Sapienza
Università di Roma

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Ph.D. Thesis

Energy instability and overdetermined elliptic problems in cones and cylinders: an approach via domain variations

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XXXVI ciclo

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Acknowledgements

To the referees, for their most valuable suggestions.

To Filomena, more than I can express,

To Filomena, again, particularly for her patience,

To Alessandro, for teaching me to be sure,

To Liliane, without whom I wouldn't be where I am,

To Gláucio and Gaetano, who guided my first steps,

To David, who brought me here,

To Andrea, Antonio, Fabiana, Francesca, Giulio, and Isabeau, for all the Amatricianas,

To Federico, Valerio, and all my colleagues and friends in the Castelnuovo, for being a pleasant obstacle to the completion of this thesis,

To Ana Laura and Waldemar, for being who they are,

To Waldemar, for buying me that astronomy book,

To Vincent, for holding my hand,

To Carol and Vitor, for being my family away from home,

To Moderno, Elvis, Felipe, Sereno, Garapa, Vitor, Lissa, and Nathália, for existing,

To Hanna, for understanding.

Abstract

In this thesis, we study semilinear elliptic problems of the type $-\Delta u = f(u)$ in domains that are constrained to be inside a fixed unbounded open set $\mathcal{C}$, with appropriate boundary conditions. Our aim is to understand how the geometry of $\mathcal{C}$ selects domains in which positive solutions of the equation have special properties, mainly related to notions of symmetry. Our arguments are primarily based on analyzing how the energy of a positive solution in a domain varies when the domain moves inside $\mathcal{C}$. We first consider the case where $\mathcal{C}$ is generic. We show how to define an energy functional T when the equation possesses more than one solution and compute the domain derivative of T. In the case when $\mathcal{C}$ is a cone or a cylinder, we show that some special domains may be unstable as critical points to the energy shape functional. This opens room for the search for nonsymmetric domains with the same special properties, to be found, for example, by local minimization of the energy functional. This is done by analyzing the sign of the second derivative of the energy functional to understand the stability/instability of its critical domains. Furthermore, we show that in a special class of domains, namely bounded cylinders, solutions other than the one-dimensional ones do exist, under fairly general assumptions on the nonlinearity f. This is accomplished by means of bifurcation theory and Morse index comparison.

Contents

Introduction

The subject of this thesis is the study of semilinear elliptic equations of the form

$$-\Delta u = f(u) \tag{i}$$

posed in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, which will be constrained to be inside a fixed unbounded domain $\mathcal{C}$. Because of this constraint, the boundary of Ω naturally divides into two parts: the one laying inside $\mathcal{C}$ and the one intersecting $\partial\mathcal{C}$. The first one is usually called the relative boundary of Ω and we will denote it by Γ, i.e.,

$$\Gamma = \partial\Omega \cap \mathcal{C}.$$

For the part of the boundary of Ω intersecting $\partial\mathcal{C}$, we set

$$\Gamma_1 = \partial\Omega \setminus \overline{\Gamma},$$

so that $\partial\Omega \cap \partial\mathcal{C} = \overline{\Gamma}_1$.

We will consider so-called relative Dirichlet problems, which means that we study equation (i) together with the boundary conditions

$$u = 0 \qquad \text{on } \Gamma, \tag{ii}$$

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \Gamma_1. \tag{iii}$$

Note that the zero Neumann condition on Γ_1 is the natural one for differential problems in domains that are in the fixed unbounded region $\mathcal{C}$.

We will analyze several questions for problem (i)-(iii). All will be about understanding how the shape of the region $\mathcal{C}$ selects particular subdomains Ω in which the positive solutions of (i)-(iii) have special properties.

In particular, we will consider unbounded cones and cylinders, in which there are some peculiar domains that, for symmetry or other geometric features, could be natural candidates to positively answer the questions we analyze. The heart of the thesis consists in showing that this is not always the case.

The first question we consider is about the existence and the properties of domains contained in $\mathcal{C}$ which admits a solution of the overdetermined problem which is obtained by adding to (*i*)-(*iii*) an extra condition on the relative boundary, namely

$$\frac{\partial u}{\partial \nu} = c \neq 0 \quad \text{on } \Gamma. \tag{iv}$$

Mathematically speaking, this problem originates from the famous paper of James Serrin, "*A symmetry problem in potential theory*" ([71], 1971), where it is proved that the only smooth bounded domains in $\mathbb{R}^N$ which admit a positive solution to the overdetermined Dirichlet problem are balls.

Since then, plenty of other results for several types of overdetermined problems have been obtained.

We will recall Serrin's theorem and other related ones, also in the relative setting, in Chapter 1.

In Chapter 2 we will study problem (*i*)-(*iv*) in the case of the torsion problem, i.e., when $f \equiv 1$, and we will show some conditions on the domains $\Omega \subset \mathcal{C}$ admitting solutions of (*i*)-(*iv*). This will be done for general unbounded sets $\mathcal{C}$, while more specific results will be obtained for cylinders. In particular, the relation with the notion of relative Cheeger set will be investigated.

The second question analyzed in this thesis is about the study of the variation of the energy of a positive solution of (*i*)-(*iii*) in $\Omega \subset \mathcal{C}$ while making small deformations of Ω which preserve its measure.

We will study this question from a variational point of view. Roughly speaking, we consider a functional $T = T(\Omega)$ which associates to each domain $\Omega \subset \mathcal{C}$ the energy of a fixed positive solution of (*i*)-(*iii*) in Ω and study its variation with respect to small volume preserving deformations. Our analysis makes use of the theory of domain derivatives. The domains Ω which are local minima of T are particularly interesting, and they could be detected by studying the stability/instability of the critical points of T (defined in a suitable way). In turn, this leads to the study of the second derivative of the functional T, which in general is not easy.

Let us point out that the domains Ω that are critical points of T are the domains for which the overdetermined problem (*i*)-(*iv*) has a solution. This shows the connection between the energy analysis and the solvability of the overdetermined problem.

In Chapter 3 and Chapter 4 we consider the case when $\mathcal{C}$ is an unbounded cone or an unbounded cylinder. In these cases, spherical sectors, resp. bounded cylinders, are good candidates to be local minima of the energy functional T, since they admit solutions to the overdetermined problem (radial ones in the case of the spherical sector, and one-dimensional ones in the case of the bounded cylinder). One of the main results of this work is to show that this is not always true but depends on the domain on the sphere $\mathbb{S}^{N-1}$ (resp. in $\mathbb{R}^{N-1}$) which spans the cone (resp. cylinder) in $\mathbb{R}^N$.

Finally, in Chapter 5 we focus on the behavior of positive solutions of (*i*)-(*iii*) in a bounded cylinder $\Omega_\omega = \omega \times (0,1)$, where ω is a smooth bounded domain in $\mathbb{R}^{N-1}$.

In Ω_ω we can define functions that inherit geometrical features of the domain. More precisely, the axis of the cylinder defines a distinctive direction which allows us to consider functions that depend only on the variable corresponding to that direction. We will call these functions "one-dimensional". Then the question is whether or not all positive solutions of (*i*)-(*iii*) are one-dimensional. In other words, we would like to understand whether all positive solutions reflect the simple geometry of Ω_ω.

The general question of understanding how the geometry of a domain determines the shape of the positive solutions of differential problems there defined is an old one. It has been addressed, in particular, in radial domains to investigate the radial symmetry of positive solutions.

In the case of the ball, the celebrated result by Gidas, Ni, and Nirenberg obtained in the paper "*Symmetry and related properties via the maximum principle*" ([40], 1979) marked a fundamental step. Since then, plenty of other radial symmetry or break of symmetry results have been obtained. We recall some of them in Chapter 1, in particular in the relative setting of spherical sectors.

In the case of cylinders, we are not aware of similar results for one-dimensional solutions. In Chapter 5 we study some classes of cylinders where not all positive solutions are one-dimensional, for many types of nonlinearities. Even more, we prove that, for some cylinders, positive solutions that are not one-dimensional bifurcate from the trivial branch of one-dimensional solutions.

A more detailed presentation of the results of the thesis is contained in Chapter 1, Section 1.3, where we also describe the methodology and the difficulties.

Several open problems arise from our study and we outline them in Section 1.3.

The results obtained are contained in the following papers:

- D. G. Afonso, A. Iacopetti, and F. Pacella. Overdetermined problems and relative Cheeger sets in unbounded domains. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. 34* (2023) pp. 531-546 ([3]);
- D. G. Afonso, A. Iacopetti, and F. Pacella. Energy stability for a class of semilinear elliptic problems. *J. Geom. Anal. 34* (2024) ([2]);
- D. G. Afonso. Semilinear equations in bounded cylinders: Morse index and bifurcation from one-dimensional solutions. *arXiv:2311.15236v1 (submitted)* (2023). ([1]).

Chapter 1

Motivations and results

The aim of this chapter is to give a brief introduction to the problems we study and describe the results of the thesis.

1.1 Physical motivations

Consider the following engineering problem: one is to design a homogeneous pipe through which there will flow a viscous incompressible fluid in a steady state. The aim is to optimize the structure, in order to reduce the costs. Critical to the design of the pipe is the tangential stress on the boundary.

It is known (see [51, Chapter 2]) that the fluid develops a velocity profile, where the velocity depends only on the coordinates on the cross-section, and, for a given point in the cross-section, does not vary along the axis of the pipe. More precisely, consider a cylinder $\Omega \times [0, L]$, where L is the length of the pipe and Ω, the cross-section, can be thought of as a domain in the (x, y)-plane, while the axis of the pipe is in the z-direction. Then the velocity u of the fluid satisfies $u(x, y, z) = u(x, y)$, i.e., is a function on Ω. The velocity profile can then be obtained by solving the partial differential equation

$$-\Delta u = \frac{\delta}{\eta L} \quad \text{in } \Omega, \tag{1.1.1}$$

where δ is the change of pressure between the two ends of the pipe and η is the viscosity, subject to the boundary condition $u = 0$ on $\partial\Omega$, which means that the fluid attaches to the wall. It is usually denominated "no-slip condition". Furthermore, it is well-known that the tangential stress τ on the boundary of the pipe is given by $\tau = \eta \left|\frac{\partial u}{\partial \nu}\right|$, where ν denotes the outer unit normal vector to $\partial\Omega$.

Then one easily sees that the optimal pipe will be the one where the tangential stress is constant. Otherwise, the material that is necessary to reinforce the pipe in the region of greater stress is wasted on the regions of lower stress. Thus one is led to consider equation (1.1.1) with both boundary conditions

$$u = 0, \quad \frac{\partial u}{\partial \nu} = \text{constant} \quad \text{on } \partial\Omega.$$

Another interesting engineering question arises when one is to design a solid cylindrical bar that will be subjected to torsion loads. It is well-known, see [73], that with the exception of the case when the cross-section is a circle, the points in the cross-section are also displaced in the direction of the axis. In other words, originally flat cross-sections bend under torsion. It can be shown, see [72, 73, 64], that the displacements can be recovered from a function $u = u(x, y)$ (again we denote by Ω the cross-section of the original bar and by (x, y) points in Ω) satisfying the equation

$$-\Delta u = 2 \quad \text{on } \Omega, \quad u = \text{constant} \quad \text{on } \partial\Omega,$$

and that the shear stress τ, which is higher on the lateral boundary of the bar, is of the form $\tau = C\left|\frac{\partial u}{\partial \nu}\right|$. Again, to avoid wasting material and unnecessarily increasing the costs, it is convenient to find bars where $\frac{\partial u}{\partial \nu}$ is constant on $\partial\Omega$.

A more complicated nonlinear problem appears in the study of capillarity phenomena. Consider a fluid rising in a straight capillary tube, whose cross-section we denote by $\Omega \subset \mathbb{R}^2$. Denoting by $u = u(x, y)$ the height of the fluid at the point $(x, y) \in \Omega$, the conditions for hydrostatic equilibrium reduce to

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) - \frac{\rho g}{\sigma} u = \text{constant} \quad \text{in } \Omega,$$

where ρ is the density, σ is the surface tension and g is the intensity of the gravitational field. The condition that the height is constant along $\partial\Omega$ is mathematically translated into

$$\frac{\partial u}{\partial \nu} = \cot\alpha = \text{constant},$$

where α is the so-called "wetting angle" between the liquid surface and the wall of the pipe. We are again led to an overdetermined problem.

For more on these matters, we refer the reader to [72].

1.2 Historical perspective

1.2.1 Overdetermined problems in bounded domains of $\mathbb{R}^N$

The questions discussed in Section 1.1 motivated James Serrin to consider in 1971 the overdetermined problem

$$\begin{cases} -\Delta u &= 1 & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \\ \dfrac{\partial u}{\partial \nu} &= -c < 0 & \text{on } \partial\Omega \end{cases}, \tag{1.2.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $c > 0$ is a constant. In 1971, he proved the following:

Theorem 1.2.1 ([71]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary of class C^2. If there exists a classical positive solution $u \in C^2(\overline{\Omega})$ to (1.2.1), then Ω is a ball and u is radial and radially decreasing.*

His proof was based on the moving planes method, introduced by Alexandrov in 1956 to show that the only closed constant mean curvature surfaces embedded in $\mathbb{R}^N$ are spheres (see [4]). This method consists in moving hyperplanes in $\mathbb{R}^N$ up to a critical position and then comparing the solution u with its reflection with respect to the critical plane. With the aid of the maximum principle (this is where the positivity plays a role), one concludes that Ω has to be symmetric with respect to the critical plane. The radial symmetry follows by repeating the argument for hyperplanes orthogonal to every direction in $\mathbb{R}^N$. A key step is a version of Hopf's lemma for domains with corners, which was proved in [71].

In the same paper, Serrin generalized the method to prove an analogous result for nonlinear equations.

At the same time, Weinberger devised another proof of Theorem 1.2.1, using a special function written in terms of the solution u of (1.2.1). More precisely, given u, he defined what is now called the P-function:

$$P(x) = |\nabla u(x)|^2 + \frac{2}{N} u(x). \tag{1.2.2}$$

By considering the differential equation solved by P and combining the maximum principle with some integral identities and matrix inequalities, he obtained with a short proof the same radial symmetry result. See [75].

While Weinberger's proof has the advantage of being much simpler and avoiding the quite technical adaptation of Hopf's lemma for domains with corners, it has the drawback of being applicable only to the case $-\Delta u = 1$, but not to general semilinear problems.

In the years that followed, the problem attracted much interest from the mathematical community, with a number of alternative proofs being found.

Payne and Schaefer proved Theorem 1.2.1 by what they called a "duality" argument. The argument consists of showing by direct computation that the existence of a solution of (1.2.1) is equivalent to

$$\int_\Omega h \, dx = c \int_{\partial\Omega} h \, d\sigma \quad \forall h \text{ harmonic in } \Omega, \tag{1.2.3}$$

and then proving that (1.2.3) implies that the domain is a ball. This last step follows the ideas of Weinberger, in the sense that they consider a special function for which they combine integral identities, matrix inequalities, and the maximum principle. See [63] for the details.

Another proof was discovered by Choulli and Henrot in [21]. They present a shape optimization argument: a domain functional is built, for which domains Ω where (1.2.1) has a solution are local minima, and then the expression for the derivative, together with the optimality condition, yields that the boundary is a surface of constant mean curvature. Then the result follows from Alexandrov's theorem ([4]).

Yet another proof can be found in [15], where Brandolini and coauthors argue employing integral identities related to the curvature of the level sets. In fact, their argument is more general and includes also the Monge-Àmpere equation. It is interesting to notice that the same authors also discovered a stability result: if the normal derivative is "almost" constant on the boundary, then the domain is close to a disjoint union of balls (see [14]). The question of stability was also addressed in other papers, see [59] and the references therein.

Still considering the equation $-\Delta u = 1$, with homogeneous Dirichlet condition but nonconstant Neumann condition (i.e., $|\nabla u| = g$ for some nonconstant g on $\partial\Omega$), Bianchini, Henrot, and Salani proved the existence and regularity of domains where a solution exists, under suitable hypotheses on g. They perform a shape optimization analysis for the existence and then study the regularity of the free boundary (see [13]).

There were also developments concerning overdetermined problems for more general nonlinear equations.

Garofalo and Lewis ([39]) generalized the approach of Weinberger based on integral identities for a class of quasilinear equations (which includes the much-studied p-Laplace operator) of the type

$$-\operatorname{div}(|\nabla u|^{-1} f'(|\nabla u|)\nabla u) = 1,$$

where the function f, whose derivative appears in the equation, is positive and increasing, and some growth conditions are imposed on the derivatives. In particular, their result is stated in weak form, i.e., they assume the existence of a weak solution whose gradient converges to a constant on the boundary. Remarkably, they assume no regularity on $\partial\Omega$.

Combining shape derivative techniques and rearrangement inequalities, Brock and Henrot ([16]) studied overdetermined problems for quasilinear equations of the form

$$-\operatorname{div}(g(|\nabla u|)|\nabla u|^{-1}\nabla u) = f(x, u).$$

Both [39] and [16] show that, under appropriate hypotheses, the only domains where a solution to the overdetermined Dirichlet problem exists are balls.

Fragalà, Gazzola, and Kawohl ([37]) investigated the slightly different problem

$$-\operatorname{div}(A(|\nabla u|)\nabla u) = 1,$$

with quite general assumptions on A. In particular, the operator is allowed to be degenerate elliptic. They were able to show an analogous of Theorem 1.2.1 in $\mathbb{R}^2$, but in $\mathbb{R}^N$, $N \geq 3$, they could only show that the domains where a solution of the overdetermined problem exist are self-Cheeger (see Section 2.3). The approach of [37] consists of studying a suitable P-function associated with the problem, and makes heavy use of the maximum principle.

Recently, Ruiz showed that the positivity of the solution is indeed necessary for the rigidity theorem of Serrin to hold. More precisely, in [69] he studied the overdetermined problem for the equation $-\Delta u = u - (u^+)^3$ and showed, in dimension $2 \leq N \leq 4$, the existence of domains bifurcating from a ball where a sign-changing solution exists.

1.2.2 Overdetermined problems in cones

The isoperimetric problem is among the oldest and most studied problems in the calculus of variations: one wants to understand how much land can be enclosed by a given amount of fence. See [66] for an overview.

It is linked to the question of characterizing closed surfaces of constant mean curvature, i.e., to Alexandrov's result. Indeed, if one considers the function $\mathcal{P}(\Omega)$ which associates to each smooth bounded domain in $\mathbb{R}^N$ the measure of its boundary, then the critical points of $\mathcal{P}$ (with a volume constraint) in the sense of domain derivative are sets whose boundary is a CMC surface.

Apart from its geometric interest, the isoperimetric problem is also related to several important questions in Analysis, in particular in the field of PDEs. Indeed, symmetry properties of positive solutions of some differential equations with Dirichlet boundary conditions can be detected by symmetrization methods, which rely on the isoperimetric inequality.

Similarly, the symmetry of minimizers in the study of the best constant for the Sobolev embedding of the space $W_0^{1,p}(\Omega)$ into $L^{\frac{Np}{N-p}}(\Omega)$, $\Omega \subset \mathbb{R}^N$, $p < N$, can be obtained by symmetrization.

However, when considering functions that do not vanish on the whole boundary of the domain where they are defined, the classical isoperimetric inequality is not appropriate. This happens, for example, in the study of mixed boundary value problems or Sobolev inequalities which do not involve functions in $W_0^{1,p}(\Omega)$. This was one of the motivations to introduce a new isoperimetric inequality, obtained by Lions and Pacella for domains in convex cones ([55]). Their theorem reads as follows:

Theorem 1.2.2. *Let $\Sigma_D = \{tx \in \mathbb{R}^N \ : \ x \in D,\ t \in (0,\infty)\}$, where $D \subset \mathbb{S}^{N-1}$ is a smooth domain in the unit sphere $\mathbb{S}^{N-1}$. If the cone Σ_D is convex, then the following isoperimetric inequality holds:*

$$P_{\Sigma_D}(E) \geq N\alpha_N^{1/N}|E|^{\frac{N-1}{N}}$$

for every measurable set $E \subset \Sigma_D$ with $|E| < +\infty$, where P_{Σ_D} is the relative perimeter and $\alpha_N = |\Sigma \cap B_1(0)|$ is the measure of the unit sector. Moreover, equality holds if, and only if E is a spherical sector $\Sigma_R = \Sigma_D \cap B_R(0)$ of radius $R > 0$ (or a translation of Σ_R if Σ_D contains lines).

Other proofs of this isoperimetric inequality were obtained later by Ritoré and Rosales in [65], Figalli and Indrei in [36], and Cabré, Ros-Oton and Serra in [18]. An extension to almost convex cones is proved in [**BaerFigalli2017**].

The radial symmetry result inspired the study of related questions, namely the characterization of constant mean curvature surfaces with boundary inside a cone Σ_D and of domains in Σ_D admitting solutions to an overdetermined problem.

To the aim of proceeding with a more precise discussion, let us set the terminology.

As before, we denote by Σ_D the open cone spanned by a smooth domain $D \subset \mathbb{S}^{N-1}$. Given a bounded domain $\Omega \subset \Sigma_D$, the relative boundary is denoted by $\Gamma := \partial\Omega \cap \Sigma_D$. We also set $\Gamma_1 = \partial\Omega \setminus \overline{\Gamma}$. We assume that $\mathcal{H}^{N-1}(\Gamma_1) > 0$, $\mathcal{H}^{N-1}(\Gamma) > 0$ and that Γ is a smooth $(N-1)$-dimensional manifold, while $\partial\Gamma = \partial\Gamma_1 \subset \partial\Sigma_D \setminus \{0\}$ is a smooth $(N-2)$-dimensional manifold. It is said that Ω is a *sector-like domain*. If $\Omega = \Sigma_D \cap B_R(0)$ is the intersection of the cone with a ball of radius R, then Ω is said to be the spherical sector of radius R.

It is then seen from Theorem 1.2.2 that spherical sectors play in the relative isoperimetric problem the same role as balls in the whole $\mathbb{R}^N$. This fact, together with the results of Serrin ([71]) and Alexandrov ([4]), motivates the consideration of the partially overdetermined problem

$$\begin{cases} -\Delta u &= 1 & \text{in } \Omega \\ u &= 0 & \text{on } \Gamma \\ \dfrac{\partial u}{\partial \nu} &= 0 & \text{on } \Gamma_1 \setminus \{0\} \\ \dfrac{\partial u}{\partial \nu} &= -c < 0 & \text{on } \Gamma \end{cases}, \tag{1.2.4}$$

as well as the characterization of constant mean curvature (CMC in short) surfaces inside cones. A natural question is whether spherical sectors, respectively spherical caps, share the same properties relative to cones as balls, respectively spheres, in $\mathbb{R}^N$. Pacella and Tralli started a research program on these matters ([60, 61]).

Concerning the overdetermined problem, in [60] the authors prove the following:

Theorem 1.2.3 ([60], Theorem 1.1)**.** *Let $c > 0$ be fixed and assume that Σ_D is a convex cone such that $\partial\Sigma_D \setminus \{0\}$ is smooth. If Ω is a sector-like domain and there exists a classical $C^2(\Omega) \cap C^1(\Gamma \cup \Gamma_1 \setminus \{0\})$ solution u of problem (1.2.4) such that $u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$, then*

$$\Omega = \Sigma_D \cup B_R(p_0) \quad \text{and} \quad u(x) = \frac{N^2c^2 - |x - p_0|^2}{2N},$$

where $B_R(p_0)$ denotes the ball centered at the point $p_0 \in \mathbb{R}^N$ and radius $R = Nc$. Moreover, one of the following two possibilities holds:

- *$p_0 = 0$;*
- *$p_0 \in \partial\Sigma_D$ and Γ is a half-sphere lying over a flat portion of $\partial\Sigma_D$.*

In the same paper, an analogous result for CMC surfaces in convex cones was proved ([60, Theorem 1.2])

Later, Ciraolo and Roncoroni showed in [23] an analogous result for quasilinear operators.

In the subsequent paper [61], Pacella and Tralli analyzed the overdetermined problem (1.2.4) from a shape optimization perspective. They showed that, in any cone, a domain admits a solution to (1.2.4) if, and only if, it is a critical point of the shape functional

$$\Omega \mapsto T(\Omega) = \frac{1}{2}\int_\Omega |\nabla u_\Omega|^2 \, dx - \int_\Omega u_\Omega \, dx,$$

where u_Ω is the torsion function of Ω, that is, the unique solution of

$$\begin{cases} \Delta u &= 1 & \text{in } \Omega \\ u &= 0 & \text{on } \Gamma \\ \dfrac{\partial u}{\partial \nu} &= 0 & \text{on } \Gamma_1 \setminus \{0\} \end{cases}$$

in Ω, for Ω in the class of sector-like domains.

All these rigidity results cited so far concern convex cones. In fact, Iacopetti, Pacella, and Weth showed in [46], via shape optimization techniques, the existence of sets in nonconvex cones which admit a solution to the overdetermined problem (1.2.4) and which are not spherical sectors. More precisely, we have the following:

Theorem 1.2.4 ([46], Theorem 1.2)**.** *Let $D \subset \mathbb{S}^{N-1}$ be a smooth domain such that*

$$\lambda_1(D) < N-1 \quad \textit{and} \quad \mathcal{H}^{N-1}(D) < \mathcal{H}^{N-1}(\mathbb{S}^{N-1}_+),$$

where $\lambda_1(D)$ is the first nontrivial Neumann eigenvalue of the Laplace-Beltrami operator on D and $\mathbb{S}^{N-1}_+$ is a half unit sphere. Let Σ_D be the cone spanned by the domain D. Then there exists a bounded domain $\Omega^* \subset \Sigma_D$ which is a minimizer for the functional T with a fixed volume, but Ω^* is not a spherical sector. Moreover, there exists a critical dimension d^*, which can be either 5, 6, or 7, such that for the relative boundary Γ_{Ω^*} of Ω^* it holds that

- *Γ_{Ω^*} is smooth if $N < d^*$;*
- *Γ_{Ω^*} can have countably many isolated singularities if $N = d^*$;*
- *Γ_{Ω^*} can have a singular set of dimension $N - d^*$, if $N > d^*$.*

In addition, on the regular part of Γ_{Ω^*} the normal derivative $\frac{\partial u_{\Omega^*}}{\partial \nu}$ is constant, where u_{Ω^*} is the torsion function of Ω^*.

We recall that when D is convex, i.e., when Σ_D is a convex subset of $\mathbb{R}^N$, then $\lambda_1(D) \geq N-1$, as is proved in [31, Theorem 4.3] (see also [5, Theorem 4.1]).

The strategy of the proof goes as follows.

First, one sees that sector-like domains where a solution of (1.2.4) exists are critical points (under a volume constraint) of the shape functional given by the torsional energy $T(\Omega) = -\frac{1}{2}\int_\Omega u_\Omega \, dx$ (this was already known from [61]). In particular, the spherical sector is a critical point. Then, studying the second shape derivative, one obtains that the spherical sector is an *unstable* critical point whenever $\lambda_1(D) < N-1$. In particular, it is not a minimizer for the energy.

The next step is to show that a minimizer does exist. This is done by minimization in the broader class of quasi-open sets (see [44] for capacity theory), making use of the Concentration-Compactness principle of P.-L. Lions ([54], see also [17]) together with the hypothesis $\mathcal{H}^{N-1}(D) < \mathcal{H}^{N-1}(\mathbb{S}^{N-1}_+)$ and heavily exploiting the geometry of the cone. More precisely, one takes a minimizing sequence $(\Omega_n)_{n\in\mathbb{N}^+}$ and applies the Concentration-Compactness principle to the sequence of torsion functions $u_n := u_{\Omega_n}$ in order to prove convergence. To this end, one has to analyze three possible scenarios, usually denominated vanishing, dichotomy, and compactness.

By energetic considerations and exploiting classical properties of Sobolev spaces, one proves that the vanishing case cannot occur. After that, making use of the fact that the cone is invariant by dilations, it is possible to show that the dichotomy case cannot occur either. One then concludes that there exists a sequence $(y_n)_{n\in\mathbb{N}^+}$ of points in the cone such that the torsional energy of Ω_n is concentrated around a ball $B(y_n)$, of fixed radius, centered in y_n, in the sense that the contribution to the torsional energy of the part of Ω_n outside $B(y_n)$ is negligible.

Now, once more one exploits the geometry of the cone, this time to show that the sequence (y_n) is bounded. Roughly speaking, since the cone "flattens out" at infinity, if y_n escapes to infinity, then the torsional energy of Ω_n "approaches" that of the half-sphere. However, this is not possible, as a consequence of the hypothesis $\mathcal{H}^{N-1}(D) < \mathcal{H}^{N-1}(\mathbb{S}^{N-1})$.

The boundedness of the sequence y_n provides the necessary compactness for us to extract convergence of the torsion functions u_n, first by proving the relative compactness of u_n in $L^2(\Sigma_D)$, and then exploiting the compactness of the Sobolev embedding. Letting $\bar{u} = \lim_n u_n$ and $\Omega^* = \{\bar{u} > 0\}$, applying Fatou's Lemma we obtain that Ω^* satisfies the volume constraint. Then one concludes that Ω^* is a minimizer by some other easy estimates.

Furthermore, it can be shown that Ω^* is a bounded, open, and connected subset of Σ_D. The regularity statements are classical, and that $\frac{\partial u_{\Omega^*}}{\partial \nu}$ is constant on the regular part of Γ_{Ω^*} follows by exploiting the minimality of Ω^* and the expression for the shape derivative of the torsional energy functional at Ω^*

A similar break of symmetry result holds for the relative isoperimetric problem:

Theorem 1.2.5 ([46], Theorem 1.3)**.** *If D is a smooth domain of $\mathbb{S}^{N-1}$ such that*

$$\lambda_1(D) < N-1 \quad \textit{and} \quad \mathcal{H}^{N-1}(D) < \mathcal{H}^{N-1}(\mathbb{S}^{N-1}_+),$$

then, for any fixed volume there exists a bounded set E^* of finite perimeter inside Σ_D which minimizes the relative perimeter $P_{\Sigma_D}(E)$, and E^* is not a spherical sector. Moreover, for the relative boundary $\Gamma_{E^*} = \partial E^* \cap \Sigma_D$, it holds that

- *Γ_{E^*} can have a closed singular set $\widetilde{\Gamma}_{E^*}$ of Hausdorff dimension less than or equal to $N-7$;*
- *$\Gamma_{E^*} \setminus \widetilde{\Gamma}_{E^*}$ is a smooth embedded hypersurface with constant mean curvature;*

- *if $x \in \overline{\Gamma_{E^*} \setminus \widetilde{\Gamma}_{E^*}} \cap \partial\Sigma_D$, then $\Gamma_{E^*} \setminus \widetilde{\Gamma}_{E^*}$ is a smooth CMC embedded hypersurface with boundary in a neighborhood of x and meets $\partial\Sigma_D$ orthogonally.*

1.2.3 Symmetry and symmetry-breaking

An old and natural question in the qualitative theory of elliptic PDEs is whether the solutions inherit symmetry properties of the domain or not.

Concerning radial symmetry of positive solutions of semilinear Dirichlet problems, a fundamental step was the celebrated result of Gidas, Ni, and Nirenberg:

Theorem 1.2.6 ([40], Theorem 1)**.** *Let B be a ball and let $u \in C^2(\overline{\Omega})$ be a positive solution of*

$$\begin{cases} -\Delta u &= f(u) \quad \text{in } B \\ u &= 0 \quad \text{on } \partial B, \end{cases}$$

where f is of class C^1. Then u is radially symmetric and radially decreasing.

Building on the ideas of [40], more symmetry results were proved in [41] and [9, 11, 10]. The core of the arguments is the method of moving planes introduced by Alexandrov. As in Serrin's proof, the maximum principle is applied to compare the solution of the equation with its reflection with respect to the critical hyperplane.

In the relative setting of the spherical sector, the question of radial symmetry of positive solutions was first addressed by Berestycki and Pacella in [12], where, by a sophisticated adaptation of the moving planes method, they proved symmetry results in dimension $N = 2$. In higher dimensions, a similar symmetry result was recently obtained in [30] for convex spherical sectors and only for nonnegative nonlinearities.

On the other hand, many symmetry-breaking results are available in the literature. Morse theory plays a key role, as described in [26] (see the references therein). In the relative setting, Ciraolo, Pacella, and Polvara ([22]) showed that a break of symmetry occurs for certain classes of nonconvex spherical sectors.

Regarding one-dimensional solutions for the Dirichlet problem in the half-plane $\mathbb{R}^{N-1}$, it can be shown that under certain conditions on the nonlinearity f, the positive solution is one-dimensional. See [**FarinaSciunzi2017**, 24, 35].

