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A Novel Bäcklund Invariance of a Nonlinear Differential Equation

Sandra Carillo

Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università "La Sapienza," Rome, Italy

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Here the nonlinear ordinary differential equation $yy'' = S(x)$ is investigated. The interest of the proposed study is twofold: indeed, the high nonlinearity exhibited by the considered equation does not allow the application of any linearization method; on the other hand, it turns out, under suitable conditions, to be equivalent to a nonlinear integral equation arising in extended kinetic theory. The equivalence between the two nonlinear problems is exploited; in particular, conditions which need to be prescribed to establish such an equivalence are considered. Bäcklund transformations are applied to study the problem of interest. Specifically, it is proved that the nonlinear differential equation enjoys an invariance property when the "source term" $S(x)$ is represented by a solution of a suitable functional equation. The latter is discussed and some solutions are explicitly written; thus, the corresponding Bäcklund charts are depicted to show the obtained new invariances.

Key Words: nonlinear ordinary differential equations; Bäcklund transformations; invariance properties.

1. INTRODUCTION

Bäcklund and reciprocal transformations have been applied in investigating boundary value problems, as well as initial value problems in various applications. They have been first introduced in connection with models mainly in gas dynamics and fluid mechanics: an extensive bibliography can be found in [15, 16]. Subsequently (see Refs. [15, 16]), such transformations have been proved to represent a powerful tool in investigating mathematical problems arising from applied mathematics.

Indeed, in the case of nonlinear evolution equations, such transformations have been shown to play a key role in revealing structural properties

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such as Hamiltonian and/or bi-Hamiltonian structure, symmetry properties, etc. [6, 7, 9, 11].

Here, conversely, Bäcklund and reciprocal transformations are applied in investigating a nonlinear ordinary differential equation.

The nonlinear ordinary differential equation under investigation is

$$yy'' = S(x), \tag{1.1}$$

where $S(x)$ is a suitably regular given function. The interest of the proposed study is twofold: the high nonlinearity exhibited by Eq. (1.1) does not allow us to apply any linearization method; on the other hand, on imposing suitable initial conditions on Eq. (1.1), the obtained Cauchy problem turns out to be equivalent to a nonlinear integral equation which arises in extended kinetic theory (see for instance Ref. [4]).

In the opening Section 2, the model is briefly recalled. Specifically, the particular case under investigation is specified; the connection between the nonlinear differential equation (1.1) and a nonlinear integral Boltzmann equation is considered. The Cauchy problem equivalent to the nonlinear integral equation is written.

In the subsequent Section 3, Bäcklund and reciprocal transformations are employed to study the nonlinear differential equation (1.1).

A functional equation is shown to single out those given functions $S(x)$ in (1.1) which correspond to nonlinear differential equations exhibiting an invariance property. Such a novel invariance is proved on application of Bäcklund and reciprocal transformations; the results are depicted in a Bäcklund chart constructed to summarize the invariance properties as well as all the links among the equations which have been considered.

Section 4 is devoted to the investigation of the functional equation obtained in the previous section; some solutions of this functional equation are exhibited as well as the corresponding invariances of the nonlinear differential equation.

The paper closes with a remark concerning the general solution of the functional equation in the case when all the real variables of the problem, namely the independent as well as the dependent variable, are replaced by complex ones.

2. MODEL EQUATIONS

The nonlinear differential problem to be discussed in this paper originates from the nonlinear stationary spatially homogeneous, isotropic particle transport theory. The physics of the problem is exhaustively described in the general case in Ernst's review paper [8] and this model is studied by

Boffi and co-workers [2–5]; the relevant goal is the determination of certain given test particles, diffusing in an unbounded medium by binary interactions between themselves. The source-removal problem here investigated is governed by the extended Boltzmann equation,

$$f(x)\int_0^\infty K(x,\xi)f(\xi)\,d\xi = QS(x), \qquad S, f : A \subset \mathbb{R} \to \mathbb{R}^+ \qquad Q \in \mathbb{R}^+, \tag{2.1}$$

wherein x denotes the speed modulus, $f(x)$ the sought isotropic distribution function (nonnegative valued), and the kernel $K(x, \xi)$ is given by[1]

$$K(x,\xi) = \frac{\gamma_R}{2x\xi}\int_{|x-\xi|}^{x+\xi} u g_R(u)\,du; \qquad x > 0,\ \xi > 0,\ \delta_R > 0; \tag{2.2}$$

in which γ_R measures the strength of the removal and $QS(x)$ represents an external source (nonnegative valued function) of the considered test particles of intensity Q and shape $S(x)$ with $\int_0^\infty S(x)\,dx = 1$.

