible*, if there exist a placement p_c such that

$$\nabla p_c^\top\nabla p_c = \mathbf{C}_o, \quad \text{with } \mathbf{C}_o = \mathbf{F}_o^\top\mathbf{F}_o. \tag{4.2}$$

In this case, $\mathbf{C}_o$ is an Euclidean metric tensor, and an unique configuration $p_c(\mathcal{B})$ is realizable, provided p_c satisfies any possible boundary condition; more important, as shown in the next, $p_c(\mathcal{B})$ is relaxed, *i.e.* stress free.[2] The multiplicative decomposition of ∇p_c and $\mathbf{F}_o$ into symmetric and orthogonal components through the polar decomposition theorem reveals that

$$\nabla p_c = \mathbf{R}_c\mathbf{U}_c \quad \text{and} \quad \mathbf{F}_o = \mathbf{R}_o\mathbf{U}_o. \tag{4.3}$$

Hence, the vanishing of the Riemann curvature associated to $\mathbf{C}_o$ grants for the existence of an unique placement p_c such that only the symmetric (positive definite) components of ∇p_c and $\mathbf{F}_o$ are equals

$$\mathbb{R}(\mathbf{C}_o) = 0 \quad \Leftrightarrow \quad (\nabla p^\top\nabla p_c)^{1/2} = \mathbf{U}_c = \mathbf{U}_o = (\mathbf{F}_o^T\mathbf{F}_o)^{1/2}. \tag{4.4}$$

Thus, the compatibility of $\mathbf{F}_o$ has two straightforward consequences, below highlighted (see Fig. 4).

- The elastic deformation $\mathbf{F}_e$ is a rotation

  $$\mathbb{R}(\mathbf{C}_o) = 0 \quad \Rightarrow \quad \mathbf{F}_e = \nabla p_c\mathbf{F}_o^{-1} = \mathbf{R}_c\mathbf{U}_c\mathbf{U}_o^{-1}\mathbf{R}_o^\top = \mathbf{R}_c\mathbf{R}_o^\top \in \mathbb{R}\mathrm{ot}; \tag{4.5}$$

  thus, the elastic metric is the identity and the elastic stress is identically null

  $$\mathbf{F}_e = \mathbf{R}_c\mathbf{R}_o^\top \in \mathbb{R}\mathrm{ot} \quad \Rightarrow \quad \mathbf{C}_e = \mathbf{F}_e^\top\mathbf{F}_e = \mathbf{I} \quad \Rightarrow \quad \frac{\partial\psi_o(\mathbf{I})}{\partial\mathbf{F}_e} = 0. \tag{4.6}$$

- The set of all the compatible distortions sharing a same metric $\mathbf{C}_o$ constitutes an equivalence class [14,27]: if $\mathbf{F}_o$ is compatible, then, for any $\mathbf{Q}\in\mathbb{R}\mathrm{ot}$, also $\mathbf{Q}\mathbf{F}_o$ is compatible, as both yield the same intrinsic metric $\mathbf{C}_o$.

In the end, given a compatible distortion field $\mathbf{F}_o$, only its symmetric component $\mathbf{U}_o$ determines the unique placement p_c satisfying (4.2) [34]. Nevertheless, both its components have a fundamental role: the stretch $\mathbf{U}_o$ provides the unique relaxed configuration, whereas the rotation $\mathbf{R}_o$ determines its material response. The equation $\mathbb{R}(\mathbf{C}_o) = 0$ yields a tool to characterize a compatible fields $\mathbf{F}_o$. We do not write explicitly the formula relating a metric tensor to its Riemann curvature; this formula, that can be easily found on many differential geometry books (see [28,35] and references therein), does not add any further comprehension to our discussions. We only note here that the Riemann curvature $\mathbb{R}$ has six strict components in a three-dimensional space: the equation $\mathbb{R}(\mathbf{C}_o) = 0$ consists of a system of six partial differential equations, involving the six strict components of $\mathbf{U}_o$. We note that although the six equations are linearly independent, they are not differentially independent, being constrained by the three Bianchis identities. This fact agree with the heuristic consideration that the six differential conditions identify, in the manifold of the six unknown scalar fields, a sub-manifold parametrized only by the three components of the placement p_c [36].

5. Material response

As largely discussed in the previous sections, the *local state* of a body element at $y\in\mathcal{B}$ is known when the pair $(\mathbf{F}(y), \mathbf{F}_o(y))$ of tensors describing its visible and relaxed state, respectively, is given. At any point, given the energy function ϕ_o depending on $\mathbf{C}_e$, the equivalence class of local states to which there correspond the same value of the energy defines the material symmetries of the local response at that point. Precisely, fixed a point $y\in\mathcal{B}$ and considered a transformation of the local state $(\mathbf{F}, \mathbf{F}_o)$ such that

$$(\mathbf{F}, \mathbf{F}_o) \mapsto (\hat{\mathbf{F}}, \hat{\mathbf{F}}_o), \tag{5.1}$$

it holds $\mathbf{F}_e = \mathbf{F}\mathbf{F}_o^{-1}$ and $\hat{\mathbf{F}}_e = \hat{\mathbf{F}}\hat{\mathbf{F}}_o^{-1}$; then, if

$$\phi_o(\mathbf{C}_e) = \phi_o(\hat{\mathbf{C}}_e) \quad \text{with } \mathbf{C}_e = \mathbf{F}_e^T\mathbf{F}_e \quad \text{and} \quad \hat{\mathbf{C}}_e = \hat{\mathbf{F}}_e^\top\hat{\mathbf{F}}_e, \tag{5.2}$$

the transformation defined by the map (5.1) is called a material symmetry of the body element. We first discuss the material symmetry with reference to the local state $(\mathbf{F}, \mathbf{I})$, corresponding to

[2] In the following, we shall mention the uniqueness of p_c without the further specification "up to a global isometry"

a trivially relaxed body element ($\mathbf{F}_o=\mathbf{I}$); hence, given $\mathbf{Q}\in\mathbb{O}\mathrm{rth}$, being $\mathbb{O}\mathrm{rth}$ the group of orthogonal transformations of $\mathcal{VE}$, consider the transformation

$$(\mathbf{F},\mathbf{I})\mapsto(\mathbf{FQ},\mathbf{I}), \tag{5.3}$$

to whom it corresponds $\mathbf{F}_e=\mathbf{F}$ and $\hat{\mathbf{F}}_e=\mathbf{FQ}$. Moreover, being $\mathbf{F}_o\equiv\mathbf{I}$, it holds $\mathbf{C}_e=\mathbf{C}$ and $\hat{\mathbf{C}}_e=\mathbf{Q}^\top\mathbf{CQ}$. With this, we define the *group of material symmetry* $\mathcal{G}_{(y,\mathbf{I})}$ of ϕ_o, relative to the body element at y relaxed by $\mathbf{F}_o=\mathbf{I}$, as the group of orthogonal tensors $\mathbf{Q}$ such that

$$\phi_o(\mathbf{C})=\phi_o(\mathbf{Q}^\top\mathbf{CQ}),\quad \forall\mathbf{C}\in\mathbb{S}\mathrm{ym}^+. \tag{5.4}$$

When we consider a body element y relaxed by a generic $\mathbf{F}_o$, natural questions arise about the corresponding group $\mathcal{G}_{(y,\mathbf{F}_o)}$ of material symmetry relative to the body element at y relaxed by $\mathbf{F}_o$. The next two subsections present a first discussion on this subject.