On the other hand, del Pino and coauthors discovered counterexamples in dimension $N \geq 9$ to a famous conjecture of De Giorgi which states that the level sets of bounded solutions of $-\Delta u = u - u^3$ in $\mathbb{R}^N$ with $\frac{\partial u}{\partial x_N} > 0$ are all hyperplanes (and thus the solution is one-dimensional). See [29].

1.2.4 Other related questions

See [72] for other physical motivations for overdetermined problems, also in the case of exterior domains.

Concerning overdetermined problems in unbounded domains, there is now a solid body of results showing that rigidity is much less strong than in the case of bounded domains. See [67, 68, 70, 33].

1.3 Description of our results and open problems

We begin by introducing the notations and setting the problems we consider in this thesis in a general form.

Let $\mathcal{C} \subset \mathbb{R}^N$ be an unbounded uniformly Lipschitz domain, and $\Omega \subset \mathcal{C}$ be a bounded Lipschitz domain. We denote by Γ_Ω the relative boundary of Ω, that is, $\Gamma_\Omega := \partial\Omega \cap \mathcal{C}$. We also set the notation $\Gamma_{1,\Omega} = \partial\Omega \setminus \overline{\Gamma}_\Omega$, and assume that $\mathcal{H}^{N-1}(\Gamma_{1,\Omega}) > 0$, where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure. In other words, Ω and $\mathcal{C}$ share a nontrivial portion of the boundary.

We consider elliptic problems of the type

$$\begin{cases} -\Delta u &= f(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma_\Omega \\ \dfrac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma_{1,\Omega} \end{cases} \tag{1.3.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a $C^{1,\alpha}$ nonlinearity and ν denotes the exterior unit normal vector to $\partial\Omega$, and their overdetermined counterparts, where in addition to (1.3.1) we require that the solution u_Ω satisfies the overdetermined condition

$$\frac{\partial u_\Omega}{\partial \nu} = -c < 0 \quad \text{on } \Gamma_\Omega, \tag{1.3.2}$$

where $c > 0$ is a positive constant.

The main inspiration for our investigations is the series of papers [60, 61] and [46] by Pacella and collaborators. In these papers, the authors study, among other things, the relatively overdetermined torsion problem, i.e., the case where $f \equiv 1$ in (1.3.1), in the case when $\mathcal{C}$ is a cone.

1.3.1 Chapter 2: domains inside general unbounded sets

Our first investigations were motivated by [60, 61], in which the problem (1.3.1) with $f \equiv 1$ is studied in the relative setting of convex cones.

We began by trying to understand more of the behavior of solutions to the relatively overdetermined torsion problem (1.2.4) in general unbounded domains. Intuitively, the geometry of $\mathcal{C}$ should play a role, and thus some kind of hypothesis on it is to be required. The most general result we obtain is under a convexity hypothesis on $\mathcal{C}$, which is inspired by [37]:

Theorem (Theorem 2.3.17)**.** *Let $\mathcal{C} \subset \mathbb{R}^N$ be a convex uniformly Lipschitz unbounded domain and let $\Omega \subset \mathcal{C}$ be a bounded domain with Lipschitz relative boundary Γ_Ω. If* *a solution u of the relatively overdetermined torsion problem exists, then Ω is relatively self-Cheeger*[1].

The proof of this theorem is accomplished by taking the P-function introduced by Weinberger (see (1.2.2)) and showing that it satisfies

$$\begin{cases} \Delta P & \geq & 0 & \text{in } \Omega \\ P & = & c^2 & \text{on } \Gamma_\Omega \\ \dfrac{\partial P}{\partial \nu} & \leq & 0 & \text{on } \Gamma_{1,\Omega} \end{cases}.$$

The proof is completed by applying the maximum principle to the function P. The convexity of $\mathcal{C}$ plays a key role in showing that $\frac{\partial P}{\partial \nu} \leq 0$ on $\Gamma_{1,\Omega}$.

In the course of our investigations, we discovered other properties of relatively Cheeger sets when $\mathcal{C}$ is a cone or a cylinder. See Section 2.3.1.

In the special case when $\mathcal{C}$ is a cylinder

$$\mathcal{C} := \{x = (x', x_N) \in \mathbb{R}^N \ : \ x' \in \omega, \ x_N > 0\} = \omega \times (0, +\infty),$$

where $\omega \subset \mathbb{R}^{N-1}$ is a smooth bounded domain, we consider domains whose relative boundary Γ_Ω is the graph of some positive function φ, i.e., a cartesian graph. In this context we highlight Proposition 2.3.11, where we obtain a bound on the curvature of Γ_Ω if Ω is self-Cheeger, and Proposition 2.3.12, where a necessary condition on Γ_Ω for Ω to be self-Cheeger is obtained. In both cases, we proceed by taking small perturbations of the function φ and using the definition of Cheeger set.

Always in the relative setting of cartesian graphs, inspired by the parallel between overdetermined problems and CMC surfaces, we proved in Proposition 2.3.20 that the only CMC graphs intersecting the walls of the cylinder (i.e., the lateral boundary $\partial\omega \times (0, +\infty)$) orthogonally are flat graphs, i.e., graphs of constant functions. On this matter, see also the comment in Section 2.3.3.1.

In the same chapter, we also set the preliminaries for the study of semilinear problems, to the aim of generalizing the results of [46].

The first step is to consider the energy functional

$$(\Omega \subset \Sigma_D) \mapsto T(\Omega) = J(u_\Omega), \tag{1.3.3}$$

where u_Ω is a positive solution of (1.3.1) in Ω and J is the nonlinear energy functional

$$J(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx - \int_\Omega F(v) \, dx, \quad v \in H_0^1(\Omega \cup \Gamma_{1,\Omega}),$$

and where $F(s) = \int_0^s f(\tau) \, d\tau$.

A fundamental issue is that the semilinear problem (1.3.1) can have, in general, many solutions. One sees immediately that this feature makes ambiguous the definition of the

[1]See Section 2.3 for the definition of a relatively self-Cheeger set and related notions.

energy functional T: which solution to choose in the definition? First, one could think of working under the hypothesis that (1.3.1) had only one solution. However, a turnaround for multiplicity is to consider nondegenerate solutions, where nondegeneracy is meant in the sense of critical points of the energy functional J (in the appropriate space). The nondegeneracy hypothesis yields a "local uniqueness" property (Proposition 2.2.1), which allows us to well-define the functional T, at least in a neighborhood of each fixed domain. Since our aim is to take domain derivatives (i.e., derivatives of domain functionals[2]), that are local in nature, this is sufficient for our purposes.

We then proceed to compute the derivative of T and show that, under a volume constraint, the critical points of T are precisely the domains Ω where u_Ω satisfies (1.3.1) together with the relative overdetermined condition

$$\frac{\partial u_\Omega}{\partial \nu} = \text{constant} \qquad \text{on } \Gamma_\Omega.$$

The described analysis was first performed in the special cases of cones and cylinders, but we later realized that the results hold for any geometry of $\mathcal{C}$ (provided that it is uniformly Lipschitz), and thus we collect them in Section 2.2.

1.3.2 Chapter 3: stability analysis in cones

Our next step was to try to develop a study of the semilinear problem (1.3.1) in bounded domains inside cones. We chose to follow the shape optimization approach of [46], with the aim of finding domains different from spherical sectors where the overdetermined problem has a solution.

Denote by Σ_D the cone spanned by the domain $D \subset \mathbb{S}^{N-1}$. We work, as do [46], in the class of star-shaped domains (star-shaped with respect to the vertex of the cone). This class of domains can be parametrized by functions in $C^2(\overline{D})$. We refer to Section 3.2 for the precise definitions.

The first observation is that spherical sectors always admit a radial solution to (1.3.1), which then satisfies (1.3.2) and thus is a critical point for the shape functional T (see (1.3.3)) in the class of radial graphs (see Definition 2.2.4 for a better understanding of what we call a critical point of the energy functional T).

Proceeding further, we comput the second derivative of T and find a threshold for the first Neumann eigenvalue $\lambda_1(D)$ of the Laplace-Beltrami operator on D which yields the stability/instability of spherical sectors (the intersection between the cone and a ball centered on the vertex) as critical points of T under a volume constraint. We have the following theorem:

Theorem (Theorem 3.4.7)**.** *Let Σ_D be the cone spanned by the smooth domain $D \subset \mathbb{S}^{N-1}$. Let $\lambda_1(D)$ be the first nonzero Neumann eigenvalue of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^{N-1}}$ on D and let $\widehat{\nu}_1$ be the first eigenvalue of the singular eigenvalue problem* (3.3.5)*. Let u_D be a nondegenerate radial solution of* (3.1.1) *in the spherical sector Ω_D. We have:*

[2] The interested reader may consult [44] and [74].

(i) If $-\widehat{\nu}_1 < \lambda_1(D) < N-1$*, then* Ω_D *is an unstable critical point for* T*;*

(ii) If $\lambda_1(D) > N-1$*, then* Ω_D *is a stable critical point for* T*.*

The meaning of this result is that there exists a threshold for the shape of the cones where the spherical sectors are (or are not) local minimizers of the energy functional T. In particular, when instability holds, we obtain the existence of domains which are small perturbations of the spherical sectors that carry positive solutions with smaller energy than the radial ones.

Remarkably, this threshold is the same as the one obtained in [46], and depends only on the dimension, holding for every (suitable) nonlinearity.

The proof is based on an analysis of the sign of the second shape derivative of the energy functional T, which leads us to consider the eigenvalue problems (3.3.4) and (3.3.5).

1.3.3 Chapter 4: stability analysis in cylinders

To comment on Chapters 4 and 5, we need to introduce some notation.

We denote a point in $\mathbb{R}^N$ by $x = (x', x_N)$, where $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$. Let $\omega \subset \mathbb{R}^{N-1}$, $N \geq 2$, be a smooth bounded domain. We consider in $\mathbb{R}^N$ the cylinder spanned by ω,

$$\Sigma_\omega := \omega \times (0, +\infty) = \{x = (x', x_N) \in \mathbb{R}^N \ : \ x' \in \omega,\ x_N > 0\},$$

and set

$$\Gamma := \{x = (x', x_N) \in \Sigma_\omega \ : \ x_N = 1\}.$$

The bounded cylinder Ω_ω is defined as

$$\Omega_\omega := \{x = (x', x_N) \in \Sigma_\omega \ : \ x_N < 1\}.$$

Finally, we set $\Gamma_1 := \partial\Omega_\omega \setminus \overline{\Gamma}$.

The set Γ is usually said to be the relative boundary of Ω_ω with respect to the cylinder Σ_ω.

There are special solutions to (1.3.1) in Ω_ω that inherit the geometry of the cylinder, in the sense that they depend solely on the "height" x_N and thus are said to be "one-dimensional". They can be obtained by solving, for example by some standard variational method, the ordinary differential equation

$$\begin{cases} -u'' = f(u) & \text{in } (0,1) \\ u'(0) = u(1) = 0 \end{cases} \tag{1.3.4}$$

and then setting

$$u_\omega(x', x_N) = \widetilde{u}(x_N), \quad (x', x_N) \in \Omega_\omega, \tag{1.3.5}$$

where $\widetilde{u}$ is a solution of (1.3.4).

One-dimensional solutions can be viewed as the analogous of the radial ones in radially symmetric domains, and therefore it is natural to ask whether or not all positive solutions of (5.1.2) are one-dimensional.

The main results we obtain are analogous to the stability/instability Theorem 3.4.7, in the sense that we analyze the stability/instability of the special domain Ω_ω as a critical point of the energy shape functional T.

We first prove a general stability/instability condition that, in contrast to what is obtained in cones, strongly depends on the nonlinearity.

As in the cone, we study the sign of the second shape derivative of T, which is based on a decomposition of the spectrum of the linearized operator $L_{u_\omega} = -\Delta - f'(u_\omega)$. This, in turn, leads us to consider the Neumann eigenvalue problem on ω and the one-dimensional eigenvalue problem (4.3.5).

The instability Theorem 4.4.4 tells us some conditions under which we have more hope of finding domains other than Ω_ω which admit a solution to the relatively overdetermined problem: if Ω_ω is not a stable critical point, then in particular it is not a minimizer. If we do find a minimizer, as in [46] (see Theorem 1.2.4), then we will have found such a domain.

We also obtain a precise numerical threshold for stability/instability in the case $f \equiv 1$ (see Theorem 4.4.5). It is interesting to notice the connection with the overdetermined problem. Indeed, it can be inferred from the results of [34] (see the comments at the end of Chapter 2 and after Theorem 4.4.5) that, for some classes of cylinders, there exist domains whose relative boundary is not flat where a solution to the overdetermined problem exists. In [34], these domains are obtained by bifurcation from the bounded cylinder using as a parameter the height of the cylinder, and bifurcation arises when the height is sufficiently small. Our result, in turn, shows that instability occurs when we scale the base, making it sufficiently large. Together, these results tell us that when the bounded cylinder is unstable as a critical domain of the energy functional, then there is another critical domain at a lower energy level, close to the bounded cylinder.

1.3.4 Chapter 5: bounded cylinders and existence of solutions that are not one-dimensional

In Chapter 5 we consider only bounded cylinders Ω_ω, and we are interested in understanding whether or not solutions to (1.3.1) which are not one-dimensional exist.

The analysis of the spectrum of the linearized operator $L_{u_\omega} = -\Delta - f'(u_\omega)$ performed in Chapter 4 comes into play, for it allows us to easily compute the Morse index of one-dimensional solutions (see Theorem 5.2.2)

From this, we deduce a general existence result of solutions of Morse index one (see Chapter 5 for the definitions) which are not one dimensional (Theorem 5.3.3). See [22] for a similar argument in cones.

Our approach is inspired by [43] (see also [26]), where the authors prove symmetry-breaking results for the Lane-Emden problem in annuli. In our case, we are able to deal with more general nonlinearities.

Thus we obtain the existence of solutions that are not one-dimensional, provided that we consider a large enough scaling $t\omega = \{tx' \,:\, x' \in \omega\}$ of the domain $\omega \subset \mathbb{R}^{N-1}$ which spans the cylinder. This is linked to the decomposition of the spectrum of L_{u_ω} obtained in Lemma 4.3.2 and the fact that the Neumann eigenvalues scale as

$$\lambda_j(t\omega) = \frac{1}{t^2}\lambda_j(\omega). \tag{1.3.6}$$

More precisely, we know that an eigenvalue of the linearized operator L_{u_ω} is the sum of an eigenvalue of a one-dimensional eigenvalue problem (see (4.3.5)), which depends only on the nonlinearity f, and a Neumann eigenvalue of $-\Delta_{\mathbb{R}^{N-1}}$ in ω, which does not depend on f. Then, it is simple to compute the Morse index of one-dimensional solutions (see Theorem 5.2.2). Furthermore, under the hypothesis that some eigenvalue α of the one-dimensional problem is negative, by scaling ω we can find $\bar{t} > 0$ such that $\alpha + \lambda(\bar{t}\omega) = 0$, for some Neumann eigenvalue λ of $-\Delta\mathbb{R}^{N-1}$ in ω. As a consequence, the positive one-dimensional solution in the scaled domain $\Omega_{\bar{t}\omega}$ is degenerate. However, it can be shown that such a solution is nondegenerate in the subspace of one-dimensional functions (see Proposition 5.3.6).

Scalings also allow us to detect changes on the Morse index of u_ω. Moreover, if $-\Delta_{\mathbb{R}^{N-1}}$ has simple Neumann eigenvalues in ω, we can also detect a branch of positive solutions that are not one-dimensional bifurcating from the one-dimensional ones (see Theorem 5.3.8).

1.3.5 Open problems

In this section, we outline some interesting questions that arise from our results.

Minimization of the nonlinear energy functional in cones

In view of Theorem 3.4.7 and the examples presented in [46, Appendix A], we know that there are cones where we can find domains close to a spherical sector and which are at a lower energy level. It would be interesting to show that a (local or global) minimizer exists, for it would give rise to a new domain where the overdetermined problem has a solution (in view of Proposition 2.2.5).

First one can think of trying to adapt the arguments of [46], where this is done for the case $f \equiv 1$. However, the nonlinearity prevents one from exploiting the flattening of the cone at infinity, which is crucial for the proof presented in [46].

Characterization of stability/instability in cylinders and minimization of the energy functional

In Chapter 4, we prove an abstract result (Theorem 4.4.4) concerning the stability/instability of a bounded cylinder as a critical domain of the energy functional. A relevant question would be to find more explicit characterizations of stability/instability, given general conditions on the nonlinearity and/or on the domain ω which spans the cylinder.

A more concrete result we obtained in this direction is a threshold on the first Neumann eigenvalue of $-\Delta_{\mathbb{R}^{N-1}}$ in ω (depending on the nonlinearity and on the solution u_ω) that yields a sufficient condition for stability (see Theorem 4.4.6). A drawback of our result is that it does not imply that, below the threshold, instability occurs.

This question is related to whether or not there are minimizers (either local or global) for the energy functional. Even in the case $f \equiv 1$ this problem is completely open.

Again, one could think of reproducing the shape optimization approach of [46]. However, in that article, the invariance of the cone by dilation is also heavily exploited. Such a feature is not available in the case of the cylinder. More generally, it could be interesting to investigate the problem of determining suitable geometric conditions on a given unbounded container $\mathcal{C}$ such that a minimizer of the relative torsional energy, under a volume constraint, exists.

Interestingly enough, it can be inferred from our results and those of [34] that other critical domains do exist. See Section 2.3.3.1 and the comments after Theorem 4.4.5.

Self-Cheeger domains inside nonconvex sets

In Theorem 2.3.17 we showed that domains Ω which admit solutions to the overdetermined torsion problem are self-Cheeger (See Section 2.3.2). Our arguments depend strongly on the hypothesis that the unbounded set $\mathcal{C}$ which contains Ω is convex. It would be constructive to understand if this always holds, also in nonconvex containers, or how the possible concavities lower the Cheeger quotient.

Sign-changing solutions

The main results of this thesis concern positive solutions of semilinear equations. Positivity plays a key role in our arguments, since they are based mostly on understanding the sign of some integrals, where we strongly exploit the positivity of the solution.

A relevant question would be to extend our stability/instability results to sign-changing solutions.

While the geometry of positive solutions of semilinear problems in bounded domains of $\mathbb{R}^N$ tends to reflect the symmetry of the domain (See Section 1.2.3), sign-changing solutions may exhibit a much more complex behavior. Furthermore, while positivity is fundamental in proving rigidity results for overdetermined problems (e.g. Serrin's theorem), there are domains different from a ball where an overdetermined problem admits a sign-changing solution (see [69]).

Thus, since we already have instability in the more rigid case of positive solutions, it is expected that similar phenomena occur also for sign-changing solutions.

Chapter 2

General unbounded sets

2.1 Introduction

Let $\mathcal{C} \subset \mathbb{R}^N$, $N \geq 2$, be an unbounded uniformly Lipschitz domain and let $\Omega \subset \mathcal{C}$ be a bounded Lipschitz domain with smooth relative boundary $\Gamma_\Omega := \partial\Omega \cap \mathcal{C}$. More precisely, we assume that Γ_Ω is a smooth $(N-1)$-dimensional manifold with Lipschitz boundary $\partial\Gamma_\Omega \subset \partial\mathcal{C}$. We set $\Gamma_{1,\Omega} := (\partial\Omega \setminus \overline{\Gamma}_\Omega)$ and assume that $\mathcal{H}^{N-1}(\Gamma_{1,\Omega}) > 0$, where $\mathcal{H}^{N-1}$ stands for the $(N-1)$-dimensional Hausdorff measure. Note that we can write $\partial\Omega = \Gamma_\Omega \cup \Gamma_{1,\Omega} \cup \partial\Gamma_\Omega$.

We consider semilinear elliptic problems with mixed boundary conditions:

$$\begin{cases} -\Delta u &= f(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma_\Omega \\ \dfrac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma_{1,\Omega} \end{cases}, \tag{2.1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a $C^{1,\alpha}$ nonlinearity and ν denotes the exterior unit normal vector to $\partial\Omega$.

The homogeneous Neumann boundary condition on $\Gamma_{1,\Omega}$ makes the problem analogous to pure Dirichlet problems.

The appropriate functional space to deal with this kind of problem is the subspace of functions in $H^1(\Omega)$ whose trace vanishes on Γ_Ω, which we denote by $H_0^1(\Omega \cup \Gamma_{1,\Omega})$. To be precise, $H_0^1(\Omega \cup \Gamma_{1,\Omega})$ is defined as the closure in $H^1(\Omega)$ of the space $C_c^\infty(\Omega \cup \Gamma_{1,\Omega})$. This space enjoys all the classic properties of Sobolev spaces and provides us with a functional framework to study the problem (2.1.1) that mimics the structure of the pure Dirichlet problem (the case in which $\partial\Omega = \Gamma_\Omega$). We refer to [26] and the references therein for more properties of those spaces and related integral identities.

By standard variational methods, under suitable assumptions on f we obtain that a weak solution $u_\Omega \in H_0^1(\Omega \cup \Gamma_{1,\Omega})$ exists and is a critical point of the energy functional

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\, dx - \int_\Omega F(v)\, dx, \quad v \in H_0^1(\Omega \cup \Gamma_{1,\Omega}), \tag{2.1.2}$$

where $F(s) = \int_0^s f(\tau)\, d\tau$.

A classical example of a nonlinearity for which a positive solution exists for any domain Ω in $\mathcal{C}$ is the Lane-Emden nonlinearity, namely

$$f(u) = u^p, \ \text{ where } \begin{cases} 1 < p < \frac{N+2}{N-2}, & N \geq 3 \\ 1 < p < \infty, & N = 2 \end{cases}.$$

Given the unbounded set $\mathcal{C}$, an interesting question is to understand how the energy $J(u_\Omega)$ behaves with respect to variations of the domain Ω inside $\mathcal{C}$. In particular, one could ask whether the energy increases or decreases when Ω is deformed into a domain $\widetilde{\Omega}$ close to Ω and with the same measure.

Loosely speaking, one could consider the shape functional $\Omega \mapsto T(\Omega) = J(u_\Omega)$ and study it in a suitable "neighborhood" of Ω. This question can be attacked by differentiating the functional $T(\Omega)$ with respect to variations of Ω that leave the volume invariant and studying the stability or instability of its critical points.

One significant issue is that, in general, the nonlinear problem (2.1.1) could have many weak solutions, so in principle it is not clear a priori how to well define the energy $T(\Omega)$. The first result of this chapter is a "local uniqueness" result for solutions of (2.1.1) in domains which are “small perturbations" of Ω, as long as we start from a nondegenerate solution u_Ω in Ω. This yields that the functional $T(\Omega)$ is well-defined for domains obtained by small deformations of Ω induced by vector fields which leave $\mathcal{C}$ invariant.

Next, we obtain a characterization of the critical points of T as domains where $\frac{\partial u_\Omega}{\partial \nu}$ = constant on Γ_Ω. This result sheds light on the partially overdetermined problem for (2.1.1) where the overdetermined condition is posed only on Γ_Ω. Under this aspect, unstable critical points of T become of particular interest. Indeed, for some design regions $\mathcal{C}$ there are trivial domains where nondegenerate solutions for the overdetermined problem exist, and these domains and the respective solutions enjoy strong symmetry properties. Conditions that tell us that such domains are not local minima for the energy open room for the search of nonsymmetric domains where solutions to the overdetermined problem exist, in contrast to the classical result of Serrin (see [71]) where $\mathcal{C} = \mathbb{R}^N$.

In the second part of the chapter we study (2.1.1) with $f \equiv 1$, under the additional assumption that $\mathcal{C}$ is convex. We obtain a bound on the mean curvature of the relative boundary and show that domains that admit a solution to the partially overdetermined problem for the relative torsion problem are relatively self-Cheeger, by making use of the so-called P-function introduced by Weinberger in [75], together with the maximum principle. See Section 2.3.1 for the precise definitions and preliminary results.

2.2 Semilinear problems

In this section, we consider problem (2.1.1) in a bounded Lipschitz domain $\Omega \subset \mathcal{C}$ with smooth relative boundary Γ_Ω. Starting from a positive nondegenerate solution of (2.1.1) in Ω we show how to define an energy functional for small variations of Ω that preserve the volume.

We consider a positive weak solution u_Ω of (2.1.1) in the Sobolev space $H_0^1(\Omega \cup \Gamma_{1,\Omega})$. Under suitable assumptions on the growth of f, by standard elliptic regularity theory we have that u_Ω is a classical solution of (2.1.1) inside Ω and on the regular part of the boundary, and that u_Ω is bounded (see [22, Proposition 3.1]).

We assume that u_Ω is nondegenerate, i.e., the linearized operator

$$L_{u_\Omega} = -\Delta - f'(u_\Omega)$$

does not have zero as an eigenvalue in the space $H_0^1(\Omega \cup \Gamma_{1,\Omega})$. In other words, L_{u_Ω} defines an isomorphism between $H_0^1(\Omega \cup \Gamma_{1,\Omega})$ and its dual space. We consider small deformations of Ω that leave $\mathcal{C}$ invariant and would like to show that the nondegeneracy of u_Ω induces a local uniqueness result for solutions of (2.1.1) in the deformed domains.

By an *admissible vector field*, we mean a smooth vector field V such that $V(x) \in T_x\partial\mathcal{C}$ for every $x \in \partial\mathcal{C}^{\text{reg}}$, $V(x) = 0$ for $x \in \partial\mathcal{C} \setminus \partial\mathcal{C}^{\text{reg}}$, where $\partial\mathcal{C}^{\text{reg}}$ denotes the regular part of the boundary of $\mathcal{C}$ and $T_x\partial\mathcal{C}$ denotes the tangent space to $\partial\mathcal{C}$ at the point $x \in \partial\mathcal{C}$. We denote by ξ_t the associated one-parameter family of diffeomorphisms and set $\Omega_t = \xi_t(\Omega)$ for $t \in (-\eta, \eta)$, $\eta > 0$ small (in particular, $\Omega_0 = \Omega$). To simplify the notations we set

$$\Gamma_t := \Gamma_{\Omega_t}, \qquad \Gamma_{1,t} := \Gamma_{1,\Omega_t}.$$

Proposition 2.2.1. *Let u_Ω be a positive nondegenerate solution of (2.1.1), with $u_\Omega \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. Let V be an admissible vector field and let ξ_t be the associated one-parameter family of diffeomorphisms. Then there exists $\delta > 0$ such that for any $t \in (-\delta, \delta)$ there exists a unique solution $u_t \in H_0^1(\Omega_t \cup \Gamma_{1,t})$ of the problem*

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega_t \\ u = 0 & \text{on } \Gamma_t \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_{1,t} \end{cases} . \tag{2.2.1}$$

in a neighborhood of the function $u_\Omega \circ \xi_t^{-1}$ in the space $H_0^1(\Omega_t \cup \Gamma_{1,t})$. Moreover, the map $t \in (-\delta, \delta) \mapsto u_t \circ \xi_t \in H_0^1(\Omega \cup \Gamma_{1,\Omega})$ is differentiable at $t = 0$.

Proof. By using the diffeomorphism ξ_t we can pass from the space $H_0^1(\Omega_t \cup \Gamma_{1,t})$ to the space $H_0^1(\Omega \cup \Gamma_{1,\Omega})$. Indeed,

$$H_0^1(\Omega \cup \Gamma_{1,\Omega}) = \{v \circ \xi_t \ : \ v \in H_0^1(\Omega_t \cup \Gamma_{1,t})\}.$$

Moreover, u_t is a weak solution of (2.2.1), i.e.,

$$\int_{\Omega_t} \nabla u_t \nabla v \, dx - \int_{\Omega_t} f(u_t) v \, dx = 0 \quad \forall v \in H_0^1(\Omega_t \cup \Gamma_{1,t})$$

if and only if the function $\widehat{u}_t = u_t \circ \xi_t \in H_0^1(\Omega \cup \Gamma_{1,\Omega})$ satisfies

$$\int_\Omega (M_t \nabla \widehat{u}_t) \nabla w J_t \, dx - \int_\Omega f(\widehat{u}_t) w J_t \, dx = 0 \quad \forall w \in H_0^1(\Omega \cup \Gamma_{1,\Omega}),$$

where

$$J_t(x) = |\det \operatorname{Jac} \xi_t(x)|$$

and

$$M_t = [\operatorname{Jac} \xi_t^{-1}(\xi_t(x))][\operatorname{Jac} \xi_t^{-1}(\xi_t(x))]^T.$$

In other words, setting $\widehat{M_t} := M_t J_t$ we have that $\widehat{u}_t$ is a solution of

$$-\operatorname{div}(\widehat{M_t} \nabla \widehat{u}_t) - f(\widehat{u}_t) J_t = 0 \tag{2.2.2}$$

in the space $H_0^1(\Omega \cup \Gamma_{1,\Omega})$.

Now we consider the map

$$\mathcal{F} : (-\eta, \eta) \times H_0^1(\Omega \cup \Gamma_{1,\Omega}) \to H_0^1(\Omega \cup \Gamma_{1,\Omega})^*$$

defined as

$$\mathcal{F}(t, v) = -\operatorname{div}(\widehat{M_t} \nabla v) - f(v) J_t.$$

Since u_Ω is a solution to (2.1.1) in Ω and ξ_0 is the identity map, we have

$$\mathcal{F}(0, u_\Omega) = 0.$$

Notice that $\mathcal{F}$ is differentiable with respect to v, and

$$\partial_v \mathcal{F}(0, u_\Omega) = -\Delta - f'(u_\Omega). \tag{2.2.3}$$

Indeed, for any $h \in H_0^1(\Omega \cup \Gamma_{1,\Omega})$ we have

$$\begin{aligned}
\frac{\mathcal{F}(t, v + \varepsilon h) - \mathcal{F}(t, v)}{\varepsilon} &= \frac{-\operatorname{div}(\widehat{M_t}(\nabla v + \varepsilon \nabla h)) - (-\operatorname{div}(\widehat{M_t} \nabla v))}{\varepsilon} \\
&\quad - \frac{f(v + \varepsilon h) J_t - f(v) J_t}{\varepsilon} \\
&= -\frac{\operatorname{div}(\varepsilon \widehat{M_t} \nabla h)}{\varepsilon} - \frac{f(v + \varepsilon h) J_t - f(v) J_t}{\varepsilon} \\
&\to -\operatorname{div}(\widehat{M_t} \nabla h) - f'(v) J_t h
\end{aligned}$$

as $\varepsilon \to 0$. Then $\mathcal{F}(t, \cdot)$ is differentiable, and evaluating $\partial_v \mathcal{F}$ at $(0, u_\Omega)$ we obtain (2.2.3).

By the nondegeneracy assumption on the solution u_Ω, we infer that (2.2.3) gives an isomorphism between $H_0^1(\Omega \cup \Gamma_{1,\Omega})$ and $H_0^1(\Omega \cup \Gamma_{1,\Omega})^*$. Then, by the Implicit Function Theorem, there exists an interval $(-\delta, \delta)$ and a neighborhood $\mathcal{B}$ of u_Ω in $H_0^1(\Omega \cup \Gamma_{1,\Omega})$ such that for every $t \in (-\delta, \delta)$ there exists a unique function $\widehat{u}_t \in H_0^1(\Omega \cup \Gamma_{1,\Omega})$ in $\mathcal{B}$ such that $\mathcal{F}(t, \widehat{u}_t) = 0$, that is, $\widehat{u}_t$ is the unique solution (in $\mathcal{B}$) of (2.2.2). It follows that $u_t = \widehat{u}_t \circ \xi_t^{-1}$ is the unique solution of (2.2.1) in a neighborhood of $u_\Omega \circ \xi_t^{-1}$ in $H_0^1(\Omega_t \cup \Gamma_{1,t})$, as claimed.

Finally, the map $t \mapsto \widehat{u}_t = u_t \circ \xi_t$ is smooth. In addition, for $x \in \Omega$,

$$\widetilde{u}(x) := \frac{d}{dt}\bigg|_{t=0} u_t(x) = \left(\frac{d}{dt}\bigg|_{t=0} \widehat{u}_t(x) \right) - \langle \nabla u_\Omega(x), V(x) \rangle. \tag{2.2.4}$$

The proof is complete. □

We observe that the gradient behaves well with respect to the derivative of $t \mapsto u_t$. More precisely, we have the following:

Lemma 2.2.2. *Let u_Ω, V and ξ_t be as in Proposition 2.2.1. Then*

$$\frac{d}{dt}\bigg|_{t=0} (\nabla u_t) = \nabla \left(\frac{d}{dt}\bigg|_{t=0} u_t \right). \tag{2.2.5}$$

Proof. Writing $u_t = \widehat{u}_t \circ \xi_t^{-1}$, we obtain (2.2.5) by the chain rule and direct computations.

