The use of GeoGebra for exploring some constructions of Euclid, Archimedes and Apollonius

Maria Chiara Cibien[1], Agnese Del Zozzo[2] and Enrico Rogora[3]

[1]University of Trento, PopMat Laboratory, Italy; mariachiara.cibien@gmail.com

[2]University of Trento, Italy

[3]Sapienza University of Rome, Italy

This paper describes a mathematical laboratory path designed and tested as part of a university course of History of Mathematics. This laboratory path was developed in collaboration with PopMat[1] laboratory of the University of Trento. For carrying out the activities that make up the laboratory, we used the dynamic geometry software GeoGebra[2], which, thanks to its characteristics, facilitated the participants in the process of reconstructing the mathematical thought of Euclid, Archimedes and Apollonius. The purely geometric approach to the problems considered with GeoGebra makes it possible to design teaching/learning opportunities in which it is required a drastic change of point of view (compared to the usual one, that is analytical), which we consider particularly stimulating and useful in the training process of future teachers.

Keywords: History of mathematics teaching and learning, learning mathematics in digital environments, mathematics teacher education.

Introduction

The mathematics laboratory we describe in this article aims to facilitate the reconstruction process of some aspects of the thought of Euclid, Archimedes and Apollonius through the use of GeoGebra (Brigaglia et al., 2021) and to develop a geometric intuition on conics based on their synthetic properties, closer to that of ancient geometers. This process requires a conscious use of software. If it is not accompanied by an appropriate historical contextualization and a comparison with the original texts, it can instead contribute to raising further barriers. Examples will be provided in the following.

The teaching/learning material that we have prepared for these activities has been used during the 2022 spring semester at Sapienza University of Rome, in an undergraduate course in the History of Mathematics. In the context of a History of Mathematics course, the purpose of this mathematics laboratory was to assist students to retrace some fundamental steps in the development of conics in antiquity and to familiarize with the synthetic geometric approach employed by Archimedes and Apollonius to investigate their properties. In the context of perspective teacher's training, the purpose was also that of getting students (that may will become teachers) used to employ materials taken from the history of mathematics to design laboratories that stimulate the learning of mathematics within manifold cultural contexts and to become aware of the possibilities to observe the cognitive processes of students by watching the way they use a dynamic geometry software.

The activities are distributed along seven meetings, performed using the GeoGebra Classroom[3] environment, that is "a virtual platform through which teachers can assign interactive and engaging tasks for students" (*GeoGebra*[4]). This platform has made it possible to make activities accessible at

[1] https://webmagazine.unitn.it/ricerca/80304/pop-mat-popularization-of-mathematics-lab
[2] https://www.geogebra.org/
[3] https://www.geogebra.org/m/hncrgruu
[4] https://www.geogebra.org/

the most convenient times and places for students. For each activity, the teachers let two weeks pass before discussing with the class the results of student's work, recorded on the platform. Participants could perform the proposed activities, individually or in small groups, according to personal preferences. There were about sixty students enrolled in the course and participation in laboratory activities was not obligatory. At least thirty students attended each meeting. The seven meetings are aimed at deepening some of the topics covered in the course. More precisely:

- Euclid: application of areas; construction of the mean proportional between two segments.
- Archimedes: the elementary properties of the parabola exposed in the first propositions of his work "Quadrature of the parabola".
- Apollonius: geometric algebra used for studying cone sections.

All the activities are available online[5]. The current version has been modified as a result of the data collected after the first experimentation. Comments on students' answers and observations, which we will comment on below, refer to the original version. This paper provides an example of history as a tool for preparing teaching/learning resources, in particular the principle to suitably modify historical arguments to reproduce the genesis of key mathematical concepts and techniques in order to help students to overcome epistemological obstacles attested to by history (Brousseau, 1997; Jankvist, 2008; Brigaglia et al., 2021). The data collected provide some empirical evidence about the adequacy with respect to the goals that we had in mind, but we plan to make further research on this in the future, under a suitably chosen theoretical framework.[6]

Design criteria: Contents and educational purposes

With the works of Euclid, Archimedes and Apollonius, ancient mathematics reached the apex of its development. After them, an inexorable decline began. Only during the XVII century did the substantial recovery of ancient knowledge be completed. Between the XVII and XVIII centuries new original research directions appeared, leading to the discovery of differential calculus and the use of the new calculus to solve, albeit not yet rigorously, an enormous amount of problems that remained out of reach of ancient mathematics. Only in the XIX century, however, we witnessed a real methodological advance, achieved through a profound critical revision of the foundations of Mathematics, that, with Set theory, finally allowed a rigorous treatment of infinity and provided solid foundations to differential calculus.