The specialization of the removal collision frequency $g_R(u)$ and, consequently, of the kernel $K(x, \xi)$, produces thus a whole series of problems of physical interest.

Restricting to the case of a one dimensional box of length $\beta - \alpha$ and adopting the so-called "maximum" approximation to the "hard sphere" model, that is, when the kernel is chosen of the form,

$$K(x,\xi) = \begin{cases} x, & \xi < x \\ \xi, & \xi > x, \end{cases} \tag{2.3}$$

Eq. (2.1) reads

$$f(x)\left\{x\int_\alpha^x f(\xi)\,d\xi + \int_x^\beta \xi f(\xi)\,d\xi\right\} = \frac{QS(x)}{\gamma_R}, \qquad 0 < \alpha < \beta < +\infty. \tag{2.4}$$

If, furthermore, for sake of simplicity, the parameters Q and γ_R are both set equal to 1, the nonlinear integral equation simplifies to

$$f(x)\left\{x\int_\alpha^x f(\xi)\,d\xi + \int_x^\beta \xi f(\xi)\,d\xi\right\} = S(x) \tag{2.5}$$

in the unknown function $f(x)$ and where $S(x)$ is the same given function of Eq. (2.1) which represents the "source" term, and the kernel $K(x, \xi)$

[1] In the following, definition of the kernel $K(x, \xi)$ when x or ξ equal to zero is not required.

has been assumed of the form (2.3). In Refs. [2–5] nonlinear integral equations of the form (2.1) have been studied when investigating the particular case of extended kinetic theory of gases. The mathematical problem (2.5) models a "low density" [5] gas of particles, which are annihilated under collision, contained in a one dimensional box of length $\beta - \alpha$.

On substituting the expression of the kernel (2.3) in Eq. (2.5) the nonlinear integral problem of interest reads

$$f(x)\left\{x\int_{\alpha}^{x} f(\xi)\,d\xi + \int_{x}^{\beta} \xi f(\xi)\,d\xi\right\} = S(x) \tag{2.6}$$

in the unknown function $f(x)$ and where $S(x)$ is the "source" term. A corresponding nonlinear differential problem can be constructed on setting

$$y(x) := x\int_{\alpha}^{x} f(\xi)\,d\xi + \int_{x}^{\beta} \xi f(\xi)\,d\xi, \tag{2.7}$$

which, after derivation with respect to x, in turn, once and twice, gives

$$y'(x) = \int_{\alpha}^{x} f(\xi)\,d\xi, \qquad y''(x) = f(x). \tag{2.8}$$

Thus, the Cauchy problem, which is equivalent to the integral problem (2.6), is obtained

$$\begin{cases} yy'' = S(x), & S:[\alpha,\beta] \subset \mathbb{R}^+ \to \mathbb{R}^+ \\ y'(\alpha) = 0, & y:[\alpha,\beta] \subset \mathbb{R}^+ \to \mathbb{R}^+ \\ y(\alpha) = M_1, & M_1 \in \mathbb{R}^+, \end{cases} \tag{2.9}$$

where $S(x)$ represents the "source" term which appears in (2.6).

3. BÄCKLUND CHART

This section is devoted to investigating the nonlinear differential equation

$$yy'' = S(x), \qquad S:\mathbb{R}^+ \to \mathbb{R}^+, \tag{3.1}$$

in the unknown function $y:\mathbb{R}^+ \to \mathbb{R}^+$ where S is a given function. It should be preliminarily remarked that no solution of (3.1) can be determined via similarity reduction methods [13, 14] since Eq. (3.1) does not enjoy any non-trivial symmetry property.[2] In this section, Bäcklund transformations are employed to relate the nonlinear equation (3.1) to other nonlinear ordinary differential equations.

The idea is to obtain a nonlinear equation which enjoys an invariance property under some transformations. Since the nonlinear ordinary differential equation here considered is a non-homogeneous one an invariance result can be obtained only corresponding to "suitable" given functions $S(x)$. Accordingly, the applied Bäcklund transformations induce a functional equation for the function $S(x)$. Specifically, it will be shown that, provided $S(x)$ satisfies the mentioned functional equation, the differential equation (3.1) is related to a differential equation which is invariant under the Möbius group of transformations. Such Bäcklund transformations and functional equations are explicitly written in the following proposition.