5.1. Symmetry-preserving distortions

An approach largely followed in literature [3,4,20,14,27,37–39] is based on the assumption that all the symmetries of a local state $(\mathbf{F},\mathbf{I})$ are preserved once the body element has been relaxed by $\mathbf{F}_o\neq\mathbf{I}$, or, equivalently, $\mathcal{G}_{(y,\mathbf{F}_o)}=\mathcal{G}_{(y,\mathbf{I})}$. From the point of view of the transformations (5.1), this assumption implies that the action of a material symmetry $\mathbf{Q}$ on the local state $(\mathbf{F},\mathbf{F}_o)$ is such that

$$(\mathbf{F},\mathbf{F}_o)\mapsto(\mathbf{FQ},\mathbf{Q}^\top\mathbf{F}_o\mathbf{Q}); \tag{5.5}$$

with this, it holds $\mathbf{F}_e=\mathbf{F}\mathbf{F}_o^{-1}$ and $\hat{\mathbf{F}}_e=\mathbf{F}_e\mathbf{Q}$. As $\hat{\mathbf{C}}_e=\mathbf{Q}^\top\mathbf{C}_e\mathbf{Q}$, the material symmetry group $\mathcal{G}_{(y,\mathbf{F}_o)}$ is defined as the set of $\mathbf{Q}\in\mathbb{O}\mathrm{rth}$ such that

$$\phi_o(\mathbf{C}_e)=\phi_o(\mathbf{Q}^\top\mathbf{C}_e\mathbf{Q}),\quad \forall\,\mathbf{C}_e\in\mathbb{S}\mathrm{ym}^+, \tag{5.6}$$

i.e. comparing Eqs. (5.5) and (5.6), what turns out is that the function ϕ_o has the same symmetries when evaluated with respect to a trivially relaxed state or not.

5.2. Symmetry-varying distortions

A different approach is possible, based on the assumption that any transformation of the local state does not affect the relaxed state, *i.e.* the transformation (5.1) takes the form

$$(\mathbf{F},\mathbf{F}_o)\mapsto(\mathbf{FQ},\mathbf{F}_o); \tag{5.7}$$

with this, it holds $\mathbf{F}_e=\mathbf{F}\mathbf{F}_o^{-1}$ and $\hat{\mathbf{F}}_e=\mathbf{F}_e\hat{\mathbf{Q}}$. Hence, as

$$\hat{\mathbf{C}}_e=\hat{\mathbf{Q}}^\top\,\mathbf{C}_e\,\hat{\mathbf{Q}},\quad \hat{\mathbf{Q}}=\mathbf{F}_o\mathbf{Q}\mathbf{F}_o^{-1}, \tag{5.8}$$

the *group of material symmetry* $\hat{\mathcal{G}}_{(y,\mathbf{F}_o)}$ of ϕ_o at y is defined as the group of linear transformations $\hat{\mathbf{Q}}=\mathbf{F}_o\,\mathbf{Q}\,\mathbf{F}_o^{-1}$, with $\mathbf{Q}\in\mathcal{G}_{(y,\mathbf{I})}$, such that

$$\phi_o(\mathbf{C}_e)=\phi_o(\hat{\mathbf{Q}}^\top\mathbf{C}_e\hat{\mathbf{Q}}),\quad \forall\,\mathbf{C}_e\in\mathbb{S}\mathrm{ym}^+. \tag{5.9}$$

From here, a special representation problem should be solved, whose solution depends on $\mathbf{F}_o$. It is beyond the aims of this paper. We only note that spherical distortions such as $\mathbf{F}_o=g\mathbf{I}$ do not vary the symmetry of the material response relative to $(y,\mathbf{I})$, whichever be $\mathcal{G}_{(y,\mathbf{I})}$; moreover, if $\mathcal{G}_{(y,\mathbf{I})}=\mathbb{R}\mathrm{ot}$ and $\mathbf{F}_o\in\mathbb{R}\mathrm{ot}$, then the material symmetry is still unchanged.

6. Discussion

We discuss a few examples of elastic constitutive prescriptions based upon the assumption of symmetry-preserving distortions, for which (5.4) holds, and close the section with an example dealing with the response of relaxed configurations realized through compatible distortions. We consider two well-known material responses: isotropic and transversely isotropic.

6.1. Isotropic material response

The material response is isotropic when $\mathcal{G}_{(y,\mathbf{I})}=\mathcal{G}_{(y,\mathbf{F}_o)}=\mathbb{R}\mathrm{ot}$; the representation theorem for scalar-valued isotropic functions [40] dictates that the elastic energy ϕ_o is a function of the three principal invariants (I_1,I_2,I_3) of $\mathbf{C}_e$

$$I_1(\mathbf{C}_e)=\mathbf{C}_e\cdot\mathbf{I},\quad I_2(\mathbf{C}_e)=\tfrac{1}{2}[(\mathbf{C}_e\cdot\mathbf{I})^2-\mathbf{C}_e^2\cdot\mathbf{I}],\quad I_3(\mathbf{C}_e)=\det(\mathbf{C}_e). \tag{6.1}$$

