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Output containment via multi-consensus for heterogeneous linear systems on digraphs

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***Abstract*—In this paper we show how the problem of containment on arbitrary output trajectories for a group of linear systems connected by a digraph can be analyzed and solved based on topological results on multi-consensus concerning the network structure. Specifically, we apply this approach to the case of heterogeneous agents, for which output containment is the relevant task. We assume that each agent only measures the output of the corresponding exosystem, that defines the consensus trajectory. Indeed, the latter one is asymptotically generated by a set of linear systems (the exosystems) suitably exchanging information through the network. The approach proposed allows for the distributed, consensus-based stabilization of the containment dynamics without requiring the agents to have any non-local knowledge of the network structure.**

***Index Terms*—Multi agent systems; Multi-consensus; Containment control; Output feedback control; Linear systems.**

I. Introduction

Multi-consensus or *cluster consensus*, where parts of a multi-agent system simultaneously reach different consensus states, has attracted the interest of the research community [1], [2] for its relevance in many application fields as, among many others, opinion dynamics, game theory, biology, robotics, communication systems (e.g., [3]–[8] to cite a few).

Previous results on cluster consensus and multi-consensus include, among many others, [9], where conditions for establishing multi-consensus are derived for multi-agent systems with fixed and switching topology. In [10], the clustering of a set of discrete-time dynamical agents is guaranteed thanks to the introduction of different inputs to different clusters.

An interesting application of multi-consensus is the *containment problem*; namely, given a multi-agent system one seeks for a control constraining the behavior of the network to the one dictated by a subset of agents, that might be leaders or, more in general, roots of the graph. This task can be formulated as a multi-consensus problem, in which, for example, some clusters play the role of leader agents that converge toward some desired trajectories, whereas other clusters converge to an appropriate convex combination of the clusters containing the leaders.

Many previous works on containment are available. In general, they are not based on multi-consensus. For example, in [11] distributed containment control is considered for a second-order multi-agent system guided by multiple leaders with random switching topology. In this case, distinct dynamics are explicitly assigned to the leaders and the agents. In [12] the distributed control containment problem for heterogeneous first-order and second-order agents on a digraph is solved. [13] studies containment for heterogeneous agents. The leaders are assumed to be exosystems and the approach relies upon designing a distributed output-feedback control based on a multi-leader following output regulation framework.

In this paper we consider the situation in which a set of heterogeneous agents endowed with uniform reference trajectory generators are connected by a general directed graph. We emphasize that the idea of enforcing consensus on a desired reference trajectory via a network of exosystems is typically used in many applications such as, for instance, attitude and formation tracking control of robots, swarms of satellites or trains (see for instance [14]–[18]). Also, it is worth to note that such an approach also embeds leader-following consensus control [19]. We show that in this situation by suitably designing in a distributed way the coupling gains and the local controllers it is possible to make the agents evolve in clusters that are determined only by the network topology. The clusters are partitioned in "autonomous" clusters $\mathcal{H}_i$, in which all the agents evolve together on trajectories determined by their respective leaders, and "dependent" clusters $\mathcal{C}_i$ that evolve together on a convex combination of the $\mathcal{H}_i$. The role of leaders, followers, autonomous and dependent clusters depend only on the network topology. In other words, a distinctive feature of this approach is that the role of the agents depends only on their position in the network and not on their inherent structure or dynamics. Thus, the topological characterization described in the paper is useful either to analyze the behavior of a given network or design a specific topology to arrange the configuration of the multi-agent system in a pre-determined way. These results pave the way to a new control approach for possibly heterogeneous multi-agent systems, *topological control*, in which the collective behavior of the agents is controlled by only switching on/off their interactions without any modification of the single agents' dynamics.

The reference trajectories are those that can be generated by an arbitrary observable and controllable linear system. The structure of the reference trajectory generators is uniform across the graph, but their initial conditions at each agent can be arbitrary. We describe two solutions, one for the case when the state of the local generator is available to design the control, and the other one for the case when only the output is available. We remark that this is the first work to investigate multi-consensus-based containment on general trajectories for heterogeneous agents, where the clusters are determined only

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by the network, that is, homogeneity hypotheses are not required for agents in the same cluster.

In order to solve the problem we exploit recent results on the characterization of the multi-consensus behavior induced by features of the network, namely the presence of symmetries [20] or of almost (or external) equitable partitions ([21]–[25]). In particular, in [24], [26] it was shown that on digraphs the network topology naturally induces a clustering of the dynamics that can be exploited to reach cluster synchronization to a static point, where the cluster trajectories are distinguished by the initial conditions of the root agents of each cluster. Interestingly, one feature of the proposed solution is that the clustering is induced by the network, i.e., the agents are unaware of the cluster they belong, and the control is designed without any knowledge of the structure of the clusters. The technical contributions can be summarized as follows.

1) We extend the distributed internal model principle of [27] to multi-consensus, that is, to the case of graphs that are not uniformly connected.
2) We extend the multi-consensus containment design of [26] to the case of a class of time-varying reference trajectories, rather than constant values; namely, we consider output references that can be generated by a suitably defined LTI finite-dimensional system.
3) We apply these results to solve the output containment problem with general trajectories of multi-agent systems over arbitrary digraphs when the agents have access only to the output, and not to the whole state, of their neighbors.

The rest of the paper is organized in the following way. In Section II, preliminary notions are given. In Section III, we formulate the problem of output multi-consensus design for heterogeneous linear systems on general unweighted digraphs. In Section IV, the theory of output regulation and the results on multi-consensus are applied to the design of the reference generators. In Section V, we discuss two cases, namely when the state of the reference generators is available in the feedback and when only their output is available. In Section VI, we provide two simulated examples: the first one ensures containment on a set of systems driven by heterogeneous actuators toward periodic trajectories; in the second one, the proposed approach is used to enforce formation tracking of a group of unicycles. Concluding remarks in Section VII end the paper.

II. Preliminaries

A. Notations, definitions and basic graph properties

$\mathbb{C}^+$ and $\mathbb{C}^-$ represent, respectively, the open right and left sides of the complex plane. $\mathcal{R}_+ \subset \mathcal{R}$ denotes the set of non-negative real. $|\mathcal{S}|$ denotes the cardinality of the set $\mathcal{S}$. Given a matrix $A \in \mathbb{C}^{n\times n}$, $A^\star$ denotes its conjugate transpose. Given $\lambda \in \mathbb{C}$, we define by $Re(\lambda)$ the corresponding real-part. We denote by $\mathbf{0}$ either the zero vector or the zero matrix of suitable dimensions. $\mathbb{1}_c$ denotes the c-dimensional column vector whose elements are all ones while I_n is the identity matrix of dimension n. $\sigma(A) \subset \mathbb{C}$ denotes the spectrum of a matrix $A \in \mathbb{R}^{n\times n}$ whereas $\mathrm{Im}A$ is its image (or range space). The Kronecker product is denoted by $A\otimes B$. Given N matrices $E_i \in \mathbb{R}^{n_i\times m_i}$ of suitable dimensions, we denote

$$\mathrm{col}_{i=1}^N(E_i) = \begin{pmatrix} E_1^\top & \dots & E_n^\top \end{pmatrix}^\top \in \mathbb{R}^{(n_1+\dots+n_N)\times m}$$

when $m_i = m$ for all $i = 1,\dots,N$,

$$\mathrm{row}_{i=1}^N(E_i) = \left(\mathrm{col}_{i=1}^N(E_i^\top)\right)^\top \in \mathbb{R}^{n\times(m_1+\dots+m_N)}$$

when $n_i = n$ for all $i = 1,\dots,N$ and, finally,

$$\mathrm{diag}_{i=1}^N(E_i) = \begin{pmatrix} E_1 & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & E_M \end{pmatrix} \in \mathbb{R}^{n\times m}$$

with $n := \sum_{i=1}^N n_i$ and $m := \sum_{i=1}^N m_i$.

Consider a digraph (that is a simple unweighted directed graph) $\mathcal{G} = \{\mathcal{V},\mathcal{E}\}$ with $\mathcal{V}$ the set of vertices, $|\mathcal{V}| = N$ and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ the set of edges. $(j,r) \in \mathcal{E}$ if there exists an edge from j to r or, equivalently, j is a neighbour of r. The set of neighbours associated to $j \in \mathcal{V}$ is denoted $\mathcal{N}_j$ and $d_j = |\mathcal{N}_j|$ is the in-degree. The in-degree matrix is defined as $D = \mathrm{diag}_{j=1,\dots,N}(d_j) \in \mathbb{R}^{N\times N}$ whereas the adjacency matrix is $A = \{a_{jr}\} \in \mathbb{R}^{N\times N}$ with $a_{jj} = 0$ and $a_{jr} = 1$ if $(r,j) \in \mathcal{E}$ and $a_{jr} = 0$ otherwise for $j \neq r$. We say that $\mathcal{G}_u = \{\mathcal{V},\mathcal{E}_u\}$ is the undirected version of the digraph $\mathcal{G} = \{\mathcal{V},\mathcal{E}\}$ if $\mathcal{E} \subseteq \mathcal{E}_u$ and, in addition, for all $(j,r) \in \mathcal{E}$ then $(r,j) \in \mathcal{E}_u$.

$\mathcal{G}$ is said to be: *weakly connected* if its undirected version is connected; *strongly connected* if there always exists a directed path between every pair of nodes and there is no unreachable node. Given $j \in \mathcal{V}$, $\mathcal{R}(j)$ denotes a *node reach*, that is, the set of nodes that are reachable from j. If $\mathcal{R}(j)$ is not contained in any other $\mathcal{R}(r)$, $j \neq r$, then $\mathcal{R}(j)$ is called a *graph reach* of $\mathcal{G}$ and denoted $\mathcal{R}_j$. Thus, $\mathcal{R}_i$ for $i = 1,\dots,\mu$ are the distinct graph reaches of $\mathcal{G}$ with $\mu \leq N$. For each reach $\mathcal{R}_i$, $\mathcal{H}_i = \mathcal{R}_i \setminus \cup_{j=1,j\neq i}^{\mu} \mathcal{R}_j$ with $h_i = |\mathcal{H}_i|$ defines the exclusive part while $\overline{\mathcal{R}}_i = \mathcal{R}_i\setminus\mathcal{H}_i$ is the corresponding common part whose union defines $\mathcal{C} = \cup_{i=1}^{\mu}\overline{\mathcal{R}}_i$ with $c = |\mathcal{C}|$, see Example 2.1 below.