In our opinion, the demanding process necessary to recover the ancient vision of geometry has good reasons to be pursued, especially in the training of future teachers. For example: it helps to practice the good habit of changing point of view on a problem and searching for connections with other problems; it promotes geometric intuition and highlights its heuristic value; it helps to perceive mathematics as a cultural object and to appreciate the historical process of its development and the effectiveness of using it for assisting the teaching/learning process. As we said before we did not properly validate yet the adequacy of our activities against these goals but the data we collected this year helped us to formulate precise research questions to be answered and we plan to do further research on this under a suitable theoretical framework.

[5] https://docs.google.com/document/d/1mI1Sy8Yp8Eg70Zs-kXbt8KnDgxrVujow/edit

[6] For allowing the presentation of the activities in a workshop at CERME 13 we are translating part of them in English. This work in progress can also be viewed online (https://www.geogebra.org/m/ypvrk2jc).

This process of approaching the way of thinking of ancient geometers is facilitated by a dynamic geometry software but it requires a conscious use of it. Otherwise, as we said already in the introduction, it can raise further misunderstanding. For example, in the fifth meeting we face the classical problem of inserting two segments in continuous proportion between two assigned segments. To do this, new geometric operations are required that cannot be performed with ruler and compass (Cox, 2012). Menechmus noted that it is possible to perform the construction by intersecting conic sections. This is easily done with GeoGebra Classic's default tools. A non-conscious use of software, which puts ruler and compass constructions on the same level as constructions with conics, can move us away from the point of view of the ancient geometers and from the understanding of their problems. We have tried to highlight the risks of this uncritical use of tools throughout all the activities, especially those of seventh meeting, aimed at confronting the synthetic geometric point of view with the analytic one. The typical problem that we posed in this meeting was “Reformulate this exercise as Apollonio could have posed it and solve it as he could have solved it. Compare the analytic solution of the original question with the geometric solution of the new one”.

Observing how students use GeoGebra (choice of tools, ways of interacting with the software, etc.) helps to shed light on their attitudes towards the geometrical objects and constructions they are dealing with. This may help the understanding of their learning processes. Students' interactions with dynamic geometry software may provide teachers with simple and useful benchmarks for monitoring their learning process in the field of History of Mathematics, for example about their understanding of the process of development of key mathematical concepts and techniques. This makes it possible to discuss the diversity between the ancient and the modern point of view more concretely. For example, in the first meeting, dedicated to the exploration of some properties of the arbelos, it is shown how the extension of a plane figure can be represented both as a number (as is natural from our point of view) and as a simpler equivalent figure, which can be a square or a circle, as was natural for Archimedes (see Figure 1). The approach of the ancient geometers shifts our attention from algebraic manipulations to the geometric constructions of the equivalent figures and the observation of the tools used by the students allows us to easily highlight the register in which they work.

Figure 1: Example of how extension is treated by students as a number (left) or directly as an equivalent figure (right)

Another example which, in our opinion, is even more striking, is the following. The start screen of GeoGebra Classic, at least up to version 5, shows the Cartesian grid by default. Initially, this does not bother students: it even seems that they do not see the axes or at least they do not feel the need to remove them. However, at a certain point in the laboratory process, many students remove the axes or ask how to do it (see Figure 2). They have reached the awareness that it is an element unrelated to the mathematics they are doing, and they are annoyed by it.

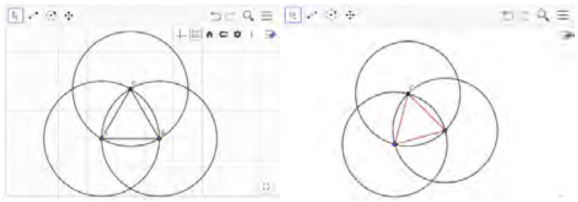

Figure 2: Example of how, initially, some students prefer to keep the Cartesian axes as a reference (left) and then delete them later (right)

Here is a simple example of a benchmark (that shows a difference in attitude) that can help the teacher in monitoring students' learning processes. Proper validation of this claim and a suitable understanding of the learning processes behind it is left as a future research question.

