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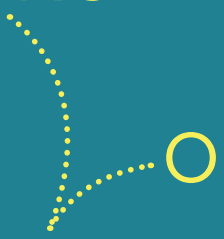
FRACTIONAL VOLTERRA INTEGRAL EQUATIONS



A NEURAL NETWORK APPROACH

RomaTrE-Press
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Fractional Volterra integral equations: a neural network approach
Marisa Cenci, Maria Alessandra Congedo, Antonio Luciano Martire and Barbara Rogo

Index

FRACTIONAL VOLTERRA INTEGRAL EQUATIONS: A NEURAL NETWORK APPROACH

Marisa Cenci, Maria Alessandra Congedo, Antonio Luciano Martire[*] and Barbara Rogo[**]

ABSTRACT

Il Calcolo Frazionario ha recentemente guadagnato un crescente interesse nella letteratura economica e finanziaria. Per quanto riguarda i modelli economici, quelli di crescita sono stati modellati utilizzando una rappresentazione tramite derivate frazionarie. Questo tipo di equazioni non consente soluzioni in forma chiusa e quindi è necessario ricorrere a metodi numerici appropriati per ottenere approssimazioni accurate delle soluzioni. Per questo motivo, in questo contributo, proponiamo un approccio basato sulle cosiddette *Physics Informed Neural Network* per risolvere le equazioni integrali di Volterra di ordine frazionario. Alcuni esperimenti numerici mostrano l'accuratezza dell'algoritmo suggerito.

Fractional calculus has recently gained increasing interest in the economic and financial literature. As for economic models, economic growth has been modeled using a state space representation of fractional derivatives. These kinds of equations do not allow closed-form solutions and therefore require appropriate numerical methods to obtain accurate approximations of the solutions. For this reason, in this paper, we propose an approach based on Physics Informed Neural Network to solve Volterra fractional-order integral equations. Some numerical experiments show the accuracy of the suggested algorithm.

KEYWORDS: Fractional differential equation, Fractional Volterra integral equations, Physics Informed Neural Network.

PAROLE CHIAVE: Equazioni differenziali frazionarie, Equazioni integrali frazionarie, Physics Informed Neural Network.

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1. Introduction

Fractional Calculus has recently received increasing attention in the economic and financial literature (see, e.g., Tarasova et al. [9]). For instance, economic growth has been modelled using a state space representation of fractional derivatives (see, e.g., Hu et al. [7] and Marom et al. [4]). Problems requiring fractional calculus use differential and/or integral operators whose order generally is a non-integer number. Differential or integral equations of fractional order are defined in different ways depending on the operator involved. One can than consider the fractional derivatives of Caputo, Riemann-Liouville, Atangana-Baleanu and Caputo-Fabrizio (see, e.g., Mainardi [10], Atangana and Baleanu [8] and Caputo and Fabrizio [6]).

For this reason, regardless of the type of operator involved in the modelling, the theory of differential and integral equations of fractional order has attracted much attention among scholars. A crucial point is that, generally, these kinds of equations do not allow closed-form solutions and thus require the use of appropriate numerical methods to obtain accurate approximations of the solutions.

Several techniques have been proposed in the literature to solve such problems numerically. Although an exhaustive survey is beyond the scope of this work, we wish to mention the Adomian decomposition method, the He variational iteration method, the homotopy perturbation method, the collocation method, and the Galerkin method and the mean-value theorem approach (see e.g. Ervin et al. [3], De Angelis et al. [11] and references therein).

Existing numerical methods generally require that the functions involved in the integral equation, i.e. the kernel and the unknown solution satisfy some specific properties such as boundedness, Lipschitz continuity, or high order differentiability. From this, the integral equations are transformed so that the approximate solution is obtained by solving a system of linear or nonlinear equations involved in the transformation used. However, from the computational viewpoint, such an aspect could be very restrictive.

One possible way to tackle this problem is to use Physics-Informed Neural Networks (PINNs, see, e.g. Yuan et at. [13]). These are approximation methodologies of universal function which, in the learning process, can embed the a-priori knowledge of any physical law that describes a dataset. The methods allow for dealing with the lack of data availability that can occur in some real-world problems.