The Laplacian $\mathcal{L} = D - A$ of $\mathcal{G}$, $\mathcal{L} \in \mathbb{R}^{N\times N}$ has one eigenvalue $\lambda = 0$ with algebraic multiplicity μ equal to the number of reaches of $\mathcal{G}$ and all other eigenvalues with positive real parts [21], [28]. After suitably reordering the graphs nodes, the Laplacian $\mathcal{L}$ admits the upper triangular form

$$\mathcal{L} = \begin{pmatrix} \mathrm{diag}_{i=1}^{\mu}(\mathcal{L}_1) & \mathbf{0} \\ \mathrm{row}_{i=1}^{\mu}(\mathcal{M}_i) & \mathcal{M} \end{pmatrix},\ \mathcal{L}_i = \begin{pmatrix} P_i & \mathbf{0} \\ R_i & Q_i \end{pmatrix} \qquad (1)$$

where $\mathcal{L}_i \in \mathbb{R}^{h_i\times h_i}$ is the Laplacian associated to $\mathcal{H}_i$ and $\mathcal{M} \in \mathbb{R}^{c\times c}$ associated to the common part $\mathcal{C}$ of the digraph. P_i is a square matrix of size $p_i = |\mathcal{P}_i|$ with $\mathcal{P}_i \subseteq \mathcal{H}_i$ the set of root nodes in $\mathcal{H}_i$. Q_i is non-singular of dimension $h_i - p_i = |\mathcal{Q}_i|$ with $\mathcal{Q}_i = \mathcal{H}_i\setminus\mathcal{P}_i$. As a consequence, each $\mathcal{L}_i$ possesses an eigenvalue $\lambda = 0$ with algebraic multiplicity 1 and $\sigma(\mathcal{M}) \subset \mathbb{C}^+$. If $\mathcal{M}_i = \mathbf{0}$ then the reach $\mathcal{R}_i$ defines a disconnected

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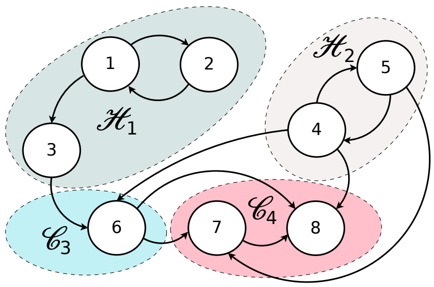

Fig. 1. A weakly connected digraph and its coarsest AEP. $\mathcal{R}_1 = \mathcal{R}(1) = \mathcal{R}(2) = \{1,2,3,6,7,8\}$ and $\mathcal{R}_2 = \mathcal{R}(4) = \mathcal{R}(5) = \{4,5,6,7,8\}$ are the maximal sets of reachable nodes and they thus correspond to the two reaches of the graph. Their exclusive parts are $\mathcal{H}_1 = \{1,2,3\}$ and $\mathcal{H}_2 = \{4,5\}$, while the common part $\mathcal{C} = \{6,7,8\}$ contains two cells, $\mathcal{C} = \mathcal{C}_3 \cup \mathcal{C}_4$.

component of the graph $\mathcal{G}$. The eigenspace associated to the zero eigenvalue of $\mathcal{L}$ is $E_s = \text{span}\{z_1,\dots,z_\mu\}$ with

$$z_1 = \begin{pmatrix} \mathbb{1}_{h_1} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \gamma^1 \end{pmatrix}, \quad z_2 = \begin{pmatrix} \mathbf{0} \\ \mathbb{1}_{h_2} \\ \vdots \\ \mathbf{0} \\ \gamma^2 \end{pmatrix}, \quad \dots, \; z_\mu = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbb{1}_{h_\mu} \\ \gamma^\mu \end{pmatrix} \quad (2)$$

with $\sum_{i=1}^{\mu} \gamma^i = \mathbb{1}_c$, $\mathcal{L}_i \mathbb{1}_{h_i} = 0$ and $\mathcal{M}_i \mathbb{1}_{h_i} + \mathcal{M}\gamma^i = 0$ for all $i = 1,\dots,\mu$. The left eigenvectors associated to the zero eigenvalue of $\mathcal{L}$ are given by

$$\tilde{v}_1^\top = \begin{pmatrix} v_1^\top & \dots & \mathbf{0} & \mathbf{0} \end{pmatrix}, \dots, \; \tilde{v}_\mu^\top = \begin{pmatrix} \mathbf{0} & \dots & \nu_\mu^\top & \mathbf{0} \end{pmatrix}. \quad (3)$$

All $\nu_i^\top \in \mathbb{R}^{1\times h_i}$ satisfy $\nu_i^\top \mathcal{L}_i = 0$ and are assumed, without loss of generality, such that $\nu_i^\top \mathbb{1}_{h_i} = 1$ for all $i = 1,\dots,\mu$. Moreover, it can be checked that ν_i is non-negative with non-zero entries on $\mathcal{P}_i$, *i.e.* $\nu_i^\top = (\nu_{p_i}^\top \; \mathbf{0}_{h_i - p_i})$.

Definition 2.1: A *partition* $\pi = \{\rho_1,\dots,\rho_r\}$ of $\mathcal{V}$ is a collection of disjoint cells $\rho_i \subseteq \mathcal{V}$ such that $\cup_{i=1}^r \rho_i = \mathcal{V}$. The *characteristic vector* of a cell $\rho \subseteq \mathcal{V}$ is $p(\rho) = (p_1(\rho) \; \dots p_N(\rho))^\top \in \mathbb{R}^N$ with for $i = 1,\dots,N$, $p_i(\rho) = 1$ if $v_i \in \rho$ and 0 otherwise. The *characteristic matrix* of a partition $\pi = \{\rho_1,\dots,\rho_r\}$ of $\mathcal{V}$ is $P(\pi) = \text{row}_i(p(\rho_i))$. π_1 is said to be *coarser* than π_2 ($\pi_1 \succeq \pi_2$) if all cells of π_2 are a subset of some cell of π_1. We name $\pi = \mathcal{V}$ the *trivial partition* containing a unique cell with all nodes.

Definition 2.2: A partition $\pi_{AE} = \{\rho_1,\rho_2,\dots,\rho_r\}$ is said to be an *almost equitable partition (AEP)* of $\mathcal{G}$ if, for each $i,j \in \{1,2,\dots,r\}$, with $i \neq j$, there exists an integer d_{ij} such that $|\mathcal{N}(v,\rho_j)| = d_{ij}$ for all $v \in \rho_i$, where $\mathcal{N}(v,\rho)$ denotes the set of neighbors of v in the cell ρ.
Equivalently and in terms of Laplacian, π is an AEP for $\mathcal{G}$ if and only if $\text{Im}P(\pi)$ is $\mathcal{L}$-invariant [22]. A non trivial partition $\pi^\star$ is the coarsest AEP of $\mathcal{G}$ if for all non trivial π AEP of $\mathcal{G}$ then $\pi^\star \succeq \pi$.

The coarsest AEP can be represented, exploiting (1), as

$$\pi^\star = \{\mathcal{H}_1,\dots,\mathcal{H}_\mu,\mathcal{C}_{\mu+1},\dots,\mathcal{C}_{\mu+k}\}. \quad (4)$$

Thus, a further partition within the common $\mathcal{C}$ is revealed: nodes in $\mathcal{C}$ belong to the same cell $\mathcal{C}_{\mu+\ell}$ of the AEP if for all $i = 1,\dots,\mu$ the corresponding components of γ^i coincide. We denote $c_\ell = |\mathcal{C}_{\mu+\ell}|$, $\ell = 1,\dots,k$.

Example 2.1: In the digraph of Fig.1 there are two reaches $\mathcal{R}_1$ and $\mathcal{R}_2$ that correspond to the two maximal sets of reachable nodes $\mathcal{R}(1) = \mathcal{R}(2)$ and $\mathcal{R}(4) = \mathcal{R}(5)$[1]. Thus, $\mu = 2$ is the multiplicity of 0 as an eigenvalue of the Laplacian. The exclusive part of $\mathcal{R}_1$ is $\mathcal{H}_1 = \{1,2,3\}$ and its root nodes are $\mathcal{P}_1 = \{1,2\}$. The exclusive part of $\mathcal{R}_2$ is $\mathcal{H}_2 = \{4,5\}$ and its root nodes are $\mathcal{P}_2 = \{4,5\}$. Notice that the sub-graphs associated to $\mathcal{P}_1$ and $\mathcal{P}_2$ are strongly connected. The common part of the reaches is $\mathcal{C} = \overline{\mathcal{R}}_1 = \overline{\mathcal{R}}_2 = \{6,7,8\}$ which is composed by the two cells $\mathcal{C}_3$ and $\mathcal{C}_4$ of the AEP. Nodes in $\mathcal{C}_3$ have one neighbor in $\mathcal{H}_1$ and one neighbor in $\mathcal{H}_2$, whereas nodes in $\mathcal{C}_4$ have one neighbor in $\mathcal{C}_3$ and one neighbor in $\mathcal{H}_2$.

