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Multi-year stochastic modelling for market and underwriting risks in non-life insurance

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Dedicated to Lucrezia

Abstract

Nowadays, all EU insurance companies are requested by Solvency II to calculate capital requirements to prevent the risk of insolvency. This can be achieved either in accordance with the Standard Formula or using a full or partial Internal Model. An Internal Model is based on a market-consistent valuation of balance sheet items at a one-year time span, where a real-world probabilistic structure is used for the first projection year. In this thesis, we examine the main risks of a non-life insurer, i.e. the non-life underwriting risk and market risk, and their interactions, focusing on the non-life premium and reserve risk, equity risk, property risk, and interest rate risk. Consequently, we quantify the risk profile either with the Standard Formula or using a partial Internal Model. This analysis is performed using some reference stochastic models in the practical insurance business, i.e. the Collective Risk Model for non-life premium risk, the Re-Reserving Model for non-life reserve risk, the Geometric Brownian Motion for equity and property risk, and a real-world version of the G2++ Model for interest rate risk. In this thesis, we also consider a stochastic model for inflation, explicitly affecting the non-life underwriting items. We extend a real-world version of the Jarrow-Yildirim model, assuming that both the nominal rate and real rate evolve according to a two-factor process. It is indeed quite common that a two-factor model is preferred in order to admit some correlations between different rates. Finally, we illustrate a case study and several analyses on a multi-line insurance company in order to see how the risk drivers behave in both a stand-alone and an aggregate framework, and we calibrate the parameters on current and real market data.

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Introduction

A few observation and much reasoning lead to error; many observations and a little reasoning to truth.

Alexis Carrel

Since the introduction of Solvency II, the management of risks has become more important, compared to the pre-existing system of Solvency 0 and Solvency I. Now, insurance companies are requested to calculate capital requirements to face all the quantifiable risks for existing business and new business expected to be written over the following twelve months. They are allowed to calculate the so-called Solvency Capital Requirement (SCR) under the Standard Formula (SF) or a partial or full Internal Model (IM), and they are required to derive it as the Value-at-Risk of the basic own funds, subject to a confidence level of 99.5% over a one-year period. In this regard, there is an important debate about the fairness of the Value-at-Risk. Indeed, many authors argue that other risk measures, such as the Expected Shortfall or Tail Value-at-Risk, are preferable, notwithstanding the more difficult calibration of the distribution tails of the various risks. Nowadays, another important part of the risk management system is the so-called Own Risk and Solvency Assessment (ORSA) (Directive 2009/138/EC, 2009). It means that insurance companies must evaluate their overall solvency needs on a continuous basis, with the compliance of both capital requirements and requirements on technical provisions, and also the compliance of their risk profile with the assumptions underlying the SCR. In addition, the insurance companies must have a forward-looking perspective with a three to five-year view (see EIOPA, 2013a; EIOPA, 2013b). As a result, not only are insurance companies encouraged to have a short-term view but also a medium and long-term perspective. Indeed, a short-term view only is not desirable, because in this case the risk is not properly managed, since problems might arise in the future, and it could be too late.

The inversion of the production cycle regarding revenues and costs is a peculiar and important feature of insurance companies, and it means that policyholders pay premiums in advance, and contractual benefits or indemnities are paid later only if unfavourable events occur. This characteristic implies that insurance companies have significant financial resources to be invested in order to properly face future liabilities. This is certainly true in the majority of life insurers, for which asset management is often the main business providing profitable margins, but it is also true in non-life insurance, where financial profits can be quite sizeable when the insurance is issued with a relatively long duration in the technical liabilities. The resources are larger if we consider that insurance companies have also their own equity and cumulative profit margins produced year by year. Consequently, while insurance companies must focus on underwriting risk, which is the most representative risk of the insurance business, they also must focus on market risk, which in the Solvency II framework is often one of the most material risks in terms of capital requirements. Indeed, the resources of the non-life insurance business (i.e. the claims reserve and the premiums net of the claim amounts and expenses), that is risky by its nature, are invested in financial markets and create an additional risk, which must be estimated and monitored. Hence, it is important to put in place a risk management system covering both financial and underwriting items.

In recent years, the literature has been mainly focused on stand-alone analyses of these two sources of risk. The aim of this thesis is thus the examination of market risk, non-life underwriting risk (from now on, denoted as non-life risk) and their interactions under Solvency II (Directive 2009/138/EC, 2009), in order to quantify a reasonable risk profile either with the SF or an IM based on coherent stochastic models under real-world probabilities and applied to actual data. We thus try to give our support to the literature of real-world models. In doing this, we compare the SF and our IM, trying to explain their differences and similarities.

In order to manage uncertainty in insurance, a wide range of stochastic models can be used. The non-life premium risk (from now on, denoted as premium risk) will be described by a Collective Risk Model (that is well-known in the risk theory literature), i.e. a frequency-severity approach in which the number of claims and the single claim amount are separately described by some suitable distributions, under some independence assumptions between the claim count and claim size. A popular choice is the Negative Binomial for the number of claims and the Lognormal for the single claim amount, as proposed in practical analyses for instance by Beard et al. (1984) and Daykin et al. (1994). The non-life reserve risk (from now on, denoted as reserve risk) is usually described either by Bootstrapping (see Lowe, 1994; England and Verrall, 1999) or Bayesian models (see Scollnik, 2001; De Alba, 2002). We will use the so-called Re-Reserving Model (of the former group), that is a technique to estimate the sampling distribution of the claims reserve insufficiency using a random sampling method. On the other hand, the market risk is usually described by financial models based on stochastic differential equations, i.e. mathematical equations describing stochastic processes in the continuous time. Our choice is the popular Geometric Brownian Motion for stocks and a short-rate model for the term structure of interest rates, as proposed for instance by Ballotta and Savelli (2006). A short-rate model is an interest rate model, based on stochastic differential equations, that describes the behaviour of the instantaneous short rate, and it is able to describe the entire term structure. In general, interest rate models are distinguished in two main categories, i.e. equilibrium models and arbitrage-free models. Equilibrium models produce a term structure as output, and hence they do not match the current term structure observed in the market. Arbitrage-free models take the observed term structure as an input, and hence they match the current term structure observed in the market. Some well-known equilibrium models have been introduced by Vasicek (1977), Cox et al. (1985), and Duffie and Kan (1996), and some well-known arbitrage-free models have been introduced by Hull and White (1990) and Heath et al. (1992). Furthermore, we will also consider a stochastic model for inflation, explicitly affecting both the total claim amount and claims development result. This is because, in very recent years, inflation rates and volatility have drastically increased, therefore we include and analyse this element in the comprehensive non-life insurance framework of this thesis. Nowadays, this is not anymore a minor point, and we do not want to ignore it (or leave it implicit in the model) as often has been done in the non-life insurance literature. The most famous inflation model has been introduced by Jarrow and Yildirim (2003), and it is still the main reference technique adopted in the inflation market (Cipollini and Canty, 2013). This is a nominal risk-neutral arbitrage-free model that, at the same time, describes the nominal short rate, real short rate and Consumer Price Index (CPI), using a one-factor process for each of them, so that it is possible to derive the entire nominal, real, and inflation term structures. Other popular models have been introduced by Mercurio (2005). For our purpose, we will extend the original Jarrow-Yildirim model, in a way that both the nominal short rate and real short rate are a two-factor process (this is especially useful when a two-factor model for nominal rate or real rate is preferred, so that the original Jarrow-Yildirim model is no more suitable), and we will adapt it to the real-world context. We will also derive, in a clear and usable form, both the pricing formulas for inflation-indexed derivatives and real-world expectation, that we will need to calibrate the model.

As already mentioned, an objective of this thesis is to support the literature in the area of models under real-world probabilities. Unfortunately, over the past few years, only moderate attention has been directed to such financial models, because the literature has been primarily devoted to the pricing of interest rate derivatives, where risk-neutral probabilities are typically preferred (see Brigo and Mercurio, 2006). As explained by Giordano and Siciliano (2015), risk-neutral probabilities are acceptable for pricing, but not to forecast the future value of an asset. Real-world probabilities should instead be used for risk management purposes. It is worth mentioning that Solvency II requires a market-consistent valuation for technical provisions to be made using risk-neutral probabilities. In the same way, the new international accounting principles for insurance contract valuation, i.e. IFRS 17 (IASB, 2017), effective from 2023 in EU regulation, require a market-consistent valuation for technical provisions that is usually carried out by insurance companies using risk-neutral probabilities as well. On the other hand, Solvency II requires that capital requirements according to an IM are calculated using real-world probabilities to predict the risk drivers' behaviour for the first year of projection, even though risk-neutral probabilities must be used again to determine the market-consistent value of the basic own funds of the insurer after the first year. We point out that, by specifying the models under both real-world and risk-neutral probabilities, it is possible to carry out either risk management analysis or market-consistent valuations, as proposed by Gambaro et al. (2018) or Berninger and Pfeiffer (2021). Such model specification can also be used for planning and management purposes.

This thesis is organised as follows. In Chapter 1, we describe the theoretical framework underlying our analysis. More precisely, we present the risk reserve equation and the related quantities, such as annual net cash flows originated by the insurance business and investments portfolio. Moreover, we describe the measures of dependence and copula functions that we need for our analysis. In Chapter 2, we present the theoretical framework underlying Solvency II, focusing on quantitative requirements, such as SCR, SF, and IM, and qualitative requirements, such as the supervisory review process, system of governance, and risk management system. We point out that we only go into detail of the sub-modules of the SF for market and non-life risk, because we need them for our analysis. In Chapter 3, we describe the risk-neutral and real-world economic scenario generator, we present the related financial derivatives and their pricing formulas. Here, we also describe the extended version of the Jarrow-Yildirim model, and we derive the pricing formulas of the main inflation-indexed derivatives. In Chapter 4, we present the market model used to describe the annual rate of return, the non-life premium model (from now on, denoted as premium model) used to describe the total claim amount, and also the non-life reserve model (from now on, denoted as reserve model) used to describe the claims reserve and claims development result. Moreover, in Chapter 5 we estimate the general parameters of our insurance company, and we calibrate the models just mentioned and their dependence structure. In Chapter 6, we propose a numerical analysis in which we determine the capital requirements for market and non-life risk, according to a partial IM and the SF. This case study has been performed on a multi-line insurance company, using current and available market data. Finally, we report the main conclusions of our research and further steps to be investigated.

Chapter 1

Analysis framework

In this chapter, we describe the analysis framework of our research and we introduce the main variables, parameters and coefficients involved. Furthermore, we describe some instruments and concepts that will be useful later on.

As a general important rule of this thesis, the random variables are indicated by an uppercase Latin letter, while the volume parameters and coefficients are specified with lowercase or Greek letters.

1.1 Risk reserve

As explained by Daykin et al. (1994), the risk reserve represents the own funds accumulated by the insurer over time to face the business uncertainty and avoid the risk of insolvency. For simplicity, in this thesis we ignore reinsurance mitigation, taxes, and dividends. As a result, we only consider the market and non-life risk, assuming that the stochastic risk reserve at the end of time t is given by:

$$U_t = (1 + J_t)\,(U_{t-1} + L_{t-1}) + (1 + \bar{J}_t)\,(b_t - C_t - e_t) - L_t$$

where J_t is the stochastic annual rate of return (annually compounded) of the initial resources of the insurance company, L_t is the stochastic claims reserve (also called loss reserve), b_t is the gross premium amount, C_t is the stochastic incremental amount for paid claims, and e_t is the expense amount, including both acquisition and general expenses. Premiums, paid claims, and expenses are assumed to be equally realised at the beginning, in the middle, and at the end of the year, so that $\bar{J}_t$ is their stochastic average rate of return. The premium reserve is not accounted for, because the premium amount is assumed to refer to a single calendar year only, and consequently earned premiums and written premiums are identical (for this reason, from now on, we simply denote the gross premium amount as GPW). If the coverage were not referred to the single calendar year, it would be enough to introduce the rule to calculate the premium reserve, considering the dynamic nominal growth of earned premiums and the initial premium reserve portion released in each year.

The claims reserve can be distinguished based on the claims exactly occurred (denoted by the superscript inside the square brackets) in the current year and the claims already occurred (denoted by the superscript inside the round brackets) in the previous year. We thus have:

$$L_t = L_t^{[t]} + L_t^{(t-1)}$$

In the same way, the amount for paid claims can be distinguished based on the claims exactly occurred in the current year and settled in the same year, and the claims already occurred in the previous year and settled in the current year (again denoted by the superscript inside the square or round brackets). We thus have:

$$C_t = C_t^{[t]} + C_t^{(t-1)}$$

Claims reserve and amount for paid claims can be assigned to two different random variables. The stochastic total claim amount (i.e. the relevant random variable for premium risk) is given by the following equation:

$$X_t = C_t^{[t]} + L_t^{[t]} \tag{1.1}$$

The stochastic claims development result related to claims reserve (i.e. the relevant random variable for reserve risk) is given by the following equation:

$$Y_t = L_{t-1} - C_t^{(t-1)} - L_t^{(t-1)} \tag{1.2}$$

However, the claims reserve is calculated discounting by the stochastic regulatory nominal term structure required by the supervisory authority, hence the random variables above also implicitly include some market risk.

As a consequence, the risk reserve is found to be:

$$U_t = (1 + J_t)\, U_{t-1} + (1 + \bar{J}_t)\, (b_t - e_t) - X_t + Y_t + J_t\, L_{t-1} - \bar{J}_t\, C_t$$

We now assume that the GPW is calculated based on the following formula:

$$b_t = \pi_t + \lambda\, \pi_t + c\, b_t \tag{1.3}$$

where π_t is the risk premium amount, λ is the explicit safety loading coefficient, and c is the expense loading coefficient. Note that the insurance portfolio is dynamic, because the risk and gross premium amounts are time dependent.

The expense amount is assumed to be deterministic and equal to the expense loadings included in the GPW. Notwithstanding that the amount of expenses is linked to the GPW (and hence it is not fixed over time), empirically it has a small volatility in the non-life insurance business, in particular where acquisition costs are high. This is supported by the Solvency II calibration, where expense risk in non-life insurance is roughly incorporated in the volatility of the loss ratio. We have:

$$e_t = c\, b_t \tag{1.4}$$

As a result, the risk reserve is found to be:

$$U_t = (1 + J_t)\, U_{t-1} + (1 + \bar{J}_t)\, (1 + \lambda)\, \pi_t - X_t + Y_t + J_t\, L_{t-1} - \bar{J}_t\, C_t \tag{1.5}$$

According to the classical risk theory, the risk premium amount is defined as the expected total claim amount of the insurance company:

$$\pi_t = E(X_t) \tag{1.6}$$

The solution of this expected value will be provided in Section 4.2.3, and it will depend on a deterministic annual real growth rate and the expected claims inflation of the insurance portfolio. In order to emphasise the contribution of the claims inflation to the non-life risk, we assume that the future risk premium amounts are not adjusted for the stochastic claims inflation realised over time (i.e. they are calculated at time zero and kept fixed over time). We also prefer not to add more complexity into the model introducing stochastic premium amounts. Anyway, in our numerical analysis we will consider a time horizon of three years, meaning that only after one and two years the insurance company could adjust the premium amounts and does not do it. Furthermore, the premiums could be adjusted both if the claims inflation is smaller and bigger than its expectation, resulting both in underestimation and of overestimation of the risk.

We assume that the insurance company is multi-line, because it has underwriting business in Motor Third-Party Liability (MTPL), Motor Other Damages (MOD), and General Third-Party Liability (GTPL). It means that premiums, paid claims, expenses and claims reserves come from three different sources, and their overall value is equal to their sum. Moreover, we assume a hierarchical structure of copula functions (see Savelli and Clemente, 2011, for more details), either based on linear dependence (i.e. Gaussian copulas) or non-linear dependence (i.e. Gumbel copulas in our analysis). Therefore, we firstly join MTPL, MOD, and GTPL total claim amounts (i.e. the sources of premium risk) and claims development results (i.e. the sources of reserve risk), and then we aggregate them together. We also join the stock, property, zero-coupon bond components of the annual rate of return and discount factors (i.e. sources of market risk), and we finally aggregate market and non-life variables. In case of Gumbel copula, we join MTPL and MOD at the first step, and we add GTPL later on.

In conclusion, we point out that the risk reserve is an absolute amount, that in the short term depends more on the initial capital position of the insurance company than on economic results. Actually, we might have a significant risk reserve, which is low compared with the premium volume of the insurance company and vice versa. Hence, it is preferable, also for comparative analyses, to consider relative figures (see Savelli, 2003).

1.1.1 Annual net cash flows

The stochastic annual net cash flows originated by the insurance business at the end of time t are given by:

$$F_t = b_t - C_t - e_t$$

Using equations (1.1) and (1.2), we have that the stochastic amount for paid claims is given by:

$$C_t = C_t^{[t]} + C_t^{(t-1)} = X_t - L_t^{[t]} + L_{t-1} - L_t^{(t-1)} - Y_t = X_t - L_t + L_{t-1} - Y_t \tag{1.7}$$

then, using equations (1.3) and (1.4), the stochastic annual net cash flows originated by the insurance business are found to be:

$$F_t = (1 + \lambda)\,\pi_t - X_t + L_t - L_{t-1} + Y_t \tag{1.8}$$

1.1.2 Asset portfolio

Even if there are a lot of other investments in the market, in this thesis we consider a stock and property investment and five investments in nominal zero-coupon bonds with time to maturity $w = 1, 2, 3, 5, 10$. We also assume that the asset allocation is kept constant year by year.

First of all, we point out that the initial asset value of the portfolio is given by:

$$A_0 = U_0 + L_0$$

The stochastic asset value of the portfolio at the end of time t is obtained from the combination of the stochastic values of the stock, property and bond portfolios:

$$A_t = A_t^S + A_t^H + A_t^{P_n} + (1 + \bar{J}_t)\,F_t$$

The stochastic value of the stock portfolio is given by:

$$A_t^S = \alpha\,A_{t-1}\,\frac{S(t)}{S(t-1)}$$

the stochastic value of the property portfolio is given by:

$$A_t^H = \beta\,A_{t-1}\,\frac{H(t)}{H(t-1)}$$

and the stochastic value of the nominal zero-coupon bond portfolio is given by:

$$A_t^{P_n} = (1 - \alpha - \beta)\,A_{t-1} \sum_{w \in \{1,2,3,5,10\}} \gamma_w\,\frac{P_n(t, t-1+w)}{P_n(t-1, t-1+w)}$$

so that the stochastic value of the single bond investment with time to maturity $w = 1, 2, 3, 5, 10$ is found to be:

$$A_t^{P_{n,w}} = (1 - \alpha - \beta)\,\gamma_w\,A_{t-1}\,\frac{P_n(t, t-1+w)}{P_n(t-1, t-1+w)}$$

where α is the percentage invested in the stock portfolio, β is the percentage invested in the property portfolio, and $(1 - \alpha - \beta)$ is the percentage invested in the nominal zero-coupon bond portfolio, while γ_w is the percentage of bond portfolio invested in the bond with time to maturity w.

The stochastic annual rate of return of the initial resources of the insurance company is given by:

$$J_t = \frac{A_t^S + A_t^H + A_t^{P_n}}{A_{t-1}} - 1$$

so that we have:

$$J_t = \alpha \frac{S(t)}{S(t-1)} + \beta \frac{H(t)}{H(t-1)} + (1-\alpha-\beta) \sum_{w\in\{1,2,3,5,10\}} \gamma_w \frac{P_n(t,t-1+w)}{P_n(t-1,t-1+w)} - 1 \quad (1.9)$$

Furthermore, assuming to invest in the same assets of the portfolio, the stochastic rate of return (semi-annually compounded) of the annual net cash flows originated by the insurance business in the middle of the year is given by:

$$J_t^{(2)} = \alpha \frac{S(t)}{S(t-1/2)} + \beta \frac{H(t)}{H(t-1/2)} + (1-\alpha-\beta) \sum_{w\in\{1,2,3,5,10\}} \gamma_w \frac{P_n(t,t-1+w)}{P_n(t-1/2,t-1+w)} - 1$$

so that the stochastic average rate of return of the annual net cash flows equally originated by the insurance business at the beginning, in the middle, and at the end of the year is found to be:

$$\bar{J}_t = \frac{J_t + J_t^{(2)}}{3} \quad (1.10)$$

1.2 Copula functions

We finally introduce copula functions, in order to describe the dependence structure between random variables. Copula functions are very popular in insurance, because they are able to describe a wide range of dependence structures, including but not limited to linear dependence (see for example Embrechts et al., 2003; Nelsen, 2006).

A n-dimensional copula $C : [0,1]^n \to [0,1]$ is a multivariate cumulative distribution function of uniformly distributed marginals, and it satisfies the following properties:

1. $C(u_1, \dots, u_n)$ is non-decreasing in each component u_i;
2. $C(u_1, \dots, u_n)$ is null if at least one component u_i is null;
3. $C(u_1, \dots, u_n)$ is equal to u_i if all the components are equal to one, except u_i.

According to the Sklar's theorem (Sklar, 1959), every n-dimensional multivariate cumulative distribution function F can be expressed using its marginal distributions $F_1, \dots, F_n$ and a n-dimensional copula C as follows:

$$F(x_1, \dots, x_n) = C[F_1(x_1), \dots, F_n(x_n)]$$

Furthermore, if the marginals are continuous, then the copula is unique, and we have:

$$C(u_1, \dots, u_n) = F[F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)]$$

Consequently, the steps to generate pseudo-random samples from general classes of multivariate distributions, using copula functions, are:

1. generate a sample $(u_1^{(1)}, \dots, u_n^{(1)})$ from the copula;
2. obtain the required sample $(x_1^{(1)}, \dots, x_n^{(1)})$ by the inverse of the marginals:

$$(x_1^{(1)}, \dots, x_n^{(1)}) = [F_1^{-1}(u_1^{(1)}), \dots, F_n^{-1}(u_n^{(1)})]$$

Furthermore, through the relationship between the probability density function and cumulative distribution function and through the Sklar's theorem, the multivariate probability density function related to the copula is found to be:

$$c(u_1, \dots, u_n) = \frac{f[F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)]}{\prod_{i=1}^{n} f_i[F_i^{-1}(u_i)]}$$

The most important families of copulas are the Elliptical and Archimedean ones. Elliptical copulas are a function of the Pearson correlation coefficient. Archimedean copulas are quite easily manageable from a mathematical point of view.

Elliptical copulas are based on multivariate elliptical distributions, which have some properties in common with the multivariate Normal distribution. There is no simple analytical formula for the elliptical copulas, hence they can be approximated using numerical integration. Some popular elliptical copulas in insurance are the Gaussian and Student's t copula functions. Unlike the first one, the Student's t copula can also be used to model the extreme dependence, i.e. the dependence on the distribution tails.

Archimedean copulas are based on the generator function $\psi : [0, 1] \to [0, \infty)$, i.e. a continuous, strictly decreasing and convex function, such that $\psi(1) = 0$. Its pseudo-inverse is given by:

$$\psi^{[-1]}(t) = \begin{cases} \psi^{-1}(t) & \text{if} \quad 0 \leq t \leq \psi(0) \\ 0 & \text{if} \quad \psi(0) \leq t \leq \infty \end{cases}$$

A copula is said to be Archimedean if it can be written as follows:

$$C(u_1, ..., u_n) = \psi^{[-1]}[\psi(u_1) + ... + \psi(u_n)]$$

and:

$$(-1)^k \frac{\partial^k \psi^{-1}(t)}{\partial t^k} \geq 0 \quad \text{for} \quad k \in \mathbb{N}$$

Archimedean copulas are able to describe a lot of dependence structures. Moreover, there are simple analytical formulas for them. Some popular Archimedean copulas in insurance are the Gumbel and Clayton copula functions.

In this thesis, the dependence structure is described by using the Gaussian copula (without tail dependence) and Gumbel copula (with upper tail dependence only), which are the benchmarks in the practical insurance modelling. EU insurance supervisory authorities are aware that some tail dependence is very often present, and it cannot be disregarded, but they often prefer that the IM is based on Elliptical copulas, such as the Gaussian one, because it is quite manageable. However, they require more conservative linear correlation coefficients. We point out that this point has been introduced also in the SF, where the correlation matrix includes some prudence to implicitly incorporate the tail dependence. Consequently, in this thesis we consider both the Gaussian and Gumbel copulas for the aggregation, in order to assess the effect of having a tail dependence and the consequent change in capital requirements.

1.2.1 Measures of dependence

The main measures of dependence are the Pearson correlation coefficient, the Kendall's and Spearman's rank correlation coefficients and the coefficients of tail dependence.

Let (X, Y) be a random vector. Its Pearson correlation coefficient is given by:

$$\rho = \operatorname{Corr}(X, Y) = \frac{\operatorname{Cov}(X, Y)}{\operatorname{Std}(X)\operatorname{Std}(Y)}$$

The Pearson correlation coefficient is the linear correlation coefficient, and it is the canonical measure for spherical and elliptical distributions. Since we need finite variance for its calculation, heavy-tailed distributions present some computational difficulties. Note that the Pearson correlation coefficient is invariant under positive linear transformations, but not under general strictly increasing ones. Moreover, a null Pearson correlation coefficient (i.e. uncorrelation) implies full independence in the case of Normal random variables only.

The Kendall's rank correlation coefficient of the random vector is given by:

$$\tau = \operatorname{Corr}[\operatorname{sign}(X_1 - X_2), \operatorname{sign}(Y_1 - Y_2)]$$

where (X_1, Y_1) and (X_2, Y_2) are two independent random vectors with the same distribution as the original one. Moreover, if the random vector has continuous marginals, i.e. the bivariate copula C is unique, then we have:

$$\tau = 4 \int_0^1 \int_0^1 C(u, v)\, dC(u, v) - 1$$

For a given sample of n observations from the random vector, an estimate of the Kendall's rank correlation coefficient is found to be:

$$\hat{\tau} = \frac{2}{n\,(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \operatorname{sign}(x_i - x_j) \operatorname{sign}(y_i - y_j)$$

The Spearman's rank correlation coefficient of the random vector is given by:

$$\rho_s = \operatorname{Corr}[F_X(x), F_Y(y)]$$

where F_X and F_Y are the marginal cumulative distribution functions. Once again, if the random vector has continuous marginals, i.e. the bivariate copula C is unique, then we have:

$$\rho_s = 12 \int_0^1 \int_0^1 C(u, v) - u\, v\, du\, dv$$

For a given sample of n observations from the random vector, an estimate of the Spearman's rank correlation coefficient is found to be:

$$\hat{\rho}_s = 1 - \frac{6}{n\,(n^2-1)} \sum_{i=1}^{n} (\operatorname{rank}(x_i) - \operatorname{rank}(y_i))^2$$

In conclusion, the upper and lower coefficients of tail dependence of the random vector are given by:

$$\lambda_U = \lim_{u \to 1^-} P[F_X(x) > u \mid F_Y(y) > u]$$

and:

$$\lambda_L = \lim_{u \to 0^+} P[F_X(x) \leq u \mid F_Y(y) \leq u]$$

1.2.2 Gaussian copula

For a given linear correlation matrix Σ, the Gaussian copula is given by:

$$C_{\Sigma}^{gaussian}(u_1, \dots, u_n) = \Phi_{\Sigma}\left[\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n)\right]$$

where Φ_{Σ} and Φ are respectively the multivariate and univariate standard Normal cumulative distribution functions.

The multivariate probability density function related to the Gaussian copula is found to be:

$$c_{\Sigma}^{gaussian}(u_1, \dots, u_n) = |\Sigma|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\zeta^T\left(\Sigma^{-1} - \mathrm{I}\right)\zeta\right]$$

where:

$$\zeta^T = \left(\Phi^{-1}(u_1) \cdots \Phi^{-1}(u_n)\right)$$

For the bivariate case, the Kendall's rank correlation coefficient is found to be:

$$\tau = \frac{2}{\pi}\arcsin\rho \tag{1.11}$$

where ρ is the Pearson correlation coefficient between the two distributions.

Moreover, unless the correlation matrix exhibits perfect positive or negative dependence, the upper and lower coefficients of tail dependence of the Gaussian copula are found to be null. As a result, the Gaussian copula is used to describe dependence structures without tail dependence. Figure 1.1 shows an example of bivariate probability density function related to the Gaussian copula.

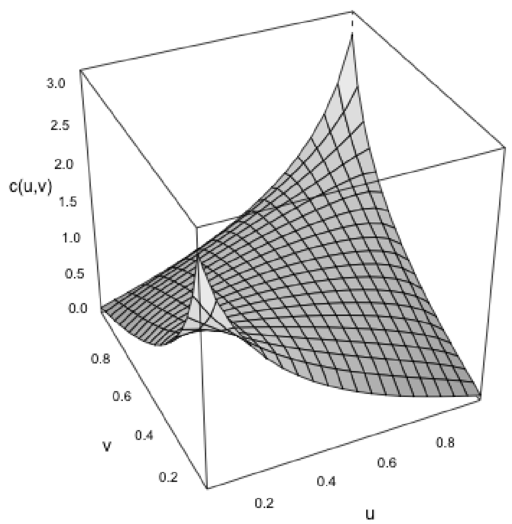


Figure 1.1. Bivariate probability density function of a Gaussian copula with Pearson correlation coefficient 0.5

1.2.3 Gumbel copula

For a given parameter $\theta \geq 1$, the bivariate Gumbel copula is given by:

$$C_{\theta}^{gumbel}(u_1, u_2) = \exp\left\{-\left((-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}\right)^{\frac{1}{\theta}}\right\}$$

Moreover, the Kendall's rank correlation coefficient is found to be:

$$\tau = 1 - \frac{1}{\theta} \tag{1.12}$$

The lower coefficient of tail dependence of the Gumbel copula is found to be null, while the upper coefficient of tail dependence is found to be:

$$\lambda_U = 2 - 2^{\frac{1}{\theta}}$$

Figure 1.2 shows an example of bivariate probability density function related to the Gumbel copula.

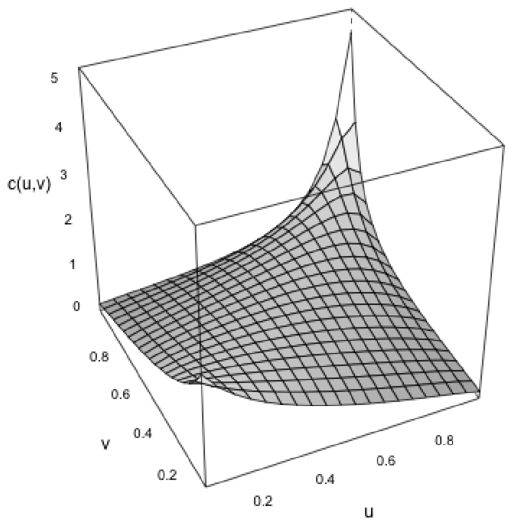


Figure 1.2. Bivariate probability density function of a Gumbel copula with parameter 1.5

Chapter 2

Solvency II

Solvency II (Directive 2009/138/EC, 2009) is an EU directive which codifies and harmonises the EU insurance regulation, and it was implemented in Italy in 2015 by D. Lgs. 74/2015. Primarily, this concerns the amount of capital that EU insurance companies must hold to reduce the risk of insolvency.

Note that different parts of this chapter are taken from Solvency II and the Delegated Regulation (Commission Delegated Regulation (EU) 2015/35, 2009).

2.1 Regulatory harmonisation

The process of regulatory harmonisation distinguishes the following main levels:

1. Solvency II
2. Delegated Regulation

2.5. Technical standards, proposed by EIOPA, involving:
   - Regulatory technical standard
   - Implementing technical standard

3. Guidelines, issued by EIOPA and implemented in Italy by IVASS Regulations
4. Rigorous enforcement of community legislation by the Commission

Solvency II has a three-pillar structure with the following involved items:

1. Quantitative requirements
   - 1.1. Economic balance sheet
   - 1.2. Eligible own funds
   - 1.3. Solvency Capital Requirement (SCR)
   - 1.4. Minimum Capital Requirement (MCR)
2. Qualitative requirements
   - 2.1. Supervisory review process, involving:

- Capital add-on

2.2. System of governance

2.3. Risk management system, involving:
   - Own Risk and Solvency Assessment (ORSA)

2.4. Control functions, involving:
   - Risk management function
   - Compliance function
   - Internal audit function
   - Actuarial function

3. Reporting and disclosures requirements

3.1. Solvency and Financial Condition Report (SFCR)

3.2. Regular Supervisory Report (RSR)

3.3. Quantitative Reporting Templates (QRTs)

2.2 Technical provisions

Solvency II defines a market-consistent valuation of technical provisions and makes a distinction for hedgeable and non-hedgeable technical provisions. The following part is taken from Art. 76 and 77 of Solvency II.

> *The value of technical provisions shall correspond to the current amount insurance and reinsurance undertakings would have to pay if they were to transfer their insurance and reinsurance obligations immediately to another insurance or reinsurance undertaking.*
>
> *The value of technical provisions shall be equal to the sum of a best estimate and a risk margin. The best estimate shall correspond to the probability-weighted average of future cash-flows, taking account of the time value of money (expected present value of future cash-flows), using the relevant risk-free interest rate term structure.*
>
> *Where future cash flows associated with insurance or reinsurance obligations can be replicated reliably using financial instruments for which a reliable market value is observable, the value of technical provisions associated with those future cash flows shall be determined on the basis of the market value of those financial instruments. In this case separate calculations of the best estimate and the risk margin shall not be required.*

Note that the hedgeable technical provisions are for example related to unit-linked or index-linked contracts without guarantees. The best estimate is calculated gross of recoverables, and it takes account of all the cash flows of the insurance and reinsurance obligations over their lifetime. Furthermore, the basic risk-free interest rates are derived on the basis of interest-rate swap rates, adjusted to take account of credit risk. When this is not possible, they are derived on the basis of government bonds. Solvency II allows the application either of a volatility adjustment or a matching adjustment to the basic risk-free interest rates. The risk margin is calculated by determining the cost of capital for providing an amount of eligible own funds equal to the SCR necessary to support the insurance and reinsurance obligations over their lifetime. The risk margin is based on the whole portfolio, and the Cost-of-Capital rate is the same for each insurance company. We point out that Solvency II allows some simplifications in the calculation of the risk margin.

2.3 Solvency Capital Requirement

Solvency II demands some requirements for the calculation of the SCR (see Figure 2.1). The following part is taken from Art. 101 of Solvency II.

> *The Solvency Capital Requirement shall be calculated on the presumption that the undertaking will pursue its business as a going concern.*
>
> *The Solvency Capital Requirement shall be calibrated so as to ensure that all quantifiable risks to which an insurance or reinsurance undertaking is exposed are taken into account. It shall cover existing business, as well as the new business expected to be written over the following 12 months. With respect to existing business, it shall cover only unexpected losses.*
>
> *It shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99.5% over a one-year period.*

The SCR shall cover at least the following risks:

1. Non-life underwriting risk
2. Life underwriting risk
3. Health underwriting risk
4. Market risk
5. Credit risk
6. Operational risk

2.4 Standard formula

The SCR calculated on the basis of the SF shall be equal to the following:

$$SCR = BasicSCR + SCR_{op} - Adj$$

where $BasicSCR$ is the Basic SCR, SCR_{op} is the SCR for the operational risk, and Adj is the adjustment for the loss-absorbing capacity of technical provisions and deferred taxes.

The Basic SCR shall consist of at least the following risk modules:

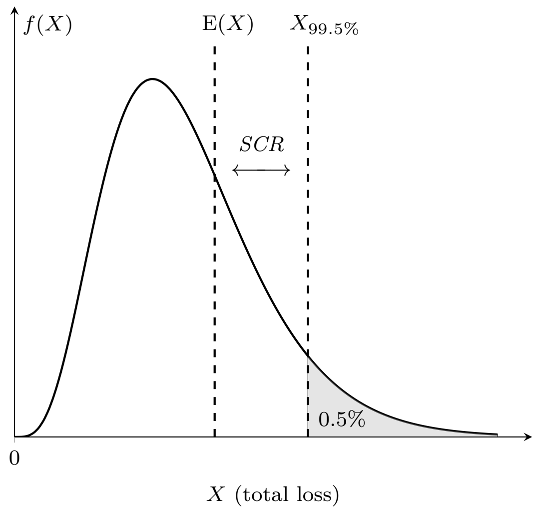


Figure 2.1. Representation of SCR calculation

1. Non-life underwriting risk, involving:
   - Non-life premium and reserve risk
   - Non-life catastrophe risk
   - Non-life lapse risk
2. Life underwriting risk, involving:
   - Mortality risk
   - Longevity risk
   - Disability risk
   - Life expense risk
   - Revision risk
   - Lapse risk
   - Life catastrophe risk
3. Health underwriting risk
4. Market risk, involving:
   - Interest rate risk
   - Equity risk
   - Property risk
   - Spread risk
   - Currency risk
   - Market risk concentrations
5. Counterparty default risk

We point out that the value of a risk module or sub-module of the Basic SCR cannot be negative. The Basic SCR shall be equal to the following:

$$BasicSCR = \sqrt{\sum_{i,j} Corr_{(i,j)}\, SCR_i\, SCR_j} + SCR_{intangibles}$$

where $Corr_{(i,j)}$ is the correlation coefficient for the Basic SCR for modules i and j (see Table 2.1), SCR_i and SCR_j are the capital requirements for modules i and j, respectively, and $SCR_{intangibles}$ is the SCR for intangible asset risk.

Table 2.1. Correlation matrix for Basic SCR

$i \setminus j$	Market	Default	Life	Health	Non-life
Market	1	0.25	0.25	0.25	0.25
Default	0.25	1	0.25	0.25	0.5
Life	0.25	0.25	1	0.25	0
Health	0.25	0.25	0.25	1	0
Non-life	0.25	0.5	0	0	1

We point out that the capital requirements aggregated in the SF are not standard deviations but quantiles of probability distributions. On one hand, the aggregation based on a correlation matrix produces a correct aggregate of quantiles only for marginal Normal distributions (more in general, for elliptical distributions). On the other hand, the aggregation based on a correlation matrix is correct only if linear dependence exists. Nevertheless, the shape of marginal distributions could be significantly different from the Normal one (e.g. for skewed distributions), and the dependence between distributions could be non-linear (e.g. when tail dependence is present).

The SCR is calculated separately for each risk module or sub-module through a factor-based or scenario-based approach. In the first case, the capital requirements are determined by using single risk exposures and risk factors, which are calibrated considering the tail of the distribution and taking into account both volatility and trend effects. In the second case, the capital requirements shall be equal to the loss in basic own funds given by a stressed scenario.

The Delegated Regulation demands some requirements for the scenario-based approach. The following part is taken from Art. 83 of the Delegated Regulation.

> *Where the calculation of a module or sub-module of the Basic Solvency Capital Requirement is based on the impact of a scenario on the basic own funds of insurance and reinsurance undertakings, all of the following assumptions shall be made in that calculation:*
>
> *(a) the scenario does not change the amount of the risk margin included in technical provisions;*
>
> *(b) the scenario does not change the value of deferred tax assets and liabilities;*

(c) *the scenario does not change the value of future discretionary benefits included in technical provisions;*

(d) *no management actions are taken by the undertaking during the scenario.*

2.4.1 Non-life underwriting risk

The non-life risk module shall consist of all of the following sub-modules:

1. Non-life premium and reserve risk
2. Non-life catastrophe risk
3. Non-life lapse risk

The SCR for non-life risk shall be equal to the following:

$$SCR_{non\text{-}life} = \sqrt{\sum_{i,j} CorrNL_{(i,j)}\, SCR_i\, SCR_j}$$

where $CorrNL_{(i,j)}$ is the correlation parameter for non-life risk for sub-modules i and j (see Table 2.2), and SCR_i and SCR_j are the capital requirements for risk sub-modules i and j, respectively.

Table 2.2. Correlation matrix for non-life risk

$i \setminus j$	Non-life premium and reserve	Non-life catastrophe	Non-life lapse
Non-life premium and reserve	1	0.25	0
Non-life catastrophe	0.25	1	0
Non-life lapse	0	0	1

Non-life premium and reserve risk

Solvency II defines the premium and reserve risk sub-module. The following part is taken from Art. 105 of Solvency II.

> *The risk of loss, or of adverse change in the value of insurance liabilities, resulting from fluctuations in the timing, frequency and severity of insured events, and in the timing and amount of claim settlements.*

The SCR for premium and reserve risk shall be equal to the following:

$$SCR_{nl\ prem\ res} = 3\,\sigma_{nl}\, V_{nl} \tag{2.1}$$

where σ_{nl} and V_{nl} are the standard deviation in relative terms (also called volatility factor) and the volume measure for premium and reserve risk.

The volatility factor for premium and reserve risk is given by:

$$\sigma_{nl} = \frac{1}{V_{nl}} \sqrt{\sum_{s,t} CorrS_{(s,t)}\, \sigma_s\, V_s\, \sigma_t\, V_t}$$

where $CorrS_{(s,t)}$ is the correlation parameter for premium and reserve risk for segments s and t (see Table 2.4), and σ_s and σ_t are the volatility factors for premium and reserve risk of segments s and t, respectively. They are given by:

$$\sigma_s = \frac{\sqrt{\sigma^2_{(prem,s)}\, V^2_{(prem,s)} + \sigma_{(prem,s)}\, V_{(prem,s)}\, \sigma_{(res,s)}\, V_{(res,s)} + \sigma^2_{(res,s)}\, V^2_{(res,s)}}}{V_{(prem,s)} + V_{(res,s)}}$$

where $\sigma_{(prem,s)}$ and $\sigma_{(res,s)}$ are the volatility factors for premium risk and reserve risk of segment s (see Table 2.3), and $V_{(prem,s)}$ and $V_{(res,s)}$ are the volume measures for premium risk and reserve risk of segment s. Note that the volatility factors for premium and reserve risk were amended for some minor segments in 2019, after the approval of the Solvency II review of 2018.

We point out that Table 2.3 contains volatility factors, estimated through the market-wide approach. Table 2.3 and 2.4 take into account the proportional and non-proportional direct reinsurance. The proportional direct reinsurance is treated as direct insurance, since the relative volatility of a particular reinsurance segment is the same as in the case of the insurance company at gross level. Moreover, the volatility factor for premium risk of a segment shall be equal to the product of the volatility factor for gross premium risk of the segment and the adjustment factor for non-proportional excess of loss and stop loss reinsurance. For segments 1, 4 and 5 the adjustment factor for non-proportional reinsurance shall be equal to 80%. For all the other segments the adjustment factor for non-proportional reinsurance shall be equal to 100%. All these adjustment factors may be changed only after an undertaking-specific parameter approval process.