To be more precise, first, we claim that

$$\frac{d}{dt}\bigg|_{t=0} (\nabla \widehat{u}_t) = \nabla \left(\frac{d}{dt}\bigg|_{t=0} \widehat{u}_t \right). \tag{2.2.6}$$

To prove the claim, let us first consider the following map: $\widehat{\Phi} : (-\delta, \delta) \to H_0^1(\Omega \cup \Gamma_{1,\Omega})$ given by

$$\widehat{\Phi}(t) = \widehat{u}_t = u_t \circ \xi_t.$$

Now let $G : H_0^1(\Omega \cup \Gamma_{1,\Omega}) \to L^2(\Omega \cup \Gamma_{1,\Omega}, \mathbb{R}^N)$ be the gradient operator: $G(f) := \nabla f$. Since G is a bounded linear operator we have $G'(f)[g] = \nabla g$, for $g \in H_0^1(\Omega \cup \Gamma_{1,\Omega})$.

By the differentiability of $\widehat{\Phi}$ it follows that $G \circ \widehat{\Phi} : (-\delta, \delta) \to L^2(\Omega \cup \Gamma_{1,\Omega}, \mathbb{R}^N)$ is differentiable and it holds that

$$(G \circ \widehat{\Phi})'(t) = G'(\widehat{\Phi}(t))[\widehat{\Phi}'(t)] = \nabla \widehat{\Phi}'(t), \quad t \in (-\delta, \delta),$$

which is precisely (2.2.6).

Next, we observe that

$$\frac{\partial u_t}{\partial x_i} = \frac{\partial}{\partial x_i}(\widehat{u}_t \circ \xi_t^{-1}) = \nabla \widehat{u}_t(\xi_t^{-1}) \cdot \frac{\partial \xi_t^{-1}}{\partial x_i},$$

from which we deduce that

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial}{\partial x_i} u_t \right) &= \left(\nabla \left(\frac{d}{dt} \widehat{u}_t \right)(\xi_t^{-1}) + D^2 \widehat{u}_t(\xi_t^{-1}) \frac{d\xi_t^{-1}}{dt} \right) \cdot \frac{\partial \xi_t^{-1}}{\partial x_i} \\ &\quad + \nabla \widehat{u}_t(\xi_t^{-1}) \cdot \frac{\partial}{\partial x_i} \frac{d\xi_t^{-1}}{dt} \\ &= \frac{\partial}{\partial x_i}\left(\frac{d}{dt} u_t \right). \end{aligned}$$

This holds for every $i = 1, \ldots, N$, and thus the result follows. □

Note that, as for u_Ω, elliptic regularity theory implies that u_t is a classical solution to (2.2.1) in Ω_t and on the regular part of the boundary $\partial\Omega_t$.

The importance of Proposition 2.2.1 is that, given a solution u_Ω to (2.1.1), the functional

$$T(\Omega_t) = J(u_t) = \frac{1}{2}\int_{\Omega_t} |\nabla u_t|^2 \, dx - \int_{\Omega_t} F(u_t)\, dx, \tag{2.2.7}$$

where $F(s) = \int_0^s f(\tau)\, d\tau$, is well-defined for every t small enough.

Observe that since u_t is a solution to (2.2.1) we have

$$\int_{\Omega_t} |\nabla u_t|^2 \, dx = \int_{\Omega_t} f(u_t) u_t \, dx,$$

so we can also write

$$T(\Omega_t) = \frac{1}{2}\int_{\Omega_t} f(u_t) u_t \, dx - \int_{\Omega_t} F(u_t)\, dx. \tag{2.2.8}$$

In the next proposition, we show that T is differentiable with respect to t and compute its derivative at $t = 0$, that is, at the initial domain Ω.

Proposition 2.2.3. *Assume that $u_\Omega \in H_0^1(\Omega \cup \Gamma_{1,\Omega})$ is a nondegenerate positive solution of (2.1.1) such that $u_\Omega \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. Then*

$$\left.\frac{d}{dt}\right|_{t=0} T(\Omega_t) = -\frac{1}{2}\int_{\Gamma_\Omega} |\nabla u_\Omega|^2 \langle V, \nu\rangle \, d\sigma. \tag{2.2.9}$$

Proof. Recall from Proposition 2.2.1 that $t \mapsto u_t$ is smooth and (2.2.4) holds. Then

$$\widetilde{u} + \langle \nabla u_\Omega, V\rangle = \left(\left.\frac{d}{dt}\right|_{t=0} \widehat{u}_t \right) \in H_0^1(\Omega \cup \Gamma_{1,\Omega})$$

and therefore

$$\widetilde{u} = -\frac{\partial u_\Omega}{\partial \nu}\langle V, \nu\rangle \quad \text{ on } \Gamma_\Omega$$

in the weak sense.

Since ξ_t maps $\partial\mathcal{C}$ into itself, then for every small t and $x \in \Gamma_{1,\Omega}^{\text{reg}}$ it holds

$$\langle \nabla u_t(\xi_t(x)), \nu(\xi_t(x))\rangle = 0.$$

Differentiating this relation with respect to t and evaluating at $t = 0$ we obtain

$$0 = \langle \nabla \widetilde{u}(x), \nu(x)\rangle + \underbrace{d_x(\langle \nabla u_\Omega, \nu\rangle)[V(x)]}_{=0},$$

where $d_x(\langle \nabla u_\Omega, \nu\rangle)[V(x)]$ is the differential of the function $\langle \nabla u_\Omega, \nu\rangle|_{\Gamma_{1,\Omega}^{\text{reg}}}$ computed at x, along $V(x)$, and vanishes because of the Neumann condition on u_Ω on $\Gamma_{1,\Omega}$.

Finally, writing the equation $-\Delta u_t = f(u_t)$ in the weak form and differentiating with respect to t we obtain $-\Delta\widetilde{u} = f'(u_\Omega)\widetilde{u}$. More precisely, let $v \in C_c^\infty(\Omega \cup \Gamma_{1,\Omega})$. For all sufficiently small t, we also have $v \in C_c^\infty(\Omega_t \cup \Gamma_{1,\Omega_t})$. Hence, since u_t is a weak solution to (2.2.1) we have

$$0 = \int_{\Omega_t} \nabla u_t \nabla v \, dx - \int_{\Omega_t} f(u_t) v \, dx = \int_{\Omega} \nabla u_t \nabla v \, dx - \int_{\Omega} f(u_t) v \, dx. \tag{2.2.10}$$

Now, by Lemma 2.2.2 it holds that

$$\left.\frac{d}{dt}\right|_{t=0} \nabla u_t = \nabla \widetilde{u}.$$

Then, taking the derivative with respect to t in (2.2.10) and evaluating at $t = 0$, since v is arbitrary, we easily conclude.

In the end, we obtain that $\widetilde{u} \in H^1(\Omega)$ is a weak solution to

$$\begin{cases} -\Delta\widetilde{u} &= f'(u_\Omega)\widetilde{u} & \text{in } \Omega \\ \widetilde{u} &= -\dfrac{\partial u_\Omega}{\partial \nu}\langle V, \nu\rangle & \text{on } \Gamma_\Omega \\ \dfrac{\partial \widetilde{u}}{\partial \nu} &= 0 & \text{on } \Gamma_{1,\Omega} \end{cases}. \tag{2.2.11}$$

By elliptic regularity theory, we also have that $\widetilde{u}$ is a classical solution in the interior of Ω and on the regular part of the boundary.

Recalling (2.2.8) we can write

$$T(\Omega_t) = \int_{\Omega_t} \left(\frac{1}{2} f(u_t) u_t - F(u_t)\right) \, dx.$$

Since $t \mapsto \frac{1}{2} f(u_t)u_t - F(u_t)$ is differentiable at $t = 0$, $\partial\Omega$ is Lipschitz and taking into account that $u_\Omega \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$, then, applying [44, Theorem 5.2.2], we can compute

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} T(\Omega_t) &= \frac{1}{2}\int_\Omega (f'(u_\Omega)\widetilde{u}u_\Omega + f(u_\Omega)\widetilde{u}) \, dx - \int_\Omega f(u_\Omega)\widetilde{u} \, dx \\ &\quad + \int_{\partial\Omega} \left(\frac{1}{2} f(u_\Omega)u_\Omega - F(u_\Omega)\right) \langle V, \nu\rangle \, d\sigma \\ &= \frac{1}{2}\int_\Omega (f'(u_\Omega)\widetilde{u}u_\Omega - f(u_\Omega)\widetilde{u}) \, dx \\ &= \frac{1}{2}\int_\Omega ((-\Delta\widetilde{u})u_\Omega + (\Delta u_\Omega)\widetilde{u}) \, dx \\ &= \frac{1}{2}\int_{\partial\Omega} \left(\widetilde{u}\frac{\partial u_\Omega}{\partial \nu} - u_\Omega \frac{\partial \widetilde{u}}{\partial \nu}\right) \, d\sigma \\ &= -\frac{1}{2}\int_{\Gamma_\Omega} |\nabla u_\Omega|^2 \langle V, \nu\rangle \, d\sigma. \end{aligned}$$

Here we have used the Gauss-Green Theorem ([32, Theorem 5.16]) together with the regularity hypotheses on u_Ω and $\partial\Omega$.

The proof is complete. $\square$

Let us now consider domains $\Omega \subset \mathcal{C}$ of fixed measure $c > 0$ and define

$$\mathcal{A} := \{\Omega \subset \mathcal{C} : \ \Omega \text{ is admissible and } |\Omega| = c\}, \tag{2.2.12}$$

where by *admissible domain* we mean that $\Omega \subset \mathcal{C}$ is a bounded Lipschitz domain with smooth relative boundary $\Gamma_\Omega = \partial\Omega \cap \mathcal{C}$, $\partial\Gamma_\Omega$ is a smooth $(N-2)$-dimensional manifold and $\Gamma_{1,\Omega} = \partial\Omega \setminus \overline{\Gamma}_\Omega$ is such that $\mathcal{H}^{N-1}(\Gamma_{1,\Omega}) > 0$. We consider vector fields that induce deformations that preserve the volume. More precisely, we take a one-parameter family of diffeomorphisms ξ_t, $t \in (-\eta, \eta)$, associated to an admissible smooth vector field V, satisfying the condition $|\Omega_t| = |\Omega|$ for all $t \in (-\eta, \eta)$, where $\Omega_t = \xi_t(\Omega)$.

By Proposition 2.2.3 we can give the following definition:

Definition 2.2.4. Let $\Omega \in \mathcal{A}$ and $u_\Omega \in H^1_0(\Omega \cup \Gamma_{1,\Omega})$ be a positive nondegenerate solution of (2.1.1) such that $u_\Omega \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. The pair (Ω, u_Ω) is said to be *energy-stationary* if

$$\frac{d}{dt}\Big|_{t=0} T(\Omega_t) = 0 \tag{2.2.13}$$

for every smooth admissible vector field such that the associated one-parameter family of diffeomorphisms preserves the volume.

A characterization of energy-stationary domains in $\mathcal{C}$ is the following:

Proposition 2.2.5. *Let $\Omega \in \mathcal{A}$ and assume that $u_\Omega \in H^1_0(\Omega \cup \Gamma_{1,\Omega})$ is a positive nondegenerate solution (2.1.1) such that $u_\Omega \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. Then (Ω, u_Ω) is energy-stationary under the volume constraint if and only if u_Ω satisfies the overdetermined condition $|\nabla u_\Omega| =$ constant on Γ_Ω.*

Proof. Let ξ_t be the one-parameter family of diffeomorphisms associated with an admissible vector field V that preserves the volume.

Since the volume is preserved, by [44, Theorem 5.2.2] we have

$$0 = \frac{d}{dt}\Big|_{t=0} |\Omega_t| = \int_{\partial\Omega} \langle V, \nu \rangle \, d\sigma = \int_{\Gamma_\Omega} \langle V, \nu \rangle \, d\sigma. \tag{2.2.14}$$

If $|\nabla u_\Omega|$ is constant on Γ_Ω, then (Ω, u_Ω) is energy stationary, in view of (2.2.9) and (2.2.14).

On the other hand, if (Ω, u_Ω) is energy-stationary, then

$$\int_{\Gamma_\Omega} (|\nabla u_\Omega|^2 - a)\langle V, \nu \rangle \, d\sigma = 0$$

for every constant $a \in \mathbb{R}$ and every admissible vector field V. Assume by contradiction that $|\nabla u_\Omega|$ is not constant on Γ_Ω. Then there exists a compact set $K \subset \Gamma_\Omega$, with a nonempty interior, such that $|\nabla u_\Omega|$ is not constant on K. Take a nonnegative cutoff Θ such that $\Theta \equiv 1$ in K and choose

$$a = \frac{\int_{\Gamma_\Omega} \Theta |\nabla u_\Omega|^2 \, d\sigma}{\int_{\Gamma_\Omega} \Theta \, d\sigma}.$$

Then we can build a deformation starting from the vector field $V = (|\nabla u_\Omega|^2 - a)\Theta\nu$, and in this case, since (Ω, u_Ω) is energy-stationary, we would have

$$\int_K (|\nabla u_\Omega|^2 - a)^2 \, d\sigma = 0,$$

which contradicts the fact that $|\nabla u_\Omega|$ is not constant on K. □

Remark 2.2.6. It is relevant to observe that all concepts introduced in this section apply when $\Gamma_{1,\Omega}$ is empty, or, equivalently, when $\mathcal{C} = \mathbb{R}^N$. Thus all the aforementioned results hold for Dirichlet problems in domains in the whole space. In this case, it is known by Serrin's Theorem (see [71]) that if a positive solution for the overdetermined problem

$$\begin{cases} -\Delta u &= f(u) & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \\ \dfrac{\partial u}{\partial \nu} &= \text{constant} & \text{on } \partial\Omega \end{cases}$$

exists, then Ω is a ball. Therefore, in view of Proposition 2.2.5, it follows that the only energy-stationary pairs in $\mathbb{R}^N$ are (B, u_B) where B is a ball and u_B is a positive nondegenerate solution.

Remark 2.2.7. We observe that all results in this section hold true also for nondegenerate sign-changing solutions u_Ω to (2.1.1).

2.3 Torsion problem

Our focus now is on the case when $f \equiv 1$, which amounts to the "relative torsion problem"

$$\begin{cases} -\Delta u &= 1 & \text{in } \Omega \\ u &= 0 & \text{on } \Gamma_\Omega \\ \dfrac{\partial u}{\partial \nu} &= 0 & \text{on } \Gamma_{1,\Omega} \end{cases} . \tag{2.3.1}$$

We will show that there are connections between the related overdetermined problem and the Cheeger problem, which can be posed in the relative setting. We follow [42], referring the interesting reader also to [8].

2.3.1 Relative Cheeger sets

We begin by briefly describing the geometric measure theory tools we need in the relative setting.

Given the unbounded region $\mathcal{C} \subset \mathbb{R}^N$, which we assume to be a uniformly Lipschitz domain, and a bounded Lipschitz open set Ω, we can define the relative (to $\mathcal{C}$) total variation of a function $u \in L^1(\Omega)$ as

$$|Du|_{\mathcal{C}}(\Omega) \coloneqq \sup \left\{ \int_{\mathcal{C}} u \operatorname{div} \psi \, dx \;:\; \psi \in C^1_c(\Omega, \mathbb{R}^N),\; \|\psi\|_\infty \leq 1 \right\}.$$

If $|Du|_{\mathcal{C}}(\Omega) < +\infty$, then u is said to be of bounded variation. The space of functions of bounded variation is denoted by $BV_{\mathcal{C}}(\Omega)$, where we keep the subscript $\mathcal{C}$ to emphasize the relative nature of our objects. The space $BV_{\mathcal{C}}(\Omega)$ is a Banach space when endowed with the norm

$$\|u\|_{BV_{\mathcal{C}}} = \|u\|_1 + |Du|_{\mathcal{C}}(\Omega).$$

If $u \in BV_{\mathcal{C}}(\Omega)$, then letting $\widetilde{u}$ be the extension of u by 0 in $\mathcal{C} \setminus \Omega$ we can define a trace of u on Γ_Ω, and it holds

$$\widetilde{u} \in BV_{\mathcal{C}}(\mathcal{C}), \quad |Du|_{\mathcal{C}}(\mathcal{C}) = |Du|_{\mathcal{C}}(\Omega) + \int_{\Gamma_\Omega} |u| \, d\sigma.$$

The definition of total variation can be extended to more general cases when Ω is not necessarily open. Then the relative perimeter of $\Omega \subset \mathcal{C}$ can be defined in terms of the relative total variation of the characteristic function of the set Ω:

Definition 2.3.1. Let $\Omega \subset \mathcal{C}$ be a Borel set. We denote by $P_{\mathcal{C}}(\Omega)$ the *relative (to $\mathcal{C}$) De Giorgi perimeter* of Ω:

$$P_{\mathcal{C}}(\Omega) \coloneqq |D\chi_\Omega|_{\mathcal{C}}(\mathcal{C}) = \sup \left\{ \int_{\Omega} \operatorname{div} \psi \, dx \;:\; \psi \in C^1_c(\mathcal{C}, \mathbb{R}^N),\; \|\psi\|_\infty \leq 1 \right\}. \tag{2.3.2}$$

In other words, $P_{\mathcal{C}}$ is the total variation in $\mathcal{C}$ of the characteristic function χ_Ω. A set Ω is said to be of *finite (relative) perimeter* if $P_{\mathcal{C}}(\Omega) < +\infty$.

Observe that if $\Gamma_\Omega \coloneqq \partial\Omega \cap \mathcal{C}$ is Lipschitz, then the relative perimeter coincides with the $(N-1)$-dimensional Hausdorff measure of the relative boundary:

$$P_{\mathcal{C}}(\Omega) = \mathcal{H}^{N-1}(\Gamma_\Omega) = \int_{\Gamma_\Omega} 1 \, d\sigma,$$

where $d\sigma$ denotes the surface measure.

Definition 2.3.2. The *relative (to $\mathcal{C}$) Cheeger constant* of a bounded domain $\Omega \subset \mathcal{C}$ with Lipschitz boundary is

$$h_{\mathcal{C}}(\Omega) \coloneqq \inf \left\{ \frac{P_{\mathcal{C}}(E)}{|E|} \;:\; E \subset \overline{\Omega} \text{ is a Borel set, } |E| \neq 0 \right\}. \tag{2.3.3}$$

If this infimum is attained by some set E, then this set is said to be a *Cheeger set* of Ω. If Ω attains the infimum, that is, if $h_{\mathcal{C}}(\Omega) = \frac{P_{\mathcal{C}}(\Omega)}{|\Omega|}$, then Ω is said to be *self-Cheeger*.

The Cheeger constant has many interesting motivations and applications. Indeed, it first appeared in a lower bound for the first eigenvalue of the Dirichlet-Laplacian on Riemannian manifolds ([20]), being later generalized for the Neumann problem ([52]); see also [48] for the link with the spectral theory for the p-Laplacian. In [50], the author shows how the Cheeger constant appears naturally in the problem of the fracture of a plate under pressure. See [47] for a landslide problem and [19] for a generalization of the Cheeger problem applied to image processing. The problem of finding the Cheeger constant and Cheeger sets is remarkably difficult, and there are interesting results only for planar convex sets: see [49] for a characterization of the Cheeger sets and [38] for sharp estimates of the Cheeger constant in terms of other geometric quantities.

For the sake of completeness, we now report some properties of functions of bounded variation which will be needed in our discussion of relative Cheeger sets.

Proposition 2.3.3. *Let $\{u_j\}$ be a sequence of functions in $BV_{\mathcal{C}}(\Omega)$ such that $u_j \to u$ in $L^1_{loc}(\Omega)$. Then*

$$|Du|_{\mathcal{C}}(\Omega) \leq \liminf_{j\to\infty} |Du_j|_{\mathcal{C}}(\Omega).$$

Proof. Let $\psi \in C^1_c(\Omega, \mathbb{R}^N)$ be such that $\|\psi\|_\infty \leq 1$. Observe that

$$\int_\Omega u \operatorname{div} \psi \, dx = \lim_{j\to\infty} \int_\Omega u_j \operatorname{div} \psi \, dx \leq \liminf_{j\to\infty} |Du_j|_{\mathcal{C}}(\Omega).$$

We then conclude by taking the supremum over all such ψ. □

Proposition 2.3.4. *Let $\Omega \subset \mathcal{C}$ be an open bounded domain with Lipschitz boundary. Let $\{u_j\}$ be a uniformly bounded sequence in $BV_{\mathcal{C}}(\Omega)$. Then there exists a function $u \in BV_{\mathcal{C}}(\Omega)$ such that $u_j \to u$ in $L^1(\Omega)$, up to subsequences.*

Proof. See [42, Theorem 1.19]. □

Proposition 2.3.5. *Let $u \in BV_{\mathcal{C}}(\Omega)$ and define*

$$E_t = \{x \in \Omega \ : \ u(x) > t\}.$$

Then

$$|Du|_{\mathcal{C}}(\Omega) = \int_{-\infty}^{+\infty} P_{\mathcal{C}}(E_t) \, dt.$$

Proof. See [42, Theorem 1.23]. □

Now we give some properties of Cheeger sets in the relative setting.

Proposition 2.3.6 ([62], Proposition 3.1)**.** *Let $\Omega \subset \mathcal{C}$ be a bounded domain such that Γ_Ω is Lipschitz. Then there exists at least one Cheeger set for Ω.*

Proof. Define
$$\widetilde{h}_{\mathcal{C}}(\Omega) = \inf_{v\in BV_{\mathcal{C}}(\Omega)\setminus\{0\}} \frac{|Dv|_{\mathcal{C}}(\mathcal{C})}{\|v\|_1}.$$

Since Ω is Lipschitz, we can extend the functions in $BV_{\mathcal{C}}(\Omega)$ by 0 to obtain functions in $BV_{\mathcal{C}}(\mathcal{C})$. Then, by definition,
$$\widetilde{h}_{\mathcal{C}}(\Omega) \leq h_{\mathcal{C}}(\Omega). \tag{2.3.4}$$
Taking a minimizing sequence and applying Propositions 2.3.3 and 2.3.4, together with the direct method of the Calculus of Variations, we obtain that there exists $u \in BV_{\mathcal{C}}(\Omega)$, $u \not\equiv 0$, such that
$$\frac{|Du|_{\mathcal{C}}(\mathcal{C})}{\|u\|_1} = \widetilde{h}_{\mathcal{C}}(\Omega).$$

From [8, Exercise 3.12] it follows that $|D|u||_{\mathcal{C}}(\mathcal{C}) \leq |Du|_{\mathcal{C}}(\mathcal{C})$, and therefore we can assume $u \geq 0$. Let
$$E_t := \{x \in \Omega \ : \ u(x) > t\}.$$
From Proposition 2.3.5 and Cavalieri's principle (layer cake representation, see [53, Section 1.13]), we obtain
$$\begin{aligned} 0 &= |Du|_{\mathcal{C}}(\mathcal{C}) - \widetilde{h}_{\mathcal{C}}(\Omega)\|u\|_1 \\ &= \int_0^{+\infty} (P_{\mathcal{C}}(E_t) - \widetilde{h}_{\mathcal{C}}(\Omega)|E_t|)\, dt \\ &\geq \int_0^{+\infty} (P_{\mathcal{C}}(E_t) - h_{\mathcal{C}}(\Omega)|E_t|)\, dt \\ &\geq 0. \end{aligned}$$

Then, for almost every t (in the sense of the Lebesgue measure in $\mathbb{R}$) it holds
$$P_{\mathcal{C}}(E_t) - \widetilde{h}_{\mathcal{C}}(\Omega)|E_t| = 0. \tag{2.3.5}$$
Since $u \not\equiv 0$, then there exists $\bar{t} > 0$ such that $E_{\bar{t}} > 0$ and such that (2.3.5) holds. But then, for the set $E_{\bar{t}}$ it holds that
$$h_{\mathcal{C}}(\Omega) \leq \frac{P_{\mathcal{C}}(E_{\bar{t}})}{|E_{\bar{t}}|} = \widetilde{h}_{\mathcal{C}}(\Omega).$$

In view of (2.3.4), we conclude that $h_{\mathcal{C}}(\Omega) = \widetilde{h}_{\mathcal{C}}(\Omega)$ and that $E_{\bar{t}}$ is a Cheeger set for Ω. □

Remark 2.3.7. If Γ_Ω is Lipschitz, then the Cheeger constant can be approximated by relatively smooth sets, in the sense that
$$h_{\mathcal{C}}(\Omega) = \inf_{E\subset\Omega} \frac{|\Gamma_E|}{|E|}$$
where the infimum is taken over the subsets of Ω such that the relative boundary $\partial E \cap \mathcal{C}$ is smooth. See [62].

2.3.1.1 Special geometries

Proposition 2.3.8. *Let $\mathcal{C}$ be a cone, that is, let $D \subset \mathbb{S}^{N-1}$ be a smooth domain and consider*

$$\mathcal{C} = \Sigma_D = \{x \in \mathbb{R}^N \ : \ x = tq, \ q \in D, \ t > 0\}.$$

Let $\Omega \subset \Sigma_D$ be a bounded domain with Lipschitz relative boundary Γ_Ω. If $E \subset \Omega$ is a Cheeger set, then $\partial E \cap \overline{\Gamma_\Omega} \neq \emptyset$.

Proof. For the sake of contradiction, suppose that $d(E, \Gamma_\Omega) \geq \delta$ for some $\delta > 0$. Then we can find some $t > 1$ such that the set

$$tE \coloneqq \{x \in \Sigma_D \ : \ t^{-1}x \in E\}$$

is still contained in Ω. By the formula for change of variables we have

$$|tE| = t^N |E|,$$

while for the relative perimeter, we have

$$\begin{aligned} P_{\Sigma_D}(tE) &= \sup\left\{ \int_{tE} \operatorname{div} \psi(x) \, dx \ : \ \psi \in C_c^1(\Sigma_D, \mathbb{R}^N), \ \|\psi\|_\infty \leq 1 \right\} \\ &= \sup\left\{ \int_{E} \operatorname{div} \psi(t^{-1}x) t^N \, dx \ : \ \psi \in C_c^1(\Sigma_D, \mathbb{R}^N), \ \|\psi\|_\infty \leq 1 \right\} \\ &= t^{N-1} P_{\Sigma_D}(E). \end{aligned}$$

But then

$$\frac{P_{\Sigma_D}(tE)}{|tE|} = \frac{t^{N-1}}{t^N} \frac{P_{\Sigma_D}(E)}{|E|} < h_{\Sigma_D}(\Omega),$$

which contradicts the definition of the Cheeger set E. □

Let $\omega \subset \mathbb{R}^{N-1}$ be a smooth bounded domain and let $\Sigma_\omega = \omega \times (0, +\infty)$ be the cylinder spanned by ω.

Proposition 2.3.9. *Assume that $\Omega \subset \Sigma_\omega$ is a bounded domain such that Γ_Ω is a connected surface whose projection onto $\mathbb{R}^{N-1}$ is exactly ω. If some relative Cheeger set E_φ is defined by the graph of a positive function φ on ω, that is, $E_\varphi = \{(x', x_N) \in \Sigma_\omega \ : \ 0 < x_N < \varphi(x')\}$, then $\overline{\Gamma_{E_\varphi}} \cap \overline{\Gamma_\Omega} \neq \emptyset$.*

Proof. Note that Γ_{E_φ} is precisely the graph of φ. If $\overline{\Gamma_{E_\varphi}} \cap \overline{\Gamma_\Omega} = \emptyset$, we could consider the function $\varphi + \delta$, where $\delta > 0$ is small enough so that $E_{\varphi+\delta} \subset \Omega$, and we would have

$$\frac{P_{\Sigma_\omega}(E_{\varphi+\delta})}{|E_{\varphi+\delta}|} = \frac{P_{\Sigma_\omega}(E_\varphi)}{|E_{\varphi+\delta}|} < \frac{P_{\Sigma_\omega}(E_\varphi)}{|E_\varphi|},$$

contradicting the fact that E_φ is a relative Cheeger set. □

The mean curvature of Γ_Ω, which is denoted by H_{Γ_Ω}, is defined as

$$H_{\Gamma_\Omega}(x) = -(N-1)\operatorname{div}\nu(x), \tag{2.3.6}$$

where ν is the outer unit normal vector and div here means the tangential divergence on the surface Γ_Ω.

Now we consider domains inside Σ_ω defined by graphs of smooth functions. More precisely, for a positive function $\varphi \in C^2(\overline{\omega})$ we consider the domain

$$\Omega_\varphi = \{(x', x_N) \in \Sigma_\omega \ : \ 0 < x_N < \varphi(x')\}.$$

To simplify the notation we set $\Gamma_\varphi := \Gamma_{\Omega_\varphi}$.

In the case when $\Omega = \Omega_\varphi$, t is well-known that

$$H_{\Gamma_\varphi} = -\frac{1}{N-1}\operatorname{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right).$$

Remark 2.3.10. If $\overline{\Gamma}_\varphi$ and $\partial\Sigma_\omega$ intersect orthogonally, we have that

$$\nu_{\Gamma_\varphi}\cdot\nu_{\partial\Sigma_\omega} = 0 \quad \text{on } \overline{\Gamma}_\varphi \cap \partial\Sigma_\omega,$$

where the subscript indicates the surface in which we are considering the normal vector. Since Σ_ω is a cylinder, then $\nu_{\partial\Sigma_\omega} = (\nu_{\partial\omega}, 0)$ on the lateral part, that is, on $\partial\Sigma_\omega \setminus \overline{\omega}$. Therefore $\nabla\varphi\cdot\nu_{\partial\omega} \equiv 0$ on $\partial\omega$.

Proposition 2.3.11. *Assume that $\overline{\Gamma}_\varphi$ intersects $\partial\Sigma_\omega$ orthogonally. If Ω_φ is self-Cheeger, then*

$$H_{\Gamma_\varphi} \le \frac{1}{N-1} h_{\Sigma_\omega}(\Omega_\varphi). \tag{2.3.7}$$

Proof. Let $v \in C^2(\overline{\omega})$ such that $v \le 0$ in $\overline{\omega}$. Observe that, for every $t > 0$ small enough we have that $\Omega_{\varphi+tv} \subset \Omega_\varphi$.