Furthermore, unlike existing numerical methods. NNs do not require any kind of transformation of the integral equation. In this paper, we propose to apply PINNs to solve the Volterra integral equation of fractional order. For validation and verification, we also present some numerical examples that show promising results in terms of accuracy and computational efficiency.
The rest of this article is organized as follows. Section 2 introduces some preliminary concepts of Fractional Calculus. Section 3 describes the numerical approach used to solve Volterra integral equation of fractional order. Section 4 provides some numerical experiments. Section 5, finally, concludes.

2. Volterra integral equations of fractional order

In this section we introduce some preliminary concepts on fractional differential and integral equations, and we provide some definitions and theoretical results useful for solving the Volterra integral equations of fractional orders.
First, we define the Riemann-Liouville operator.

Definition 2.1. Let f be a continuous, real-valued function defined over the interval [a, b] $\subseteq \mathbb{R}$ and set $\alpha > 0$. For any $t \in [a, b]$, the Riemann-Liouville integral of order α is defined as

$$J_a^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, \qquad s > 0, \tag{2.1}$$

where $\Gamma(\alpha)$ is the Gamma function.
Let us define $m := \lceil \alpha \rceil$, where $\lceil \cdot \rceil$ represents the ceiling function.

Definition 2.2. *Let $a \in \mathbb{R}$ be a fixed point and set $\alpha > 0$. The* Caputo differential operator *of order α is*

$$D_a^{\alpha}[h(t)] = J_a^{m-\alpha} h^{(m)}. \tag{2.2}$$

where $h^{(m)}$ is the ordinary mth derivative of h.
In the following we report the generalization of the Fundamental Theorem of Calculus in the fractional framework. For more details we refer the interest reader to Diethelm et al. [1] and references therein.

Theorem 2.3. *Let $h : [a, b] \to \mathbb{R}$ be a continuous function and set $\alpha > 0$. Then,*

$$D_a^{\alpha} J_a^{\alpha}[h(t)] = h(t). \tag{2.3}$$

A fractional differential equation can be defines as

$$D_a^{\alpha}[h(t)] = F(t, h(t)), \tag{2.4}$$

where $\alpha > 0$, F is a sufficiently regular function and $h_0^{(k)}(0)$ with k = 0,1,2,...,m – 1, represent some initial conditions.

From Theorem 2.3, is straightforward to show that Expression (2.4) is equivalent to the following Volterra fractional integral equation

$$h(t) = \sum_{k=0}^{\lceil \alpha \rceil - 1} h_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F\left(t, h(s)\right) ds \tag{2.5}$$

In the next section, we provide an algorithm to solve the integral equation (2.5).

3. A neural network approach

In this section, we discuss how to apply a neural network approach for solving ordinary differential equation or, more in general, partial differential equations. Solving these types of mathematical problems using Neural Networks (NNs) consists in solving an optimization problem where the residuals of the differential equation are minimised. NNs are mathematical models inspired by biological neural networks and their elaboration process. The fundamental elements of a NN are:

- neurons: elementary units apt to receive an input, elaborate it, producing an output;
- layers: objects containing neurons;
- synaptic connections: connections between neurons in different layers. Each connection is pondered by a weight, so that each neuron receives as input a weighted information, namely activation;
- activation function: a differentiable function applied to the activation, so that each neuron generates an output.

Typically, the description of NNs is provided by a graph-like representation:

- Neurons are the nodes of the graph, represented with circles.
- Synaptic connections are the edges of the graph. Such edges are then pondered by appropriate weights.
- Layers represent a collection of nodes operating of nodes operating together at a specific depth within a NN. Layers can be of three types: the input layer, the hidden layer, the output layer.
  The input layer is the initial data for the neural network. The hidden layer is the intermediate layer between input and output layer and is the place where all calculations are performed. The output layer is the last layer of an NN and produces output for given inputs.
- Activation functions allow each neuron to transform an input to an output. The input received is the activation.