Being $\mathbf{C}_e=\mathbf{F}_o^{-T}\mathbf{C}\mathbf{F}_o^{-1}$, it holds

$$I_1(\mathbf{C}_e)=\mathbf{C}\cdot\mathbf{C}_o^{-1},\quad I_2(\mathbf{C}_e)=\frac{1}{2}[(\mathbf{C}\cdot\mathbf{C}_o^{-1})^2-\mathbf{C}_o\mathbf{C}\cdot\mathbf{C}\mathbf{C}_o],\quad I_3(\mathbf{C}_e)=\left(\frac{J}{J_o}\right)^2=J_e^2; \tag{6.2}$$

hence, the orthogonal component $\mathbf{R}_o$ of $\mathbf{F}_o$ does not affect the isotropic material response, that is, isotropic materials do not distinguish within the equivalence class of the distortions sharing the same intrinsic metric $\mathbf{C}_o$. Indeed, for any $\mathbf{Q}\in\mathbb{R}\mathrm{ot}$, given the elastic metric $\mathbf{C}_e=\mathbf{F}_e^T\mathbf{F}_e$ with $\mathbf{F}_e=\mathbf{F}\mathbf{F}_o^{-1}$, the elastic metric corresponding to $\mathbf{F}(\mathbf{Q}\mathbf{F}_o)^{-1}$ is $\mathbf{Q}\mathbf{C}_e\mathbf{Q}^T$: $\mathbf{C}_e$ and $\mathbf{Q}\mathbf{C}_e\mathbf{Q}^T$ share the same three principal invariants. A well-known example is the NH energy, defined for elastically incompressible materials as

$$\phi_{oNH}=\tfrac{1}{2}\mu(\mathbf{C}_e\cdot\mathbf{I}-3)=\tfrac{1}{2}\mu(\mathbf{C}\cdot\mathbf{C}_o^{-1}-3),\quad J_e=1. \tag{6.3}$$

6.2. Transversely isotropic material response

The material response is *transversely isotropic* with respect to the direction $\mathbf{a}$ when $\mathcal{G}_{(y,\mathbf{I})}=\mathcal{G}_{(y,\mathbf{F}_o)}=\{\mathbb{R}\mathrm{ot}(\mathbf{a}),\mathbb{R}\mathrm{ef}(\mathbf{a})\}$, where $\mathbb{R}\mathrm{ot}(\mathbf{a})$ is the subgroup of $\mathbb{R}\mathrm{ot}$ whose elements are the rotations of axis $\mathbf{a}$, and $\mathbb{R}\mathrm{ef}(\mathbf{a})=\mathbf{I}-2\mathbf{a}/|\mathbf{a}|\otimes\mathbf{a}/|\mathbf{a}|$ are the reflections in any plane containing $\mathbf{a}$; in such a case, ϕ_o is a function of five invariants of $\mathbf{C}_e$, namely, the three listed in (6.2), plus two additional ones [41]

$$I_4(\mathbf{C}_e,\mathbf{a})=\mathbf{C}_e\mathbf{a}\cdot\mathbf{a},\quad I_5(\mathbf{C}_e,\mathbf{a})=\mathbf{C}_e\mathbf{a}\cdot\mathbf{C}_e\mathbf{a}. \tag{6.4}$$

The request (5.6) can be easily verified checking that: for any $\mathbf{Q}\in\mathbb{R}\mathrm{ot}(\mathbf{a})$,

$$I_4(\mathbf{Q}^\top\mathbf{C}_e\mathbf{Q},\mathbf{a})=\mathbf{Q}^\top\mathbf{C}_e\mathbf{Q}\mathbf{a}\cdot\mathbf{a}=\mathbf{C}_e\mathbf{Q}\mathbf{a}\cdot\mathbf{Q}\mathbf{a}=\mathbf{C}_e\mathbf{a}\cdot\mathbf{a}=I_4(\mathbf{C}_e,\mathbf{a}); \tag{6.5}$$

and $I_5(\mathbf{Q}^\top\mathbf{C}_e\mathbf{Q},\mathbf{a})=I_5(\mathbf{C}_e,\mathbf{a})$. Importantly, a transversely isotropic response does depend on $\mathbf{R}_o$ as, with reference to the I_4-example,

$$I_4(\mathbf{C}_e,\mathbf{a})=\mathbf{C}_e\mathbf{a}\cdot\mathbf{a}=\mathbf{U}_o^{-1}\,\mathbf{C}\,\mathbf{U}_o^{-1}\cdot\mathbf{R}_o^\top(\mathbf{a}\otimes\mathbf{a})\mathbf{R}_o; \tag{6.6}$$

hence, distortions sharing the same metric $\mathbf{C}_o$ can be distinguished by the material response. A well known example of transversely isotropic response is obtained by adding two anisotropic terms to a Neo-Hookean energy [42]. A second example, which will be discussed in the following, is the Kirchhoff–Saint Venant (KSV) material response, effective when displacements are large but strains are small. The KSV strain energy is a positive-definite quadratic form of the Green–Saint Venant strain $\mathbf{E}_e$, based on a symmetric, positive-definite, fourth-order tensor $\mathbb{C}$, called the *elasticity tensor*

$$\psi_{oKSV}(\mathbf{E}_e)=\tfrac{1}{2}\mathbb{C}\mathbf{E}_e\cdot\mathbf{E}_e. \tag{6.7}$$

Such an energy yields a linear relation between the second Piola–Kirchhoff stress and strain measure $\mathbf{E}_e$

$$\frac{\partial\phi_o}{\partial\mathbf{E}_e}=\mathbb{C}\mathbf{E}_e. \tag{6.8}$$

From (3.5) and (3.7), it follows:

$$\mathbf{S}_{oe}=\frac{\partial\psi_o}{\partial\mathbf{F}_e}=\mathbf{F}_e\frac{\partial\psi_o}{\partial\mathbf{E}_e}=\mathbf{F}_e\mathbb{C}\mathbf{E}_e,\quad \mathbf{S}_e=\mathbf{S}_{oe}\mathbf{F}_o^*=\mathbf{F}_e(\mathbb{C}\mathbf{E}_e)\mathbf{F}_o^*. \tag{6.9}$$

The material symmetries of the KSV energy are expressed in terms of the properties of the elasticity tensor: $\mathbf{Q}\in\mathbb{O}\mathrm{rth}$ is a material symmetry for $\psi_{oKSV}(\mathbf{E}_e)$ if and only if

$$\mathbb{C}(\mathbf{Q}\mathbf{E}_e\mathbf{Q}^\top) = \mathbf{Q}(\mathbb{C}\mathbf{E}_e)\mathbf{Q}^\top; \tag{6.10}$$

the response is isotropic or transversely isotropic if (6.10) holds with $\mathbf{Q}\in\mathbb{R}\mathrm{ot}$ or $\mathbf{Q}\in\{\mathbb{R}\mathrm{ot}(\mathbf{a}), \mathbb{R}\mathrm{ef}(\mathbf{a})\}$, respectively. The KSV elastic energy (6.7) can also be represented as a quadratic form of the strain measure $(\mathbf{C}-\mathbf{C}_o)/2$, by using the fourth-order tensor[3] $\tilde{\mathbb{C}}$, conjugate of $\mathbb{C}$ with respect to $\mathbf{F}_o^{-1}\boxtimes\mathbf{F}_o^{-1}$

$$\tilde{\mathbb{C}} = (\mathbf{F}_o^{-1}\boxtimes\mathbf{F}_o^{-1})\mathbb{C}(\mathbf{F}_o^{-\top}\boxtimes\mathbf{F}_o^{-\top}). \tag{6.11}$$