The relative perimeter of the perturbed domain is given by

$$P_v(t) := P_{\Sigma_\omega}(\Omega_{\varphi+tv}) = |\Gamma_{\varphi+tv}| = \int_\omega \sqrt{1+|\nabla\varphi + t\nabla v|^2}\,dx',$$

while the volume is given by

$$V_v(t) := |\Omega_{\varphi+tv}| = \int_\omega \varphi + tv\,dx'.$$

Since Ω_φ is self-Cheeger, then

$$\left.\frac{d}{dt}\right|_{t=0} \frac{P_v(t)}{V_v(t)} = \left.\frac{P_v'(t)V_v(t) - P_v(t)V_v'(t)}{(V_v(t))^2}\right|_{t=0} \ge 0, \tag{2.3.8}$$

and thus we obtain that

$$P_v'(0)V_v(0) - P_v(0)V_v'(0) = |\Omega_\varphi| \int_\omega \frac{\nabla\varphi\nabla v}{\sqrt{1+|\nabla\varphi|^2}}\, dx' - |\Gamma_\varphi| \int_\omega v\, dx' \geq 0.$$

Now, integrating by parts and using that $\nabla\varphi \cdot \nu_{\partial\omega} = 0$, since $\overline{\Gamma}_\varphi$ intersects $\partial\Sigma_\omega$ orthogonally, we have, using Remark 2.3.10:

$$\begin{aligned}
\int_\omega \frac{\nabla\varphi\nabla v}{\sqrt{1+|\nabla\varphi|^2}}\, dx' &= \int_\omega \operatorname{div}\left(v\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right)\, dx' - \int_\omega v \operatorname{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right)\, dx' \\
&= \int_{\partial\omega} v\frac{\nabla\varphi\cdot\nu_{\partial\omega}}{\sqrt{1+|\nabla\varphi|^2}}\, d\sigma_{\partial\omega} - \int_\omega v \operatorname{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right)\, dx' \\
&= -\int_\omega v \operatorname{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right)\, dx'
\end{aligned}$$

We then conclude that

$$\int_\omega v\left(-|\Omega_\varphi|(N-1)H_{\Gamma_\varphi} + |\Gamma_\varphi|\right)\, dx' \leq 0.$$

Since this holds for every $v \leq 0$, then (2.3.7) immediately follows. □

Proposition 2.3.12. *Suppose that $|\nabla\varphi| \geq \delta$ in ω, for some $\delta > 0$. Then Ω_φ is not self-Cheeger.*

Proof. Again we apply the idea of perturbing Ω_φ, but now the aim is to find a specific $v \leq 0$ such that

$$\left.\frac{d}{dt}\right|_{t=0} \frac{P_v(t)}{V_v(t)} < 0.$$

Take

$$v(x') = -e^{\alpha\varphi(x')}, \quad x' \in \overline{\omega},$$

where $\alpha > 0$ is a constant to be chosen later. Since

$$\nabla v(x') = -\alpha e^{\alpha\varphi(x')}\nabla\varphi(x'),$$

arguing as in (2.3.8), we have

$$P_v'(0)V_v(0) - P_v(0)V_v'(0) = \int_\omega e^{\alpha\varphi}\left(|\Gamma_\varphi| - \alpha|\Omega_\varphi|\frac{|\nabla\varphi|^2}{\sqrt{1+|\nabla\varphi|^2}}\right)\, dx'.$$

Since $|\nabla\varphi|$ is bounded in $\overline{\omega}$ (since φ is smooth) and $|\nabla\varphi| > \delta$, choosing α big enough such that

$$|\Gamma_\varphi| - \alpha|\Omega_\varphi|\frac{\delta^2}{\sqrt{1+\|\nabla\varphi\|_\infty^2}} < 0$$

we easily conclude. □

Returning to a generic uniformly Lipschitz unbounded domain $\mathcal{C}$, we can consider the eigenvalue problem with mixed boundary conditions in a domain Ω inside $\mathcal{C}$:

$$\begin{cases} -\Delta\varphi & = & \Lambda\varphi & \text{in } \Omega \\ \varphi & = & 0 & \text{on } \Gamma_\Omega \\ \dfrac{\partial\varphi}{\partial\nu} & = & 0 & \text{on } \Gamma_{1,\Omega} \end{cases}. \tag{2.3.9}$$

The spectral theory for this kind of problem is analogous to the one for the pure Dirichlet-Laplacian, see [26, 52] and the references therein for the details. In particular, the eigenvalues form an increasing sequence

$$0 < \Lambda_1 < \Lambda_2 \le \Lambda_3 \le \ldots \to +\infty,$$

the first eigenfunction does not change sign and we have the following variational characterization for the first eigenvalue:

$$\Lambda_1(\Omega) = \inf_{v \in H_0^1(\Omega\cup\Gamma_{1,\Omega})\setminus\{0\}} \frac{\int_\Omega |\nabla v|^2\, dx}{\int_\Omega v^2\, dx}. \tag{2.3.10}$$

Then we have the following theorem, analogous to the original result of Cheeger ([20]):

Theorem 2.3.13. *Let $\Omega \subset \mathcal{C}$ be a bounded domain with Lipschitz relative boundary Γ_Ω. Then*

$$\Lambda_1(\Omega) \ge \left(\frac{h_\mathcal{C}}{2}\right)^2. \tag{2.3.11}$$

Proof. Let $w \in C_c^\infty(\Omega\cup\Gamma_{1,\Omega})$. By the coarea formula ([42, Theorem 1.23]), the definitions of relative perimeter and of $h_\mathcal{C}(\Omega)$ and the Cavalieri principle (layer cake representation) we have

$$\begin{aligned} \int_\Omega |\nabla w|\, dx &= \int_{-\infty}^{+\infty} \mathcal{H}^{N-1}(\{w=t\})\, dt \\ &= \int_{-\infty}^{+\infty} \frac{P_\mathcal{C}(\{w>t\})}{|\{w>t\}|}|\{w>t\}|\, dt \\ &\ge h_\mathcal{C}(\Omega)\int_\Omega |w|\, dx \end{aligned}$$

and therefore

$$h_\mathcal{C}(\Omega) \le \frac{\int_\Omega |\nabla w|\, dx}{\int_\Omega |w|\, dx} \quad \forall w \in C_c^\infty(\Omega\cup\Gamma_{1,\Omega}). \tag{2.3.12}$$

Since $C_c^\infty(\Omega\cup\Gamma_{1,\Omega})$ is dense in $W_0^{1,1}(\Omega\cup\Gamma_{1,\Omega})$, then (2.3.12) holds also in this space.

Let $v \in H_0^1(\Omega\cup\Gamma_{1,\Omega})$. By Hölder's inequality it follows that $v^2 \in W^{1,1}(\Omega\cup\Gamma_{1,\Omega})$. Indeed:

$$\int_\Omega |\nabla v^2|\, dx = 2\int_\Omega |v||\nabla v|\, dx \le 2\|v\|_2\|\nabla v\|_2.$$

Then, from (2.3.12) we infer that

$$h_{\mathcal{C}}(\Omega) \leq \frac{\int_\omega |\nabla v^2|\, dx}{\int_\Omega v^2\, dx} \leq 2\frac{\|v\|_2\|\nabla v\|_2}{\|v\|_2^2} = 2\frac{\|\nabla v\|_2}{\|v\|_2}.$$

We conclude the proof by taking into account the variational characterization of $\Lambda_1(\Omega)$, see (2.3.10). □

2.3.2 Cheeger sets and the overdetermined torsion problem

Throughout this section, we assume that the set $\mathcal{C}$ is a uniformly Lipschitz convex unbounded domain of $\mathbb{R}^N$.

We are concerned with the study of the partially overdetermined problem for the torsion equation with mixed boundary conditions (2.3.1). More precisely, let $\Omega \subset \mathcal{C}$ be a bounded domain with smooth relative boundary $\Gamma_\Omega := \partial\Omega \cap \mathcal{C}$. We study the following problem:

$$\begin{cases} -\Delta u &= 1 & \text{in } \Omega \\ u &= 0 & \text{on } \Gamma_\Omega \\ \dfrac{\partial u}{\partial \nu} &= 0 & \text{on } \Gamma_{1,\Omega} \\ \dfrac{\partial u}{\partial \nu} &= -c < 0 & \text{on } \Gamma_\Omega \end{cases}, \tag{2.3.13}$$

where $c > 0$ is a constant.

Our approach consists of using the so-called P-function, which was introduced by Weinberger in [75] to provide another proof of Serrin's theorem via integral identities. This technique was successfully used in [60] to characterize all domains that admit solutions to (2.3.13) in convex cones.

We are inspired by the approach of [37], where the authors combine the use of P-functions with the maximum principle to study more general, possibly degenerate, elliptic equations with overdetermined boundary conditions in the case when $\mathcal{C} = \mathbb{R}^N$. We are able to obtain a bound on the curvature of the relative boundary of domains where a solution for (2.3.13) exists and characterize such domains as self-Cheeger sets.

Suppose that a weak solution $u \in H_0^1(\Omega \cup \Gamma_{1,\Omega})$ to (2.3.13) exists. By standard elliptic regularity theory, we know that it is a classical solution up to the regular part of the boundary.

Definition 2.3.14. Let u be the solution to (2.3.13). The P-function associated to u is defined as

$$P(x) = |\nabla u(x)|^2 + \frac{2}{N} u(x), \quad x \in \overline{\Omega}. \tag{2.3.14}$$

Under the hypothesis that there exists a solution for the overdetermined problem (2.3.13) we can give a bound on the curvature of Γ_Ω. A preliminary step is the following

Lemma 2.3.15. *If a solution for the overdetermined problem (2.3.13) exists, then either $P \equiv c^2$ in $\overline{\Omega}$ or $\frac{\partial P}{\partial \nu} > 0$ on Γ_Ω.*

Proof. In what follows, D^2u denotes the Hessian matrix of u, and $\|D^2u\|^2$ denotes the sum of the square of its elements.

Direct computations yield

$$\nabla P = 2D^2u\nabla u + \frac{2}{N}\nabla u$$

and

$$\Delta P = 2\|D^2u\|^2 + 2\langle \nabla u, \nabla(\Delta u)\rangle - \frac{2}{N} = 2\left(\|D^2u\|^2 - \frac{(\Delta u)^2}{N}\right),$$

since $-\Delta u = 1$. From the well-known matrix inequality

$$\|A\|^2 \geq \frac{1}{N}(\text{trace}(A))^2,$$

which holds for any symmetric matrix A, and since $\Delta u = \text{trace}(D^2u)$, it follows that

$$\Delta P \geq 0 \quad \text{in } \Omega.$$

Moreover, from the boundary conditions for u we obtain the following boundary conditions for P:

$$P \equiv c^2 \text{ on } \Gamma_\Omega, \qquad \frac{\partial P}{\partial \nu} = 2\langle D^2u\nabla u, \nu\rangle \text{ on } \Gamma_{1,\Omega}.$$

Consider the vector field differentiating along the direction ∇u on $\Gamma_{1,\Omega}$:

$$X_u = \sum_{k=1}^{N} \partial_k u(x)\partial_k,$$

where for simplicity of notation we set $\partial_k = \frac{\partial}{\partial x_k}$. By the Neumann condition $\frac{\partial u}{\partial \nu} = 0$ on $\Gamma_{1,\Omega}$ we deduce that ∇u is tangent to the regular part of $\Gamma_{1,\Omega}$, whereas from the convexity assumption on $\mathcal{C}$ we have that the second fundamental form $h(\cdot,\cdot)$ on $\partial\mathcal{C}$ is positive semidefinite at any regular point. Then differentiating the function $\langle \nabla u, \nu\rangle \equiv 0$ along the vector field X_u on $\Gamma_{1,\Omega}$ we obtain

$$0 = X_u(\langle \nabla u, \nu\rangle) = \underbrace{\sum_{j,k=1}^{N} \partial_k u \partial_{jk} u \nu_j}_{=\langle D^2u\nabla u, \nu\rangle} + h(\nabla u, \nabla u) \geq \langle D^2u\nabla u, \nu\rangle.$$

Hence the P-function satisfies

$$\begin{cases} \Delta P \geq 0 & \text{in } \Omega \\ P = c^2 & \text{on } \Gamma_\Omega \\ \dfrac{\partial P}{\partial \nu} \leq 0 & \text{on } \Gamma_{1,\Omega} \end{cases}. \tag{2.3.15}$$

By the maximum principle for mixed boundary value problems (see [26, Section 1.2.1] or [60, Corollary 2.3]) we obtain that $P \leq c^2$ in Ω. Then the strong maximum principle applies, and we obtain that either $P \equiv c^2$ in Ω or $P < c^2$ in Ω. In this last case, by Hopf's Lemma, we get $\frac{\partial P}{\partial \nu} > 0$ on Γ_Ω. □

Let $H(x)$ denote the mean curvature at a point $x \in \Gamma_\Omega$ (see (2.3.6)).

Proposition 2.3.16. *If there exists a solution for* (2.3.13)*, then either*

$$H(x) < \frac{1}{Nc} \qquad \forall x \in \Gamma_\Omega \tag{2.3.16}$$

or

$$H \equiv \frac{1}{Nc} \tag{2.3.17}$$

Proof. It holds that (see [44, Section 5.4])

$$\Delta u = \Delta_{\Gamma_\Omega} u + (N-1)H\frac{\partial u}{\partial \nu} + \frac{\partial^2 u}{\partial \nu^2} \qquad \text{on } \Gamma_\Omega,$$

where Δ_{Γ_Ω} is the Laplace-Beltrami operator. Then, since u is a solution of (2.3.13), we obtain

$$\frac{\partial^2 u}{\partial \nu^2} - (N-1)cH(x) = -1 \qquad \text{on } \Gamma_\Omega. \tag{2.3.18}$$

Consider the two possible cases given by Lemma 2.3.15. In the inequality case,

$$\frac{\partial P}{\partial \nu} = 2\frac{\partial u}{\partial \nu}\frac{\partial^2 u}{\partial \nu^2} + \frac{2}{N}\frac{\partial u}{\partial \nu} > 0. \tag{2.3.19}$$

Dividing (2.3.19) by $2\frac{\partial u}{\partial \nu}$ we obtain

$$\frac{\partial^2 u}{\partial \nu^2} = -\frac{1}{N}.$$

Substituting into (2.3.18) we obtain (2.3.16).

The case of equality in Lemma 2.3.15 develops similarly and we obtain (2.3.17). □

As anticipated, we show that domains that admit a solution to the overdetermined problem (2.3.13) are relatively self-Cheeger.

Theorem 2.3.17. *Let $\mathcal{C} \subset \mathbb{R}^N$ be a convex uniformly Lipschitz unbounded domain and let $\Omega \subset \mathcal{C}$ be a bounded domain with Lipschitz relative boundary Γ_Ω. If a solution u to* (2.3.13) *exists, then $|\nabla u| \le c$ and Ω is relatively self-Cheeger.*

Proof. Recall from the proof of Lemma 2.3.15 that

$$P(x) = |\nabla u(x)|^2 + \frac{2}{N}u(x) \le c^2$$

in $\overline{\Omega}$. By the maximum principle, we have that u is positive in Ω, and therefore we obtain that $|\nabla u| \le c$ in $\overline{\Omega}$. In particular, $|\nabla u| < c$ in Ω.

It follows that for any smooth subdomain $E \subset \Omega$ (recall Remark 2.3.7) we have

$$|E| = \int_E -\Delta u \, dx = -\int_{\partial E} \frac{\partial u}{\partial \nu} \, d\sigma \le c\mathcal{H}^{N-1}(\partial E \cap \mathcal{C})$$

On the other hand, integrating (2.3.13) in Ω we have

$$|\Omega| = \int_\Omega -\Delta u \, dx = -\int_{\Gamma_\Omega} \frac{\partial u}{\partial \nu} \, d\sigma = \int_{\Gamma_\Omega} c \, d\sigma = c\mathcal{H}^{N-1}(\Gamma_\Omega).$$

Hence

$$\frac{P_\mathcal{C}(\Omega)}{|\Omega|} = \frac{1}{c} \leq \frac{P_\mathcal{C}(E)}{|E|}$$

for every smooth subdomain $E \subset \Omega$. Taking the definition of the Cheeger constant into account we complete the proof. □

2.3.3 Some results in cylinders

We conclude this chapter with some results in the particular setting of cylinders. Recall that we denote by Σ_ω the cylinder spanned by the smooth bounded domain $\omega \subset \mathbb{R}^{N-1}$:

$$\Sigma_\omega := \omega \times (0, +\infty).$$

For a positive function $\varphi \in C^2(\overline{\omega})$ we consider the domain

$$\Omega_\varphi = \{(x', x_N) \in \Sigma_\omega \ : \ 0 < x_N < \varphi(x')\}.$$

To simplify the notation we set $\Gamma_\varphi := \Gamma_{\Omega_\varphi}$.

Recall that

$$H_{\Gamma_\varphi} = -\frac{1}{N-1} \operatorname{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right). \tag{2.3.20}$$

We begin with a preliminary geometrical lemma.

Lemma 2.3.18. *Assume that $\overline{\Gamma}_\varphi$ and $\partial\Sigma_\omega$ intersect orthogonally. Then*

$$\int_{\Gamma_\varphi} H_{\Gamma_\varphi} \langle x_N e_N, \nu\rangle \, d\sigma = \frac{1}{N-1} \int_\omega \frac{|\nabla\varphi|^2}{\sqrt{1+|\nabla\varphi|^2}} \, dx', \tag{2.3.21}$$

where $e_N = (0, \dots, 0, 1) \in \mathbb{R}^N$.

Proof. Since Γ_φ is the graph of φ we have

$$d\sigma = \sqrt{1+|\nabla\varphi|^2} \, dx',$$

while

$$\nu|_{\Gamma_\varphi} = \frac{(-\nabla\varphi, 1)}{\sqrt{1+|\nabla\varphi|^2}}.$$

Using (2.3.20) and integrating by parts we obtain

$$\int_{\Gamma_\varphi} H_{\Gamma_\varphi} \langle x_N e_N, \nu\rangle \, d\sigma = -\frac{1}{N-1} \int_\omega \operatorname{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right) \varphi \, dx'$$

$$= \frac{1}{N-1}\left(\int_\omega \frac{|\nabla\varphi|^2}{\sqrt{1+|\nabla\varphi|^2}}\,dx' - \int_{\partial\omega}\varphi\frac{\nabla\varphi\cdot\nu_{\partial\omega}}{\sqrt{1+|\nabla\varphi|^2}}\,d\sigma_{\partial\omega}\right).$$

We then conclude by making use of Remark 2.3.10. □

Proposition 2.3.19. *If $\overline{\Gamma}_\varphi$ intersects $\partial\Sigma_\omega$ orthogonally and Ω_φ is such that there exists a solution to the overdetermined problem* (2.3.13)*, then*

$$\frac{|\Gamma_\varphi|}{N} \le \int_\omega \frac{1}{\sqrt{1+|\nabla\varphi|^2}}\,dx'.$$

Proof. From (2.3.21) we have

$$\begin{aligned}\int_{\Gamma_\varphi} H_{\Gamma_\varphi}\langle x_N e_N,\nu\rangle\,d\sigma &= \frac{1}{N-1}\int_\omega \frac{|\nabla\varphi|^2}{\sqrt{1+|\nabla\varphi|^2}}\,dx' \\ &= \frac{1}{N-1}\int_\omega \frac{1+|\nabla\varphi|^2}{\sqrt{1+|\nabla\varphi|^2}}\,dx' - \frac{1}{N-1}\int_\omega \frac{1}{\sqrt{1+|\nabla\varphi|^2}}\,dx' \\ &= \frac{|\Gamma_\varphi|}{N-1} - \frac{1}{N-1}\int_\omega \frac{1}{\sqrt{1+|\nabla\varphi|^2}}\,dx'. \end{aligned} \tag{2.3.22}$$

Now, from the bound on the mean curvature given by Proposition 2.3.16 we obtain

$$\begin{aligned}\int_{\Gamma_\varphi} H_{\Gamma_\varphi}\langle x_N e_N,\nu\rangle\,d\sigma &\le \int_{\Gamma_\varphi}\frac{\langle x_N e_N,\nu\rangle}{Nc}\,d\sigma \\ &= \frac{1}{Nc}\int_\omega \varphi\,dx' \\ &= \frac{|\Omega_\varphi|}{Nc} \\ &= \frac{|\Gamma_\varphi|}{N}, \end{aligned} \tag{2.3.23}$$

by the proof of Theorem 2.3.17.

Substituting (2.3.23) into (2.3.22), the result immediately follows. □

Proposition 2.3.20. *Assume that $\overline{\Gamma}_\varphi$ has constant mean curvature and intersects $\partial\Sigma_\omega$ orthogonally. Then Γ_φ is a minimal hypersurface and φ is a constant function.*

Proof. Recall that

$$H_{\Gamma_\varphi} = -\frac{1}{N-1}\,\mathrm{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right).$$

Integrating over ω we obtain

$$(N-1)H_{\Gamma_\varphi}|\omega| = -\int_\omega \mathrm{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right)\,dx'$$

$$= -\int_{\partial\omega} \frac{\nabla\varphi \cdot \nu_{\partial\omega}}{\sqrt{1+|\nabla\varphi|^2}}\, d\sigma_{\partial\omega}$$
$$= 0,$$

whence $H_{\Gamma_\varphi} = 0$. Moreover,

$$0 = \int_\omega H_{\Gamma_\varphi}\varphi\, dx'$$
$$= -\frac{1}{N-1}\int_\omega \operatorname{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right)\varphi\, dx',$$

and it follows that

$$\int_\omega \frac{|\nabla\varphi|^2}{\sqrt{1+|\nabla\varphi|^2}}\, dx' = 0.$$

The conclusion immediately follows. □

2.3.3.1 On the relation between the overdetermined problem and CMC surfaces in cylinders

There is a strong parallel between overdetermined problems and the problem of constant mean curvature surfaces. Indeed, in $\mathbb{R}^N$, the solutions to both problems are "the same", in the sense that the only bounded domains that admit a solution to the overdetermined problem are balls ([71]) and the only closed CMC surfaces are spheres ([4]). Analogous results are true in the relative setting of convex cones, see [60].

The papers [71], [4] and [60], together with our Proposition 2.3.20, constitute a strong body of evidence for the flat cylinders, given by $\varphi \equiv$ constant, being the only domains in cylinders where the overdetermined problem (2.3.13) has a solution. Considerable effort was put into proving this fact, which turns out to be false, since, as was recently brought to our attention, in 2017, Fall, Minlend, and Weth proved the following:

Theorem 2.3.21 ([34], Theorem 1.1). *For each $m, n \geq 1$ and $\alpha \in (0,1)$, there exists $\lambda_* = \lambda_*(n) > 0$ and a smooth map*

$$\begin{array}{rcl} (-\varepsilon_0, \varepsilon_0) & \to & (0,+\infty) \times C^{2,\alpha}(\mathbb{R}^m) \\ s & \mapsto & (\lambda_s, \varphi_s) \end{array}$$

with $\varphi_0 \equiv 0$, $\lambda_0 = \lambda_*$ and such that for all $s \in (-\varepsilon_0, \varepsilon_0)$, letting $\phi_s = \lambda_s + \varphi_s$, there exists a solution $u \in C^{2,\alpha}(\overline{\Omega}_{\phi_s})$ of the overdetermined problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega_{\phi_s} \\ u = 0 & \text{on } \partial\Omega_{\phi_s} \\ \dfrac{\partial u}{\partial \nu} = -\dfrac{\lambda_s}{n} & \text{on } \partial\Omega_{\phi_s} \end{cases}$$

in the domain

$$\Omega_s = \{(t,z) \in \mathbb{R}^m \times \mathbb{R}^n \ : \ |z| < \phi_s(t)\}.$$

Moreover, the function φ_s is even and 2π-periodic in $t_1, \ldots, t_m$ and invariant with respect to permutations of these variables.

We can interpret Theorem 2.3.21 in the following way: there exist periodic perturbations of an infinite cylinder $(-\infty, +\infty) \times B_{\lambda_*}(0)$ where the overdetermined problem for $-\Delta u = 1$ has a solution, which is also periodic along the axis and radial on the cross-section. We remark that the statement of Theorem 2.3.21 is somewhat more general since the authors consider the so-called generalized cylinders, whose "axes" are multi-dimensional, being represented by the coordinate $t \in \mathbb{R}^m$.

Now, $m = n = 1$ in Theorem 2.3.21, and consider the domain (written in the notation of the theorem)

$$\Omega_{\phi_s} = \{(t, z) \,:\, t \in \mathbb{R},\ 0 < z < \phi_s(t)\}.$$

By the evenness with respect to $t = 0$ and the periodicity in the variable t, we know that $\phi_s'(2k\pi) = 0$ for every $k \in \mathbb{Z}$. Moreover, the solution u is symmetric with respect to $t = 0$ and (2π-periodic in t.

We can then restrict u to the set $\{(t, z) \in \mathbb{R}^2 \,:\, t \in (0, 2\pi),\ 0 < z < \phi_s(t)\}$, which is precisely the domain Ω_{ϕ_s} in $\Sigma = (0, 2\pi) \times (0, +\infty)$, and we obtain a solution of (2.3.13) in Ω_{ϕ_s}, whose relative boundary Γ_{ϕ_s} is not flat, since ϕ_s is not constant. Observe, furthermore, that $\phi_s'(0) = \phi_s'(2\pi) = 0$, by the periodicity of ϕ_s, and thus Γ_{ϕ_s} intersects Γ_{1,ϕ_s} orthogonally, where

$$\Gamma_{1,\phi_s} = (0, 2\pi) \cup (\{0, 2\pi\} \times (0, +\infty))$$

From these considerations, we conclude that the parallel between overdetermined problems and CMC surfaces breaks in our relative setting of cartesian graphs, since the only CMC graphs intersecting the wall orthogonally are flat, that is, graphs of constant functions (Proposition 2.3.20).

Let us observe that the regular CMC surfaces intersecting the boundary of the cylinder orthogonally are the critical points of the relative perimeter functional under a volume constraint (in particular, could be minimizers). Therefore, from Proposition 2.3.20 we obtain that if a minimizer of the relative perimeter, under a volume constraint, is a regular graph, then it is just the flat graph. Hence there is a correspondence between the perimeter functional and the torsional energy functional and their critical points (with a fixed volume).

Moreover, in view of Theorem 2.3.17, we see that there are relatively self-Cheeger domains in the cylinder different from the flat ones, which is also somewhat surprising.

Chapter 3

Cones

3.1 Introduction

Let $D \subset \mathbb{S}^{N-1}$ be a smooth domain of the unit sphere and let Σ_D be the cone spanned by D:

$$\Sigma_D := \{x \in \mathbb{R}^N \; : \; x = tq, \; q \in D, \; t > 0\}.$$

In Σ_D we consider the spherical sector Ω_D obtained by intersecting the cone with the unit ball centered at the origin, i.e., $\Omega_D = \Sigma_D \cap B_1$. We consider nonlinearities f such that there exists a radially symmetric solution u_D to the problem

$$\begin{cases} -\Delta u &= f(u) & \text{in } \Omega_D \\ u &= 0 & \text{on } \Gamma_{\Omega_D} = D \\ \dfrac{\partial u}{\partial \nu} &= 0 & \text{on } \Gamma_{1,\Omega_D} \setminus \{0\} \end{cases} . \tag{3.1.1}$$

It is clear that u_D is the restriction of the analogous Dirichlet problem in the unit ball B_1.

In this chapter, we show that whenever u_D is a nondegenerate solution of (3.1.1) the pair (Ω_D, u_D) is energy-stationary in the sense of Definition 2.2.4, and we investigate its "stability" as a critical point of the energy functional T (see (2.2.7)), which is well-defined for small perturbations of Ω_D (by Proposition 2.2.1).

The main result we get is that the stability of (Ω_D, u_D) depends on the first nontrivial Neumann eigenvalue of the Laplace-Beltrami operator on $-\Delta_{\mathbb{S}^{N-1}}$ on the domain D of the unit sphere which spans the cone. In particular, we obtain a precise threshold for stability/instability which is independent of the nonlinearity, and on the radial positive solution considered, if multiple positive solutions exist. Let us remark that for several nonlinearities the radial positive solution is unique (see [57]), as, for instance, the Lane-Emden nonlinearity $f(s) = s^p$.

3.2 Energy functional for star-shaped domains

Given the cone Σ_D, we work in the class of admissible domains, in the sense of (2.2.12), that are strictly star-shaped with respect to the vertex of the cone, which, for simplicity, is set to be the origin 0 in $\mathbb{R}^N$. In other words, we consider domains whose relative boundary is the radial graph in Σ_D associated to a function in $C^2(\overline{D})$. Namely, for $\varphi \in C^2(\overline{D})$ we set

$$\Gamma_\varphi := \{x \in \mathbb{R}^N \;:\; x = e^{\varphi(q)} q,\; q \in D\} \tag{3.2.1}$$

and consider the strictly star-shaped domain Ω_φ defined as

$$\Omega_\varphi := \{x \in \mathbb{R}^N \;:\; x = tq,\; 0 < t < e^{\varphi(q)}, q \in D\}.$$

To simplify the notation we set

$$\Gamma_{1,\varphi} := \Gamma_{1,\Omega_\varphi} = (\partial\Omega_\varphi \setminus \overline{\Gamma}_\varphi) \cap \partial\Sigma_D.$$

In Ω_φ we consider the semilinear elliptic problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega_\varphi \\ u = 0 & \text{on } \Gamma_\varphi \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_{1,\varphi} \setminus \{0\} \end{cases} \tag{3.2.2}$$

and assume throughout the chapter that nondegenerate positive solution $u_\varphi := u_{\Omega_\varphi}$ exists and belongs to $W^{1,\infty}(\Omega_\varphi) \cap W^{2,2}(\Omega_\varphi)$. Then we can apply the results of Chapter 2 and define the functional T as in (2.2.7) for small variations of Ω_φ. Since Ω_φ is strictly star-shaped, this property also holds for the domains obtained by small regular deformations. Thus it is convenient to parametrize the domains and their variations by C^2 functions on $\overline{D}$. Hence, for $v \in C^2(\overline{D})$ and $t \in (-\eta, \eta)$, where $\eta > 0$ is a fixed number sufficiently small, we consider the domain variations $\Omega_{\varphi+tv} \subset \Sigma_D$.

Let us now introduce the diffeomorphisms that are associated with this kind of domain variations. Let $\xi : (-\eta, \eta) \times (\overline{\Sigma_D} \setminus \{0\}) \to (\overline{\Sigma_D} \setminus \{0\})$ be the map defined by

$$\xi(t, x) = e^{tv\left(\frac{x}{|x|}\right)} x.$$

It is clear that for all $t \in (-\eta, \eta)$ the map $\xi|_{\Omega_\varphi}(t, \cdot) : \Omega_\varphi \to \Omega_{\varphi+tv}$ is a diffeomorphism, whose inverse is

$$(\xi|_{\Omega_\varphi})^{-1}(t, x) = e^{-tv\left(\frac{x}{|x|}\right)} x = \xi(-t, x).$$

By definition, $\xi(t, x) \in \partial\Sigma_D \setminus \{0\}$ for all $(t, x) \in (-\eta, \eta) \times (\partial\Sigma_D \setminus \{0\})$ and ξ is the flow associated to the vector field

$$V(x) = v\left(\frac{x}{|x|}\right) x, \tag{3.2.3}$$

since $\xi(0, x) = x$ and

$$\frac{d}{dt}\xi(t, x) = e^{tv\left(\frac{x}{|x|}\right)} v\left(\frac{x}{|x|}\right) x = V(\xi(t, x)).$$

For simplicity we set $\xi_t(x) := \xi(t,x)$.

The energy functional T becomes a functional defined on functions in $C^2(\overline{D})$:

$$T(\varphi) := T(\Omega_\varphi) = J(u_\varphi) = \frac{1}{2}\int_{\Omega_\varphi} |\nabla u_\varphi|^2\, dx - \int_{\Omega_\varphi} F(u_\varphi)\, dx. \tag{3.2.4}$$

Then, for every $v \in C^2(\overline{D})$ we have

$$T(\varphi + tv) := T(\Omega_{\varphi+tv}) = J(u_{\varphi+tv}),$$

for $t \in (-\delta, \delta)$, with $\delta > 0$ small, where

$$u_{\varphi+tv} := u_{\Omega_{\varphi+tv}}$$

is the unique positive solution of (3.2.2) in the domain $\Omega_{\varphi+tv}$ in a neighborhood of $u_\varphi \circ \xi_t^{-1}$.

We now compute the first derivative of the functional T at φ along a direction $v \in C^2(\overline{D})$, i.e., the derivative with respect to t of $T(\varphi + tv)$ at $t = 0$.

Lemma 3.2.1. *Let $\varphi \in C^2(\overline{D})$ and assume that u_φ is a nondegenerate positive solution to (3.2.2) and that u_φ belongs to $W^{1,\infty}(\Omega_\varphi) \cap W^{2,2}(\Omega_\varphi)$. Then for any $v \in C^2(\overline{D})$ it holds that*

$$T'(\varphi)[v] = -\frac{1}{2}\int_D \left(\frac{\partial u_\varphi}{\partial \nu}(e^\varphi q)\right)^2 e^{N\varphi} v\, d\sigma. \tag{3.2.5}$$

Proof. The result follows from Proposition 2.2.3. Indeed, since the exterior unit normal to Γ_φ is given by

$$\nu(x) = \frac{\frac{x}{|x|} - \nabla_{\mathbb{S}^{N-1}}\varphi\left(\frac{x}{|x|}\right)}{\sqrt{1 + \left|\nabla_{\mathbb{S}^{N-1}}\varphi\left(\frac{x}{|x|}\right)\right|^2}}, \qquad x \in \Gamma_\varphi,$$

where $\nabla_{\mathbb{S}^{N-1}}$ is the gradient in $\mathbb{S}^{N-1}$ (see [46, Section 2]), then, from (3.2.3) it follows that

$$\langle V, \nu\rangle = \frac{|x|}{\sqrt{1 + \left|\nabla_{\mathbb{S}^{N-1}}\varphi\left(\frac{x}{|x|}\right)\right|^2}}\, v\left(\frac{x}{|x|}\right) \qquad \text{on } \Gamma_\varphi.$$

Then, using the parametrization $x = e^{\varphi(q)}q$, for $q \in D$, taking into account that induced $(N-1)$-dimensional area element on Γ_φ is given by

$$d\sigma_{\Gamma_\varphi} = e^{(N-1)\varphi}\sqrt{1 + |\nabla_{\mathbb{S}^{N-1}}\varphi|^2},$$

and since $u_\varphi = 0$ on Γ_φ, then, from (2.2.9) we obtain (3.2.5). □

The next step is to compute the second derivative of T at Ω_φ along directions $v, w \in C^2(\overline{D})$. The proof is essentially the same as in [46, Lemma 3.2], but we present it here for the sake of completeness.