The volume measure for premium and reserve risk is given by:

$$V_{nl} = \sum_s V_s$$

where V_s is the volume measure for premium and reserve risk of segment s, after adjustment for the geographical diversification. It is given by:

$$V_s = \left(V_{(prem,s)} + V_{(res,s)}\right)(0.75 + 0.25\, DIV_s)$$

where $V_{(prem,s)}$ and $V_{(res,s)}$ are the volume measures for premium risk and reserve risk of segment s, and DIV_s is the geographical diversification factor of segment s. The volume measure for premium risk of segment s is given by:

$$V_{(prem,s)} = \max\left(P_s, DIV_{(last,s)}\right) + FP_{(existing,s)} + FP_{(future,s)}$$

where P_s is an estimate of the premiums to be earned by the insurance or reinsurance undertaking in the segment s during the following 12 months, $P_{(last,s)}$ are the premiums earned by the insurance or reinsurance undertaking in the segment s during the last 12 months, $FP_{(existing,s)}$ is the expected present value of premiums

Table 2.3. Segmentation of non-life insurance and reinsurance obligations and volatility factors for premium and reserve risk

	Segment s	$\sigma_{(prem,s)}$	$\sigma_{(res,s)}$
1	Motor vehicle liability insurance and proportional reinsurance	10%	9%
2	Other motor insurance and proportional reinsurance	8%	8%
3	Marine, aviation and transport insurance and proportional reinsurance	15%	11%
4	Fire and other damage to property insurance and proportional reinsurance	8%	10%
5	General liability insurance and proportional reinsurance	14%	11%
6	Credit and suretyship insurance and proportional reinsurance	19%	17.2%
7	Legal expenses insurance and proportional reinsurance	8.3%	5.5%
8	Assistance and its proportional reinsurance	6.4%	22%
9	Miscellaneous financial loss insurance and proportional reinsurance	13%	20%
10	Non-proportional casualty reinsurance	17%	20%
11	Non-proportional marine, aviation and transport reinsurance	17%	20%
12	Non-proportional property reinsurance	17%	20%

Table 2.4. Correlation matrix for premium and reserve risk

$s \backslash t$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	0.5	0.5	0.25	0.5	0.25	0.5	0.25	0.5	0.25	0.25	0.25
2	0.5	1	0.25	0.25	0.25	0.25	0.5	0.5	0.5	0.25	0.25	0.25
3	0.5	0.25	1	0.25	0.25	0.25	0.25	0.5	0.5	0.25	0.5	0.25
4	0.25	0.25	0.25	1	0.25	0.25	0.25	0.5	0.5	0.25	0.5	0.5
5	0.5	0.25	0.25	0.25	1	0.5	0.5	0.25	0.5	0.5	0.25	0.25
6	0.25	0.25	0.25	0.25	0.5	1	0.5	0.25	0.5	0.5	0.25	0.25
7	0.5	0.5	0.25	0.25	0.5	0.5	1	0.25	0.5	0.5	0.25	0.25
8	0.25	0.5	0.5	0.5	0.25	0.25	0.25	1	0.5	0.25	0.25	0.5
9	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	0.25	0.5	0.25
10	0.25	0.25	0.25	0.25	0.5	0.5	0.5	0.25	0.25	1	0.25	0.25
11	0.25	0.25	0.5	0.5	0.25	0.25	0.25	0.25	0.5	0.25	1	0.25
12	0.25	0.25	0.25	0.5	0.25	0.25	0.25	0.5	0.25	0.25	0.25	1

to be earned by the insurance or reinsurance undertaking in the segment s after the following 12 months for existing contracts, and $FP_{(future,s)}$ refers to contracts where the initial recognition date falls in the following 12 months. For all such contracts whose initial term is one year or less, it is the expected present value of premiums to be earned by the insurance or reinsurance undertaking in the segment s, but excluding the premiums to be earned during the 12 months after the initial recognition date. For all such contracts whose initial term is more than one year, it is the amount equal to 30% of the expected present value of premiums to be earned by the insurance or reinsurance undertaking in the segment s after the following 12 months. Note that the definition of the last item was amended in 2019, after the approval of the Solvency II review of 2018.

The volume measure for reserve risk of segment s is given by:

$$V_{(res,s)} = PCO_s$$

where PCO_s is the best estimate (without risk margin) of the provisions for claims outstanding for the segment s, after deduction of the amounts recoverable from reinsurance contracts and special purpose vehicles.

The geographical diversification factor of segment s is given by:

$$DIV_s = \frac{\sum_j \left(V_{(prem,j,s)} + V_{(res,j,s)} \right)^2}{\left(\sum_j V_{(prem,j,s)} + V_{(res,j,s)} \right)^2}$$

where $V_{(prem,j,s)}$ and $V_{(res,j,s)}$ are the volume measures for premium risk and reserve risk of segment s and region j. We point out that the geographical diversification is ignored for credit and suretyship, non-proportional reinsurance, and if the insurers use an undertaking-specific parameter for the volatility factor for premium risk or reserve risk.

2.4.2 Market risk

The market risk module shall consist of all of the following sub-modules:

1. Interest rate risk
2. Equity risk
3. Property risk
4. Spread risk
5. Currency risk
6. Market risk concentrations

The SCR for market risk shall be equal to the following:

$$SCR_{market} = \sqrt{\sum_{i,j} CorrM_{(i,j)}\, SCR_i\, SCR_j}$$

Table 2.5. Correlation matrix for market risk

$i \setminus j$	Interest rate	Equity	Property	Spread	Currency	Concentration
Interest rate	1	A	A	A	0.25	0
Equity	A	1	0.75	0.75	0.25	0
Property	A	0.75	1	0.5	0.25	0
Spread	A	0.75	0.5	1	0.25	0
Currency	0.25	0.25	0.25	0.25	1	0
Concentration	0	0	0	0	0	1

where $CorrM_{(i,j)}$ is the correlation parameter for market risk for sub-modules i and j (see Table 2.5), and SCR_i and SCR_j are the capital requirements for risk sub-modules i and j, respectively.

We point out that the parameter A in Table 2.5 is equal to 0 where the SCR for interest rate risk depends on the risk of an increase in the term structure of interest rates. In all other cases, the parameter A is equal to 0.5.

Note that the market risk module may affect both assets and liabilities. Hence, the effect on the asset side can be partially compensated by the effect on the liability side, and vice versa. Consider the example of an increase in the term structure of interest rates. The bond market value decreases as well as the technical provisions, because of a bigger discounting effect. Consequently, the basic own funds either increase or decrease, depending on which effect prevails. Furthermore, it is worth mentioning that EU government bonds under the SF are not exposed to spread risk and market risk concentrations.

Interest rate risk

Solvency II defines the interest rate risk sub-module. The following part is taken from Art. 105 of Solvency II.

> *The sensitivity of the values of assets, liabilities and financial instruments to changes in the term structure of interest rates, or in the volatility of interest rates.*

The SCR for interest rate risk shall be equal to the larger of the following:

- the sum, over all currencies, of the capital requirements for the risk of an increase in the term structure of interest rates;
- the sum, over all currencies, of the capital requirements for the risk of a decrease in the term structure of interest rates.

We point out that the scenario shall be coherent with the scenario of the largest net Basic SCR, related to the adjustment for the loss-absorbing capacity of technical provisions, described in Art. 206 of the Delegated Regulation.

The capital requirement for the risk of a shock of the term structure of interest rates for a given currency shall be equal to the loss in the basic own funds that

Table 2.6. Shocks in the term structure of interest rates

Maturity (years)	Shock up	Shock down
1	70%	75%
2	70%	65%
3	64%	56%
4	59%	50%
5	55%	46%
6	52%	42%
7	49%	39%
8	47%	36%
9	44%	33%
10	42%	31%
11	39%	30%
12	37%	29%
13	35%	28%
14	34%	28%
15	33%	27%
16	31%	28%
17	30%	28%
18	29%	28%
19	27%	29%
20	26%	29%
90	20%	20%

would result from an instantaneous increase or decrease in basic risk-free interest rates for that currency at different maturities in accordance with Table 2.6.

For maturities not specified in Table 2.6, the shock shall be linearly interpolated. For maturities shorter than 1 year, the shock up shall be 70%, and the shock down shall be equal to 75%. For maturities longer than 90 years, the shock shall be 20%. Furthermore, the shock up at any maturity shall be at least one percentage point. For negative basic risk-free interest rates the shock down shall be nil.

Equity risk

Solvency II defines the equity risk sub-module. The following part is taken from Art. 105 of Solvency II.

> *The sensitivity of the values of assets, liabilities and financial instruments to changes in the level or in the volatility of market prices of equities.*

The equity risk sub-module shall consist of all of the following sub-modules:

1. Type 1 equities
2. Type 2 equities
3. Qualifying infrastructure equities
4. Qualifying infrastructure corporate equities

On one hand, type 1 equities shall comprise equities listed in regulated markets in the countries which are members of the European Economic Area (EEA) or the Organisation for Economic Cooperation and Development (OECD), or traded on multilateral trading facilities, whose registered office or head office is in EU Member States. On the other hand, type 2 equities shall comprise equities listed in stock exchanges in countries which are not members of the EEA or the OECD, equities which are not listed, commodities and other alternative investments. They shall also comprise all assets other than those covered in the interest rate risk sub-module, the property risk sub-module or the spread risk sub-module, including the assets and indirect exposures where a look-through approach is not possible. Furthermore, qualifying infrastructure equities or qualifying infrastructure corporate equities shall comprise equity investments in infrastructure project entities.

The SCR for equity risk shall be equal to the following:

$$SCR_{equity} = \sqrt{SCR_{equ1}^2 + 3/2\, SCR_{equ1}\, SCR_{equ} + SCR_{equ}^2}$$

and we have:

$$SCR_{equ} = SCR_{equ2} + SCR_{quinf} + SCR_{quinfc}$$

where SCR_{equ1}, SCR_{equ2}, SCR_{quinf} and SCR_{quinfc} are the capital requirements for type 1, type 2, qualifying infrastructure and qualifying infrastructure corporate equities, respectively.

The SCR for type 1 equities shall be equal to the loss in the basic own funds that would result from the following instantaneous decreases:

- an instantaneous decrease equal to 22% in the value of type 1 equities in related undertakings, where these investments are of a strategic nature;
- an instantaneous decrease equal to 22% in the value of type 1 equities that are treated as long-term equities;
- an instantaneous decrease equal to the sum of 39% and the whole symmetric adjustment in the value of other type 1 equities.

The SCR for type 2 equities shall be equal to the loss in the basic own funds that would result from the following instantaneous decreases:

- an instantaneous decrease equal to 22% in the value of type 2 equities in related undertakings, where these investments are of a strategic nature;
- an instantaneous decrease equal to 22% in the value of type 2 equities that are treated as long-term equities;

- an instantaneous decrease equal to the sum of 49% and the whole symmetric adjustment in the value of other type 2 equities.

The SCR for qualifying infrastructure equities shall be equal to the loss in the basic own funds that would result from the following instantaneous decreases:

- an instantaneous decrease equal to 22% in the value of qualifying infrastructure equities in related undertakings, where these investments are of a strategic nature;
- an instantaneous decrease equal to 22% in the value of qualifying infrastructure equities that are treated as long-term equities;
- an instantaneous decrease equal to the sum of 30% and 77% of the symmetric adjustment in the value of qualifying infrastructure equities.

The SCR for qualifying infrastructure corporate equities shall be equal to the loss in the basic own funds that would result from the following instantaneous decreases:

- an instantaneous decrease equal to 22% in the value of qualifying infrastructure corporate equities in related undertakings, where these investments are of a strategic nature;
- an instantaneous decrease equal to 22% in the value of qualifying infrastructure corporate equities that are treated as long-term equities;
- an instantaneous decrease equal to the sum of 36% and 92% of the symmetric adjustment in the value of qualifying infrastructure corporate equities.

Furthermore, in some cases, equities can be subject to the duration-based equity risk. Note that the criteria for the calculation of the capital requirements for type 1, type 2, qualifying infrastructure and qualifying infrastructure corporate equities were amended in 2019, after the approval of the Solvency II review of 2018.

The Delegated Regulation describes the equity investments of a strategic nature. The following part is taken from Art. 171 of the Delegated Regulation.

> *Equity investments of a strategic nature shall mean equity investments for which the participating insurance or reinsurance undertaking demonstrates the following:*
>
> *(a) that the value of the equity investment is likely to be materially less volatile for the following 12 months than the value of other equities over the same period as a result of both the nature of the investment and the influence exercised by the participating undertaking in the related undertaking;*
>
> *(b) that the nature of the investment is strategic, taking into account all relevant factors, including:*
>
> *(i) the existence of a clear decisive strategy to continue holding the participation for long period;*
>
> *(ii) the consistency of the strategy referred to in point (a) with the main policies guiding or limiting the actions of the undertaking;*

(iii) *the participating undertaking's ability to continue holding the participation in the related undertaking;*
(iv) *the existence of a durable link;*
(v) *where the insurance or reinsurance participating company is part of a group, the consistency of such strategy with the main policies guiding or limiting the actions of the group.*

In addition, the Delegated Regulation describes the long-term equity investments. The following part is taken from Art. 171a (introduced in 2019) of the Delegated Regulation.

Equity investments may be treated as long-term equity investments if the insurance or reinsurance undertaking demonstrates, to the satisfaction of the supervisory authority, that all of the following conditions are met:

(a) *the sub-set of equity investments as well as the holding period of each equity investment within the sub-set are clearly identified;*
(b) *the sub-set of equity investment is included within a portfolio of assets which is assigned to cover the best estimate of a portfolio of insurance or reinsurance obligations corresponding to one or several clearly identified businesses, and the undertaking maintains that assignment over the lifetime of the obligations;*
(c) *the portfolio of insurance or reinsurance obligations, and the assigned portfolio of assets referred to in point (b) are identified, managed and organised separately from the other activities of the undertaking, and the assigned portfolio of assets cannot be used to cover losses arising from other activities of the undertaking;*
(d) *the technical provisions within the portfolio of insurance or reinsurance obligations referred to in point (b) only represent a part of the total technical provisions of the insurance or reinsurance undertaking;*
(e) *the average holding period of equity investments in the sub-set exceeds 5 years, or where the average holding period of the sub-set is lower than 5 years, the insurance or reinsurance undertaking does not sell any equity investments within the sub-set until the average holding period exceeds 5 years;*
(f) *the sub-set of equity investments consists only of equities that are listed in the EEA or of unlisted equities of companies that have their head offices in countries that are members of the EEA;*
(g) *the solvency and liquidity position of the insurance or reinsurance undertaking, as well as its strategies, processes and reporting procedures with respect to asset-liability management, are such as to ensure, on an ongoing basis and under stressed conditions, that it is able to avoid forced sales of each equity investments within the sub-set for at least 10 years;*

(h) *the risk management, asset-liability management and investment policies of the insurance or reinsurance undertaking reflects the undertaking's intention to hold the sub-set of equity investments for a period that is compatible with the requirement of point (e) and its ability to meet the requirement of point (g).*

Note that the treatment of equity investments as long-term equity investments shall not be reverted back to an approach that does not include long-term equity investments. Furthermore, where an insurance or reinsurance undertaking that treats a sub-set of equity investments as long-term equity investments is no longer able to comply with the conditions, it shall immediately inform the supervisory authority.

The symmetric adjustment is an anti-procyclicality measure, because the stress parameters are reduced when the market drops, in order to avoid that insurers sell equities and make the market drop further. The symmetric adjustment is given by:

$$SA = \frac{1}{2}\left(\frac{CI - AI}{AI} - 8\%\right)$$

where CI is the current level of the equity index, and AI is the weighted average of the daily levels of the equity index over the last 36 months, where the weights for all daily levels shall be equal. We point out that the symmetric adjustment shall not be lower than -10% or higher than $+10\%$.

We point out that a transitional measure for standard equity risk shall only be applied to type 1 equities that were purchased on or before January 1, 2016, and which are not subject to the duration-based equity risk.

Property risk

Solvency II defines the property risk sub-module. The following part is taken from Art. 105 of Solvency II.

The sensitivity of the values of assets, liabilities and financial instruments to changes in the level or in the volatility of market prices of real estate.

The capital requirement for property risk shall be equal to the loss in the basic own funds that would result from an instantaneous decrease equal to 25% in the value of real estate. This coefficient was calibrated using historical time series for properties in different EU countries, with specific focus on the UK market data.

2.5 Internal models

Solvency II describes the full or partial IM. The following several parts are taken from Section 3 (Art. 112-127) of Solvency II.

The Solvency Capital Requirement shall be calculated, either in accordance with the standard formula [...] or using an internal model [...].

Member States shall ensure that insurance or reinsurance undertakings may calculate the Solvency Capital Requirement using a full or partial internal model as approved by the supervisory authorities.

Insurance and reinsurance undertakings may use partial internal models for the calculation of one or more of the following:

(a) one or more risk modules, or sub-modules, of the Basic Solvency Capital Requirement [...];

(b) the capital requirement for operational risk [...];

(c) the adjustment [...].

In addition, partial modelling may be applied to the whole business of insurance and reinsurance undertakings, or only to one or more major business units.

After having received approval [...], insurance and reinsurance undertakings shall not revert to calculating the whole or any part of the Solvency Capital Requirement in accordance with the standard formula, [...] except in duly justified circumstances and subject to the approval of the supervisory authorities.

Where it is inappropriate to calculate the Solvency Capital Requirement in accordance with the standard formula, [...] because the risk profile of the insurance or reinsurance undertaking concerned deviates significantly from the assumptions underlying the standard formula calculation, the supervisory authorities may, by means of a decision stating the reasons, require the undertaking concerned to use an internal model to calculate the Solvency Capital Requirement, or the relevant risk modules thereof.

The requirements for an IM shall consist of the following:

1. Use test

   Insurance and reinsurance undertakings shall demonstrate that the internal model is widely used in and plays an important role in their system of governance.

2. Statistical quality standards

   The methods used to calculate the probability distribution forecast shall be based on adequate, applicable and relevant actuarial and statistical techniques and shall be consistent with the methods used to calculate technical provisions.

3. Calibration standards

> *Insurance and reinsurance undertakings may use a different time period or risk measure [...] for internal modelling purposes as long as the outputs of the internal model can be used by those undertakings to calculate the Solvency Capital Requirement in a manner that provides policy holders and beneficiaries with a level of protection equivalent [...].*

4. Profit and loss attribution

   > *They shall demonstrate how the categorisation of risk chosen in the internal model explains the causes and sources of profits and losses. The categorisation of risk and attribution of profits and losses shall reflect the risk profile of the insurance and reinsurance undertakings.*

5. Validation standards

   > *The model validation process shall include an effective statistical process for validating the internal model which enables the insurance and reinsurance undertakings to demonstrate to their supervisory authorities that the resulting capital requirements are appropriate.*

6. Documentation standards

   > *The documentation shall provide a detailed outline of the theory, assumptions, and mathematical and empirical bases underlying the internal model.*

Note that a full IM considers all the risk modules and sub-modules, differently from a partial IM. The choice to adopt a full or partial IM is not as simple as it could seem. This depends on several reasons, such as commercial reasons and the need of human resources (mainly for quantitative modelling) involved in the project for a considerable time.

2.6 Supervisory review process

Solvency II describes the supervisory review process. The following part is taken from Art. 36 of Solvency II.

> *Member States shall ensure that the supervisory authorities review and evaluate the strategies, processes and reporting procedures which are established by the insurance and reinsurance undertakings to comply with the laws, regulations and administrative provisions adopted pursuant to this Directive.*
>
> *The supervisory authorities shall in particular review and evaluate compliance with the following:*
>
> *(a) the system of governance, including the own-risk and solvency assessment [...];*
>
> *(b) the technical provisions [...];*

(c) the capital requirements [...];

(d) the investment rules [...];

(e) the quality and quantity of own funds [...];

(f) where the insurance or reinsurance undertaking uses a full or partial internal model, on-going compliance with the requirements for full and partial internal models [...].

The supervisory authorities shall approve an IM (see Section 2.5) and evaluate compliance with their requirements. We point out that the supervisory review process is important to ensure that insurance companies comply with the law. In the case that insurance companies do not comply with the law, the supervisory authorities may set a capital add-on.

2.6.1 Capital add-on

Solvency II describes the capital add-on. The following part is taken from Art. 37 of Solvency II.

> *Following the supervisory review process supervisory authorities may in exceptional circumstances set a capital add-on for an insurance or reinsurance undertaking by a decision stating the reasons. That possibility shall exist only in the following cases:*
>
> *(a) the supervisory authority concludes that the risk profile of the insurance or reinsurance undertaking deviates significantly from the assumptions underlying the Solvency Capital Requirement, as calculated using the standard formula [...];*
>
> *(b) the supervisory authority concludes that the risk profile of the insurance or reinsurance undertaking deviates significantly from the assumptions underlying the Solvency Capital Requirement, as calculated using an internal model or partial internal model [...], because certain quantifiable risks are captured insufficiently and the adaptation of the model to better reflect the given risk profile has failed within an appropriate timeframe;*
>
> *(c) the supervisory authority concludes that the system of governance of an insurance or reinsurance undertaking deviates significantly from the standards [...], that those deviations prevent it from being able to properly identify, measure, monitor, manage and report the risks that it is or could be exposed to and that the application of other measures is in itself unlikely to improve the deficiencies sufficiently within an appropriate time frame;*
>
> *[...]*

We point out that the supervisory review process and the capital add-on prevent insurance companies to have low attention when using the SF if the underlying assumptions are not respected, and they also prevent the risk of a weak governance.

2.7 System of governance

Solvency II describes the system of governance. The following part is taken from Art. 41 of Solvency II.

> *Member States shall require all insurance and reinsurance undertakings to have in place an effective system of governance which provides for sound and prudent management of the business.*

We point out that the system of governance is important to ensure that insurance companies work well, avoiding the above mentioned supervisory review process and capital add-on.

2.8 Risk management system

Solvency II describes the risk management system. The following part is taken from Art. 44 of Solvency II.

> *Insurance and reinsurance undertakings shall have in place an effective risk-management system comprising strategies, processes and reporting procedures necessary to identify, measure, monitor, manage and report, on a continuous basis the risks, at an individual and at an aggregated level, to which they are or could be exposed, and their interdependencies.*
>
> *That risk-management system shall be effective and well integrated into the organisational structure and in the decision-making processes of the insurance or reinsurance undertaking with proper consideration of the persons who effectively run the undertaking or have other key functions.*
>
> *The risk-management system shall cover the risks to be included in the calculation of the Solvency Capital Requirement [...] as well as the risks which are not or not fully included in the calculation thereof.*
>
> *For insurance and reinsurance undertakings using a partial or full internal model [...] the risk-management function shall cover the following additional tasks:*
>
> *(a) to design and implement the internal model;*
>
> *(b) to test and validate the internal model;*
>
> *(c) to document the internal model and any subsequent changes made to it;*
>
> *(d) to analyse the performance of the internal model and to produce summary reports thereof;*
>
> *(e) to inform the administrative, management or supervisory body about the performance of the internal model, suggesting areas needing improvement, and up-dating that body on the status of efforts to improve previously identified weaknesses.*

We point out that insurance company work with risks. Hence, in a parallel way than the system of governance, the risk management system is also important to ensure that insurance companies work well, avoiding the supervisory review process and capital add-on.

2.8.1 Own Risk and Solvency Assessment

Solvency II describes the ORSA. The following part is taken from Art. 45 of Solvency II.

> *As part of its risk-management system every insurance undertaking and reinsurance undertaking shall conduct its own risk and solvency assessment.*
>
> *That assessment shall include at least the following:*
>
> *(a) the overall solvency needs taking into account the specific risk profile, approved risk tolerance limits and the business strategy of the undertaking;*
>
> *(b) the compliance, on a continuous basis, with the capital requirements [...] and with the requirements regarding technical provisions [...];*
>
> *(c) the significance with which the risk profile of the undertaking concerned deviates from the assumptions underlying the Solvency Capital Requirement [...], calculated with the standard formula [...] or with its partial or full internal model [...].*
>
> *The own-risk and solvency assessment shall be an integral part of the business strategy and shall be taken into account on an ongoing basis in the strategic decisions of the undertaking.*
>
> *Insurance and reinsurance undertakings shall perform the assessment [...] regularly and without any delay following any significant change in their risk profile.*
>
> *The insurance and reinsurance undertakings shall inform the supervisory authorities of the results of each own-risk and solvency assessment [...].*

We point out that the SCR shall be calculated over a one-year period, hence the view according to Solvency II is short-term. In this situation, the risk is not properly managed, since problems could emerge in the future. Nevertheless, the ORSA encourages insurance companies to have a medium-term view as well.

Forward Looking Assessment of Own Risks

Solvency II requires the insurance and reinsurance undertakings to perform a regular Forward Looking Assessment of the undertaking's Own Risks (FLAOR) as part of the risk management system. The main purpose of the FLAOR is to ensure that the undertaking engages in the process of assessing all the risks inherent to its business and determines the corresponding capital needs. To achieve this, an undertaking needs adequate and robust processes to assess, monitor and measure its risks and overall solvency needs, and also to ensure that the output from the assessment forms an important part of the decision making processes of the undertaking. We point out that different parts of this paragraph are taken from the EIOPA Explanatory Text (EIOPA, 2013a).

The EIOPA Guidelines on FLAOR demand some provisions. The following part is taken from Guideline 13 of the EIOPA Guidelines (EIOPA, 2013b).

> *In accordance with Article 45 of Solvency II, national competent authorities should ensure that the undertaking's assessment of the overall solvency needs is forward-looking, including a medium term or long term perspective as appropriate.*

The analysis of the undertaking's ability to continue as a going concern and the financial resources needed to do so over a time horizon of more than one year is an important part of the FLAOR.

Unless an undertaking is in a winding-up situation, it has to consider how it can ensure that it can continue as a going concern. In order to do this successfully, not only does it have to assess its current risks, but also the risks it will or could face in the long term. That means that, depending on the complexity of the undertaking's business, it may be appropriate to perform long term projections of the business, which are in any case a key part of any undertaking's financial planning. This might include business plans and projections of the economic balance sheet as well as variation analysis to reconcile these two items. These projections are required to feed into the FLAOR in order to enable the undertaking to form an opinion on its overall solvency needs and own funds in a forward looking perspective.

An undertaking also identifies and takes into account external factors that could have an adverse impact on its overall solvency needs or on its own funds. Such external factors could include changes in the economic conditions, the legal framework, the fiscal environment, the insurance market, technical developments that have an impact on underwriting risk, or any other probable relevant event. The undertaking will need to consider as part of its capital management plans and capital projections how it might respond to unexpected changes in external factors.

Chapter 3

Economic scenario generator

In this chapter, we present the economic scenario generator used to simulate future market returns in our numerical analysis. The models here considered are based on differential equations. Once we have described over time the real-world distributions of the stock, property and bond prices, we are able to obtain the distribution of the annual or average rate of return time by time by using equation 1.9 or 1.10. Furthermore, the CPI will contribute to describe the paid claims. We assume that the market is frictionless, meaning that all securities are perfectly divisible and that no short-sale restrictions, transaction costs, or taxes are present. The security trading is continuous, and there are no riskless arbitrage opportunities.

The random variables directly involved in the analysis framework and non-life model are indicated by uppercase Latin letters. However, due to the complexity of the mathematical notation of this chapter, this general rule to distinguish random variables, parameters, and coefficients is not valid. Here, the random variables are introduced by a stochastic differential equation, or they are a function of it.

3.1 Risk-neutral economic scenario generator

Risk-neutral probabilities are preferred for pricing of interest rate derivatives and, more in general, for market-consistent valuation.

3.1.1 Nominal and real interest rates and CPI

We now introduce short-rate models (see among others Brigo and Mercurio, 2006), in order to describe over time the distribution of the short-term nominal and real interest rate at time t (also called instantaneous short rate, since it applies to an infinitesimally short period of time at time t). Once it is specified, we are able to compute the zero-coupon bond price and determine the initial zero curve and its future evolution. As a matter of fact, the price at time t of a zero-coupon bond that provides a terminal payoff equal to 1 at maturity date $T > t$ is given by:

$$P(t,T) = e^{-R(t,T)(T-t)}$$

As a result, the zero-coupon interest rate (continuously compounded) at time t for a term of $T - t$ is given by:

$$R(t,T) = -\frac{\ln P(t,T)}{T-t} \tag{3.1}$$

Be aware that the continuously compounded zero-coupon interest rate is different to the short-term interest rate. Contrary to the former, the latter applies to an infinitesimally short period of time. Once we have the zero-coupon bond price, we can use the equation above to get the continuously compounded zero-coupon interest rate. This equation will be useful to calibrate the risk-premium parameters of the real-world short-rate models.

We assume that the nominal short rate follows a two-additive-factor Gaussian model (i.e. G2++ model) that is given by:

$$n(t) = x_n(t) + y_n(t) + \varphi_n(t)$$

where $x_n(t)$ and $y_n(t)$ are the nominal state variables, and $\varphi_n(t)$ is a deterministic function of time that allows the model to fit perfectly the nominal term structure observed in the market.

The nominal state variables under the nominal risk-neutral measure Q_n satisfy the following stochastic differential equations:

$$dx_n(t) = -a_n\, x_n(t)\, dt + \sigma_n\, dW_n^{x_n}(t) \quad \text{with} \quad x_n(0) = 0$$
$$dy_n(t) = -b_n\, y_n(t)\, dt + \eta_n\, dW_n^{y_n}(t) \quad \text{with} \quad y_n(0) = 0$$

where a_n, b_n, σ_n and η_n are positive constants, and $W_n^{x_n}(t)$ and $W_n^{y_n}(t)$ are standard Brownian motions with instantaneous correlation $-1 \leq \rho_n \leq 1$.

The deterministic function of time that allows the model to fit perfectly the nominal term structure observed in the market is given by:

$$\varphi_n(t) = f_n^M(0,t) + \frac{\sigma_n^2}{2a_n^2}\left(1 - e^{-a_n t}\right)^2 + \frac{\eta_n^2}{2b_n^2}\left(1 - e^{-b_n t}\right)^2 + \frac{\rho_n\,\sigma_n\,\eta_n}{a_n\, b_n}\left(1 - e^{-a_n t}\right)\left(1 - e^{-b_n t}\right)$$

where $f_n^M(0,t)$ is the instantaneous forward rate at initial time for the maturity t implied by the nominal term structure observed in the market.

The stochastic differential equations have explicit solutions that are given by:

$$x_n(t) = x_n(s)\, e^{-a_n(t-s)} + \sigma_n \int_s^t e^{-a_n(t-u)}\, dW_n^{x_n}(u)$$
$$y_n(t) = y_n(s)\, e^{-b_n(t-s)} + \eta_n \int_s^t e^{-b_n(t-u)}\, dW_n^{y_n}(u)$$

Hence, the vector of the nominal state variables under the nominal risk-neutral measure Q_n and conditional on the sigma-field $\mathcal{F}_s$ is normally distributed, with mean vector and variance-covariance matrix given by:

$$\begin{bmatrix} x_n(s)\, e^{-a_n(t-s)} \\ y_n(s)\, e^{-b_n(t-s)} \end{bmatrix}, \quad \begin{bmatrix} \dfrac{\sigma_n^2}{2a_n}\left(1 - e^{-2a_n(t-s)}\right) & \dfrac{\rho_n\,\sigma_n\,\eta_n}{a_n + b_n}\left(1 - e^{-(a_n+b_n)(t-s)}\right) \\ \cdot & \dfrac{\eta_n^2}{2b_n}\left(1 - e^{-2b_n(t-s)}\right) \end{bmatrix}$$

The price at time t (conditional on the sigma-field $\mathcal{F}_t$) of a nominal zero-coupon bond with maturity in $T > t$ is thus found to be:

$$P_n(t,T) = \exp\left\{ -\int_t^T \varphi_n(u)\,du - \frac{1-e^{-a_n(T-t)}}{a_n}\,x_n(t) \right. \\ \left. - \frac{1-e^{-b_n(T-t)}}{b_n}\,y_n(t) + \frac{1}{2}\,V_n(t,T) \right\}$$

The integral has an explicit solution that is given by:

$$\exp\left\{ -\int_t^T \varphi_n(u)\,du \right\} = \frac{P_n^M(0,T)}{P_n^M(0,t)}\,\exp\left\{ -\frac{1}{2}\,V_n(0,T) + \frac{1}{2}\,V_n(0,t) \right\}$$

where $P_n^M(0,t)$ is the price at initial time of a zero-coupon bond with maturity in t implied by the nominal term structure observed in the market. Moreover, we have:

$$\begin{aligned} V_n(t,T) &= \frac{\sigma_n^2}{a_n^2}\left[T-t+\frac{2}{a_n}\,e^{-a_n(T-t)}-\frac{1}{2a_n}\,e^{-2a_n(T-t)}-\frac{3}{2a_n}\right] \\ &+ \frac{\eta^2}{b_n^2}\left[T-t+\frac{2}{b_n}\,e^{-b_n(T-t)}-\frac{1}{2b_n}\,e^{-2b_n(T-t)}-\frac{3}{2b_n}\right] \\ &+ \frac{2\rho_n\,\sigma_n\,\eta_n}{a_n\,b_n}\left[T-t+\frac{e^{-a_n(T-t)}-1}{a_n}+\frac{e^{-b_n(T-t)}-1}{b_n}-\frac{e^{-(a_n+b_n)(T-t)}-1}{a_n+b_n}\right] \end{aligned} \tag{3.2}$$

We can write the price at time t of a nominal zero-coupon bond with maturity in $T > t$ in this useful compact form:

$$P_n(t,T) = A_n(t,T)\,e^{-B(a_n,t,T)\,x_n(t)-B(b_n,t,T)\,y_n(t)} \tag{3.3}$$

where:

$$A_n(t,T) = \frac{P_n^M(0,T)}{P_n^M(0,t)}\,\exp\left\{\frac{1}{2}\left[V_n(t,T)-V_n(0,T)+V_n(0,t)\right]\right\}$$

$$B(z,t,T) = \frac{1-e^{-z(T-t)}}{z}$$

We assume that also the real short rate follows a G2++ model that is given by:

$$r(t) = x_r(t) + y_r(t) + \varphi_r(t)$$

where $x_r(t)$ and $y_r(t)$ are the real state variables, and $\varphi_r(t)$ is a deterministic function of time that allows the model to fit perfectly the real term structure observed in the market.

The real state variables under the real risk-neutral measure Q_r satisfy the following stochastic differential equations:

$$dx_r(t) = -a_r\,x_r(t)\,dt + \sigma_r\,dW_r^{x_r}(t) \quad \text{with} \quad x_r(0) = 0$$

$$dy_r(t) = -b_r\,y_r(t)\,dt + \eta_r\,dW_r^{y_r}(t) \quad \text{with} \quad y_r(0) = 0$$

where a_r, b_r, σ_r and η_r are positive constants, and $W_r^{x_r}(t)$ and $W_r^{y_r}(t)$ are standard Brownian motions with instantaneous correlation $-1 \le \rho_r \le 1$.

The deterministic function of time that allows the model to fit perfectly the real term structure observed in the market is given by:

$$\varphi_r(t) = f_r^M(0,t) + \frac{\sigma_r^2}{2a_r^2}\left(1 - e^{-a_r t}\right)^2 + \frac{\eta_r^2}{2b_r^2}\left(1 - e^{-b_r t}\right)^2 + \frac{\rho_r\,\sigma_r\,\eta_r}{a_r\,b_r}\left(1 - e^{-a_r t}\right)\left(1 - e^{-b_r t}\right)$$

where $f_r^M(0,t)$ is the instantaneous forward rate at initial time for the maturity t implied by the real term structure observed in the market.

The explicit solutions, mean vector and variance-covariance matrix of the real state variables and the price of a real zero-coupon bond are analogous to the nominal case, and they can be found by replacing the sub and superscripts n with r.

We now assume that the instantaneous correlations between the nominal and real state variables are given by:

$$d\begin{bmatrix} W_n^{x_n} \\ W_n^{y_n} \\ W_r^{x_r} \\ W_r^{y_r} \end{bmatrix} d\begin{bmatrix} W_n^{x_n} & W_n^{y_n} & W_r^{x_r} & W_r^{y_r} \end{bmatrix} = \begin{bmatrix} 1 & \rho_n & \rho_{x_n,x_r} & \rho_{x_n,y_r} \\ \cdot & 1 & \rho_{y_n,x_r} & \rho_{y_n,y_r} \\ \cdot & \cdot & 1 & \rho_r \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} dt$$

The CPI under the nominal risk-neutral measure Q_n satisfies the following stochastic differential equation:

$$dI(t) = (n(t) - r(t))\,I(t)\,dt + \sigma_I\,I(t)\,dW_n^I(t) \qquad \text{with} \qquad I(0) = I_0$$

where σ_I and I_0 are positive constants, and $W_n^I(t)$ is a standard Brownian motion.

The stochastic differential equation has an explicit solution that is given by:

$$I(t) = I(s)\,\exp\left\{\int_s^t (n(u) - r(u))\,du - \frac{1}{2}\,\sigma_I^2\,(t-s) + \sigma_I\left(W_n^I(t) - W_n^I(s)\right)\right\} \tag{3.4}$$

We assume that the instantaneous correlations between the CPI and real state variables are given by:

$$d\begin{bmatrix} W_n^{I} \\ W_r^{x_r} \\ W_r^{y_r} \end{bmatrix} d\begin{bmatrix} W_n^{I} & W_r^{x_r} & W_r^{y_r} \end{bmatrix} = \begin{bmatrix} 1 & \rho_{x_r,I} & \rho_{y_r,I} \\ \cdot & 1 & \rho_r \\ \cdot & \cdot & 1 \end{bmatrix} dt$$

We now apply the change-of-numeraire technique, and we obtain that the real state variables under the nominal risk-neutral measure Q_n satisfy the following stochastic differential equations:

$$dx_r(t) = (-a_r\,x_r(t) - \rho_{x_r,I}\,\sigma_r\,\sigma_I)\,dt + \sigma_r\,dW_n^{x_r}(t)$$
$$dy_r(t) = (-b_r\,y_r(t) - \rho_{y_r,I}\,\eta_r\,\sigma_I)\,dt + \eta_r\,dW_n^{y_r}(t)$$

where the standard Brownian motions keep the same correlation structure described above. Furthermore, the instantaneous correlations that are not listed above can be found by replacing the respective sub and superscripts.

3.1.2 Stock and property prices

We assume that the non-dividend-paying stock price under the nominal risk-neutral measure Q_n follows a Geometric Brownian Motion with stochastic drift that satisfies the following stochastic differential equation:

$$dS(t) = n(t)\,S(t)\,dt + \sigma_S\,S(t)\,dW_n^S(t) \quad \text{with} \quad S(0) = S_0$$

where σ_S and S_0 are positive constants, and $W_n^S(t)$ is a standard Brownian motion.

The stochastic differential equation has an explicit solution that is given by:

$$S(t) = S(s)\,\exp\left\{\int_s^t n(u)\,du - \frac{1}{2}\,\sigma_S^2\,(t-s) + \sigma_S\,\left(W_n^S(t) - W_n^S(s)\right)\right\} \tag{3.5}$$

Moreover, we assume that also the property price under the nominal risk-neutral measure Q_n follows a Geometric Brownian Motion with stochastic drift that satisfies the following stochastic differential equation:

$$dH(t) = n(t)\,H(t)\,dt + \sigma_H\,H(t)\,dW_n^H(t) \quad \text{with} \quad H(0) = H_0$$

where σ_H and H_0 are positive constants, and $W_n^H(t)$ is a standard Brownian motion.

The stochastic differential equation has an explicit solution that is given by:

$$H(t) = H(s)\,\exp\left\{\int_s^t n(u)\,du - \frac{1}{2}\,\sigma_H^2\,(t-s) + \sigma_H\,\left(W_n^H(t) - W_n^H(s)\right)\right\}$$

We point out that there are some instantaneous correlations between the nominal and real state variables, CPI, stock and property prices, and they can be found by replacing the respective sub and superscripts.

3.2 Financial derivatives

Financial derivatives are financial instruments that are linked to a specific financial instrument or indicator or commodity. They are typically used for the calibration of risk-neutral models. In this thesis, we are only interested in (nominal) interest rate derivatives, inflation-indexed derivatives, and asset-price derivatives.

The main interest rate derivatives are the interest-rate cap or floor (IRCF) and the European payer or receiver swaption (ES). These instruments are typically quoted in terms of their volatility. On the other side, the main inflation-indexed derivatives are the zero-coupon inflation-indexed swap (ZCIIS), the year-on-year inflation-indexed swap (YYIIS), and the inflation-indexed cap or floor (IICF). The most important asset-price derivative for our purpose is the European stock-price option (EO).

3.2.1 Interest-rate cap or floor

An interest-rate cap (IRC) is a call option, which depends on the simply-compounded spot interest rate. The corresponding put option is an interest-rate floor (IRF).

An IRCF can be decomposed in a stream of interest-rate caplets or floorlets with the set of payment dates $T_{\alpha+1}, ..., T_\beta$. Their payoff at time T_i is given by:

$$N\,\tau_i \left[\omega\left(\frac{1-P_n(T_{i-1},T_i)}{\tau_i\,P_n(T_{i-1},T_i)} - K\right)\right]^+$$

where $\omega = 1$ ($\omega = -1$) for a cap (floor), N is the nominal value, K is the strike rate of the contract, and τ_i is the year fraction between T_i and T_{i-1} (we thus have that $\tau_i = T_i - T_{i-1}$ when T_i and T_{i-1} are real numbers).

According to the Bachelier's model (Bachelier, 1900), the price at time $t \leq T_\alpha$ (conditional on the sigma-field $\mathcal{F}_t$) of an IRCF is given by:

$$\begin{aligned}
&\mathrm{IRCF}^{\mathrm{Bachelier}}(t,\mathcal{T},\tau,N,K,\sigma_{\alpha,\beta}) \\
&\quad = \sum_{i=\alpha+1}^{\beta}\left[\omega\left(\frac{P_n(t,T_{i-1})-P_n(t,T_i)}{\tau_i\,P_n(t,T_i)} - K\right)\Phi\left(\omega\,\frac{\frac{P_n(t,T_{i-1})-P_n(t,T_i)}{\tau_i\,P_n(t,T_i)} - K}{\sigma_{\alpha,\beta}\sqrt{T_{i-1}-t}}\right)\right. \\
&\quad\quad \left. + \sigma_{\alpha,\beta}\sqrt{T_{i-1}-t}\,\phi\left(\frac{\frac{P_n(t,T_{i-1})-P_n(t,T_i)}{\tau_i\,P_n(t,T_i)} - K}{\sigma_{\alpha,\beta}\sqrt{T_{i-1}-t}}\right)\right] N\,\tau_i\,P_n(t,T_i)
\end{aligned}$$

where $\mathcal{T} = \{T_\alpha, ..., T_\beta\}$ is the set of payment and/or reset dates, $\tau = \{\tau_{\alpha+1}, ..., \tau_\beta\}$ is the set of corresponding year fractions, and $\sigma_{\alpha,\beta}$ is the volatility parameter for the IRCF. Moreover, Φ and ϕ are respectively the standard Normal cumulative and probability distribution functions.