A NN architecture shows how neurons are arranged within the layers and how neurons are interconnected. The most popular architecture in the so-called Feed-Forward Neural Network (FFNN), where the elaboration process moves forward from the input layer to the output layer.
The simplest FFNN model is the well-known perceptron:

- It is composed by an input layer and an output layer.
- Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \in \mathbb{R}^n$ the vector of weights connecting the inputs from the imput layer to the output layer.
- Let $A: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a linear function allowing for the activation computation and le $\sigma: \mathbb{R} \longrightarrow \mathbb{R}$

be the activation function for the output layer.
A perceptron can be defined as a function $f_{NN}: \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$y = f_{NN}(x; \omega) = \sigma(A(x)) = \sigma(\omega^T x). \tag{3.1}$$

If we denote by u the unknown solution of a differential equation, then the idea behind this numerical method is to parametrize the solution u by means of u_θ to approximate the ground truth (i.e. the theoretical unknown exact solution).
In the case of the PINN (see, e.g., Yuan et al. [13]), u_θ is an artificial neural network as the surrogate where the parameter θ contains weights and biases. More precisely, $u_\theta(\cdot)$ is a multi-dimensional map and is typically modelled using a Multi-Layer Perceptron (MLP), which is a class of fully connected feed-forward artificial neural networks. MLP consists of at least three layers of nodes: an input layer, a hidden layer, and an output layer. Except for the input nodes, each node is a neuron that uses a non-linear activation function $\sigma(\cdot)$.

The perceptron model can be enriched adding hidden layers, thus obtaining the so-called multi-layer perceptron. Let us consider a MLP with one hidden layer with N neurons; let ω_{ij} denote the NN parameter weighting the information moving from the j-th neuron of the input layer to the i-th neuron of the hidden layer. It can be seen as the element of the following matrix:

$$W^{(H)} = \begin{pmatrix} (\omega_1^{(H)})^T \\ (\omega_2^{(H)})^T \\ \vdots \\ (\omega_N^{(H)})^T \end{pmatrix} = \begin{pmatrix} \omega_{1,1}^{(H)} & \omega_{1,2}^{(H)} & \dots & \omega_{1,n}^{(H)} \\ \omega_{1,2}^{(H)} & \omega_{2,2}^{(H)} & \dots & \omega_{2,n}^{(H)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{N,1}^{(H)} & \omega_{N,2}^{(H)} & \dots & \omega_{N,n}^{(H)} \end{pmatrix} \in \mathbb{R}^{N \times n},$$

where *n* is the length of the input vector

To estimate the parameters involved in the NN model, a training procedure is used through the appropriate number of steps. Furthermore, a loss function must be defined to evaluate the discrepancy between the NN prediction and the target. Hence, the loss function is minimized with respect to the NN parameters. The most popular algorithm used to solve such an optimization problem is the Adam algorithm developed by Kingma et al. [5].

If we consider the problem of solving a PDE, then the most commonly used loss function is defined as a suitable combination of the initialization domain, the boundary domain, and the equation itself.

Physics Informed Neural Networks (PINNs) are, generally, difficult to implement in practice.

For this reason, we implement our PINN using IDRLnet provided by Peng et al. [12].

It is a Python toolbox for modelling and solving problems through PINN and employs as an activation function $\sigma(\cdot) := \text{swish}(\cdot)$, i.e.

$$\sigma(\mathrm{x}) = \frac{\mathrm{x}}{1+\mathrm{e}^{-\mathrm{x}}},$$

where x is a generic argument.

As mentioned in the previous section, our aim is to solve the Volterra fractional integral equation (2.5), with we report in the following for convenience

$$h(t) = f\left(t\right) + \frac{1}{\Gamma\left(\alpha\right)} \int_a^t \left(t-s\right)^{\alpha-1} F\left(t, h\left(s\right)\right) ds \tag{3.2}$$

where

$$f(t) = \sum_{k=0}^{\lceil \alpha \rceil - 1} h_0^{(k)} \frac{t^k}{k!}$$

For this scope, we use a PINN, where its loss function is defined as the difference between the left and right hand side of Equation (3.2). The numerical integration is performed by Gauss-Legendre quadrature with a suitable degree.