PROPOSITION 3.1. *Whenever the function $S(x)$ is a solution of the functional equation*

$$S(x) = \frac{ad - bc}{[cx + d]^2} S\left(\frac{ax + b}{cx + d}\right);$$

$$S\colon [\alpha, \beta] \subset \mathbb{R}^+ \to \mathbb{R}^+;\ a, b, c, d \in \mathbb{R};\ ad - bc \neq 0 \qquad (3.2)$$

then the nonlinear equation $yy'' = S(x)$ is invariant under all the transformations defined by

$$I: \begin{cases} \bar{x} = \dfrac{ax + b}{cx + d}, & ad - bc \neq 0 \\ [\bar{y}(\bar{x})]^2 = \dfrac{(cx + d)^2}{ad - bc} [y(x)]^2. \end{cases} \qquad (3.3)$$

Proof. Consider the differential equation (1.1) and apply the transformation

$$T\colon v = y^2 \qquad (3.4)$$

which has an empty kernel since the solution of the differential equation is looked for in the set of positive valued functions. It follows

$$vv'' - \frac{1}{2}v'^2 = 2vS(x). \qquad (3.5)$$

[2] The symmetry analysis of the nonlinear equation $y'' = f(x)y^{-n}$ ($f(x)$ given regular function) comprised in [12] does not apply in the present case; indeed, the authors of [12] are interested in the case $n = 2$ which they exploit extensively and, in addition, they obtain further symmetry results for other values of n; however, their analysis does not apply when $n = 1$, which is the case considered here.

$$\boxed{y\,y'' = S(x)} \overset{T}{\text{————}} \boxed{v\,v'' - \frac{1}{2}v'^2 = 2v\,S(x)} \overset{R}{\text{————}} \boxed{\{\Phi;t\} = 2\Phi_t\,S(\Phi)}$$

FIGURE 1

The subsequent application of the reciprocal transformation [15, 16]

$$R\begin{cases} x = \Phi, & D_t = vD_x \\ v = D_t\Phi, & D_x = [\Phi_t]^{-1}D_t, \end{cases} \tag{3.6}$$

wherein

$$D_t := \frac{d}{dt}, \qquad D_x := \frac{d}{dx}, \qquad \Phi_t = \frac{d}{dt}\Phi, \tag{3.7}$$

gives

$$\{\Phi;t\} = 2\Phi_t S(\Phi), \tag{3.8}$$

where $\{\Phi;t\}$ denotes the Schwartzian derivative

$$\{\Phi;t\} := \left(\frac{\Phi_{tt}}{\Phi_t}\right)_t - \frac{1}{2}\left(\frac{\Phi_{tt}}{\Phi_t}\right)^2. \tag{3.9}$$

The links among Eqs. (3.1), (3.5), and (3.8) are summarized in the Bäcklund chart shown in Fig. 1. This shows how a new invariance property is obtained. Since, as is well known, the Schwartzian derivative $\{\Phi;t\}$ is invariant under the Möbius group of transformations,

$$M:\Phi \to \Psi = \frac{a\Phi + b}{c\Phi + d}, \qquad a,b,c,d \in \mathbb{R}; \qquad ad - bc \neq 0, \tag{3.10}$$

it follows that if also the right hand side of (3.8) enjoys the same property, namely if

$$\Psi_t S(\Psi) = \Phi_t S(\Phi), \tag{3.11}$$

where Ψ is given by (3.10), then Eq. (3.1) is invariant under the all transformations,

$$I:\begin{cases} \bar{x} = \dfrac{ax+b}{cx+d}, & ad - bc \neq 0 \\ [\bar{y}(\bar{x})]^2 = \dfrac{(cx+d)^2}{ad-bc}[y(x)]^2. \end{cases} \tag{3.12}$$

$$
\begin{tikzcd}
\boxed{y\,y'' = S(x)} \arrow[r, dash, "T"] \arrow[d, dash, "I_2"] & \boxed{v\,v'' - \frac{1}{2}\,v'^2 = 2v\,S(x)} \arrow[r, dash, "R"] \arrow[d, dash, "I_1"] & \boxed{\{\Phi; t\} = 2\Phi_t\,S(\Phi)} \arrow[d, dash, "M"] \\
\boxed{\bar{y}\,\bar{y}'' = S(\bar{x})} \arrow[r, dash, "T"] & \boxed{\bar{v}\,\bar{v}'' - \frac{1}{2}\,\bar{v}'^2 = 2\bar{v}\,S(\bar{x})} \arrow[r, dash, "R"] & \boxed{\{\Psi; t\} = 2\Psi_t\,S(\Psi)}
\end{tikzcd}
$$