For $w \in C^2(\overline{D})$, we denote by $\widetilde{u}_w$ the solution of (2.2.11) with $V = w\left(\frac{x}{|x|}\right)x$:

$$\begin{cases} -\Delta \widetilde{u}_w = f'(u_\varphi)\widetilde{u}_w & \text{in } \Omega \\ \widetilde{u}_w = -\dfrac{\partial u_\varphi}{\partial \nu} w\left(\dfrac{x}{|x|}\right) x \cdot \nu & \text{on } \Gamma_\Omega \\ \dfrac{\partial \widetilde{u}_w}{\partial \nu} = 0 & \text{on } \Gamma_{1,\Omega} \end{cases}$$

Lemma 3.2.2. *Let φ and u_φ be as in Lemma 3.2.1. Then for any $v, w \in C^2(\overline{D})$ it holds*

$$\begin{aligned} T''(\varphi)[v,w] = &-\frac{N}{2}\int_D e^{N\varphi} vw \left(\frac{\partial u_\varphi}{\partial \nu}(e^\varphi q)\right)^2 d\sigma \\ &- \int_D e^{N\varphi} v \frac{\partial u_\varphi}{\partial \nu}(e^\varphi q) \frac{\partial \widetilde{u}_w}{\partial \nu}(e^\varphi q)\, d\sigma \\ &- \int_D e^{N\varphi} vw \frac{\partial u_\varphi}{\partial \nu}(e^\varphi q)(D^2 u_\varphi(e^\varphi q) e^\varphi q) \cdot \nu \, d\sigma \\ &+ \int_D e^{N\varphi} v \frac{\partial u_\varphi}{\partial \nu}(e^\varphi q) \frac{\nabla u_\varphi(e^\varphi q) \cdot \nabla_{\mathbb{S}^{N-1}} w}{\sqrt{1+|\nabla_{\mathbb{S}^{N-1}}\varphi|^2}}\, d\sigma \\ &+ \int_D e^{N\varphi} \left(\frac{\partial u_\varphi}{\partial \nu}(e^\varphi q)\right)^2 \frac{\nabla_{\mathbb{S}^{N-1}}\varphi \cdot \nabla_{\mathbb{S}^{N-1}} w}{1+|\nabla_{\mathbb{S}^{N-1}}\varphi|^2}\, d\sigma \end{aligned} \tag{3.2.6}$$

Proof. Let $v, w \in C^2(\overline{D})$. Then, by definition and Lemma 3.2.1 we have:

$$\begin{aligned} T''(\varphi)[v,w] &= \frac{d}{ds}\bigg|_{s=0} \left(-\frac{1}{2}\int_D e^{N(\varphi+sw)} v \left(\frac{\partial u_{\varphi+sw}}{\partial \nu}(e^{\varphi+sw} q)\right)^2 d\sigma \right) \\ &= -\frac{1}{2}\int_D e^{N\varphi} N vw \left(\frac{\partial u_\varphi}{\partial \nu}(e^\varphi q)\right)^2 d\sigma \\ &\quad - \int_D e^{N\varphi} \frac{\partial u_\varphi}{\partial \nu}(e^\varphi q)\, \frac{d}{ds}\bigg|_{s=0} \left(\frac{\partial u_{\varphi+sw}}{\partial \nu}(e^{\varphi+sw} q)\right) d\sigma. \end{aligned} \tag{3.2.7}$$

Since $\Gamma_{\varphi+sw}$ is a radial graph we have

$$\frac{\partial u_{\varphi+sw}}{\partial \nu}(e^{\varphi+sw} q) = \nabla u_{\varphi+sw}(e^{\varphi+sw} q) \cdot \frac{q - \nabla_{\mathbb{S}^{N-1}}(\varphi+sw)}{\sqrt{1+|\nabla_{\mathbb{S}^{N-1}}(\varphi+sw)|^2}}. \tag{3.2.8}$$

Then we can compute the derivative in the second integral on the right-hand side of (3.2.7). We have:

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=0} \left(\frac{\partial u_{\varphi+sw}}{\partial \nu}(e^{\varphi+sw} q)\right) &= \frac{d}{ds}\bigg|_{s=0} \left(\nabla u_{\varphi+sw}(e^{\varphi+sw} q) \cdot \nu_{\varphi+sw}\right) \\ &= \frac{d}{ds}\bigg|_{s=0} \left(\nabla u_{\varphi+sw}(e^{\varphi+sw} q)\right) \cdot \nu_\varphi \\ &\quad + \nabla u_\varphi(e^\varphi q) \cdot \frac{d}{ds}\bigg|_{s=0} \nu_{\varphi+sw} \end{aligned} \tag{3.2.9}$$

By Lemma 2.2.2 we have

$$\left.\frac{d}{ds}\right|_{s=0} (\nabla u_{\varphi+sw}(e^{\varphi+sw}q)) = \nabla \widetilde{u}_w(e^{\varphi}q) + D^2 u_{\varphi}(e^{\varphi}q)e^{\varphi}wq.$$

On the other hand, a direct computation yields

$$\left.\frac{d}{ds}\right|_{s=0} \nu_{\varphi+sw} = -\frac{\nabla_{\mathbb{S}^{N-1}}w}{\sqrt{1+|\nabla_{\mathbb{S}^{N-1}}\varphi|^2}} - \frac{\nabla_{\mathbb{S}^{N-1}}\varphi \cdot \nabla_{\mathbb{S}^{N-1}}w}{\sqrt{1+|\nabla_{\mathbb{S}^{N-1}}\varphi|^2}},$$

and then (3.2.6) follows. □

In view of Definition 2.2.4, we are interested in studying pairs $(\Omega_\varphi, u_\varphi)$ which are energy-stationary under a volume constraint.

We recall that the volume of the domain defined by the radial graph of a function $\varphi \in C^2(\overline{D})$ is given by

$$\mathcal{V}(\varphi) := \mathcal{V}(\Omega_\varphi) = |\Omega_\varphi| = \frac{1}{N}\int_D e^{N\varphi}\, d\sigma.$$

Standard computations yield, for $v, w \in C^2(\overline{D})$,

$$\mathcal{V}'(\varphi)[v] = \int_D e^{N\varphi} v\, d\sigma, \qquad \mathcal{V}''(\varphi)[v,w] = N\int_D e^{N\varphi} vw\, d\sigma. \tag{3.2.10}$$

Then, for $c > 0$ we define the manifold

$$M := \{\varphi \in C^2(\overline{D}) \ : \ \mathcal{V}(\varphi) = c\}, \tag{3.2.11}$$

whose tangent space at any point $\varphi \in M$ is given by

$$T_\varphi M = \left\{ v \in C^2(\overline{D}) \ : \ \int_d e^{N\varphi} v\, d\sigma = 0 \right\}.$$

We restrict the energy functional to the manifold M and denote it by

$$I(\varphi) = T|_M(\varphi). \tag{3.2.12}$$

Clearly, if the pair $(\Omega_\varphi, u_\varphi)$ is energy stationary under a volume constraint, in the sense of Definition 2.2.4, then $\varphi \in M$ is a critical point of I and therefore there exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$T'(\varphi) = \mu\mathcal{V}'(\varphi). \tag{3.2.13}$$

Moreover, the following result holds true:

Proposition 3.2.3. *Let $\varphi \in M$ be such that $(\Omega_\varphi, u_\varphi)$ is energy-stationary under the volume constraint. Then the Lagrange multiplier μ is negative and*

$$\frac{\partial u_\varphi}{\partial \nu} = -\sqrt{-2\mu} \quad \text{on} \quad \Gamma_\varphi. \tag{3.2.14}$$

Proof. By the expressions for $T'(\varphi)$ and $\mathcal{V}'(\varphi)$ we have, for any $v \in C^2(\overline{D})$:

$$-\frac{1}{2}\int_D e^{N\varphi} v \left(\frac{\partial u_{\Omega_\varphi}}{\partial \nu}(e^\varphi q)\right)^2 \, d\sigma = \mu \int_D e^{N\varphi} v \, d\sigma.$$

Then

$$\int_D e^{N\varphi} v \left(\left(\frac{\partial u_{\Omega_\varphi}}{\partial \nu}(e^\varphi q)\right)^2 + 2\mu\right) \, d\sigma = 0.$$

Since v is arbitrary, it immediately follows that $\mu < 0$ and

$$\left(\frac{\partial u_{\Omega_\varphi}}{\partial \nu}\right)^2 = -2\mu \quad \text{on } \Gamma_\varphi.$$

Then, since u_{Ω_φ} is smooth and positive, by Hopf's Lemma we have that $\frac{\partial u_{\Omega_\varphi}}{\partial \nu} < 0$ on Γ_φ, and then we easily conclude. □

For the second derivative of I, the following holds:

Lemma 3.2.4. *Let $\varphi \in M$ and let $v, w \in T_\varphi M$. If $(\Omega_\varphi, u_\varphi)$ is energy-stationary under the volume constraint, then*

$$I''(\varphi)[v, w] = T''(\varphi)[v, w] - \mu \mathcal{V}''(\varphi)[v, w]. \tag{3.2.15}$$

Proof. By definition, the second variation of I at the point $\varphi \in M$ along the directions $v, w \in T_\varphi M$ is given by

$$I''(\varphi)[v, w] = \left.\frac{\partial^2 I(\Psi(t, s))}{\partial t \partial s}\right|_{(t,s)=(0,0)},$$

where $\Psi : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \to M$ is a smooth map such that

$$\Psi(0, 0) = \varphi, \quad \frac{\partial \Psi}{\partial t}(0, 0) = v, \quad \frac{\partial \Psi}{\partial s}(0, 0) = w.$$

Since $I(\Psi(t, s)) = T(\Psi(t, s))$ we have

$$\frac{\partial}{\partial s}(T(\Psi(t, s))) = T'(\Psi(t, s))\left[\frac{\partial \Psi}{\partial s}(t, s)\right]$$

and therefore

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}(T(\Psi(t, s))) = T''(\Psi(t, s))\left[\frac{\partial \Psi}{\partial s}(t, s), \frac{\partial \Psi}{\partial t}(t, s)\right] + T'(\Psi(t, s))\left[\frac{\partial^2 \Psi}{\partial t \partial s}(t, s)\right]. \tag{3.2.16}$$

On the other hand, since $\mathcal{V}(\Psi(t, s)) = c$ for every $(t, s) \in (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$, differentiating we obtain

$$\mathcal{V}'(\Psi(t, s))\left[\frac{\partial \Psi}{\partial s}(t, s)\right] = 0$$

and

$$\mathcal{V}''(\Psi(t,s))\left[\frac{\partial\Psi}{\partial s}(t,s),\frac{\partial\Psi}{\partial t}(t,s)\right]+\mathcal{V}'(\Psi(t,s))\left[\frac{\partial^2\Psi}{\partial t\partial s}(t,s)\right]=0. \tag{3.2.17}$$

Taking $t=s=0$ and substituting (3.2.13) and (3.2.17) into (3.2.16), since φ is a critical point, we immediately conclude. □

3.3 Spherical sectors and nondegeneracy of radial solutions

Given the cone Σ_D, the spherical sector Ω_D is obtained by intersecting the cone with the unit ball B_1. It is clear that in this case the relative boundary Γ_{Ω_D} is the radial graph obtained by taking $\varphi\equiv 0$ in (3.2.1). In fact, $\Gamma_{\Omega_D}=D$.

In the spherical sector Ω_D we would like to consider a nondegenerate positive radial solution $u_D:=u_{\Omega_D}$ of (3.1.1). One way to find such a solution is, for example, to find a solution u of

$$\begin{cases} -\Delta u &= f(u) & \text{in } B_1\\ u &= 0 & \text{on } \partial B_1\end{cases}. \tag{3.3.1}$$

The function u is radial, by the Gidas-Ni-Nirenberg theorem, and thus we find u_D by just considering the restriction of u to the domain Ω_D.

We begin by recalling conditions on the nonlinearity f that ensure that a positive radial solution of (3.3.1) exists.

Proposition 3.3.1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a C^1 function. Assume that f satisfies one of the following conditions:*

(i) $|f(s)|\le a|s|+b$ *for all* $s>0$, *where* $b>0$ *and* $a\in(0,\mu_1(B_1))$ *where* $\mu_1(B_1)$ *is the first eigenvalue of the operator* $-\Delta$ *in* $H_0^1(B_1)$.

(ii) $f:[0,+\infty)\to[0,+\infty)$ *is non-increasing.*

(iii)
- $|f(s)|<c|s|^p+d$, *where* $c,d>0$ *and* $p\in(1,\frac{N+2}{N-2})$ *if* $N\ge 3$, $p>1$ *if* $N=2$;
- $f(s)=o(s)$ *as* $s\to 0$;
- *There exist* $\gamma>2$, $\kappa>0$ *such that for* $|s|>\kappa$ *it holds*

$$0<\gamma F(s)<sf(s);$$

- $f'(s)>\frac{f(s)}{s}$ *for all* $s>0$.

Then there exists a positive radial solution of (3.3.1) *in* B_1, *and hence of* (3.1.1) *in* Ω_D.

Proof. In cases *(i)* and *(ii)*, the corresponding functional

$$J(u)=\frac{1}{2}\int_{B_1}|\nabla u|^2\,dx-\int_{B_1}F(u)\,dx$$

is coercive and weakly lower semicontinuous in the space $H^1_{0,rad}(B_1)$, which is the subspace of $H^1_0(B_1)$ of radial functions, and so it has a minimum which is a solution of (3.3.1). In the case *(iii)*, by standard variational methods, such as minimization on the Nehari manifold or Mountain Pass type theorems, we obtain the existence of a positive solution of (3.3.1) in $H^1_0(B_1)$, which is then radial by the Gidas-Ni-Nirenberg theorem (see [40]). We refer to [7] and [26] for the details. □

Now let u_D be a weak solution of (3.1.1). Consider the function $u(r,q) = u_D(r)$, for $r \in (0,1)$ and $q \in \mathbb{S}^{N-1}$. It is immediately seen that $u \in H^1_0(B_1)$ and u is a weak solution of (3.3.1). Then, by elliptic regularity theory, we know that u is a classical solution. In particular, it follows that u_D is bounded in Ω_D.

Now we analyze the nondegeneracy of a radial solution u_D of (3.1.1) in Ω_D. We need conditions that ensure that zero is not an eigenvalue of the linearized operator

$$L_{u_D} = -\Delta - f'(u_D) \tag{3.3.2}$$

in the space $H^1_0(\Omega_D \cup \Gamma_{1,0})$, where $\Gamma_{1,0} = (\partial\Omega_D \setminus \overline{D}) \cap \partial\Sigma_D$. Clearly, if the linearized operator L_{u_D} admits only positive eigenvalues, then u_D is nondegenerate. This is the case of stable solutions to (3.1.1), which happens when f satisfies conditions *(i)* or *(ii)* in Proposition 3.3.1. In particular, this is the case when f is a positive constant.

In general, L_{u_D} could have negative eigenvalues, so to detect the nondegeneracy of u_D we have to study the spectrum of the operator L_{u_D} in $H^1_0(\Omega_D \cup \Gamma_{1,0})$. As we will see, the fact that Ω_D is a spherical sector in the cone Σ_D (and not the ball B_1) plays a role.

The first remark is that zero is an eigenvalue for L_{u_D} if and only if it is an eigenvalue for the singular problem

$$\begin{cases} -\Delta\psi - f'(u_D)\psi &= \dfrac{\widehat{\Lambda}}{|x|^2}\psi \quad \text{in } \Omega_D \\ \psi &= 0 \quad \text{on } D \\ \dfrac{\partial\psi}{\partial\nu} &= 0 \quad \text{on } \Gamma_{1,0} \setminus \{0\} \end{cases}. \tag{3.3.3}$$

The advantage of considering this singular eigenvalue problem is that, since u_D is radial, its eigenfunctions can be obtained by separation of variables, using polar coordinates in $\mathbb{R}^N$, and therefore we investigate its eigenvalues.

We consider the eigenvalue problem

$$\begin{cases} -\Delta\psi &= \lambda\psi \quad \text{on } D \\ \dfrac{\partial\psi}{\partial\nu} &= 0 \quad \text{on } \partial D. \end{cases} \tag{3.3.4}$$

and denote its eigenvalues, counted with multiplicity: $0 = \lambda_0(D) < \lambda_1(D) \le \lambda_2(D) \le \ldots \to +\infty$. The corresponding $L^2(D)$-normalized eigenfunctions are denoted by ψ_j, $j \in \mathbb{N}$. Then

$$\int_D \psi_j^2 \, d\sigma = 1 \quad \forall j = 0,1,\ldots, \qquad \psi_0 = \text{constant}, \qquad \int_D \psi_j \, d\sigma = 0 \quad \forall j \ge 1.$$

Then we consider the following singular eigenvalue problem in the interval $(0,1)$:

$$\begin{cases} -z'' - \frac{N-1}{r} z' - f'(u_D) z = \frac{\widehat{\nu}}{r^2} z & \text{in } (0,1) \\ z(1) = 0 \end{cases} . \tag{3.3.5}$$

It is shown in [6] (see also [22]) that nonpositive eigenvalues for (3.3.5) can be defined. They are in finite number and we denote them by $\widehat{\nu}_i$, $i = 1, \dots, k$. It is immediate to check that the eigenvalues of (3.3.5) are the eigenvalues of (3.3.3) that correspond to radial eigenfunctions. In particular, we consider the first eigenvalue $\widehat{\nu}_1$ of (3.3.5), referring to [6] for a variational definition and a study of its main properties.

Proposition 3.3.2. *Problem* (3.3.3) *admits zero as an eigenvalue if and only if there exist* $i \in \{1, \dots, k\}$ *and* $j \in \mathbb{N}$ *such that*

$$\widehat{\nu}_i + \lambda_j(D) = 0. \tag{3.3.6}$$

Proof. The proof follows by [22, Proposition 2.6], where it is proved that the nonpositive eigenvalues of (3.3.3) are obtained by summing the eigenvalues of the one-dimensional problem (3.3.5) and the Neumann eigenvalues of $-\Delta_{\mathbb{S}^{N-1}}$ on D. We refer also to [28] for another approach, which consists in approximating the ball by annuli in order to avoid the singularity at 0. □

From Proposition 3.3.2 we obtain the following sufficient condition for a radial solution u_D to be nondegenerate:

Corollary 3.3.3. *A radial solution* u_D *of* (3.1.1) *is nondegenerate if both the following conditions are satisfied:*

(I) The eigenvalue problem (3.3.5) *does not admit zero as an eigenvalue;*

(II) $\lambda_1(D) > -\widehat{\nu}_1$.

Proof. From condition *(I)* we have

$$\widehat{\nu}_i \neq 0 \quad \forall i \in \{1, \dots, k\}, \tag{3.3.7}$$

what means that zero is not an eigenvalue of (3.3.3) corresponding to a radial eigenfunction. This, in turn, is equivalent to saying that zero is not a "radial" eigenvalue of the linearized operator L_{u_D}, i.e., u_D is a radial solution that is nondegenerate in the space $H^1_{0,rad}(\Omega_D \cup \Gamma_{1,0})$, which is the subspace of $H^1_0(\Omega_D \cup \Gamma_{1,0})$ constituted by the radial functions.

Now, since $\lambda_0(D) = 0$, $\lambda_1(D) > 0$ and since $\widehat{\nu}_1$ is the smallest eigenvalue of (3.3.5), then, from *(II)* and (3.3.7) we infer that the sum (3.3.6) can never be zero. Hence, thanks to Proposition 3.3.2, we have that zero is not an eigenvalue of (3.3.3) and so cannot be an eigenvalue of the linearized operator L_{u_D} in the whole space $H^1_0(\Omega_D \cup \Gamma_{1,0})$, i.e., u_D is a nondegenerate solution to (3.1.1). □

Remark 3.3.4. Condition *(I)* in Corollary 3.3.3, i.e., the nondegeneracy of u_D in the subspace of radial functions in $H^1_0(\Omega_D \cup \Gamma_{1,0})$, is satisfied by positive radial solutions of (3.1.1) corresponding to many kinds of nonlinearities. It holds if f satisfies conditions *(i)* or *(ii)* of Proposition (3.3.1), because in this case all eigenvalues of (3.3.3) and L_{u_D} are positive, and then *(II)* holds as well. More precisely, in case *(i)*, since $0 < a < \mu_1(B_1)$, the first eigenvalue of L_{u_D} is positive, so

$$\lambda_0(D) + \widehat{\nu}_1 > 0.$$

In the case *(ii)*, since $f'(u_D) \leq 0$, it follows that $\widehat{\nu}_1 > 0$.

Among the nonlinearities satisfying condition *(iii)* of Proposition 3.3.1 we could consider $f(u) = u^p$, $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$. Then it is known that the positive radial solution of (3.3.1) is unique and nondegenerate (see [25, 40]), so *(I)* holds. It is also well-known that for this nonlinearity it holds that $\widehat{\nu}_1 < 0$ and $\widehat{\nu}_1$ is the only negative eigenvalue of (3.3.5), because u_D can be obtained by the Mountain Pass Theorem or by minimization on the Nehari manifold, and thus u_D has Morse index one. Then the validity of *(II)* depends on the cone since it depends on $\lambda_1(D)$. However, once p is fixed, since $\widehat{\nu}_1$ does not depend on the cone, it is obvious that by varying D there are many cones for which *(II)* holds. Moreover, it has been proved in [22] that $\widehat{\nu}_1 > -(N-1)$ for every autonomous nonlinearity, so that whenever $\lambda_1(D) > N-1$ all radial solutions of (3.1.1) are nondegenerate.

3.4 Instability of spherical sectors

Let us first observe that if u_D is a positive nondegenerate solution of (3.1.1) belonging to $W^{1,\infty}(\Omega_D) \cap W^{2,2}(\Omega_D)$, then (Ω_D, u_D) is energy-stationary in the sense of Definition 2.2.4. Indeed, since u_D is radial, we have that $\frac{\partial u_D}{\partial \nu}$ = constant on D, and then we conclude making use of Proposition 2.2.5.

To investigate the stability of (Ω_D, u_D) we analyze the quadratic form corresponding to the second derivative $I''(\varphi)$ at $\varphi = 0$ (recall the definition of the functional I in (3.2.12)). Fixing the constant c in the definition of M (see (3.2.11)) as $c = |\Omega_D|$, we have that the tangent space to M at $\varphi = 0$ is given by

$$T_0 M = \left\{ v \in C^2(\overline{D}) \ : \ \int_D v \, d\sigma = 0 \right\}. \tag{3.4.1}$$

Writing $u_D(r) = u_D(|x|)$, we denote by u'_D and u''_D the derivatives of u_D with respect to r, so that

$$u'_D(1) = \left.\frac{\partial u_D}{\partial \nu}\right|_D, \qquad u''_D(1) = [(D^2 u_D \nu) \cdot \nu]|_D. \tag{3.4.2}$$

By Hopf's Lemma we know that $u'_D(1) < 0$ and actually

$$u'_D(1) = -\sqrt{-2\mu_D}, \tag{3.4.3}$$

where μ_D denotes the Lagrange multiplier in the case $\varphi = 0$ (see Proposition 3.2.3).

For $v \in T_0M$ we will denote by $\widetilde{u}_v$ the solution of

$$\begin{cases} -\Delta\widetilde{u}_v - f'(u_D)\widetilde{u}_v &= 0 & \text{in } \Omega_D \\ \widetilde{u}_v &= -u_D'(1)v & \text{on } D \\ \dfrac{\partial\widetilde{u}_v}{\partial\nu} &= 0 & \text{on } \Gamma_{1,0}\setminus\{0\} \end{cases}. \tag{3.4.4}$$

Let us remark that for every $q \in D$ the outer unit normal vector $\nu(q)$ is precisely q, hence (3.4.4) corresponds to (2.2.11) in Ω_D.

Note that, since u_D is a nondegenerate radial solution, then the weak solution $\widetilde{u}_v$ of (3.4.4) is unique for every v.

Our next result shows that the quadratic form corresponding to the second derivative of I at $\varphi = 0$ has a simple expression.

Lemma 3.4.1. *For any $v \in T_0M$ it holds*

$$I''(0)[v,v] = -u_D'(1)\left(\int_D v\frac{\partial\widetilde{u}_v}{\partial\nu}\,d\sigma + u_D''(1)\int_D v^2\,d\sigma\right), \tag{3.4.5}$$

where $\widetilde{u}_v$ is the solution of (3.4.4)*.*

Proof. From Lemma 3.2.2, (3.2.10) and Lemma 3.2.4, by simple substitutions and elementary computations we obtain

$$\begin{aligned} I''(0)[v,v] = &-\frac{N}{2}\int_D (u_D'(1))^2 v^2\,d\sigma - \int_D u_D'(1)v\frac{\partial\widetilde{u}_v}{\partial\nu}\,d\sigma \\ &- \int_D u_D'(1)v^2(D^2u_D\nu)\cdot\nu\,d\sigma - N\mu_D\int_D v^2\,d\sigma. \end{aligned} \tag{3.4.6}$$

Since $\widetilde{u}_v = -u_D'(1)v$ on D, by (3.4.2) and (3.4.3) we deduce that

$$-\frac{N}{2}\int_D (u_D'(1))^2 v^2\,d\sigma = -\frac{N}{2}\int_D \widetilde{u}_v^2\,d\sigma; \tag{3.4.7}$$

$$-N\mu_D\int_D v^2\,d\sigma = \frac{N}{2}\int_D \widetilde{u}_v^2\,d\sigma. \tag{3.4.8}$$

Then (3.4.5) easily follows. □

To investigate the stability of (Ω_D, u_D) as an energy-stationary pair for I we need to study the solution $\widetilde{u}_v$ of (3.4.4) for any $v \in T_0M$, namely, for functions in $C^2(\overline{D})$ with mean value zero. In the case when $v = \psi_j$, where ψ_j is an L^2-normalized Neumann eigenfunction of the Laplace-Beltrami operator on D corresponding to the eigenvalue $\lambda_j(D)$, we can proceed by separation of variables. More precisely, we have the following:

Theorem 3.4.2. *Let $j \geq 1$ and let $\widetilde{u}_j$ be the solution of* (3.4.4) *for $v = \psi_j$. Then, writing $\widetilde{u}_j = \widetilde{u}_j(r,q)$, the function*

$$h_j(r) = \int_D \widetilde{u}_j(r,q)\psi_j(q)\,d\sigma, \quad r \in (0,1) \tag{3.4.9}$$

satisfies

$$\begin{cases} -h_j'' - \frac{N-1}{r} h_j' - f'(u_D) h_j = -\frac{\lambda_j(D)}{r^2} h_j \\ h_j(1) = -u_D'(1) \end{cases}. \tag{3.4.10}$$

Proof. It is immediate to check that $h_j(1) = -u_D'(1)$. Moreover, since we can bring the radial derivative inside the integral on D, for every $r \in (0,1)$, we have:

$$\begin{aligned} -h''(r) - \frac{N-1}{r} h_j'(r) &= \int_D \left(-\frac{\partial^2 \widetilde{u}_j}{\partial r^2}(r,q) - \frac{N-1}{r} \frac{\partial \widetilde{u}_j}{\partial r}(r,q) \right) \psi_j(q)\, d\sigma \\ &= \int_D \left(-\Delta \widetilde{u}_j + \frac{1}{r^2} \Delta_{\mathbb{S}^{N-1}} \widetilde{u}_j \right) \psi_j\, d\sigma \\ &= \int_D f'(u_D(r)) \widetilde{u}_j \psi_j\, d\sigma + \frac{1}{r^2} \int_D (\Delta_{\mathbb{S}^{N-1}} \widetilde{u}_j) \psi_j\, d\sigma. \end{aligned} \tag{3.4.11}$$

Now, on the one hand,

$$\int_D f'(u_D(r)) \widetilde{u}_j \psi_j\, d\sigma = f'(u_D(r)) h_j(r).$$

On the other hand, applying Green's formula, taking into account the Neumann conditions on ψ_j and $\widetilde{u}_j$, we infer that

$$\begin{aligned} \frac{1}{r^2} \int_D (\Delta_{\mathbb{S}^{N-1}} \widetilde{u}_j) \psi_j\, d\sigma &= \frac{1}{r^2} \int_D \widetilde{u}_j \Delta_{\mathbb{S}^{N-1}} \psi_j\, d\sigma \\ &= -\frac{\lambda}{r^2} \int_D \widetilde{u}_j \psi_j\, d\sigma \\ &= -\frac{\lambda}{r^2} h_j(r). \end{aligned} \tag{3.4.12}$$

Substituting these expressions into (3.4.11) we conclude the proof. □

Remark 3.4.3. Note that, for $\widetilde{u}_j$ and h_j as in Theorem 3.4.2, we have that

$$\widetilde{u}_j(r,q) = h_j(r) \psi_j(q).$$

Indeed, the function $h_j \psi_j$ satisfies the boundary conditions of (3.4.4), and it holds that

$$\begin{aligned} -\Delta(h_j \psi_j) &= -h_j'' \psi_j - \frac{N-1}{r} h_j' \psi_j - \frac{h_j}{r^2} \Delta_{\mathbb{S}^{N-1}} \psi_j \\ &= f'(u_D) h_j \psi_j - \frac{\lambda_j(D)}{r^2} h_j \psi_j + \frac{\lambda_j(D)}{r^2} h_j \psi_j \\ &= f'(u_D) h_j \psi_j. \end{aligned} \tag{3.4.13}$$

Proposition 3.4.4. *Let h_j be as in Theorem 3.4.2. Then for any $j \in \mathbb{N}^+$ we have*

$$\int_0^1 r^{N-3} h_j^2\, dr < +\infty \tag{3.4.14}$$

and

$$\int_0^1 r^{N-1} (h_j')^2\, dr < +\infty. \tag{3.4.15}$$

Moreover, $h_j \in L^\infty(0,1)$ and $h_j(0) = 0$.

Proof. Since $\widetilde{u}_j \in H^1(\Omega_D)$, writing $\widetilde{u}_j = \widetilde{u}_j(r,q)$ and recalling that ψ_j is a $L^2(D)$-normalized Neumann eigenfunction, it follows that

$$\begin{aligned} +\infty &> \int_{\Omega_D} |\nabla \widetilde{u}_j|^2 \, dx \\ &= \int_0^1 r^{N-1}(h_j')^2 \int_D \psi_j^2 \, d\sigma \, dr + \int_0^1 r^{N-3} h_j^2 \int_{\Omega_D} |\nabla_{\mathbb{S}^{N-1}} \psi_j|^2 \, d\sigma \, dr \\ &= \int_0^1 r^{N-1}(h_j')^2 \, dr + \lambda_j \int_0^1 r^{N-3} h_j^2 \, dr, \end{aligned} \tag{3.4.16}$$

what proves (3.4.14) and (3.4.15). Once we have these estimates, we can proceed as in [27, Lemma A.9] to get the boundness of h_j and that $h_j(0) = 0$. ☐

Proposition 3.4.5. *Let $\lambda_j(D)$, $j \in \mathbb{N}$, be a Neumann eigenvalue of $-\Delta_{\mathbb{S}^{N-1}}$ on D. Assume that*

$$-\widehat{\nu}_1 < \lambda_j(D),$$

where $\widehat{\nu}_1$ is the smallest eigenvalue of (3.3.5). Then

$$h_j > 0 \quad \textit{in } (0,1). \tag{3.4.17}$$

Proof. Let z_1 be an L^2-normalized eigenfunction of (3.3.5). From [6, Section 3.1] we know that z_1 does not change sign.

Writing the equations satisfied by h_j and z_1 in Sturm-Liouville form we have:

$$(r^{N-1}h_j')' + r^{N-1}(f'(u_D) - r^{-2}\lambda_j(D))h_j = 0, \tag{3.4.18}$$

$$(r^{N-1}z_1')' + r^{N-1}(f'(u_D) + r^{-2}\widehat{\nu}_1)z_1 = 0. \tag{3.4.19}$$

Recall that $h_j(1) = -u_D'(1) > 0$ and $h_j(0) = 0$. Assume by contradiction that h_j changes sign in $(0,1)$. Then there would exist $r_0 \in (0,1)$ such that $h_j(r_0) = 0$. Since $-\widehat{\nu}_1 < \lambda_j(D)$, then, by the Sturm-Picone comparison theorem it would follow that z_1 has a zero in $(0, r_0)$. This is a contradiction because z_1 does not change sign. Hence the only possibility is that h_j is strictly positive in $(0,1)$. ☐

Remark 3.4.6. Note that $-\Delta u_D = f(u_D)$ in polar coordinates reads as

$$-u_D'' - \frac{N-1}{r}u_D' - f(u_D) = 0,$$

where $r = |x|$. Differentiating with respect to r we obtain

$$-(u_D')'' - \frac{N-1}{r}(u_D')' - f'(u_D)u_D' = -\frac{N-1}{r^2}u_D'. \tag{3.4.20}$$

Moreover, by the Gidas-Ni-Nirenberg theorem we know that $u_D'(0) = 0$ and $u_D' < 0$ in $(0,1)$.