B. Multiconsensus of scalar integrators

As proved in [24] the notion of AEP is linked to the characterization of *multi-consensus* for multi-agent systems. Consider a set of N of scalar integrators $\dot{x}_j = u_j, \quad x_j \in \mathbb{R}$ that exchange information based on a communication graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ whose vertices $j \in \mathcal{V}$ correspond to the agent (or, equivalently, node) x_j. Then, under the coupling rule

$$u_j = -\sum_{r\in\mathcal{N}_j} (x_j - x_r) = -\sum_{r=1}^{N} \mathcal{L}_{jr} x_r \quad (5)$$

nodes asymptotically cluster into $r = \mu + k$ consensuses with such clusters uniquely defined by the coarsest AEP $\pi^\star$: the states of all agents belonging to the same cell of the AEP converge to the same consensus state. Considering the aggregate network dynamics $\dot{x} = -\mathcal{L}x$ with $x = \text{col}_{j=1}^N\{x_j\}$, $\mathcal{L}$ as in (1), and

$$\boldsymbol{x}_i = \text{col}_{j\in\mathcal{H}_i}(x_j) \in \mathbb{R}^{h_i}, \; \boldsymbol{x}_{\mu+\ell} = \text{col}_{j\in\mathcal{C}_{\mu+\ell}}(x_j) \in \mathbb{R}^{c_\ell},$$

for $i = 1,\dots,\mu$ and $\ell = 1,\dots,k$, then each $\gamma^i \in \mathbb{R}^c$ in (2) reads

$$\gamma^i = \text{col}_{\ell=1}^k(\mathbb{1}_{c_\ell} \otimes \gamma_\ell^i), \quad \gamma_\ell^i \in \mathbb{R}. \quad (6)$$

Thus, as $t \to \infty$, the following holds true:

1) nodes in $\mathcal{H}_i$ converge to a consensus value; i.e., for all $i = 1,\dots,\mu$ and $j \in \mathcal{H}_i$
$$\boldsymbol{x}_i(t) \to \mathbb{1}_{h_i} \otimes x_i^{ss}, \quad x_i^{ss} := \nu_i^\top \boldsymbol{x}_i(0)$$
with $\nu_i^\top$ being the left eigenvector (3);
2) nodes in $\mathcal{C}_{\mu+\ell}$ for $\ell = 1,\dots,k$ converge to a convex combination of the consensuses induced by the $\mathcal{H}_i$,
$$\boldsymbol{x}_{\mu+\ell}(t) \to \mathbb{1}_{c_\ell} \otimes x_{c_\ell}^{ss}, \quad x_{c_\ell}^{ss} := \sum_{i=1}^{\mu} \gamma_\ell^i x_i^{ss}.$$

We note that, due to the structure (3), the consensus values x_i^{ss} and $x_{c_\ell}^{ss}$ (with $i = 1,\dots,\mu$ and $\ell = 1,\dots,k$) depend only on the initial condition of the root nodes of $\mathcal{H}_i$. These results were extended to general linear homogeneous systems in [26]. From now on, for a general digraph $\mathcal{G}$ with AEP $\pi^\star$ as

[1]The node reaches $\mathcal{R}(3)$, $\mathcal{R}(6)$, $\mathcal{R}(7)$ and $\mathcal{R}(8)$ are not graph reaches because they are contained in $\mathcal{R}_1$ and $\mathcal{R}_2$.

in (4), $\mu > 1$ and $k > 0$ will denote the number of exclusive reaches $\mathcal{H}_i$ and the number of cells of the AEP partitioning the common $\mathcal{C}$.

III. Problem statement and overview of the approach

Consider the case of heterogeneous agents connected by a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$. Let each agent $j \in \mathcal{V}$ be in the form

$$\dot{x}_j = A_j x_j + B_j u_j \tag{7a}$$
$$y_j = C_j x_j \tag{7b}$$

with $x_j \in \mathbb{R}^{n_j}$, $y_j \in \mathbb{R}^q$, $u_j \in \mathbb{R}^{p_j}$. All agents are assumed stabilizable and detectable and the nodes of the digraph suitably sorted so that the communication digraph exhibits the triangular form (1). Let $\boldsymbol{x}_i = \mathrm{col}_{j\in\mathcal{H}_i}(x_j) \in \mathbb{R}^{\boldsymbol{h}_i}$, $\boldsymbol{u}_i = \mathrm{col}_{j\in\mathcal{H}_i}(u_j) \in \mathbb{R}^{\boldsymbol{p}_i}$, $\boldsymbol{y}_i = \mathrm{col}_{j\in\mathcal{H}_i}(y_j) \in \mathbb{R}^{qh_i}$ be, respectively, the stack of the state, input and output variables in the reach $\mathcal{H}_i$ and $\boldsymbol{x}_{\mu+\ell} = \mathrm{col}_{j\in\mathcal{C}_{\mu+\ell}}(x_j) \in \mathbb{R}^{\boldsymbol{c}_\ell}$, $\boldsymbol{u}_{\mu+\ell} = \mathrm{col}_{j\in\mathcal{C}_{\mu+\ell}}(u_j) \in \mathbb{R}^{\boldsymbol{p}_\ell}$, $\boldsymbol{y}_{\mu+\ell} = \mathrm{col}_{j\in\mathcal{C}_{\mu+\ell}}(u_j)(y_j) \in \mathbb{R}^{qc_\ell}$ be, respectively, the stack of the state, input and output variables in the cell $\mathcal{C}_{\mu+\ell}$, for $i = 1, \ldots, \mu$ and $\ell = 1, \ldots, k$. In the following, we address the problem of designing a distributed control law driving the output of all agents to multi-consensus trajectories induced by the digraph $\mathcal{G}$ via the almost equitable partition (4). More in detail, we seek for a distributed feedback ensuring the following properties.

P1. The output evolutions of all nodes belonging to the same reach $\mathcal{H}_i$ converge to a desired suitably defined trajectory $\theta_{s,i}(t)$ (as introduced in Section IV below); namely, for $i = 1, \ldots, \mu$, as $t \to \infty$

$$\boldsymbol{y}_i(t) \to (\mathbb{1}_{h_i} \otimes I_q)\theta_{s,i}(t).$$

P2. The output evolutions of all nodes in $\mathcal{C}$ within the same cell of the AEP in (4) converge to a suitably defined convex combination of $\theta_{s,i}(t)$; namely, for all $\mathcal{C}_{\mu+\ell} \in \pi^\star$, γ_i^ℓ as in (6), $\ell = 1, \ldots, k$ and $t \to \infty$

$$\boldsymbol{y}_{\mu+\ell}(t) \to (\mathbb{1}_{c_\ell} \otimes I_q) \sum_{i=1}^{\mu} \gamma_\ell^i \theta_{s,i}(t).$$

As the intuition suggests, an internal model principle condition as proved in [27] for single consensus should also hold to get linear output multi-consensus.

The approach that we propose is the following. In the first place we associate to each node $j \in \mathcal{V}$ a set of N homogeneous (i.e., identical) reference generators of the form

$$\dot{w}_j = S w_j + K v_j \tag{8a}$$
$$\theta_j = Q w_j \tag{8b}$$

with $w_j \in \mathbb{R}^{n_0}$ for some n_0, $\theta_j \in \mathbb{R}^q$, $S \in \mathbb{R}^{n_0 \times n_0}$ and $Q \in \mathbb{R}^{q \times n_0}$ with (S, Q) observable and such that the resonance condition below holds for all $j \in \mathcal{V}$

$$\mathrm{rank}\begin{pmatrix} A_j - \lambda I_n & B_j \\ C_j & 0 \end{pmatrix} = n_j + q, \text{ for all } \lambda \in \sigma(S). \tag{9}$$

The second step is to assume that each node may access the output of the generators of its neighbors and to induce output multi-consensus on the generators. To this aim $v_j \in \mathbb{R}^q$ is designed as a coupling induced by $\mathcal{G}$ of the form

$$v_j = -\sum_{r\in\mathcal{N}_j}(\theta_j - \theta_r) = -\sum_{r=1}^{N} \mathcal{L}_{jr}(\theta_j - \theta_r), \tag{10}$$

with the coupling strength matrix $K \in \mathbb{R}^{n_0 \times q}$ designed to guarantee that the outputs of all agents in the same cell synchronize; i.e., as $t \to \infty$

$$\theta_j(t) \to \theta_{s,i}(t), \quad j \in \mathcal{H}_i \tag{11a}$$
$$\theta_j(t) \to \sum_{i=1}^{\mu} \gamma_\ell^i \theta_{s,i}(t), \quad j \in \mathcal{C}_{\mu+\ell} \subseteq \mathcal{C}. \tag{11b}$$

The third step is to design a local and distributed regulator so that the output of each agent is forced to track the output $\theta_j(t)$ of the local reference generator (8) that is, for all $j \in \mathcal{V}$ and as $t \to \infty$ one gets $y_j(t) \to \theta_j(t)$, with the constraint that each agent can access only measures of the output of the corresponding generator with no global knowledge of the network configuration.

Remark 3.1: The matrices (S, Q) can be suitably designed to generate the desired consensus trajectories $\theta_{s,i} : \mathbb{R}_+ \to \mathbb{R}^q$. Their choice allows the designer to impose a family of desired consensuses (e.g., constant, sinusoidal, etc). We shall prove that the coupling strength K can be the same for all the agents and it is possible to compute it in a distributed way. The local regulator is specific to each agent, due to the heterogeneous nature of the agents.

Remark 3.2: In the following, we assume that nodes of the network are ordered so that the Laplacian is of the form (1) with AEP as in (4). In addition, agents within the common $\mathcal{C}$ are assumed sorted so that the last component of the eigenvectors in (2) get the form (6). Such a sorting and the spectral information of (1) are only used in the proofs of the results to come. No knowledge of the aforementioned quantities is used for implementing the control laws we will design. All results hold true for the case of generally labeled nodes of $\mathcal{G}$.

IV. Design of the reference generators

In this section we illustrate how to design K so that the output of agents in the same cell of the AEP $\pi^\star$ in (4) converge to the same consensus trajectory, independent from K, the so-called *mean-field dynamics* of the cluster of generators (8) with the corresponding state the so-called *mean-field unit* [29]. The section also provides the expression of these consensus trajectories. In particular, for the cells $\mathcal{H}_i$ the mean-field dynamics depends only on the pair (S, Q) and on the initial conditions of the root nodes in $\mathcal{P}_i \subseteq \mathcal{H}_i$. Conversely, for the cells $\mathcal{C}_{\mu+\ell}$ the consensus trajectory is a convex combination of the mean-field dynamics of the cells $\mathcal{H}_i$. The results in this section extend those of [26] to the case when only the output of the neighbors is available.