With GeoGebra it is possible to deactivate any subset of all the available tools and this allows, for example, to ask for the solution of the same exercise using different sets of tools, in order to become aware of the different ways of looking at a problem and its solution. This opportunity was used in almost every meeting. In the seventh meeting, for example, we proposed some problems on the parabola taken from a high school textbook, designed to be solved with analytic geometry. As we said already, we discussed whether and how it is possible to reformulate them in a geometric way and what tools can be used to solve them. We just report two samples of students' answers that highlight their perception of how the choice of tools is connected to the way of understanding the solution and of considering the objects with which they are working:

Student 1: I was initially bewildered by not having the tool "Polygon" available, but when I noticed the availability of the "Drawing circle" tool I got the right idea about how to complete the task.

Student 2: Having few tools available may appear as a restriction but actually forced me to consider the problem from the right perspective and conceive more quickly a proper solution.

This mathematics laboratory offers a way to better understand the way of thinking of the ancient geometers. But a question naturally arises: can this elementary mathematics from an ancient point of view be useful and interesting also outside the context of the history of mathematics, in particular in teacher training? On the basis of a first analysis of empirical data, and of intensive and informal interviews, we assume an affirmative answer, that we plan to investigate by well-designed experiments within a suitable theoretical framework. We think that teaching and learning elementary mathematics from an ancient point of view is useful for teacher training for several reasons, including the following (see also Brigaglia et al., 2021 for the specific role of GeoGebra for this):

- expanding the perspective from which one looks at the objects being studied can also be useful for applying analytical techniques more effectively and consciously, suggesting how to save calculations and how to control them geometrically;
- develops a geometric intuition which is useful not only for solving problems but also for posing new ones and for finding hidden links between different areas of mathematics.

As confirmation of our confidence in the usefulness of practicing the synthetic point of view to imagine new problems, we bring an example that we observed using the materials of the first meeting in a course for in-service teachers. We asked to imagine a problem that generalized the one considered by Archimedes of circling the arbelos and we got the suggestion, among others, to consider the triangular arbelos, shown in Figure 3, and to find a construction to "triangling" it. Here is the solution they found (the two red equilateral triangles on the right are equivalent to triangular arbelos, i.e. to the light brown parallelogram).

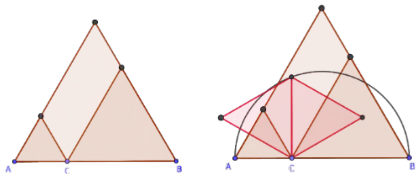

Figure 3: Triangular arbelos [first meeting in a course for in-service teachers].

For the purpose of the numerical calculation of triangular arbelos' extension, the construction of the equivalent figure, that is of the two red equilateral triangles shown in Figure 3, is useless. The algebraic manipulation of the formulas for involved figures' areas is very simple and has no need for the construction of that particular equivalent figure, but it cannot be denied that we observed something more from the geometric solution that cannot be reduced to numbers. We found it interesting to observe that the geometric solution was generally liked better than the numerical one and was perceived as a true and unexpected discovery. Why? Because it allowed in-service teachers to see beyond the original figure and produce a new way with a strong geometric link with the original. Moreover, this new figure may suggest other figures and unexpected links with other problems. For example, it can be suggested further explorations asking questions such as "what is the locus traced by one of the vertices of the crimson triangles?" and discover that it is an arc of an ellipse; "what are the principal axes of this ellipse?" and discover that they can be traced with a simple geometric construction; etc… The analytical approach would have reduced the problem to a solely algebraic calculation. The geometric interpretation allowed us to connect the initial figure to many others, opening the possibility for the formulation of conjectures.

So far we have presented a general overview of the laboratory and the design criteria, providing examples of some students' production. In the next section we propose a more detailed description of each meeting content.

Meetings content

Meeting 1[7] - *Exploring the arbelos with Archimedes:* The geometric approach to measuring the extension of figures

The goal of this meeting is to familiarize with GeoGebra. From a historical point of view, the activities center around the construction of a simpler figure which is equivalent to a given one and to highlight the difference with the problem of computing the numerical value of an area.