It holds

$$\mathbb{C}\mathbf{E}_e\cdot\mathbf{E}_e = \tilde{\mathbb{C}}(\mathbf{C}-\mathbf{C}_o)/2\cdot(\mathbf{C}-\mathbf{C}_o)/2. \tag{6.12}$$

When the material response is isotropic (yet it does not distinguish between distortions sharing the same intrinsic metric), the representation form of the elasticity tensor is

$$\mathbb{C} = \mu\mathbb{I}_s + \lambda\mathbf{I}\otimes\mathbf{I}, \tag{6.13}$$

with μ and λ the Lamé moduli, and $\mathbb{I}_s$ the forth-order tensor projecting $\mathbb{L}\mathrm{in}$ onto $\mathbb{S}\mathrm{ym}$; thus, the isotropic KSV energy writes as

$$\psi_{oKSV} = \mu|\mathbf{E}_e|^2 + \lambda(\mathrm{Tr}(\mathbf{E}_e))^2. \tag{6.14}$$

The isotropic KSV response is largely applied in literature [3,1], typically, by representing the energy as a quadratic form of the strain $(\mathbf{C}-\mathbf{C}_o)/2$; in such a case, (6.15) with (6.13) yields

$$\tilde{\mathbb{C}} = \mu(\mathbf{C}_0^{-1}\boxtimes\mathbf{C}_o^{-1})_s + \lambda\mathbf{C}_o^{-1}\otimes\mathbf{C}_o^{-1}. \tag{6.15}$$

Let us now consider a response transversely isotropic with respect to the direction $\mathbf{a}$; given the orthonormal basis $\{\mathbf{e}_i, i=1,2,3\}$ of $\mathcal{V}\mathcal{E}$, with $\mathbf{e}_3=\mathbf{a}$, we can build upon it an appropriate orthonormal basis of $\mathbb{S}\mathrm{ym}$ to represent the elasticity tensor $\mathbb{C}$ [31]: $\{\mathbf{C}_i, i=1,4;\ \mathbf{D}_\alpha, \alpha=1,2\}$ with $\sqrt{2}\mathbf{C}_\alpha = \mathbf{e}_\alpha\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_\alpha$, $\sqrt{2}\mathbf{C}_3 = \mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1$, $\sqrt{2}\mathbf{C}_4 = \mathbf{e}_1\otimes\mathbf{e}_1-\mathbf{e}_2\otimes\mathbf{e}_2)$, and $\mathbf{D}_1=\mathbf{e}_3\otimes\mathbf{e}_3$, $\sqrt{2}\mathbf{D}_2 = \mathbf{e}_1\otimes\mathbf{e}_1+\mathbf{e}_2\otimes\mathbf{e}_2)$. The transversely isotropic elasticity tensor is represented in terms of five material moduli γ_1, γ_2, and $\delta_1,\delta_2,\delta_3$ as follows:

$$\mathbb{C} = \gamma_1(\mathbf{C}_1\otimes\mathbf{C}_1+\mathbf{C}_2\otimes\mathbf{C}_2)+\gamma_2(\mathbf{C}_3\otimes\mathbf{C}_3+\mathbf{C}_4\otimes\mathbf{C}_4)+\mathbb{D},$$
$$\text{with}\quad \mathbb{D} = \delta_1(\mathbf{D}_1\otimes\mathbf{D}_1)+\delta_2(\mathbf{D}_1\otimes\mathbf{D}_2+\mathbf{D}_2\otimes\mathbf{D}_1)+\delta_3(\mathbf{D}_2\otimes\mathbf{D}_2). \tag{6.16}$$

The transverse isotropic KSV energy writes as

$$\psi_{oKSV} = \gamma_1(E_{e13}^2+E_{e23}^2)+\gamma_2E_{e12}^2+\frac{1}{2}\delta_1E_{e33}^2+\frac{1}{\sqrt{2}}\delta_2E_{e33}(E_{e11}+E_{e22})$$
$$+\frac{1}{2}(\delta_3+\gamma_2)(E_{e11}^2+E_{e22}^2)+(\delta_3-\gamma_2)E_{e11}E_{e22}, \tag{6.17}$$

with $E_{eij} = \mathbf{E}_e\mathbf{e}_i\cdot\mathbf{e}_j$. As done for the isotropic response, we can compute the fourth-order tensor $\tilde{\mathbb{C}}$ conjugate of (6.16); in such a case, each of the summands of $\mathbb{C}$ is transformed as follows:

$$\mathbf{C}_i\otimes\mathbf{C}_i\mapsto\mathbf{F}_o^{-1}\mathbf{C}_i\mathbf{F}_o^{-\top}\otimes\mathbf{F}_o^{-1}\mathbf{C}_i\mathbf{F}_o^{-\top},\quad \mathbf{D}_i\otimes\mathbf{D}_i\mapsto\mathbf{F}_o^{-1}\mathbf{D}_i\mathbf{F}_o^{-\top}\otimes\mathbf{D}_o^{-1}\mathbf{D}_i\mathbf{F}_o^{-\top}, \tag{6.18}$$

6.3. *Finite bending of a block*

We discuss the finite bending of a rectangular parallelepiped, which is initially bended and relaxed, and then elastically stretched. Given the orthonormal frame of $\mathcal{E}$, $\{o\in\mathcal{E}; \mathbf{e}_i\in\mathcal{V}\mathcal{E}, |\mathbf{e}_i\cdot\mathbf{e}_j=\delta_{ij}, i=1,2,3\}$, let us identify the body $\mathcal{B}$ with the region

$$\mathcal{B} = \{\mathcal{E}\ni y = o + y_i\mathbf{e}_i; y_1\in(-w/2,w/2), y_2\in(-l/2,l/2), y_3\in(-h/2,h/2)\}. \tag{6.19}$$