The network composed of the N identical reference generators (8) can be compactly rewritten as

$$\dot{\boldsymbol{w}}_i = \big((I_{h_i} \otimes S) - (\mathcal{L}_i \otimes KQ)\big)\boldsymbol{w}_i \tag{12a}$$

$$\dot{\boldsymbol{w}}_{\mathcal{C}} = \big((I_c \otimes S) - (\mathcal{M} \otimes KQ)\big)\boldsymbol{w}_{\mathcal{C}} - \sum_{i=1}^{\mu}(\mathcal{M}_i \otimes KQ)\boldsymbol{w}_i \tag{12b}$$

$$\boldsymbol{\theta}_i = (I_{h_i} \otimes Q)\boldsymbol{w}_i \tag{12c}$$

$$\boldsymbol{\theta}_{\mathcal{C}} = (I_c \otimes Q)\boldsymbol{w}_{\mathcal{C}} \tag{12d}$$

with, $i = 1, \dots, \mu$ and $\ell = 1, \dots, k$, $\boldsymbol{w}_i = \mathrm{col}_{j\in\mathcal{H}_i}(w_j) \in \mathbb{R}^{n_0 h_i}$, $\boldsymbol{\theta}_i = \mathrm{col}_{j\in\mathcal{H}_i}(\theta_j) \in \mathbb{R}^{q h_i}$, $\boldsymbol{w}_{\mu+\ell} = \mathrm{col}_{j\in\mathcal{C}_{\mu+\ell}}(w_j) \in \mathbb{R}^{n_0 c_\ell}$, $\boldsymbol{\theta}_{\mu+\ell} = \mathrm{col}_{j\in\mathcal{C}_{\mu+\ell}}(\theta_j) \in \mathbb{R}^{q c_\ell}$, $\boldsymbol{w}_{\mathcal{C}} = \mathrm{col}_{\ell=1}^{k}(\boldsymbol{w}_{\mu+\ell}) \in \mathbb{R}^{n_0 c}$, $\boldsymbol{\theta}_{\mathcal{C}} = \mathrm{col}_{\ell=1}^{k}(\boldsymbol{\theta}_{\mu+\ell}) \in \mathbb{R}^{qc}$.

Proposition 4.1: Consider a digraph $\mathcal{G}$ that connects the homogeneous reference generators (8). Let $\pi^\star$ as in (4) be an AEP of $\mathcal{G}$. Then, the following holds.

1) In each $\mathcal{H}_i$ the functions $w_{s,i} : \mathbb{R}_+ \to \mathbb{R}^{n_0}$ and $\theta_i : \mathbb{R}_+ \to \mathbb{R}^q$ defined by

$$w_{s,i} = (\nu_i^\top \otimes I_{n_0})\boldsymbol{w}_i \tag{13a}$$

$$\theta_{s,i} = (\nu_i^\top \otimes I_q)\boldsymbol{\theta}_i \tag{13b}$$

are the mean-field units and evolve according to

$$\dot{w}_{s,i} = S w_{s,i} \tag{14a}$$

$$\theta_{s,i} = Q w_{s,i}, \tag{14b}$$

that is, $w_{s,i}(t) = e^{St}(\nu_i^\top \otimes I_{n_0})\boldsymbol{w}_i(0)$.

2) In each cell $\mathcal{C}_{\mu+\ell} \subseteq \mathcal{C}$ the mean-field units given by

$$w_{s,\mu+\ell} = \sum_{i=1}^{\mu}\gamma_\ell^i w_{s,i} \tag{15a}$$

$$\theta_{s,\mu+\ell} = \sum_{i=1}^{\mu}\gamma_\ell^i \theta_{s,i} \tag{15b}$$

evolve according to (14).

3) If $\boldsymbol{w}_i = \mathbb{1}_{h_i} \otimes w_{s,i}$ and $\boldsymbol{w}_{\mu+\ell} = \mathbb{1}_{c_\ell} \otimes w_{s,\mu+\ell}$, that is, if there is consensus in each cell, then

$$\boldsymbol{w}_{\mathcal{C}} = \sum_{i=1}^{\mu}\gamma^i \otimes w_{s,i} \tag{16a}$$

$$\boldsymbol{\theta}_i = \mathbb{1}_{h_i} \otimes \theta_{s,i}, \quad i = 1, \dots, \mu \tag{16b}$$

$$\boldsymbol{\theta}_{\mu+\ell} = \mathbb{1}_{c_\ell} \otimes \theta_{s,\mu+\ell}, \quad \ell = 1, \dots, k. \tag{16c}$$

Proof. Point (1). Since $\nu_i^\top \mathcal{L}_i = 0$ and $\nu_i^\top \mathbb{1}_{h_i} = 1$,

$$\begin{aligned}
\dot{w}_{s,i} &= (\nu_i^\top \otimes I_{n_0})\big((I_{h_i} \otimes S) - (\mathcal{L}_i \otimes KQ)\big)\boldsymbol{w}_i \\
&= (\nu_i^\top \otimes S)\boldsymbol{w}_i = S(\nu_i^\top \otimes I_{n_0})\boldsymbol{w}_i = S w_{s,i} \\
\theta_{s,i} &= (\nu_i^\top \otimes I_q)(I_{h_i} \otimes Q)(\mathbb{1}_{h_i} \otimes I_{n_0})w_{s,i} \\
&= (\nu_i^\top \otimes Q)(\mathbb{1}_{h_i} \otimes I_{n_0})w_{s,i} = Q w_{s,i}.
\end{aligned}$$

Point (2). Immediate by using (14) into (15).

Point (3). It can be proven as follows.

$$\begin{aligned}
\boldsymbol{w}_{\mathcal{C}} &= \mathrm{col}_{\ell=1}^{k}(\mathbb{1}_{c_\ell} \otimes w_{s,\mu+\ell}) \\
&= \mathrm{col}_{\ell=1}^{k}\left(\mathbb{1}_{c_\ell} \otimes \sum_{i=1}^{\mu}\gamma_\ell^i w_{s,i}\right) = \sum_{i=1}^{\mu}\gamma^i \otimes w_{s,i} \\
\boldsymbol{\theta}_i &= (I_{h_i} \otimes Q)(\mathbb{1}_{h_i} \otimes w_{s,i}) = \mathbb{1}_{h_i} \otimes (Q w_{s,i}) \\
&= \mathbb{1}_{h_i} \otimes \theta_{s,i} \\
\boldsymbol{\theta}_{\mu+\ell} &= (I_{c_\ell} \otimes Q)(\mathbb{1}_{c_\ell} \otimes w_{s,\mu+\ell}) = \mathbb{1}_{c_\ell} \otimes (Q w_{s,\mu+\ell}) \\
&= \mathbb{1}_{c_\ell} \otimes \theta_{s,\mu+\ell}
\end{aligned}$$

■

From the result above, it is clear that the mean-field dynamics are invariant for all generators; namely, when the corresponding initial conditions satisfy

$$\boldsymbol{w}_i(0) = (\mathbb{1}_{h_i}\nu_i^\top \otimes I_{n_0})\boldsymbol{w}_i(0)$$

$$\boldsymbol{w}_{\mu+\ell}(0) = \sum_{i=1}^{\mu}\gamma_\ell^i(\mathbb{1}_{c_\ell}\nu_i^\top \otimes I_{n_0})\boldsymbol{w}_i(0)$$

for $i = 1, \dots, \mu$ and $\ell = 1, \dots, k$, then all generators in the same cell of $\pi^\star$ in (4) evolve with the same trajectory provided by a combination of the outputs of the generators (8) associated to the root nodes of the communication digraph $\mathcal{G}$. More in details, for nodes in $\mathcal{H}_i$, the consensus trajectory is defined by a weighted mean of the initial states of the root nodes and governed by the mean-field dynamics (14). On the other side, the consensus of nodes in the same cell $\mathcal{C}_{\mu+\ell}$ is a convex combination of the consensuses over the reaches.

At this point, for inducing output multi-consensus, the coupling matrix K must be fixed to make the mean-field dynamics attractive for all agents, that is to asymptotically stabilize the multi-consensus error dynamics

$$\dot{\boldsymbol{e}}_i = \big((I_{h_i} \otimes S) - (\mathcal{L}_i \otimes KQ)\big)\boldsymbol{e}_i \tag{17a}$$

$$\dot{\boldsymbol{e}}_{\mathcal{C}} = \big((I_c \otimes S) - (\mathcal{M} \otimes KQ)\big)\boldsymbol{e}_{\mathcal{C}} - \sum_{i=1}^{\mu}(\mathcal{M}_i \otimes KQ)\boldsymbol{e}_i \tag{17b}$$

with the error components being

$$\boldsymbol{e}_i = \boldsymbol{w}_i - (\mathbb{1}_{h_i} \otimes I_{n_0})w_{s,i}, \quad i = 1, \dots, \mu \tag{18a}$$

$$\boldsymbol{e}_{\mu+\ell} = \boldsymbol{w}_{\mu+\ell} - \mathbb{1}_{c_\ell} \otimes \sum_{i=1}^{\mu}\gamma_\ell^i w_{s,i}, \quad \ell = 1, \dots, k. \tag{18b}$$

This can be achieved by exploiting [26, Theorem 3.3] for multi-consensus of homogeneous networks over weakly connected digraphs as detailed below.