The price at time $t \leq T_\alpha$ (conditional on the sigma-field $\mathcal{F}_t$) of an IRCF under the G2++ model is given by (Brigo and Mercurio, 2006):

$$\begin{aligned}
&\mathrm{IRCF}^{\mathrm{G2++}}(t,\mathcal{T},\tau,N,K) \\
&\quad = \sum_{i=\alpha+1}^{\beta}\omega\left[P_n(t,T_{i-1})\,N\,\Phi\left(\omega\,\frac{\ln\frac{P_n(t,T_{i-1})}{(1+K\,\tau_i)\,P_n(t,T_i)} + \frac{1}{2}\,\Sigma(t,T_{i-1},T_i)^2}{\Sigma(t,T_{i-1},T_i)}\right)\right. \\
&\quad\quad \left. - N\,(1+K\,\tau_i)\,P_n(t,T_i)\,\Phi\left(\omega\,\frac{\ln\frac{P_n(t,T_{i-1})}{(1+K\,\tau_i)\,P_n(t,T_i)} - \frac{1}{2}\,\Sigma(t,T_{i-1},T_i)^2}{\Sigma(t,T_{i-1},T_i)}\right)\right]
\end{aligned}$$

where:

$$\begin{aligned}
\Sigma(t,T_{i-1},T_i)^2 &= \frac{\sigma_n^2}{2a_n^3}\left(1-e^{-a_n\tau_i}\right)^2\left(1-e^{-2a_n(T_{i-1}-t)}\right) \\
&\quad + \frac{\eta_n^2}{2b_n^3}\left(1-e^{-b_n\tau_i}\right)^2\left(1-e^{-2b_n(T_{i-1}-t)}\right) \\
&\quad + \frac{2\rho_n\,\sigma_n\,\eta_n}{a_n\,b_n\,(a_n+b_n)}\left(1-e^{-a_n\tau_i}\right)\left(1-e^{-b_n\tau_i}\right)\left(1-e^{-(a_n+b_n)(T_{i-1}-t)}\right)
\end{aligned}$$

3.2.2 European payer or receiver swaption

A European payer swaption (EPS) is an option giving the right to enter at a given future time (i.e. the swaption maturity) in a payer interest-rate swap with a given length (i.e. the swaption tenor). The option giving the right to enter in a receiver interest-rate swap is a European receiver swaption (ERS).

Given a set of payment dates $T_{\alpha+1}, ..., T_\beta$ for the underlying interest-rate swap, the payoff of a ES at time T_α is given by:

$$N \left[\omega \sum_{i=\alpha+1}^{\beta} P_n(T_\alpha, T_i)\, \tau_i \left(\frac{P_n(T_\alpha, T_{i-1}) - P_n(T_\alpha, T_i)}{\tau_i\, P_n(T_\alpha, T_i)} - K \right) \right]^+$$

where $\omega = 1$ ($\omega = -1$) for a payer (receiver) swaption, N is the nominal value, K is the strike rate of the contract, and τ_i is the year fraction between T_i and T_{i-1} (we thus have that $\tau_i = T_i - T_{i-1}$ when T_i and T_{i-1} are real numbers).

According to the Bachelier's model (Bachelier, 1900), the price at time $t \leq T_\alpha$ (conditional on the sigma-field $\mathcal{F}_t$) of a ES is given by:

$$\begin{aligned}
&\mathrm{ES}^{\text{Bachelier}}(t, \mathcal{T}, \tau, N, K, \sigma_{\alpha,\beta}) \\
&\quad = \left[\omega \left(\frac{P_n(t, T_\alpha) - P_n(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i\, P_n(t, T_i)} - K \right) \Phi \left(\omega\, \frac{\frac{P_n(t,T_\alpha) - P_n(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i\, P_n(t,T_i)} - K}{\sigma_{\alpha,\beta} \sqrt{T_\alpha - t}} \right) \right. \\
&\qquad \left. + \sigma_{\alpha,\beta} \sqrt{T_\alpha - t}\, \phi \left(\frac{\frac{P_n(t,T_\alpha) - P_n(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i\, P_n(t,T_i)} - K}{\sigma_{\alpha,\beta} \sqrt{T_\alpha - t}} \right) \right] \sum_{i=\alpha+1}^{\beta} N\, \tau_i\, P_n(t, T_i)
\end{aligned}$$

where $\mathcal{T} = \{T_\alpha, ..., T_\beta\}$ is the set of payment and/or reset dates for the underlying interest-rate swap, $\tau = \{\tau_{\alpha+1}, ..., \tau_\beta\}$ is the set of corresponding year fractions, and $\sigma_{\alpha,\beta}$ is the volatility parameter for the ES. Moreover, Φ and ϕ are respectively the standard Normal cumulative and probability distribution functions.

According to Schrager and Pelsser (2006), an approximation of the price at time $t \leq T_\alpha$ (conditional on the sigma-field $\mathcal{F}_t$) of a ES under the G2++ model is given by:

$$\begin{aligned}
&\mathrm{ES}^{\text{G2++}}(t, \mathcal{T}, \tau, N, K) \\
&\quad = \left[\omega \left(\frac{P_n(t, T_\alpha) - P_n(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i\, P_n(t, T_i)} - K \right) \Phi \left(\omega\, \frac{\frac{P_n(t,T_\alpha) - P_n(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i\, P_n(t,T_i)} - K}{S(t, T_\alpha, T_\beta)} \right) \right. \\
&\qquad \left. + S(t, T_\alpha, T_\beta)\, \phi \left(\frac{\frac{P_n(t,T_\alpha) - P_n(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i\, P_n(t,T_i)} - K}{S(t, T_\alpha, T_\beta)} \right) \right] \sum_{i=\alpha+1}^{\beta} N\, \tau_i\, P_n(t, T_i)
\end{aligned}$$

where:

$$\begin{aligned}
S(t, T_\alpha, T_\beta)^2 = {} & \frac{\sigma_n^2}{2a_n} \left(e^{2a_n(T_\alpha - t)} - 1 \right) C_{x_n}(t, T_\alpha, T_\beta)^2 \\
& + \frac{\eta_n^2}{2b_n} \left(e^{2b_n(T_\alpha - t)} - 1 \right) C_{y_n}(t, T_\alpha, T_\beta)^2 \\
& + \frac{2\rho_n\, \sigma_n\, \eta_n}{a_n + b_n} \left(e^{(a_n + b_n)(T_\alpha - t)} - 1 \right) C_{x_n}(t, T_\alpha, T_\beta)\, C_{y_n}(t, T_\alpha, T_\beta)
\end{aligned}$$

and:

$$C_{x_n}(t, T_\alpha, T_\beta) = \frac{1}{a_n} \left[\frac{e^{-a_n(T_\alpha - t)}\, P_n(t, T_\alpha)}{\sum_{i=\alpha+1}^{\beta} \tau_i\, P_n(t, T_i)} - \frac{e^{-a_n(T_\beta - t)}\, P_n(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i\, P_n(t, T_i)} \right.$$

$$- \frac{P_n(t,T_\alpha) - P_n(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i \, P_n(t,T_i)} \sum_{i=\alpha+1}^{\beta} \frac{\tau_i \, e^{-a_n(T_i-t)} \, P_n(t,T_i)}{\sum_{i=\alpha+1}^{\beta} \tau_i \, P_n(t,T_i)} \Bigg]$$

$$C_{y_n}(t,T_\alpha,T_\beta) = \frac{1}{b_n} \left[\frac{e^{-b_n(T_\alpha-t)} \, P_n(t,T_\alpha)}{\sum_{i=\alpha+1}^{\beta} \tau_i \, P_n(t,T_i)} - \frac{e^{-b_n(T_\beta-t)} \, P_n(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i \, P_n(t,T_i)} \right.$$
$$\left. - \frac{P_n(t,T_\alpha) - P_n(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i \, P_n(t,T_i)} \sum_{i=\alpha+1}^{\beta} \frac{\tau_i \, e^{-b_n(T_i-t)} \, P_n(t,T_i)}{\sum_{i=\alpha+1}^{\beta} \tau_i \, P_n(t,T_i)} \right]$$

3.2.3 Zero-coupon inflation-indexed swap

In a ZCIIS, at the final time T_M (assuming $T_M = M$ years), one party pays a fixed amount:

$$N\left[(1+K)^M - 1\right] \tag{3.6}$$

where N is the nominal value, and K is the fixed rate of the contract.

In exchange for the fixed payment, at the final time T_M, another party pays a floating amount:

$$N\left[\frac{I(T_M)}{I_0} - 1\right]$$

The no-arbitrage price at time t, $0 \le t < T_M$, of the ZCIIS floating leg under the nominal risk-neutral measure Q_n is given by:

$$\text{ZCIIS}(t,T_M,I_0,N) = N\, E_n\left\{ e^{-\int_t^{T_M} n(u)du} \left[\frac{I(T_M)}{I_0} - 1\right] \mid \mathcal{F}_t \right\}$$

By the foreign-currency analogy, for each $t < T$, we have the following relation:

$$I(t)\, P_r(t,T) = I(t)\, E_r\left\{ e^{-\int_t^{T} r(u)du} \mid \mathcal{F}_t \right\} = E_n\left\{ e^{-\int_t^{T} n(u)du}\, I(T) \mid \mathcal{F}_t \right\}$$

Therefore, we have:

$$\text{ZCIIS}(t,T_M,I_0,N) = N\left[\frac{I(t)}{I_0}\, P_r(t,T_M) - P_n(t,T_M)\right] \tag{3.7}$$

which at time $t = 0$ simplifies to:

$$\text{ZCIIS}(0,T_M,N) = N\left[P_r(0,T_M) - P_n(0,T_M)\right]$$

ZCIISs can be used to easily derive the real term structure. Let $K = K(T_M)$ be the fixed rate of the contract for a given maturity T_M. The nominal discounted value of equation 3.6 shall be equal to equation 3.7, so that the price at time $t = 0$ of a real zero-coupon bond with maturity in T_M is found to be:

$$P_r(0,T_M) = P_n(0,T_M)\,(1 + K(T_M))^M \tag{3.8}$$

We can observe that the price of a ZCIIS does not depend on the assumptions on the evolution of the interest rate market.

3.2.4 Year-on-year inflation-indexed swap

Given a set of payment dates $T_1, ..., T_M$, in a YYIIS, at each time T_i, one party pays a fixed amount:

$$N\,\varphi_i\,K$$

where N is the nominal value, K is the fixed rate of the contract, and φ_i is the contract fixed-leg year fraction between T_i and T_{i-1} (we thus have that $\varphi_i = T_i - T_{i-1}$ when T_i and T_{i-1} are real numbers).

In exchange for the fixed payment, at each time T_i, another party pays a floating amount:

$$N\,\psi_i\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right]$$

where ψ_i is the contract floating-leg year fraction between T_i and T_{i-1} (we thus have that $\psi_i = T_i - T_{i-1}$ when T_i and T_{i-1} are real numbers).

The no-arbitrage price at time $t < T_{i-1}$ of the payoff at time T_i of the YYIIS floating leg under the nominal risk-neutral measure Q_n is found to be:

$$\begin{aligned}
\text{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) &= N\,\psi_i\,E_n\Big\{e^{-\int_t^{T_i} n(u)du}\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right] \mid \mathcal{F}_t\Big\} \\
&= N\,\psi_i\,E_n\Big\{e^{-\int_t^{T_{i-1}} n(u)du}\,P_r(T_{i-1}, T_i) \mid \mathcal{F}_t\Big\} - N\,\psi_i\,P_n(t, T_i)
\end{aligned}$$

We apply the change-of-numeraire technique, so that under the nominal forward measure $Q_n^{T_{i-1}}$ we have:

$$\begin{aligned}
\text{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N\,\psi_i\,P_n(t, T_{i-1})\,E_n^{T_{i-1}}\{P_r(T_{i-1}, T_i) \mid \mathcal{F}_t\} \\
- N\,\psi_i\,P_n(t, T_i)
\end{aligned} \tag{3.9}$$

The expected value above depends on the assumptions on the evolution of the interest rate market, because real rates are stochastic. According to the model described in Section 3.1, we have:

$$\begin{aligned}
\text{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N\,\psi_i\,P_n(t, T_{i-1})\,\frac{P_r(t, T_i)}{P_r(t, T_{i-1})}\,e^{C(t, T_{i-1}, T_i)} \\
- N\,\psi_i\,P_n(t, T_i)
\end{aligned}$$

where (see Appendix A.1):

$$\begin{aligned}
C(t, T_{i-1}, T_i) &= \sigma_r\,B(a_r, T_{i-1}, T_i)\Big[B(a_r, t, T_{i-1})\Big(\rho_{x_r,I}\,\sigma_I - \frac{\sigma_r}{2}\,B(a_r, t, T_{i-1}) \\
&+ \frac{\rho_{x_n,x_r}\,\sigma_n}{a_n + a_r}\,(1 + a_r\,B(a_n, t, T_{i-1})) + \frac{\rho_{y_n,x_r}\,\eta_n}{b_n + a_r}\,(1 + a_r\,B(b_n, t, T_{i-1}))\Big) \\
&- \frac{\rho_{x_n,x_r}\,\sigma_n}{a_n + a_r}\,B(a_n, t, T_{i-1}) - \frac{\rho_{y_n,x_r}\,\eta_n}{b_n + a_r}\,B(b_n, t, T_{i-1})\Big] \\
&+ \eta_r\,B(b_r, T_{i-1}, T_i)\Big[B(b_r, t, T_{i-1})\Big(\rho_{y_r,I}\,\sigma_I - \frac{\eta_r}{2}\,B(b_r, t, T_{i-1})
\end{aligned}$$

$$+ \frac{\rho_{x_n,y_r}\,\sigma_n}{a_n + b_r}\,(1 + b_r\,B(a_n,t,T_{i-1})) + \frac{\rho_{y_n,y_r}\,\eta_n}{b_n + b_r}\,(1 + b_r\,B(b_n,t,T_{i-1}))\Big)$$

$$- \frac{\rho_{x_n,y_r}\,\sigma_n}{a_n + b_r}\,B(a_n,t,T_{i-1}) - \frac{\rho_{y_n,y_r}\,\eta_n}{b_n + b_r}\,B(b_n,t,T_{i-1})\Bigg]$$

$$- \frac{\rho_r\,\sigma_r\,\eta_r}{a_r\,b_r}\Bigg[(B(a_r,T_{i-1},T_i) + B(b_r,T_{i-1},T_i) - B(a_r,t,T_i) - B(b_r,t,T_i)$$

$$+ B(a_r,t,T_{i-1}) + B(b_r,t,T_{i-1})) + \frac{1}{a_r + b_r}\,(a_r\,b_r\,B(a_r,t,T_{i-1})\,B(b_r,t,T_{i-1})$$

$$- a_r\,B(a_r,t,T_{i-1}) - b_r\,B(b_r,t,T_{i-1}))\big(a_r\,B(a_r,T_{i-1},T_i) + b_r\,B(b_r,T_{i-1},T_i)\big)\Bigg]$$

If real rates were not stochastic, the parameters σ_r and η_r would be equal to zero, and the correction term C would result to be null.

3.2.5 Inflation-indexed cap and floor

An inflation-indexed cap (IIC) is a call option, which depends on the CPI. The corresponding put option is an inflation-indexed floor (IIF).

A year-on-year inflation-indexed cap or floor (YYIICF) can be decomposed in a stream of year-on-year inflation-indexed caplets or floorlets (YYIICFlts) with the set of payment dates $T_1, ..., T_M$. Their payoff at time T_i is given by:

$$N\,\zeta_i\left[\omega\left(\frac{I(T_i)}{I(T_{i-1})} - K\right)\right]^+ \quad \text{with} \quad K = 1 + \kappa$$

where $\omega = 1$ ($\omega = -1$) for a cap (floor), N is the nominal value, κ is the strike rate of the contract, and ζ_i is the year fraction between T_i and T_{i-1} (we thus have that $\zeta_i = T_i - T_{i-1}$ when T_i and T_{i-1} are real numbers).

The no-arbitrage price at time $t \leq T_{i-1}$ of the YYIICFlt payoff at time T_i under the nominal risk-neutral measure Q_n is given by:

$$\begin{aligned}&\text{YYIICFlt}(t, T_{i-1}, T_i, \zeta_i, K, N, \omega)\\ &\quad = N\,\zeta_i\,E_n\Big\{e^{-\int_t^{T_i} n(u)du}\left[\omega\left(\frac{I(T_i)}{I(T_{i-1})} - K\right)\right]^+ \mid \mathcal{F}_t\Big\}\end{aligned}$$

We apply the change-of-numeraire technique, so that under the nominal forward measure $Q_n^{T_i}$ we have:

$$\begin{aligned}&\text{YYIICFlt}(t, T_{i-1}, T_i, \zeta_i, K, N, \omega)\\ &\quad = N\,\zeta_i\,P_n(t,T_i)\,E_n^{T_i}\Big\{\left[\omega\left(\frac{I(T_i)}{I(T_{i-1})} - K\right)\right]^+ \mid \mathcal{F}_t\Big\}\end{aligned} \tag{3.10}$$

The expected value above depends on the assumptions on the evolution of the interest rate market, because nominal and real rates are stochastic.

The ratio $I(T_i)/I(T_{i-1})$ (i.e. CPI ratio) under the nominal forward measure $Q_n^{T_i}$ and conditional on the sigma-field $\mathcal{F}_t$ is lognormally distributed. For this reason, equation 3.10 can be solved using the expected value of the CPI ratio and the variance of its logarithm. Let X be a Lognormal random variable with $E(X) = m$ and $\mathrm{Std}(\ln X) = v$, we thus have:

$$E\big[(\omega\,(X-K))^{+}\big] = \omega\, m\, \Phi\bigg(\omega\, \frac{\ln \frac{m}{K} + \frac{v^2}{2}}{v}\bigg) - \omega\, K\, \Phi\bigg(\omega\, \frac{\ln \frac{m}{K} - \frac{v^2}{2}}{v}\bigg) \tag{3.11}$$

where Φ is the standard Normal cumulative distribution function.

According to the model described in Section 3.1, the expected value of the CPI ratio is given by:

$$E_n^{T_i}\bigg\{\frac{I(T_i)}{I(T_{i-1})} \mid \mathcal{F}_t\bigg\} = \frac{P_n(t,T_{i-1})}{P_n(t,T_i)}\,\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}\, e^{C(t,T_{i-1},T_i)}$$

Furthermore, the variance of the logarithm of the CPI ratio is given by (see Appendix A.2):

$$\begin{aligned}
\mathrm{Var}_n^{T_i}\bigg\{\ln \frac{I(T_i)}{I(T_{i-1})} \mid \mathcal{F}_t\bigg\} &= \frac{\sigma_n^2}{2a_n^3}\left(1-e^{-a_n\zeta_i}\right)^2\left(1-e^{-2a_n(T_{i-1}-t)}\right)\\
&+ \frac{\sigma_n^2}{a_n^2}\left[\zeta_i + \frac{2}{a_n}\,e^{-a_n\zeta_i} - \frac{1}{2a_n}\,e^{-2a_n\zeta_i} - \frac{3}{2a_n}\right]\\
&+ \frac{\eta_n^2}{2b_n^3}\left(1-e^{-b_n\zeta_i}\right)^2\left(1-e^{-2b_n(T_{i-1}-t)}\right)\\
&+ \frac{\eta_n^2}{b_n^2}\left[\zeta_i + \frac{2}{b_n}\,e^{-b_n\zeta_i} - \frac{1}{2b_n}\,e^{-2b_n\zeta_i} - \frac{3}{2b_n}\right]\\
&+ \frac{\sigma_r^2}{2a_r^3}\left(1-e^{-a_r\zeta_i}\right)^2\left(1-e^{-2a_r(T_{i-1}-t)}\right)\\
&+ \frac{\sigma_r^2}{a_r^2}\left[\zeta_i + \frac{2}{a_r}\,e^{-a_r\zeta_i} - \frac{1}{2a_r}\,e^{-2a_r\zeta_i} - \frac{3}{2a_r}\right]\\
&+ \frac{\eta_r^2}{2b_r^3}\left(1-e^{-b_r\zeta_i}\right)^2\left(1-e^{-2b_r(T_{i-1}-t)}\right)\\
&+ \frac{\eta_r^2}{b_r^2}\left[\zeta_i + \frac{2}{b_r}\,e^{-b_r\zeta_i} - \frac{1}{2b_r}\,e^{-2b_r\zeta_i} - \frac{3}{2b_r}\right] + \sigma_I^2\,\zeta_i\\
&+ \frac{2\rho_n\,\sigma_n\,\eta_n}{a_n\,b_n\,(a_n+b_n)}\left(1-e^{-a_n\zeta_i}\right)\left(1-e^{-b_n\zeta_i}\right)\left(1-e^{-(a_n+b_n)(T_{i-1}-t)}\right)\\
&+ \frac{2\rho_n\,\sigma_n\,\eta_n}{a_n\,b_n}\left[\zeta_i - \frac{1-e^{-a_n\zeta_i}}{a_n} - \frac{1-e^{-b_n\zeta_i}}{b_n} + \frac{1-e^{-(a_n+b_n)\zeta_i}}{a_n+b_n}\right]\\
&+ \frac{2\rho_r\,\sigma_r\,\eta_r}{a_r\,b_r\,(a_r+b_r)}\left(1-e^{-a_r\zeta_i}\right)\left(1-e^{-b_r\zeta_i}\right)\left(1-e^{-(a_r+b_r)(T_{i-1}-t)}\right)\\
&+ \frac{2\rho_r\,\sigma_r\,\eta_r}{a_r\,b_r}\left[\zeta_i - \frac{1-e^{-a_r\zeta_i}}{a_r} - \frac{1-e^{-b_r\zeta_i}}{b_r} + \frac{1-e^{-(a_r+b_r)\zeta_i}}{a_r+b_r}\right]\\
&- \frac{2\rho_{x_n,x_r}\,\sigma_n\,\sigma_r}{a_n\,a_r\,(a_n+a_r)}\left(1-e^{-a_n\zeta_i}\right)\left(1-e^{-a_r\zeta_i}\right)\left(1-e^{-(a_n+a_r)(T_{i-1}-t)}\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{2\rho_{x_n,x_r}\,\sigma_n\,\sigma_r}{a_n\,a_r}\left[\zeta_i - \frac{1-e^{-a_n\zeta_i}}{a_n} - \frac{1-e^{-a_r\zeta_i}}{a_r} + \frac{1-e^{-(a_n+a_r)\zeta_i}}{a_n+a_r}\right] \\
&- \frac{2\rho_{x_n,y_r}\,\sigma_n\,\eta_r}{a_n\,b_r\,(a_n+b_r)}\left(1-e^{-a_n\zeta_i}\right)\left(1-e^{-b_r\zeta_i}\right)\left(1-e^{-(a_n+b_r)(T_{i-1}-t)}\right) \\
&- \frac{2\rho_{x_n,y_r}\,\sigma_n\,\eta_r}{a_n\,b_r}\left[\zeta_i - \frac{1-e^{-a_n\zeta_i}}{a_n} - \frac{1-e^{-b_r\zeta_i}}{b_r} + \frac{1-e^{-(a_n+b_r)\zeta_i}}{a_n+b_r}\right] \\
&- \frac{2\rho_{y_n,x_r}\,\eta_n\,\sigma_r}{b_n\,a_r\,(b_n+a_r)}\left(1-e^{-b_n\zeta_i}\right)\left(1-e^{-a_r\zeta_i}\right)\left(1-e^{-(b_n+a_r)(T_{i-1}-t)}\right) \\
&- \frac{2\rho_{y_n,x_r}\,\eta_n\,\sigma_r}{b_n\,a_r}\left[\zeta_i - \frac{1-e^{-b_n\zeta_i}}{b_n} - \frac{1-e^{-a_r\zeta_i}}{a_r} + \frac{1-e^{-(b_n+a_r)\zeta_i}}{b_n+a_r}\right] \\
&- \frac{2\rho_{y_n,y_r}\,\eta_n\,\eta_r}{b_n\,b_r\,(b_n+b_r)}\left(1-e^{-b_n\zeta_i}\right)\left(1-e^{-b_r\zeta_i}\right)\left(1-e^{-(b_n+b_r)(T_{i-1}-t)}\right) \\
&- \frac{2\rho_{y_n,y_r}\,\eta_n\,\eta_r}{b_n\,b_r}\left[\zeta_i - \frac{1-e^{-b_n\zeta_i}}{b_n} - \frac{1-e^{-b_r\zeta_i}}{b_r} + \frac{1-e^{-(b_n+b_r)\zeta_i}}{b_n+b_r}\right] \\
&+ \frac{2\rho_{x_n,I}\,\sigma_n\,\sigma_I}{a_n}\left[\zeta_i - \frac{1-e^{-a_n\zeta_i}}{a_n}\right] + \frac{2\rho_{y_n,I}\,\eta_n\,\sigma_I}{b_n}\left[\zeta_i - \frac{1-e^{-b_n\zeta_i}}{b_n}\right] \\
&- \frac{2\rho_{x_r,I}\,\sigma_r\,\sigma_I}{a_r}\left[\zeta_i - \frac{1-e^{-a_r\zeta_i}}{a_r}\right] - \frac{2\rho_{y_r,I}\,\eta_r\,\sigma_I}{b_r}\left[\zeta_i - \frac{1-e^{-b_r\zeta_i}}{b_r}\right]
\end{aligned}
$$

In a zero-coupon inflation-indexed cap or floor (ZCIICF), at the final time T_M (assuming $T_M = M$ years), the payoff is given by:

$$
N\left[\omega\left(\frac{I(T_M)}{I_0} - K\right)\right]^+ \quad \text{with} \quad K = (1+\kappa)^M
$$

The no-arbitrage price at time t, $0 \le t < T_M$, of the ZCIICF payoff under the nominal risk-neutral measure Q_n is given by:

$$
\begin{aligned}
&\mathrm{ZCIICF}(t, T_M, I_0, K, N, \omega) \\
&\quad = N\,E_n\left\{e^{-\int_t^{T_M} n(u)du}\left[\omega\left(\frac{I(T_M)}{I_0} - K\right)\right]^+ \mid \mathcal{F}_t\right\}
\end{aligned}
$$

We apply the change-of-numeraire technique, so that under the nominal forward measure $Q_n^{T_M}$ we have:

$$
\begin{aligned}
&\mathrm{ZCIICF}(t, T_M, I_0, K, N, \omega) \\
&\quad = N\,P_n(t, T_M)\,E_n^{T_M}\left\{\left[\omega\left(\frac{I(T_M)}{I_0} - K\right)\right]^+ \mid \mathcal{F}_t\right\}
\end{aligned}
$$

The CPI ratio related to the ZCIICF under the nominal forward measure $Q_n^{T_M}$ and conditional on the sigma-field $\mathcal{F}_t$ is again lognormally distributed, and the solution form is the same as in equation 3.11. The expected value of the CPI ratio is now given by:

$$
E_n^{T_M}\left\{\frac{I(T_M)}{I_0} \mid \mathcal{F}_t\right\} = \frac{I(t)}{I_0}\,E_n^{T_M}\left\{\frac{I(T_M)}{I(t)} \mid \mathcal{F}_t\right\}
$$

Moreover, the variance of the logarithm of the CPI ratio is now given by:

$$\mathrm{Var}_n^{T_M}\left\{\ln \frac{I(T_M)}{I_0} \mid \mathcal{F}_t\right\} = \mathrm{Var}_n^{T_M}\left\{\ln \frac{I(T_M)}{I(t)} \mid \mathcal{F}_t\right\}$$

We can easily obtain analytical solutions for the expected value and variance above, by replacing T_i with T_M and T_{i-1} with t (consequently ζ_i becomes the year fraction between T_M and t) in the formulas related to the YYIICFlt.

3.2.6 European stock-price option

The payoff of a EO at time T is given by:

$$N\left[\omega\left(S(T) - K\right)\right]^+$$

where $\omega = 1$ ($\omega = -1$) for a call (put), N is the nominal value, and K is the strike price of the contract.

The no-arbitrage price at time $t \leq T$ of the EO payoff at time T under the nominal risk-neutral measure Q_n is given by:

$$\mathrm{EO}(t, T, K, N, \omega) = N\, E_n\left\{e^{-\int_t^T n(u)du}\left[\omega\left(S(T) - K\right)\right]^+ \mid \mathcal{F}_t\right\}$$

We apply the change-of-numeraire technique, so that under the nominal forward measure Q_n^T we have:

$$\mathrm{EO}(t, T, K, N, \omega) = N\, P_n(t, T)\, E_n^T\left\{\left[\omega\left(S(T) - K\right)\right]^+ \mid \mathcal{F}_t\right\} \tag{3.12}$$

The expected value above depends on the assumptions on the evolution of the stock market, because the non-dividend-paying stock price is stochastic.

The non-dividend-paying stock price under the nominal forward measure Q_n^T and conditional on the sigma-field $\mathcal{F}_t$ is lognormally distributed. For this reason, equation 3.12 can be solved using equation 3.11.

According to the nominal forward measure Q_n^T and the no-arbitrage pricing theory, the expected value of the non-dividend-paying stock price is found to be:

$$E_n^T\{S(T) \mid \mathcal{F}_t\} = \frac{S(t)}{P_n(t, T)}$$

Furthermore, according to the model described in Section 3.1, the variance of the logarithm of the non-dividend-paying stock price is given by (see Appendix A.3):

$$\begin{aligned}\mathrm{Var}_n^T\{\ln S(T) \mid \mathcal{F}_t\} = V_n(t, T) + \sigma_S^2\,(T - t) + \frac{2\rho_{x_n,S}\,\sigma_n\,\sigma_S}{a_n}\left[T - t - \frac{1 - e^{-a_n(T-t)}}{a_n}\right] \\ + \frac{2\rho_{y_n,S}\,\eta_n\,\sigma_S}{b_n}\left[T - t - \frac{1 - e^{-b_n(T-t)}}{b_n}\right]\end{aligned} \tag{3.13}$$

3.3 Real-world economic scenario generator

Risk-neutral probabilities are acceptable for pricing, but not to forecast the future value of an asset. Real-world probabilities should instead be used for risk management purposes. Note that the variance, covariance, and correlation are the same both in the real-world and risk-neutral model.

We point out that the standard Brownian motions do not have any subscript, because they are now referred to the real-world measure P. Also, note that in this section we present the solution of some expected values that will be used in the calibration of our numerical analysis.

3.3.1 Nominal and real interest rates and CPI

According to Berninger and Pfeiffer (2021), the nominal state variables under the real-world measure P satisfy the following stochastic differential equations:

$$dx_n(t) = a_n\,(d_{x_n}(t) - x_n(t))\,dt + \sigma_n\,dW^{x_n}(t) \quad \text{with} \quad x_n(0) = 0$$

$$dy_n(t) = b_n\,(d_{y_n}(t) - y_n(t))\,dt + \eta_n\,dW^{y_n}(t) \quad \text{with} \quad y_n(0) = 0$$

where $d_{x_n}(t)$ and $d_{y_n}(t)$ are nominal deterministic time-dependent mean reversion levels. Their sum can be interpreted as the local risk premium of the nominal short rate, i.e. the amount which is added in the real world to the nominal short rate under the nominal risk-neutral measure, allowing the change of measure according to the Girsanov theorem.

The stochastic differential equations have explicit solutions that are given by:

$$x_n(t) = x_n(s)\,e^{-a_n(t-s)} + a_n \int_s^t e^{-a_n(t-u)}\,d_{x_n}(u)\,du + \sigma_n \int_s^t e^{-a_n(t-u)}\,dW^{x_n}(u)$$

$$y_n(t) = y_n(s)\,e^{-b_n(t-s)} + b_n \int_s^t e^{-b_n(t-u)}\,d_{y_n}(u)\,du + \eta_n \int_s^t e^{-b_n(t-u)}\,dW^{y_n}(u)$$

Hence, the vector of the nominal state variables under the real-world measure P and conditional on the sigma-field $\mathcal{F}_s$ is again normally distributed, with mean vector given by:

$$\begin{bmatrix} x_n(s)\,e^{-a_n(t-s)} + a_n \int_s^t e^{-a_n(t-u)}\,d_{x_n}(u)\,du \\ y_n(s)\,e^{-b_n(t-s)} + b_n \int_s^t e^{-b_n(t-u)}\,d_{y_n}(u)\,du \end{bmatrix}$$

The nominal zero-coupon bond price formula is found to be exactly the same as in the nominal risk-neutral case, even if the nominal state variables correspond now to the values of the processes under the real-world measure. As described by Berninger and Pfeiffer (2021), using the explicit solutions of the differential equations both in the real and risk-neutral world, and using equation 3.1 and 3.3, the relation between the expected nominal zero-coupon interest rate (continuously compounded and conditional on the sigma-field $\mathcal{F}_0$) at time t for a term of $T - t$

under the real-world measure P and the nominal risk-neutral measure Q_n is found to be:

$$
\begin{aligned}
E\{R_n(t,T) \mid \mathcal{F}_0\} = E_n\{R_n(t,T) \mid \mathcal{F}_0\} + \frac{1}{T-t}\bigg[&\frac{1-e^{-a_n(T-t)}}{a_n}\, RP_{x_n}(0,t) \\
&+ \frac{1-e^{-b_n(T-t)}}{b_n}\, RP_{y_n}(0,t)\bigg]
\end{aligned}
\tag{3.14}
$$

where $RP_{x_n}(0,t)$ and $RP_{y_n}(0,t)$ are the actual risk premiums of the nominal short rate between time zero and t for the nominal state variables:

$$RP_{x_n}(0,t) = a_n \int_0^t e^{-a_n(t-u)}\, d_{x_n}(u)\, du$$

$$RP_{y_n}(0,t) = b_n \int_0^t e^{-b_n(t-u)}\, d_{y_n}(u)\, du$$

We now assume that the deterministic nominal time-dependent mean reversion levels are given by step functions:

$$d_{x_n}(t) = \mathbb{1}_{t\le\tau}\, d_{x_n} + \mathbb{1}_{t>\tau}\, l_{x_n}$$

$$d_{y_n}(t) = \mathbb{1}_{t\le\tau}\, d_{y_n} + \mathbb{1}_{t>\tau}\, l_{y_n}$$

where d_{x_n}, l_{x_n}, d_{y_n} and l_{y_n} are constants, τ is a constant time parameter, and $\mathbb{1}$ is the indicator function.

As a result, the actual risk premiums of the nominal short rate between time zero and t for the nominal state variables are found to be:

$$
\begin{aligned}
RP_{x_n}(0,t) = {} & \mathbb{1}_{t\le\tau}\, d_{x_n}\left(1-e^{-a_n t}\right) + \mathbb{1}_{t>\tau}\, d_{x_n}\left(e^{-a_n(t-\tau)} - e^{-a_n t}\right) \\
& + \mathbb{1}_{t>\tau}\, l_{x_n}\left(1-e^{-a_n(t-\tau)}\right)
\end{aligned}
$$

$$
\begin{aligned}
RP_{y_n}(0,t) = {} & \mathbb{1}_{t\le\tau}\, d_{y_n}\left(1-e^{-b_n t}\right) + \mathbb{1}_{t>\tau}\, d_{y_n}\left(e^{-b_n(t-\tau)} - e^{-b_n t}\right) \\
& + \mathbb{1}_{t>\tau}\, l_{y_n}\left(1-e^{-b_n(t-\tau)}\right)
\end{aligned}
$$

The real state variables under the real-world measure P satisfy the following stochastic differential equations:

$$dx_r(t) = a_r\left(d_{x_r}(t) - x_r(t)\right) dt + \sigma_r\, dW^{x_r}(t) \quad \text{with} \quad x_r(0) = 0$$

$$dy_r(t) = b_r\left(d_{y_r}(t) - y_r(t)\right) dt + \eta_r\, dW^{y_r}(t) \quad \text{with} \quad y_r(0) = 0$$

where $d_{x_r}(t)$ and $d_{y_r}(t)$ are real deterministic time-dependent mean reversion levels, and their sum can be interpreted as the local risk premium of the real short rate.

We now assume that the deterministic real time-dependent mean reversion levels are given by step functions:

$$d_{x_r}(t) = \mathbb{1}_{t\le\tau}\, d_{x_r} + \mathbb{1}_{t>\tau}\, l_{x_r}$$

$$d_{y_r}(t) = \mathbb{1}_{t\le\tau}\, d_{y_r} + \mathbb{1}_{t>\tau}\, l_{y_r}$$

where d_{x_r}, l_{x_r}, d_{y_r} and l_{y_r} are constants, and τ is the same constant time parameter as for the nominal short rate.

The explicit solutions and mean vector of the real state variables are analogous to the nominal case, and they can be found by replacing the sub and superscripts n with r. The real zero-coupon bond price formula is found to be exactly the same as in the real risk-neutral case, even if the real state variables correspond now to the values of the processes under the real-world measure. Furthermore, the actual risk premiums and the the relation between the expected real zero-coupon interest rate under the real-world measure P and the real risk-neutral measure Q_r are also analogous to the nominal case, and once again they can be found by replacing the sub and superscripts n with r.

Finally, the CPI under the real-world measure P satisfies the following stochastic differential equation:

$$dI(t) = (n(t) - r(t))\, I(t)\, dt + \sigma_I\, I(t)\, dW^I(t) \qquad \text{with} \quad I(0) = I_0$$

We point out that the nominal and real short rates are now the variables under the real-world measure P.

The stochastic differential equation has an explicit solution that is given by:

$$I(t) = I(s)\, \exp\left\{ \int_s^t (n(u) - r(u))\, du - \frac{1}{2}\, \sigma_I^2\, (t-s) + \sigma_I\, \left(W^I(t) - W^I(s)\right) \right\}$$

Once again, the ratio of the CPI at time t and zero under the real-world measure P and conditional on the sigma-field $\mathcal{F}_0$ is lognormally distributed. For this reason, its expected value can be solved using the expected value and variance of the logarithm of this CPI ratio. Let X be a Lognormal random variable with $E(\ln X) = m$ and $\mathrm{Std}(\ln X) = v$, we thus have:

$$E(X) = \exp\left\{ m + \frac{v^2}{2} \right\} \tag{3.15}$$

The expected value of the logarithm of the ratio of the CPI at time t and zero (conditional on the sigma-field $\mathcal{F}_0$) under the real-world measure P is found to be:

$$\begin{aligned}
E\left\{ \ln \frac{I(t)}{I_0} \mid \mathcal{F}_0 \right\} &= \ln \frac{P_r^M(0,t)}{P_n^M(0,t)} + \frac{1}{2}\, V_n(0,t) - \frac{1}{2}\, V_r(0,t) \\
&+ \mathbb{1}_{t \le \tau}\, d_{x_n} \left[t - \frac{1 - e^{-a_n t}}{a_n} \right] - \mathbb{1}_{t \le \tau}\, d_{x_r} \left[t - \frac{1 - e^{-a_r t}}{a_r} \right] \\
&+ \mathbb{1}_{t > \tau}\, d_{x_n} \left[\tau - \frac{e^{-a_n(t-\tau)} - e^{-a_n t}}{a_n} \right] - \mathbb{1}_{t > \tau}\, d_{x_r} \left[\tau - \frac{e^{-a_r(t-\tau)} - e^{-a_r t}}{a_r} \right] \\
&+ \mathbb{1}_{t > \tau}\, l_{x_n} \left[t - \tau - \frac{1 - e^{-a_n(t-\tau)}}{a_n} \right] - \mathbb{1}_{t > \tau}\, l_{x_r} \left[t - \tau - \frac{1 - e^{-a_r(t-\tau)}}{a_r} \right] \\
&+ \mathbb{1}_{t \le \tau}\, d_{y_n} \left[t - \frac{1 - e^{-b_n t}}{b_n} \right] - \mathbb{1}_{t \le \tau}\, d_{y_r} \left[t - \frac{1 - e^{-b_r t}}{b_r} \right]
\end{aligned}$$

$$+ \mathbb{1}_{t>\tau}\, d_{y_n} \left[\tau - \frac{e^{-b_n(t-\tau)} - e^{-b_n t}}{b_n}\right] - \mathbb{1}_{t>\tau}\, d_{y_r} \left[\tau - \frac{e^{-b_r(t-\tau)} - e^{-b_r t}}{b_r}\right]$$
$$+ \mathbb{1}_{t>\tau}\, l_{y_n} \left[t - \tau - \frac{1 - e^{-b_n(t-\tau)}}{b_n}\right] - \mathbb{1}_{t>\tau}\, l_{y_r} \left[t - \tau - \frac{1 - e^{-b_r(t-\tau)}}{b_r}\right] - \frac{1}{2}\,\sigma_I^2\, t$$

Further, we can easily obtain an analytical solution for the variance of the logarithm of this CPI ratio, by replacing T_i with t, and T_{i-1} and t with zero (consequently ζ_i becomes the year fraction between t and zero) in the formula of the variance of the logarithm of the CPI ratio related to the YYIICFlt in Section 3.2.5 (see also Appendix A.2).

3.3.2 Stock and property prices

The non-dividend-paying stock price under the real-world measure P satisfies the following stochastic differential equation:

$$dS(t) = (n(t) + d_S(t))\, S(t)\, dt + \sigma_S\, S(t)\, dW^S(t) \quad \text{with} \quad S(0) = S_0$$

where $d_S(t)$ is a deterministic time-dependent parameter, and it can be interpreted as the additional local risk premium of the stock price. We point out that the nominal short rate is now the variable under the real-world measure P.

The stochastic differential equation has an explicit solution that is given by:

$$S(t) = S(s)\, \exp\left\{ \int_s^t (n(u) + d_S(u))\, du - \frac{1}{2}\,\sigma_S^2\,(t-s) + \sigma_S\,(W^S(t) - W^S(s)) \right\}$$

We now assume that the deterministic time-dependent parameter of the stock price is given by a step function:

$$d_S(t) = \mathbb{1}_{t\le\tau_1}\, d_S + \mathbb{1}_{\tau_1<t\le\tau_2}\, l_S + \mathbb{1}_{t>\tau_2}\, q_S \quad \text{with} \quad \tau_1 < \tau_2$$

where d_S, l_S, and q_S are constants, while τ_1 and τ_2 are constant time parameters.