4. Numerical results

In this section we apply the numerical procedure described in Secion 3 on two case studies. All the procedures have been implemented in Python 3.7 and have been executed on a MacBook Pro with processor 2.6 GHz Intel Core i7 with 16 GB RAM. Furthermore, in both the numerical experiments, we consider 4000 epochs for the neural network and 20 degrees for the Gauss-Legendre quadrature.

Example 4.1 (see Diethelm et al. [1]). Let us consider the following equation where $\alpha = 2/3$ and $t \in [0,1]$. As shown in Diethelm et al. [1],

$$h(t) = \frac{1}{27\Gamma(2/3)} \int_0^t s\left(t-s\right)^{\alpha-1} h\left(s\right) ds + \Gamma\left(2/3\right)t - \frac{1}{40} t^{\frac{8}{3}} \tag{4.1}$$

Equation (4.1) has the exact solution $h(t) = \Gamma(2/3)t$. To evaluate the accuracy of our proposed algorithm, in Figure 1 we report the diffecence between the exact and the approximate solution. Note that by varying $t \in [0,1]$ the errors are on the order of 10e-4.

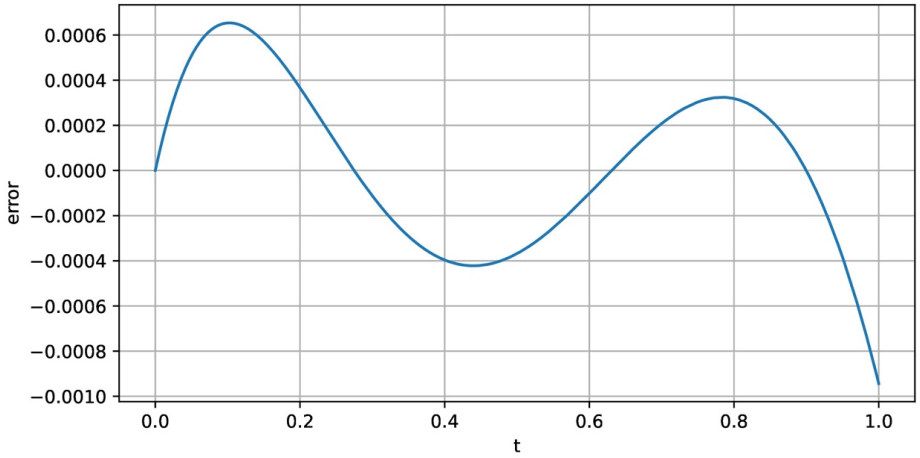

Figure 1: Difference between the exact and approximate solution for Example 4.1.

Example 4.2 (see Diethelm et al. [1]). Let us consider the following fractional differential equation

$$D^{\alpha}h(t)=\frac{40320}{\Gamma(9-\alpha)}t^{8-\alpha}-3\frac{\Gamma\left(5+\frac{\alpha}{2}\right)}{\Gamma\left(5-\frac{\alpha}{2}\right)}t^{4-\frac{\alpha}{2}}+\frac{9}{4}\Gamma(\alpha+1)+\left(\frac{3}{2}t^{\frac{\alpha}{2}}-t^{4}\right)^{3}-\left[h(t)\right]^{\frac{3}{2}} \qquad (4.2)$$

where $\alpha = 1$, $t \in [0,1]$, h(0)=0 and h′(0) = 0.
Its exact solution is $h(t) = t^8 - 3t^{4+\alpha/2} + (9/4)^{\alpha}$.
Note that equation (4.2) can be reformulated as a Volterra integral equation of fractional order. Similar to experiment 4.1, to evaluate the accuracy of our proposed algorithm, in Figure 2 we show the difference between the exact and the approximate solution. Again, by varying $t \in [0,1]$ the errors are on the order of 10e-4.