Then, we consider a special distortion field $\mathbf{F}_o(y) = \mathbf{R}_o(y_2)\mathbf{U}_o(y_3)$ with

$$\mathbf{R}_o(y_2) = \cos(\alpha_o(y_2))\bar{\mathbf{I}} + \sin(\alpha_o(y_2))\overline{\mathbf{W}} + \mathbf{e}_1\otimes\mathbf{e}_1,$$
$$\mathbf{U}_o(y_3) = \lambda(y_3)\bar{\mathbf{I}} + \mathbf{e}_1\otimes\mathbf{e}_1, \tag{6.20}$$

being $\bar{\mathbf{I}} = \mathbf{I}-\mathbf{e}_1\otimes\mathbf{e}_1$, and $\overline{\mathbf{W}} = \mathbf{e}_3\wedge\mathbf{e}_2$.[4] Let us note that this special distortion is plane, *i.e.* $\mathbf{F}_o\mathbf{e}_1 = \mathbf{e}_1$, and yields a transversely isotropic intrinsic metric $\mathbf{C}_o$, parameterized by a single scalar field

$$\mathbf{C}_o(y) = \lambda^2(y_3)\bar{\mathbf{I}} + \mathbf{e}_1\otimes\mathbf{e}_1. \tag{6.21}$$

We use the compatibility condition $\mathbb{R}(\mathbf{C}_o)=0$ to characterize a class of functions $y_3\mapsto\lambda(y_3)$, delivering a compatible metric. Being the distortion plane, the only non-vanishing component of the Riemann curvature is the sectional curvature with respect to the plane of unit normal $\mathbf{e}_1$, which identifies the Gaussian curvature $K(\bar{\mathbf{I}}\,\mathbf{C}_o\,\bar{\mathbf{I}})$ of the plane metric $\bar{\mathbf{I}}\,\mathbf{C}_o\,\bar{\mathbf{I}} = \lambda^2\bar{\mathbf{I}}$. Hence, we look for solutions of the equation

$$\mathbb{R}(\mathbf{C}_o) = K(\bar{\mathbf{I}}\mathbf{C}_o\bar{\mathbf{I}}) = 0. \tag{6.22}$$

The Brioschi's formula [28] gives a representation of $K(\bar{\mathbf{I}}\mathbf{C}_o\bar{\mathbf{I}})$ which, used into the equation (6.22), yields the following ODE for the function λ:

$$\frac{\lambda'(y_3)^2-\lambda(y_3)\lambda''(y_3)}{\lambda(y_3)^4} = 0, \tag{6.23}$$

whose solution is

$$\lambda(y_3) = a\exp(by_3), \tag{6.24}$$

with a,b arbitrary constants. It is easy to verify that the unique placement p_c which realizes the distortion field (6.20), made compatible by (6.24), is

$$p_c(y) = y_1\mathbf{e}_1 + \frac{1}{a}\lambda(y_3)(\sin(by_2)\mathbf{e}_2+\cos(by_2)\mathbf{e}_3). \tag{6.25}$$

Then, we can use the polar decomposition of $\nabla p_c = \mathbf{R}_c\mathbf{U}_c$, where $\mathbf{U}_c = \mathbf{U}_o$, to compute the orthogonal part $\mathbf{R}_c\neq\mathbf{R}_o$

$$\mathbf{R}_c(y_2) = \cos\alpha_c(y_2)\bar{\mathbf{I}} + \sin\alpha_c(y_2)\overline{\mathbf{W}}$$
$$+\mathbf{e}_1\otimes\mathbf{e}_1,\quad \alpha_c(y_2) = -by_2. \tag{6.26}$$

In Fig. 5 we sketch the proposed example. At left, we draw a cross-section of $\mathcal{B}$ in the plane $(\mathbf{e}_2,\mathbf{e}_3)$, with a superimposed Cartesian grid; at the bottom we show a cut view of the same grid, deformed by the distortion field (6.20) (depicted here with $\mathbf{R}_o=\mathbf{I}$ for the sake of simplicity): it represents as the body elements would like to stay once relaxed by the selected distortion field, *i.e.* as they would stay once cut away one from each other.

The only way they can be glued together is represented at middle: we have a global bending. It is worth noting that the bending curvature of this relaxed configuration is entirely determined by $\mathbf{U}_o$, and cannot depend on the local orientation of any single piece. Thus, in general, the elastic deformation $\mathbf{F}_e$ is a rotation: $\mathbf{F}_e = \nabla p_c\mathbf{F}_o^{-1} = \mathbf{R}_c\mathbf{R}_o^\top = \mathbf{R}_e$; in our case, is a rotation with axis $\mathbf{e}_1$ and angle $\alpha_e = \alpha_c-\alpha_o$, which admits the following representation:

$$\mathbf{R}_e = \cos(\alpha_c-\alpha_o)\bar{\mathbf{I}} + \sin(\alpha_c-\alpha_o)\overline{\mathbf{W}} + \mathbf{e}_1\otimes\mathbf{e}_1. \tag{6.27}$$

As previously stated, this relative rotation does not determine the global relaxed configuration described by p_c, but affects its material response. This can be experimented by superimposing a further deformation $\tilde{\mathbf{F}}$; indeed, the new elastic deformation measured from the relaxed state determined by $\mathbf{F}_o$, is

$$\tilde{\mathbf{F}}_e = \tilde{\mathbf{F}}\mathbf{R}_e. \tag{6.28}$$

We choose a plane, isochoric, and uniaxial deformation $\tilde{\mathbf{F}}$

$$\tilde{\mathbf{F}} = \tilde{\lambda}\mathbf{e}_2\otimes\mathbf{e}_2 + \tilde{\lambda}^{-1}(\mathbf{e}_3\otimes\mathbf{e}_3) + \mathbf{e}_1\otimes\mathbf{e}_1; \tag{6.29}$$

the corresponding elastic metric $\tilde{\mathbf{C}}_e$ is

$$\tilde{\mathbf{C}}_e = \mathbf{R}_e^\top\tilde{\mathbf{C}}\mathbf{R}_e \quad\text{with } \tilde{\mathbf{C}} = \tilde{\lambda}^2\mathbf{e}_2\otimes\mathbf{e}_2 + \tilde{\lambda}^{-2}(\mathbf{e}_3\otimes\mathbf{e}_3) + \mathbf{e}_1\otimes\mathbf{e}_1. \tag{6.30}$$

[3] See the Appendix for further details.

[4] Shortly, $\mathbf{e}_3\wedge\mathbf{e}_2 = 2\,\mathrm{skw}\,(\mathbf{e}_3\otimes\mathbf{e}_2)$.