FIGURE 2

The latter are induced by the Bäcklund chart shown in Fig. 2. The thesis is readily proved, since on substitution of Ψ in (3.10) into (3.11), the functional equation (3.2) is obtained. ▮

4. FUNCTIONAL EQUATION

This section is devoted to the discussion of the functional equation which determines those "source" terms such that the invariance property stated in the previous section holds.

Indeed, the invariance property (3.3) is fulfilled by (1.1) whenever the function S represents a solution of the functional equation (3.2), here rewritten, only for convenience,

$$S(x) = \frac{ad - bc}{[cx + d]^2} S\left(\frac{ax + b}{cx + d}\right), \qquad S : [\alpha, \beta] \subset \mathbb{R}^+ \to \mathbb{R}^+,$$

wherein

$$0 < \alpha < x < \beta < +\infty, \qquad a, b, c, d \in \mathbb{R}; ad - bc \neq 0. \tag{4.1}$$

Hence, a brief discussion concerning solutions of (3.2) is convenient, indeed, the general solution [1] of Eq. (3.2). In what follows the Bäcklund charts which correspond to particular choices of the parameters $a, b, c, d \in \mathbb{R}$ are depicted.

(1) Let $a \in \mathbb{R}^+$, $b = c = 0$, $d = 1$; namely the functional equation (3.2) reduces to

$$S(x) = aS(ax), \tag{4.2}$$

$$
\begin{tikzcd}
\boxed{y\,y'' = \frac{1}{ax}} \arrow[r, dash, "\tilde{R}"] \arrow[d, dash, "I_2"] & \boxed{\{\Phi;t\} = -2\,\frac{\Phi_t}{\Phi}} \arrow[d, dash, "\Psi = a\,\Phi"] \\
\boxed{\bar y\,\bar y'' = \frac{1}{a\bar x}} \arrow[r, dash, "\tilde{R}"] & \boxed{\{\Psi;t\} = -2\,\frac{\Psi_t}{\Psi}}
\end{tikzcd}
$$

FIGURE 3

which admits solution

$$S(x) = \frac{1}{ax}. \tag{4.3}$$

Correspondingly, the Bäcklund chart takes the form shown in Fig. 3 where $\tilde{R}$ denotes the transformation obtained when the transformations T and R, defined, in turn via (3.4) and (3.6), are combined.

(2) Let $d = a = 0$, $b = c = 1$ in (3.2); then the functional equation reads

$$S(x) = -\frac{1}{x^2} S\left(\frac{1}{x}\right). \tag{4.4}$$

The latter admits the solution

$$S(x) = -\frac{1}{x}\ln(x). \tag{4.5}$$

Hence, the Bäcklund chart can be specialized in this case, shown in Fig. 4.

$$
\begin{tikzcd}
\boxed{y\,y'' = -\frac{1}{x}\ln x} \arrow[r, dash, "\tilde{R}"] \arrow[d, dash, "I_2"] & \boxed{\{\Phi;t\} = -2\,\frac{\Phi_t}{\Phi}\ln\Phi} \arrow[d, dash, "\Psi = \frac{1}{\Phi}"] \\
\boxed{\bar y\,\bar y'' = -\frac{1}{\bar x}\ln\bar x} \arrow[r, dash, "\tilde{R}"] & \boxed{\{\Psi;t\} = -2\,\frac{\Psi_t}{\Psi}\ln\Psi}
\end{tikzcd}
$$

FIGURE 4

$$\begin{tikzcd}
\boxed{y\, y'' = \dfrac{\gamma}{x}\,,} \arrow[r, dash, "\widetilde{R}"] \arrow[d, dash, "I_2"] & \boxed{\{\Phi; t\} = -2\dfrac{\Phi_t}{\Phi}} \arrow[d, dash, "\Psi = -\dfrac{1}{\Psi}"] \\
\boxed{\bar{y}\, \bar{y}'' = \dfrac{\gamma}{\bar{x}}} \arrow[r, dash, "\widetilde{R}"] & \boxed{\{\Psi; t\} = -2\dfrac{\Psi_t}{\Psi}}
\end{tikzcd}$$