We can now state the main theorem of this chapter:

Theorem 3.4.7. *Let Σ_D be the cone spanned by the smooth domain $D \subset \mathbb{S}^{N-1}$ and let $\lambda_1(D)$ be the first nonzero Neumann eigenvalue of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^{N-1}}$ on D. Let u_D be a nondegenerate radial solution of (3.1.1) in the spherical sector Ω_D. We have:*

(i) If $-\widehat{\nu}_1 < \lambda_1(D) < N-1$, then the pair (Ω_D, u_D) is an unstable energy-stationary pair;

(ii) If $\lambda_1(D) > N-1$, then (Ω_D, u_D) is a stable energy-stationary pair.

Proof. Let us fix the domain $D \subset \mathbb{S}^{N-1}$ that spans the cone and denote $\lambda_j(D)$ simply by λ_j.

For *(i)*, let $\widetilde{u}_1 = h_1\psi_1$ be the solution of (3.4.4) with $v = \psi_1$. Then

$$I''(0)[\psi_1, \psi_1] = -u_D'(1)(h_1'(1) + u_D''(1)). \tag{3.4.21}$$

Putting (3.4.10) in Sturm-Liouville form we obtain

$$-(r^{N-1}h_1')' - r^{N-1}f'(u_D)h_1 = -r^{N-3}\lambda_1 h_1. \tag{3.4.22}$$

On the other hand, putting (3.4.20) in Sturm-Liouville form yields

$$-(r^{N-1}(u_D')')' - r^{N-1}f'(u_D)u_D' = r^{N-3}(N-1)u_D'. \tag{3.4.23}$$

Multiplying (3.4.22) by u_D' and integrating by parts in $(\bar{r}, 1)$ we obtain

$$\int_{\bar{r}}^1 r^{N-1}h_1'u_D''\,dr - (r^{N-1}h_1'u_D')|_{\bar{r}}^1 - \int_{\bar{r}}^1 r^{N-1}f'(u_D)h_1u_D'\,dr = -\lambda_1 \int_{\bar{r}}^1 r^{N-3}h_1u_D'\,dr. \tag{3.4.24}$$

Similarly, multiplying (3.4.23) by h_1 and integrating by parts we deduce that

$$\int_{\bar{r}}^1 r^{N-1}h_1'u_D''\,dr - (r^{N-1}h_1u_D'')|_{\bar{r}}^1 - \int_{\bar{r}}^1 r^{N-1}f'(u_D)h_1u_D'\,dr = -(N-1)\int_{\bar{r}}^1 r^{N-3}h_1u_D'\,dr. \tag{3.4.25}$$

Notice that, in view of Proposition 3.4.4, the right-hand sides of (3.4.24) and (3.4.25) remain finite when taking the limit as $\bar{r} \to 0^+$. In addition, we claim that

$$\lim_{\bar{r}\to 0^+} r^{N-1}h_1'(\bar{r})u_D'(\bar{r}) = 0. \tag{3.4.26}$$

Indeed, integrating (3.4.22) and taking the absolute value we obtain

$$\left|\int_{\bar{r}}^1 -(r^{N-1}h_1')'\,dr\right| = |\bar{r}^{N-1}h_1'(\bar{r}) - h_1'(1)|$$

$$\leq \int_{\bar{r}}^{1} r^{N-1}|f'(u_D)|h_1\, dr + \int_{\bar{r}}^{1} r^{N-3}\lambda_1 h_1\, dr$$
$$\leq C_1$$

for some $C_1 > 0$. Hence
$$\limsup_{\bar{r}\to 0^+} \bar{r}^{N-1}|h_1'(\bar{r})| \leq C_2$$
for some $C_2 > 0$ and thus, since $\lim_{\bar{r}\to 0^+} u_D'(\bar{r}) = u_D'(0) = 0$, the claim (3.4.26) follows.

Now, subtracting (3.4.25) from (3.4.24) and taking the limit as $\bar{r} \to 0^+$, in view of (3.4.26) and since $h_1(0) = 0, h_1(1) = -u_D'(1)$ we obtain

$$-u_D'(1)(h_1(1) + u_D''(1)) = (N - 1 - \lambda_1) \int_0^1 r^{N-3} h_1 u_D'\, dr. \tag{3.4.27}$$

Since $\lambda_1 > -\widehat{\nu}_1$, then, by Proposition 3.4.17 we have that $h_1 > 0$ in $(0, 1)$. On the other hand, $u_D' < 0$ in $(0, 1)$ and $\lambda_1 < N - 1$ by assumption. Hence by (3.4.21) and (3.4.27) we obtain
$$I''(0)[\psi_1, \psi_1] < 0,$$
what proves *(i)*.

For *(ii)*, we choose an orthonormal basis $(\psi_j)_j$ of $L^2(D)$-normalized eigenfunctions of (3.3.4). Then any $v \in T_0M$ can be written as

$$v = \sum_{j=1}^{\infty} (v, \psi_j)\psi_j,$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(D)$. We assume without loss of generality that $\int_D v^2\, d\sigma = 1$.

Let $\widetilde{u}_j$ be the solution of (3.4.4) with $v = \psi_j$. Then we can check that

$$\widetilde{v} = \sum_{j=1}^{\infty} (v, \psi_j)\widetilde{u}_j$$

is the solution of (3.4.4). By Remark 3.4.3, $\widetilde{u}_j(r, q) = h_j(r)\psi_j(q)$ for every $j \in \mathbb{N}$, so

$$\frac{\partial \widetilde{u}_j}{\partial \nu}(1, q) = h_j'(1)\psi_j(q) \quad \text{ on } D.$$

Arguing as in the proof of *(i)*, we have that if $k > j$, then $h_k'(1) \geq h_j'(1)$ and in fact $h_k'(1) > h_j'(1)$ if $\lambda_k > \lambda_j$. Indeed, by writing the equations for h_j and h_k in Sturm-Liouville form, integrating and subtracting we get obtain

$$-u_D'(1)(h_k'(1) - h_j'(1)) = (-\lambda_j + \lambda_k) \int_0^1 r^{N-3} h_j h_k\, dr \geq 0.$$

Exploiting the orthogonality of the basis $(\psi_j)_j$ and (3.4.27) we deduce

$$\begin{aligned}
I''(0)[v,v] &= -u'_D(1)\left(\int_D \left(\sum_{j=1}^{\infty}(v,\psi_j)\psi_j\right)\left(\sum_{k=1}^{\infty}(v,\psi_k)h'_k(1)\psi_k\right)\,d\sigma \right.\\
&\qquad\qquad \left. +u''_D(1)\int_D v^2\,d\sigma\right)\\
&= -u'_D(1)\left(\left(\sum_{j=1}^{\infty}(v,\psi_j)^2 h'_j(1)\right)+u''_D(1)\right)\\
&\geq -u'_D(1)\left(h'_1(1)\left(\sum_{j=1}^{\infty}(v,\psi_j)^2\right)+u''_D(1)\right)\\
&= -u'_D(1)(h'_1(1)+u''_D(1))\\
&= (N-1-\lambda_1)\int_0^1 r^{N-3}h_1u'_D\,dr\\
&> 0,
\end{aligned} \tag{3.4.28}$$

since $h_1 > 0$ and $u'_D < 0$ in $(0,1)$ and $\lambda_1 > N-1$ by assumption. The proof is complete. □

Chapter 4

Cylinders

4.1 Introduction

Let $\omega \subset \mathbb{R}^{N-1}$ be a smooth bounded domain and let Σ_ω be the half-cylinder spanned by ω, that is,

$$\Sigma_\omega := \omega \times (0, +\infty).$$

We denote by $x = (x', x_N)$ the points in $\overline{\Sigma_\omega}$, where $x' = (x_1, \dots, x_{N-1}) \in \overline{\omega}$ and $x_N \geq 0$.

In Σ_ω we consider the bounded cylinder

$$\Omega_\omega := \omega \times (0, 1),$$

and study so-called one-dimensional solutions to

$$\begin{cases} -\Delta u & = & f(u) & \text{in } \Omega_\omega \\ u & = & 0 & \text{on } \Gamma_{\Omega_\omega} = (\{x_N = 1\} \cap \Sigma_\omega) \\ \dfrac{\partial u}{\partial \nu} & = & 0 & \text{on } \Gamma_{1,\Omega_\omega} \end{cases}, \tag{4.1.1}$$

which are solutions of the form $u(x', x_N) = u(x_N)$.

We show that $(\Omega_\omega, u_\omega)$ is an energy-stationary pair and study its stability as a critical point of the energy functional.

As in the case of the cone, we find out that the stability depends on the first nontrivial of the Neumann-Laplacian on ω. However, here the threshold for stability depends both on the nonlinearity and on the solution u_ω we consider, in case there are many, see Theorem 4.4.4. In particular, for the case of the torsion problem (i.e., when $f \equiv 1$), we obtain an explicit threshold. We also report the results of a numerical analysis that corroborates and complements our theoretical study. This is the content of Section 4.4.

4.2 Energy functional for cylindrical domains

In analogy with the case of the cone, we consider domains whose relative boundaries are the cartesian graphs of functions in $C^2(\overline{\omega})$. More precisely, for $\varphi \in C^2(\overline{\omega})$ we set

$$\Gamma_\varphi := \{(x', x_N) \in \Sigma_\omega \ : \ x_N = e^{\varphi(x')}\} \tag{4.2.1}$$

and consider the domain

$$\Omega_\varphi := \{(x', x_N) \in \Sigma_\omega \ : \ x_N < e^{\varphi(x')}\}. \tag{4.2.2}$$

Finally, we set

$$\Gamma_{1,\varphi} := \partial\Omega_\varphi \setminus \overline{\Gamma}_\varphi. \tag{4.2.3}$$

Remark 4.2.1. There are similar definitions in Chapter 2. Notice, however, that here the domain Ω_φ is delimited by the graph of the function e^φ, while in Chapter 2 it was delimited by the graph of the function φ itself. In other words, here we denote by Ω_φ what in the notation of Chapter 2 would be Ω_{e^φ}. We make use of the exponential function because it allows us to parametrize graphs of positive functions in the whole space of $C^2(\overline{\omega})$ functions, and we need to have this whole space at hand to take derivatives of the energy functional later on. Such need was not present in Chapter 2.

We study the semilinear mixed boundary value problem

$$\begin{cases} -\Delta u &= f(u) & \text{in } \Omega_\varphi \\ u &= 0 & \text{on } \Gamma_\varphi \\ \dfrac{\partial u}{\partial \nu} &= 0 & \text{on } \Gamma_{1,\varphi} \end{cases} \tag{4.2.4}$$

and consider bounded positive weak solutions of (4.2.4) in the Sobolev space $H^1_0(\Omega_\varphi \cup \Gamma_{1,\varphi})$, that is the space of functions in $H^1(\Omega_\varphi)$ whose trace vanishes on Γ_φ.

As before, we assume that a positive nondegenerate solution u_φ of (4.2.4) exists and belongs to $W^{1,\infty}(\Omega_\varphi) \cap W^{2,2}(\Omega_\varphi)$, so that we can apply the results of Chapter 2.

We consider variations of the domain Ω_φ in the class of cartesian graphs, namely, we consider perturbed domains of the form $\Omega_{\varphi+tv}$, for $v \in C^2(\overline{\omega})$ and t small. This corresponds to considering a one-parameter family of diffeomorphisms $\xi : (-\eta, \eta) \times \overline{\Sigma}_\omega \to \overline{\Sigma}_\omega$ of the type

$$\xi(t, x) = (x', e^{tv(x')}x_N),$$

whose inverse, for any fixed $t \in (-\eta, \eta)$ is given by

$$\xi(t, x)^{-1} = (x', e^{-tv(x')}x_N) = \xi(-t, x).$$

This one-parameter family of diffeomorphisms is generated by the vector field

$$V(x) = (0', v(x')x_N), \tag{4.2.5}$$

where $0' = (0, \dots, 0) \in \mathbb{R}^{N-1}$. Indeed, $\xi(0,x) = x$ for every $x \in \overline{\Sigma}_\omega$,

$$\frac{d\xi}{dt}(t,x) = (0', e^{tv(x')}v(x')x_N) = V(\xi(t,x)) \quad \forall (t,x) \in (-\eta,\eta) \times \overline{\Sigma}_\omega$$

and $\xi(t,x) \in \partial\Sigma_\omega$ for all $(t,x) \in (-\eta,\eta) \times \partial\Sigma_\omega$. For simplicity, we set $\xi_t(x) := \xi(t,x)$. We also observe that the outer unit normal vector on Γ_φ at a point $(x', e^{\varphi(x')})$ is given by

$$\nu = \nu_\varphi = \frac{(-e^{\varphi(x')}\nabla_{\mathbb{R}^{N-1}}\varphi(x'), 1)}{\sqrt{1 + |e^{\varphi(x')}\nabla_{\mathbb{R}^{N-1}}\varphi(x')|^2}}, \tag{4.2.6}$$

where $\nabla_{\mathbb{R}^{N-1}}$ denotes the gradient with respect to the variables $x_1, \dots, x_{N-1}$. Then it holds

$$\langle V, \nu \rangle = \left\langle (0', ve^\varphi), \frac{(-e^\varphi \nabla_{\mathbb{R}^{N-1}}\varphi, 1)}{\sqrt{1 + |e^\varphi \nabla_{\mathbb{R}^{N-1}}\varphi|^2}} \right\rangle = \frac{ve^\varphi}{\sqrt{1 + |e^\varphi \nabla_{\mathbb{R}^{N-1}}\varphi|^2}} \quad \text{on } \Gamma_\varphi. \tag{4.2.7}$$

The energy functional T defined in (2.1.2) becomes a functional depending only on functions in $C^2(\overline{\omega})$. More precisely, for every $v \in C^2(\overline{\omega})$, in view of Proposition 2.2.1, there exists $\delta > 0$ sufficiently small such that for all $t \in (-\delta, \delta)$

$$T(\varphi + tv) := T(\Omega_{\varphi+tv}) = J(u_{\varphi+tv}) \tag{4.2.8}$$

is well-defined, where $u_{\varphi+tv} = u_{\Omega_{\varphi+tv}}$ is the unique positive solution of (4.2.4) in the domain $\Omega_{\varphi+tv}$, in a neighborhood of $u_\varphi \circ \xi_t^{-1}$.

Since we are assuming that u_φ is nondegenerate, by the results of Chapter 2 we know that $t \mapsto u_{\varphi+tv}$ is differentiable at $t = 0$ and the derivative $\widetilde{u}$ is a weak solution of

$$\begin{cases} -\Delta \widetilde{u} = f'(u_\varphi)\widetilde{u} & \text{in } \Omega_\varphi \\ \widetilde{u} = -\dfrac{\partial u_\varphi}{\partial \nu} \dfrac{ve^\varphi}{\sqrt{1 + |e^\varphi \nabla_{\mathbb{R}^{N-1}}\varphi|^2}} & \text{on } \Gamma_\varphi \\ \dfrac{\partial \widetilde{u}}{\partial \nu} = 0 & \text{on } \Gamma_{1,\varphi} \end{cases}. \tag{4.2.9}$$

We now compute the derivatives of T at Ω_φ, i.e., for $t = 0$, with respect to variations in $C^2(\overline{\omega})$.

Lemma 4.2.2. *Let $\varphi \in C^2(\overline{\omega})$ and assume that u_φ is a positive nondegenerate solution of (4.2.4) that belongs to $W^{1,\infty}(\Omega_\varphi) \cap W^{2,2}(\Omega_\varphi)$. Then for any $v \in C^2(\overline{\omega})$ we have*

$$T'(\varphi)[v] = -\frac{1}{2} \int_\omega \left(\frac{\partial u_\varphi}{\partial \nu}(x', e^{\varphi(x')}) \right)^2 v(x') e^{\varphi(x')} \, dx'. \tag{4.2.10}$$

Proof. The proof is similar to that of Lemma 3.2.1. It suffices to observe that for the parametrization of Γ_φ given by $x = (x', e^{\varphi(x')})$, for $x' \in \omega$, the induced $(N-1)$-dimensional area element on Γ_φ is expressed by

$$d\sigma_{\Gamma_\varphi} = \sqrt{1 + |e^\varphi \nabla_{\mathbb{R}^{N-1}}\varphi|^2} \, dx'.$$

Then the result follows immediately from Proposition 2.2.3, taking into account (4.2.7). □

Lemma 4.2.3. *Let φ and u_φ be as in Lemma 4.2.2. Then for any $v, w \in C^2(\overline{\omega})$ it holds*

$$\begin{aligned} T''(\varphi)[v,w] = &-\frac{1}{2}\int_\omega \left(\frac{\partial u_\varphi}{\partial \nu}(x', e^\varphi)\right)^2 e^\varphi v w \, dx' \\ &- \int_\omega \frac{\partial \widetilde{u}_w}{\partial \nu}(x', e^\varphi)\frac{\partial u_\varphi}{\partial \nu}(x', e^\varphi) e^\varphi v \, dx' \\ &- \int_\omega \frac{\partial u_\varphi}{\partial \nu}(x', e^\varphi)[(D^2 u_\varphi(x', e^\varphi)(0', e^\varphi)) \cdot \nu] v w \, dx' \\ &+ \int_\omega \frac{\partial u_\varphi}{\partial \nu}(x', e^\varphi) e^{2\varphi} v \frac{\nabla u_\varphi(x', e^\varphi) \cdot (w \nabla_{\mathbb{R}^{N-1}}\varphi + \nabla_{\mathbb{R}^{N-1}} w, 0)}{\sqrt{1 + |e^\varphi \nabla_{\mathbb{R}^{N-1}}\varphi|^2}} \\ &+ \int_\omega \left(\frac{\partial u_\varphi}{\partial \nu}(x', e^\varphi)\right)^2 e^{3\varphi} v \frac{\nabla_{\mathbb{R}^{N-1}}\varphi \cdot (w \nabla_{\mathbb{R}^{N-1}}\varphi + \nabla_{\mathbb{R}^{N-1}} w)}{1 + |e^\varphi \nabla_{\mathbb{R}^{N-1}}\varphi|^2}, \end{aligned} \tag{4.2.11}$$

where $\widetilde{u}_w$ is the solution of (4.2.9) *with w in place of v.*

Proof. Let $v, w \in C^2(\overline{\omega})$. By definition, Lemma 4.2.2 and using the Leibniz rule, we have:

$$\begin{aligned} T''(\varphi)[v,w] &= \frac{d}{ds}\bigg|_{s=0} \left(-\frac{1}{2}\int_\omega \left(\frac{\partial u_{\varphi+sw}}{\partial \nu}(x', e^{\varphi+sw})\right)^2 e^{\varphi+sw} v \, dx'\right) \\ &= -\int_\omega e^\varphi v \frac{\partial u_\varphi}{\partial \nu}(x', e^{\varphi(x')}) \frac{d}{ds}\bigg|_{s=0} \left(\frac{\partial u_{\varphi+sw}}{\partial \nu}(x', e^{\varphi+sw})\right) dx' \\ &\quad - \frac{1}{2}\int_\omega \left(\frac{\partial u_\varphi}{\partial \nu}(x', e^\varphi)\right)^2 e^\varphi v w \, dx' \end{aligned} \tag{4.2.12}$$

Let us now compute the derivative in the first integral of the right-hand side of (4.2.12). Observe that

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=0} \left(\frac{\partial u_{\varphi+sw}}{\partial \nu}(x', e^{\varphi+sw})\right) &= \frac{d}{ds}\bigg|_{s=0} (\nabla u_{\varphi+sw}(x', e^{\varphi+sw}) \cdot \nu_{\varphi+sw}) \\ &= \frac{d}{ds}\bigg|_{s=0} (\nabla u_{\varphi+sw}(x', e^{\varphi+sw})) \cdot \nu_\varphi \\ &\quad + \nabla u_\varphi(x', e^\varphi) \cdot \frac{d}{ds}\bigg|_{s=0} \nu_{\varphi+sw}, \end{aligned} \tag{4.2.13}$$

where ν_φ is given by (4.2.6) and

$$\nu_{\varphi+sw} = \frac{(-e^{\varphi+sw}\nabla_{\mathbb{R}^{N-1}}(\varphi+sw), 1)}{\sqrt{1 + |e^{\varphi+sw}\nabla_{\mathbb{R}^{N-1}}(\varphi+sw)|^2}}.$$

For the first term in the right-hand side of (4.2.13), thanks to Lemma 2.2.2 we have

$$\frac{d}{ds}\bigg|_{s=0} (\nabla u_{\varphi+sw}) = \nabla\left(\frac{d}{ds}\bigg|_{s=0} u_{\varphi+sw}\right),$$

whence

$$\left.\frac{d}{ds}\right|_{s=0}(\nabla u_{\varphi+sw}(x',e^{\varphi+sw})) = \nabla\widetilde{u}_w(x',e^{\varphi}) + D^2u_{\varphi}(x',e^{\varphi})(0',we^{\varphi}). \tag{4.2.14}$$

On the other hand, for the second term on the right-hand side of (4.2.13), a direct computation yields

$$\begin{aligned}\left.\frac{d}{ds}\right|_{s=0}\nu_{\varphi+sw} = &-\frac{e^{\varphi}}{\sqrt{1+|e^{\varphi}\nabla_{\mathbb{R}^{N-1}}\varphi|^2}}(\nabla_{\mathbb{R}^{N-1}}w + w\nabla_{\mathbb{R}^{N-1}}\varphi, 0)\\ &-\frac{(e^{\varphi})^2(w|\nabla_{\mathbb{R}^{N-1}}\varphi|^2 + \nabla_{\mathbb{R}^{N-1}}\varphi\cdot\nabla_{\mathbb{R}^{N-1}}w)}{1+|e^{\varphi}\nabla_{\mathbb{R}^{N-1}}\varphi|^2}\nu_{\varphi}.\end{aligned} \tag{4.2.15}$$

Finally, substituting (4.2.13)-(4.2.15) into (4.2.12) we readily obtain (4.2.11). □

In view of Definition 2.2.4, we consider a volume constraint. In the case of cartesian graphs, the volume of the domain Ω_{φ} associated to $\varphi \in C^2(\overline{\omega})$ is expressed by

$$\mathcal{V}(\varphi) := |\Omega_{\varphi}| = \int_{\omega} e^{\varphi}\, dx'. \tag{4.2.16}$$

The functional V is of class C^2 and for every $v, w \in C^2(\overline{\omega})$ it holds

$$\mathcal{V}'(\varphi)[v] = \int_{\omega} e^{\varphi}v\, dx', \qquad \mathcal{V}''(\varphi)[v,w] = \int_{\omega} e^{\varphi}vw\, dx'. \tag{4.2.17}$$

For $c > 0$ we define the manifold

$$M := \left\{\varphi \in C^2(\overline{\omega}) \ :\ \int_{\omega} e^{\varphi}\, dx' = c\right\}, \tag{4.2.18}$$

whose tangent space at any point $\varphi \in M$ is given by

$$T_{\varphi}M = \left\{v \in C^2(\overline{\omega}) \ :\ \int_{\omega} e^{\varphi}v\, dx' = 0\right\}. \tag{4.2.19}$$

We consider the restricted functional

$$I(\varphi) = T|_M(\varphi), \quad \varphi \in M. \tag{4.2.20}$$

If $\varphi \in M$ is a critical point for I, then there exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$T'(\varphi) = \mu\mathcal{V}'(\varphi). \tag{4.2.21}$$

Proposition 4.2.4. *Let $\varphi \in M$ and let $(\Omega_{\varphi}, u_{\varphi})$ be energy-stationary under a volume constraint. Then the Lagrange multiplier μ is negative and*

$$\frac{\partial u_{\varphi}}{\partial \nu} = -\sqrt{-2\nu} \quad \text{on } \Gamma_{\varphi} \tag{4.2.22}$$

Proof. Analogous to the proof of Proposition 3.2.3. □

For the second derivative of I, we have the following:

Lemma 4.2.5. *Let $\varphi \in M$ and let $v, w \in T_\varphi M$. If $(\Omega_\varphi, u_\varphi)$ is energy-stationary, under a volume constraint, then*

$$I''(\varphi)[v,w] = T''(\varphi)[v,w] - \mu \mathcal{V}''(\varphi)[v,w]. \tag{4.2.23}$$

Proof. Analogous to the proof of Lemma 3.2.4. □

4.3 Bounded cylinders and nondegeneracy of one-dimensional solutions

When $\varphi \equiv 0$, $\Gamma_\varphi = \Gamma_0$ is the intersection of the cylinder with the plane $x_N = 1$ and the domain Ω_0 is just the finite cylinder

$$\Omega_\omega := \Omega_0 = \omega \times (0,1). \tag{4.3.1}$$

Then, if f is a $C^{1,\alpha}$ function, any weak solution of (4.2.4) in Ω_ω is also a classical solution up to the regular part of the boundary. This follows by standard regularity theory by considering the boundary conditions and that $\partial\Omega_\omega$ is made by the union of three $(N-1)$-dimensional planes intersecting orthogonally (see also [60, Proposition 6.1]).

For suitable nonlinearities, we can find a solution of (4.2.4) in Ω_ω that depends only on x_N in the following way: first we apply some variational method to find a solution u of the ordinary differential equation

$$\begin{cases} -u'' = f(u) & \text{in } (0,1) \\ u'(0) = u(1) = 0 \end{cases} \tag{4.3.2}$$

and then set

$$u_\omega(x', x_N) := u(x_N), \quad (x', x_N) \in \Omega_\omega. \tag{4.3.3}$$

Remark 4.3.1. Recall that there is no critical Sobolev exponent in the real line. Therefore, the class of nonlinearities for which one-dimensional solutions exist is very wide, since there is no need to assume any growth condition for the ODE (4.3.2) to have a solution (which then yields a one-dimensional solution of (4.1.1) in Ω_ω via (4.3.2)). However, since we want to consider also nonsymmetric solutions, we always assume a subcritical growth on f, which is necessary for the variational arguments that yield general existence results in the N-dimensional domain Ω_ω.

As examples of suitable nonlinearities, we can take those of Proposition 3.3.1.

For our purposes, we need to consider one-dimensional solutions u_ω of (4.2.4) in Ω_ω that are nondegenerate, which means that the linearized operator

$$L_{u_\omega} = -\Delta - f'(u_\omega)$$

does not admit zero as an eigenvalue. In other words, u_ω is nondegenerate if the problem

$$\begin{cases} -\Delta\phi - f'(u_\omega)\phi &= \tau\phi & \text{in } \Omega_\omega \\ \phi &= 0 & \text{on } \Gamma_0 \\ \dfrac{\partial\phi}{\partial\nu} &= 0 & \text{on } \Gamma_{1,0} \end{cases} . \tag{4.3.4}$$

does not admit zero as an eigenvalue.

To analyze the spectrum of L_{u_ω} it is convenient to consider the following auxiliary one-dimensional eigenvalue problem:

$$\begin{cases} -z'' - f'(u_\omega)z = \alpha z & \text{in } (0,1) \\ z'(0) = z(1) = 0 \end{cases} \tag{4.3.5}$$

We denote the eigenvalues of (4.3.5) by α_i, for $i \in \mathbb{N}$. They correspond to the eigenvalues of the linear operator

$$\widehat{L}_{u_\omega}(z) = -z'' - f'(u_\omega)z$$

with the boundary conditions of (4.3.5).

We also consider the following Neumann eigenvalue problem in the domain $\omega \subset \mathbb{R}^{N-1}$:

$$\begin{cases} -\Delta_{\mathbb{R}^{N-1}}\psi &= \lambda\psi & \text{in } \omega \\ \dfrac{\partial\psi}{\partial\nu} &= 0 & \text{on } \partial\omega \end{cases} , \tag{4.3.6}$$

where $-\Delta_{\mathbb{R}^{N-1}} = -\sum_{i=1}^{N-1} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in $\mathbb{R}^{N-1}$, i.e., with respect to the variables $x_1, \dots, x_{N-1}$. We denote its eigenvalues by

$$0 = \lambda_0(\omega) < \lambda_1(\omega) \le \lambda_2(\omega) \le \dots \nearrow +\infty. \tag{4.3.7}$$

Lemma 4.3.2. *The spectra of L_{u_ω}, $\widehat{L}_{u_\omega}$ and $-\Delta_{\mathbb{R}^{N-1}}$ are related by*

$$\sigma(L_{u_\omega}) = \sigma(\widehat{L}_{u_\omega}) + \sigma(-\Delta_{\mathbb{R}^{N-1}}). \tag{4.3.8}$$

Proof. We begin by showing that $\sigma(L_{u_\omega}) \subset \sigma(\widehat{L}_{u_\omega}) + \sigma(-\Delta_{\mathbb{R}^{N-1}})$. Let $\tau \in \sigma(L_{u_\omega})$ and let $\phi \in H_0^1(\Omega_\omega \cup \Gamma_{1,0})$ be an associated eigenfunction, that is, ϕ is a weak solution to (4.3.4). Observe that, by standard regularity theory, considering the boundary conditions and since $\partial\Omega_\omega$ is made by the union of three $(N-1)$-dimensional planes intersecting orthogonally (see [60, Proposition 6.1]), we have that ϕ is a classical solution of (4.3.4) in $\overline{\Omega}_\omega$. Let λ be an eigenvalue of $-\Delta_{\mathbb{R}^{N-1}}$ with homogeneous Neumann boundary on ω and let ψ be an associated eigenfunction. Define

$$z(x_N) = \int_\omega \phi(x', x_N)\psi(x')\,dx', \quad x_N \in (0,1). \tag{4.3.9}$$

Then, differentiating with respect to x_N, using Green's formulas and the boundary conditions we have

$$-z'' = \int_\omega -\frac{\partial^2\phi}{\partial x_N^2}\psi\,dx'$$

$$
\begin{aligned}
&= \int_\omega (-\Delta\phi + \Delta_{\mathbb{R}^{N-1}}\phi)\psi \, dx' \\
&= \int_\omega f'(u_\omega)\phi\psi \, dx' + \int_\omega \tau\phi\psi \, dx' + \int_\omega \Delta_{\mathbb{R}^{N-1}}\psi\phi \, dx' \\
&= f'(u_\omega)z + \tau z - \lambda z, \qquad (4.3.10)
\end{aligned}
$$

and therefore $\tau - \lambda \in \sigma(\widehat{L}_{u_\omega})$. It then follows that

$$\tau = (\tau - \lambda) + \lambda \in \sigma(\widehat{L}_{u_\omega}) + \sigma(-\Delta_{\mathbb{R}^{N-1}}).$$

To show the reverse inclusion, let $\alpha \in \sigma(\widehat{L}_{u_\omega})$, $\lambda \in \sigma(-\Delta_{\mathbb{R}^{N-1}})$ and let z, ψ be, respectively, the associated eigenfunctions. Let

$$\phi(x', x_N) = z(x_N)\psi(x'), \quad x = (x', x_N) \in \Omega_\omega. \qquad (4.3.11)$$

Note that

$$
\begin{aligned}
-\Delta\phi &= -z''\psi - \Delta_{\mathbb{R}^{N-1}}\psi z \\
&= f'(u_\omega)z\psi + \alpha z\psi + \lambda z\psi \\
&= f'(u_\omega)\phi + (\alpha + \lambda)\phi. \qquad (4.3.12)
\end{aligned}
$$

By construction, it is immediate to check that ϕ satisfies the boundary conditions in (4.3.4). As a consequence, we deduce that

$$\alpha + \lambda \in \sigma(L_{u_\omega}),$$

and this concludes the proof. □

Corollary 4.3.3. *The problem* (4.3.4) *admits zero as an eigenvalue if and only if there exist* $i \in \mathbb{N}^+$, $j \in \mathbb{N}$ *such that*

$$\alpha_i + \lambda_j(\omega) = 0. \qquad (4.3.13)$$

Proof. Follows immediately from Lemma 4.3.2. □

Corollary 4.3.4. *A one-dimensional solution* u_ω *of* (4.2.4) *in* Ω_ω *is nondegenerate if both the following conditions are satisfied:*

(I) the eigenvalue problem (4.3.5) *does not admit zero as an eigenvalue;*

(II) $\lambda_1(\omega) > -\alpha_1$.