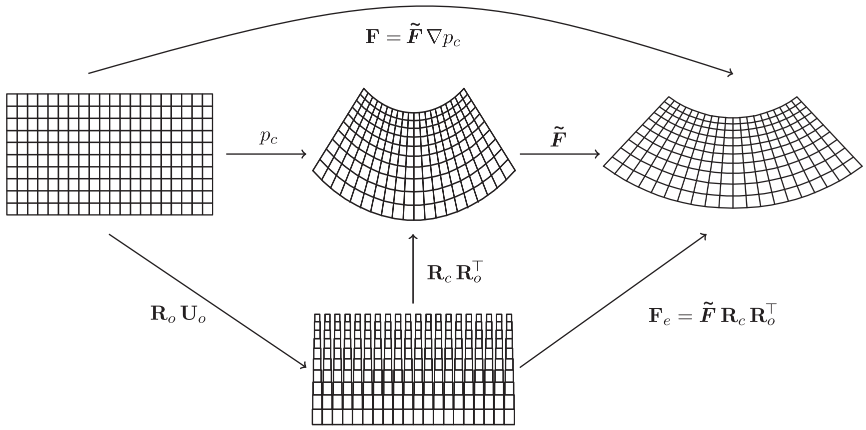

Fig. 5. A cross-section of the bar-like body in the plane of unit normal $\mathbf{e}_1$ is represented, with a superimposed Cartesian grid (left); a cut view of the same grid is shown, as deformed by the distortion field (6.20) (corresponding to $\mathbf{R}_o = \mathbf{I}$ for the sake of simplicity) (middle down); the gluing together operation delivers a relaxed configuration through a global finite bending (middle top); the configuration realized through a further deformation is superimposed on the relaxed configuration (right).

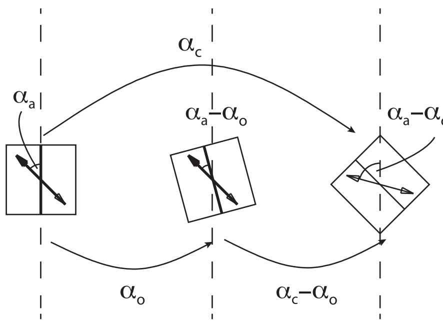

Fig. 6. The transversely isotropic direction is represented as a double arrow, having angle α_a with the vertical direction (left); the distortion $\mathbf{F}_o = \mathbf{R}_o\mathbf{U}_o$ would realize a rotation of the same fiber: the angle shared with the vertical is $\alpha_a - \alpha_o$ (middle); the relaxed configuration which is realized determines a further rotation $\alpha_e = \alpha_c - \alpha_o$ of this direction which in the end share the angle $\alpha_a + \alpha_e$ with the vertical direction (right).

We now assume a transversely isotropic material response, namely, the aforementioned example [42] based on the augmented Neo-Hookean energy

$$\phi_{oTI}(\mathbf{C}_e) = \phi_{oNH}(\mathbf{C}_e) + (\phi_4(I_4(\mathbf{C}_e)) + \phi_5(I_5(\mathbf{C}_e)))f(\lambda_e(\mathbf{a})), \quad J_e = 1, \tag{6.31}$$

with the anisotropic contributions expressed through the quadratic functions

$$\phi_i(I_i(\mathbf{C}_e)) = \tfrac{1}{2}\mu_i(I_i(\mathbf{C}_e) - 1)^2, \quad i = 4, 5, \tag{6.32}$$

and the step function f defined as:[5]

$$f(\lambda_e) = \begin{cases} 0, & \lambda_e \le 1, \\ 1, & \lambda_e > 1. \end{cases} \tag{6.33}$$

From (3.18) and (3.20), and being the anisotropic contribution effective only for positive stretches along $\mathbf{a}$, i.e., for $\lambda_e(\mathbf{a}) > 1$, it holds

$$\mathbf{S}_{oe} = \mu\mathbf{F}_e + \left(2\mathbf{F}_e\frac{\partial\phi_4}{\partial\mathbf{C}_e} + 2\mathbf{F}_e\frac{\partial\phi_5}{\partial\mathbf{C}_e}\right)f(\lambda_e(\mathbf{a})) = \mu\mathbf{F}_e + (\mathbf{S}_{oe4} + \mathbf{S}_{oe5})f(\lambda_e(\mathbf{a})). \tag{6.34}$$

Let the transverse isotropy axis defined by the vector field

$$\mathbf{a} = -\sin\alpha_a\,\mathbf{e}_2 + \cos\alpha_a\,\mathbf{e}_3. \tag{6.35}$$

[5] Typically, the step function makes the anisotropic contribution effective only when the elastic stretch of the material fiber $\mathbf{a}$ is positive.

Then, with $\mathbf{a}_e = \mathbf{R}_e\mathbf{a}$, it holds

$$I_1 = \mathbf{C}_e\cdot\mathbf{I} = \tilde{\mathbf{C}}\cdot\mathbf{I} = \frac{1 + \tilde{\lambda}^2 + \tilde{\lambda}^4}{\tilde{\lambda}^2}, \tag{6.36}$$

$$I_4 = \mathbf{C}_e\mathbf{a}\cdot\mathbf{a} = \tilde{\mathbf{C}}\mathbf{a}_e\cdot\mathbf{a}_e = \frac{1 + \tilde{\lambda}^4 + (1 - \tilde{\lambda}^4)\cos(2(\alpha_a + \alpha_e))}{2\tilde{\lambda}^2}, \tag{6.37}$$

$$I_5 = \mathbf{C}_e\mathbf{a}\cdot\mathbf{C}_e\mathbf{a} = \tilde{\mathbf{C}}\mathbf{a}_e\cdot\tilde{\mathbf{C}}\mathbf{a}_e = \frac{1 + \tilde{\lambda}^8 + (1 - \tilde{\lambda}^8)\cos(2(\alpha_a + \alpha_e))}{2\tilde{\lambda}^4}. \tag{6.38}$$

It is evident from (6.36)–(6.38) that the isotropic part of the material response is not affected by α_o; on the contrary, the anisotropic response is sensitive to an *actual anisotropy direction* determined by the angle $\alpha = \alpha_a + \alpha_e$, between the transverse isotropy direction assumed in the realized compatible configuration and the vertical direction (see Fig. 6).

From Eqs. (3.5), (3.7), and (6.34), the Cauchy stress follows as $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_4 + \mathbf{T}_5$ with:

$$\mathbf{T}_1 = \mu(\tilde{\mathbf{B}} - \tfrac{1}{3}\mathrm{tr}(\tilde{\mathbf{B}})\mathbf{I}), \tag{6.39}$$

$$\mathbf{T}_4 = 2\mu_4(I_4 - 1)(\tilde{\mathbf{F}}\mathbf{e}_e\otimes\mathbf{e}_e\tilde{\mathbf{F}}^\top - \tfrac{1}{3}I_4\mathbf{I}), \tag{6.40}$$

$$\mathbf{T}_5 = 2\mu_5(I_5 - 1)(\tilde{\mathbf{F}}(\tilde{\mathbf{C}}\mathbf{e}_e\otimes\mathbf{e}_e + \mathbf{e}_e\otimes\tilde{\mathbf{C}}\mathbf{e}_e)\tilde{\mathbf{F}}^\top - \tfrac{2}{3}I_5\mathbf{I}), \tag{6.41}$$

and $\tilde{\mathbf{B}} = \tilde{\mathbf{F}}\tilde{\mathbf{F}}^T$.