Meeting 2[8] - *Designing a dynamic geometry software with Euclid:T*he axiomatic method

In the first part of this meeting proposes activities aimed at highlighting different types of interaction with the figures produced by GeoGebra through dragging, in order to show how different attitudes towards the solution of geometric problems are reflected in different ways of dragging the figures. For the activities proposed in this part we have drawn inspiration from the literature (e.g. Baccaglini-Frank & Mariotti, 2010). Conversely, by observing these attitudes, it is possible to obtain useful information on the students' learning path. Dragging is only one aspect of the possible didactic use of

[7] https://www.geogebra.org/m/jtfu5rsr
[8] https://www.geogebra.org/m/yzwgyadv

the software, but they are not the only possible examples. The class discussion of the observations collected in the first part of the activities of this second meeting allowed a fruitful disquisition in this regard. In the second part of the meeting, instead, some features of the software are used to link the analysis of Euclid's postulates to GeoGebra architecture. This software, in fact, gives many opportunities for stimulating reflections on the technological importance of the axiomatic point of view for the construction of useful models of reality (Russo, 2004). By customizing GeoGebra's toolbar, students are invited to build constructions using sets of different tools and then to reflect on the compatibility of these tools and on the usefulness of having a minimal and efficient set at their disposal. As for the other activities, students are asked to give motivations and written reflections.

Meeting 3[9]- *Discovering the geometrical properties of the rectangular cone sections with Apollonius:* The geometric properties of the points which belong to the plane section of a circular cone

Shifting the focus from Euclid to Apollonius, the third meeting involves the construction of the sections of a rectangular cone, using GeoGebra 3D. In this meeting the students approach two novelties at the same time: on the one hand the work of Apollonius and, on the other hand, a new functionality of GeoGebra along with the analysis of the subtile learning difficulties one might run into when working with three-dimensional objects represented on the screen of a computer (Accascina & Rogora, 2006). GeoGebra 3D offers useful tools to help the process of moving from two to three dimensions. From the historical point of view, the characteristic elements of a conical section (latus rectum, axis, etc.) are introduced as done by Apollonius, and the activities allow to study the properties that link them (symptoms).

Meeting 4[10] - *Making constructions with Greek geometers, without verbal indications:* Constructions with ruler and compass relating to the conics present in the works of Archimedes and Apollonius

In this meeting, students are asked to use the [Locus] tool of GeoGebra for building a conic starting from a set of characteristic points and straight lines that define it (e.g. focus and directrix) and, vice versa, construct with rule and compass the special points and lines associated to a given conic. In all these constructions the most of the attention is on the instruments that are allowed to be used and the discussion focuses on which were available to the Greeks and on which are available in GeoGebra, and the different nature of problems when different sets of instruments are permitted. In this meeting, students are not bound to follow pre-established steps as in the previous activities. During the laboratory, we hoped that observing their spontaneous way in reconstructing the provided figures would allow us to set benchmarks that marked different phases in the process of transition from an operational understanding to a structural understanding of the concept of locus. However, we did not get conclusive evidence that such a benchmark should be found and we think that further investigation with younger students, for which the idea of geometric locus has not yet crystallized, are needed.

Meeting 5[11] - *Duplicate the cube with Menaechmus:* The connection between the duplication of the cube and the Hippocrates problem concerning the insertion of two segments in continuous proportion with two assigned segments; Menechmus' solution of Hippocrates' problem with the use of conics

[9] https://www.geogebra.org/m/e2cynnwr
[10] https://www.geogebra.org/m/rnkbpcfe
[11] https://www.geogebra.org/m/bhq7ns4e

To address the problem of duplicating the cube, the fifth meeting includes a series of guided activities to explain Menechmus' approach which solved the problem by intersecting conics. During the laboratory students had to recognize, without using analytic geometry, the relationship between the geometric construction of two segments in continuous proportion with two assigned ones, obtained by intersecting suitable conics, and the corresponding algebraic operations (extraction of roots).

Meeting 6[12]- *Identifying conic sections and geometrical loci with Apollonius:* An algebraic approach, without formulas and without numbers, to geometric problems

In this meeting, returning to Apollonius and using GeoGebra and GeoGebra 3D through *guided* and no longer *free* exploration activities, the plane constructions of parabola, hyperbola and ellipse are related to the sections of a cone. Students discover that Apollonius' construction allows for the construction of a conic and its supplementary conic at the same time with respect to a given direction. The importance of the concept of supplementary conic to develop the geometric theory of conics without introducing complex numbers, but using the ideal elements defined through the use of supplementary conics, will be recognized only in the first half of the XIX century with the works of Poncelet (Poncelet, 1822). The activities planned for this meeting represent the conclusion of the topic *Apollonius and the Conics* and aim at shed light on the need to develop an algebraic approach to deal with identification problems.