Theorem 4.1: Consider a digraph $\mathcal{G}$ and a network of homogeneous reference generators (8) with (S, Q) observable and the coupling as in (10). For an arbitrary $a > 0$, let $P \in \mathbb{R}^{n_0 \times n_0}$ be the unique $P = P^\top \succ 0$ solution to the Riccati equation

$$SP + PS^\top + aI - PQ^\top QP = 0. \tag{19}$$

Then, setting

$$\begin{aligned}
&K = \kappa P Q^\top \\
&\kappa \geq \frac{1}{2\lambda_m}, \quad \lambda_m = \min_{\lambda \neq 0}\{Re(\lambda) : \lambda \in \sigma(\mathcal{L})\}.
\end{aligned} \tag{20}$$

the outputs of all generators (8) converge to a multi-consensus trajectory; i.e., for all $i = 1, \dots, \mu$, $\ell = 1, \dots, k$ and $\theta_{s,i} \in \mathbb{R}^q$ as in (13b)

$$\lim_{t\to\infty} \boldsymbol{\theta}_i(t) = \mathbb{1}_{h_i} \otimes \theta_{s,i}(t) \tag{21a}$$

$$\lim_{t\to\infty} \boldsymbol{\theta}_{\mu+\ell}(t) = \mathbb{1}_{c_\ell} \otimes \sum_{i=1}^{\mu} \gamma_\ell^i \theta_{s,i}(t). \tag{21b}$$

Proof. The result is proved if we show that $\boldsymbol{e}_i$ and $\boldsymbol{e}_{\mathcal{C}}$ are asymptotically stable. As proved in [26, Lemma A.1], since the $\mathcal{L}_i$ admit a spanning tree the error dynamics (17) is asymptotically stable if and only if the matrices $S_\lambda = S - \lambda K Q$ are Hurwitz for all $\lambda \in \sigma(\mathcal{L})\backslash\{0\}$. Accordingly, it is enough to show that for $P \succ 0$ then for all $x \in \mathbb{C}^{n_0}\backslash\{0\}$ and with K as in (20) $x^\star\Big(PS_\lambda^\star + S_\lambda P\Big)x < 0$. To this end, we compute

$$\begin{aligned}
&x^\star\Big(PS_\lambda^\star - S_\lambda P\Big)x\\
=&x^\star\Big(PS^\top + SP - \lambda KQP - \lambda^\star PQ^\top K^\top\Big)x\\
=&x^\star\Big(PS^\top + SP - (\lambda+\lambda^\star)\kappa PQ^\top QP\Big)x\\
=&x^\star\Big(PS^\top + SP - 2\lambda_m \kappa PQ^\top QP\Big)x\\
\leq&x^\star\Big(PS^\top + SP - PQ^\top QP\Big)x\\
=&-a\|x\|^2 < 0, \quad \text{for all } x \neq 0
\end{aligned}$$

so concluding the proof. ■

Remark 4.1: The distributed computation of K is possible. As a matter of fact, each $P \succ 0$ is solution to the Riccati equation (19) which depends only on the pair (S, Q) with no required information on the network topology. In addition, the bound $\kappa \geq 1/(2\lambda_m)$ in (20) can be satisfied can be computed locally, even if κ depends on the structure of the network. This can be made by choosing κ sufficiently large (thanks to the high-gain nature of the condition) or, if necessary, in a fully distributed way by using, e.g., the adaptive algorithm in [30].

V. Multi-consensus control via local regulation

Given a network of N identical generators (8) under the hypotheses of Theorem 4.1, we construct a distributed and local regulator to force the output of all agents (7) to asymptotically track the one of the corresponding generator (8); this corresponds to zeroing the regulation error

$$\varepsilon_j = y_j - \theta_j = C_j x_j - Q w_j, \quad j = 1, \dots, N. \tag{22}$$

In Section V-A, the regulation problem is solved by Theorem 5.1 when the state of the generators is accessible and all dynamics are known. In Section V-B, the local and distributed control law is designed for the case of partial information, that is, each agent (7) only possesses information on the output of the corresponding generator (8).

A. The case of full-information feedback

We consider the agent $j \in \mathcal{V}$ embedded with the corresponding set of h_i reference generators (12a) whose state is measured with apriori known matrices (S, Q).

Theorem 5.1: Consider a network of heterogeneous LTI systems of the form (7) being stabilizable and detectable with communication digraph $\mathcal{G}$. Consider the exosystem of reference generators (12a) under the hypotheses of Theorem 4.1 and assume that the resonance conditions (9) hold. Then, under the feedback law

$$u_j = F_j(x_j - \Pi_j w_j) + \Psi_j w_j \tag{23}$$

with F_j such that $\sigma(A_j + B_j F_j) \subset \mathbb{C}^-$ and (Π_j, Ψ_j) solution to the Francis equation

$$\begin{aligned}
\Pi_j S &= A_j \Pi_j + B_j \Psi_j\\
C_j \Pi_j &= Q
\end{aligned} \tag{24}$$

the outputs of all nodes converge to the consensus trajectory

$$\boldsymbol{y}_i(t) \to \mathbb{1}_{h_i} \otimes \theta_{s,i}(t), \quad \text{for all } j \in \mathcal{H}_i \tag{25a}$$

$$\boldsymbol{y}_\ell(t) \to \mathbb{1}_{c_\ell} \otimes \Big(\sum_{i=1}^{\mu} \gamma_\ell^i \theta_{s,i}(t)\Big), \quad \text{for all } j \in \mathcal{C}_{\mu+\ell} \tag{25b}$$

with $\theta_{s,i} \in \mathbb{R}^q$ as in (13b).

Proof. We use a coordinate transformation on $\boldsymbol{w}^i$ with arbitrarily chosen first component, $\boldsymbol{w}_1^i$, and $\boldsymbol{w}_2^i = \mathrm{col}_{j\in\mathcal{H}_i,\, j\neq i}(w_j - \boldsymbol{w}_1^i)$, representing the generators' disagreement within $\mathcal{H}_i$. That is,

$$\begin{pmatrix}\boldsymbol{w}_1^i\\ \boldsymbol{w}_2^i\end{pmatrix} = (T_i \otimes I_{n_0})\boldsymbol{w}_i, \quad T_i = \begin{pmatrix} 1 & \mathbf{0}\\ -\mathbb{1}_{h_1-1} & I_{h_1-1}\end{pmatrix} \tag{26}$$

where it is easy to see that T_i satisfy, for all $i = 1, \dots, \mu$

$$T_i \mathcal{L}_i T_i^{-1} = \begin{pmatrix} 0 & L_{i,1}\\ \mathbf{0} & L_{i,2}\end{pmatrix}.$$

For the nodes in the common, introduce the transformation

$$\tilde{\boldsymbol{w}}_{\mathcal{C}} = \boldsymbol{w}_{\mathcal{C}} - \sum_{i=1}^{\mu} (\gamma^i \otimes I_{n_0})\boldsymbol{w}_1^i \tag{27}$$

to all generators (12b) in $\mathcal{C}$ with $\boldsymbol{w}_1^i$ as in (26). Accordingly, for all $j \in \mathcal{H}_i$ we consider a local regulation problem over the extended system

$$\dot{\boldsymbol{w}}_1^i = S\boldsymbol{w}_1^i - (L_{i,1} \otimes KQ)\boldsymbol{w}_2^i \tag{28a}$$
$$\dot{\boldsymbol{w}}_2^i = \big((I_{h_i-1} \otimes S) - (L_{i,2} \otimes KQ)\big)\boldsymbol{w}_2^i \tag{28b}$$
$$\dot{x}_j = A_j x_j + B_j u_j \tag{28c}$$
$$\varepsilon_j = C_j x_j - Q(\boldsymbol{w}_1^i + E_j \boldsymbol{w}_2^i) \tag{28d}$$

with $E_1 = 0$, $E_j = e_j^\top \otimes I_{n_0}$ with $e_j \in \mathbb{R}^{h_i-1}$ the canonical vector. Introducing the coordinate transformation $z_j = x_j - \Pi_j \boldsymbol{w}_1^i$ and applying the feedback in (23), (28) reads

$$\begin{aligned}
\dot{\boldsymbol{w}}_1^i &= S\boldsymbol{w}_1^i - (L_{i,1} \otimes KQ)\boldsymbol{w}_2^i\\
\dot{\boldsymbol{w}}_2^i &= \big((I_{h_i-1} \otimes S) - (L_{i,2} \otimes KQ)\big)\boldsymbol{w}_2^i\\
\dot{z}_j &= (A_j + B_j F_j) z_j\\
\varepsilon_j &= C_j z_j - Q E_j \boldsymbol{w}_2^i.
\end{aligned}$$

Since $A_j + B_j F_j$ and $(I_{h_i-1} \otimes S) - (L_{i,2} \otimes KQ)$ are Hurwitz by construction, one gets $z_j(t) \to 0$ and $\boldsymbol{w}_2^i(t) \to 0$ as $t \to \infty$. Accordingly, $\varepsilon_j(t) \to 0$ and, hence, $y_j(t) \to \theta_j(t)$ as $t \to \infty$. Finally, from Theorem 4.1, $\theta_j(t) \to \theta_{s,i}(t)$ so that (25a) follows.