Let us assume as transverse isotropy direction $\alpha_a = 0$; with this, Fig. 7 shows the plane components T_{22}, T_{33}, T_{23} of the Cauchy stress versus λ for different values of α_o, at the center of the body $\mathcal{B}$; with $\alpha_o(y) = -\overline{\alpha}_o y$, we obtain an actual transverse isotropic direction $(-(b - \overline{\alpha}_o)y)$. For $\lambda = 1$, $\tilde{\mathbf{F}} = \mathbf{I}$ and, being the starting configuration a relaxed configuration, it holds $\mathbf{T} = \mathbf{0}$, as it is shown in Fig. 7.

Fig. 8 shows the T_{22}, T_{33}, T_{23} patterns along the beam axis, for $\lambda = 0.8$, and for different values of $\overline{\alpha}_o$.

7. Conclusions and future directions

We discussed the notion of distortions and the definition of an appropriate elastic metric, a strain measure upon which an elastic energy density can be defined, focussing on two issues: the compatibility of distortions, and the symmetries of the material response. We also presented a simple, yet non-trivial example, on how a distortion field, even if compatible, may affect the response of anisotropic materials.

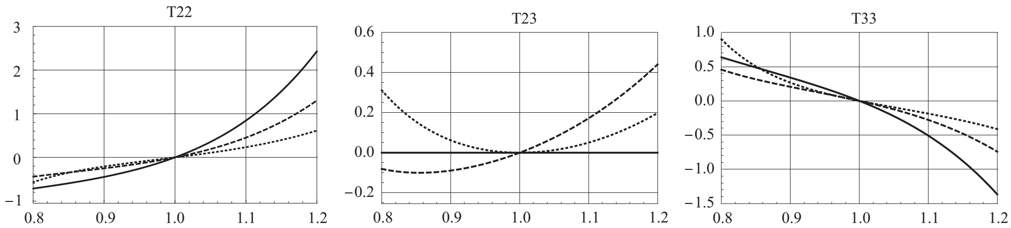

Fig. 7. In plane component of the Cauchy stress tensor T_{22}, T_{33}, T_{23} in the center of the beam versus the intensity $\tilde{\lambda}$ of the superimposed uniaxial deformation for different values of the preferred orientation: $\overline{\alpha}_o = 0$ (solid), $\overline{\alpha}_o = \pi/6$ (dashed), $\overline{\alpha}_o = \pi/4$ (dotted).

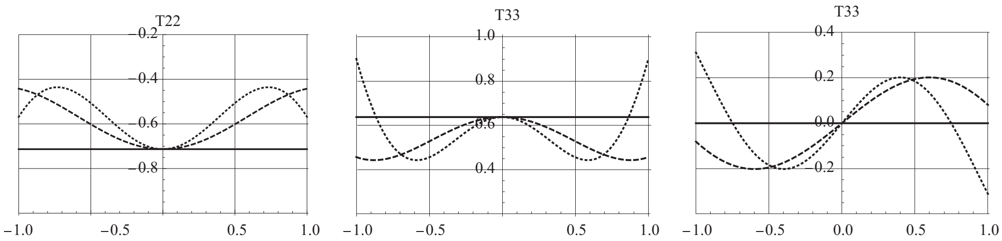

Fig. 8. Patterns of T_{22}, T_{33}, T_{23} along the beam axis for $\tilde{\lambda} = 0.8$, and for different values of the preferred orientation: $\overline{\alpha}_o = 0$ (solid), $\overline{\alpha}_o = \pi/6$ (dashed), $\overline{\alpha}_o = \pi/4$ (dotted).

Our work aimed at highlighting a few questions related to compatibility issues which, even if largely discussed in Literature in the past, have now got a new position within the limit of the morphing of soft responsive materials. The authors are involved in a long term research on the mathematical model of active biological tissues, especially in the modelling of muscle, where the question concerning the compatibility of muscle contractions may have important biological implications; as example: does a muscle store elastic energy when it shortens upon activation? Thus, future points of interest deal with design of biomimetic devices, and the theoretical framework of elasticity with distortions appears now to be mature enough to improve our ability to produce controlled motions of soft materials, by mimicking natural muscles.

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Appendix A

Given the 3D Euclidean ambient space $\mathcal{E}$, let $V\mathcal{E}$ be the associated vector space, $\mathbb{L}\mathrm{in} = V\mathcal{E} \otimes V\mathcal{E} = \mathbb{S}\mathrm{ym} \oplus \mathbb{S}\mathrm{kw}$ be the space of (second-order) tensors on $V\mathcal{E}$ (linear maps of $V\mathcal{E}$ into itself), and $\mathbb{LIN} = \mathbb{L}\mathrm{in} \otimes \mathbb{L}\mathrm{in}$ be the space of (fourth-order) tensors on $\mathbb{L}\mathrm{in}$ (linear maps of $\mathbb{L}\mathrm{in}$ into itself). For $\mathbf{a}, \mathbf{b} \in V\mathcal{E}$, we introduce the dyadic product of $\mathbf{a}$ and $\mathbf{b}$ as the element of $\mathbb{L}\mathrm{in}$ defined by

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{v} = (\mathbf{b} \cdot \mathbf{v})\mathbf{a}, \quad \forall \mathbf{v} \in V\mathcal{E}. \tag{A.1}$$