The expected value of the logarithm of the ratio of the non-dividend-paying stock price at time t and zero (conditional on the sigma-field $\mathcal{F}_0$) under the real-world measure P is found to be:

$$\begin{aligned} E\left\{\ln \frac{S(t)}{S_0} \mid \mathcal{F}_0\right\} &= \ln \frac{1}{P_n^M(0,t)} + \frac{1}{2}\, V_n(0,t) + \mathbb{1}_{t\le\tau}\, d_{x_n} \left[t - \frac{1 - e^{-a_n t}}{a_n}\right] \\ &+ \mathbb{1}_{t>\tau}\, d_{x_n} \left[\tau - \frac{e^{-a_n(t-\tau)} - e^{-a_n t}}{a_n}\right] + \mathbb{1}_{t>\tau}\, l_{x_n} \left[t - \tau - \frac{1 - e^{-a_n(t-\tau)}}{a_n}\right] \\ &+ \mathbb{1}_{t\le\tau}\, d_{y_n} \left[t - \frac{1 - e^{-b_n t}}{b_n}\right] + \mathbb{1}_{t>\tau}\, d_{y_n} \left[\tau - \frac{e^{-b_n(t-\tau)} - e^{-b_n t}}{b_n}\right] \\ &+ \mathbb{1}_{t>\tau}\, l_{y_n} \left[t - \tau - \frac{1 - e^{-b_n(t-\tau)}}{b_n}\right] + \mathbb{1}_{t\le\tau_1}\, d_S\, t + \mathbb{1}_{t>\tau_1}\, d_S\, \tau_1 \\ &+ \mathbb{1}_{\tau_1<t\le\tau_2}\, l_S\,(t-\tau_1) + \mathbb{1}_{t>\tau_2}\, l_S\,(\tau_2-\tau_1) + \mathbb{1}_{t>\tau_2}\, q_S\,(t-\tau_2) - \frac{1}{2}\,\sigma_S^2\, t \end{aligned} \tag{3.16}$$

Moreover, the property price under the real-world measure P satisfies the following stochastic differential equation:

$$dH(t) = (n(t) + d_H(t))\, H(t)\, dt + \sigma_H\, H(t)\, dW^H(t) \quad \text{with} \quad H(0) = H_0$$

where $d_H(t)$ is a deterministic time-dependent parameter, and it can be interpreted as the additional local risk premium of the property price. We point out that the nominal short rate is again the variable under the real-world measure P.

We now assume that also the deterministic time-dependent parameter of the property price is given by a step function:

$$d_H(t) = \mathbb{1}_{t \leq \tau_1}\, d_H + \mathbb{1}_{\tau_1 < t \leq \tau_2}\, l_H + \mathbb{1}_{t > \tau_2}\, q_H$$

where d_H, l_H, and q_H are constants, while τ_1 and τ_2 are the same constant time parameters as for the stock price.

The explicit solution of the stochastic differential equation and the expected value of the logarithm of the ratio of the property prices are analogous to the stock case, and they can be found by replacing the character S with H. Note that property will have a small weight in the asset allocation of our insurance company.

Chapter 4

Market and non-life underwriting model

The inversion of the production cycle implies that insurance companies have a lot of resources to invest in order to make profits. The inversion of the production cycle also implies that insurance companies have to measure and manage future claims, in order to control losses and determine insurance premiums and reserves.

In this chapter, we present the market model, that we will use to describe over time the distribution of the annual or average rate of return, and the non-life model, that we will use to describe over time the distribution of the total claim amount and claims development result. Note that the model for the annual or average rate of return is completely determined by the real-world economic scenario generator, the model for the total claim amount is based on a compound process, and the model for the claims development result is given by a Bootstrapping model.

4.1 Market model

As anticipated in equation 1.9 and 1.10, the annual rate of return is given by:

$$J_t = \alpha\,\frac{S(t)}{S(t-1)} + \beta\,\frac{H(t)}{H(t-1)} + (1-\alpha-\beta)\sum_{w\in\{1,2,3,5,10\}} \gamma_w\,\frac{P_n(t,t-1+w)}{P_n(t-1,t-1+w)} - 1$$

and the average rate of return is given by:

$$\bar{J}_t = \frac{J_t + J_t^{(2)}}{3}$$

where the rate of return (semi-annually compounded) of the annual net cash flows originated by the insurance business in the middle of the year is given by:

$$J_t^{(2)} = \alpha\,\frac{S(t)}{S(t-1/2)} + \beta\,\frac{H(t)}{H(t-1/2)} + (1-\alpha-\beta)\sum_{w\in\{1,2,3,5,10\}} \gamma_w\,\frac{P_n(t,t-1+w)}{P_n(t-1/2,t-1+w)} - 1$$

We point out that the real-world stock, property and bond prices are involved in the equations above. As already anticipated, the real-world non-dividend-paying stock price is given by:

$$S(t) = S(s)\,\exp\left\{\int_s^t (n(u) + d_S(u))\,du - \frac{1}{2}\,\sigma_S^2\,(t-s) + \sigma_S\,(W^S(t) - W^S(s))\right\}$$

The real-world property price is analogous to the stock case, and it can be found by replacing the character S with H.

Furthermore, we remind that the real-world price of a nominal zero-coupon bond is given by:

$$\begin{aligned} P_n(t,T) = \exp\Bigg\{ - \int_t^T \varphi_n(u)\,du - \frac{1-e^{-a_n(T-t)}}{a_n}\,x_n(t) \\ - \frac{1-e^{-b_n(T-t)}}{b_n}\,y_n(t) + \frac{1}{2}\,V_n(t,T) \Bigg\} \end{aligned}$$

and the real-world nominal state variables are given by:

$$x_n(t) = x_n(s)\,e^{-a_n(t-s)} + a_n \int_s^t e^{-a_n(t-u)}\,d_{x_n}(u)\,du + \sigma_n \int_s^t e^{-a_n(t-u)}\,dW^{x_n}(u)$$

$$y_n(t) = y_n(s)\,e^{-b_n(t-s)} + b_n \int_s^t e^{-b_n(t-u)}\,d_{y_n}(u)\,du + \eta_n \int_s^t e^{-b_n(t-u)}\,dW^{y_n}(u)$$

It is worth mentioning that further details on the elements involved in the equations above can be found in Section 3.1 and 3.3.

4.2 Non-life premium model

We now assume that the distribution of the total claim amount is described by a Collective Risk Model, which is a well-know model based on a compound process (see among others Daykin et al., 1994). This is very popular in non-life insurance modelling, because each risk can produce claims of different severity, therefore a unique distribution for the claim size of homogeneous groups is used (collective approach). On the contrary, in the individual approach a different distribution is defined for each contract. We now consider the entire LoB portfolio, composed of homogeneous risks, and we separately analyse the number of claims and the single claim amount, that is assumed to be independent of the contract that generated it.

The stochastic total claim amount at the end of time t is given by:

$$X_t = \sum_{n=1}^{N_t} Z_{n,t}$$

where N_t is the stochastic number of claims, and $Z_{n,t}$ is the stochastic amount for the n-th claim, for which we assume that:

1. the random variables $Z_{n,t}$ are independent;
2. the random variables $Z_{n,t}$ are identically distributed;
3. the random variables $Z_{n,t}$ and N_t are independent.

The first and second assumption are satisfied, in particular, in limited homogenous portfolios and also only in some certain time period. In order to solve this kind of problem, the portfolio can be split into different more homogenous sub-portfolios, where the mentioned conditions are more properly satisfied (the sub-portfolios must be then aggregated to come back to the full portfolio, assuming linear or non-linear dependence as appropriate). The third assumption is usually satisfied, but it might be refuted in some situations. In case of windstorm or hurricane, for example, the number of claims and single claim amount variables influence each other.

4.2.1 Number of claims

There are several distributions associated with the number of claims, such as the Poisson and Negative Binomial. We point out that the total claim amount is found to be zero, when the number of claims is zero.

Since the insurance portfolio is dynamic, we assume that the expected number of claims increases or decreases every year:

$$E(N_t) = k_t = k_0\,(1+g)^t \quad \text{with} \quad k_0 > 0$$

where g is the annual real growth rate parameter, which is the reference indicator for the new expected number of policyholders. It is useful to observe that empirically this rate usually differs by LoB, according to the practical commercial strength in that particular segment.

The expected number of claims increases every year in the same way as the insurance portfolio. As a result, the claims frequency of the portfolio is assumed to remain the same over time. This assumption could be refuted in some situations, such as in case of considerable modification of the portfolio, where new policyholders have significantly different claim frequency (e.g. young drivers or policyholders with high-power vehicles). Moreover, we point out that not only does the initial expected number of claims depend on the insurance portfolio size but also on the individual claims frequency of the people insured. Later on, the parameter of the expected number of claims will influence the so called size factor, which is implicitly taken into account in an IM, rather surprisingly, differently from what is done in the SF of Solvency II.

We now assume that the number of claims is distributed as a Mixed Poisson, i.e. a Poisson with stochastic (and not deterministic) parameter $K_t > 0$, such that:

$$K_t = k_t\,Q$$

where Q is the stochastic structure variable, which denotes the multiplicative noise term, representing the parameter uncertainty embedded in the distribution. The cumulant generating function of the Mixed Poisson distribution is given by:

$$\Psi_{N_t}(s) = \Psi_Q\big[k_t\,(e^s-1)\big] \tag{4.1}$$

In order to describe the number of claims, considering the short-term fluctuations only, we must assume the structure variable to have an expected value equal to one.

In the literature, this distribution is frequently assumed to be a Gamma with equal parameters (h, h), that is defined by the following probability density function:

$$f_Q(q) = \frac{h^h q^{h-1}}{\Gamma(h)} e^{-hq} \quad \text{with} \quad q > 0 \;\text{ and }\; h > 0$$

with mean, standard deviation, and skewness that are given by:

$$E(Q) = 1 \quad \text{and} \quad \text{Std}(Q) = \sigma_Q = \frac{1}{\sqrt{h}} \quad \text{and} \quad \text{Sk}(Q) = \gamma_Q = \frac{2}{\sqrt{h}} = 2\sigma_Q$$

As a result, the Mixed Poisson distribution is found to be a Negative Binomial with parameters (h, p_t) and to be defined by the following probability mass function:

$$\Pr(N_t = n) = \binom{n+h-1}{h-1} p_t^h (1-p_t)^n$$

$$\text{with} \quad n = 0, 1, ... \;\text{ and }\; h > 0 \;\text{ and }\; 0 < p_t = \frac{h}{h+k_t} < 1$$

4.2.2 Single claim amount

Since the insurance portfolio is dynamic, we assume that the single claim amount distribution (dropping the index k because of the identical distribution assumption) is identically distributed to the initial single claim amount, rescaled using the ratio of the CPI at time t and zero under the real-world measure P (which is assumed to be independent of the single claim amount):

$$Z_t \sim Z_0 \, \vartheta^t \, \frac{I(t)}{I_0}$$

where ϑ is the claims inflation percentage, which is a percentage to inflate or deflate the CPI ratio (which is based on the economic inflation) and obtain the so-called claims inflation index (which is based on the segment inflation). This percentage rely on the practical dependence of the specific segment on the economic inflation.

The mean of the single claim amount at time t is found to be:

$$E(Z_t) = m_t = m_0 \, \xi_t$$

where ξ_t is the expected claims inflation index, which is the reference indicator for the new expected claim size:

$$\xi_t = \vartheta^t \, E\left\{ \frac{I(t)}{I_0} \mid \mathcal{F}_0 \right\} \tag{4.2}$$

The expected value of the ratio of the CPI at time t and zero is based on equation 3.15 and on the expected value and variance of the logarithm of the ratio of the CPI at time t and zero, presented in Section 3.3.1.

Therefore, we start by describing the initial single claim distribution and, by using the relation above, we obtain the subsequent ones. Furthermore, we assume that the initial single claim amount is distributed as a Lognormal with constant parameters (μ, σ), that is defined by the following probability density function:

$$f_{Z_0}(z) = \frac{1}{z\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{1}{2}\left(\frac{\ln(z)-\mu}{\sigma}\right)^2\right\}$$

$$\text{with} \quad z > 0 \;\text{ and }\; -\infty < \mu < +\infty \;\text{ and }\; \sigma > 0$$

We remind that the CPI ratio is lognormally distributed, hence the single claim amount is also lognormally distributed (the product of Lognormal distributions is again a Lognormal). Therefore, the skewness of the single claim amount at time t is given by:

$$\text{Sk}(Z_t) = 3c_{Z_t} + c_{Z_t}^3$$

where c_{Z_t} is the coefficient of variation of the single claim amount.

This relation is quite interesting, because it clearly provides an insight into the obvious relation between the coefficient of variation and skewness of a Lognormal distribution. The higher the relative volatility, the higher the skewness. For this reason, the difference between a quantile and the mean cannot be expressed in terms of the same multiplier of the standard deviation, since the multiplier changes every time the relative volatility changes. However, in the SF for the calculation of the capital requirement for non-life premium and reserve risk, the underlying distribution is assumed to be a Lognormal, and the multiplier is kept constant and equal to 3.

There are also other distributions associated with the single claim amount, such as the Gamma, Weibull, Inverse Normal, Pareto, and many other potential distributions. The first three distributions, together with the Lognormal, usually fit attritional claims well, i.e. the most frequent and least expensive claims (large frequency and low severity). The last distribution fits large claims well, i.e. the least frequent and most expensive claims. The Lognormal is a distribution that often serves as a reference for the single claim amount. A popular alternative is to use distributions where attritional and large claims are described by different random variables. The Lognormal distribution is a basic assumption in the literature of the single claim amount behaviour. The latter can be represented by many other distributions, as explained above, and clearly the most appropriate one should be revealed using fitting procedures on appropriate empirical data. In addition, mixture, composite or spliced distribution approaches can be used to have a proper distinction between the behaviour of small-size amounts and medium-large amounts (see for instance Clemente et al., 2014). It is worth pointing out that we follow a total approach, without any distinction between small or medium-large claims. In some standard procedures, the simple attritional-large approach is followed (e.g. SF of the Swiss Solvency Test), but for an IM the total approach is highly recommended, in order to have more powerful information, in particular when we want to analyse the benefits coming from non-proportional reinsurance strategies (e.g. excess of loss for different priorities and layers).

4.2.3 Total claim amount

Using the definitions and assumptions above, we are able to determine the total claim amount distribution. Indeed, the cumulant generating function of the total claim amount distribution is found to be:

$$\Psi_{X_t}(s) = \Psi_{N_t}\big[\Psi_{Z_t}(s)\big]$$

Using equation (4.1), the cumulant generating function of the total claim amount distribution according to the mixed compound Poisson process is found to be:

$$\Psi_{X_t}(s) = \Psi_Q\big[k_t\,(M_{Z_t}(s) - 1)\big]$$

As a result, the mean, variance, coefficient of variation, and skewness of the total claim amount distribution are found to be:

$$E(X_t) = k_t\, m_t = \pi_0\,(1+g)^t\,\xi_t \tag{4.3}$$

$$\mathrm{Var}(X_t) = k_t\,\mu_{2,Z_t} + k_t^2\, m_t^2\,\sigma_Q^2$$

$$\mathrm{CV}(X_t) = \sqrt{\frac{1+c_{Z_t}^2}{k_t} + \sigma_Q^2} \tag{4.4}$$

$$\mathrm{Sk}(X_t) = \frac{1}{\mathrm{Std}(X_t)^3}\left(k_t\,\mu_{3,Z_t} + 3k_t^2\, m_t\,\mu_{2,Z_t}\,\sigma_Q^2 + k_t^3\, m_t^3\,\gamma_Q\,\sigma_Q^3\right)$$

where μ_{n,Z_t} is the n-th raw moment of the single claim amount.

Consequently, it may be easily proven that the coefficient of variation of the total claim amount approaches the standard deviation of the structure variable as k_0 or g increase, i.e. when the dimension of the insurance company increases, and the skewness approaches the skewness of the structure variable.

4.3 Non-life reserve model

We now assume that the distribution of the claims development result is described by the Re-Reserving Model (see for instance Ohlsonn and Lauzenings, 2008; Diers, 2009). This is based on the Bootstrapping method, that is a technique to estimate the sampling distribution of a statistic using random sampling methods. When a set of observations can be assumed to come from an independent and identically distributed population, the sampling distribution can be obtained by constructing a number of resamples of the observed dataset, each of them with replacement and having equal size to the observed data set.

The incremental amounts for paid claims are independent, but not identically distributed. For this reason, we apply the Bootstrapping method on the adjusted Pearson residuals. The reserving calculation approach used is the well-known Paid Chain-Ladder method. Furthermore, we assume that each incremental amount for paid claims is distributed as an independent Over-Dispersed Poisson. We hence consider the development triangle at (calendar) time zero, with $\upsilon + 1$ accident and development years, and without tail, as shown in Table 4.1.

Table 4.1. Development triangle (without tail) of incremental amounts for paid claims

		development year					
		0	1	2	3	...	v
accident year	$-v$	$C_{-v,0}$	$C_{-v,1}$	$C_{-v,2}$	$C_{-v,3}$	...	$C_{-v,v}$
	...	...	...	...	...	...	
	-3	$C_{-3,0}$	$C_{-3,1}$	$C_{-3,2}$	$C_{-3,3}$		
	-2	$C_{-2,0}$	$C_{-2,1}$	$C_{-2,2}$			
	-1	$C_{-1,0}$	$C_{-1,1}$				
	0	$C_{0,0}$					

Considering both the in-force and new business (i.e. business underwritten after the initial time), the stochastic claims reserve at the end of time t can be written as follows:

$$L_t = \sum_{k=1}^{t} L_t^{[k]} + L_t^{(0)} \tag{4.5}$$

Furthermore, the stochastic claims development result at the end of time t is given by equation 1.2 and can be written as follows:

$$Y_t = \sum_{k=1}^{t-1} L_{t-1}^{[k]} + L_{t-1}^{(0)} - \sum_{k=1}^{t-1} C_t^{[k]} - C_t^{(0)} - \sum_{k=1}^{t-1} L_t^{[k]} - L_t^{(0)}$$

However, the risk of the new business is addressed by the premium risk, and we assume that there is not any further fluctuation in the cash flows of the additional claims reserve. For the new business only, we also do not consider the effect of the interest rate unwinding, assuming that it is completely absorbed by the incremental amount for paid claims. Consequently, the claims development result arising from the new business is null, and we have:

$$Y_t = L_{t-1}^{(0)} - C_t^{(0)} - L_t^{(0)}$$

4.3.1 Claims reserve

According to the Paid Chain-Ladder method, the estimate of the age-to-age factor for each development year j is given by:

$$\hat{f}_j = \frac{\sum_{i=-v}^{-j-1} \sum_{h=0}^{j+1} C_{i,h}}{\sum_{i=-v}^{-j-1} \sum_{h=0}^{j} C_{i,h}}$$

where $C_{i,h}$ is the observation of the incremental amount for paid claims of accident year i and development year h with value of money as of the current calendar year of the development triangle, meaning that it incorporates the realised historical claims inflation until the current calendar year. Note that the observed incremental amount for paid claims just introduced (distinguished by the the subscripts for accident and development years) is used for modelling purposes only (the true incremental amount for paid claims has the correct value of money, depending on the specific calendar year).

As a consequence, the estimate of the incremental amount for paid claims for each cell of the lower part of the development triangle is given by:

$$\hat{C}_{i,j} = \sum_{h=0}^{-i} C_{i,h} \prod_{k=-i}^{j-1} \hat{f}_k - \sum_{h=0}^{-i} C_{i,h} - \sum_{h=-i+1}^{j-1} \hat{C}_{i,h} \quad \text{with} \quad i+j>0$$

We assume that the discounting and expected claims inflation index are considered in the reserve calculation. On the opposite side, we do not consider the risk margin to avoid any discretion to solve the well-known circularity issue (i.e. the risk margin depends on the capital requirements and vice versa). We remind that the goal of this thesis is the capital requirements modelling, therefore the risk margin can be ignored (on the contrary, if we wanted to make a correct estimate of the own funds of the insurance company, we would need also to correctly consider the risk margin). The estimate of the claims reserve of the in-force business at the end of time zero is then found to be:

$$\hat{L}_0^{(0)} = \sum_{i=-v+1}^{0} \sum_{j=-i+1}^{v} \hat{C}_{i,j}\, \xi_{i+j}\, P_{n_s}^{M}(0, i+j)$$

where the subscript n_s refers to the regulatory nominal term structure required by the supervisory authority, and the superscript M refers to the fact that it is observed in the market.

If the reserving cash flows are deterministic (as well as the claims inflation effect), then the estimate of the claims reserve of the in-force business at the end of year t is given by:

$$\hat{L}_t^{(0)} = \sum_{i=-v+t+1}^{0} \sum_{j=-i+t+1}^{v} \hat{C}_{i,j}\, \xi_{i+j}\, P_{n_s}(t, i+j) \tag{4.6}$$

and the claims development result at the end of year t is given by:

$$\hat{Y}_t = \hat{L}_{t-1}^{(0)} - \sum_{i=-v+t}^{0} \sum_{j=-i+t}^{v} \hat{C}_{i,j}\, \xi_{i+j}\, P_{n_s}(t, i+j) \tag{4.7}$$

where the superscript M does not appear anymore, as the regulatory nominal term structure is now stochastic.

We remind that the total claim amount is equal to the sum of the incremental paid claims and claims reserve of the new business, as shown in equation 1.1. Since we assumed that the claims development result of the new business is null, the equation can be extended to a later year t than the occurring-claims year k, so that we have:

$$X_k = \sum_{h=k}^{t} C_h^{[k]} + L_t^{[k]} \quad \text{with} \quad 0<k\leq t$$

We assume that the estimate of the age-to-age factors of the new business always refers to the end of time zero. It means that the nominal term structure is the only element to change over time, and we have:

$$\hat{L}_t^{[k]} = X_k \, \frac{\sum_{h=t-k}^{\upsilon-1} \xi_{h+k+1} \, P_{n_s}(t, h+k+1) \, (\hat{f}_h - 1) \, \prod_{u=0}^{h-1} \hat{f}_u}{1 + \sum_{h=0}^{\upsilon-1} \xi_{h+k+1} \, P_{n_s}(k, h+k+1) \, (\hat{f}_h - 1) \, \prod_{u=0}^{h-1} \hat{f}_u} \quad \text{with} \quad 0 < k \leq t \tag{4.8}$$

We remind that the effect of the interest rate unwinding of the new business is completely absorbed by the incremental amount for paid claims. We also remind that the claims development result of the new business is null, so that we have:

$$\hat{C}_t^{[k]} = \hat{L}_{t-1}^{[k]} - \hat{L}_t^{[k]} \quad \text{with} \quad 0 < k \leq t$$

Note that the stochastic claims inflation index affects the total claim amount, and this is reflected in the reserving and incremental claim amount calculation.

4.3.2 Claims development result

We can now fill the cells of the upper part of a new development triangle with the back-fitted incremental amounts for paid claims, as shown in Table 4.2. These back-fitted incremental amounts for paid claims are given by:

$$\dot{C}_{i,j} = \begin{cases} \dfrac{\sum_{h=0}^{-i} C_{i,h}}{\prod_{h=0}^{-i-1} \hat{f}_h} & \text{if } j = 0 \\[2ex] \dfrac{\sum_{h=0}^{-i} C_{i,h}}{\prod_{h=j}^{-i-1} \hat{f}_h} - \dfrac{\sum_{h=0}^{-i} C_{i,h}}{\prod_{h=j-1}^{-i-1} \hat{f}_h} & \text{if } 0 < j < -i \\[2ex] \sum_{h=0}^{-i} C_{i,h} - \dfrac{\sum_{h=0}^{-i} C_{i,h}}{\hat{f}_{-i-1}} & \text{if } j = -i \end{cases}$$

For the first accident year and last development year, and for the vice versa, the back-fitted incremental amounts for paid claims are equal to the observed ones.

Table 4.2. Back-fitted incremental amounts for paid claims

		development year					
		0	1	2	3	...	υ
accident year	$-\upsilon$	$\dot{C}_{-\upsilon,0}$	$\dot{C}_{-\upsilon,1}$	$\dot{C}_{-\upsilon,2}$	$\dot{C}_{-\upsilon,3}$	...	$C_{-\upsilon,\upsilon}$
	...	...	...	...	...	...	
	-3	$\dot{C}_{-3,0}$	$\dot{C}_{-3,1}$	$\dot{C}_{-3,2}$	$\dot{C}_{-3,3}$		
	-2	$\dot{C}_{-2,0}$	$\dot{C}_{-2,1}$	$\dot{C}_{-2,2}$			
	-1	$\dot{C}_{-1,0}$	$\dot{C}_{-1,1}$				
	0	$C_{0,0}$					

We can hence fill the cells of the upper part of a new development triangle with the adjusted Pearson residuals, as shown in Table 4.3. These adjusted Pearson residuals are given by:

$$r_{i,j} = \frac{C_{i,j} - \dot{C}_{i,j}}{\sqrt{\dot{C}_{i,j}}} \sqrt{\frac{(\upsilon+1)\,(\upsilon+2)}{\upsilon\,(\upsilon-1)}}$$

where the second factor is the adjustment made to the unscaled Pearson residual.

On one side, if there are not outliers, the adjusted Pearson residuals are small. On the other side, when both the individual age-to-age factors and the age-to-age factor of each development year are always equal, the adjusted Pearson residuals are null. We here do not include the over-dispersion parameter in the calculation of the adjusted Pearson residuals, because we are now producing estimation variance only. Furthermore, for the first accident year and last development year, and for the vice versa, the adjusted Pearson residuals are null.

Table 4.3. Adjusted Pearson residuals

		development year					
		0	1	2	3	...	v
accident year	$-v$	$r_{-v,0}$	$r_{-v,1}$	$r_{-v,2}$	$r_{-v,3}$	...	–
	...	...	...	...	...	...	
	-3	$r_{-3,0}$	$r_{-3,1}$	$r_{-3,2}$	$r_{-3,3}$		
	-2	$r_{-2,0}$	$r_{-2,1}$	$r_{-2,2}$			
	-1	$r_{-1,0}$	$r_{-1,1}$				
	0	–					

We now begin an iterative loop, and we repeat it a sufficient number of times. In this regard, for each sample u, we resample the adjusted Pearson residuals with replacement, to fill the cells of the upper part of a new development triangle, as shown in Table 4.4.

Table 4.4. Resampled adjusted Pearson residuals

		development year					
		0	1	2	3	...	v
accident year	$-v$	$r_{-v,0,u}$	$r_{-v,1,u}$	$r_{-v,2,u}$	$r_{-v,3,u}$	...	$r_{-v,v,u}$
	...	...	...	...	...	...	
	-3	$r_{-3,0,u}$	$r_{-3,1,u}$	$r_{-3,2,u}$	$r_{-3,3,u}$		
	-2	$r_{-2,0,u}$	$r_{-2,1,u}$	$r_{-2,2,u}$			
	-1	$r_{-1,0,u}$	$r_{-1,1,u}$				
	0	$r_{0,0,u}$					

Hence, we compute the pseudo-incremental amounts for paid claims from the resampled adjusted Pearson residuals, to fill the cells of the upper part of a new development triangle, as shown in Table 4.5. These pseudo-incremental amounts for paid claims are given by:

$$C_{i,j,u} = \dot{C}_{i,j} + r_{i,j,u} \sqrt{\dot{C}_{i,j}}$$

We point out that when the resampled adjusted Pearson residual is negative and high, some pseudo-incremental amount for paid claims can be negative. One simple solution to this problem is to repeat the sampling of the adjusted Pearson residuals in case of negative values. Another possible solution is to introduce some constraints for getting positive results from the process distribution simulation.

Table 4.5. Pseudo-incremental amounts for paid claims

		development year					
		0	1	2	3	...	v
accident year	$-v$	$C_{-v,0,u}$	$C_{-v,1,u}$	$C_{-v,2,u}$	$C_{-v,3,u}$	...	$C_{-v,v,u}$
	...	...	...	...	...	...	
	-3	$C_{-3,0,u}$	$C_{-3,1,u}$	$C_{-3,2,u}$	$C_{-3,3,u}$		
	-2	$C_{-2,0,u}$	$C_{-2,1,u}$	$C_{-2,2,u}$			
	-1	$C_{-1,0,u}$	$C_{-1,1,u}$				
	0	$C_{0,0,u}$					

We can again calculate the estimate of the age-to-age factors, according to the Paid Chain-Ladder method, in order to estimate the pseudo-incremental amounts for paid claims to fill the cells of the next diagonal of the development triangle (i.e. first calendar year), as shown in Table 4.6.

Table 4.6. Estimated pseudo-incremental amounts for paid claims

		development year					
		0	1	2	3	...	v
accident year	$-v$	$C_{-v,0,u}$	$C_{-v,1,u}$	$C_{-v,2,u}$	$C_{-v,3,u}$	...	$C_{-v,v,u}$
	...	...	...	...	...	...	...
	-3	$C_{-3,0,u}$	$C_{-3,1,u}$	$C_{-3,2,u}$	$C_{-3,3,u}$	...	
	-2	$C_{-2,0,u}$	$C_{-2,1,u}$	$C_{-2,2,u}$	$\hat{C}_{-2,3,u}$		
	-1	$C_{-1,0,u}$	$C_{-1,1,u}$	$\hat{C}_{-1,2,u}$			
	0	$C_{0,0,u}$	$\hat{C}_{0,1,u}$				

In order to produce some process variance, for each cell of the next diagonal of each u-th development triangle, we simulate an estimate of the pseudo-incremental amount for paid claims from an Over-Dispersed Poisson distribution, as shown in Table 4.7. This distribution is characterised by a variance equal to the mean multiplied by an over-dispersion parameter. For this reason, we simply consider a Negative Binominal with parameters such that the mean is equal to the estimated pseudo-incremental amount for paid claims, and the variance is equal to the mean multiplied by the Pearson scale parameter, i.e. an estimate of the over-dispersion parameter that is given by:

$$\phi = \frac{(v+1)\,(v+2)}{2} \sum_{i=-v}^{0} \sum_{j=0}^{-i} r_{i,j}^2 \tag{4.9}$$

The unscaled Pearson residuals (i.e. the Pearson residuals, before the adjustment) should be independent and distributed as standard Normal random variables. We here include the over-dispersion parameter in the calculation of the Pearson residuals (either unscaled or not). The sum of squares of independent standard Normal distributions is a Chi-square, hence by applying the method of moments we can obtain the formula above.

Table 4.7. Simulated pseudo-incremental amounts for paid claims

		development year					
		0	1	2	3	...	v
accident year	$-v$	$C_{-v,0,u}$	$C_{-v,1,u}$	$C_{-v,2,u}$	$C_{-v,3,u}$	...	$C_{-v,v,u}$
	...	...	...	...	...	...	...
	-3	$C_{-3,0,u}$	$C_{-3,1,u}$	$C_{-3,2,u}$	$C_{-3,3,u}$	...	
	-2	$C_{-2,0,u}$	$C_{-2,1,u}$	$C_{-2,2,u}$	$\hat{C}_{-2,3,u^*}$		
	-1	$C_{-1,0,u}$	$C_{-1,1,u}$	$\hat{C}_{-1,2,u^*}$			
	0	$C_{0,0,u}$	$\hat{C}_{0,1,u^*}$				

We now fill the cells of the upper part of a new development triangle with the observed incremental amounts for paid claims and the cells of the next calendar year with the simulations of the estimated pseudo-incremental amounts for paid claims, as shown in Table 4.8.

Table 4.8. Observed and simulated pseudo-incremental amounts for paid claims

		development year					
		0	1	2	3	...	v
accident year	$-v$	$C_{-v,0}$	$C_{-v,1}$	$C_{-v,2}$	$C_{-v,3}$	...	$C_{-v,v}$
	...	...	...	...	...	...	...
	-3	$C_{-3,0}$	$C_{-3,1}$	$C_{-3,2}$	$C_{-3,3}$	...	
	-2	$C_{-2,0}$	$C_{-2,1}$	$C_{-2,2}$	$\hat{C}_{-2,3,u^*}$		
	-1	$C_{-1,0}$	$C_{-1,1}$	$\hat{C}_{-1,2,u^*}$			
	0	$C_{0,0}$	$\hat{C}_{0,1,u^*}$				

We finally calculate the new estimate of the age-to-age factors, according to the Paid Chain-Ladder method, in order to estimate the pseudo-incremental amount for paid claims of each cell of the next calendar years:

$$\hat{C}_{i,j|u^*} \quad \text{with} \quad i+j>1$$

The estimate of the u-th total pseudo-claims reserve at the end of the first year is then given by:

$$\hat{L}^{(0)}_{1,u^*} = \sum_{i=-v+2}^{0} \sum_{j=-i+2}^{v} \hat{C}_{i,j|u^*}\, \xi_{i+j}\, P_{n_s}(1, i+j)$$

and it can be seen as a simulation from the random variable $L^{(0)}_1$ involved in the calculation of the claims development result at the end of time one and two. More precisely, it is the stochastic claims reserve at the end of the first year for the claims already occurred at the end of time zero. It is worth pointing out that the regulatory nominal term structure is now stochastic. As explained in Section 1.1 for the risk premium amounts, we also assume that the future claims reserves are not adjusted for the stochastic claims inflation realised over time (i.e. the expected claims inflation index is calculated at time zero and kept fixed over time). This is because, as already mentioned, we want to emphasise the contribution of the claims inflation to the non-life risk, and we prefer not to complicate the model. We remind that in our numerical analysis we will consider a time horizon of three years, which implies the calculation of just three future claims reserves for which the insurer does not adjust the expected claims inflation index. It results both in underestimation and overestimation of the risk.

Moreover, the simulation of the u-th total estimated pseudo-incremental amount for paid claims of the next calendar year, rescaled using the claims inflation index described in Section 4.2.2, is given by:

$$\hat{C}^{(0)}_{1,u^*} = \sum_{i=-v+1}^{0} \hat{C}_{i,-i+1,u^*}\, \vartheta \, \frac{I(1)}{I_0}$$

and it can be seen as a simulation from the random variable $C^{(0)}_1$ involved in the calculation of the claims development result at the end of time one. More precisely, it is the stochastic amount for claims already occurred at the end of time zero and settled in the first year. It is worth pointing out that this variable produces both estimation and process variance. On the other side, the future claims reserve produces estimation variance only, but it is affected by the process variance of the other random variable. We are now able to derive some simulations of the claims development result at the end of time one.

In order to describe the claims development result at the end of time two, we start from the situation shown in Table 4.5, and we estimate the pseudo-incremental amounts for paid claims, not only to fill the cells of the next diagonal of the development triangle, but also to fill the cells of the following calendar year. Once again, we simulate an estimate of the pseudo-incremental amounts for paid claims from the Over-Dispersed Poisson distribution described above, and we build the development triangle shown in Table 4.9.

Table 4.9. Observed and simulated pseudo-incremental amounts for paid claims

		development year					
		0	1	2	3	...	v
accident year	$-v$	$C_{-v,0}$	$C_{-v,1}$	$C_{-v,2}$	$C_{-v,3}$	...	$C_{-v,v}$
	...	...	...	...	...	...	...
	-3	$C_{-3,0}$	$C_{-3,1}$	$C_{-3,2}$	$C_{-3,3}$	...	...
	-2	$C_{-2,0}$	$C_{-2,1}$	$C_{-2,2}$	$\hat{C}_{-2,3,u^*}$	...	
	-1	$C_{-1,0}$	$C_{-1,1}$	$\hat{C}_{-1,2,u^*}$	$\hat{C}_{-1,3,u^*}$		
	0	$C_{0,0}$	$\hat{C}_{0,1,u^*}$	$\hat{C}_{0,2,u^*}$			

Once again, we apply the Paid Chain-Ladder method. The estimate of the u-th total pseudo-claims reserve at the end of the second year is given by:

$$\hat{L}^{(0)}_{2,u^*} = \sum_{i=-v+3}^{0} \sum_{j=-i+3}^{v} \hat{C}_{i,j|u^*}\, \xi_{i+j}\, P_{n_s}(2, i+j)$$

and the simulation of the u-th total estimated pseudo-incremental amount for claims paid in the second year is given by:

$$\hat{C}^{(0)}_{2,u^*} = \sum_{i=-v+2}^{0} \hat{C}_{i,-i+2,u^*}\, \vartheta^2\, \frac{I(2)}{I_0}$$

which can be seen as a simulation from the random variables $L^{(0)}_2$ and $C^{(0)}_2$ involved in the calculation of the claims development result at the end of time two and three.

In general, we can simply repeat the same procedure just reported, in order to describe the claims development result at the end of any time. The estimate of the u-th total pseudo-claims reserve at the end of year t is given by:

$$\hat{L}^{(0)}_{t,u^*} = \sum_{i=-v+t+1}^{0} \sum_{j=-i+t+1}^{v} \hat{C}_{i,j|u^*}\, \xi_{i+j}\, P_{n_s}(t, i+j)$$

and the simulation of the u-th total estimated pseudo-incremental amount for claims paid in year t is given by:

$$\hat{C}^{(0)}_{t,u^*} = \sum_{i=-v+t}^{0} \hat{C}_{i,-i+t,u^*}\, \vartheta^t\, \frac{I(t)}{I_0}$$

Chapter 5

Calibration and parameters

In this chapter, we present all the parameters of our numerical analysis, and we calibrate the previously described model on Italian historical data and Euro market data. First of all, we present some general parameters of our insurance company to determine the initial capital and reserve, the business mix and investments. We also estimate the pricing elements for each LoB. We remind that the parameters of the risk-neutral model are in common with the real-world model. They are calibrated on derivative instruments (this is the reason why we need risk-neutral models). Once we estimate the risk-neutral parameters, we calibrate the additional real-world ones (i.e. risk premiums), using some real-world expectations (e.g. some interest rate forecasts published by OECD). Furthermore, we calibrate the parameters of the premium and reserve model, including the development triangles of incremental amounts for paid claims. Finally, we describe the hierarchical structure of copula functions, and we determine their parameters.

5.1 Insurance company

We now estimate the parameters of our insurance company on Italian insurance market data until the end of 2020, provided by ANIA (2021). Note that sometimes the parameters are based on initial assumptions (justified by market trends).

The general parameters of our insurance company are presented in Table 5.1. We point out that the initial ratio of risk reserve to GPW (25%) is roughly one and a half times the Required Solvency Margin for non-life insurance, required by Solvency 0 and Solvency I. The initial GPW is 100 mln, hence the initial asset value of the total portfolio (i.e. the sum of initial risk reserve and claims reserve) is equal to 205 mln. We remind that there is no premium reserve, since we assume exposure referred to the full calendar year only.

Going into detail, the composition of GPW and claims reserve by LoB is shown in Figure 5.1. Note that the business mix is 50% for MTPL, 25% for MOD, and 25% for GTPL. Furthermore, the ratio of initial claims reserve to GPW is in line with the Italian insurance historical data provided by IVASS (2020-21) and IVASS (2017-22), and it is 150% for MTPL, 20% for MOD, and 400% for GTPL.

The asset allocation of our insurance company is presented in Table 5.2. The stock (40% of the total) and property portfolios (5% of the total) are smaller than

Table 5.1. General parameters (amounts in millions) of our insurance company

Parameter	Value
U_0	25
L_0	180
b_0	100

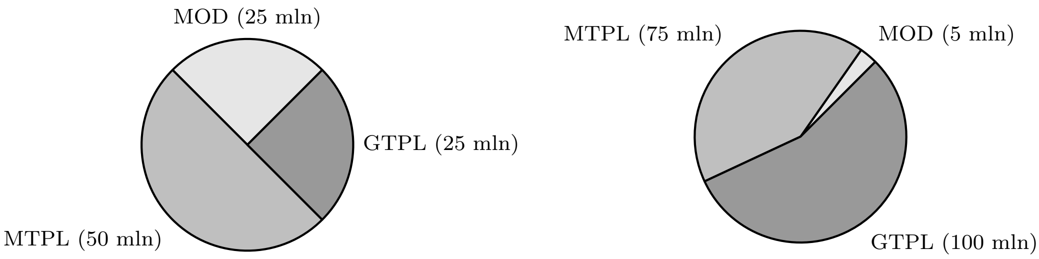


Figure 5.1. Initial GPW (left) and claims reserve (right) by LoB

the bond portfolio (55% of the total). The latter is more invested in medium and long-term assets (e.g. 40% of the bond portfolio is invested in a five-year zero-coupon bond). Using the asset allocation parameters, it is thus possible to obtain the initial value of each investment.

Table 5.2. Asset allocation parameters (expressed in %) of our insurance company

Parameter	Value
α	40
β	5
$1-\alpha-\beta$	55
γ_1	5
γ_2	15
γ_3	20
γ_5	40
γ_{10}	20

The expense loading coefficient is estimated by the average of the last five annual observations of the Italian expense ratio (see Table 5.3). The role of the expense loading indeed is to cover future expenses, both acquisition and general ones.

On the other hand, the safety loading coefficient is estimated by the complement of 100% of the average of the last five annual observations of the Italian combined ratio without claims reserving run-off (see Table 5.4). The complement of 100% of the combined ratio indeed represents the expected profit of the insurance company included in the pricing process. This safety loading coefficient is a multiplier of the GPW, hence we can write:

$$b_t = \pi_t + \lambda_b\, b_t + c\, b_t$$

Table 5.3. Historical Italian expense ratio (expressed in %)

LoB	2016	2017	2018	2019	2020
MTPL	21.44	21.19	21.09	21.26	21.47
MOD	30.51	30.74	31.52	32.06	31.61
GTPL	32.80	33.14	32.58	32.15	32.86

where λ_b is the safety loading coefficient described above.

Using the equation above, the safety loading coefficient can be transformed into a coefficient of the risk premium amount:

$$\lambda = \frac{\lambda_b}{1 - \lambda_b - c}$$

Table 5.4. Historical Italian combined ratio net of run-off (expressed in %)

LoB	2016	2017	2018	2019	2020
MTPL	101.96	102.49	101.42	101.69	89.63
MOD	90.00	93.12	91.84	101.06	87.39
GTPL	95.41	94.82	92.99	88.47	94.10

The estimated pricing parameters of our insurance company (i.e. expense and safety loading coefficients) are finally shown in Table 5.5. They correspond to a total safety loading coefficient around 5% and a total expense loading coefficient around 26%, slightly increasing over time as the insurance company size increases.

Table 5.5. Estimated pricing parameters (expressed in %) of our insurance company

Parameter	MTPL	MOD	GTPL
c	21.29	31.29	32.71
λ	0.72	11.92	11.32

5.2 Risk-neutral economic scenario generator

We now calibrate the risk-neutral model described in Section 3.1 on Euro market data on December 31, 2021, therefore we obtain a market-consistent result. We assume that the interest-rate swap term structure is our reference risk-free nominal interest rate curve (see Table 5.6). Our reference risk-free real interest rate curve is instead derived using ZCIIS fixed rates (i.e. our inflation curve) and equation 3.8.

It is worth mentioning that in our numerical analysis we will not consider the interest-rate swap term structure presented in this section. Instead, we will take into account an economic scenario generator with same parameters (as those derived from such term structure), but different nominal interest rates. On the asset side, we will use the average of all EEA government nominal rates on December 31, 2021.