Analogously, for all $j \in \mathcal{C}_{\mu+\ell}$, we consider a regulation problem over the extended system

$$\begin{aligned}
\dot{\boldsymbol{w}}_1^i =& S\boldsymbol{w}_1^i - (L_{i,1} \otimes KQ)\boldsymbol{w}_2^i && \text{(29a)}\\
\dot{\boldsymbol{w}}_2^i =& \big((I_{h_i-1} \otimes S) - (L_{i,2} \otimes KQ)\big)\boldsymbol{w}_2^i && \text{(29b)}\\
\dot{\tilde{\boldsymbol{w}}}_{\mathcal{C}} =& \big((I_c \otimes S) - (\mathcal{M} \otimes KQ)\big)\tilde{\boldsymbol{w}}_{\mathcal{C}}\\
&- \sum_{i=1}^{\mu} \big((\gamma^i L_{i,1} \otimes KQ) + (M_i \otimes KQ)\big)\boldsymbol{w}_2^i && \text{(29c)}\\
\dot{x}_j =& A_j x_j + B_j u_j && \text{(29d)}\\
\varepsilon_j =& C_j x_j - Q\sum_{i=1}^{\mu} \gamma_\ell^i \boldsymbol{w}_1^i - QE_j\tilde{\boldsymbol{w}}_{\mathcal{C}} && \text{(29e)}
\end{aligned}$$

with $M_i = \mathcal{M}_i \begin{pmatrix} \mathbf{0}^\top & I_{c-1}\end{pmatrix}^\top$ and selection matrix $E_j = e_j^\top \otimes I_{n_0}$, with $e_j \in \mathbb{R}^c$ the canonical vector. Using again the coordinate transformation $z_j = x_j - \Pi_j w_j$ and (26)-(27) with the feedback (23), the system above gets the form

$$\begin{aligned}
\dot{\boldsymbol{w}}_1^i =& S\boldsymbol{w}_1^i - (L_{i,1} \otimes KQ)\boldsymbol{w}_2^i\\
\dot{\boldsymbol{w}}_2^i =& \big((I_{h_i-1} \otimes S) - (L_{i,2} \otimes KQ)\big)\boldsymbol{w}_2^i\\
\dot{\tilde{\boldsymbol{w}}}_{\mathcal{C}} =& \big((I_c \otimes S) - (\mathcal{M} \otimes KQ)\big)\tilde{\boldsymbol{w}}_{\mathcal{C}}\\
&- \sum_{i=1}^{\mu} \big((\gamma^i L_{i,1} \otimes KQ) + (M_i \otimes KQ)\big)\boldsymbol{w}_2^i\\
\dot{z}_j =& (A_j + B_jF_j)z_j\\
\varepsilon_j =& C_j z_j - QE_j\tilde{\boldsymbol{w}}_{\mathcal{C}}.
\end{aligned}$$

Since $A_j + B_jF_j$, $(I_{h_i-1} \otimes S) - (L_{i,2} \otimes KQ)$ and $(I_c \otimes S) - (\mathcal{M} \otimes KQ)$ are Hurtwitz, one gets asymptotically $z_j(t) \to 0$, $\boldsymbol{w}_2^i(t) \to 0$ and $\tilde{\boldsymbol{w}}_{\mathcal{C}}(t) \to 0$. Accordingly, $\varepsilon_j(t) \to 0$ and, hence, $y_j(t) \to \theta_j(t)$. In addition, by construction of the exosystem in Theorem 4.1, it is guaranteed that $\theta_j(t) \to \sum_{i=1}^{\mu} \gamma_\ell^i \theta_{s,i}(t)$ for all $j \in \mathcal{C}_{\mu+\ell}$, so that (25b) follows. ■

From Theorem 5.1 it follows that the distributed control laws (23) ensure output multi-consensus for the network of heterogeneous systems (7). For such a feedback to be implementable, all agents need to access the state of the corresponding generator and the matrices (S, Q).

B. The case of partial information feedback

Assuming only the output of the corresponding signal generator accessible to each agent, we construct a local and distributed feedback composed of:

(i) a (pre-processing) internal model of the exosystem injecting a copy of the multi-consensus behavior on all agents of the network;

(ii) a stabilizer making the output consensus trajectory attractive.

In order to prove the result we consider separately the problem over the reaches $\mathcal{H}_i$ $(i = 1, \dots, \mu)$ and then in $\mathcal{C}_{\mu+\ell} \subset \mathcal{C}$ $(\ell = 1, \dots, k)$. However, we shall conclude that the controller structure is the same in both cases and consequently each node can implement the control law without any knowledge about its role in the network. In the reaches $\mathcal{H}_i$ the exosystem driving the regulator is given by (28a). For all $j \in \mathcal{H}_i$ the corresponding local feedback is a system of the form

$$\begin{aligned}
\dot{\xi}_j =& L_j\xi_j + M_j(y_j - \theta_j) && \text{(30a)}\\
\dot{\eta}_j =& \Phi\eta_j + G\Big(N_j\xi_j + H_j(y_j - \theta_j)\Big) && \text{(30b)}\\
u_j =& \Gamma\eta_j + N_j\xi_j + H_j(y_j - \theta_j) && \text{(30c)}
\end{aligned}$$

where (30a) is the consensus stabilizer at each node and (30b) is the internal model of the exosystem (28a). This latter component allows to handle the case in which agents use distinct realizations of the output consensus trajectories. More in detail, denoting by $p(s) = c_0 + c_1 s + \dots + c_{n_0-1}s^{n_0-1} + s^{n_0}$ the minimal polynomial of S (that can be assumed known apriori to all agents with no loss of generality), a state realization of the output of each generator can be assumed of the form[2]

$$\Phi = S_0 \otimes I_q, \quad G = B_0 \otimes I_q \tag{31}$$

$$S_0 = \begin{pmatrix} 0 & 1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 1\\ -c_0 & -c_1 & \dots & -c_{n_0-1}\end{pmatrix}, \quad B_0 = \begin{pmatrix} 0\\ \vdots\\ 0\\ 1\end{pmatrix}$$

Exploiting the results in [31, Section 4.6] the following result can be proved.

Theorem 5.2: Consider a network of heterogeneous LTI systems of the form (7) under the hypotheses of Theorem 5.1 with (Π_j, Ψ_j) the solution to the Francis equation (24). Then, the control law (30) ensures that all outputs converge to the consensus trajectory in (25) provided that (Φ, G) are fixed as in (31), Γ and (L_j, M_j, N_j, H_j) are such that, respectively, $\sigma(\Phi - G\Gamma) \subset \mathbb{C}^-$ and

$$\sigma\begin{pmatrix} A_j + B_jH_jC_j & B_j\Gamma & B_jN_j\\ GH_jC_j & \Phi & GN_j\\ M_jC_j & \mathbf{0} & L_j\end{pmatrix} \subset \mathbb{C}^-. \tag{32}$$

Proof. By construction of (Φ, G) as in (31), one can pick Γ to make $\Phi - G\Gamma$ Hurwitz. Thus, there exists Ω_j satisfying the Francis equation $\Omega_j S = \Phi\Omega_j, \quad \Psi_j = \Gamma\Omega_j$ with Ψ_j the solution to (24). Accordingly, one must show that the feedback (30) ensures that, as $t \to \infty$, $x_j(t) \to \Pi_j\boldsymbol{w}_1^i(t)$ and $\eta(t) \to \Omega_j\boldsymbol{w}_1^i(t)$. To this end, let us first consider $j \in \mathcal{H}_i$ and the coordinate transformation $z_j = x_j - \Pi_j\boldsymbol{w}_1^i, \quad \zeta_j = \eta_j - \Omega_j\boldsymbol{w}_1^i$ and the augmented system

$$\begin{aligned}
\dot{\boldsymbol{w}}_1^i =& S\boldsymbol{w}_1^i - (L_{i,1} \otimes KQ)\boldsymbol{w}_2^i && \text{(33a)}\\
\dot{\boldsymbol{w}}_2^i =& \big((I_{h_i-1} \otimes S) - (L_{i,2} \otimes KQ)\big)\boldsymbol{w}_2^i && \text{(33b)}\\
\dot{z}_j =& -\big(B_jH_jQE_j - \Pi_j(L_{i,1} \otimes QK)\big)\boldsymbol{w}_2^i\\
&+ (A_j + B_jH_jC_j)z_j + B_j\Gamma\zeta_j + B_jN_j\xi_j && \text{(33c)}\\
\dot{\zeta}_j =& -\big(GH_jQE_r^i + \Omega_j(L_{i,1} \otimes QK)\big)\boldsymbol{w}_2^i\\
&+ GH_jC_jz_j + \Phi\zeta_j + GN_j\xi_j && \text{(33d)}\\
\dot{\xi}_j =& -M_jQE_j\boldsymbol{w}_2^i + M_jC_j\zeta_j + L_j\xi_j && \text{(33e)}\\
\varepsilon_j =& C_jz_j - QE_r\boldsymbol{w}_2^i && \text{(33f)}
\end{aligned}$$

[2] The realization in canonical controllable form is assumed the same for all nodes for the sake of clarity. However, different realizations can be associated to each node.

exhibiting a lower block triangular form with $E_1 = \mathbf{0}$, $E_j = e_j^\top \otimes I_{n_0}$ and $e_j \in \mathbb{R}^{h_i-1}$ the canonical vector. By Theorem 4.1, one gets that $\boldsymbol{w}_2^i(t) \to 0$ asymptotically because $(I_{h_i-1}\otimes S)-(L_{i,2}\otimes KQ)$ is Hurwitz. Accordingly, $z_j(t) \to 0$ and $\zeta_j(t) \to 0$ if and only if (32) is Hurwitz. To this end, because (A_j, B_j, C_j) are stabilizable and detectable and (9) holds, then there exist (L_j, M_j, N_j, H_j) ensuring that the matrix (32) possesses all eigenvalues with negative real part. Accordingly, one gets $\varepsilon_j(t) \to 0$ as $t \to \infty$ and thus the result for the reaches. Let us consider now nodes in the common $j \in \mathcal{C}_{\mu+\ell}$ and the dynamics composed by (33a), (33b) and

$$\begin{aligned}
\dot{\tilde{\boldsymbol{w}}}_{\mathcal{C}} =& \big((I_c \otimes S) - (\mathcal{M} \otimes KQ)\big)\tilde{\boldsymbol{w}}_{\mathcal{C}} \\
&- \sum_{i=1}^{\mu} \big((\gamma^i L_{i,1} \otimes KQ) + (M_i \otimes KQ)\big)\boldsymbol{w}_2^i \\
\dot{z}_j =& -\sum_{i=1}^{\mu} \Pi_j^i (L_{i,1} \otimes QK)\big)\boldsymbol{w}_2^i - B_j H_j Q E_{\bar{r}}^{\ell} \boldsymbol{w}_{\mathcal{C}} \\
&+ (A_j + B_j H_j C_j) z_j + B_j(\Gamma \zeta_j + B_j N_j \xi_j) \\
\dot{\zeta}_j =& -\sum_{i=1}^{\mu} \Omega_j (L_{i,1} \otimes QK)\boldsymbol{w}_2^i - G H_j Q E_{\bar{r}}^{\mu+\ell} \boldsymbol{w}_{\mathcal{C}} \\
&+ G H_j C_j z_j + \Phi \zeta_j + G N_j \xi_j \\
\dot{\xi}_j =& - M_j Q E_{\bar{r}} \tilde{\boldsymbol{w}}_{\mathcal{C}} + M_j C_j \zeta_j + L_j \xi_j \\
\varepsilon_j =& C_j z_j - Q E_{\bar{r}}^{\ell} \tilde{\boldsymbol{w}}_{\mathcal{C}}
\end{aligned}$$

with selection matrix $E_j = e_j^\top \otimes I_{n_0}$ and $e_j \in \mathbb{R}^c$ the canonical vector. By Theorem 4.1, one gets that $\boldsymbol{w}_2^i(t) \to 0$ and $\boldsymbol{w}_{\mathcal{C}}(t) \to 0$ asymptotically because $(I_{h_i-1}\otimes S)-(L_{i,2}\otimes KQ)$ and $(I_c\otimes S)-(\mathcal{M}\otimes KQ)$ are Hurwitz. Accordingly, $z_j(t) \to 0$ and $\zeta_j(t) \to 0$ if and only if (32) holds. To this end, because (A_j, B_j, C_j) are stabilizable and detectable and (9) holds, there exist (L_j, M_j, N_j, H_j) making the matrix in (32) Hurwitz. Accordingly, one gets $\varepsilon_j(t) \to 0$ and the result. ■