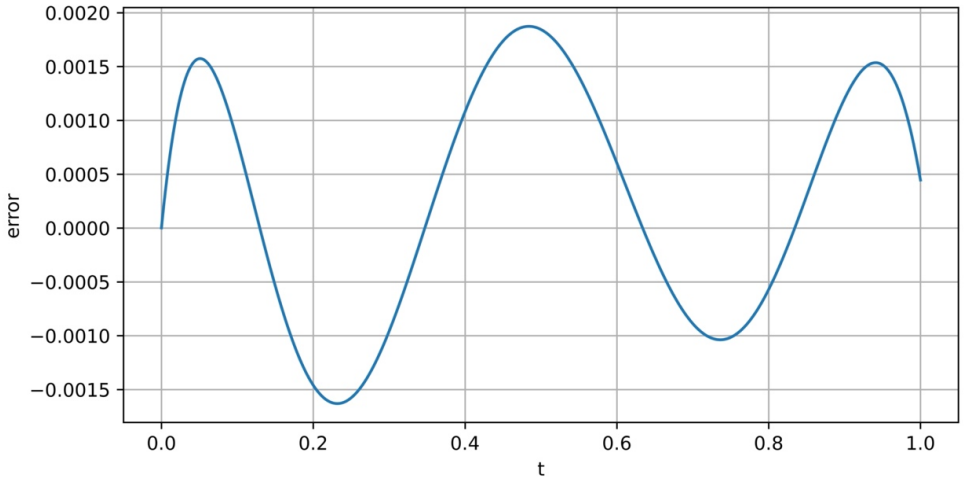

Figure 2: Difference between the exact and approximate solution for Example 4.2.

5. Conclusion

In this paper we have proposed a numerical approach based on Physics Informed NeuralNetworks to solve the Volterra integral equation of fractional order.

Differently from several existing methods in the literature typically require specific properties of the functions involved in the model, our procedure is less restrictive in this respect. Thus, applying PINN-based method and using the IDRLnet toolbox, we have solved Volterra integral equations of fractional order with a high degree of accuracy in two case studies taken from the literature.

Further future research might be directed to extend this numerical technique to multi-dimensional problems.

References

[1] Diethelm K., Ford N.J., Freed A.D., (2002). A Predictor-Corrector Approach for the Numerical Solution of Fractional Differential Equations, Nonlinear Dynamics 29(1-4), 3-22.
[2] Cesarone F., Caputo M., Cametti C., (2005). Memory Formalism in the Passive Diffusion Across Highly Heterogeneous Systems, Journal of Membrane Scienze 250(1-2), 79-84.
[3] Ervin V.J., Roop J.P. (2005) Variational Formulation for the Stationary Fractional Advection Dispersion Equation, Numer. Methods Partial Differential Equations 22, 558-576.
[4] Marom O., Mammoniat E., (2009). A Comparison of Numerical Solutions of Fractional Diffusion Models in Finance, Nonlinear Analysis: Real World Applications, 10, 3435-3442.
[5] Kingma D. P., Ba J., (2014). Adam: A Method for Stochastic Optimization. arXiv:1412.6980v9.
[6] Caputo M., Fabrizio M., (2015). A New Definition of Fractional Derivative Without Singular Kernel, Progress in Fractional Differentiation and Applications, 1(2), 73-85.
[7] Hu Z., Tu X, (2015). A New Discrete Economic Model Involving Generalized Fractal Derivative, Advances in Differerence Equations, 65, 1-11.
[8] Atangana A, Baleanu D., (2016). New Fractional Derivatives with Non Local and Nonsingular Kernel: Theory and Application to Heat Transfer Model, Therm Sci 20(2), 763-769.
[9] Tarasova V.V., Tarasov V.E. (2016) Non-local measures of risk aversion in the economic process, Economics: Theory and Practice 44, 54-58.
[10] Mainardi F., (2018). Fractional calculus: theory and applications. Mathematics, Mathematics, 6, 476-484.
[11] De Angelis P., De Marchis R., Martire A. L., Oliva I., (2020). A mean-value Approach to solve fractional differential and integral equations, Chaos Solitons and Fractals 138, 109895.
[12] Peng W., Zhang J., Zhou W., Zhao X., Yao W., Chen X., (2021). IDRLnet: a Physics Informed Neural Network Library, arXiv:2108.06905v2.
[13] Yuan L., Ni Y.-Q., Deng Y.-X., Hao S., (2022). A-PINN: Auxiliary Physics Informed Neural Networks for Forward and Inverse Problems of Nonlinear Integro-Differential Equations, Journal of Computational Physics, Volume 462, 111260.

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