Meeting 7[13]- *Moving the point of view on a geometric problem with Descartes:* Comparisons between the modern and the ancient point of view

The laboratory ends with activities that relate the vision of ancient mathematicians with what is usually done at school (in Italy). The activities concern simple exercises taken from a current school manual, which can be transposed into the language of ancient mathematics, in order to understand the relationships between analytical and synthetic geometry. A link is created between the topics of the History of Mathematics course and some familiar topics faced at school, so that prospective teachers can appreciate the importance of knowing the historical development of the theory of conics for their future teaching practice. A question systematically proposed to students is: "How Apollonio would have formulated the exercise and how it would have solved the problem". We think that the answers collected with this question are very interesting to be discussed in class. Proper validation of this claim is left as a future research question.

Feedback collection

Various phases of feedback collection were planned during the mathematics laboratory we are describing, to provide students with useful information regarding the activities carried out, to inform them on their progress, and to let us collect their observations, comments and suggestions. The observation of students' work was done by analyzing their answers and comments in the GeoGebra activity sheets and feedback was shared discussing them with the class every two weeks. At the conclusion of the laboratory, a form[14] was created using Google Forms[15], through which we wanted to investigate students' perspective on their experience of the laboratory regarding: the usefulness of the meetings in relation to the entire course; the level of integration between the theoretical part and

[12] https://www.geogebra.org/m/gmy6ymj2
[13] https://www.geogebra.org/m/mqpw86uy
[14] https://docs.google.com/forms/d/1UfuhP4hW8Qt_GNqLnteyHThwkWGSR4iSZiKDWCiw4-E/edit?pli=1
[15] https://www.google.com/forms/about/

the laboratory one, etc. Analyzing both the data collected after each activity and the students' final comments, we can say that the goal of facing the classics of Greek mathematics through the lens of a dynamic geometry software has been satisfactorily achieved. As an example, we report here two paradigmatic examples of students' answers:

Student 1: The laboratory experiences were very useful both for better understanding the topics covered in the course and for understanding the way of thinking of ancient mathematicians and understanding the difficulties encountered in solving problems in relation to the mathematical tools known at the time.

Student 2: In most cases, especially for the first activities, the laboratory was useful for better understanding how Greek mathematicians could reason. Working only with a ruler and compass has helped me a lot, both in understanding the subject and in trying to broaden my way of thinking about certain problems. In this sense, the activities with tools "excluded" from the bar have been very useful.

Conclusions

We believe that a mathematics laboratory that takes advantage of GeoGebra for an university course on the history of Greek mathematics can be a very effective tool both to assist and to motivate students' learning processes and to provide useful elements for the evaluation of such a process. We plan to offer this mathematics laboratory again in the next course of History of Mathematics with the aim to transform a part of the summative assessment (so far composed of a written exam and an oral exam) into a formative assessment practice, by exploiting the feedback possibilities about the learning process provided by the software and the teacher-student interaction mediated by GeoGebra Classroom environment. Finally, in collaboration with a group of secondary school teachers, we have begun to adapt the activities to propose them in a secondary school class, in order to bring students closer to a vision of mathematics capable of recognizing and appreciating its historical evolution and its being a cultural and social object closely linked to other disciplinary fields.

References

Accascina, G., & Rogora, E. (2006). Using Cabri3D diagrams for teaching geometry. *International Journal for Technology in Mathematics Education*, *13*(1), 11–22.

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practices in Cabri Environments. *ZDM, 34(3),* 66–72. https://doi.org/10.1007/BF02655708

Baccaglini-Frank, A. & Mariotti, M. A. (2010). Generating conjectures in dynamic geometry: The maintaining dragging model. *International Journal of Computers for Mathematical Learning*, *15*(3), 225–253. https://doi.org/10.1007/s10758-010-9169-3

Brigaglia, A., Raspanti, M. A. & Rogora, E. (2021). L'uso di un software di geometria dinamica nella formazione dei futuri insegnanti [The use of dynamic geometry software in the training of future teachers]. *Matematica, Cultura e Società. Rivista dell'Unione Matematica Italiana, 6*(1), 37–67.

Brousseau, G. (1997). *Theory of didactical situation in mathematics.* Kluwer Academic.

Cox, D. A. (2012). *Galois Theory.* Wiley.

Jankvist, U. (2009). A categorization of the "whys" and "how's" of using history in mathematics education. *Edu. Stud. Math, 71*(3), 235–261. https://doi.org/10.1007/s10649-008-9174-9

Poncelet, J. V. (1822). *Traité des propriété projectives des figures* [A treatise on the projective properties of figures]. Bachelier.

Russo, L. (2004). *The forgotten revolution.* Springer.