Theorem 5.2 characterizes, through the resonance conditions (9), the class of output multi-consensuses which are admissible for a network of heterogeneous dynamics (7) over a general communication digraph $\mathcal{G}$. As a matter of fact, the assignable multi-consensuses output evolutions are the ones whose modes (associated to a suitably defined dynamical matrix S) are compatible with all agents' dynamics. For this reason, the internal model (30a)-(30b) that is injected in all agents under feedback is homogeneous.

Remark 5.1: The dynamical matrix in (32) is the one governing the regulation error (22). Accordingly, multi-consensus can be enforced over the network as Theorem 5.2 ensures the existence of a controller making the corresponding dynamics asymptotically stable that is equivalent to satisfy (32).

Remark 5.2: For the design of the local control law (30), the knowledge of the spectral and structural properties of the Laplacian (and hence of the network) are not required. Also, no agent must possess the information of the cell of the partition it belongs to. The corresponding decomposition is used only for proving the results in a structural manner. Accordingly, the feedback (30) can be computed in a decentralized and distributed way.

Thanks to the regulation framework we have settled the problem into [31], assuming all nodes only possess measures the output of the corresponding generator, the solution specified in Theorem 5.2 can be extended to handle robust design, with due modifications. This includes, for instance, the following cases: (i) agents cannot measure the corresponding state; (ii) measures of the state of each generator are not available to the corresponding node; (iii) agents do not possess information on the network configuration (and thus, of the corresponding exosystem); (iv) the model of each agent (i.e., the matrices A_j, B_j, C_j) are known with uncertainties; (v) the matrix S characterizing all generators (and the multi-consensus state dynamics) is not known to all agents. This latter case includes different further scenarios often occurring in practice as, for instance, the following ones: the output of the generator is known to the corresponding agent; for a fixed output trajectory, the exosystem (and thus the realization of the output trajectories) is not the same for all agents.

Accordingly, the general stabilizers (30a) allow the inclusion of adaptive laws or robust observers for the stabilization of the whole dynamics independently on the knowledge of the corresponding model and of the network structure. For similar reasons, the internal model (30b) defines a realization of the output evolutions of the multi-consensus (whose nature is deduced from the measures of the reference generator) when the specific matrix S and the network structure are available with uncertainty.

As the intuition suggests, under stronger hypotheses the design can be notably simplified as highlighted in the following remarks.

Remark 5.3: If each agent $j \in \mathcal{V}$ knows the corresponding (A_j, B_j, C_j) with no uncertainty, the control (30) can be simplified setting

$$\begin{aligned}
&N_j + H_j C_j = F_j, \quad L_j - B_j N_j = A_j - K_j C_j \\
&K_j = M_j - B_j H_j
\end{aligned}$$

so that (32) holds whenever M_j and H_j guarantee

$$\sigma \begin{pmatrix} A_j - K_j C_j & B_j \Gamma \\ -G N_j & \Phi \end{pmatrix} \subset \mathbb{C}^-.$$

Remark 5.4: If each agent $j \in \mathcal{V}$ knows the corresponding (A_j, B_j, C_j) and measures the state of the corresponding generator, the feedback (30) reduces to the output controller

$$\begin{aligned}
\dot{\xi}_j =& A_j \xi_j + B_j u_j + K_j (y_j - C_j \xi_j) \\
u_j =& \Psi_j w_j + F_j (\xi_j - \Pi_j w_j)
\end{aligned}$$

with K_j making $\sigma(A_j - K_j C_j) \subseteq \mathbb{C}^-$ and the dynamical component being a local state observer. This case represents an extension of the solution proposed in [27] for single consensus.

Remark 5.5: When the model of each agent and the matrix S are known to all agents, the feedback (30) can be modified as follows: the component (30a) is replaced by the corresponding observers presented in Remark 5.4; the internal models (30b) constitute an observer of the state of the corresponding generator. In the latter case, the generator observer can be designed as proposed in [26, Section 4] depending on the information that is available to each node.

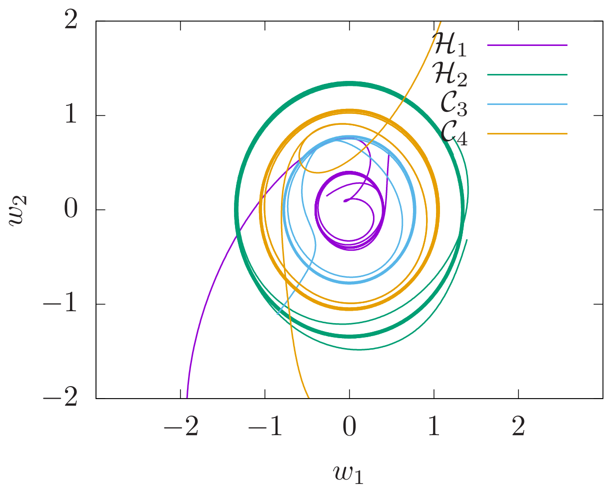

Fig. 2. Clustering of the reference generators in the four cells of the AEP π^* in the state space $(w_1,\, w_2)$ of the generators.

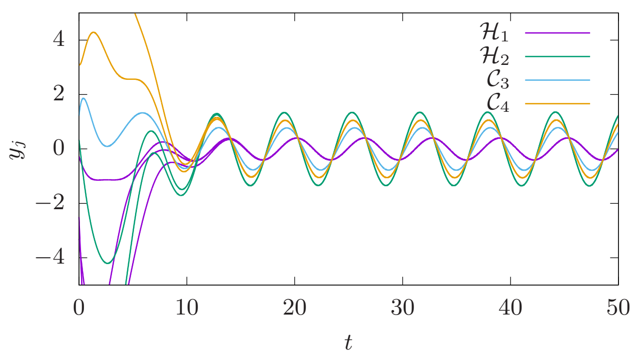

Fig. 3. Containment of the output $y_j(t)$ as a function of time.

VI. Examples

A. An academic example

We consider the containment problem for the example proposed in [27] of a set of interconnected double integrators driven by different types of actuators,

$$A_j = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & c_j \\ 0 & -d_j & -a_j \end{pmatrix} \; B_j = \begin{pmatrix} 0 \\ 0 \\ b_j \end{pmatrix} \; C_j = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{34}$$

with $a_j,\, b_j,\, c_j > 0$ and $d_j \geq 0$. The systems are observable and controllable, the first two components are a double integrator and the third component can be considered as the actuator state, with speed a_j and gains $b_j,\, c_j$. In the simulations these parameters are randomly chosen with uniform distribution in the interval $[0,\, 5]$.

We want to confine the first component with a sinusoidal function, thus we choose

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{35}$$

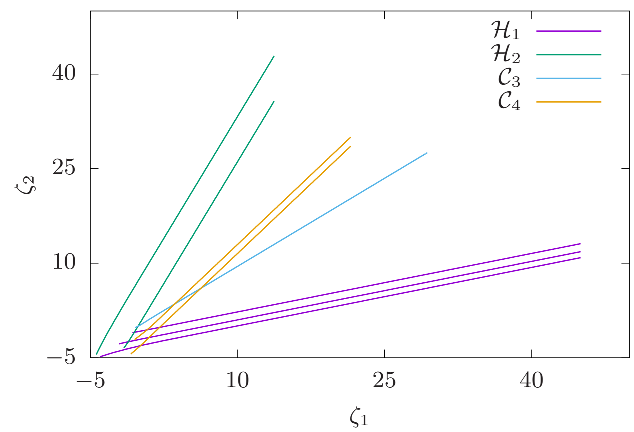

Fig. 4. Clustered trajectories of the unicycles.

The regulator equations (24) are satisfied by

$$\Pi_j = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1/c_j & 0 \end{pmatrix}, \quad \Psi_j = \begin{pmatrix} \frac{a_j}{b_j c_j} & \frac{d_j}{b_j} \end{pmatrix}. \tag{36}$$

The graph in Fig. 1 contains $|\mathcal{N}| = 8$ nodes and the coarsest equitable partition is $\pi^* = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{C}_3, \mathcal{C}_4\}$ where $\mathcal{H}_1 = \{1,\, 2,\, 3\}$, $\mathcal{H}_2 = \{4,\, 5\}$, $\mathcal{C}_3 = \{6\}$ and $\mathcal{C}_4 = \{7,\, 8\}$. The multiplicity of 0 as an eigenvalue of $\mathcal{L}$ is $\mu = 2$, thus there are two reaches, whose exclusive parts are $\mathcal{H}_1$ and $\mathcal{H}_2$ with $h_1 = 3$ and $h_2 = 2$. The common part contains $c = 3$ nodes and it is the union of $\mathcal{C}_3$ (with $c_3 = 1$) and $\mathcal{C}_4$ (with $c_4 = 2$). Thus the number of cells in the common part is $k = 2$.