Table 5.6. Risk-free nominal interest rates, real interest rates, and inflation rates (annually compounded and expressed in %) on December 31, 2021

Maturity (years)	Nominal rate	Real rate	Inflation rate
1	−0.488	−3.826	3.470
2	−0.299	−2.859	2.635
3	−0.150	−2.452	2.360
5	0.015	−2.107	2.168
7	0.128	−1.925	2.094
10	0.302	−1.727	2.065
15	0.496	−1.598	2.128
20	0.552	−1.589	2.175

On the liability side, we will consider instead the Eiopa interest rate term structure without volatility adjustment on December 31, 2021. Both the interest rate term structures are shown in Table 5.7. With this approach we are implicitly assuming to have a deterministic spread over the risk-free nominal interest rates and also that a change in the nominal term structure does not affect the model parameters. The first assumption is what we also have in a regulatory risk-neutral framework (e.g. Solvency II and IFRS 17). The second assumption is necessary, as we do not have Euro market derivatives based on the alternative interest rate term structure. Another possibility could be to assume that a change in the nominal term structure does not affect the Euro market derivatives volatility, but there is not any reason to prefer this approach instead of ours. We finally assume that the inflation term structure remains always the same, meaning that the real rates absorb the change in the nominal interest rate term structure. This is because the inflation rates are under the control of the ECB, and they should be on average more stable than the real rates.

Table 5.7. Average of all EEA government nominal rates and Eiopa interest rates without volatility adjustment (annually compounded and expressed in %) on December 31, 2021

Maturity (years)	EEA rate	Eiopa rate	Eiopa rate (shock up)	Eiopa rate (shock down)
1	−0.626	−0.585	0.415	−0.585
2	−0.529	−0.395	0.605	−0.395
3	−0.421	−0.246	0.754	−0.246
5	−0.193	−0.084	0.916	−0.084
7	0.030	0.030	1.030	0.018
10	0.319	0.205	1.205	0.141
15	0.662	0.399	1.399	0.291
20	0.874	0.456	1.456	0.324

5.2.1 Nominal and real interest rates and CPI

First of all, we calibrate the nominal parameters that we will use as an input to calibrate the remaining ones. We look for the set of parameters that minimises the sum of squared differences between market and model nominal interest rate derivative prices. Therefore, the optimisation problem can be formalised as follows:

$$\operatorname{argmin}_{a_n, b_n, \sigma_n, \eta_n, \rho_n} \sum_i \left(\text{price}_i^{\text{market}} - \text{price}_i^{\text{model}}\right)^2$$

In this exercise, we only consider the derived at-the-money (ATM) IRC prices with maturity from one to twenty years (see Table 5.8) and the derived ATM EPS prices with maturity and tenor combination from one to ten years (see Table 5.9).

Table 5.8. ATM IRC prices (expressed in %) on December 31, 2021

Maturity (years)	Market price
1	0.05
2	0.32
3	0.70
5	1.89
7	3.28
10	5.74
15	10.09
20	14.55

Table 5.9. ATM EPS prices (expressed in %) on December 31, 2021

Maturity / Tenor (years)	1	2	3	4	5	6	7	8	9	10
1	0.19	0.44	0.70	0.96	1.22	1.48	1.74	2.00	2.25	2.51
2	0.35	0.72	1.08	1.43	1.76	2.12	2.48	2.82	3.16	3.49
3	0.47	0.94	1.38	1.79	2.18	2.60	3.02	3.43	3.82	4.21
5	0.60	1.20	1.76	2.29	2.80	3.32	3.84	4.33	4.81	5.27
7	0.69	1.38	2.03	2.66	3.26	3.85	4.43	4.98	5.53	6.06
10	0.77	1.54	2.27	2.98	3.66	4.33	5.00	5.65	6.28	6.91

Given the nominal parameters, the remaining ones are calibrated looking for the set of parameters that minimises the squared differences between market and model inflation-indexed derivative quotes. We now must ensure that the correlation matrix is positive definite. The optimisation problem can be formalised as follows:

$$\operatorname{argmin}_{\substack{a_r, b_r, \sigma_r, \eta_r, \rho_r, \rho_{x_n, x_r}, \rho_{x_n, y_r}, \rho_{y_n, x_r}, \rho_{y_n, y_r} \\ \sigma_I, \rho_{x_n, I}, \rho_{y_n, I}, \rho_{x_r, I}, \rho_{y_r, I}}} \sum_i \left(\text{quote}_i^{\text{market}} - \text{quote}_i^{\text{model}}\right)^2$$

$$\text{s.t.} \quad \text{positive definite correlation matrix}$$

In this exercise, we only consider the YYIIS fixed rates with maturity from one to twenty years (see Table 5.10) and the IIC prices with different strike rates and with maturity from one to twenty years (see Table 5.11).

Table 5.10. YYIIS fixed rates (expressed in %) on December 31, 2021

Maturity (years)	Market price
1	3.470
2	2.637
3	2.360
5	2.168
7	2.094
10	2.065
15	2.126
20	2.172

Table 5.11. IIC prices (expressed in %) on December 31, 2021

	Zero-coupon				Year-on-year			
Maturity / Strike	1.00%	2.00%	3.00%	4.00%	1.00%	2.00%	3.00%	4.00%
1	2.49	1.49	0.59	0.11	2.49	1.49	0.59	0.11
2	3.38	1.47	0.31	0.07	3.42	1.81	0.69	0.14
3	4.32	1.59	0.32	0.08	4.36	2.13	0.80	0.19
5	6.38	1.89	0.33	0.09	6.50	3.01	1.21	0.44
7	8.58	2.39	0.45	0.13	8.84	4.13	1.81	0.82
10	12.45	4.15	0.88	0.18	12.74	6.16	2.97	1.60
15	20.44	7.16	1.58	0.33	19.99	9.95	4.98	2.89
20	29.72	10.85	2.52	0.59	27.27	13.74	7.00	4.23

As an alternative to the full calibration process, we assume that the CPI diffusion coefficient and correlations are null. It means that the CPI process is not stochastic, except for the drift, which depends on other dynamics, therefore it produces anyway some correlation with nominal or real rates. In this way we reduce the parameters to calibrate, but we still allow to obtain good results. We point out that the model is market-consistent whenever the current derivative quotes are well quantified, and it is true even if we have some parameter constraints.

The calibrated parameters for the nominal and real interest rate model and CPI model are shown in Table 5.12, and the differences between resulting model quotes and market quotes are shown in Figure 5.2, 5.3, 5.4 and 5.5.

Both in the full and constrained calibration processes, we obtain the same nominal parameters, because the first optimisation problem is equivalent. In the cases of IRCs and EPSs, the absolute value of the differences between market and model prices is always lower than 0.25% and 0.10%, respectively, and its highest peaks are

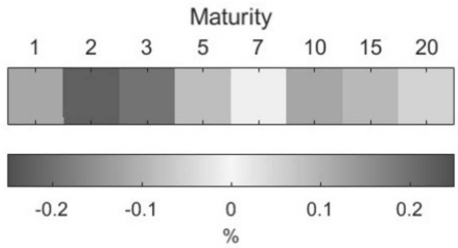


Figure 5.2. ATM IRC model errors (expressed in %) on December 31, 2021

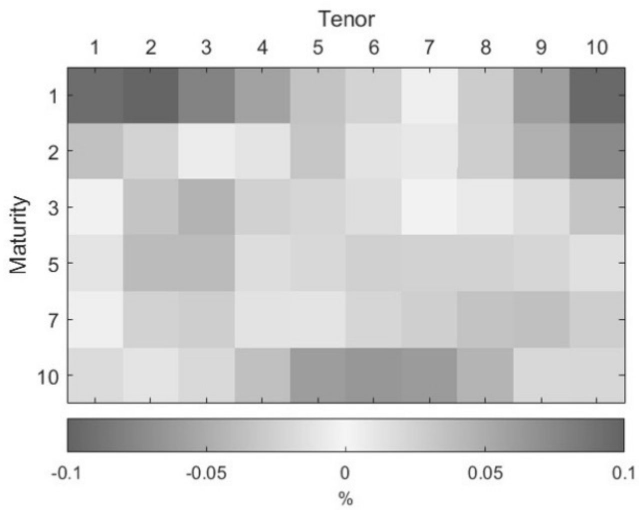
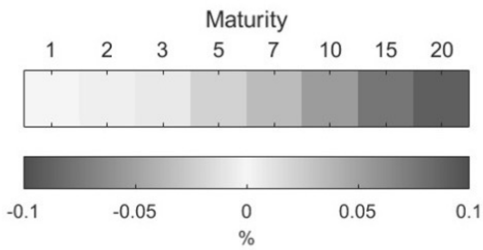
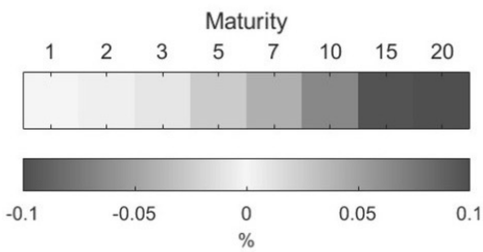


Figure 5.3. ATM EPS model errors (expressed in %) on December 31, 2021

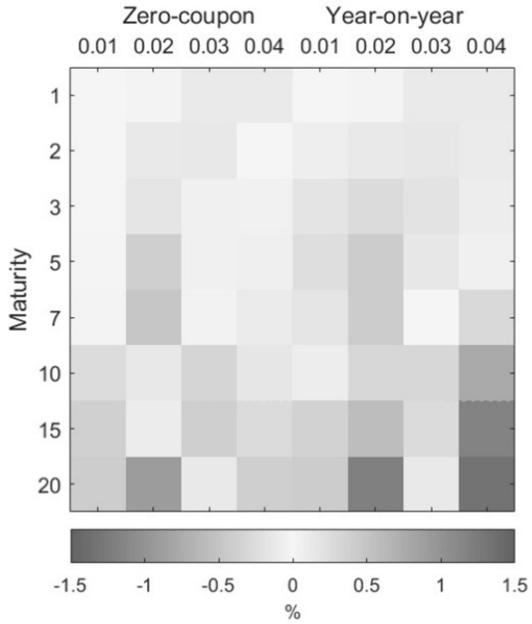


(a) Full calibration

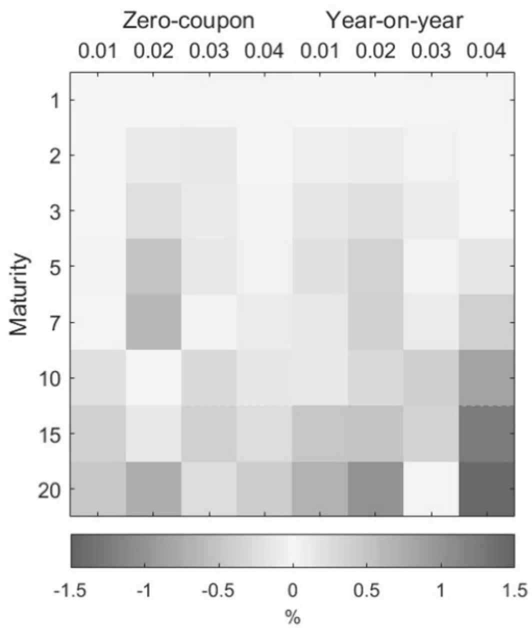


(b) Constrained calibration

Figure 5.4. YYIIS model errors (expressed in %) on December 31, 2021



(a) Full calibration



(b) Constrained calibration

Figure 5.5. IIC model errors (expressed in %) on December 31, 2021

Table 5.12. Calibrated nominal and real parameters on December 31, 2021

Parameter	Full calibration	Constrained calibration
a_n	0.45114	0.45114
b_n	0.03631	0.03631
σ_n	0.01172	0.01172
η_n	0.00872	0.00872
ρ_n	−0.60984	−0.60984
a_r	0.36384	0.12021
b_r	0.15195	0.68688
σ_r	0.00318	0.01244
η_r	0.01364	0.02001
ρ_r	−0.14675	0.00057
ρ_{x_n,x_r}	−0.16914	−0.19394
ρ_{x_n,y_r}	0.03920	0.13850
ρ_{y_n,x_r}	0.05096	0.66771
ρ_{y_n,y_r}	0.69111	0.48630
σ_I	0.00892	0
$\rho_{x_n,I}$	−0.02798	0
$\rho_{y_n,I}$	−0.57244	0
$\rho_{x_r,I}$	−0.27672	0
$\rho_{y_r,I}$	−0.40128	0

found in proximity to the shortest maturities. In the cases of YYIISs and IICs, the absolute value of the differences between market and model quotes is always lower than 0.10% and 1.50%, respectively (both in the full and constrained calibration processes), and its highest peaks are found in proximity to the longest maturities.

The magnitude order of the errors in IICs is bigger if compared to the other derivatives. This is because the IICs are the most parametrised instruments we have and because errors in absolute terms are affected by the magnitude order of the quotes. If errors in relative terms were examined, their values in IICs would still be bigger if compared to the other derivatives, but less than above.

In addition, the value of the errors in ZCIICs is lower than in the corresponding YYIICs, because less elements are involved in the pricing formula. In conclusion, the value of the errors is slightly smaller in the full calibration process than in the constrained one, because of more degrees of freedom. We believe that parsimony should be used whenever possible, hence from now on we will take into account only the parameters coming from the constrained calibration process. This decision will also avoid problems in the definition of a positive definite correlation matrix in the next section.

5.2.2 Stock and property prices

Given the constrained nominal parameters, the stock-price ones are calibrated looking for the set of parameters that minimises the squared differences between market and model EO prices. We must again ensure that the entire correlation matrix is positive definite. We point out that this time the correlation matrix also includes the coefficients of the nominal and real interest rates and CPI. Also, note that we only calibrate the unique volatility parameter, assuming that the other correlation parameters are null. This is done for parsimony and because the instruments used for the calibration do not really depend on correlations. The optimisation problem can be formalised as follows:

$$\operatorname{argmin}_{\sigma_S} \sum_i \left(\text{price}_i^{\text{market}} - \text{price}_i^{\text{model}}\right)^2$$

$$\text{s.t.} \quad \text{positive definite correlation matrix}$$

In this exercise, we only consider the ATM EO price on the Euro Stoxx 50 Index with maturity of one year (see Table 5.13). The annual divided yield is the expectation of the market on December 31, 2021.

Table 5.13. ATM EO prices on December 31, 2021

Maturity (years)	Market price	Underlying price	Strike price	Annual dividend yield
1	243.25	4,306.07	4,300.00	0.0274

The property-price volatility parameter is calibrated on historical data, because we do not have a sufficient liquid property-price option quoted in the market. In particular, the diffusion parameter of the property-price model is estimated solving a system based on equation 3.13 (we point out that the variance of the logarithm of the property price is analogous to the stock case, and it can be found by replacing the character S with H) and with continuously compounded daily interest rates of the Italy Real Residential Property Price Index between December 31, 2011, and December 31, 2021. We remind that the nominal parameters are given, and the variance is not affected by the change-of-numeraire technique. Also, we assume 252 trading days in a year. We finally assume that the correlation coefficient between stock and property is equal to the SF assumption (+0.75).

The calibrated parameters for the stock-price and property-price models are finally shown in Table 5.14. Note that we do not have any calibration error, because we have problems with unique analytical solutions.

5.3 Real-world economic scenario generator

We now calibrate the real-world model described in Section 3.3 on Euro forecast data on December 31, 2021, and Euro historical data between December 31, 2011, and December 31, 2021. However, we only need to estimate the local risk premiums, because the remaining parameters are already calibrated for the risk-neutral model in Section 5.2, and they are also the same in the real-world model.

Table 5.14. Calibrated stock and property-price parameters on December 31, 2021

Parameter	Constrained calibration
σ_S	0.17953
$\rho_{x_n,S}$	0
$\rho_{y_n,S}$	0
$\rho_{x_r,S}$	0
$\rho_{y_r,S}$	0
$\rho_{I,S}$	0
σ_H	0.01469
$\rho_{x_n,H}$	0
$\rho_{y_n,H}$	0
$\rho_{x_r,H}$	0
$\rho_{y_r,H}$	0
$\rho_{I,H}$	0
$\rho_{S,H}$	0.75000

5.3.1 Nominal and real interest rates and CPI

The nominal local risk premiums are estimated solving a system based on equation 3.14 and with real-world expectations of continuously compounded nominal interest rates (see Table 5.15) obtained from the data published by OECD (OECD, 2021a; OECD, 2021b). For the longest available projection horizon (i.e. fourth quarter of 2023), the real-world expectations are the latest forecasts of the three-month and ten-year EEA nominal interest rates on December 31, 2021. For a forty-year projection horizon, the real-world expectations are assumed to be the average of monthly three-month and ten-year EEA nominal interest rates between December 31, 2011, and December 31, 2021. In this exercise, we assume that τ is equal to two years, which is the time horizon from December 31, 2021, to the fourth quarter of 2023.

Table 5.15. Real-world nominal interest rate expectations (continuously compounded and expressed in %) on December 31, 2021

Forecast horizon (years)	Maturity (years)	Nominal rate
2	0.25	−0.346
2	10.00	0.176
40	0.25	0.752
40	10.00	2.301

Given the nominal local risk premiums, the real ones are estimated solving a system based on equation 3.15 (see also the other equations presented in Section 3.3.1) and with some real-world expectations of the ratio of the CPI at a given time and time zero (see Table 5.16). One of these expectations is obtained from a forecast published again by OECD (OECD, 2021a). This real-world expectation is the EEA forecast on December 31, 2021, for the longest available projection horizon (i.e. fourth quarter of 2023). Moreover, for a five, ten and eighty-year projection horizon, we assume a real-world expectation given by the accumulation of an average inflation rate of 2% (i.e. inflation target of the ECB in the medium term) over years.

Table 5.16. Real-world expectations of ratio of the CPI at a given time and time zero on December 31, 2021

Forecast horizon (years)	Initial CPI ratio
2	1.03694
5	1.10408
10	1.21899
80	4.87544

The calibrated parameters for the nominal and real interest rate model are shown in Table 5.17.

Table 5.17. Calibrated nominal and real local risk premiums on December 31, 2021

Parameter	Constrained calibration
τ	2
d_{x_n}	0.01430
l_{x_n}	−0.01918
d_{y_n}	−0.12428
l_{y_n}	−0.00273
d_{x_r}	−0.02384
l_{x_r}	0.03927
d_{y_r}	0.02394
l_{y_r}	−0.04330

5.3.2 Stock and property prices

Given the constrained nominal parameters and nominal local risk premiums, the stock-price and property-price additional local risk premiums are estimated solving a system based on equation 3.16 and with real-world expectations of continuously compounded stock and property-returns (see Table 5.18) obtained from the Euro Stoxx 50 Index and Italy Real Residential Property Price Index (once again, we assume 252 trading days in a year). In detail, the real-world expectations are calculated as one-year, two-year, and three-year rolling returns between December 31, 2011, and December 31, 2021. As a consequence, in this exercise we assume that τ_1 and τ_2 are respectively equal to one and two years.

The calibrated parameters for the stock-price and property-price models are shown in Table 5.19.

Table 5.18. Real-world stock and property-return expectations (continuously compounded and expressed in %) on December 31, 2021

Forecast horizon (years)	Stock return	Property return
1	3.507	0.759
2	1.335	0.705
3	1.612	0.712

Table 5.19. Calibrated stock and property-price additional local risk premiums on December 31, 2021

Parameter	Constrained calibration
τ_1	1
τ_2	2
d_S	0.05690
l_S	0.01161
q_S	0.04520
d_H	0.01341
l_H	0.01048
q_H	0.01478

5.4 Non-life premium model

We now calibrate the premium model on Italian insurance market data until the end of 2020, provided by ANIA (2021), and Euro historical data.

Firstly, the value of the initial mean and coefficient of variation of the single claim amount, as well as the real growth rate parameter, are determined through empirical evidence. Furthermore, the initial expected number of claims is obtained in function of the size of the insurance company. Indeed, according to equation 1.6, the risk premium amount is equal to the expected value of the total claim amount. Moreover, the latter is solved by equation 4.3. Combining equation 1.3 with the preceding ones, the initial expected number of claims is found to be:

$$k_0 = \frac{b_0\,(1-c)}{m_0\,(1+\lambda)}$$

The standard deviation of the structure variable (i.e. the parameter uncertainty of the number of claims distribution) is given by the product of the standard deviation of the loss ratio on accrual basis (see Table 5.20) and the ratio of gross to risk premium amount. The latter can be simplified using equation 1.3, so that we have:

$$\sigma_Q = \mathrm{Std}\left(\frac{X_t}{b_t}\right)\frac{1+\lambda}{1-c}$$

This result holds because we can see the Italian market data as the portfolio history of a huge-sized insurance company, and thus the standard deviation of the pure loss ratio (i.e. both the formula above and the coefficient of variation of the total claim amount) approaches the standard deviation of the structure variable, as as shown in equation 4.4. Currently, there is no method in the literature to estimate this relevant parameter excluding the variability coming from the underwriting cycle.

Table 5.20. Historical Italian loss ratio on accrual basis (expressed in %)

LoB	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
MTPL	76.85	68.42	68.54	71.83	76.35	80.52	81.30	80.32	80.43	68.16
MOD	64.86	62.52	68.09	63.02	60.93	59.48	62.38	60.32	69.00	55.78
GTPL	77.99	73.91	72.55	68.81	63.99	62.61	61.68	60.41	56.32	61.24

The main indicators of the premium model (i.e. initial mean of the single claim amount, initial coefficient of variation of the single claim amount, initial expected number of claims, real growth rate, and standard deviation of the structure variable) are shown in Table 5.21. These are now used to calibrate the model parameters.

Table 5.21. Main indicators of the premium model

Parameter	MTPL	MOD	GTPL
m_0	4,000	2,500	10,000
c_{Z_0}	7	2	12
k_0	9,768	6,139	1,511
g	2%	2%	2%
σ_Q	0.07066	0.06439	0.11554

We consider the ratio of Italian annual claims inflation index (see Table 5.22) to Euro annual CPI (see Table 5.23) adjusted over time for the claims inflation percentage. We then consider the average of the last five annual observations of this ratio, and we estimate the claims inflation percentage as the value that makes such average equal to one.

Table 5.22. Historical Italian annual claims inflation index (unit value as of 2015)

LoB	2016	2017	2018	2019	2020
MTPL	0.97921	0.96845	0.97640	0.97332	1.10098
MOD	1.03076	1.06494	1.04990	1.13124	1.16473
GTPL	1.01447	1.00402	0.96130	0.94269	1.22180

Moreover, the parameters of the Lognormal distribution (for the initial single claim amount) are calibrated by using the method of moments:

$$\sigma = \sqrt{\ln\left(1 + c_{Z_0}^2\right)} \quad \text{and} \quad \mu = \ln m_0 - \frac{\sigma^2}{2}$$

The parameters of the Negative Binomial distribution (for the number of claims)

Table 5.23. Historical Euro annual CPI (unit value as of 2015)

Year	CPI
2016	1.00531
2017	1.02028
2018	1.03925
2019	1.04953
2020	1.04524

are calibrated by using the following relations (see Section 4.2.1):

$$h = \frac{1}{\sigma_Q^2} \quad \text{and} \quad p_t = \frac{h}{h + k_0 (1+g)^t}$$

The calibrated parameters for the premium model (i.e. claims inflation percentage, parameters of the Lognormal distribution, and parameters of the Negative Binomial distributions) are finally shown in Table 5.24.

Table 5.24. Calibrated parameters of the premium model

Parameter	MTPL	MOD	GTPL
ϑ	0.98959	1.01765	0.99903
μ	6.33804	7.01933	6.72197
σ	1.97788	1.26864	2.23086
h	200.27	241.16	74.91
p_1	0.01970	0.03708	0.04634
p_2	0.01932	0.03638	0.04548
p_3	0.01895	0.03569	0.04463

5.5 Non-life reserve model

We now determine or calibrate the input for the reserve model by Italian insurance market data until the end of 2020, provided by IVASS (2020-21) and IVASS (2017-22). We assume that the Eiopa interest rate term structure without volatility adjustment is our regulatory nominal interest rate curve (see Table 5.7).

Each development triangle of incremental amounts for paid claims is determined by the historical Italian incremental settlement speed for amount and a reasonable assumption of its coefficient of variation by development year (see Table 5.25, 5.26, and 5.27). We point out that we ignore the development triangle tail, and for MOD and GTPL, we applied a linear interpolation in the last development years of the cumulative settlement speed for amount, because of the lack of historical data.

It is worth noticing that the historical data of an entire country has low volatility. In order to get some reasonable data for the dimension of the company considered

Table 5.25. Historical Italian settlement speed for amount and coefficient of variation (expressed in %) of MTPL

		Development year											
		0	1	2	3	4	5	6	7	8	9	10	11
Accident year	2009	36.81	32.13	11.95	5.93	3.53	2.53	1.82	1.48	0.92	0.67	0.47	0.32
	2010	36.66	32.20	12.20	5.71	3.33	2.49	2.02	1.38	0.92	0.74	0.49	
	2011	36.96	32.06	11.99	5.52	3.44	2.52	1.88	1.34	0.93	0.64		
	2012	35.47	31.84	12.50	5.88	3.46	2.62	1.72	1.33	0.97			
	2013	35.34	31.23	12.82	5.88	3.24	2.29	1.81	1.15				
	2014	36.04	31.55	12.09	5.23	3.15	2.33	1.57					
	2015	36.36	31.49	12.23	5.30	3.28	2.14						
	2016	37.46	30.92	12.15	5.50	2.93							
	2017	37.98	31.32	12.16	5.39								
	2018	38.70	31.65	11.60									
	2019	40.18	30.67										
	2020	42.32											
Coefficient of variation		0.16	0.13	0.10	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95

Table 5.26. Historical Italian settlement speed for amount and coefficient of variation (expressed in %) of MOD

		Development year							
		0	1	2	3	4	5	6	7
Accident year	2013	76.90	20.20	1.00	0.40	0.20	0.20	0.16	0.13
	2014	78.10	19.30	0.90	0.30	0.30	0.20	0.16	
	2015	78.00	19.90	0.90	0.30	0.20	0.20		
	2016	78.10	19.60	1.10	0.40	0.10			
	2017	75.80	21.70	0.90	0.30				
	2018	77.00	20.20	0.90					
	2019	72.30	24.20						
	2020	77.90							
Coefficient of variation		0.03	0.02	0.01	0.03	0.05	0.07	0.09	0.11

Table 5.27. Historical Italian settlement speed for amount and coefficient of variation (expressed in %) of GTPL

Accident year	Development year 0	1	2	3	4	5	6	7	8	9	10	11
2009	11.00	19.40	12.40	8.20	7.30	5.80	4.64	3.71	2.97	2.38	1.90	1.52
2010	11.00	19.40	12.40	8.20	7.30	5.80	4.64	3.71	2.97	2.38	1.90	
2011	12.20	19.30	11.30	8.00	6.20	5.50	4.40	3.52	2.82	2.25		
2012	13.30	20.60	10.60	7.60	6.20	5.50	4.40	3.52	2.82			
2013	12.80	19.70	10.70	7.90	6.60	5.80	4.64	3.71				
2014	13.70	18.80	10.80	8.40	6.40	5.50	4.40					
2015	13.20	18.70	10.90	7.90	7.00	4.80						
2016	14.20	18.60	11.50	7.80	6.10							
2017	14.70	19.80	12.10	7.10								
2018	15.30	20.00	10.80									
2019	15.30	20.20										
2020	17.60											
Coefficient of variation	0.20	0.15	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00

in our numerical analysis, we simulate from a Beta distribution having mean equal to the Italian incremental settlement speed for amount and variance based on the assumed coefficient of variation. We must ensure that the incremental amounts for paid claims give the claims reserve in Figure 5.1. In this context, we assume that the expected ultimate cost of any accident year is either equal, bigger or smaller by 5% of the elder expected ultimate cost. Hence, we change the latter to determine the incremental amounts for paid claims (using the simulated settlement speed for amount), so that the Paid Chain-Ladder claims reserve (see Section 4.3.2) is equal to our initial assumption. Note that before applying the Paid Chain-Ladder method, we appreciate the incremental amounts for paid claims of the upper triangle until the current calendar year, using the historical Italian annual claims inflation index (see Table 5.22), so that all the amounts for paid claims have value of money as of the current calendar year. Furthermore, we remind that we consider the discounting effect and expected claims inflation index, but we do not consider the risk margin in the reserve calculation. Note that the expected claims inflation index is estimated according to equation 4.2, based on the calibrated parameters of the real-world economic scenario generator. Each resulting development triangle of incremental amounts for paid claims (with value of money as of the current calendar year) is finally shown in Table 5.28, 5.29, and 5.30.

The Pearson scale parameter is calibrated according to equation 4.9. Moreover, we remind that we have already calibrated the claims inflation percentage for the premium model. The calibrated parameters for the reserve model (i.e. expected claims inflation indices, claims inflation percentage, and Pearson scale parameter) are finally shown in Table 5.31.

5.6 Copula parameters

We now estimate the copula parameters on the SF linear correlation coefficients (see Table 5.32) which are defined in the Delegated Regulation (Commission Delegated Regulation (EU) 2015/35, 2009) and already presented in Section 2.4. We remind that in our numerical analysis, we will consider a hierarchical structure of copula functions, either based on linear dependence, i.e. Gaussian copulas, or non-linear one. We will only consider Gumbel copulas to inject non-linear dependence in the aggregation process. However, we will use Gaussian copulas in the aggregation of the sources of market risk, both in the linear and non-linear context. In our numerical analysis, we will show that the market risk depends almost completely on equity risk and almost nothing on property risk (we remind that property is only 5% of the asset allocation). We will also prove that our portfolio is exposed to an increase in interest rates (i.e. positive portfolio duration), and in this case the SF linear correlation coefficient between equity and interest rate risk (i.e. parameter A of Table 5.32c) is null. In this context, the additional value of using a Student's t copula (the main non-linear benchmark for market risk) is rather limited. Moreover, if we used both a Gaussian and Student's t copula, we would multiply the number of results and increase the complexity of our analysis. As a result, we prefer to keep it simple, considering Gaussian copula only for market risk aggregation. Finally, note that the correlation matrix of our economic scenario generator (see mainly Table

Table 5.28. Incremental paid amounts (amounts in millions, appreciated until current calendar year) of MTPL

		Development year											
		−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0
Accident year	−11	21.32	22.79	7.79	3.53	1.21	1.14	0.70	0.75	0.93	0.69	0.16	0.05
	−10	23.78	19.88	6.93	2.59	1.87	1.31	1.69	2.84	0.14	0.11	0.05	0.05
	−9	19.05	15.24	8.39	3.77	2.16	2.47	1.38	0.23	0.60	0.14	0.09	0.04
	−8	14.76	20.12	9.11	4.03	2.84	1.39	1.41	1.27	0.07	0.29	0.10	0.05
	−7	23.51	17.08	7.81	3.21	2.04	1.58	0.86	0.21	0.43	0.30	0.10	0.05
	−6	18.89	19.30	7.26	2.44	1.69	1.53	0.65	0.98	0.40	0.28	0.09	0.04
	−5	19.19	21.46	6.66	2.09	2.51	1.54	1.11	1.04	0.43	0.30	0.10	0.05
	−4	24.52	19.79	7.21	3.66	0.97	1.68	1.20	1.12	0.46	0.32	0.11	0.05
	−3	23.35	16.64	5.47	2.88	1.82	1.50	1.07	1.00	0.41	0.29	0.09	0.05
	−2	23.35	18.98	4.62	3.10	1.88	1.55	1.11	1.04	0.43	0.30	0.10	0.05
	−1	23.12	16.24	6.96	3.06	1.86	1.53	1.09	1.02	0.42	0.30	0.10	0.05
	0	17.57	15.53	5.85	2.57	1.56	1.29	0.92	0.86	0.35	0.25	0.08	0.04

Age-to-age	1.884	1.177	1.066	1.038	1.030	1.021	1.019	1.008	1.005	1.002	1.001

Table 5.29. Incremental paid amounts (amounts in millions, appreciated until current calendar year) of MOD

Accident year	Development year 0	1	2	3	4	5	6	7
−7	14.32	4.00	0.19	0.08	0.04	0.04	0.02	0.02
−6	15.69	3.92	0.18	0.06	0.06	0.04	0.03	0.03
−5	14.10	3.58	0.15	0.05	0.03	0.03	0.03	0.02
−4	14.47	3.54	0.20	0.07	0.01	0.03	0.03	0.02
−3	14.31	4.20	0.16	0.05	0.04	0.04	0.03	0.02
−2	13.23	3.44	0.14	0.05	0.03	0.03	0.02	0.02
−1	12.43	3.98	0.15	0.05	0.03	0.03	0.02	0.02
0	13.30	3.60	0.16	0.06	0.03	0.03	0.02	0.02

Age-to-age	1.271	1.009	1.003	1.002	1.002	1.001	1.001

Table 5.30. Incremental paid amounts (amounts in millions, appreciated until current calendar year) of GTPL

		Development year											
		−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0
Accident year	−11	4.47	10.63	6.69	4.12	1.84	2.11	1.37	1.45	2.46	2.11	0.52	0.16
	−10	5.09	9.20	5.70	2.70	3.26	2.30	2.97	6.12	0.29	0.23	0.12	0.16
	−9	4.72	7.34	6.47	4.67	3.10	4.34	2.50	0.40	1.48	0.31	0.30	0.15
	−8	4.01	10.26	6.16	4.02	3.89	2.06	2.85	2.73	0.11	0.89	0.32	0.16
	−7	7.31	8.96	5.29	3.22	3.24	3.34	1.77	0.42	1.04	0.85	0.30	0.15
	−6	6.15	9.70	5.15	2.97	2.88	3.15	1.33	2.15	1.04	0.85	0.30	0.15
	−5	5.38	9.91	4.42	2.19	4.55	2.53	2.05	2.13	1.03	0.84	0.30	0.15
	−4	7.79	9.90	5.97	4.71	1.40	3.11	2.33	2.42	1.17	0.95	0.34	0.17
	−3	7.24	8.59	4.52	2.86	2.87	2.73	2.04	2.12	1.03	0.84	0.30	0.15
	−2	7.48	9.67	2.93	3.38	2.90	2.76	2.06	2.14	1.04	0.84	0.30	0.15
	−1	6.46	6.95	4.65	3.04	2.61	2.48	1.86	1.92	0.93	0.76	0.27	0.13
	0	4.30	6.58	3.77	2.46	2.12	2.01	1.51	1.56	0.76	0.62	0.22	0.11

Age-to-age	2.530	1.347	1.168	1.124	1.105	1.071	1.069	1.031	1.025	1.009	1.004

Table 5.31. Calibrated parameters of the reserve model

(a) Expected claims inflation index

Year	MTPL	MOD	GTPL
1	1.01912	1.04802	1.02885
2	1.01546	1.07386	1.03493
3	1.01880	1.10794	1.04824
4	1.03206	1.15418	1.07202
5	1.04780	1.20501	1.09876
6	1.06191	1.25586	1.12418
7	1.07458	1.30688	1.14846
8	1.08455	1.35640	1.17017
9	1.09151	1.40381	1.18891
10	1.09788	1.45205	1.20727
11	1.10228	1.49920	1.22368
12	1.10503	1.54555	1.23843
13	1.10848	1.59434	1.25416
14	1.11117	1.64352	1.26920

(b) Other parameters

Parameter	MTPL	MOD	GTPL
ϑ	0.98959	1.01765	0.99903
ϕ	333,849	10,483	529,090

5.14) is already in line with the SF, and we do not need to change it.

Using equation 1.11, we can calculate the Kendall's rank correlation coefficient of the Gaussian copula corresponding to any linear correlation coefficient of Table 5.32. Therefore, using equation 1.12, we calculate the parameter of our Gumbel copulas corresponding to the Kendall's rank correlation coefficients. We make this calculation in order to have the same rank correlation structure in case of SF and linear or non-linear copula aggregation. Moreover, in order to obtain the parameter of the Gumbel copula used to join GTPL to the aggregation of MTPL and MOD, we consider the implicit linear correlation coefficient between GTPL and the sum of MTPL and MOD. Also, note that the Gumbel copula is used on variables with the same sign convention (i.e. either all profits or all losses), otherwise a rotated Gumbel copula can be applied (e.g. for the total claim amount and annual rate of return). We do not specify which version is used in each case.

The linear and non-linear copula hierarchical structures and their parameters are finally shown in Table 5.33 and 5.34. We again point out that the first one is based on Gaussian copulas only.

We finally remind that there are two random variables depending on reserve risk, i.e. claims development result and claims reserve. When we aggregate a generic

Table 5.32. Linear correlation matrices

(a) Premium or reserve risk

	MTPL	MOD	GTPL
MTPL	1	0.50	0.50
MOD	·	1	0.25
GTPL	·	·	1

(b) Premium and reserve risk

	Premium	Reserve
Premium	1	0.50
Reserve	·	1

(c) Market risk

	Interest rate	Equity	Property
Interest rate	1	A	A
Equity	·	1	0.75
Property	·	·	1

(d) Market and non-life risk

	Market	Non-life
Market	1	0.25
Non-life	·	1

variable (e.g. total claim amount, which is the source of premium risk) to reserve risk variables, we simulate from a linear or non-linear copula, we apply the simulations to the claims development result, and we look for the corresponding claims reserve (i.e. the claims reserve realisation that produced that particular simulated aggregate claims development result). This is an important point, because the two random variables affected by reserve risk strongly depend one on each other, and then we need to preserve their original relationship.

Table 5.33. Linear copula hierarchical structure

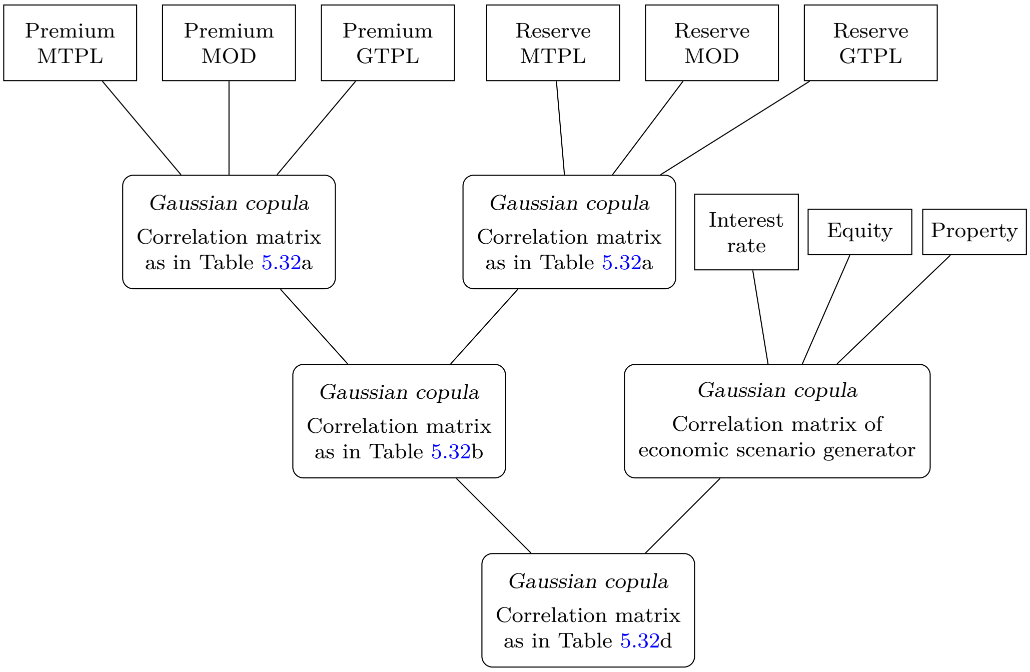
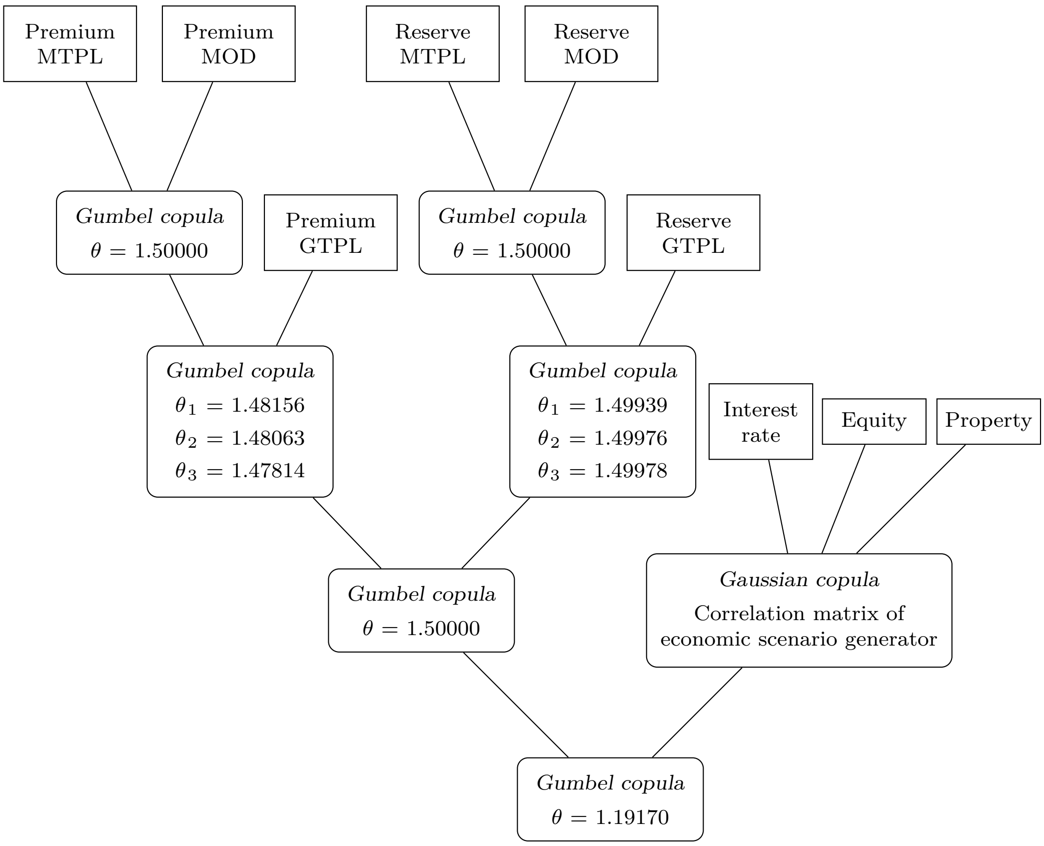


Table 5.34. Non-linear copula hierarchical structure



Chapter 6

Numerical results

In this chapter, we finally calculate the capital requirements for market and non-life risk of our insurance company, according to the partial IM described so far and the SF. We point out that the capital requirements of our IM are calculated according to the Risk-Based Capital (RBC), which is a risk measure that takes into account the expected return produced by the investment of the resources (for this reason, SCR denotes the capital requirement under the SF only). A similar case study has been proposed for example by Daykin et al. (1987), Ballotta and Savelli (2006), and Cotticelli and Savelli (2023). We utilise the following assumptions:

- the capital requirements according to our IM are calculated over a period of one, two, and three years, using a Monte Carlo simulation approach based on 100,000 simulation paths;
- the insurance contracts are not multi-annual, their exposure is referred to the full calendar year, and there is no geographical diversification for SF purposes;
- the discounting effect and expected claims inflation index are considered in the reserve calculation, while the risk margin is ignored to avoid the circularity issue in the capital requirements calculation (we remind that our goal is not the own funds calculation, but the capital requirements modelling);
- for new business, the claims development result is null, and the interest rate unwinding is completely absorbed by the incremental amount for paid claims;
- the equity risk, property risk, and interest rate risk are the only sources of market risk, while the premium risk and reserve risk are the only sources of non-life risk;
- the interest rate risk affects both assets and liabilities, while the equity risk and property risk only affect the investments;
- the security trading is continuous, all securities are perfectly divisible, and there are no transaction costs, taxes, short-sale restrictions, or riskless arbitrage opportunities;

- the bond investments are zero-coupon bonds, and their remuneration is the average of all EEA government nominal rates on December 31, 2021, presented in Table 5.7;
- the bond investments are risk-free (i.e. no spread risk, liquidity risk, or default risk), and they thus evolve according to the dynamic of the European interest rate swaps only;
- the regulatory nominal curve (i.e. the discounting interest rates) is the Eiopa interest rate term structure without volatility adjustment on December 31, 2021, presented in Table 5.7;
- the regulatory nominal curve evolves according to the dynamic of the European interest rate swaps;
- the stock (or equity) investments are listed in regulated markets of the EEA, hence they can be considered as type 1 equities under the SF;
- the stock investments are non-dividend-paying stocks without strategic nature, because of the absence of a clear decisive strategy to continue holding them for long period;
- the symmetric adjustment required by the SF is not present;
- the asset allocation is recalibrated at year end only;
- the specific assumptions for the models considered in this numerical analysis are described in the previous chapters.