FIGURE 5

(3) Let $a = d = 0$, $b = -1$, $c = 1$; namely the functional equation (3.2) reduces to

$$S(x) = \frac{1}{x^2} S\left(\frac{1}{x}\right), \tag{4.6}$$

which [1] admits the general solution[3]

$$S(x) = \frac{\gamma}{x}, \qquad \gamma \in \mathbb{R}^+. \tag{4.9}$$

Correspondingly, the Bäcklund chart takes the form shown in Fig. 5 which can be obtained from the Bäcklund chart first examined in case (1) provided $\gamma = a^{-1}$, on use of the invariance $\Phi \to -\Phi$ enjoyed by equation

$$\{\Phi; t\} = -2\frac{\Phi_t}{\Phi}. \tag{4.10}$$

[3] Equation (4.6), letting $y = x^{-2}$, can be equivalently written in the form

$$S(xy) = y^{-1} S(x), \tag{4.7}$$

which represents a special case of the family of functional equations, in the unknown function ϕ,

$$\phi(xy) = y^k \phi(x), \tag{4.8}$$

whose general solution is given by $\phi(x) = \gamma x^k$. Equation (4.6) corresponds to choosing $k = -1$ in (4.8); thus, the solution in the present case is readily written recalling that those solutions of interest are $S: \mathbb{R}^+ \to \mathbb{R}^+$.

(4) The case $d = a = 0$, $b = -1$, $c = 1$ in (3.2); then the functional equation reads

$$S(x) = \frac{1}{x^2} S\left(\frac{1}{x}\right). \tag{4.11}$$

The latter admits the solution

$$S(x) = \frac{1}{x} \ln|x|. \tag{4.12}$$

Hence, the Bäcklund chart can be specialized in this case, shown in Fig. 6. Note that this invariance can be obtained directly combining the solution of the functional equation which has been obtained in case (2) with the invariance $\Phi \to -\Phi$ which is enjoyed by the function $-2(\Phi_t/\Phi)$.

5. REMARK

The functional equation

$$F(z) = F\left(\frac{az+b}{cz+d}\right), \qquad F: \mathbb{C} \to \mathbb{C}; \qquad a, b, c, d \in \mathbb{C}, \tag{5.1}$$

when $ad - bc = 1$, characterizes the automorphic functions [10]. Hence, if analytic solutions are looked for, in the case when in (5.1) the parameters $a, b, c, d \in \mathbb{C}$ are chosen so that $ad - bc \neq 0$, the general solution is represented by the set of the rescaled automorphic functions.

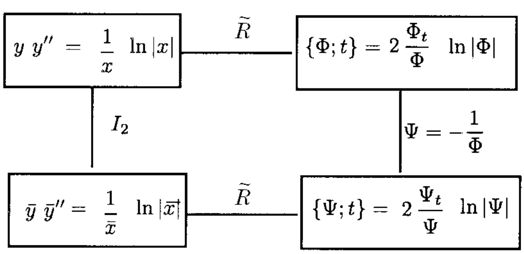

FIGURE 6

The analyticity of automorphic functions implies that, when both sides of (5.1) are derived with respect to z, the following functional equation is obtained

$$F'(z) = \frac{ad - bc}{[cz + d]^2} F'\left(\frac{az + b}{cz + d}\right), \tag{5.2}$$
$$F' : \mathbb{C} \to \mathbb{C} a, b, c, d \in \mathbb{C}; ad - bc \neq 0.$$

The latter, on introduction of the analytic functions $f(z) := F'(z)$, takes the form

$$f(z) = \frac{ad - bc}{[cz + d]^2} f\left(\frac{az + b}{cz + d}\right), \qquad f : \mathbb{C} \to \mathbb{C}; \qquad a, b, c, d \in \mathbb{C}; ad - bc \neq 0. \tag{5.3}$$

By construction, its general solution is given by the set of all analytic functions which represent derivatives of the rescaled automorphic functions.

Now, the functional equation (5.3) can be obtained if in (3.2) the real variable x and the real valued function unknown $S(x)$ are replaced, in turn, by a complex variable z and by the complex unknown function $f : \mathbb{C} \to \mathbb{C}$.

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