Fixed an orthonormal basis $\{\mathbf{e}_i, i = 1,2,3\}$ of $V\mathcal{E}$, the corresponding orthonormal basis of $\mathbb{L}\mathrm{in}$ is $\{\mathbf{e}_i \otimes \mathbf{e}_j, i,j = 1,2,3\}$ and, for any $\mathbf{A} \in \mathbb{L}\mathrm{in}$, it holds

$$A_{ij} = \mathbf{A} \cdot \mathbf{e}_i \otimes \mathbf{e}_j. \tag{A.2}$$

For $\mathbf{A}, \mathbf{B} \in \mathbb{L}\mathrm{in}$, we introduce the dyadic product of $\mathbf{A}$ and $\mathbf{B}$ as the element of $\mathbb{LIN}$ defined by

$$(\mathbf{A} \otimes \mathbf{B})[\mathbf{C}] = (\mathbf{B} \cdot \mathbf{C})\mathbf{A}, \quad \forall \mathbf{C} \in \mathbb{L}\mathrm{in}; \tag{A.3}$$

with this, we build the orthonormal basis of $\mathbb{LIN}$ induced by the basis of $V\mathcal{E}$ as $\{(\mathbf{e}_i \otimes \mathbf{e}_j) \otimes (\mathbf{e}_k \otimes \mathbf{e}_l), i,j,k,l = 1,2,3\}$. For any $\mathbb{A} \in \mathbb{LIN}$, it holds

$$\mathbb{A}_{ijhk} = \mathbb{A}[\mathbf{e}_h \otimes \mathbf{e}_k] \cdot \mathbf{e}_i \otimes \mathbf{e}_j. \tag{A.4}$$

Once defined the conjugation product of the ordered pair $(\mathbf{A}, \mathbf{B}) \in \mathbb{L}\mathrm{in} \times \mathbb{L}\mathrm{in}$ as the fourth-order tensor

$$(\mathbf{A} \boxtimes \mathbf{B})[\mathbf{C}] = \mathbf{A}\mathbf{C}\mathbf{B}^T, \quad \forall \mathbf{C} \in \mathbb{L}\mathrm{in}, \tag{A.5}$$

the identity $\mathbb{I}$ in $\mathbb{LIN}$ can be defined as $\mathbb{I} = \mathbf{I} \boxtimes \mathbf{I}$. It holds $\mathbb{I}_{ijhk} = \delta_{ih}\,\delta_{jk}$, being δ_{ij} the Kronecker symbol, whose value is 1 if $i=j$, 0 if $i \neq j$. Two special elements of $\mathbb{LIN}$ are the projectors

$$\mathrm{sym}\,[\mathbf{A}] = \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^T) \quad \text{and} \quad \mathrm{skw}\,[\mathbf{A}] = \tfrac{1}{2}(\mathbf{A} - \mathbf{A}^T). \tag{A.6}$$

Following [31], we denote as $\mathbb{I}_s = \mathbb{I} \circ 2\mathrm{sym}$; it holds $(\mathbb{I}_s)_{ijhk} = \delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}$.

A different and in general not orthonormal basis of $V\mathcal{E}$ is the one induced by the linear transformation $\mathbf{F}_o$ as $\{\mathbf{e}_{oi} = \mathbf{F}_o\mathbf{e}_i, i = 1,2,3\}$; it can be easily verified that the dual basis is $\{\mathbf{e}_o^i = \mathbf{F}_o^{-T}\mathbf{e}_i, i = 1,2,3\}$. It holds

$$\mathbf{e}_{oi} \cdot \mathbf{e}_{oj} = e_{oij} = (\mathbf{C}_o)_{ij} \quad \text{and} \quad \mathbf{e}_o^i \cdot \mathbf{e}_o^j = e_o^{ij} = (\mathbf{C}_o^{-1})_{ij}, \tag{A.7}$$

with $(\mathbf{C}_o)_{ij}$ and $(\mathbf{C}_o^{-1})_{ij}$ the components of $\mathbf{C}_o$ and $\mathbf{C}_o^{-1}$ with respect to the orthonormal basis $\{\mathbf{e}_i, i = 1,2,3\}$. Two vectors $\mathbf{u}$ and $\mathbf{v}$ allowing for the same (covariant) components with respect to the two bases, *i.e.* $\mathbf{u} \cdot \mathbf{e}_{oi} = \mathbf{v} \cdot \mathbf{e}_i$, are related through the following equation: $\mathbf{u} = \mathbf{F}_o^{-T}\mathbf{v}$. Likewise, two tensors $\mathbf{U}$ and $\mathbf{V}$ with the same (covariant) components with respect to the two bases, *i.e.*

$$\mathbf{U} \cdot \mathbf{e}_{oi} \otimes \mathbf{e}_{oj} = \mathbf{V} \cdot \mathbf{e}_i \otimes \mathbf{e}_j, \tag{A.8}$$

are related through the following:

$$\mathbf{U} = \mathbf{F}_o^{-T}\mathbf{V}\mathbf{F}_o^{-1}. \tag{A.9}$$

Finally, two fourth-order tensor $\mathbb{U}$ and $\mathbb{V}$ with the same (controvariant) components with respect to the two bases, *i.e.*

$$\mathbb{U}[\mathbf{e}_o^k \otimes \mathbf{e}_o^l] \cdot \mathbf{e}_o^i \otimes \mathbf{e}_o^j = \mathbb{V}[\mathbf{e}_k \otimes \mathbf{e}_l] \cdot \mathbf{e}_i \otimes \mathbf{e}_j, \tag{A.10}$$

are related through the equation

$$\tilde{\mathbb{C}} = (\mathbf{F}_o^{-1} \boxtimes \mathbf{F}_o^{-1})\mathbb{C}(\mathbf{F}_o^{-T} \boxtimes \mathbf{F}_o^{-T}). \tag{A.11}$$

If $\mathbb{U}$ is represented in terms of its spectral representation

$$\mathbb{U} = \sum_{i=1}^{6} \gamma_i \, \mathbf{C}_i \otimes \mathbf{C}_i, \tag{A.12}$$

being $\mathbf{C}_i \cdot \mathbf{C}_j = \delta_{ij}$ and where it is understood that proper numbers are repeated as many times as their algebraic multiplicity, then $\mathbb{V}$ admits the representation

$$\mathbb{V} = \sum_{i=1}^{6} \gamma_i \tilde{\mathbf{C}}_i \otimes \tilde{\mathbf{C}}_i, \quad \tilde{\mathbf{C}}_i = \mathbf{F}_o^{-1} \mathbf{C}_i \mathbf{F}_o^{-T}, \tag{A.13}$$

as can be easily verified using Eqs. (A.11) and (A.12). However, $\mathbb{V}\tilde{\mathbf{C}}_k \neq \gamma_k \tilde{\mathbf{C}}_k$, *i.e.* (A.13) is not a spectral representation for $\mathbb{V}$. When fourth-order tensors such as $\mathbb{U}$ and $\mathbb{V}$ (related through Eq. (A.11)) are evaluated on second-order tensor such as **U** and **V** (related through Eq. (A.8)), it holds

$$\mathbb{U}[\mathbf{U}] \cdot \mathbf{U} = \mathbb{V}[\mathbf{V}] \cdot \mathbf{V}. \tag{A.14}$$

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