As already mentioned, spread risk, liquidity risk, and default risk are not considered, and this is consistent with the SF, according to which, government bonds are assumed not to be affected by these sources of risk. Moreover, we point out that taxes and dividends usually have a rescaling impact on the stochastic result of the insurance company, but we do not consider them for comparability with the SF.

As already anticipated, the RBC is a risk measure that takes into account the expected return produced by the investment of the resources. The RBC over the time horizon $(0, t)$ within the confidence level $1 - \varepsilon$ is given by:

$$RBC(0,t) = U_0 - \frac{U_\varepsilon(t)}{\prod_{k=1}^{t}(1 + E(J_k))} \tag{6.1}$$

where $U_\varepsilon(t)$ is the ε-th order quantile of the risk reserve.

Clearly, the RBC above decreases (or increases) for expected profits (or losses) above the initial risk reserve. In this thesis, we use this approach, even though the EU insurance supervisory authorities are reluctant to include expected profits or losses in the capital requirement calculation according to an IM, because they want to target consistency with the SF framework, which does not consider this possibility.

Figure 6.1 shows the quantiles of the simulated stock price, property price, and three-year zero-coupon bond price (as representative of the bond asset class) over time. We can clearly see that the volatility of the stock or property increases as the time horizon raises. On the other hand, the increasing risk of the zero-coupon bond is offset by the convergence to its nominal value.

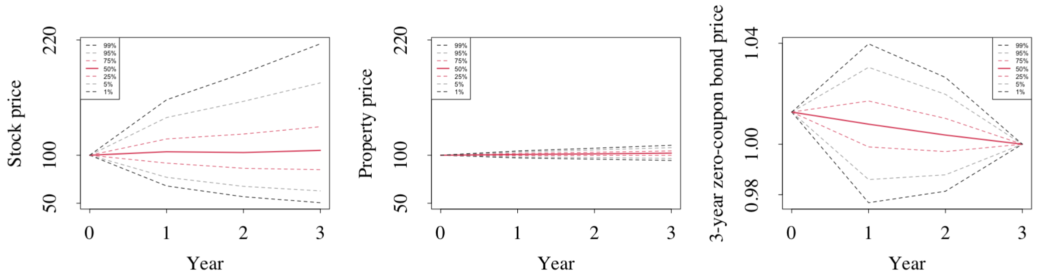


Figure 6.1. Quantiles of the simulated stock, property and three-year zero-coupon bond price over the years

We now simulate the stock price, property price, and zero-coupon bond annual and average rate of return. In addition, we simulate the total claim amount, claims reserve (both in-force reserve and new-business one), and claims development result of MTPL, MOD, and GTPL. As a consequence, Figure 6.2, 6.3, 6.4, and 6.5 show the simulated marginal distributions (except for the average rate of return) after one, two, and three years. Here, claims reserve and claims development result are affected by both market and non-life risk. Table 6.1, 6.2, 6.3, and 6.4 show some descriptive statistics.

We point out that the mean of the annual rates of return mainly depends on the real-world assumptions used to calibrate the local risk premium functions. In this regard, the two key elements are the level and slope of expected real-world returns. The former determines the magnitude of the mean, the latter determines its direction. Note that the mean of the stock annual rate of return (between ca. 0.8% and 5.3%) is always the biggest, as the real-world expected return is quite high. Moreover, it decreases between the first and second year (ca. −4.5% additive), and it increases between the second and third one (ca. +3% additive), as the real-world expected return has this pattern. The mean of the property annual rate of return has a medium value (ca. 0.7%) and remains quite stable over time. Also, the mean of the zero-coupon bond annual rate of return is the smallest (between ca. 0.1% and 0.8%) and increases over time, as the expected real-world nominal forward rates increase as well. Furthermore, the standard deviation and skewness (i.e. the main indicators of the distribution riskiness) are the biggest for the stock (ca. 19% and +0.5, respectively), they have medium values for the zero-coupon bond (ca. 2.5% and +0.1, respectively), and are the smallest for the property (ca. 1.5% and +0.05, respectively). This depends on the volatility calibration of the risk-neutral economic scenario generator, which is the same for the real-world one.

The mean and standard deviation of the total claim amount of each LoB rise over time, because of the dynamic portfolio assumption, i.e. increasing size for real growth rate and claims inflation index. However, this effect is not extreme, because of a limited increase in the GPW (the real growth rate is 2% for all the LoBs, and the expected claims inflation index is between ca. 2% and 4%). It is also noted that the distribution of MOD is heavily concentrated around the mean compared to

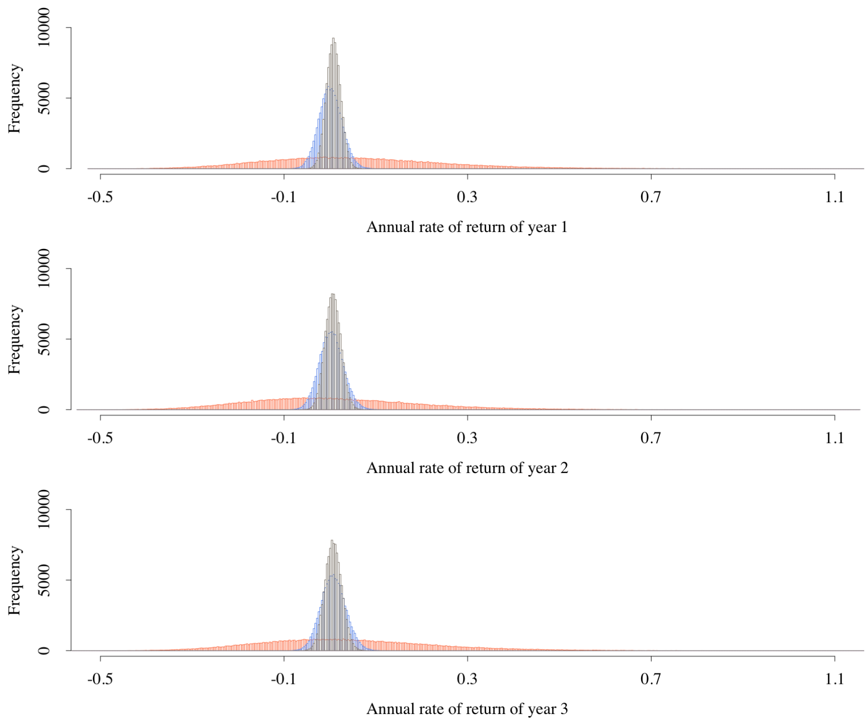


Figure 6.2. Simulated annual rates of return after one, two, and three years (stock in red, property in black, zero-coupon bond in blue, and x-axis values in %)

Table 6.1. Descriptive statistics of the simulated annual rates of return after one, two, and three years (amounts in %)

(a) Stock

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	5.27	19.13	0.5592	−52.24	−8.21	3.63	16.94	131.97
2	0.78	18.28	0.5524	−54.61	−12.19	−0.85	11.98	114.50
3	3.85	18.80	0.5491	−55.71	−9.44	2.22	15.37	133.03

(b) Property

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	0.77	1.56	0.0390	−5.51	−0.28	0.77	1.81	7.25
2	0.67	1.72	0.0542	−6.75	−0.50	0.65	1.82	9.14
3	0.74	1.83	0.0621	−7.37	−0.51	0.72	1.97	8.86

(c) Zero coupon-bond

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	0.13	2.47	0.1099	−10.60	−1.55	0.08	1.77	13.24
2	0.32	2.58	0.1273	−10.10	−1.44	0.28	2.02	11.46
3	0.76	2.67	0.1120	−10.05	−1.07	0.71	2.52	12.70

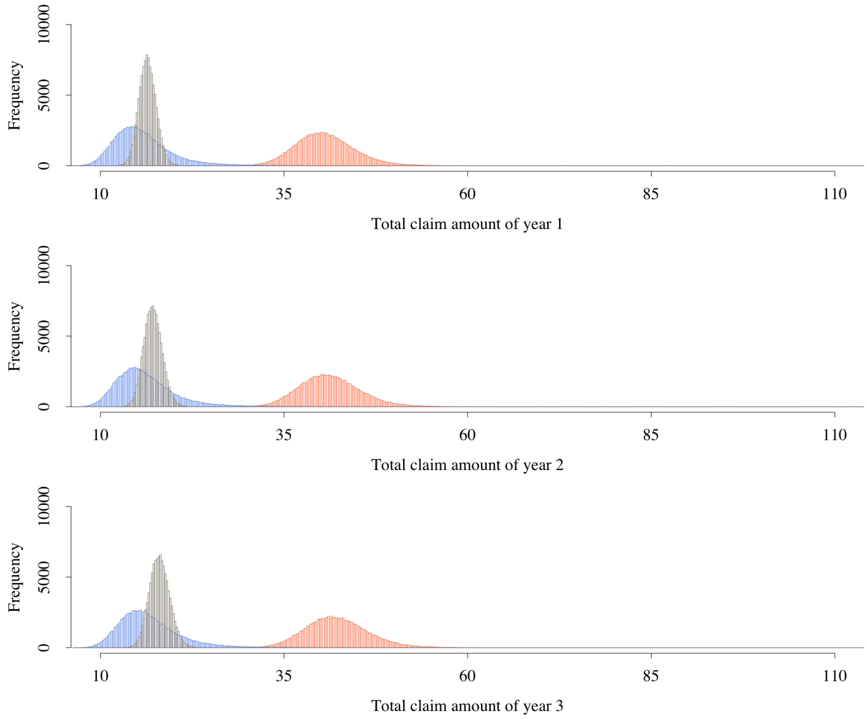


Figure 6.3. Simulated total claim amounts after one, two, and three years (MTPL in red, MOD in black, GTPL in blue, and x-axis values in millions)

Table 6.2. Descriptive statistics of the simulated total claim amounts after one, two, and three years (amounts in millions)

(a) MTPL

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	40.60	4.05	0.6998	26.74	37.84	40.34	43.04	104.68
2	41.28	4.24	2.1853	27.38	38.45	41.02	43.80	241.22
3	42.23	4.31	0.6514	26.81	39.29	41.96	44.84	108.67

(b) MOD

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	16.41	1.16	0.1358	11.44	15.61	16.38	17.17	21.82
2	17.14	1.25	0.1694	12.43	16.29	17.11	17.96	23.86
3	18.04	1.36	0.1671	12.60	17.10	18.01	18.94	24.69

(c) GTPL

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	15.87	5.06	10.4794	6.04	13.03	15.10	17.63	420.43
2	16.27	5.13	8.4902	5.99	13.38	15.47	18.08	335.16
3	16.80	5.13	6.3388	6.34	13.81	15.99	18.68	255.54

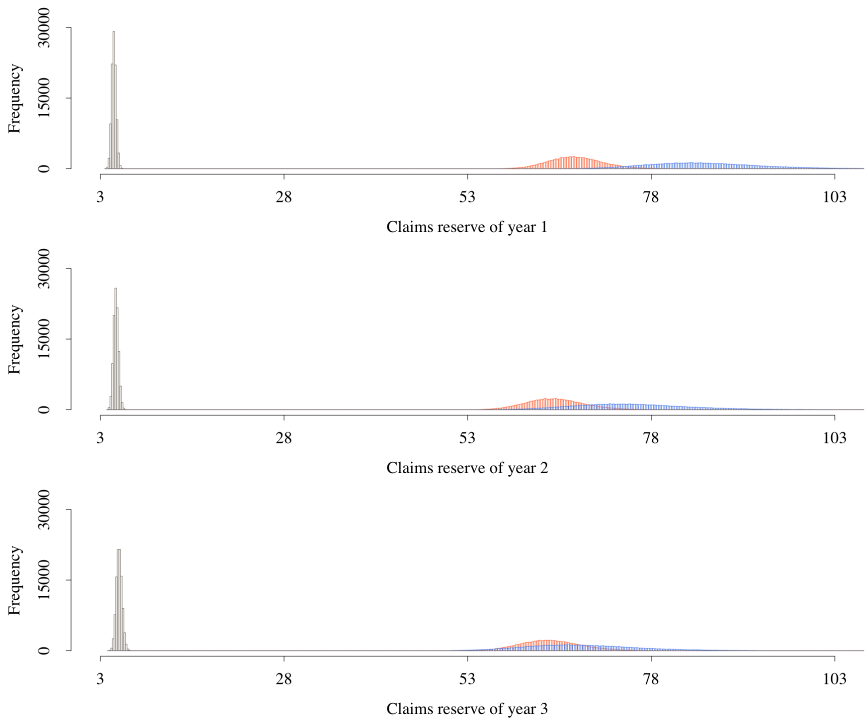


Figure 6.4. Simulated claims reserves after one, two, and three years (MTPL in red, MOD in black, GTPL in blue, and x-axis values in millions)

Table 6.3. Descriptive statistics of the simulated claims reserves after one, two, and three years (amounts in millions)

(a) MTPL

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	67.76	3.69	0.3735	51.19	65.23	67.58	70.08	107.58
2	64.91	3.99	0.7964	49.20	62.19	64.74	67.40	190.36
3	64.31	4.17	0.3831	49.34	61.45	64.12	66.96	126.12

(b) MOD

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	4.83	0.30	0.1473	3.63	4.63	4.83	5.03	6.26
2	5.12	0.34	0.1735	3.89	4.88	5.11	5.34	6.83
3	5.55	0.39	0.1859	4.00	5.28	5.53	5.80	7.50

(c) GTPL

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	85.37	8.17	1.9175	58.09	79.89	84.61	90.00	439.14
2	75.50	8.52	2.0034	47.45	69.84	74.75	80.19	347.05
3	68.77	8.38	1.8453	42.46	63.20	67.96	73.35	274.76

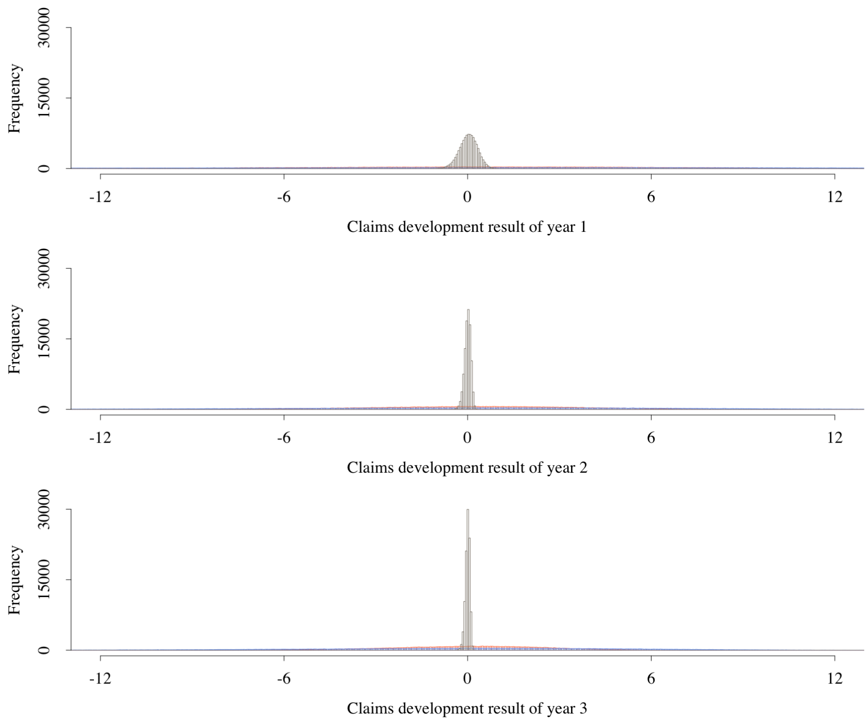


Figure 6.5. Simulated claims development results after one, two, and three years (MTPL in red, MOD in black, GTPL in blue, and x-axis values in millions)

Table 6.4. Descriptive statistics of the simulated claims development results after one, two, and three years (amounts in millions)

(a) MTPL

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	0.20	6.47	−0.2686	−39.08	−3.94	0.45	4.67	23.77
2	−0.01	3.66	−0.3182	−18.18	−2.35	0.18	2.53	15.00
3	−0.15	2.66	−0.3801	−16.51	−1.82	0.03	1.72	10.95

(b) MOD

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	0.03	0.29	−0.1942	−1.53	−0.16	0.04	0.23	1.17
2	0.00	0.10	−0.4774	−0.75	−0.06	0.01	0.07	0.36
3	0.00	0.07	−0.5691	−0.54	−0.05	0.00	0.05	0.27

(c) GTPL

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	−0.03	11.74	−0.5172	−79.58	−7.19	0.86	8.16	43.84
2	−0.16	7.09	−0.3831	−44.15	−4.61	0.30	4.79	27.41
3	−0.35	5.41	−0.4390	−34.82	−3.70	0.03	3.42	20.27

GTPL, notwithstanding that they have a similar volume. This is mainly due to the coefficient of variation of the single claim amount (2 and 12, respectively). Besides, it can be seen from the values in the table that the skewness trend is sometimes affected by some outliers. It is evident at the second year for MTPL (in order to better understand, it is useful to see the maximum value in the last column of the table). In any case, the skewness realised for the three LoBs is quite different. It is very significant for GTPL (between ca. +6 and +10), it has a medium value for MTPL (ca. +0.7), and it is less significant for MOD (ca. +0.1). The skewness is mainly influenced by the single claim amount distribution, due to the quite small portfolio size, in particular for GTPL. In case of bigger insurance companies, all these skewness values would be closer to the skewness of the structure variable (even though they are also influenced by the claims inflation index distribution). In any case, these skewness values will be helpful to properly understand the effective multipliers of our IM, in comparison to the SF.

It is noted that the mean of the claims reserve tends to remain quite stable over time. This is because the in-force claims reserve run-off is offset by the new-business reserve (only partially for GTPL). The new-business reserve indeed depends on the premium volume (see equation 4.8), and we assumed that MTPL earns a half of premiums, while MOD and GTPL earn a quarter of them. Note that the premium volume of GTPL is lower compared to the current reserving position, resulting in a reduction of its business. This is what usually happens when the insurance company wants to reduce its exposure to a particular business. Going into detail, for MTPL the new-business reserve spans from 35% of the total claims reserve on the first year to 70% on the third year. For MOD, it is already 80% on the first year and 95% on the third one, because MOD is a business with a high settlement speed. It results that the majority of its claims reserve depends on new contracts underwritten by the insurance company. Moreover, for GTPL the new business represents only 15% of the total claims reserve on the first year and 45% on the third year. This is because of the premium volume size explained above, and because GTPL is a business with quite low settlement speed, resulting in the fact that the majority of its claims reserve depends on the in-force contracts. Furthermore, the mean and standard deviation of MOD are significantly lower than in case of MTPL and GTPL (in this regard, it is useful to see the figure). Finally, we observe that the skewness is the highest for GTPL (ca. +2), it has a medium value for MTPL (between ca. +0.4 and +0.8), and it is less significant for MOD (ca. +0.2).

We point out that the mean of the claims development result of each LoB almost completely depends on the interest rate unwinding effect. It means that the initial claims reserve (i.e. claims reserve at the beginning of the year) inside the claims development result has incremental amounts for paid claims that are discounted one year more than at the end of the year and with different interest rates. Because of the negative market interest rate term structure on the first terms, the mean of the claims development result of the first year is always positive (i.e. expected profit) for the low-medium settlement speed LoBs (i.e. MTPL and MOD). However, between the first and second year, and between the second and third year, there is a drop in interest rates, and consequently an expected loss is produced. Both the mean and standard deviation for all the LoBs decrease over time, because of the claims reserve run-off. We remind that the claims development result for the new business is assumed to be null, hence we are commenting the claims development result of the in-force business only. Finally, the standard deviation is the biggest for GTPL (ca. between 5 mln and 12 mln), it has medium value for MTPL (ca. between 3 mln and 6 mln), and is the smallest for MOD (ca. between 0.1 mln and 0.3 mln). For the first year, this corresponds to a ratio of standard deviation to initial claims reserve around 8.6% for MTPL, 5.8% for MOD, and 11.8% for GTPL. This ratio is an important indicator for the reserve risk, and it allows to make a comparison with the SF. We remind that we are now considering the interest rate unwinding effect, that will be ignored if we want to isolate the reserve risk only.

We now aggregate the simulated marginal distributions, following the linear and non-linear copula hierarchical structure described in Table 5.33 and 5.34. As a consequence, Table 6.5 and 6.6 show some descriptive statistics of the final stage of aggregate distributions (except for the average rate of return) after one, two, and three years. Here, claims reserve and claims development result are affected by both market and non-life risk.

Table 6.5. Descriptive statistics of the simulated aggregate distributions after one, two, and three years, with linear copula hierarchical structure (amounts in % for annual rate of return and in millions for other)

(a) Annual rate of return

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	2.20	7.83	0.5458	−20.85	−3.31	1.57	6.94	53.60
2	0.52	7.48	0.5204	−20.79	−4.74	−0.09	5.13	46.77
3	2.02	7.74	0.5256	−23.08	−3.41	1.35	6.78	52.71

(b) Total claim amount

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	72.81	8.13	1.8810	48.92	67.53	71.96	76.99	221.29
2	74.73	8.38	1.6941	51.06	69.30	73.86	79.05	220.23
3	77.03	8.71	2.3639	52.62	71.36	76.16	81.54	269.91

(c) Claims reserve

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	157.87	7.69	0.4622	127.79	152.60	157.37	162.56	223.82
2	145.46	5.94	0.7881	123.73	141.42	145.04	149.01	238.70
3	138.67	5.47	0.6699	115.33	135.02	138.31	141.90	208.46

(d) Claims development result

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	0.36	13.91	−0.3549	−87.76	−8.40	1.13	9.89	50.64
2	−0.16	8.27	−0.3242	−42.57	−5.35	0.30	5.54	31.07
3	−0.48	6.18	−0.3297	−34.46	−4.39	−0.11	3.79	23.66

Table 6.6. Descriptive statistics of the simulated aggregate distributions after one, two, and three years, with non-linear copula hierarchical structure (amounts in % for annual rate of return and in millions for other)

(a) Annual rate of return

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	2.16	7.76	0.5303	−20.79	−3.29	1.51	6.92	47.93
2	0.58	7.53	0.5155	−21.24	−4.75	−0.07	5.25	47.21
3	1.96	7.71	0.5200	−23.13	−3.45	1.30	6.68	47.91

(b) Total claim amount

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	72.90	8.44	2.8504	51.00	67.60	71.76	76.76	278.79
2	74.69	8.60	2.3644	51.61	69.22	73.57	78.64	262.34
3	77.04	8.79	2.1464	54.60	71.44	75.87	81.17	299.10

(c) Claims reserve

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	157.96	7.83	0.9357	130.59	152.66	157.17	162.32	312.77
2	145.49	5.90	0.8863	120.78	141.54	145.03	148.88	230.10
3	138.56	5.33	0.7685	113.74	135.01	138.17	141.61	200.04

(d) Claims development result

Year	Mean	St.Dev.	Skew.	Min.	1st Qu.	Median	3rd Qu.	Max.
1	0.21	14.19	−0.7316	−99.52	−7.85	1.59	9.90	51.03
2	−0.11	8.37	−0.6329	−56.56	−4.98	0.67	5.63	36.46
3	−0.49	6.30	−0.6677	−39.39	−4.11	0.12	3.81	20.24

The aggregate annual rate of return has a mean strongly influenced by the stock behaviour, representing around a half of the asset allocation of our insurance company. It is also possible to observe the drop in the mean of the second year, as already seen in the stock case. The aggregate standard deviation is now lower that the weighted average of the marginals, and there is a diversification benefit around 15% according to the classical portfolio theory (where the standard deviation is the main target, together with the expected value). The aggregate total claim amount, claims reserve, and claims development result have mean equal to the sum for each LoB, and we can make the same comments around the pattern as for the marginal distributions. Note that for the first year we have an expected total claim amount of 72.8 mln, corresponding to a combined ratio of roughly 96% and in line with the parameters. Once again, we can observe a reduction in the aggregate standard deviation with respect to the sum of the marginals, because a diversification benefit exists. This will be effectively quantified in the context of capital requirements calculation. Furthermore, the ratio of first-year standard deviation of the claims development result to initial total claims reserve is around 7.7% for the insurance company. Finally, except for the annual rate of return (which has a dependence structure based on the Gaussian copula only), it is noted that the skewness of aggregate variables with a non-linear dependence structure is higher (in absolute values) than in the case of linear dependence. It means that the tail is longer in the non-linear dependence context, and the risk is higher. If we observed the kurtosis of aggregate distributions, we would make similar comments, but focusing on the extremes. Once again, we will effectively quantify these results in the context of capital requirements calculation.

6.1 Market risk

We now isolate the effect of the market risk, and so we leave aside the non-life risk. Using equation 1.5 and 1.7, the risk reserve is found to be:

$$U_t = (1 + J_t)\, U_{t-1} + (1 + \bar{J}_t)\, ((1 + \lambda)\, \pi_t - X_t + Y_t) + (J_t - \bar{J}_t)\, L_{t-1} + \bar{J}_t\, L_t$$

We then drop the underwriting result by the risk reserve, but we still consider the interest on its investment. Furthermore, we assume that the total claim amount is deterministic and equal to its mean, hence the underwriting result is equal to the safety loadings. We also assume that the reserving cash flows are deterministic, in order to isolate the effect of the interest rate unwinding behind the claims reserve and claims development result. Be aware that the discounting is considered in the reserve calculation, and it is stochastic. The claims development result is on average null only if the discounting is not accounted for. It is finally worth mentioning that the claims inflation affects the paid claims and is here deterministic.

We overall believe that this approach is not a simplification and is a reasonable solution to make a stand-alone analysis of the market risk with respect to non-life risk. As a consequence of our assumptions, the annual net cash flows are based on some estimated amounts (i.e. not stochastic). Starting from equation 1.8, they are found to be:

$$\hat{F}_t = \lambda\, \pi_t + \hat{L}_t - \hat{L}_{t-1} + \hat{Y}_t$$

and the risk reserve is therefore found to be:

$$U_t = (1 + J_t)\, U_{t-1} + J_t \sum_{k=1}^{t-1} \lambda\, \pi_k + \hat{Y}_t + J_t\, \hat{L}_{t-1} + \bar{J}_t\, \hat{F}_t$$

where the claims reserve of the in-force business, claims development result and claims reserve of the new business have deterministic reserving cash flows, and they are estimated based on equation 4.6, 4.7 and 4.8. We remind that the total claims reserve is equal to the sum of the in-force and new business ones (see equation 4.5).

Using the expected return given by the real-world economic scenario generator, we calculate the capital requirements over a period of one, two, and three years. In doing so, we compute the result for each source of risk (i.e. sub-module, in this case), and we determine the diversification benefit with the usual function used in the insurance industry (see Bürgi et al., 2008, for more details):

$$\mathit{Diversification} = 100\% - \frac{RBC_{\sum_i Y_i}}{\sum_i RBC_{Y_i}}$$

The results according to our IM are then presented in Table 6.7, together with the comparison with the SF (obviously over a one-year period only).

Table 6.7. Capital requirements for market risk over the initial GPW (expressed in %) and diversification benefits over a period of one, two, and three years

(a) Standard formula

Year	Equity	Property	Interest rate	Market	Diversification
1	31.98	2.56	0.45	33.95	3.00%

(b) Internal model with Gaussian copula

Year	Equity	Property	Interest rate	Market	Diversification
1	26.71	0.84	1.99	26.76	9.40%
2	39.14	0.88	2.20	37.24	11.79%
3	44.22	1.72	4.35	42.62	15.26%

We want to be consistent with equation 6.1, hence the RBC for equity, property, or interest rate risk is calculated according to the following formula:

$$RBC(0,t)_{asset\ class} = V_0 - \frac{V_\varepsilon(t) - D_t}{\prod_{k=1}^{t}(1 + E(J_k))}$$

where V_ε is the ε-th order quantiles of the variable to isolate the risk of the specific asset class, and D_t is the variable of expected profits not related to the performance of the specific asset class. For equity and property risk, the first one is given by:

$$V_t = (1 + G_t)\, V_{t-1} + G_t \sum_{k=1}^{t-1} \lambda\, \pi_k + \hat{Y}_t^M + G_t\, \hat{L}_{t-1}^M + \bar{G}_t\, \hat{F}_t^M \quad \text{with} \quad V_0 = U_0$$

and for interest rate risk, it is given by:

$$V_t = (1 + G_t)\, V_{t-1} + G_t \sum_{k=1}^{t-1} \lambda\, \pi_k + \hat{Y}_t + G_t\, \hat{L}_{t-1} + \bar{G}_t\, \hat{F}_t \quad \text{with} \quad V_0 = U_0$$

where the superscript M now refers to the fact that the regulatory nominal term structure is deterministic and based on the average values of the real-world economic scenario generator. Also, G_t and $\bar{G}_t$ are the modified variables of annual and average rate of return. Moreover, we have:

$$D_t = E(U_t - U_0)\, \frac{E(J_t) - x_a}{E(J_t)} \quad \text{and} \quad G_t = x_b + x_c + x_d \quad \text{and} \quad \bar{G}_t = x_e + x_f + x_g$$

where x_a, x_b, x_c, x_d, x_e, x_f, and x_g are the parameters shown in Table 6.8, depending on the specific asset class considered.

Note that $J_{t,S}$, $J_{t,H}$, and J_{t,P_n} are the annual rates of return of the stock, property, and zero-coupon bond portfolio, while $\bar{J}_{t,S}$, $\bar{J}_{t,H}$, $\bar{J}_{t,P_n}$ are the corresponding average rates of return. In this regard, we have:

$$J_{t,S} = \frac{S(t)}{S(t-1)} - 1 \quad \text{and} \quad J_{t,S}^{(2)} = \frac{S(t)}{S(t - 1/2)} - 1 \quad \text{and} \quad \bar{J}_{t,S} = \frac{J_{t,S} + J_{t,S}^{(2)}}{3}$$

Table 6.8. Parameters for the RBC for equity, property, or interest rate risk

Parameter	Equity	Property	Interest rate
x_a	$\alpha\, E(J_{t,S})$	$\beta\, E(J_{t,H})$	$(1-\alpha-\beta)\, E(J_{t,P_n})$
x_b	$\alpha\, J_{t,S}$	$\alpha\, E(J_{t,S})$	$\alpha\, E(J_{t,S})$
x_c	$\beta\, E(J_{t,H})$	$\beta\, J_{t,H}$	$\beta\, E(J_{t,H})$
x_d	$(1-\alpha-\beta)\, E(J_{t,P_n})$	$(1-\alpha-\beta)\, E(J_{t,P_n})$	$(1-\alpha-\beta)\, J_{t,P_n}$
x_e	$\alpha\, \bar{J}_{t,S}$	$\alpha\, E(\bar{J}_{t,S})$	$\alpha\, E(\bar{J}_{t,S})$
x_f	$\beta\, E(\bar{J}_{t,H})$	$\beta\, \bar{J}_{t,H}$	$\beta\, E(\bar{J}_{t,H})$
x_g	$(1-\alpha-\beta)\, E(\bar{J}_{t,P_n})$	$(1-\alpha-\beta)\, E(\bar{J}_{t,P_n})$	$(1-\alpha-\beta)\, \bar{J}_{t,P_n}$

The property annual and average rates of return are analogous to the stock case and can be found by replacing the character S with H. Also, we have:

$$J_{t,P_n} = \sum_{w\in\{1,2,3,5,10\}} \gamma_w \,\frac{P_n(t, t-1+w)}{P_n(t-1, t-1+w)} - 1 \quad \text{and}$$

$$J^{(2)}_{t,P_n} = \sum_{w\in\{1,2,3,5,10\}} \gamma_w \,\frac{P_n(t, t-1+w)}{P_n(t-1/2, t-1+w)} - 1 \quad \text{and} \quad \bar{J}_{t,P_n} = \frac{J_{t,P_n} + J^{(2)}_{t,P_n}}{3}$$

We have some key assumptions to properly understand the pattern of the capital requirements in our IM. On one side, we have the riskiness of the investment. On the other side, we have several forms of expected profit, i.e. the real-world return on the investments, the gain produced by the investment of resources (i.e. claims reserve, annual cash flows and safety loadings) or the profit arising from the claims development result. In contrast to the expected profit, we have the real-world discounting effect. The higher the risk or real-world discounting effect, the bigger the RBC. The higher the expected profit, the smaller the RBC. Moreover, the dynamic portfolio assumption (i.e. the presence of a claims inflation index and a real growth rate) has a scaling effect on the RBC, because it raises the dimension of the insurance company. We point out that not only do the capital requirements depend on the riskiness of the investment but also on the amount invested. The RBC for interest rate risk is overall not high. It is worth mentioning that it depends both on assets (i.e. bond portfolio) and liabilities. Focusing on the asset side, we point out that the bond portfolio is not high-risky compared to other asset classes (e.g. stocks). Usually, the risk increases as the bond time to maturity raises. It is useful to mention that for the first year, the one-year zero-coupon bond return is not stochastic, because the initial zero-coupon bond price is given, and after one year it is equal to one for sure (we remind that there is not default risk). Despite this, there is an expected loss, due to the investment of the resources at a negative EEA interest rate (-0.626%). Moreover, we point out that the asset-liability portfolio is well interest-rate immunised, because its duration is quite low (effective duration of the net cash flows on initial asset value equal to 0.26), and consequently the RBC for interest rate risk is limited. The RBC for equity risk is definitely bigger than for other asset classes. This is because stocks are high-risky investments (ca. 20% of implied volatility in the economic scenario generator), and the weight in the asset allocation is considerable (i.e. around a half of the insurance financial resources). On the contrary, the RBC for property risk is the smallest, because the risk is low (ca. 1.5% of implied volatility in the economic scenario generator) as well as the weight in the asset allocation (5%). We point out that not only are stocks high-risk but also high-return assets. It means that they have the biggest share of expected profit to mitigate the capital requirements. However, the drop of equity return on the second year implies that the resources are mostly used to mitigate the interest rate and property risk. This is why the increase in interest rate and property risk between the first and second year is smaller than between the second and third year. Finally, note that the RBC of each asset class and the overall one increase over time, because the risk increases more than the expected profits, and because the insurance company size raises.

We remind that the stock investment can be considered as type 1 equity not having a strategic nature and a long-term holding strategy, and that the symmetric adjustment is null. Consequently, in the SF framework, the SCR for equity risk and the SCR for property risk are found to be:

$$SCR_{equity} = 39\%\, \alpha\, A_0 \quad \text{and} \quad SCR_{property} = 25\%\, \beta\, A_0$$

and the SCR for interest rate risk is found to be:

$$SCR_{interest\ rate} = \max_{shock \in \{up, down\}} \left(U_0 - A_0^{n_s^{shock}} + L_0^{n_s^{shock}} \right)$$

where the initial claims reserve is again estimated by equation 4.5 and 4.6, and the superscript n_s^{shock} indicates that we need to use the shocked regulatory nominal term structure (see Table 5.7). The initial asset value of the portfolio after the shock in the regulatory nominal term structure is given by:

$$A_0^{n_s^{shock}} = (\alpha + \beta)\, A_0 + (1 - \alpha - \beta)\, A_0 \sum_{w \in \{1,2,3,5,10\}} \frac{\gamma_w}{P^M_{n_e}(0, w)}\, P^M_{n_e^{shock}}(0, w)$$

where the subscript n_e refers to the average EEA government nominal term structure (see Table 5.7), and the superscript M again refers to the fact that it is observed in the market. The subscript n_e^{shock} refers to the same EEA curve, having applied the shock described in Table 2.6 (which is usually applied to the regulatory interest rates). This means that the spread of the average EEA government nominal curve over the regulatory nominal rate term structure does not change after the shock. We point out that in this numerical analysis, we have an exposure to an interest rate shock up.

The equity, property, and interest rate risk are the only sources of market risk we have. Consequently, under the assumption of considerable correlation between equity and property risk (+0.75), and null correlation otherwise, the SCR for market risk is found to be:

$$SCR_{market} = \sqrt{SCR^2_{interest\ rate} + SCR^2_{equity} + {}^{3}\!/_{2}\, SCR_{equity}\, SCR_{property} + SCR^2_{property}}$$

The capital requirements of the SF and our IM are quite similar, because of similar assumptions. The capital requirements for interest rate risk are slightly smaller in the case of the SF (ca. 0.5%) rather than in our IM (ca. 2%), even though the contribution to market risk is rather limited in both cases. This is mainly because, as already mentioned, the asset-liability portfolio has a duration near to zero, resulting in a good interest-rate immunisation. Moreover, we point out that the SF has an understandably simple infrastructure with respect to our IM, and it is not able to properly grasp all the peculiarities of the asset-liability portfolio. For example, we remind that in our IM the asset allocation is recalibrated at year end only, hence we do not suffer from the change in the market value during the year. In addition, the SF does not consider any form of expected profit and does not distinguish for the different investments in the same asset class. Above all, consider that the capital requirements under the SF are calculated according to some interest rate shocks, not related to the market volatility and with some simple assumptions about the minimum and maximum levels. Contrary to interest rate risk, note that the capital requirements for property risk are slightly bigger in the case of the SF (ca. 2.6%) rather than in our IM (ca. 0.8%). Once again, the contribution to the market risk is rather limited in both cases. It is evident that equity risk is the main source of market risk in this numerical analysis, and it is significantly bigger in the case of the SF (ca. 32%) rather than in our IM (ca. 27%). However, we remind that the stock investment is considered as type 1 equity not having a strategic nature and a long-term holding strategy. On one hand, if we assumed a type 2 equity, the shock would pass from 39% to 49%, and the SCR would consequently increase. On the other hand, if we assumed an equity having a strategic nature or a long-term holding strategy, the shock would drop to 22%, and the SCR would decrease. As a result, the SCR could range between around 18% and 40% without considering the symmetric adjustment. Finally, note that the diversification benefits are smaller in the case of the SF (ca. 3%) rather than in our IM (ca. 9%). Be aware that the diversification benefit is usually low when the risk strongly depends on a single source. This is especially true if the aggregation is based on the SF procedure.

6.2 Non-life underwriting risk

We now isolate the effect of the non-life risk, and so we leave aside the market risk. In this regard, we drop the investment result and discounting effect by the risk reserve, assuming that all the interest rates are null. Once again, we believe that this is not a simplification and is a reasonable solution to make a stand-alone analysis of the non-life risk with respect to market risk. As a result, the risk reserve is found to be:

$$U_t = U_{t-1} + (1 + \lambda)\, \pi_t - X_t + Y_t^{n_s^{null}}$$

where the superscript n_s^{null} indicates that we need to use a null regulatory nominal term structure.

Given a null expected real-world return, we calculate once again the capital requirements over a period of one, two, and three years. The diversification benefit is calculated at source-of-risk level (i.e. premium or reserve risk) only. The results are presented in Table 6.9, 6.10 and 6.11, together with the comparison with the SF.