Proof. From condition *(I)* we have that

$$\alpha_i \neq 0 \quad \forall i \in \mathbb{N}^+, \qquad (4.3.14)$$

which means that zero is not an eigenvalue of (4.3.4) corresponding to a one-dimensional solution.

Since $\lambda_0(\omega) = 0$, $\lambda_1(\omega) > 0$ and since α_1 is the smallest eigenvalue of (4.3.5), by *(II)* and (4.3.14) it follows that $\lambda_i(\omega) + \alpha_j \neq 0$ for every pair of indices $(i, j) \in \mathbb{N}^+ \times \mathbb{N}$. Then, by Corollary 4.3.3, it follows that zero is not an eigenvalue of (4.3.4). □

4.4 Instability of bounded cylinders

We note that, in view of (4.2.19), when $\varphi \equiv 0$ the tangent space to the manifold M is given by

$$T_0 M = \left\{ v \in C^2(\overline{\omega}) \; : \; \int_\omega v \, dx' = 0 \right\}. \tag{4.4.1}$$

Since u_ω depends only on x_N, to simplify the notations we denote with a prime the derivative with respect to x_N:

$$u'_\omega(x_N) = u'_\omega(x', x_N) = \frac{\partial u_\omega}{\partial x_N}(x', x_N)$$

and

$$u''_\omega(x_N) = u''_\omega(x', x_N) = \frac{\partial^2 u_\omega}{\partial x_N^2}(x', x_N).$$

For a function $v \in T_0 M$, the function $\widetilde{u} \in H^1(\Omega_\omega)$ (see (4.2.9)) is a weak solution to

$$\begin{cases} -\Delta \widetilde{u} &= f'(u_\omega)\widetilde{u} & \text{in } \Omega_\omega \\ \widetilde{u} &= -u'_\omega(1) v & \text{on } \Gamma_0 \\ \dfrac{\partial \widetilde{u}}{\partial \nu} &= 0 & \text{on } \Gamma_{1,0} \end{cases}. \tag{4.4.2}$$

As before, by elliptic regularity, $\widetilde{u}$ is a classical solution. We also note that $\widetilde{u}$ is unique since u_ω is nondegenerate.

Theorem 4.4.1. *Let $\lambda_j > 0$ be any nontrivial eigenvalue for the Neumann problem (4.3.6) and let ψ_j be any L^2-normalized eigenfunction associated to λ_j. Let $\widetilde{u}_j \in H^1(\Omega_\omega)$ be the solution of (4.4.2) with $v = \psi_j$. Then the function*

$$h_j(x_N) = \int_\omega \widetilde{u}_j(x', x_N)\psi_j(x') \, dx', \quad x_N \in [0, 1] \tag{4.4.3}$$

satisfies

$$\begin{cases} -h''_j - f'(u_\omega)h_j = -\lambda_j h_j & \text{in } (0,1) \\ h_j(1) = -u'_\omega(1) \\ h'_j(0) = 0 \end{cases}. \tag{4.4.4}$$

Proof. We first observe that, as $\widetilde{u}_j = -u'_\omega(1)$ on Γ_0, we have

$$h(1) = \int_\omega -u'_\omega(1)\psi_j^2 \, dx' = -u'_\omega(1).$$

Now, differentiating with respect to x_N under the integral sign and using Green's formula, taking into account the boundary conditions, we get

$$-h''_j = \int_\omega -\frac{\partial^2 \widetilde{u}_j}{\partial x_N^2}\psi_j \, dx'$$

$$
\begin{aligned}
&= \int_\omega (-\Delta \widetilde{u}_j + \Delta_{\mathbb{R}^{N-1}} \widetilde{u}_j)\psi_j \, dx' \\
&= \int_\omega f'(u_\omega)\widetilde{u}_j\psi_j \, dx' + \int_\omega \Delta_{\mathbb{R}^{N-1}} \widetilde{u}_j \psi_j \, dx' \\
&= f'(u_\omega) h_j + \int_\omega \widetilde{u}_j \Delta_{\mathbb{R}^{N-1}} \psi_j \, dx' \\
&= f'(u_\omega) h_j - \lambda_j \int_\omega \widetilde{u}_j \psi_j \, dx' \\
&= f'(u_\omega) - \lambda_j h_j. \qquad (4.4.5)
\end{aligned}
$$

Finally, exploiting the Neumann condition for $\widetilde{u}_j$ on $\Gamma_{1,0}$, we readily obtain that $h'(0) = 0$. □

Remark 4.4.2. Note that for $\widetilde{u}_j$ and h_j as in Theorem 4.4.1 we have that

$$
\widetilde{u}_j(x', x_N) = h_j(x_N)\psi_j(x'). \qquad (4.4.6)
$$

Indeed:

$$
\begin{aligned}
-\Delta(h_j(x_N)\psi_j(x')) &= -h_j(x_n)\Delta_{\mathbb{R}^{N-1}}\psi_j(x') - h_j''(x_N)\psi_j(x') \\
&= \lambda_j h_j(x_N)\psi_j(x') + f'(u_\omega)h_j(x_N)\psi_j(x') - \lambda_j h_j(x_N)\psi_j(x') \\
&= f'(u_\omega)\widetilde{u}_j. \qquad (4.4.7)
\end{aligned}
$$

Moreover, the function $h_j\psi_j$ satisfies the boundary conditions in (4.4.2), therefore $h_j\psi_j$ coincides with $\widetilde{u}_j$.

Proposition 4.4.3. *Let $j \geq 1$, λ_j be a positive Neumann eigenvalue of $-\Delta_{\mathbb{R}^{N-1}}$ in ω and let h_j be as Theorem 4.4.1. Assume that $-\alpha_1 < \lambda_j$, where α_1 is the smallest eigenvalue of* (4.3.5). *Then it holds that*

$$
h_j > 0 \quad \text{in } (0,1). \qquad (4.4.8)
$$

Proof. Since $h_j'(0) = 0$, we can reflect h_j with respect to 0, obtaining an odd solution to the linear problem

$$
\begin{cases}
-h_j'' - f'(u_\omega)h_j + \lambda_j h_j = 0 & \text{in } (-1,1) \\
h_j(-1) = h_j(1) = -u_\omega'(1) > 0
\end{cases}. \qquad (4.4.9)
$$

Moreover, by reflection and thanks to (4.3.5), the first eigenvalue of the operator

$$
\widehat{L}_{u_\omega}[z] = -z'' - f'(u_\omega)z
$$

in $(-1,1)$ with boundary conditions $z(-1) = z(1) = 0$ is precisely α_1. Therefore the first eigenvalue of the linear operator

$$
\widetilde{L}_{u_\omega} g = -g'' - f'(u_\omega)g + \lambda_j g
$$

with zero boundary condition in $(-1,1)$ is $\beta_1 = \alpha_1 + \lambda_j$.

It is well-known that $\widetilde{L}_{u_\omega}$ satisfies the maximum principle whenever $\beta_1 > 0$, i.e., when $\lambda_j > -\alpha_1$. Therefore, by (4.4.9) and by the strong maximum principle we conclude that $h_j > 0$ in $(-1,1)$. □

The main result of this chapter is the following:

Theorem 4.4.4. *Let $\omega \subset \mathbb{R}^{N-1}$ be a smooth bounded domain. Let $f \in C^{1,\alpha}_{loc}(\mathbb{R})$ be such that there exists a positive one-dimensional nondegenerate solution u_ω of (4.2.4) in Ω_ω, and let h_1 be the solution to (4.4.4) with $j = 1$. Let $\lambda_1 = \lambda_1(\omega)$ be the first nontrivial Neumann eigenvalue of $-\Delta_{\mathbb{R}^{N-1}}$ in ω, let α_1 be the first eigenvalue of (4.3.5) and let ρ be the number defined as*

$$\rho := -f(u_\omega(0))h_1(0) - \lambda_1 \int_0^1 h_1 u'_\omega \, dx_N. \tag{4.4.10}$$

Assume that $\lambda_1 > -\alpha_1$. Then

(i) if $\rho < 0$, then $(\Omega_\omega, u_\omega)$ is an unstable energy-stationary pair;

(ii) if $\rho > 0$, then $(\Omega_\omega, u_\omega)$ is a stable energy-stationary pair.

Proof. We first observe that since $\frac{\partial u_\omega}{\partial \nu}$ is constant on Γ_0, then, by Proposition 2.2.5 we infer that $(\Omega_\omega, u_\omega)$ is energy-stationary.

Let $w \in T_0 M$ and assume without loss of generality that $\int_\omega w^2 \, dx' = 1$.

We first obtain a more convenient expression for $I''(0)[w,w]$.

For $j \in \mathbb{N}$, let $\widetilde{u}_j$ be the solution of (4.4.2) with $v = \psi_j$, and let h_j be the solution of (4.4.4). Then we can write

$$w = \sum_{j=1}^{\infty} (w, \psi_j)\psi_j,$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\omega)$. Moreover, we can check that

$$\widetilde{u} = \sum_{j=1}^{\infty} (w, \psi_j)\widetilde{u}_j$$

is the solution of (4.4.2) corresponding to w. Then, taking $\varphi \equiv 0$ in Lemma 4.2.5 and taking into account that, by Proposition 4.2.4 the Lagrange multiplier μ is given by

$$\mu = -\frac{1}{2}(u'_\omega(1))^2, \tag{4.4.11}$$

by Remark 4.4.2 and observing that $\nabla u_\omega \perp (\nabla_{\mathbb{R}^{N-1}} w, 0)$, we infer that

$$\begin{aligned} I''(0)[w,w] = &-\frac{1}{2}\int_\omega (u'_\omega(1))^2 w^2 \, dx' \\ &- \int_\omega u'_\omega(1) \left(\sum_{j=1}^{\infty}(w,\psi_j)h'_j(1)\psi_j\right)\left(\sum_{k=1}^{\infty}(w,\psi_k)\psi_k\right) dx' \\ &- \int_\omega u'_\omega(1)u''_\omega(1)w^2 \, dx' \\ &+ \frac{1}{2}(u'_\omega(1))^2 \int_\omega w^2 \, dx' \end{aligned}$$

$$= -u'_\omega(1) \int_\omega \left(\sum_{j=1}^{\infty} (w,\psi_j)^2 h'_j(1) \psi_j^2 \right) \, dx' - u'_\omega(1) u''_\omega(1). \tag{4.4.12}$$

Since u_ω is a solution to (4.2.4) we deduce that

$$I''(0)[w,w] = -u'_\omega(1) \int_\omega \left(\sum_{j=1}^{\infty} (w,\psi_j)^2 h'_j(1) \psi_j^2 \right) \, dx' + u'_\omega(1) f(0). \tag{4.4.13}$$

In particular, for $w = \psi_1$ we have

$$I''(0)[\psi_1,\psi_1] = -u'_\omega(1) h'_1(1) + u'_\omega(1) f(0). \tag{4.4.14}$$

Multiplying (4.4.4) (with $j = 1$) by u'_ω and integrating by parts we obtain

$$-(h'_1 u'_\omega)|_0^1 + \int_0^1 h'_1 u''_\omega \, dx_N = \int_0^1 (f'(u_\omega) - \lambda_1) h_1 u'_\omega \, dx_N. \tag{4.4.15}$$

Exploiting (4.3.2), integrating by parts and taking into account that $h_1(1) = -u'_\omega(1)$ we obtain

$$\begin{aligned}
-h'(1)u'_\omega(1) - \int_0^1 h'_1 f(u_\omega) \, dx_N &= \int_0^1 f'(u_\omega) u'_\omega h_1 \, dx_N - \lambda_1 \int_0^1 h_1 u'_\omega \, dx_N \\
&= (f(u_\omega)h_1)|_0^1 - \int_0^1 f(u_\omega) h'_1 \, dx_N \\
&\quad - \lambda_1 \int_0^1 h_1 u'_\omega \, dx_N \\
&= -f(0)u'_\omega(1) - f(u_\omega(0)) h_1(0) \\
&\quad - \int_0^1 f(u_\omega) h'_1 \, dx_N - \lambda_1 \int_0^1 h_1 u'_\omega \, dx_N,
\end{aligned} \tag{4.4.16}$$

whence

$$-h'_1(1)u'_\omega(1) = -f(0)u'_\omega(1) - f(u_\omega(0))h_1(0) - \lambda_1 \int_0^1 h_1 u'_\omega \, dx_N. \tag{4.4.17}$$

Substituting (4.4.17) into (4.4.13) we obtain

$$I''(0)[\psi_1,\psi_1] = -f(u_\omega(0))h_1(0) - \lambda_1 \int_0^1 h_1 u'_\omega \, dx_N = \rho. \tag{4.4.18}$$

The statement *(i)* immediately follows.

To prove *(ii)*, we first show that if $k > j$, then

$$h'_k(1) \geq h'_j(1), \tag{4.4.19}$$

and actually, the strict inequality holds if $\lambda_k > \lambda_j$.

Indeed, by definition, h_k and h_j satisfy, respectively,

$$-h_k'' - f'(u_\omega)h_k = -\lambda_k h_k, \tag{4.4.20}$$
$$-h_j'' - f'(u_\omega)h_j = -\lambda_j h_j. \tag{4.4.21}$$

Multiplying (4.4.20) by $_j$ and integrating on $(0,1)$ we obtain

$$\begin{aligned}\int_0^1 -h_k'' h_j \, dx_N &= \int_0^1 h_k' h_j' \, dx_N - (h_k' h_j)|_0^1 \\ &= \int_0^1 f'(u_\omega) h_k h_j \, dx_n - \lambda_k \int_0^1 h_j h_k. \end{aligned} \tag{4.4.22}$$

Multiplying (4.4.21) by h_k, integrating on $(0,1)$ and then subtracting the result from (4.4.22) we obtain

$$-(h_k' h_j - h_j' h_k)(1) = (\lambda_j - \lambda_k) \int_0^1 h_j h_k \, dx_N \leq 0, \tag{4.4.23}$$

because h_j and h_k are positive on $(0,1)$, by Proposition 4.4.3. The claim then follows by noting that $h_j(1) = h_k(1) = -u_\omega'(1) > 0$.

Now, exploiting (4.4.17) and (4.4.19) we have, for $w \in T_0 M$ such that $\int_\omega w^2 \, dx' = 1$:

$$\begin{aligned} I''(0)[w,w] &\geq -u_\omega'(1) h_1'(1) \int_\omega \left(\sum_{j=1}^\infty (w, \psi_j)^2 \psi_j^2 \right) dx' + u_\omega'(1) f(0) \\ &= -u_\omega'(1) h_1'(1) + u_\omega'(1) f(0) \\ &= -f(u_\omega(0)) h_1(0) - \lambda_1 \int_0^1 h_1 u_\omega' \, dx_N \\ &= \rho. \end{aligned} \tag{4.4.24}$$

Since w is arbitrary, *(ii)* immediately follows. □

As a direct consequence of Theorem 4.4.4 we can prove a sharp stability threshold for the relative torsion problem, that is, for $f \equiv 1$.

Theorem 4.4.5. *Let $\omega \subset \mathbb{R}^{N-1}$ be a smooth bounded domain. Let u_ω be the unique solution of (4.2.4) with $f \equiv 1$. Let λ_1 be the first nontrivial Neumann eigenvalue for $-\Delta_{\mathbb{R}^{N-1}}$ in ω. Then there exists a number $\beta \approx 1,439$ such that*

(i) if $\lambda_1(\omega) < \beta$, then $(\Omega_\omega, u_\omega)$ is an unstable energy-stationary pair;

(ii) if $\lambda_1(\omega) > \beta$, then $(\Omega_\omega, u_\omega)$ is a stable energy-stationary pair.

Proof. When $f \equiv 1$, the eigenvalue problem (4.3.5) has only positive eigenvalues, and therefore the condition $\lambda_1 > -\alpha_1$ is automatically satisfied.

The only solution of

$$\begin{cases} -\Delta u &= 1 & \text{in } \Omega_\omega \\ u &= 0 & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial \nu} &= 0 & \text{on } \Gamma_{1,0} \end{cases} \tag{4.4.25}$$

is

$$u_\omega(x_N) = \frac{1 - x_N^2}{2}. \tag{4.4.26}$$

Clearly, as $u_\omega'(1) = -1$ and $f \equiv 1$, then for any $j \in \mathbb{N}$ the problem (4.4.4) reduces to

$$\begin{cases} -h_j'' + \lambda_j h_j = 0 & \text{in } (0,1) \\ h_j(1) = 1 & \\ h_j'(0) = 0. & \end{cases}, \tag{4.4.27}$$

whose unique solution is

$$h_j(x_N) = \frac{1}{\cosh(\sqrt{\lambda_j})} \cosh(\sqrt{\lambda_j} x_N). \tag{4.4.28}$$

In particular, taking $j = 1$ and exploiting (4.4.26) we can compute ρ explicitly:

$$\rho = -\frac{1}{\cosh(\sqrt{\lambda_1})} + \frac{\lambda_1}{\cosh(\sqrt{\lambda_1})} \int_0^1 \cosh(\sqrt{\lambda_1} x_N) x_N \, dx_N \tag{4.4.29}$$

Integrating by parts we immediately check that

$$\int_0^1 \cosh(\sqrt{\lambda_1} x_N) x_N \, dx_N = \frac{\sinh(\sqrt{\lambda_1})}{\sqrt{\lambda_1}} - \frac{\cosh(\sqrt{\lambda_1})}{\sqrt{\lambda_1}} + \frac{1}{\lambda_1}, \tag{4.4.30}$$

and therefore

$$\rho = \sqrt{\lambda_1} \tanh(\sqrt{\lambda_1}) - 1. \tag{4.4.31}$$

Now let $g : [0, +\infty) \to \mathbb{R}$ be the function $g(t) = \sqrt{t} \tanh(\sqrt{t}) - 1$. We have that $g(0) = -1$, $g(t) \to +\infty$ as $t \to +\infty$, and that g is monotone. Therefore, g has a unique root β. The result then follows immediately from Theorem 4.4.4. □

We observe that if the domain ω is such that $\lambda_1(\omega) = \beta$, then $(\Omega_\omega, u_\omega)$ is a degenerate (in the sense of critical points) energy-stationary pair.

Let us fix the domain ω for a moment. It is well-known that if we consider scalings $t\omega$, for $t > 0$, we have $\lambda_1(t\omega) = \frac{1}{t^2}\lambda_1(\omega)$. Therefore, scaling the fixed domain ω we can find a critical number $\bar{t}$ such that $\lambda_1(\bar{t}\omega) = \beta$, where a change in the stability of $(\Omega_{t\omega}, u_\omega)$ as an energy-stationary pair occurs, possibly giving rise to a bifurcating domain Ω_φ, where

φ is a non-constant function. Moreover, Ω_φ, if it exists, is also energy-stationary and, consequently, a solution to the relatively overdetermined problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega_\varphi \\ u = 0 & \text{on } \Gamma_\varphi \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_{1,\varphi} \\ \dfrac{\partial u}{\partial \nu} = \text{constant} & \text{on } \Gamma_\varphi \end{cases} \tag{4.4.32}$$

exists in Ω_φ. We infer that the "shape" of ω does not prevent nor imply that a degeneracy occurs, but only affects how large the critical scaling factor $\bar{t}$ must be.

Indeed, in dimension $N = 2$, this is related with [34, Theorem 1.1], discussed in Section 2.3.3.1. Indeed, for $N = 2$ we can consider the cylinder $\Sigma = (0, 2\pi) \times (0, +\infty)$, where the interval $(0, 2\pi)$ plays the role of ω, and the domain $\Omega = (0, 2\pi) \times (0, 1)$. Scaling the interval $(0, 2\pi)$, we can find $\bar{t}$ such that $\lambda_1(0, 2\bar{t}\pi) = \beta$, as just pointed out, and a bifurcation may occur. In fact, in view of the results of [34] (translated to our setting as described in Section 2.3.3.1), we infer that a bifurcation indeed occurs when varying the height of the cylinder, giving rise to a family ϕ_s, for $|s|$ small, such that a solution of (4.4.32), with $\varphi = \phi_s$, exists, for all such s (the constant in the overdetermined condition depending on s).

Now, observe that fixing the height of the bounded domain and scaling the base $(0, 2\pi)$ is equivalent to fixing the base and varying the height. Therefore, from the threshold β we obtain in Theorem 4.4.5 we could obtain the critical value for bifurcation in [34], and vice-versa.

We infer from the previous remarks that the appearance of instability/bifurcation is related to the "thickness" of the cylinder.

Theorem 4.4.6. *Let $\omega \subset \mathbb{R}^{N-1}$ be a smooth bounded domain, and let u_ω be a positive nondegenerate one-dimensional solution of (4.2.4) in Ω_ω. Let $\lambda_1(\omega)$ be the first nontrivial Neumann eigenvalue of $-\Delta_{\mathbb{R}^{N-1}}$ in ω and let α_1 be the first eigenvalue of (4.3.5). If the nonlinearity f satisfies $f(0) = 0$ and*

$$\lambda_1(\omega) > \max\{-\alpha_1, \|f'(u_\omega)\|_\infty\}, \tag{4.4.33}$$

then $(\Omega_\omega, u_\omega)$ is a stable energy-stationary pair.

Proof. Let $w \in T_0 M$ be such that $\int_\omega w^2 \, dx' = 1$. Since $\lambda_1 > -\alpha_1$, we can argue as in the proof of Theorem 4.4.4. In particular, since $f(0) = 0$ (by assumption), we have, arguing as in (4.4.24):

$$I''(0)[w, w] \geq -u_\omega'(1) h_1'(1). \tag{4.4.34}$$

Now, since $h_1'' = (\lambda_1 - f'(u_\omega)) h_1$ in $(0, 1)$ and $h_1 > 0$ in $[0, 1]$ (by Proposition 4.4.3), then, thanks to the assumption $\lambda_1 > \sup_{(0,1)} |f'(u_\omega)|$, it follows that $h_1'' > 0$ in $[0, 1]$. In particular, since $h_1'(0) = 0$, it follows that $h_1'(1) > 0$. Then, by (4.4.34), $I''(0)[w, w] > 0$. Since w is arbitrary, it follows that $(\Omega_\omega, u_\omega)$ is a stable energy-stationary pair. □

4.4.1 Numerical analysis

In Theorem 4.4.5 we exhibited a precise threshold for stability in the case $f \equiv 1$. On the other hand, our Theorem 4.4.4 gives a theoretical threshold for stability, namely the number ρ (see (4.4.10)), for which we do not have a precise analytical characterization as in (4.4.31). In fact, it could happen that there exists some nonlinearity f for which $(\Omega_\omega, u_\omega)$ is always a stable energy-stationary pair.

In this section, we report numerical results that show, for the Lane-Emden nonlinearity $f(u) = u^p$, with $p = 3, 4, 5$, the existence of a range of values of λ_1 for which the pair $(\Omega_\omega, u_\omega)$ is an unstable energy-stationary pair.

The following algorithm was implemented in Wolfram Mathematica:

(i) given the nonlinearity f (input), obtain by numerical methods an interpolation function that solves $-u'' = f(u)$ in $(-1, 1)$ with homogeneous Dirichlet boundary condition;

(ii) compute α_1;

(iii) compute h and ρ as a function of $\lambda := \lambda_1 = -\alpha_1 + j\tau$, where $j = 1, \dots, 100$ and the incremental step τ is a parameter to be specified;

(iv) numerically interpolate the values of ρ obtained in (iii) to obtain a function $\rho(\lambda)$;

(v) compute the first root of $\rho(\lambda)$ and plot the graph of $\rho(\lambda)$.

We begin by pointing out that, in the case of the torsion problem, the numerical results agree with Theorem (4.4.5), see Figure 4.1.

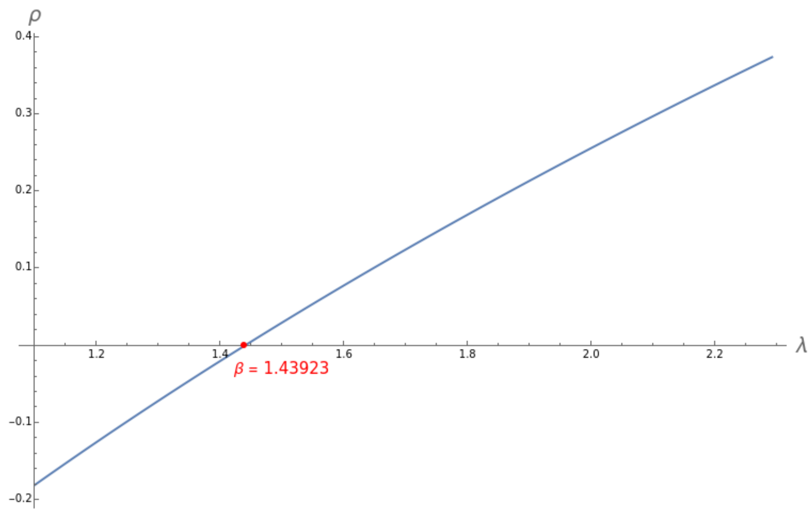

Figure 4.1: Numerical plot of the number ρ as a function of λ for $f \equiv 1$

Our numerical results show that also for the case of the Lane-Emden nonlinearity $f(u) = u^p$, with $p = 3, 4, 5$, instability may occur. Indeed, for p fixed, we obtain that

$\rho = \rho(\lambda)$ becomes singular at $\lambda = -\alpha_1 > 0$, tending to $-\infty$ as $\lambda \to -\alpha_1^+$. Moreover, in the range we considered, $\rho(\lambda)$ is strictly increasing and has a unique root β. See Figures 4.2-4.4. In the range of λ where ρ is negative, namely on $(-\alpha_1, \beta)$, Theorem 4.4.4 yields the instability of the one-dimensional solution.

Finally, it is worth noticing that the singularity at $\lambda = -\alpha_1$ becomes stronger as p increases.

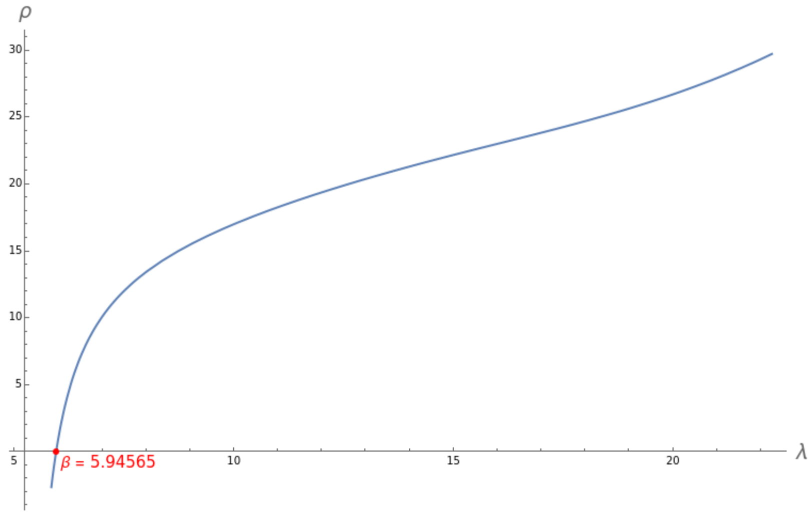

Figure 4.2: Numerical plot of the number ρ as a function of λ for $f(u) = u^3$.

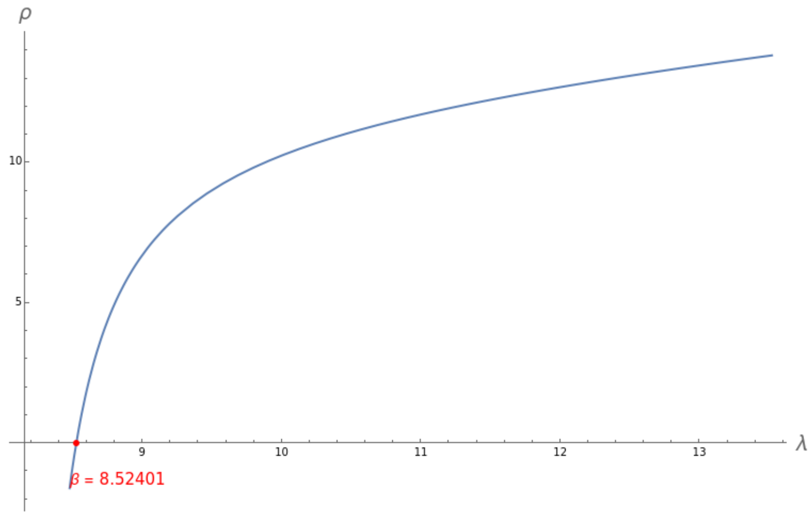

Figure 4.3: Numerical plot of the number ρ as a function of λ for $f(u) = u^4$.

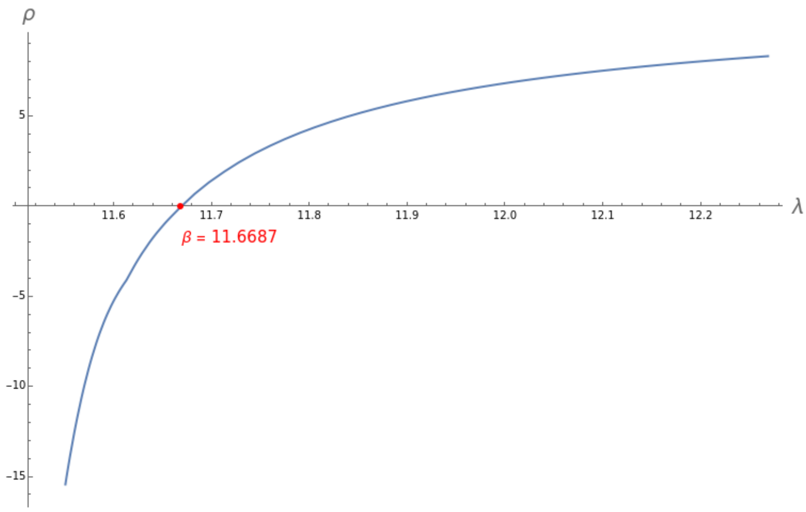

Figure 4.4: Numerical plot of the number ρ as a function of λ for $f(u) = u^5$.

Chapter 5

Morse index and bifurcation from one-dimensional solutions in bounded cylinders

5.1 Introduction

In this chapter, we focus on bounded cylinders. As we have seen in Chapter 4, there exist functions that inherit geometric features of the cylinder. More precisely, its axis defines a distinctive direction, and we study solutions that depend only on the variable corresponding to that direction. Our aim here is to show that, for some classes of cylinders, there exist other positive solutions that exhibit a more complex behavior.

We start by recalling the necessary notation. We denote a point in $\mathbb{R}^N$ by $x = (x', x_N)$, where $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$. Let $\omega \subset \mathbb{R}^{N-1}$, $N \geq 2$, be a smooth bounded domain. We consider in $\mathbb{R}^N$ the cylinder spanned by ω,

$$\Sigma_\omega := \omega \times (0, +\infty) = \{x = (x', x_N) \in \mathbb{R}^N \ : \ x' \in \omega, \ x_N > 0\},$$

and set

$$\Gamma := \{x = (x', x_N) \in \Sigma_\omega \ : \ x_N = 1\}.$$

The bounded cylinder Ω_ω of height one is defined as

$$\Omega_\omega := \{x = (x', x_N) \in \Sigma_\omega \ : \ x_N < 1\}. \tag{5.1.1}$$

Finally, we set $\Gamma_1 := \partial\Omega_\omega \setminus \overline{\Gamma}$.

The set Γ is usually said to be the relative boundary of Ω_ω with respect to the cylinder Σ_ω. We are interested in studying relative Dirichlet problems (i.e., with the Dirichlet condition assumed only on Γ) for semilinear elliptic equations in Ω_ω. More precisely, for

$f \in C^{1,\alpha}(\mathbb{R})$, $\alpha \in (0,1)$, we consider the problem

$$\begin{cases} -\Delta u &= f(u) & \text{in } \Omega_\omega \\ u &= 0 & \text{on } \Gamma \\ \dfrac{\partial u}{\partial \nu} &= 0 & \text{on } \Gamma_1 \end{cases}, \tag{5.1.2}$$

where ν denotes the outer unit normal vector to $\partial\Omega_\omega$. Under appropriate hypotheses on the nonlinearity f, this problem admits a variational formulation that is analogous to the one for pure Dirichlet problems. Indeed, the weak solutions of (5.1.2), whenever they exist, are the critical points of the functional

$$J(v) = \frac{1}{2}\int_{\Omega_\omega} |\nabla v|^2 \, dx - \int_{\Omega_\omega} F(v)\, dx, \quad v \in H_0^1(\Omega_\omega \cup \Gamma_1), \tag{5.1.3}$$

where $F(\tau) = \int_0^\tau f(s)\, ds$ and $H_0^1(\Omega_\omega \cup \Gamma_1)$ is the subspace of $H^1(\Omega_\omega)$ of functions whose trace vanishes on Γ.