The local full-information controller is implemented as

$$u_j = F_j(x_j - \Pi_j w_j) + \Psi_j w_j \tag{37}$$

with the gains F_j chosen to assign the eigenvalues $\{-0.5, -1.0, -1.5\}$ to the matrices $A_j + B_j F_j$. The consensus gain K is chosen as in (19), (20) with $a = 2$ and $\kappa = 0.5$ that yields $K = [0.9306,\, 0.3660]^\top$. With random initial conditions for the $N = 8$ agents, Fig. 2 shows the clustering of the generators state in the 4 cells of the AEP π^*. Fig. 3 plots $y_j(t)$ over time.

B. Formation control of unicycles

Consider a group of unicycles

$$\begin{aligned} \dot{\zeta}_j &= g(\vartheta_j) v_j \\ \dot{v}_j &= \frac{1}{M_j} f_j \\ \dot{\vartheta}_j &= \omega_j \\ \dot{\omega}_j &= \frac{1}{J_j} \tau_j \end{aligned}$$

with $M_j, J_j > 0$, $\zeta_j \in \mathbb{R}^2$ the vector of cartesian position coordinates, $\vartheta \in \mathbb{R}$ the orientation, $u_j := (f_j \;\; \tau_j)^\top \in \mathbb{R}^2$ the input forces and $g(\vartheta_j) = (\cos\vartheta_j \;\; \sin\vartheta_j)^\top$.

We wish to design a feedback control ensuring all nodes in the same cluster of the AEP move in the same direction (that is, distinct parallel linear trajectories). To this aim, by

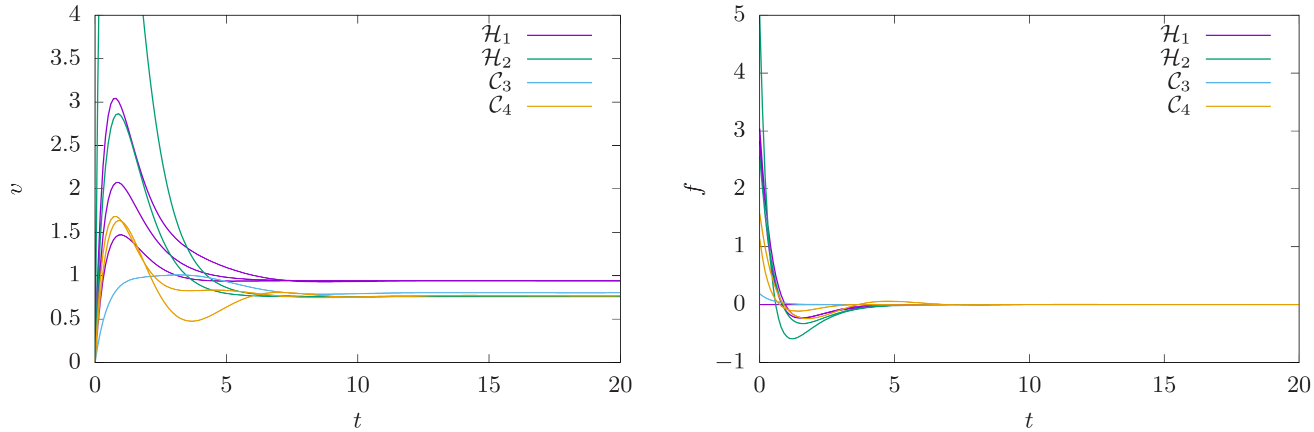

Fig. 5. Velocity (left) and control input f (right) of the unicycles.

assuming small velocities, we perform the design based on the linear tangent model at a fixed angle $\bar{\vartheta}_j \in \mathbb{R}$ (*a priori* different for all agents) that is given by

$$\dot{x}_j = A_j x_j + B_j u_j \tag{38}$$
$$y_j = C x_j \tag{39}$$

with $x_j = (\zeta_j^\top,\ v_j,\ \vartheta_j - \bar{\vartheta}_j,\ \omega_j)^\top \in \mathbb{R}^5$,

$$A_j = \begin{pmatrix} 0 & 0 & \cos\bar{\vartheta}_j & 0 & 0 \\ 0 & 0 & \sin\bar{\vartheta}_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_j = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{M_j} & 0 \\ 0 & 0 \\ 0 & \frac{1}{J_j}. \end{pmatrix}.$$

We note that the linear tangent model above is not controllable for any $\bar{\vartheta}_j \in \mathbb{R}$. Still, one can drive the agents to the required trajectory so that the problem admits a solution. To this end, we set the control objective with respect to one of the Cartesian coordinates only (say the first one, $\zeta_{j,1}$) by choosing the controlled output $y_j = (\zeta_{j,1}, v)$ with

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

In the coordinates $\tilde{x}_j = T_j x_j = (\zeta_1/\cos\bar{\vartheta}_j, v_j,\ \vartheta_j - \bar{\vartheta}_j, \omega_j)^\top \in \mathbb{R}^4$, (38)–(39) become

$$\dot{\tilde{x}}_j(t) = \tilde{A}\tilde{x}_j(t) + \tilde{B}_j u_j(t)$$
$$y_j(t) = \tilde{C}_j \tilde{x}_j(t)$$
$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{B}_j = \begin{pmatrix} 0 & 0 \\ \frac{1}{M_j} & 0 \\ 0 & 0 \\ 0 & \frac{1}{J_j} \end{pmatrix}$$
$$\tilde{C}_j = \begin{pmatrix} \cos\bar{\vartheta}_j & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and now the pair $(\tilde{A}, \tilde{B}_j)$ is controllable. Notice that the first component of $\tilde{x}$ is the abscissa along the trajectory. We can set the reference generator as $w_j = (\zeta_{j,1}^w, v_{j,1}, \vartheta_j - \bar{\vartheta}_j)$, where $\zeta_{j,1}^w$ and $v_{j,1}$ are the desired position and velocity with respect to the horizontal axis and the output is $\theta_j = (\zeta_{j,1}^w, \vartheta_j - \bar{\vartheta}_j)$

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The Francis equations on the reduced system, $\Pi_j S = \tilde{A}\Pi_j + \tilde{B}_j \Psi_j$, $\tilde{C}_j \Pi_j = Q$ are solved by $\Psi_j = 0$ and

$$\Pi_j = \begin{pmatrix} \frac{1}{\cos\bar{\vartheta}_j} & 0 & 0 \\ 0 & \frac{1}{\cos\bar{\vartheta}_j} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{40}$$

If each reference generator is initialized with $\vartheta_j - \bar{\vartheta}_j = 0$, the regulator will drive $\vartheta_j \to \bar{\vartheta}_j$; i.e. the orientation will be steered to the local linearization set-point. In order to make the orientations coincide within the same cluster, we add to the regulator design an orientation consensus algorithm on $\bar{\vartheta}_j$,

$$\dot{\bar{\vartheta}}_j = -\sum_{i \in \mathcal{N}_j} \bar{\vartheta}_i, \tag{41}$$

thus we assume that the agents exchange $y_j = (\zeta_{j,1}, v)$ and $\bar{\vartheta}_j$. With these choices, the unicycles in the same cluster will move with identical velocity, direction and horizontal coordinate $\zeta_{j,1}$ (*i.e.* they will be aligned along a vertical line). Finally, the control input is

$$u_j(t) = F_j(T_j x_j - \Pi_j w_j), \tag{42}$$

where F_j is chosen through an eigenvalue assignment algorithm so that $\tilde{A} + \tilde{B}_j F_j$ is Hurwitz. Notice that $\tilde{A}$ and $\tilde{B}_j$ do not depend on $\bar{\vartheta}_j$, thus the control gain F_j does not need to be adapted if $\bar{\vartheta}_j$ is modified by the consensus algorithm on the orientation.

Assuming the graph in Fig. 1, the unicycles will form four clusters corresponding to $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$. In $\mathcal{H}_1$, $\mathcal{H}_2$ the orientation will depend on the initial value ϑ_j of the root nodes, $j \in \{1,2\}$ and $j \in \{4,5\}$ respectively, that will determine the consensus value ϑ of the cluster. Since the regulator make $\vartheta_j \to \bar{\vartheta}$, this will steer the unicycles in the cluster to assume that orientation. Analogously, the horizontal velocity and position of the nodes in $\mathcal{H}_1$, $\mathcal{H}_2$ will depend on the initial conditions of the reference generators of the root nodes, $j \in \{1,2\}$ and $j \in \{4,5\}$ respectively. Finally, the orientation, horizontal speed and position of nodes in $\mathcal{C}_3$ and $\mathcal{C}_4$ will converge toward a suitable convex combination of the respective variables in $\mathcal{H}_1$, $\mathcal{H}_2$.

The trajectories of the unicycles for $t \in [0,50]$ are shown in Fig. 4. Initially, the unicycles' coordinates are chosen randomly in $[-5,0]\times[-5,0]$, with 0 initial speed and random orientation $\vartheta_j = \bar{\vartheta}_j$. When the control gains F_j are chosen to assign to $\tilde{A}+\tilde{B}_j F_j$ the eigenvalues $\{-1,\,-1.2,\,-1.4,1,-1.6\}$ the unicycles move toward the top right corner and form 4 clusters. Fig. 5 (left) shows the clustering of velocities of the four groups as a function of time $t \in [0,20]$ and Fig. 5 shows that the control input goes to 0 as the steady state is reached.

VII. Conclusions and Perspectives

In this paper, moving from the topology-induced containment design proposed in [26], the authors extend multi-consensus to a network of heterogeneous linear systems in the case of output information feedback, ensuring that parts of the overall multi-agent system simultaneously track different arbitrary trajectories.

Future work will be aimed, on the one hand, at further realizing the effectiveness of the proposed approach with respect to practical application scenarios, and, on the other hand, at extending the proposed output feedback containment design to networks of nonlinear interacting agents, ultimately removing the exosystem itself and embedding noise, delays and intermittent communication [32]–[35].

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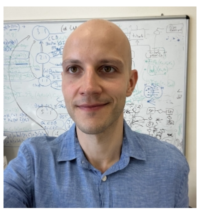

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