Table 6.9. Capital requirements for premium risk over the initial GPW (expressed in %) and diversification benefits over a period of one, two, and three years

(a) Standard formula

Year	MTPL Premium	MOD Premium	GTPL Premium	Premium	Diversification
1	15.59	6.41	11.02	26.70	19.16%

(b) Internal model with Gaussian copula

Year	MTPL Premium	MOD Premium	GTPL Premium	Premium	Diversification
1	12.35	1.16	19.85	26.09	21.77%
2	16.93	0.60	26.89	33.98	23.50%
3	20.92	−0.04	31.76	38.61	26.66%

(c) Internal model with Gumbel copula

Year	MTPL Premium	MOD Premium	GTPL Premium	Premium	Diversification
1	12.35	1.16	19.85	28.94	13.24%
2	16.93	0.60	26.89	37.49	15.60%
3	20.92	−0.04	31.76	43.64	17.09%

In order to be consistent with equation 6.1, the RBC for premium or reserve risk is calculated according to the following formula:

$$RBC(0,t)_{nl\ prem\ or\ res} = V_0 - V_{\varepsilon}(t)$$

The variable to isolate the effect of the premium risk is given by:

$$V_t = V_{t-1} + (1+\lambda)\,\pi_t - X_t \quad \text{with} \quad V_0 = U_0$$

and the variable to isolate the effect of the reserve risk is given by:

$$V_t = V_{t-1} + Y_t^{n_s^{null}} \quad \text{with} \quad V_0 = U_0$$

In order to understand the pattern of the capital requirements for premium risk in our IM, we should now focus on the riskiness of the total claim amount and on the expected profit produced by the safety loadings. The first one contributes to a bigger RBC, the second one contributes to a smaller RBC. Once again, the dynamic portfolio assumption has a scaling effect on the RBC. On the other side, in case of reserve risk, we do not have any expected profit or loss (we remind that the discounting effect is ignored in the non-life risk stand-alone analysis). The RBC is bigger than in the previous section. It is common that in non-life insurance the capital requirements for non-life risk are the biggest, because the business is mainly devoted to the underwriting side. In our numerical analysis, the capital requirements for non-life risk are considerable as we have a quite risky insurance portfolio, since the upper tail of the total claim amount and the lower tail of the claims development result are quite heavy. It is worth reminding that, on the other hand, the investment

Table 6.10. Capital requirements for reserve risk over the initial GPW (expressed in %) and diversification benefits over a period of one, two, and three years

(a) Standard formula

Year	MTPL Reserve	MOD Reserve	GTPL Reserve	Reserve	Diversification
1	20.25	1.20	33.00	47.04	13.61%

(b) Internal model with Gaussian copula

Year	MTPL Reserve	MOD Reserve	GTPL Reserve	Reserve	Diversification
1	18.38	0.81	36.77	47.37	15.35%
2	20.73	0.86	41.86	53.09	16.32%
3	21.91	0.87	45.03	56.58	16.57%

(c) Internal model with Gumbel copula

Year	MTPL Reserve	MOD Reserve	GTPL Reserve	Reserve	Diversification
1	18.38	0.81	36.77	51.75	7.52%
2	20.73	0.86	41.86	57.36	9.59%
3	21.91	0.87	45.03	60.45	10.87%

portfolio is composed of simple assets expressed in euro currency and issued by EEA governments, so that many important sources of market risk (i.e. the spread risk, liquidity risk, and default risk) are not considered. Focusing on the premium risk, the RBC of each LoB increases over time, except for MOD, in which the limited risk is more than compensated by the significant expected profit created by the safety loadings. On the other side, the RBC for reserve risk increases over time for each LoB. We point out that GTPL is the main source of risk (ca. between 70% and 75% both of premium risk and reserve risk). The aggregate total claim amount and claims development result are even riskier adopting the Gumbel copula in the aggregation process, where there is a high upper tail dependence (hence a smaller diversification benefit and higher capital requirements). Finally, we point out that the reserve risk is bigger than the premium risk (ca. 47% and 26% respectively in case of linear dependence, ca. 52% and 29% respectively in case of non-linear dependence).

As anticipated in equation 2.1 about the SF framework, the SCR for premium and reserve risk is calculated using this simplified formula:

$$SCR_{nl\ prem\ res} = 3\,\sigma_{nl}\,V_{nl}$$

where σ_{nl} and V_{nl} are the volatility factor (i.e. the standard deviation, in relative terms) and the volume measure for premium and reserve risk, respectively. We remind that they are obtained by the aggregation of the results for premium risk and reserve risk. The volume measure is roughly distinguished in earned premiums and best estimate of the provisions for claims outstanding. It is worth mentioning that until QIS 5 (i.e. the fifth Quantitative Impact Study, carried out in 2010) a very elegant and consistent solution had been adopted. It assumed the exact formula for

Table 6.11. Capital requirements for non-life risk over the initial GPW (expressed in %) and diversification benefits over a period of one, two, and three years

(a) Standard formula

Year	Premium	Reserve	Non-life	Diversification
1	26.70	47.04	64.67	12.31%

(b) Internal model with Gaussian copula

Year	Premium	Reserve	Non-life	Diversification
1	26.09	47.37	63.66	13.34%
2	33.98	53.09	71.45	17.94%
3	38.61	56.58	76.09	20.06%

(c) Internal model with Gumbel copula

Year	Premium	Reserve	Non-life	Diversification
1	28.94	51.75	74.08	8.18%
2	37.49	57.36	84.74	10.66%
3	43.64	60.45	90.12	13.42%

the quantile calculation in case of a Lognormal distribution underlying the premium and reserve risk. According to QIS 5, the higher the volatility factor, the higher the multiplier. The latter was equal to 3 only when the volatility factor was roughly 14.5%. We point out that the multiplier should approach 2.58 (i.e. the quantile of order 99.5% of a Standard Normal distribution) when the skewness is extremely close to zero.

Furthermore, since we do not have other sources of risk, the SCR for non-life risk is equal to the SCR for premium and reserve risk.

We remind that the reinsurance is not considered. Hence, the volatility factor is equal to 8.48% for premium risk, 8.71% for reserve risk, and 7.56% for non-life risk. The volume measure is instead is given by 104.9 mln (i.e. the GPW at the end of the first year) for premium risk, 180 mln (i.e. the initial claims reserve) for reserve risk, and 284.9 mln for non-life risk. Overall, there are numerous differences between our IM and the SF, due to the simplified logic behind the latter approach. Firstly, in contrast with the SF, in our IM we calibrated the parameters on specific data, using an aggregation procedure different to the linear correlation one. The expected profit produced by the safety loadings is not considered in the SF for premium risk. In addition, the size factor is not considered, so that in the SF the volatility factors for premium and reserve risk are assumed to be the same for each EU insurance company. However, in our IM for premium risk we also consider the insurance company size, coming from the expected number of claims. Moreover, as already mentioned, the multiplier 3 used in the SF for the premium and reserve risk is not fully consistent, because it is fixed regardless of the level of volatility. For this formula, EIOPA declared an underlying Lognormal distribution. In this case, when the volatility is around 14.5% we have a multiplier equal to 3, otherwise the appropriate multiplier would be different. The drawback of this formula is that it does not respect the relationship satisfied by the Lognormal distribution. In particular, when we have a quite low volatility, the skewness is small, and therefore the multiplier is closer to 2.58, which is the quantile of order 99.5% of a Standard Normal distribution (e.g. with a volatility of 5% the multiplier is equal to 2.72). On the other side, with a higher volatility, the skewness is bigger and then the multiplier is larger (e.g. with a volatility of 25% the multiplier is equal to 3.32). Clearly, in the first case the SF overestimates the capital requirement, whereas in the second case, it underestimates it. Going into detail, Table 6.12 and 6.13 show the multipliers and volatility factors for premium or reserve risk of each LoB under the SF and IM in the first year. Firstly, it is noted that the volatility factor under the SF is always lower than 14.5%, and the multiplier 3 overestimates the theoretical 99.5% percentile of the Lognormal distribution. Moreover, on one side the SF creates an inconsistent benefit for premium risk of MTPL and GTPL, and for reserve risk of GTPL, because the multipliers and volatility factors are lower in the SF than in our IM. On the other side, the SF creates an inconsistent disadvantage for premium risk of MOD, and for reserve risk of MTPL and MOD, because the multipliers and volatility factors are higher in the SF than in our IM. Also, we point out that the volatility factors under the Merz and Wüthrich formula (Merz and Wüthrich, 2008) are equal to 9.4% for MTPL, 8.6% for MOD, and 11% for GTPL, in line with our empirical results.

Table 6.12. Multipliers and volatility factors for premium risk in the first year

(a) Standard formula

Parameter	MTPL Premium	MOD Premium	GTPL Premium
multiplier	3	3	3
σ_{nl}	10%	8%	14%

(b) Internal model

Parameter	MTPL Premium	MOD Premium	GTPL Premium
multiplier	3.12	2.69	4.27
σ_{nl}	10%	7%	32%

It is important to see that, in contrast to the other LoBs, the SCR of GTPL is lower in the SF than in our IM, for both premium risk and reserve risk. Furthermore, the capital requirements for premium risk, reserve risk, and non-life risk are quite similar between the SF and IM. The SF (ca. 27%, 47%, and 65% for premium, reserve, and non-life risk, respectively) is always pretty in line with the results given by the IM based on linear dependence (ca. 26%, 47%, and 64% for premium, reserve, and non-life risk, respectively), while it always underestimates them when the IM based on non-linear dependence is used (ca. 29%, 52%, and 74% for premium, reserve, and non-life risk, respectively).

Table 6.13. Multipliers and volatility factors for reserve risk in the first year

(a) Standard formula

Parameter	MTPL Reserve	MOD Reserve	GTPL Reserve
multiplier	3	3	3
σ_{nl}	9%	8%	11%

(b) Internal model

Parameter	MTPL Reserve	MOD Reserve	GTPL Reserve
multiplier	2.86	2.77	3.14
σ_{nl}	8.5%	5.8%	11.5%

6.3 Market and non-life underwriting risk

Actually, the investment result and discounting effect are combined with the total claim amount and reserving cash flows. We now finally take into account both the market risk and non-life risk. As anticipated in equation 1.5, we have:

$$U_t = (1 + J_t)\, U_{t-1} + (1 + \bar{J}_t)\,(1 + \lambda)\, \pi_t - X_t + Y_t + J_t\, L_{t-1} - \bar{J}_t\, C_t$$

Using the expected return given by the real-world economic scenario generator, we calculate the capital requirements over a period of one, two, and three years. The diversification benefit is calculated at module level (i.e. market or non-life risk) and source-of-risk level (i.e. equity, property, or interest rate risk, and MTPL, MOD, or GTPL for both premium and reserve risk), so that we can observe both the partial and total diversification benefits. The results are presented in Table 6.14, together with the comparison with the SF.

It can be observed that the capital requirements (for market and non-life risk both separately and combined, or for premium risk, in case of linear dependence structure only) increase over time by approximately the square root of time horizon, as presented in Figure 6.6. This pattern is commonly relied upon when analysing capital requirements with respect to time. Note that it is just a rule of thumb, and there are many elements (e.g. expected profits) that can invalidate it. We point out that reserve risk only refers to the in-force business, therefore it does not respect the mentioned practical relation, because of the natural portfolio run-off. The reserve risk strongly influences the overall risk position of our insurance company, and this is why the non-life risk and its combination with market risk do not fully comply with the mentioned relation. Focusing on premium risk, we can observe that for MTLP and GTPL the expected trend is quite confirmed (there is some inconsistency as for market risk, because of some expected profit included in the models). However, this is not the case for MOD, because its capital requirement decreases, in the opposite direction from the square root of time horizon. This is because in MOD we have a highly significant expected profit, that over time is higher than the risk. Hence, the total capital requirement for premium risk is not fully consistent with the above rule of thumb. This is one situation in which the proxy described above is not supported, hence care must be taken to merely use this as a quick-and-dirty indicator.

Table 6.14. Capital requirements for market and non-life risk over the initial GPW (expressed in %) and diversification benefits over a period of one, two, and three years

(a) Standard formula

Year	Market	Non-life	Market and non-life	Diversification	Total diversification
1	33.95	64.67	80.20	18.67%	34.52%

(b) Internal model with Gaussian copula

Year	Market	Non-life	Market and non-life	Diversification	Total diversification
1	26.76	63.66	66.59	26.36%	43.97%
2	37.24	71.45	77.60	28.60%	48.29%
3	42.62	76.09	81.51	31.34%	52.27%

(c) Internal model with Gumbel copula

Year	Market	Non-life	Market and non-life	Diversification	Total diversification
1	26.76	74.08	81.96	18.73%	31.04%
2	37.24	84.74	95.36	21.82%	36.46%
3	42.62	90.12	99.07	25.37%	41.98%

Under the assumption of moderate correlation (+0.25), the SCR for market and non-life risk (that in our case refers to the premium and reserve risk only) is found to be:

$$SCR = \sqrt{SCR^2_{market} + 1/2\, SCR_{market}\, SCR_{non\text{-}life} + SCR^2_{non\text{-}life}}$$

where $SCR_{non\text{-}life}$ is the SCR for non-life risk.

Finally, we can observe that the total capital requirement is quite high, mostly because of reserve risk. This is bigger in the SF than in our IM based on linear dependence and very similar if the non-linear dependence structure is considered. Note that the SF (ca. 19% and 35% of partial and total diversification, respectively) always underestimates the diversification benefit given by the IM based on linear dependence (ca. 26% and 44% of partial and total diversification, respectively), and

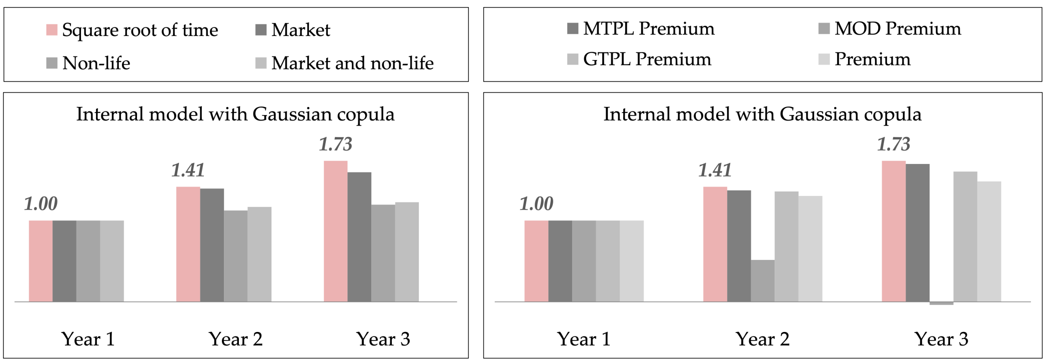


Figure 6.6. Increase in capital requirements according to our IM (with Gaussian copula) over a period of one, two, and three years, against the square root of time horizon

it is more similar if the non-linear dependence structure is used (ca. 19% and 31% of partial and total diversification, respectively). Be aware that the diversification benefit is influenced by the IM structure and simplifications made in the stand-alone analyses. The total diversification benefit is also influenced by the diversification described in the previous sections.

6.4 Analyses

Last but not least, we now describe some different investigations made to better understand the impact of some elements or assumptions involved in our numerical analysis. First of all, with some sensitivity analyses we determine the impact on our insurance company of an instantaneous change in nominal or inflation rates. In the what-if analyses, we assume a new insurance company determined by the change of a single element or assumption of our analysis framework. Moreover, we investigate the role of inflation in our context.

6.4.1 Sensitivity analyses

Here, we present the results of a couple of sensitivity analyses. We remind that in Section 5.5, we calibrated the development triangles of incremental amounts for paid claims to match our claims reserve assumption. Note that in the sensitivity analyses we do not recalibrate the development triangles (we thus apply the new assumption to the original incremental amounts for paid claims). Be also aware that we do not change the calibrated parameters of our economic scenario generator. In the first sensitivity, we evaluate the impact of using nominal rates different than the data as of year end 2021. On the asset side, we use the average of all EEA government nominal rates on December 31, 2022. On the liability side, we consider instead the Eiopa interest rate term structure without volatility adjustment on December 31, 2022. Both the interest rate term structures are shown in Table 6.15.

Table 6.15. Average of all EEA government nominal rates and Eiopa interest rates without volatility adjustment (annually compounded and expressed in %) on December 31, 2022

Maturity (years)	EEA rate	Eiopa rate	Eiopa rate (shock up)	Eiopa rate (shock down)
1	2.747	3.176	5.399	0.794
2	2.917	3.295	5.602	1.153
3	2.944	3.203	5.253	1.409
5	3.076	3.131	4.853	1.691
7	3.247	3.091	4.606	1.886
10	3.450	3.092	4.391	2.133
15	3.610	3.022	4.022	2.206
20	3.622	2.765	3.765	1.963

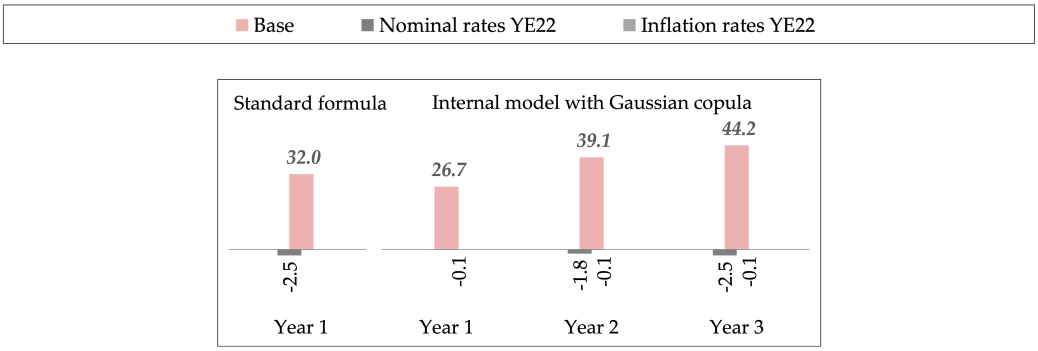
In the second sensitivity, we evaluate the impact of using an inflation curve equal to ZCIIS fixed rates on December 31, 2022 (i.e. not equal to ZCIIS fixed rates as of year end 2021) shown in Table 6.16. In this case, we have different simulations for the CPI, and we need to adjust the real-world expectation of the ratio of the CPI at time t and zero. This results in different claims inflation indices.

Table 6.16. Inflation rates (annually compounded and expressed in %) on December 31, 2022

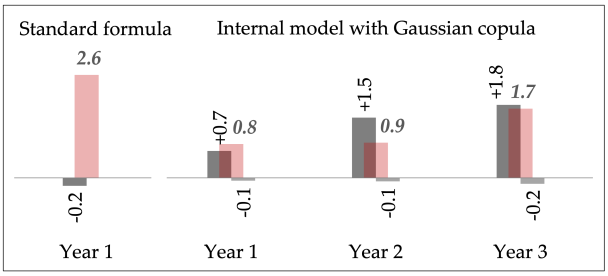
Maturity (years)	Inflation rate
1	4.523
2	3.415
3	3.053
5	2.733
7	2.601
10	2.549
15	2.584
20	2.594

On one side, Figure 6.7 shows the sensitivity impacts on capital requirements for market risk with respect to the previously described base case, according to our IM and compared to the SF. On the other side, Figure 6.8 shows the same picture for market and non-life risk.

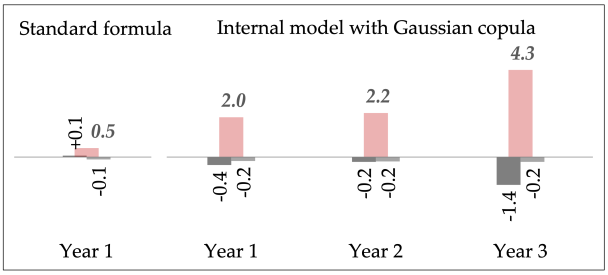
Nominal interest rates significantly raised from year end 2021 to year end 2022, and negative rates are not anymore present. Hence, in the first sensitivity analysis we observe a drop in the initial risk reserve (from 25 mln to 24.5 mln, roughly -2%) because of the reduction in claims reserves (from 180 mln to 164.3 mln, roughly -9%) and in interest-sensitive assets (from ca. 112.8 mln to 96.6 mln, roughly -14%). We point out that the original asset allocation is recalibrated after the change in nominal interest rates. This is the reason why the capital requirements for equity risk and property risk decrease under the SF. The asset-liability portfolio is now slightly less interest-rate immunised (effective duration of the net cash flows on initial asset value from 0.26 to 0.37). Be aware that the interest rate shocks under the SF are not parallel, and they are not equal in absolute value between the up and down cases. Therefore, the duration is not always able to predict the SF behaviour. In this sensitivity analysis, the capital requirements for interest rate risk raise under the SF, and the portfolio is now exposed to a drop in interest rates. In order to understand the pattern of the capital requirements for market risk in our IM, we should focus on the asset size reduction mentioned above and on the increase in nominal interest rates. Both contribute to a risk mitigation (less asset under risk and more expected profit for higher rates), but the second one also implies bigger capital requirements because of a higher real-world discounting effect in RBC calculation and an expected loss in the unwinding of claims development result. Note that the risk mitigation effects prevail for equity and interest rate risk, while the other effects prevail for property risk. We also point out that there is a different attribution of expected profits to the different risk drivers. Overall, we observe a decrease in the capital requirements for market risk, both in the SF (-2.7%) and in our IM (-1.3% over the first year, higher drops over next years), mainly



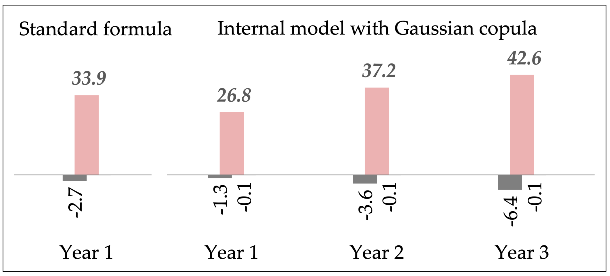
(a) Equity risk



(b) Property risk

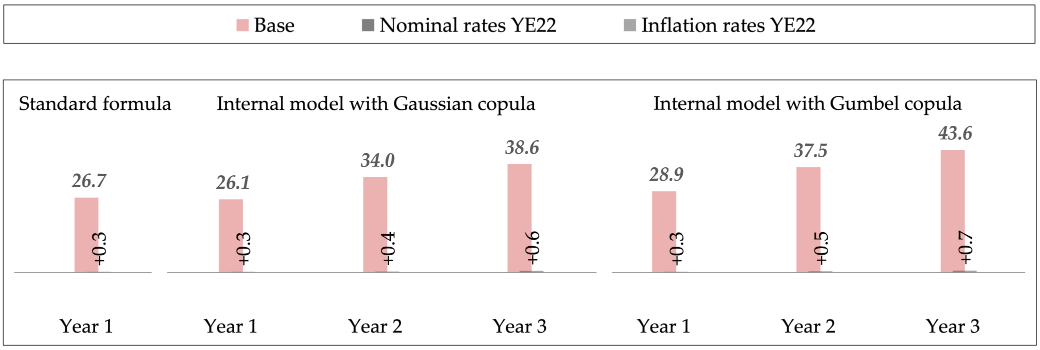


(c) Interest rate risk

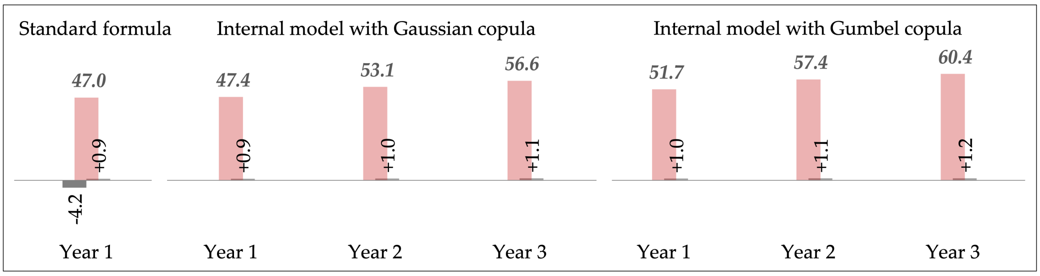


(d) Market risk

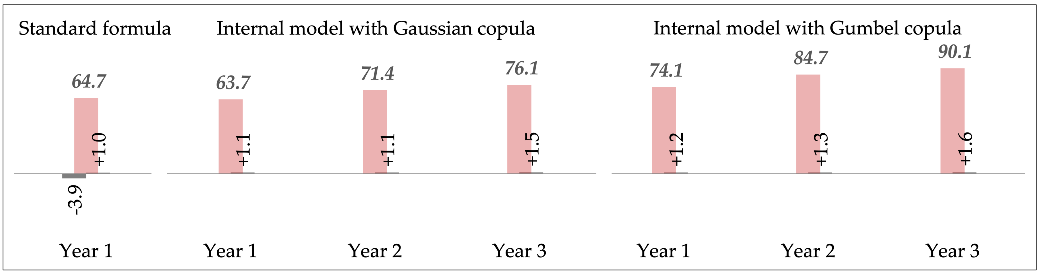
Figure 6.7. Sensitivity impacts on capital requirements for market risk with respect to the base case (results over the initial GPW and expressed in %) over a period of one, two, and three years



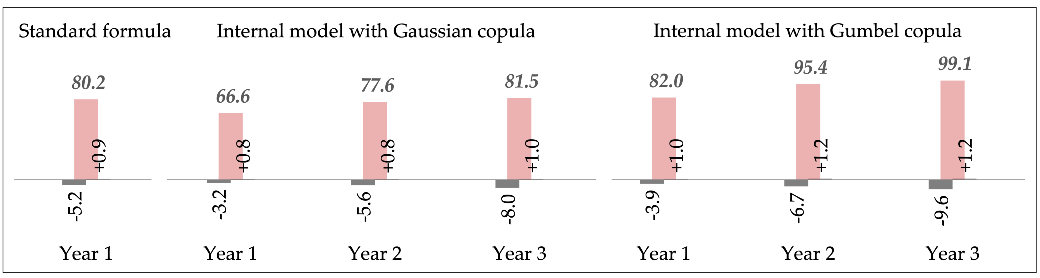
(a) Premium risk



(b) Reserve risk



(c) Non-life risk



(d) Market and non-life risk

Figure 6.8. Sensitivity impacts on capital requirements for market and non-life risk with respect to the base case (results over the initial GPW and expressed in %) over a period of one, two, and three years

driven by equity and interest rate risk. Under the SF, the capital requirements for premium risk do not change, and the capital requirements for reserve risk decrease because of the reduction in claims reserves, driving to an overall drop in the capital requirements for non-life risk (−3.9%). We remind that the real-world return and discounting effect are not considered in the calculation of RBC for premium or reserve risk. This is the reason why there is not any change according to our IM. Based on the explanations above, we observe a decrease in the capital requirements for market and non-life risk, both in the SF (−5.2%) and in our IM (−3.2% and −3.9% over the first year, based on linear and non-linear dependence, respectively, higher drops over next years).

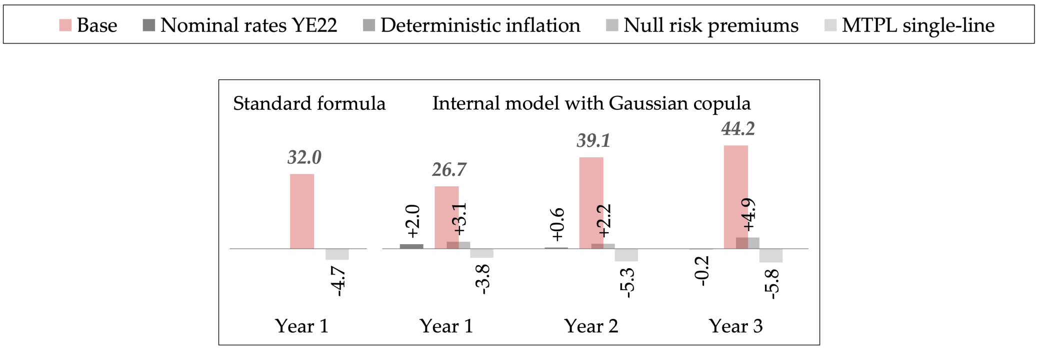
Inflation rates also raised from year end 2021 to year end 2022, especially in the short maturities. Hence, in the second sensitivity analysis we observe a drop in the initial risk reserve (from 25 mln to 21.7 mln, roughly −13%) because of an increase in claims reserves (from 180 mln to 183.3 mln, roughly +2%). Also, we observe an increase in risk or gross premium amounts (around 1 mln more at the end of the first year). The initial asset value of the total portfolio remains the same, hence the capital requirements for market risk are quite stable both in the SF and in our IM. The increase in claims reserves is the main reason why there is a small difference in capital requirements for interest rate risk. Together with the increase in premium amounts, this is also the reason why there is often an increase in capital requirements for premium or reserve risk, both in the SF and in our IM. Note that the new inflation curve slightly modifies the distribution of incremental amounts for paid claims over the various calendar years. This is an additional (minor) point to explain the change in capital requirements (this mainly affects interest rate and reserve risk). Overall, we observe an increase in capital requirements, both in the SF (+1.0% for non-life risk and +0.9% for the combination of market and non-life risk) and in our IM (for instance, over the first year we have +1.1% and +1.2% for non-life risk, based on linear and non-linear dependence, respectively, and +0.8% and +1.0% for the combination of market and non-life risk, based on linear and non-linear dependence, respectively), mainly driven by reserve risk.

6.4.2 What-if analyses

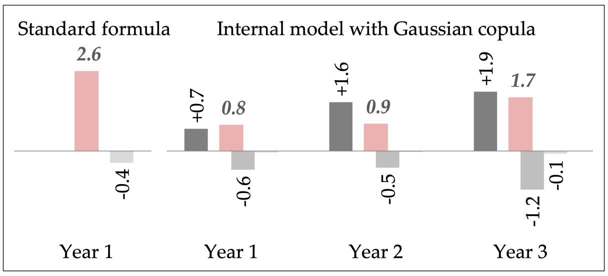
Here, we present the results of four what-if analyses. Differently from before, in the what-if analyses we recalibrate the development triangles of incremental amounts for paid claims to match our claims reserve assumption, and we also change the element or assumption under investigation. This means that we want to calculate the capital requirements of our insurance company as if new input or assumptions were available on valuation date. In the first what-if analysis, we evaluate what happens if we use nominal rates different from the data as of year end 2021. On the asset side, we use the average of all EEA government nominal rates on December 31, 2022. On the liability side, we consider instead the Eiopa interest rate term structure without volatility adjustment on December 31, 2022. Once again, both the interest rate term structures are shown in Table 6.15. In the second what-if analysis, we evaluate what happens if we assume that the inflation is not anymore stochastic, but it is deterministic. We point out that in this case no recalibration is required, and the initial figures are the same as in the base case (in other words, only projected figures are different). In the third what-if analysis, we evaluate what happens if we assume that the risk premiums of our real-world economic scenario generator are null. This means that the economic scenario generator is risk neutral. In the fourth and last what-if analysis, we evaluate what happens if we assume that the insurance company has underwriting business in MTPL only. Be aware that the GPW is the same as the multi-line total one (100 mln), and all the parameters are also the same (except the parameters depending on the premium volume, which are however estimated by the same procedure). It is noted that the volume of MTPL is bigger for the single-line insurer than the multi-line one. For this reason, the coefficient of variation of incremental settlement speeds for amount should decrease (because bigger companies have smaller relative volatility). However, we want to isolate the impact of having a different risk profile and less diversification due to the MTPL single business, keeping everything else the same and without introducing too many drivers of change.

Once again, Figure 6.9 shows the what-if impacts on capital requirements for market risk with respect to the base case, according to our IM and compared to the SF. Figure 6.10 shows the same picture for market and non-life risk.

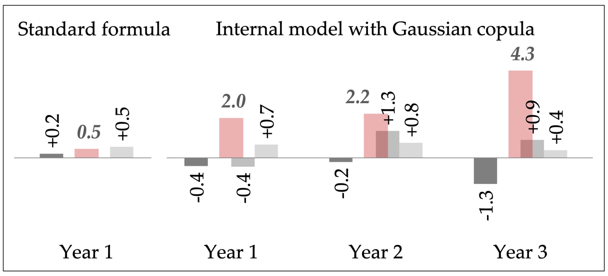
Note that the first what-if analysis (i.e. nominal interest rates as of year end 2022) seems to be identical to the corresponding previously described sensitivity analysis. However, we now have the same initial asset value of the total portfolio (205 mln) and claims reserves (180 mln) as in the base case, because we recalibrate the development triangles, raising the undiscounted incremental amounts for paid claims (their sum goes from ca. 179 mln to ca. 196 mln, roughly +9%) to offset the increase in nominal interest rates. For information purposes, it is noted that the asset-liability portfolio is now slightly less interest-rate immunised than in the base case (effective duration of the net cash flows on initial asset value from 0.26 to 0.35) and slightly more than in the corresponding sensitivity analysis. On one side, the capital requirements for equity and property risk under the SF do not change, since the asset size of our insurance company remains the same. On the other side, the capital requirements for interest rate risk raise under the SF, and the portfolio is now exposed to a drop in interest rates. It is worth noticing that the increase in nominal interest rates has different implications on the capital requirements for market risk according to our IM. Firstly, there is a risk mitigation, because higher rates create more expected profit. Moreover, we have a higher real-world discounting effect in RBC calculation and an expected loss in the unwinding of claims development result, as already explained for the corresponding sensitivity analysis. This time, the risk mitigation effect prevails for interest rate risk only, while the other effects prevail for equity and property risk. We also remind that there is a different attribution of expected profits to the different risk drivers. Overall, we have an inconsiderable change in the capital requirements for market risk under the SF and a small increase according to our IM over the first year (+0.6%), mainly driven by equity risk. Over next years, we observe some drops according to our IM, due to a higher diversification effect. Focusing on the non-life risk, the capital requirements for premium or reserve risk under the SF do not change, once again because the asset size of our insurance company remains the same. On one side, the capital requirements for premium risk according to our IM remain the same. On the other side, the RBC for reserve risk significantly raises, since the undiscounted incremen-



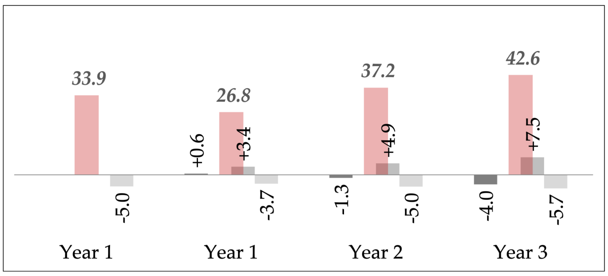
(a) Equity risk



(b) Property risk

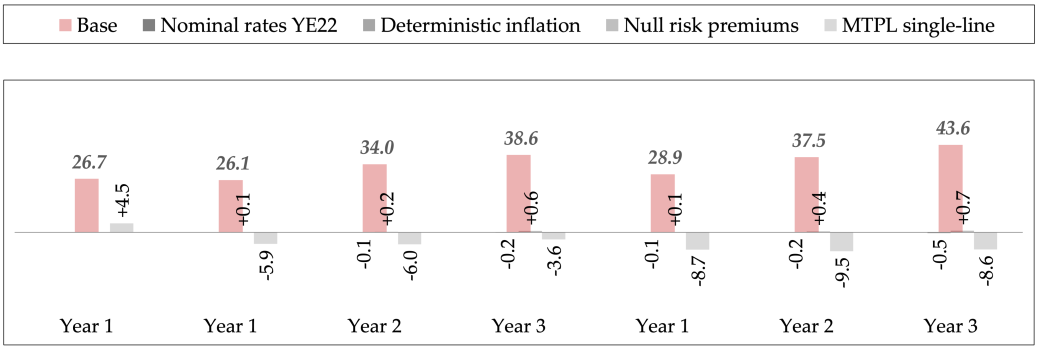


(c) Interest rate risk

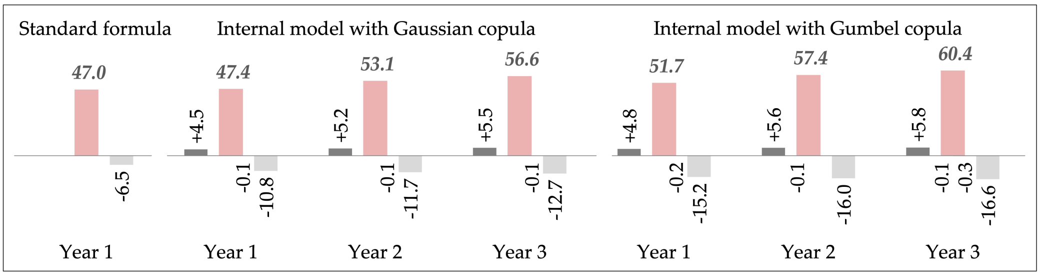


(d) Market risk

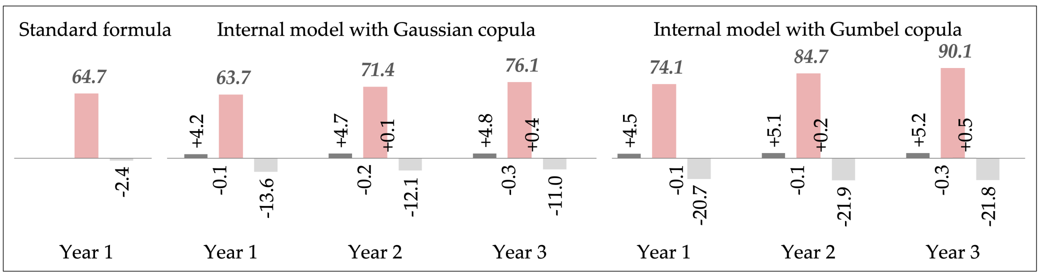
Figure 6.9. What-if impacts on capital requirements for market risk with respect to the base case (results over the initial GPW and expressed in %) over a period of one, two, and three years



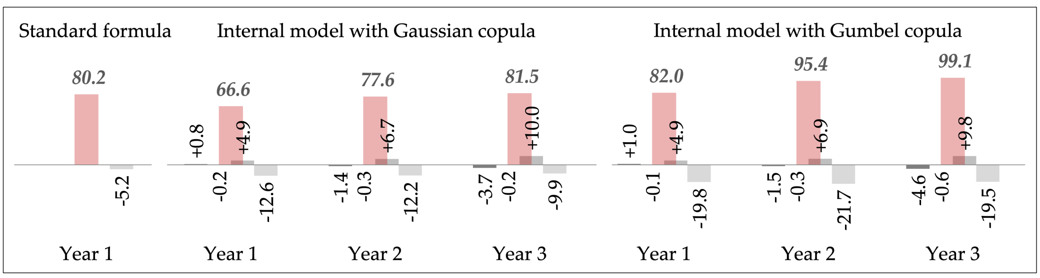
(a) Premium risk



(b) Reserve risk



(c) Non-life risk



(d) Market and non-life risk

Figure 6.10. What-if impacts on capital requirements for market and non-life risk with respect to the base case (results over the initial GPW and expressed in %) over a period of one, two, and three years

tal amounts for paid claims are now higher, driving to an overall increase in the RBC for non-life risk (+4.2% and +4.5% over the first year, based on linear and non-linear dependence, respectively, higher raises over next years). Based on the explanations above, there is an inconsiderable change in the capital requirements for market and non-life risk under the SF and an increase according to our IM over the first year (+0.8% and +1.0%, based on linear and non-linear dependence, respectively). Over next years, we have some drops according to our IM.

We point out that the inflation risk is ignored by the SF, so the what-if analysis on deterministic inflation has no impact on related capital requirements. Also, the stochastic inflation does not affect the market risk in our IM, because inflation is only included in the paid claims of the premium and reserve model. The capital requirements for premium or reserve risk according to our IM slightly drop as the overall ones (over the first year we have −0.2% and −0.1% for the combination of market and non-life risk, based on linear and non-linear dependence, respectively, higher drops over next years). This analysis shows that the stochastic inflation has no big impact on the risk quantification of our insurance company.

The third what-if analysis (i.e. null risk premiums) corresponds to a risk-neutral analysis. As a consequence, all the typical tests (e.g. curve fitting test, martingale tests, ...) could be done, but this would be beyond the goal of this thesis. We point out that the SF does not contemplate any form of expected profit, and this is the reason why the risk premiums do not have any impact on related capital requirements. Once again, we have the same initial asset value of the total portfolio and claims reserves as in the base case, and also the undiscounted incremental amounts for paid claims remain quite stable (the small change is due to the different real-world expectation of the ratio of the CPI at time t and zero, resulting in different claims inflation indices). The key assumptions to understand the pattern of the capital requirements for market risk in our IM are the same as previously described. As shown in Figure 6.11, the real-world and risk-neutral expectations are quite different. Note that for stock and property returns, the real-world expectation is significantly higher than the risk-neutral one in the first three years (where the real-world expected calibration benchmarks are explicitly defined). This difference becomes smaller over time, because it is compensated by the negative difference between the real-world and risk-neutral nominal short rates after the second year (i.e. discontinuity point). The risk-neutral expected five-year nominal rate is bigger than the real-world one. Note that the nominal rate formula under our short-rate model is based on the expected value of a function (based on equation 3.3) of the nominal state variables at a given time. This means that the difference between the risk-neutral and real worlds only depends on the level of the state variables in both cases at that given time. Consequently, the expected nominal rate (a prediction of the future) should not be confused with the realised rate of return in the years just gone by. Indeed, the risk-neutral zero-coupon bond annual rates of return realised on average after one, two, and three years (−0.63%, −0.43%, and −0.19%) are much lower than the real-world corresponding results (see Table 6.1). We point out that the risk-neutral stock and property average annual rates of return are almost equal to the zero-coupon bond ones just mentioned. The comparison between real-world and risk-neutral initial CPI ratios shows that the real-world forecasts are smaller in the first years and bigger in the last period. This has an impact on the non-life risk rather than on the market one. Considering less expected profits, less real-world discounting in the RBC formula and a different expected loss in the unwinding of claims development result (e.g. the expected loss becomes lower over the first year, because the nominal rates become now higher than in the base case), we have an increase in the capital requirements for market risk according to our IM (+3.4% over the first year, higher raises over next years), mainly driven by equity risk. Furthermore, the capital requirements for non-life risk according to our IM remain almost unchanged over the first year and slightly decrease over the second and third year, both in case of linear and non-linear dependence. This is because the RBC for premium risk has a little increase due to the raised volume (we remind that the initial CPI ratio is now a bit higher in the first years), and the RBC for reserve risk slightly drops because of the change in claims inflation indices. Overall, the effect of market risk prevails, and we observe a significant increase in the capital requirements for market and non-life risk according to our IM (+4.9% over the first year, based on both linear and non-linear dependence, higher raises over next years). This analysis shows that the risk premiums have a significant impact on the risk quantification of our insurance company. The EU insurance supervisory authorities are reluctant to allow insurance company to include an expected profit or loss in their IM for capital requirement calculation, because of the desire for consistency with the SF framework. Even if we believe that a prudent approach is suitable and desirable, we also think that a correct quantification of the solvency position of an insurance company should consider not only the sources of risk but also the risk mitigation elements (at least partially).

As already anticipated, in the last what-if analysis (i.e. MTPL single-line) we have the same GPW (100 mln) and parameters (except the parameters depending on the premium volume) as in the base case. Now, the claims reserve is equal to 150 mln (once again 150% of the GPW), and the initial asset value of the total portfolio (175 mln) is roughly four times lower than in the base case. This is the reason why the capital requirements for equity and property risk are smaller, both in the SF and in our IM. Moreover, the MTPL claims reserve has a smaller risk compensation in case of an increase in interest rates than the multi-line claims reserve, hence the capital requirements for interest rate risk raise. Overall, we observe a considerable decrease in the capital requirements for market risk, both in the SF (−5.0%) and in our IM (−3.7% over the first year, higher drops over next years), mainly driven by equity risk. Under the SF, the capital requirements for premium risk increase, and the capital requirements for reserve risk decrease, driving to an overall drop in the SCR for non-life risk (−2.4%). This is because the volume measure for premium risk (104 mln) is similar to the base one, the volume measure for reserve risk (150 mln) and for non-life risk (254 mln) decreases, and the volatility factors (10% for premium risk, 9% for reserve risk, and 8.17% for non-life risk) increase because of less diversification benefit. According to our IM, the capital requirements for premium risk or reserve risk drastically drop as well as the RBC for non-life risk (for instance, over the first year we have −13.6% and −20.7%, based on linear and non-linear dependence, respectively), because GTPL is not anymore present. Based on the explanations above, we observe a decrease in the capital requirements for market and non-life risk, both in the SF (−5.2%) and in our IM (for instance, over the first year we have −12.6% and −19.8%, based on linear and non-linear dependence, respectively).