There are special solutions that inherit the geometry of the cylinder, in the sense that they depend solely on the “height" variable x_N. We call them one-dimensional solutions and denote them by u_ω. They can be obtained by solving, for example by some standard variational method, the ordinary differential equation

$$\begin{cases} -u'' = f(u) & \text{in } (0,1) \\ u'(0) = u(1) = 0 \end{cases} \tag{5.1.4}$$

and then setting

$$u_\omega(x', x_N) = \widetilde{u}(x_N), \quad (x', x_N) \in \Omega_\omega, \tag{5.1.5}$$

where $\widetilde{u}$ is a solution of (5.1.4).

The one-dimensional solutions can be considered analogous to the radial solutions in radially symmetric domains. In particular, our relative Dirichlet problem (5.1.2) corresponds to the relative Dirichlet problem in a spherical sector,

$$\begin{cases} -\Delta u &= f(u) & \text{in } \Omega_D \\ u &= 0 & \text{on } D \\ \dfrac{\partial u}{\partial \nu} &= 0 & \text{on } \partial\Omega_D \setminus \overline{D} \end{cases}, \tag{5.1.6}$$

where Ω_D is the spherical sector of radius one in $\mathbb{R}^N$ spanned by a domain D on the unit sphere $\mathbb{S}^{N-1}$ (see [22] for more details). Thus, as for the radial solutions, it is natural to ask whether or not all positive solutions of (5.1.2) are one-dimensional.

The stability/instability results of Chapter 4 (see also [2]) obtained for the torsion problem and for other semilinear elliptic problems indicate that solutions that are not one-dimensional should exist in some classes of cylinders, depending also on the prescribed nonlinearity. The results of this chapter confirm this idea.

5.2 Computation of the Morse index for one-dimensional solutions

We begin with a remark on the regularity of the solutions of (5.1.2).

Remark 5.2.1. Since f is assumed to be of class $C^{1,\alpha}$, then every weak solution of (5.1.2) is also a classical solution up to the boundary, i.e., belongs to $C^2(\overline{\Omega_\omega})$. This follows by standard regularity theory, considering the boundary conditions and the fact that $\partial\Omega_\omega$ is made by the union of three $(N-1)$-dimensional smooth manifolds (with boundary) intersecting orthogonally (see [60, Proposition 6.1]).

Our methods are all based on the Morse index of a solution, and on the fact that the decomposition of the spectrum of the linearized operator obtained in Lemma 4.3.2 allows for an easy computation of the Morse index of one-dimensional solutions.

We recall that the Morse index $m(u_\omega)$ of u_ω is given by the number of negative eigenvalues of the linearized operator

$$L_{u_\omega} = -\Delta - f'(u_\omega) \tag{5.2.1}$$

in the space $H^1_0(\Omega_\omega \cup \Gamma_1)$ (see [26]). Note that if we consider only the eigenvalues of L_{u_ω} whose corresponding eigenfunctions are one-dimensional, then they are the eigenvalues of

$$\begin{cases} -z'' - f'(u_\omega)z = \alpha z & \text{in } (0,1) \\ z'(0) = z(1) = 0 \end{cases} . \tag{5.2.2}$$

Then, we denote by $m^{x_N}(u_\omega)$ the Morse index of u_ω in $H^1_{0,x_N}(\Omega_\omega \cup \Gamma_1)$, which is the subspace of functions in $H^1_0(\Omega_\omega \cup \Gamma_1)$ which depend only on the x_N variable.

Theorem 5.2.2. *Let u_ω be a one-dimensional solution of (5.1.2) in Ω_ω. Then it holds:*

1. *if $m^{x_N}(u_\omega) = 0$, then $m(u_\omega) = 0$;*
2. *if $m^{x_N}(u_\omega) = 1$, then*
$$m(u_\omega) = 1 + \#\{j \geq 1 \,:\, \lambda_j < -\alpha_1\} \tag{5.2.3}$$
3. *if $m^{x_N}(u_\omega) = n$, then*
$$m(u_\omega) = n + \sum_{i=1}^{n} \#\{j \geq 1 \,:\, \lambda_j < -\alpha_i\}. \tag{5.2.4}$$

Proof. We use the decomposition obtained in Lemma 4.3.2.

If $m^{x_N}(u_\omega) = 0$, then $\alpha_i \geq 0$ for every $i \in \mathbb{N}^+$. Since $\lambda_j \geq 0$ for every $j \in \mathbb{N}$, we immediately see that $\alpha_i + \lambda_j \geq 0$ for every pair of indices (i,j), and therefore no eigenvalue of L_{u_ω} in $H^1_0(\Omega_\omega \cup \Gamma_1)$ can be negative. This proves *(i)*.

For *(ii)*, we notice that, since $m^{x_N}(u_\omega) = 1$, by definition we have

$$\alpha_1 < 0 \leq \alpha_i \quad \forall i \geq 2.$$

Therefore, since $\lambda_0 = 0$ and $\lambda_j > 0$ for $j \geq 1$, we have

$$\begin{aligned} m(u_\omega) &= \#\{k \geq 1 \,:\, \lambda_k < 0\} \\ &= \#\{j \geq 0 \,:\, \alpha_1 + \lambda_j < 0\} \\ &= 1 + \#\{j \geq 1 \,:\, \lambda_j < -\alpha_1\}. \end{aligned} \tag{5.2.5}$$

For *(iii)*, since $\lambda_0 = 0$ we have

$$\begin{aligned} m(u_\omega) &= \#\{k \geq 1 \,:\, \lambda_k < 0\} \\ &= \#\{i \geq 1, j \geq 0 \,:\, \lambda_j < -\alpha_i\} \\ &= \sum_{i=1}^{n} \#\{j \geq 0 \,:\, \lambda_j < -\alpha_i\} \\ &= n + \sum_{i=1}^{n} \#\{j \geq 1 \,:\, \lambda_j < -\alpha_i\}, \end{aligned} \tag{5.2.6}$$

which completes the proof. □

Concerning the Neumann eigenvalue problem

$$\begin{cases} -\Delta_{\mathbb{R}^{N-1}}\psi = \lambda\psi & \text{in } \omega \\ \dfrac{\partial\psi}{\partial\nu} = 0 & \text{on } \partial\omega \end{cases} \tag{5.2.7}$$

we observe that if we fix a domain $\omega \subset \mathbb{R}^{N-1}$ and consider the scaled domain

$$t\omega := \{tx' \in \mathbb{R}^{N-1} \,:\, x' \in \omega\}, \quad t > 0, \tag{5.2.8}$$

then for the Neumann eigenvalues of $-\Delta_{\mathbb{R}^{N-1}}$ in $t\omega$ we have

$$\lambda_j(t\omega) = \frac{1}{t^2}\lambda_j(\omega). \tag{5.2.9}$$

Corollary 5.2.3. *Let $\omega \subset \mathbb{R}^{N-1}$ be a smooth bounded domain and let u_ω be any one-dimensional solution of (5.1.2) in Ω_ω. For $t > 0$, let u_{t_ω} be the corresponding one-dimensional solution in the scaled domain $\Omega_{t\omega}$. If $m^{x_N}(u_\omega) > 0$, then*

$$m(u_{t_\omega}) \to \infty \quad \textit{as} \quad t \to +\infty.$$

Proof. It follows immediately from (5.2.9) and Theorem 5.2.2. □

5.3 Existence of solutions which are not one-dimensional

In this section, we first prove the existence of solutions that are not one-dimensional by using the Morse index computation performed in the previous section and then focus on the bifurcation results.

5.3.1 Solutions of Morse index one

We start by presenting some conditions on the existence of positive solutions of Morse index one in $H_0^1(\Omega_\omega \cup \Gamma_1)$ for (5.1.2). The result that follows is well-known, but we report it here for completeness and to illustrate some cases in which our Theorem 5.3.3 applies. It is a straightforward adaptation to our setting of [26, Theorem 3.4], and the proof is obtained by applying the Mountain Pass Theorem to the functional J defined in (5.1.3).

Proposition 5.3.1. *Let $f \in C^{1,\alpha}(\mathbb{R})$ be a nonlinearity satisfying the following conditions:*

1. *f is superlinear:*
$$f'(s) > \frac{f(s)}{s} \quad \forall s \neq 0; \tag{5.3.1}$$
2. *f has subcritical growth: there exist $a \in L^{\frac{2N}{N+2}}(\Omega_\omega)$ and $b > 0$ such that*
$$|f(s)| < a(x) + b|s|^p \quad \forall s \in \mathbb{R}, \tag{5.3.2}$$
where $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$ and $p > 1$ if $N = 2$;
3. *$f(s) = o(|s|)$ as $s \to 0$;*
4. *f satisfies the Ambrosetti-Rabinowitz condition: there exist $\tau > 2$ and $r > 0$ such that for $|s| \geq r$ it holds*
$$0 < \tau F(s) \leq s f(s), \tag{5.3.3}$$
where F is the primitive of f;

Then there exists a positive solution of (5.1.2) of Morse index one in the space $H_0^1(\Omega_\omega \cup \Gamma_1)$.

A classical example of a function satisfying the conditions of Proposition 5.3.1 is the Lane-Emden nonlinearity $f(s) = s^p$, $s > 0$.

Under the additional condition
$$\frac{1}{2} f(s)s - F(s) \geq C \quad \forall s \in \mathbb{R}. \tag{5.3.4}$$
we can obtain a Morse index one solution of (5.1.2) which also minimizes the energy functional J defined in (5.1.3) among all possible solutions (it is then said to be a least-energy solution). This is achieved by minimizing J on the Nehari manifold, see e.g. [26].

Remark 5.3.2 (On the uniqueness of u_ω)**.** The same arguments can be carried out to yield existence of positive one-dimensional solutions of Morse index one in the space $H^1_{0,x_N}(\Omega_\omega \cup \Gamma_1)$ (or even of least energy in this space). Moreover, we observe that (with a slight abuse of notation) a positive one-dimensional solution u_ω of (5.1.4) in $(0,1)$ can be extended by evenness to $(-1,1)$ to yield a positive solution $\widetilde{u_\omega}$ of the Dirichlet problem
$$\begin{cases} -u'' = f(u) & \text{in } (-1,1) \\ u(-1) = u(1) = 0 \end{cases}. \tag{5.3.5}$$

Now, under the additional hypothesis

$$f(s) > 0 \quad \text{for} \quad s > 0, \tag{5.3.6}$$

and since we are assuming *(i)* of Proposition 5.3.1, by [57, Theorem 2.4] the extension $\widetilde{u_\omega}$ is the unique solution of (5.3.5), and thus u_ω is the unique solution of (5.1.4).

Note that, since there is no critical Sobolev exponent in $H_0^1(-1,1)$, the class of nonlinearities for which one-dimensional solutions exist is very wide. Indeed, there is no need to assume any growth condition on f in order to obtain a solution for the ODE (5.1.4) (which then yields a one-dimensional solution of (5.1.2) via (5.1.5)). However, without some growth condition on f we cannot consider other solutions in the whole space $H_0^1(\Omega_\omega \cup \Gamma_1)$.

In view of Remark 5.3.2, by Theorem 5.2.2 we can now prove the existence of solutions that are not one-dimensional.

Theorem 5.3.3. *Let us assume that the nonlinearity f is such that*

1. *there exists a unique positive one-dimensional solution u_ω of* (5.1.2)*;*
2. *there exists a positive solution $\bar{u}$ of* (5.1.2) *with Morse index one in the space $H_0^1(\Omega_\omega \cup \Gamma_1)$.*

Then, if either the first eigenvalue $\alpha_1 = \alpha_1(u_\omega)$ of (5.2.2) *is nonnegative or*

$$\lambda_1(\omega) < -\alpha_1, \tag{5.3.7}$$

then the positive solution $\bar{u}$ is not one-dimensional.

Proof. Suppose first that $\alpha_1 \geq 0$. Then $m^{x_N}(u_\omega) = 0$ and therefore, by Theorem 5.2.2, $m(u_\omega) = 0$. On the other hand, if $\alpha_1 < 0$ and $\lambda_j(\omega) < -\alpha_1$, then, again by Theorem 5.2.2, it follows that $m(u_\omega) \geq 2$.

In both cases, $m(u_\omega) \neq 1 = m(\bar{u})$, thus $\bar{u} \neq u_\omega$, and then, since u_ω is the only positive one-dimensional solution, $\bar{u}$ is not one-dimensional. □

The alternative $\alpha_1 = \alpha_1(u_\omega)$ nonnegative in (5.3.7) is due since if $\alpha_1 \geq 0$, then (5.3.7) does not make sense. However, it would be interesting to find some nonlinearities for which *1.* and *2.* hold together with $\alpha_1 \geq 0$. Indeed, the condition (5.3.1) for the uniqueness result [57, Theorem 2.4] implies that $\alpha_1 < 0$ (see [26, Section 3.2.1]).

Let us observe that the eigenvalue problem (5.2.2) depends only on the nonlinearity f and not on the domain ω which spans the cylinder. In turn, $\lambda_1(\omega)$ depends only on the domain ω but not on the nonlinearity f. Hence (5.3.7) shows that the existence of Morse index one solutions which are not one-dimensional depends on an interplay between the geometry of the basis ω of the cylinder (related to $\lambda_1(\omega)$) and the nonlinearity f.

Remark 5.3.4. It may happen that (5.1.2) has more than one positive solution with Morse index one. Then the conclusion of Theorem 5.3.3 holds for any such solution.

Recall that for the Neumann eigenvalues of $-\Delta_{\mathbb{R}^{N-1}}$ in $t\omega$ we have

$$\lambda_j(t\omega) = \frac{1}{t^2}\lambda_j(\omega).$$

Therefore, since the eigenvalue α_1 in (5.3.7) does not depend on the basis of the cylinder, by dilating ω by a sufficiently big parameter t we have that the condition (5.3.7) holds in the cylinder $\Omega_{t\omega}$. Hence we get the following result:

Corollary 5.3.5. *Let $\omega \subset \mathbb{R}^{N-1}$ be a smooth bounded domain. Let us assume that the hypotheses (i) and (ii) of Theorem 5.3.3 hold. Then there exists $\bar{t} > 0$ such that for all $t > \bar{t}$ any positive solution $\bar{u}_t$ in $\Omega_{t\omega}$ with Morse index one is not one-dimensional.*

Note that the previous result holds whatever the shape of ω is. However, how large $\bar{t}$ must be to produce positive solutions of (5.1.2) which are not one-dimensional does depend on ω, through the eigenvalue $\lambda_1(t\omega)$.

5.3.2 Bifurcation from one-dimensional solutions

In this section, we show that solutions which are not one-dimensional may arise from bifurcation from one-dimensional solutions when the $(N-1)$-dimensional domain ω is scaled.

We assume that f satisfies the conditions given in Proposition 5.3.1 and (5.3.6), and let u_ω be the unique positive one-dimensional solution of (5.1.2) (see Remark 5.3.2).

Since $m^{x_N}(u_\omega) = 1$, again by Remark 5.3.2, then the first eigenvalue α_1 of the problem

$$\begin{cases} -z'' - f'(u_\omega)z = \alpha z & \text{in } (0,1) \\ z'(0) = z(1) = 0 \end{cases} \tag{5.3.8}$$

is negative (see Section 5.2). At the same time, if we scale the domain ω by a factor $t > 0$, the Neumann eigenvalues of $-\Delta_{\mathbb{R}^{N-1}}$ in ω scale as

$$\lambda_j(t\omega) = \frac{1}{t^2}\lambda_j(\omega), \tag{5.3.9}$$

for every $j \in \mathbb{N}$. Then, in view of the decomposition of the spectrum of L_{u_ω} given by Lemma 4.3.2 and Corollary 4.3.3, scaling the domain ω we can find degenerate (in the space $H^1_0(\Omega_\omega \cup \Gamma_1)$) one-dimensional solutions.

Our aim is to make use of bifurcation theory to obtain the existence of positive solutions of (5.1.2) which are not one-dimensional. Our argument is inspired by [43] (see also [26]). The first step is to show that, if a bifurcation occurs, it does not give rise to another one-dimensional solution. This is true if we show that positive one-dimensional solutions u of (5.1.2) with Morse index $m^{x_N}(u) = 1$ are always nondegenerate in the space $H^1_{0,x_N}(\Omega_\omega \cup \Gamma_1)$.

Proposition 5.3.6. *Let $u_\omega \in H_0^1(\Omega_\omega \cup \Gamma_1)$ be a positive one-dimensional solution of (5.1.2). Assume that the conditions (i)-(iv) of Proposition 5.3.1 and (5.3.6) hold. Then u_ω is nondegenerate in the space $H^1_{0,x_N}(\Omega_\omega \cup \Gamma_1)$.*

Proof. Let α_i, $i \in \mathbb{N}^+$ be the eigenvalues of $L_u = -\Delta - f'(u_\omega)$ in $H^1_{0,x_N}(\Omega_\omega \cup \Gamma_1)$, i.e., the eigenvalues of (5.2.2). Then there are associated eigenfunctions z_i which satisfy

$$\begin{cases} -z_i'' - f'(u_\omega)z_i = \alpha_i z_i & \text{in } (0,1) \\ z_i'(0) = z_i(1) = 0 \end{cases} . \tag{5.3.10}$$

Since $m^{x_N}(u_\omega) = 1$, then only α_1 is negative. We want to show that $\alpha_2 > 0$. For the sake of contradiction, assume that $\alpha_2 = 0$. Then there exists a solution z_2 of the problem

$$\begin{cases} -z_2'' - f'(u_\omega)z_2 = 0 & \text{in } (0,1) \\ z_2'(0) = z_2(1) = 0 \end{cases} \tag{5.3.11}$$

We can assume, without loss of generality, that $z_2(0) > 0$. By the Courant nodal line theorem, since z_2 is a second eigenfunction, it changes sign only once in $(0,1)$ and therefore $z_2 < 0$ in a neighborhood of $x_N = 1$. Then, by Hopf's Lemma, $z_2'(1) > 0$, while $u_\omega'(1) < 0$. Now, using (5.1.2) and (5.3.11) we obtain:

$$\begin{aligned} 0 &> u_\omega'(1)z_2'(1) \\ &= u_\omega'(1)z_2'(1) - \underbrace{u_\omega'(0)z_2'(0)}_{=0} \\ &= \int_0^1 (u_\omega' z_2')'\, dx_N \\ &= \int_0^1 u_\omega'' z_2'\, dx_N + \int_0^1 u_\omega' z_2''\, dx_N \\ &= -\int_0^1 f(u_\omega) z_2'\, dx_N - \int_0^1 f'(u_\omega) u_\omega' z_2\, dx_N \\ &= -\int_0^1 (f(u_\omega)z_2)'\, dx_N \\ &= f(u_\omega(0))z_2(0) - f(u_\omega(1))z_2(1) \\ &= f(u_\omega(0))z_2(0) \\ &> 0, \end{aligned} \tag{5.3.12}$$

since $z_2(1) = 0$ and $f(s) > 0$ for $s > 0$. Therefore $\alpha_2 > 0$ and u_ω is nondegenerate in the space $H^1_{0,x_N}(\Omega_\omega \cup \Gamma_1)$. ☐

Now we will consider scalings $t\omega$ of the original domain ω which spans the cylinder and use t as a bifurcation parameter. As already remarked, these scalings allow us to find degenerate one-dimensional solutions u_{t_ω}, for suitable values of the scaling parameter t. This, in turn, produces changes in the Morse index of the one-dimensional solution u_{t_ω} which allows to use bifurcation theory.

Since bifurcation theory requires a fixed functional setting, independent of the bifurcation parameter, we use the original cylinder Ω_ω as the fixed domain. So, in view of Remark 5.2.1, we can work in the space $C^{1,\alpha}(\Omega_\omega \cup \Gamma_1)$, and bring back to Ω_ω the problems in the scaled domains $\Omega_{t\omega}$.

We denote by Γ_t and $\Gamma_{1,t}$ the parts of the boundary of $\Omega_{t\omega}$ corresponding to Γ and Γ_1 in Ω_ω, respectively. Moreover, we denote by $C_0^{1,\alpha}(\Omega_\omega \cup \Gamma_1)$ the subspace of functions in $C^{1,\alpha}(\overline{\Omega_{t\omega}})$ which vanish on Γ.

In what follows, points in the scaled domain $\Omega_{t\omega}$ are denoted $y = (y', x_N)$, while we write $x = (x', x_N)$ points in the original cylinder Ω_ω. It is clear that, in fact, $y' = tx'$.

For $t > 0$, consider the diffeomorphism

$$h_t : (y', x_N) \in \overline{\Omega_{t\omega}} \mapsto \left(\frac{y'}{t}, x_N\right) \in \overline{\Omega_\omega},$$

which induces the map $h_t^* : C_0^{1,\alpha}(\Omega_{t\omega} \cup \Gamma_{1,t}) \to C_0^{1,\alpha}(\Omega_\omega \cup \Gamma_1)$ given by

$$h_t^*(v)(x', x_N) = v(h_t^{-1}(x', x_N)) = v(tx', x_N). \tag{5.3.13}$$

We denote by u_ω the unique positive one-dimensional solution of (5.1.2) in Ω_ω, while the positive one-dimensional solution in $\Omega_{t\omega}$ is denoted by u_{t_ω}, for $t > 0$.

For $t > 0$, consider the function

$$u_t = h_t^*(u_{t_\omega}) = u_{t_\omega} \circ h_t^{-1} \in C_0^{1,\alpha}(\Omega_\omega \cup \Gamma_1)$$

The function u_t is the solution of

$$\begin{cases} D_t(u) &= f(u) & \text{in } \Omega_\omega \\ u &= 0 & \text{on } \Gamma \\ \dfrac{\partial u}{\partial \nu} &= 0 & \text{on } \Gamma_1 \end{cases} \tag{5.3.14}$$

where the operator D_t is defined as

$$D_t(u) = h_t^*(-\Delta((h_t^*)^{-1}(u))).$$

In fact, the boundary conditions in (5.3.14) are immediately seen to be satisfied by u_t. On the other hand, note that $u_{t_\omega} = u_t \circ h_t = (h_t^*)^{-1}(u_t)$, and thus, since $-\Delta u_{t_\omega} = f(u_{t_\omega})$ in $\Omega_{t\omega}$, we have

$$-\Delta((h_t^*)^{-1}(u_t))(y) = f((h_t^*)^{-1}(u_t))(y). \tag{5.3.15}$$

Then, writing $y = h_t^{-1}(x)$, for $x \in \overline{\Omega_\omega}$, (5.3.14) immediately follows.

Observe that the solution u_t of (5.3.14) is one-dimensional, that is, $u_t(x', x_N) = u_t(x_N)$ for all $x = (x', x_N) \in \Omega_\omega$.

We give the following definition:

Definition 5.3.7. Let $\bar{t} > 0$ and let $u_{\bar{t}\omega}$ be the unique positive one-dimensional solution of (5.1.2) in $\Omega_{\bar{t}\omega}$. We say that a not-one-dimensional bifurcation occurs at $(u_{\bar{t}\omega}, \bar{t})$ if in any neighborhood of $(h^*_{\bar{t}}(u_{\bar{t}\omega}, \bar{t})$ in $V \times (0, +\infty)$ there exists a solution of (5.1.2) which is not one-dimensional, with $h^*_{\bar{t}}$ defined in (5.3.13).

Recall that we denote by λ_j, $j \in \mathbb{N}$, the eigenvalues of

$$\begin{cases} -\Delta_{\mathbb{R}^{N-1}}\psi &= \lambda\psi & \text{in } \omega \\ \dfrac{\partial\psi}{\partial\nu} &= 0 & \text{on } \partial\omega \end{cases} . \tag{5.3.16}$$

The general bifurcation result we obtain is the following:

Theorem 5.3.8. *Assume that* (5.3.6) *and (i)-(iv) of Proposition 5.3.1 hold. Moreover, let the domain $\omega \subset \mathbb{R}^{N-1}$ be such that at least one eigenvalue $\bar{\lambda}$ of* (5.3.16) *is simple (i.e., the corresponding eigenspace has dimension one). Then there exists $\bar{t} > 0$ such that a not-one-dimensional bifurcation occurs at $(u_{\bar{t}\omega}, \bar{t})$. Moreover, the bifurcating solutions are positive.*

Proof. As anticipated, the idea is to use bifurcation theory. To this aim, we rewrite the equation $D_t(u) = f(u)$ as $u = T_t(u)$, where

$$T_t : C_0^{1,\alpha}(\Omega_\omega \cup \Gamma_1) \to C_0^{1,\alpha}(\Omega_\omega \cup \Gamma_1)$$

is the operator defined as

$$T_t(u) = D_t^{-1}(f(u)), \quad u \in C_0^{1,\alpha}(\Omega_\omega \cup \Gamma_1).$$

Observe that

$$T_t(u) = h^*_t\left((-\Delta)^{-1}\left((h^*_t)^{-1}(f(u))\right)\right)$$

Due to classical regularity results, this operator is well-defined and compact.

For any open bounded set $U \subset C_0^{1,\alpha}(\Omega_\omega \cup \Gamma_1)$ such that $I - T_t \neq 0$ on ∂U, the Leray-Schauder degree $\deg(I - T_t, U, 0)$ is well-defined. Moreover, by [58, Proposition 2, page 243] we have

$$\deg(I - T_t, U, 0) = \deg(I - P_t, (h^*_t)^{-1}(U), 0), \tag{5.3.17}$$

where $P_t(v) = (-\Delta)^{-1}(f(v))$ is defined in $C_0^{1,\alpha}(\Omega_{t\omega} \cup \Gamma_{1,t})$ and is compact. Moreover, P_t is differentiable at u_{t_ω}, with

$$P'_t(u_{t_\omega}) = (-\Delta)^{-1}(f'(u_{t_\omega})). \tag{5.3.18}$$

By the decomposition of the spectrum of $L_{u_{t_\omega}}$, given by Lemma 4.3.2, and Corollary 4.3.3, if t is such that

$$\frac{1}{t^2}\lambda_j \neq -\alpha_1 \quad \forall j \in \mathbb{N},$$

then, by definition, u_{t_ω} is nondegenerate, and thus $I - P'_t(u_{t_\omega})$ is invertible. Now let $\bar{t}$ be such that

$$\frac{1}{\bar{t}^2}\bar{\lambda} = -\alpha_1.$$

We have

$$\deg(I - P_t, V, 0) = \deg(I - P'_t(u_{t_\omega}), V, 0) = (-1)^{m(u_{t_\omega})} \tag{5.3.19}$$

for t in a small enough neighborhood of $\bar{t}$, $t \neq \bar{t}$, where $m(u_{t_\omega})$ is the Morse index of u_{t_ω} in $H_0^1(\Omega_{t\omega} \cup \Gamma_{1,t}$ and V is a neighborhood of u_{t_ω} in $C_0^{1,\alpha}(\Omega_{t\omega} \cup \Gamma_{1,t})$ such that u_{t_ω} is the unique solution of $(I - P_t)(v) = 0$ in $\overline{V}$.

Now, since $\bar{\lambda}$ is simple, $m(u_{t_\omega})$ changes by one when crossing $\bar{t}$ (see Theorem 5.2.2), and thus, in view of (5.3.17) and (5.3.19) we have, for $\varepsilon > 0$ small enough,

$$\deg(I - T_{t_k-\varepsilon}, W_{t_k-\varepsilon}, 0) = -\deg(I - T_{t_k+\varepsilon}, W_{t_k+\varepsilon}, 0), \tag{5.3.20}$$

where W_t is a neighborhood of $w_t = h_t^*(u_{t_\omega})$ in the space $C_0^{1,\alpha}(\Omega_\omega \cup \Gamma_1)$.

The change in the degree at the point $(w_{\bar{t}}, \bar{t})$ implies that a bifurcation occurs at $\bar{t}$. That the bifurcating solutions are not one-dimensional follows from the nondegeneracy of the one-dimensional solutions u_{t_ω} in the spaces $H^1_{0,x_N}(\Omega_{t\omega} \cup \Gamma_{1,t})$ (see Proposition 5.3.6).

Finally, we show that the bifurcating solutions are positive. Let us denote them by U_t. Then we have that the sequence of functions $w_t = h_t^*(U_t)$ converges to u_ω in the norm of $C_0^{1,\alpha}(\Omega_\omega \cup \Gamma_1)$ as $t \to \bar{t}$. Since $u_\omega > 0$ in $\Omega_\omega \cup \Gamma_1$ and $\frac{\partial u_\omega}{\partial \nu} < \eta < 0$ on $\overline{\Gamma}$, then there exists $\delta_1 > 0$ such that

$$\frac{\partial w_t}{\partial x_N} < 0 \quad \text{in } [\delta_1, 1]$$

and

$$w_t > 0 \quad \text{in } \omega \times [0, \delta_1],$$

for t sufficiently close to $\bar{t}$. Then, since h_t^* preserves the sign, we deduce that the bifurcating solutions U_t are positive. The proof is complete. □

The meaning of Theorem 5.3.8 is that, by dilating sufficiently the starting domain ω, we find cylinders $\Omega_{t\omega}$, close to the cylinder $\Omega_{\bar{t}\omega}$ (or possibly equal to $\Omega_{\bar{t}\omega}$) in which positive solutions w_t of (5.1.2) which are not one-dimensional exist.

If the domain $\omega \subset \mathbb{R}^{N-1}$ admits a sequence of simple Neumann eigenvalues for (5.3.16), we have a multiple bifurcation result.

Theorem 5.3.9. *Assume that* (5.3.6) *and (i)-(iv) of Proposition 5.3.1 hold. Moreover let the domain $\omega \subset \mathbb{R}^{N-1}$ be such that the problem* (5.3.16) *admits a subsequence $\{\lambda_k\}_{k\in\mathbb{N}^+}$ of simple eigenvalues. Then there exists a strictly increasing sequence $\{t_k\}_{k\in\mathbb{N}^+}$, with $t_k \to \infty$, such that a not-one-dimensional bifurcation at $(u_{t_k\omega}, t_k)$.*

Proof of Theorem 5.3.9. Recall that when we scale the domain ω which spans the cylinder, the k-th Neumann eigenvalue is $\lambda_k(t\omega) = \frac{1}{t^2}\lambda_k(\omega)$. By Theorem 5.2.2, when $\lambda_k(t\omega)$

crosses $-\alpha_1$, we have that the Morse index of the one-dimensional solution changes by one (since $\lambda_k(\omega)$ is simple). Then, setting t_k to be the scaling factor such that

$$\lambda_k(t_k\omega) = -\alpha_1,$$

we conclude the proof applying Theorem 5.3.8. □

It can be shown that the eigenvalues of the Neumann-Laplacian are all simple for "almost all" bounded smooth domains in the Euclidean space (see [45, Example 6.4] for the precise statement). Thus our Theorem 5.3.8 is meaningful, for it says that we have the existence of nonsymmetric solutions in almost every cylinder.

On the other hand, there are some relevant domains with simple geometry where there are eigenvalues with multiplicity, e.g. balls or other domains with some symmetry. Nonetheless, we still get the existence of nonsymmetric solutions by slightly modifying our arguments. In fact, it suffices to work in some space of functions that are invariant with respect to some symmetry group action.

For example, let $\omega = B$ be a ball in $\mathbb{R}^{N-1}$ and consider the subspace

$$\mathcal{E} = \{w \in C^1_0(\Omega_\omega \cup \Gamma_1) \;:\; w(x', x_N) = w(g(x'), x_N)\; \forall g \in SO(N-1)\}$$

of functions that are invariant with respect to the action of the symmetry group $SO(N-1)$. It is known that, in this case, the Neumann eigenvalues of $-\Delta_{\mathbb{R}^{N-1}}$ in $\omega = B$ are all simple in the space $\mathcal{E}$. Indeed, let λ_j be a Neumann eigenvalue and let $E_j = \{\psi_{j,1}, \dots, \psi_{j,k}\}$ be the associated eigenspace. Then every $\psi_{j,m} \in E_j$ is obtained from the action of some element of $SO(N-1)$ on one given eigenfunction, say $\psi_{j,1}$. We say that λ_j is a $SO(N-1)$-simple eigenvalue.

Considering eigenvalues that are simple with respect to some group of symmetries, we are able to repeat the proof of Theorem 5.3.8 (in the modified functional setting) and still obtain bifurcation results when ω has some symmetry. For more details on the matter of multiplicity of eigenvalues of the Laplacian in symmetric domains, we refer to [56] and the references therein.

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