6.4.3 Inflation analyses

The preceding what-if analysis on deterministic inflation showed that the impact of having a stochastic inflation model is not big in a non-life insurance framework. However, inflation is not a minor element, and its correct estimation is important to have a fair quantification of own funds, liabilities and capital requirements of the insurance company. We now present the results of three analyses to numerically show this concept. The economic scenario is the same as the base one (and it is intended as our point of truth about future realisation of financial variables), but we ignore it, and we make different (wrong) assumptions on the expected future inflation or on the explicit modelling of inflation in the claims reserve calculation. In the first situation, we either assume a null or a triple expected inflation with respect to the original assumption. In the second case, we assume to calculate the claims reserve using the classical Paid Chain-Ladder method, without appreciating the incremental amounts for paid claims of the upper part of the development triangle or adding the expected inflation to the lower part. These analyses describe the situation in which an actuary ignores a financial aspect and make a hypothesis inconsistent with the economic scenario used (e.g. to reduce liabilities). Note that we do not recalibrate the development triangles similarly to the sensitivity analyses, but we recalculate the claims reserve (with the original incremental amounts for paid claims) and the future premiums, based on the new assumption. Consequently, the initial asset value of the total portfolio remains the same, while own funds and liabilities change. Moreover, note that we do not change the estimate of the claims inflation percentage, which is still applied to the expected claims inflation index resulting from the modified assumption of the analysis.

Figure 6.12 shows the impacts of inflation assumptions on capital requirements for market risk with respect to the base case, according to our IM and compared to the SF. Figure 6.13 shows the same picture for market and non-life risk.

In the first analysis on inflation (i.e. null expected inflation), we observe a raise in the initial risk reserve (from 25 mln to 34.8 mln, roughly +39%) because of a reduction in claims reserves (from 180 mln to 170.2 mln, roughly −5%). On one side, the capital requirements for equity and property risk under the SF do not change, because the asset size of our insurance company remains the same. On the other side, the capital requirements for interest rate risk slightly raise under the SF, because the reserves have now a smaller risk compensation than in the base case. In any case, the SCR for market risk remains quite stable. We point out that we have an expected loss coming from the the fact that liabilities are on average insufficient to cover the increase in future price. This is the main reason why we have a small increase in the capital requirements for market risk according to our IM over the first year (+0.2%). Under the SF, the capital requirements for premium or reserve risk decrease, driving to an overall drop in the SCR for non-life risk (−3.1%). This is because there is a drop in all the volume measure for premium risk (101.9 mln), for reserve risk (170.2 mln), and for non-life risk (272.1 mln). We have some key assumptions to properly understand the pattern of the capital requirements for premium or reserve risk in our IM. On one side, we have the reduction of claims

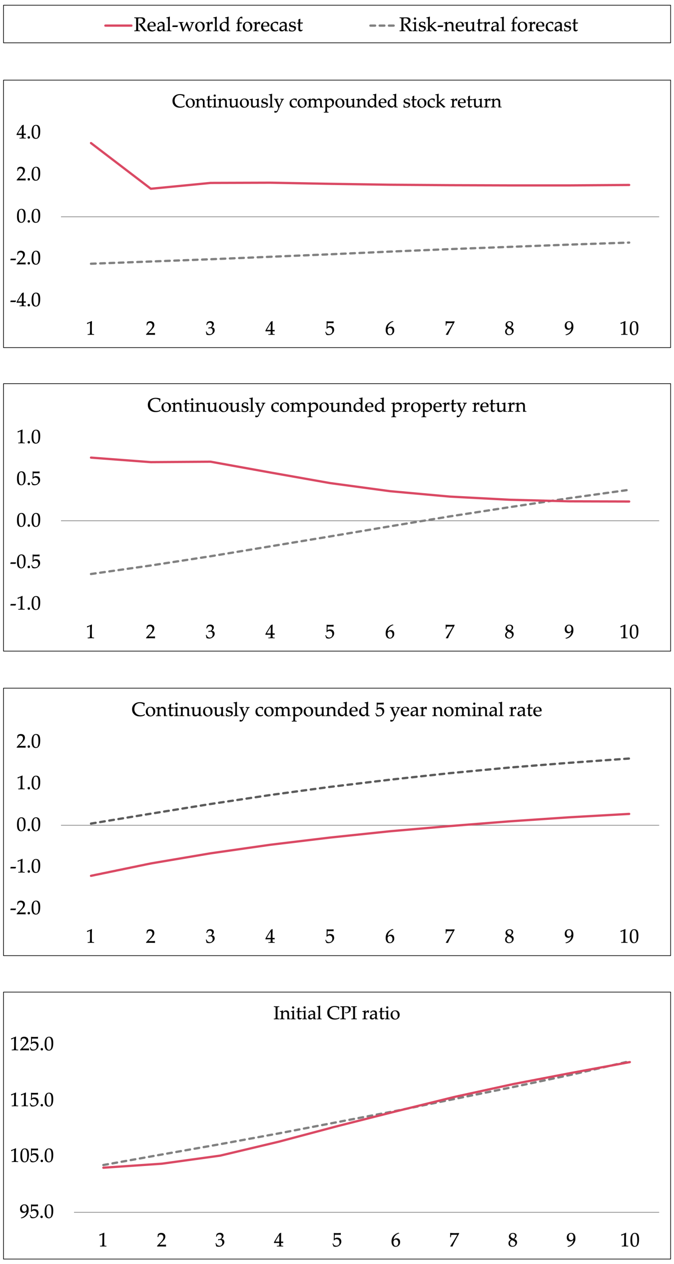
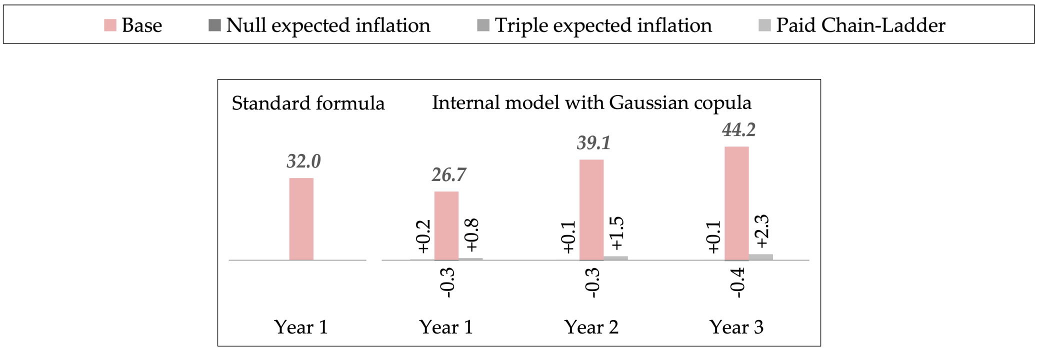
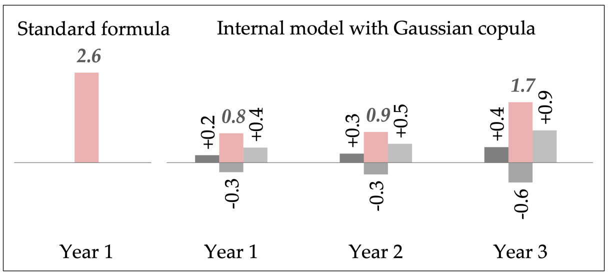


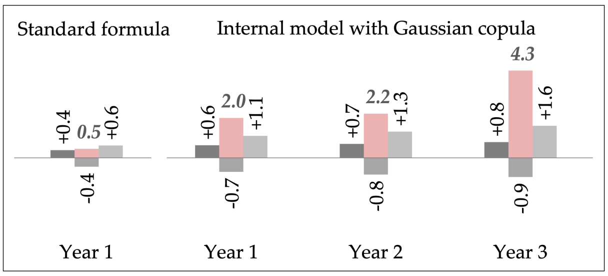
Figure 6.11. Comparison between real-world and risk-neutral forecasts (expressed in %) over ten years



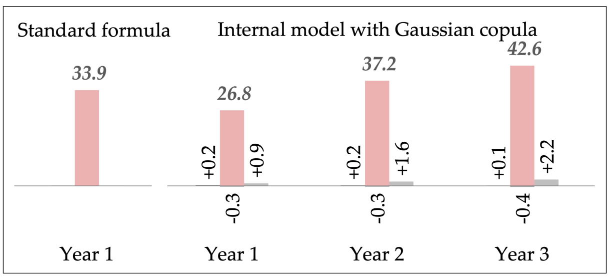
(a) Equity risk



(b) Property risk

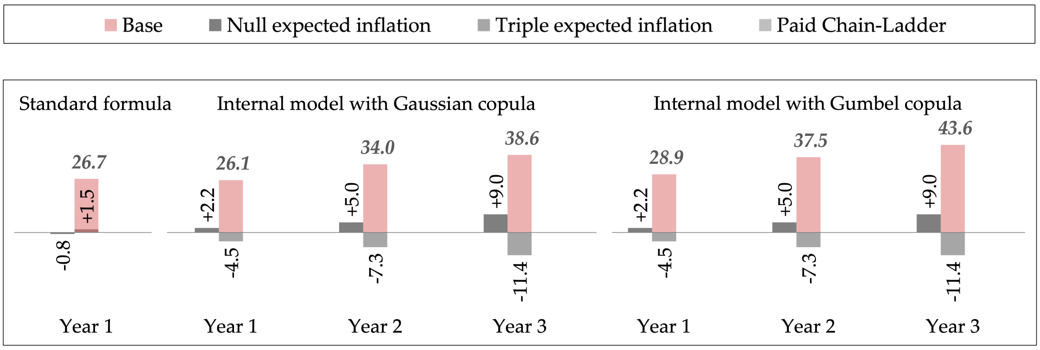


(c) Interest rate risk

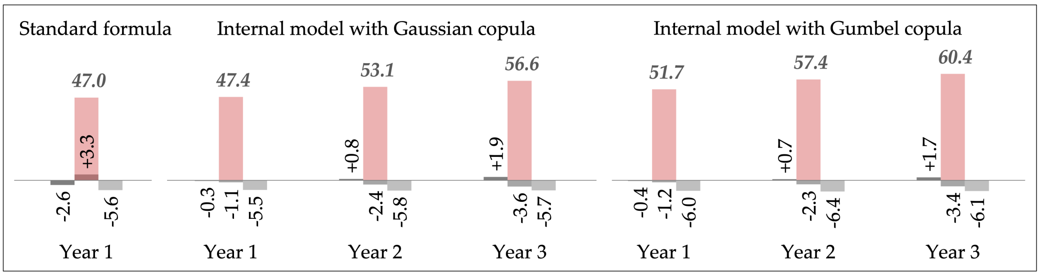


(d) Market risk

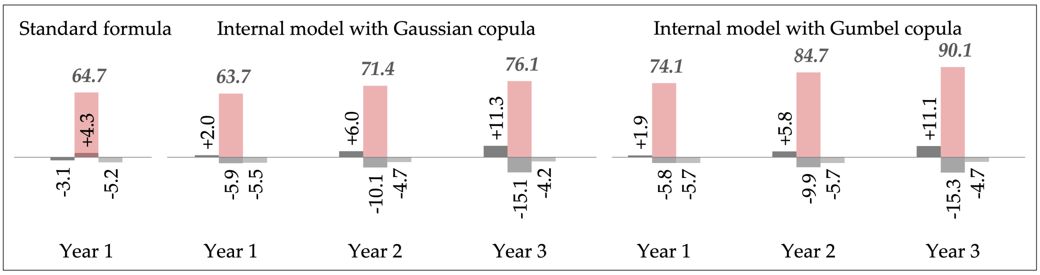
Figure 6.12. Inflation impacts on capital requirements for market risk with respect to the base case (results over the initial GPW and expressed in %) over a period of one, two, and three years



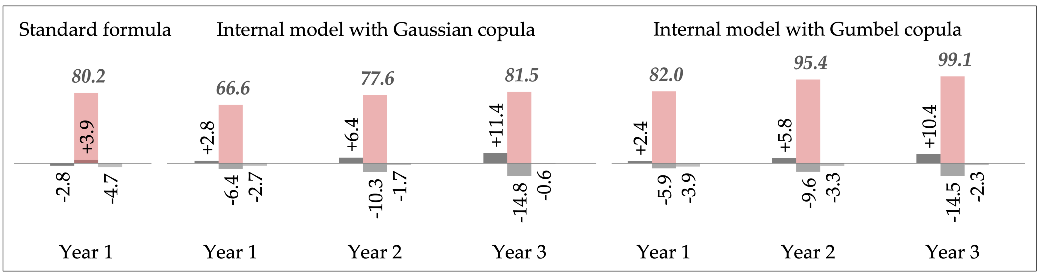
(a) Premium risk



(b) Reserve risk



(c) Non-life risk



(d) Market and non-life risk

Figure 6.13. Inflation impacts on capital requirements for market and non-life risk with respect to the base case (results over the initial GPW and expressed in %) over a period of one, two, and three years

reserves and future premiums. On the other side, we have a smaller expected inflation than the scenario realisation. The RBC for premium risk increases, because the total claim amounts (losses) are the same as in the base case, but the smaller premiums are now less effective than before in covering these losses. The RBC for reserve risk decreases over the first year and increases over the second and third one. This is because the drop in claims reserves reduces the risk size, but these reserves are now less effective than before in covering the losses produced by the incremental amounts for paid claims in the next calendar years (note that these amounts are the same as in the base case). We thus have an increase in the capital requirements for non-life risk according to our IM (+2.0% and +1.9% over the first year, based on linear and non-linear dependence, respectively, higher raises over next years). Overall, we observe a decrease in the capital requirements for market and non-life risk under the SF (−2.8%) and an increase according to our IM (+2.8% and +2.4% over the first year, based on linear and non-linear dependence, respectively, higher raises over next years). This analysis shows that an underestimation of the expected future claims inflation produces an underestimation of the claims reserve and thus of the liabilities of our insurer. At the same time, this produces an overestimation of the risk reserve and thus of the own funds of our insurer. Note that it also implies that the solvency ratio of our insurer is overestimated. This could be dangerous for policyholders, that is why the insurance supervisory authorities take care that the liability estimate is correctly determined. Consequently, notwithstanding that the liabilities are insufficient to cover the increase in future price, the SF shows a decrease of capital requirements (strongly dependent on volume measures). On the contrary, our IM captures this wrong assumption and produces an increase in capital requirements (even though it is anyway negatively influenced by the volume reduction). It is now evident that a correct inflation estimate is important to have a right balance sheet representation and capital requirement quantification.

The second analysis on inflation (i.e. triple expected inflation) is the opposite of the first one. In this case, we observe a drop in the initial risk reserve (from 25 mln to 12.6 mln, roughly −50%) because of an increase in claims reserves (from 180 mln to 192.4 mln, roughly +7%). Consequently, opposite comments to the preceding analysis apply here. In this case, we observe an increase in the capital requirements for market and non-life risk under the SF (+3.9%) and a decrease according to our IM (−6.4% and −5.9% over the first year, based on linear and non-linear dependence, respectively, higher drops over next years).

It is worth mentioning that an underestimation of the claims reserves (from 180 mln to 160.3 mln, roughly −11%) is produced if the classical Paid Chain-Ladder method is used instead of our claims reserving approach (we remind that in the latter approach, inflation is explicitly modelled). This is the first relevant point raised by the last analysis on inflation (i.e. Paid Chain-Ladder), depending on the fact that, in the classic approach, inflation is implicit in the amounts for paid claims of the upper part of the development triangle, and it is implicitly transferred to the lower part. We do not judge whether one estimate is better than the other, but we point out that the classic Paid Chain-Ladder method does not have the feature to explicitly model inflation, and this is a relevant limitation if we had high historical inflation rates, or we expect higher future inflation levels than in the past. Note that the claims reserves of this analysis (160.3 mln) are even lower than in the analysis with null expected inflation (170.2 mln), because in the latter case (according to our claims reserving approach) we added the historical inflation to the upper part of the development triangle. This was done to align the value of money to the current calendar year and to isolate the settlement process affecting the claims. We point out that our approach is equal to the classical Paid Chain-Ladder method if both the historical inflation and future expected inflation are null. Due to the decrease in claims reserves, we observe an increase in the initial risk reserve (from 25 mln to 44.7 mln, roughly +79%). Consequently, the same comments apply as applied to the analysis with null expected inflation. However, the capital requirements for premium risk are now equal to the base case, because the expected inflation is the same (we remind that we only changed the claims reserving method). Moreover, the capital requirements for reserve risk are now always lower than in the base case. Consequently, the RBC for non-life risk drops (in contrast to the analysis with null expected inflation, influenced by the premium risk), and we overall observe a decrease in the capital requirements for market and non-life risk, both in the SF (−4.7%) and in our IM (for instance, over the first year we have −2.7% and −3.9%, based on linear and non-linear dependence, respectively). Now, the impact of the reserve volume measure prevails on the IM ability to capture the different inflation modelling. This analysis remarks the importance of a correct consideration and estimation of inflation.

Conclusion

In this thesis, we showed that as the cash flows produced by the insurance business are invested, they consequently create a risk, above the non-life risk. This highlights that non-life insurers, not only face non-life risk but also market risk. We pointed out that claim amounts, claims reserves, equities, properties, and interest rates create relevant risks in non-life insurance. Consequently, we described an approach to modelling the distributions of the annual rate of return, total claim amount, claims reserve, and claims development result, in order to calculate the capital requirements for market and non-life risk and to illustrate their combined effect. We also included a stochastic model for inflation, explicitly affecting the total claim amount, claims reserve, and claims development result. We showed an extension of the Jarrow-Yildirim model, in which both the nominal short rate and real short rate are two-factor processes, adapting the formulation to the real-world measure. Consequently, we derived pricing formulas for the main inflation-indexed derivatives available in the market (used by us for calibration purposes). All these instruments depend on the CPI, and their no-arbitrage prices were obtained solving an expected value under the nominal risk-neutral measure (or other measures, by applying the change-of-numeraire technique). We finally produced a numerical analysis for a multi-line insurance company in a multi-annual dynamic perspective, using current and available market data, in order to show a realistic and heterogeneous non-life insurance context.

In this elaborate, we also showed that capital requirements are obviously more demanding from a methodological point of view when an IM is applied than when using the SF. Furthermore, notwithstanding that an IM must be approved by an insurance supervisory authority, the calibration is critical, because it influences the final result of the capital requirements. For this reason, supervisory authorities pay close attention to cases of model change.

We explained the main differences between our IM and the SF according to our numerical analysis, where market risk and non-life risk are examined in connection to each other. In particular, the SF results in higher capital requirements than our IM when using a linear dependence structure (ca. 80% against 67% as percentages of the initial GPW, see Table 6.14), while for a non-linear dependence structure it is the opposite situation (ca. 80% against 82%, see again Table 6.14). The difference in capital requirements of our IM and the SF is due to multiple reasons. Firstly, the SF is a simplified and standardised approach, and it is more conservative regarding the expected profits (not counted as a mitigation of risk). Focusing on the market risk, all equity instruments of a certain type and with a particular characteristic (e.g. strategic nature or long-term treatment) have equal capital requirements (depending on a given shock) under the SF, and the same logic is true for property instruments. Moreover, the interest rate shocks are not related to the market volatility, and the frequency of recalibration of the asset allocation during the year has no impact. Considering the non-life risk, no size factor is considered by the SF (consequently, the volatility factors are the same for each insurance company), and the multiplier 3 is fixed, irrespective of the relation between the skewness and volatility underlying the Lognormal distribution assumption. All these reasons explain why an IM can be used to better fit the various characteristics of a particular portfolio. The high upper tail dependence of the Gumbel copula explains the difference in capital requirements of our IM when using a non-linear dependence structure instead of a linear one.

We believe the analysis of multi-year modelling could be very promising, in order to derive a natural proxy as a benchmark for the capital requirement calculation on a time span longer than one year. In this thesis, we made some studies of a possible rule of thumb (represented by the square root of the time) using a period of two, and three years and the same risk measure. In our numerical analysis, we observed similar results to this rule of thumb, especially for the market risk. The differences with respect to the square root of time horizon are mainly given by the expected profits (on both market and underwriting sides) and volume increase time by time. This might be relevant in future, for instance in case EIOPA decides to modify the capital requirement metrics, lengthening the one-year time horizon (e.g. to a two-year or three-year time horizon) and introducing a multiple approach (e.g. a double approach, in which both the one-year and multi-year risk measures are taken into account). This would necessitate an update in the confidence levels and an enhancement of risk strategies to take a medium-term view.

We finally provided some additional analyses. Firstly, the sensitivity analyses allowed us to see the effect of a change in nominal or inflation rates equal to what we observed between year end 2021 and year end 2022. In second place, the what-if analyses helped us to better understand the impact of doing the numerical analysis with a modified initial assumption, namely nominal rates as of year end 2022, a deterministic inflation model instead of a stochastic one, null risk premiums, and a single-line MTPL insurer instead of a multi-line one. Furthermore, the inflation analyses allowed us to investigate the role of inflation in our framework. We showed that the increase in nominal rates observed between year end 2021 and year end 2022 has a considerable lowering effect on capital requirements of the sensitivity analysis (-5.2% under the SF with respect to the base case, -3.2% and -3.9% according to our IM over the first year, based on linear and non-linear dependence, respectively, as shown in Figure 6.8). This mainly depends on the reduction in claims reserve and interest-sensitive assets. The same analysis on nominal rates has the opposite effect on capital requirements of the what-if analysis (inconsiderable change under the SF, $+0.8\%$ and $+1.0\%$ according to our IM over the first year, based on linear and non-linear dependence, respectively, as shown in Figure 6.10). Indeed, in this case we do not have any change in the initial asset value of the total portfolio, because the development triangles are recalibrated, raising the undiscounted incremental amounts for paid claims (driving to a higher non-life risk). We also showed that the increase in inflation rates observed between year end 2021 and year end 2022 has some impact (not huge) on capital requirements of the sensitivity analysis (i.e. increase, following to a raise in the claims reserve). In the same way, the advantage (in terms of capital requirements appropriateness) of considering a stochastic model for inflation is limited (i.e. RBC only slightly increases when considering a stochastic model for inflation). It is evident that the main impact of inflation is its level, and this impact is mitigated if we make a correct inflation estimate. However, as shown in Figure 6.13, there is a considerable decrease or increase in capital requirements if we make a wrong inflation estimate incoherent with the economic scenario (for instance, with a null expected inflation instead of the correct one, we have −2.8% under the SF, +2.8% and +2.4% according to our IM over the first year, based on linear and non-linear dependence, respectively). In addition to what just described, we showed that the absence of risk premiums (i.e. expected profit coming from the economic scenario) have a considerable impact on capital requirements according to our IM (+4.9% over the first year, based on both linear and non-linear dependence, as shown again in Figure 6.10). We remind that the SF does not consider any form of expected profit, therefore in this analysis the capital requirements according to our IM based on linear dependence (we refer to linear dependence, because the SF has a dependence structure based on linear correlation coefficients) become more similar to the the capital requirements under the SF. In our view, a correct quantification of the solvency position of an insurance company should keep into account not only the risk items but also the risk mitigation elements (at least partially), and now this is only possible using an IM. Finally, we showed that the single-line MTPL insurer has much lower capital requirements than the multi-line insurance company (−5.2% under the SF, −12.6% and −19.8% according to our IM over the first year, based on linear and non-linear dependence, respectively, as shown again in Figure 6.10). Focusing on the market risk, the single-line capital requirements are lower for both the SF and our IM because of smaller claims reserve given by the absent GTPL characteristic of extremely high ratio of claims reserve to GPW (400%). Considering the premium risk, the single-line capital requirements are bigger than the multi-line ones if the SF approach is adopted, due to the lack of diversification benefit. By contrast, for our IM the single-line capital requirements are lower than in the multi-line case, because there is not anymore the high volatility and skewness of GTPL, which was not counterbalanced by either the limited values registered for MOD or the diversification benefit. Moreover, the single-line capital requirements for reserve risk are lower for both the SF and our IM because of smaller claims reserve and absence of GTPL riskiness.

We remind that our IM can only be accounted as partial, because a full IM would obviously consider all the sources of risk (e.g. cat risk, counterparty default risk, operational risk, ...). Moreover, the analysis was performed with simple assets expressed in euro currency and issued by EEA governments, which means many important sources of market risk (i.e. spread risk, liquidity risk, and default risk) were not considered. Further studies can regard, for instance, the introduction of other sources of risk or a more complex and dynamic asset allocation to better describe a real investment portfolio and to further exploit the diversification benefit of the overall market risk. Also, an additional local risk premium for the CPI could be introduced in the real-world economic scenario generator, and the parameters or correlations previously set to zero could be calibrated and considered.

Appendix A

Financial derivatives pricing

A.1 Year-on-year inflation-indexed swap pricing

In this appendix, we solve the expected value of equation 3.9, according to the model described in Section 3.1. We apply the change-of-numeraire technique, and we obtain that the real state variables under the nominal forward measure Q_n^T satisfy the following stochastic differential equations:

$$dx_r(t) = \Big[-a_r\, x_r(t) - \rho_{x_r,I}\, \sigma_r\, \sigma_I - \frac{\rho_{x_n,x_r}\, \sigma_n\, \sigma_r}{a_n}\,(1 - e^{-a_n(T-t)}) - \frac{\rho_{y_n,x_r}\, \eta_n\, \sigma_r}{b_n}\,(1 - e^{-b_n(T-t)})\Big]\, dt + \sigma_r\, dW^T_{n,x_r}(t)$$

$$dy_r(t) = \Big[-b_r\, y_r(t) - \rho_{y_r,I}\, \eta_r\, \sigma_I - \frac{\rho_{x_n,y_r}\, \sigma_n\, \eta_r}{a_n}\,(1 - e^{-a_n(T-t)}) - \frac{\rho_{y_n,y_r}\, \eta_n\, \eta_r}{b_n}\,(1 - e^{-b_n(T-t)})\Big]\, dt + \eta_r\, dW^T_{n,y_r}(t)$$

The stochastic differential equations above have explicit solutions that are given by:

$$x_r(t) = x_r(s)\, e^{-a_r(t-s)} - M^T_{n,x_r}(s,t) + \sigma_r \int_s^t e^{-a_r(t-u)}\, dW^T_{n,x_r}(u)$$

$$y_r(t) = y_r(s)\, e^{-b_r(t-s)} - M^T_{n,y_r}(s,t) + \eta_r \int_s^t e^{-b_r(t-u)}\, dW^T_{n,y_r}(u)$$

where:

$$M^T_{n,x_r}(s,t) = \frac{\rho_{x_n,x_r}\, \sigma_n\, \sigma_r}{a_n}\left[\frac{1 - e^{-a_r(t-s)}}{a_r} - \frac{e^{-a_n(T-t)} - e^{-a_n(T-s)-a_r(t-s)}}{a_n + a_r}\right] + \frac{\rho_{y_n,x_r}\, \eta_n\, \sigma_r}{b_n}\left[\frac{1 - e^{-a_r(t-s)}}{a_r} - \frac{e^{-b_n(T-t)} - e^{-b_n(T-s)-a_r(t-s)}}{b_n + a_r}\right] + \rho_{x_r,I}\, \sigma_r\, \sigma_I\, \frac{1 - e^{-a_r(t-s)}}{a_r}$$

$$M^T_{n,y_r}(s,t) = \frac{\rho_{x_n,y_r}\,\sigma_n\,\eta_r}{a_n}\left[\frac{1-e^{-b_r(t-s)}}{b_r} - \frac{e^{-a_n(T-t)} - e^{-a_n(T-s)-b_r(t-s)}}{a_n+b_r}\right]$$
$$+ \frac{\rho_{y_n,y_r}\,\eta_n\,\eta_r}{b_n}\left[\frac{1-e^{-b_r(t-s)}}{b_r} - \frac{e^{-b_n(T-t)} - e^{-b_n(T-s)-b_r(t-s)}}{b_n+b_r}\right]$$
$$+ \rho_{y_r,I}\,\eta_r\,\sigma_I\,\frac{1-e^{-b_r(t-s)}}{b_r}$$

Hence, the vector of the real state variables under the nominal forward measure Q^T_n and conditional on the sigma-field $\mathcal{F}_s$ is normally distributed, with mean vector and variance-covariance matrix given by:

$$\begin{bmatrix}\mu^T_{n,x_r}(s,t)\\ \mu^T_{n,y_r}(s,t)\end{bmatrix} = \begin{bmatrix}x_r(s)\,e^{-a_r(t-s)} - M^T_{n,x_r}(s,t)\\ y_r(s)\,e^{-b_r(t-s)} - M^T_{n,y_r}(s,t)\end{bmatrix}$$

$$\begin{bmatrix}\delta_{x_r}(s,t) & \delta_{x_r,y_r}(s,t)\\ \cdot & \delta_{y_r}(s,t)\end{bmatrix} = \begin{bmatrix}\frac{\sigma_r^2}{2a_r}\left(1-e^{-2a_r(t-s)}\right) & \frac{\rho_r\,\sigma_r\,\eta_r}{a_r+b_r}\left(1-e^{-(a_r+b_r)(t-s)}\right)\\ \cdot & \frac{\eta_r^2}{2b_r}\left(1-e^{-2b_r(t-s)}\right)\end{bmatrix}$$

Hence, the real zero-coupon bond price $P_r(T_{i-1},T_i)$ under the nominal forward measure $Q^{T_{i-1}}_n$ and conditional on the sigma-field $\mathcal{F}_t$ is lognormally distributed. For this reason, equation 3.9 can be solved using the expected value and variance of the logarithm of the real zero-coupon bond price. Let X be a Lognormal random variable with $E(\ln X) = m$ and $\mathrm{Std}(\ln X) = v$, we thus have:

$$E(X) = \exp\left\{m + \frac{v^2}{2}\right\}$$

Using equation 3.3 (replacing the sub and superscripts n with r), the expected value of the logarithm of the real zero-coupon bond price is found to be:

$$E^{T_{i-1}}_n\{\ln P_r(T_{i-1},T_i) \mid \mathcal{F}_t\} = \ln A_r(T_{i-1},T_i) - B(a_r,T_{i-1},T_i)\,\mu^{T_{i-1}}_{n,x_r}(t,T_{i-1})$$
$$- B(b_r,T_{i-1},T_i)\,\mu^{T_{i-1}}_{n,y_r}(t,T_{i-1})$$

Rearranging the terms, we obtain:

$$E^{T_{i-1}}_n\{\ln P_r(T_{i-1},T_i) \mid \mathcal{F}_t\}$$
$$= \ln\frac{P^M_r(0,T_i)}{P^M_r(0,T_{i-1})} + \frac{1}{2}\left[V_r(T_{i-1},T_i) - V_r(0,T_i) + V_r(0,T_{i-1})\right]$$
$$- B(a_r,t,T_i)\,x_r(t) + B(a_r,t,T_{i-1})\,x_r(t) + B(a_r,T_{i-1},T_i)\,M^{T_{i-1}}_{n,x_r}(t,T_{i-1})$$
$$- B(b_r,t,T_i)\,y_r(t) + B(b_r,t,T_{i-1})\,y_r(t) + B(b_r,T_{i-1},T_i)\,M^{T_{i-1}}_{n,y_r}(t,T_{i-1})$$

On the other hand, the variance of the logarithm of the real zero-coupon bond price is given by:

$$\mathrm{Var}_n^{T_{i-1}}\{\ln P_r(T_{i-1},T_i) \mid \mathcal{F}_t\} = B(a_r,T_{i-1},T_i)^2\,\delta_{x_r}(t,T_{i-1})$$
$$+\,B(b_r,T_{i-1},T_i)^2\,\delta_{y_r}(t,T_{i-1}) + 2B(a_r,T_{i-1},T_i)\,B(b_r,T_{i-1},T_i)\,\delta_{x_r,y_r}(t,T_{i-1})$$

For simplification purposes, we observe that:

$$\frac{P_r(t,T_i)}{P_r(t,T_{i-1})} = \frac{P_r^M(0,T_i)}{P_r^M(0,T_{i-1})}\,\exp\Bigg\{\frac{1}{2}\,[V_r(t,T_i) - V_r(0,T_i)$$
$$-\,V_r(t,T_{i-1}) + V_r(0,T_{i-1})] - B(a_r,t,T_i)\,x_r(t) - B(b_r,t,T_i)\,y_r(t)$$
$$+\,B(a_r,t,T_{i-1})\,x_r(t) + B(b_r,t,T_{i-1})\,y_r(t)\Bigg\}$$

As a result, we have:

$$E_n^{T_{i-1}}\{P_r(T_{i-1},T_i) \mid \mathcal{F}_t\} = \frac{P_r(t,T_i)}{P_r(t,T_{i-1})}\,e^{C(t,T_{i-1},T_i)}$$

where:

$$C(t,T_{i-1},T_i) = \Delta(t,T_{i-1},T_i) + B(a_r,T_{i-1},T_i)\,M_{n,x_r}^{T_{i-1}}(t,T_{i-1})$$
$$+\,B(b_r,T_{i-1},T_i)\,M_{n,y_r}^{T_{i-1}}(t,T_{i-1})$$

and:

$$\Delta(t,T_{i-1},T_i) = \frac{1}{2}\,[V_r(T_{i-1},T_i) - V_r(t,T_i) + V_r(t,T_{i-1})$$
$$+\,B(a_r,T_{i-1},T_i)^2\,\delta_{x_r}(t,T_{i-1}) + B(b_r,T_{i-1},T_i)^2\,\delta_{y_r}(t,T_{i-1})]$$
$$+\,B(a_r,T_{i-1},T_i)\,B(b_r,T_{i-1},T_i)\,\delta_{x_r,y_r}(t,T_{i-1})$$

In order to simplify the latter equation, we have done some straightforward algebra. Firstly, we have focused on the addends affected by the variance of each state variable and then on the remaining addends, i.e. the elements affected by the covariance between the state variables. Hence, we have:

$$\Delta(t,T_{i-1},T_i) = -\frac{\sigma_r^2}{2}\,B(a_r,t,T_{i-1})^2\,B(a_r,T_{i-1},T_i)$$
$$-\,\frac{\eta_r^2}{2}\,B(b_r,t,T_{i-1})^2\,B(b_r,T_{i-1},T_i)$$
$$-\,\frac{\rho_r\,\sigma_r\,\eta_r}{a_r\,b_r}\Bigg[(B(a_r,T_{i-1},T_i) + B(b_r,T_{i-1},T_i) - B(a_r,t,T_i) - B(b_r,t,T_i)$$
$$+\,B(a_r,t,T_{i-1}) + B(b_r,t,T_{i-1})) + \frac{1}{a_r+b_r}\,(a_r\,b_r\,B(a_r,t,T_{i-1})\,B(b_r,t,T_{i-1})$$

$$- a_r\, B(a_r,t,T_{i-1}) - b_r\, B(b_r,t,T_{i-1})\big)\big(a_r\, B(a_r,T_{i-1},T_i) + b_r\, B(b_r,T_{i-1},T_i)\big)\Bigg]$$

Rearranging the terms, we finally obtain:

$$C(t,T_{i-1},T_i) = \sigma_r\, B(a_r,T_{i-1},T_i)\Bigg[B(a_r,t,T_{i-1})\Big(\rho_{x_r,I}\,\sigma_I - \frac{\sigma_r}{2}\, B(a_r,t,T_{i-1})$$

$$+ \frac{\rho_{x_n,x_r}\,\sigma_n}{a_n+a_r}\,(1 + a_r\, B(a_n,t,T_{i-1})) + \frac{\rho_{y_n,x_r}\,\eta_n}{b_n+a_r}\,(1 + a_r\, B(b_n,t,T_{i-1}))\Big)$$

$$- \frac{\rho_{x_n,x_r}\,\sigma_n}{a_n+a_r}\, B(a_n,t,T_{i-1}) - \frac{\rho_{y_n,x_r}\,\eta_n}{b_n+a_r}\, B(b_n,t,T_{i-1})\Bigg]$$

$$+ \eta_r\, B(b_r,T_{i-1},T_i)\Bigg[B(b_r,t,T_{i-1})\Big(\rho_{y_r,I}\,\sigma_I - \frac{\eta_r}{2}\, B(b_r,t,T_{i-1})$$

$$+ \frac{\rho_{x_n,y_r}\,\sigma_n}{a_n+b_r}\,(1 + b_r\, B(a_n,t,T_{i-1})) + \frac{\rho_{y_n,y_r}\,\eta_n}{b_n+b_r}\,(1 + b_r\, B(b_n,t,T_{i-1}))\Big)$$

$$- \frac{\rho_{x_n,y_r}\,\sigma_n}{a_n+b_r}\, B(a_n,t,T_{i-1}) - \frac{\rho_{y_n,y_r}\,\eta_n}{b_n+b_r}\, B(b_n,t,T_{i-1})\Bigg]$$

$$- \frac{\rho_r\,\sigma_r\,\eta_r}{a_r\, b_r}\Bigg[\big(B(a_r,T_{i-1},T_i) + B(b_r,T_{i-1},T_i) - B(a_r,t,T_i) - B(b_r,t,T_i)$$

$$+ B(a_r,t,T_{i-1}) + B(b_r,t,T_{i-1})\big) + \frac{1}{a_r+b_r}\,\big(a_r\, b_r\, B(a_r,t,T_{i-1})\, B(b_r,t,T_{i-1})$$

$$- a_r\, B(a_r,t,T_{i-1}) - b_r\, B(b_r,t,T_{i-1})\big)\big(a_r\, B(a_r,T_{i-1},T_i) + b_r\, B(b_r,T_{i-1},T_i)\big)\Bigg]$$

A.2 Inflation-indexed cap and floor pricing

In this appendix, we get the variance of the logarithm of the CPI ratio, according to the model described in Section 3.1, which is required to solve the expected value of equation 3.10. It is important to remind that the variance is not affected by the change-of-numeraire technique, so according to equation 3.4 we have:

$$\mathrm{Var}_n^{T_i}\Big\{\ln\frac{I(T_i)}{I(T_{i-1})} \mid \mathcal{F}_t\Big\}$$

$$= \mathrm{Var}_n\Big\{\int_{T_{i-1}}^{T_i}(n(u) - r(u))\,du + \sigma_I\,(W_n^I(T_i) - W_n^I(T_{i-1})) \mid \mathcal{F}_t\Big\}$$

The solution of the latter equation is found to be to the sum of all its variance and covariance components. The demonstration is the same as in the original version of the Jarrow-Yildirim model (see Mercurio, 2005), but now we have twice the number of state variables. The variance component depending on the first state variable is given by:

$$\gamma_{x_n}(t, T_{i-1}, T_i) = \frac{\sigma_n^2}{2a_n^3} \left(1 - e^{-a_n \zeta_i}\right)^2 \left(1 - e^{-2a_n (T_{i-1} - t)}\right)$$
$$+ \frac{\sigma_n^2}{a_n^2} \left[\zeta_i + \frac{2}{a_n} e^{-a_n \zeta_i} - \frac{1}{2a_n} e^{-2a_n \zeta_i} - \frac{3}{2a_n}\right]$$

On the other hand, the variance component depending on the CPI is given by:

$$\gamma_I(t, T_{i-1}, T_i) = \sigma_I^2 \, \zeta_i$$

Moreover, the covariance component depending on the first two state variables is given by:

$$\gamma_{x_n, y_n}(t, T_{i-1}, T_i) = \frac{\rho_n \, \sigma_n \, \eta_n}{a_n \, b_n \, (a_n + b_n)} \left(1 - e^{-a_n \zeta_i}\right) \left(1 - e^{-b_n \zeta_i}\right) \left(1 - e^{-(a_n + b_n)(T_{i-1} - t)}\right)$$
$$+ \frac{\rho_n \, \sigma_n \, \eta_n}{a_n \, b_n} \left[\zeta_i - \frac{1 - e^{-a_n \zeta_i}}{a_n} - \frac{1 - e^{-b_n \zeta_i}}{b_n} + \frac{1 - e^{-(a_n + b_n)\zeta_i}}{a_n + b_n}\right]$$

On the other hand, the covariance component depending on the first state variable and the CPI is given by:

$$\gamma_{x_n, I}(t, T_{i-1}, T_i) = \frac{\rho_{x_n, I} \, \sigma_n \, \sigma_I}{a_n} \left[\zeta_i - \frac{1 - e^{-a_n \zeta_i}}{a_n}\right]$$

The remaining variance and covariance components are analogous to the preceding ones, and they can be found by replacing the parameters. Including all the terms, we finally obtain:

$$\mathrm{Var}_n^{T_i}\left\{\ln \frac{I(T_i)}{I(T_{i-1})} \mid \mathcal{F}_t\right\} = \frac{\sigma_n^2}{2a_n^3} \left(1 - e^{-a_n \zeta_i}\right)^2 \left(1 - e^{-2a_n (T_{i-1} - t)}\right)$$
$$+ \frac{\sigma_n^2}{a_n^2} \left[\zeta_i + \frac{2}{a_n} e^{-a_n \zeta_i} - \frac{1}{2a_n} e^{-2a_n \zeta_i} - \frac{3}{2a_n}\right]$$
$$+ \frac{\eta_n^2}{2b_n^3} \left(1 - e^{-b_n \zeta_i}\right)^2 \left(1 - e^{-2b_n (T_{i-1} - t)}\right)$$
$$+ \frac{\eta_n^2}{b_n^2} \left[\zeta_i + \frac{2}{b_n} e^{-b_n \zeta_i} - \frac{1}{2b_n} e^{-2b_n \zeta_i} - \frac{3}{2b_n}\right]$$
$$+ \frac{\sigma_r^2}{2a_r^3} \left(1 - e^{-a_r \zeta_i}\right)^2 \left(1 - e^{-2a_r (T_{i-1} - t)}\right)$$
$$+ \frac{\sigma_r^2}{a_r^2} \left[\zeta_i + \frac{2}{a_r} e^{-a_r \zeta_i} - \frac{1}{2a_r} e^{-2a_r \zeta_i} - \frac{3}{2a_r}\right]$$

$$
\begin{aligned}
&+ \frac{\eta_r^2}{2b_r^3}\left(1 - e^{-b_r\zeta_i}\right)^2\left(1 - e^{-2b_r(T_{i-1}-t)}\right) \\
&+ \frac{\eta_r^2}{b_r^2}\left[\zeta_i + \frac{2}{b_r}\,e^{-b_r\zeta_i} - \frac{1}{2b_r}\,e^{-2b_r\zeta_i} - \frac{3}{2b_r}\right] + \sigma_I^2\,\zeta_i \\
&+ \frac{2\rho_n\,\sigma_n\,\eta_n}{a_n\,b_n\,(a_n+b_n)}\left(1 - e^{-a_n\zeta_i}\right)\left(1 - e^{-b_n\zeta_i}\right)\left(1 - e^{-(a_n+b_n)(T_{i-1}-t)}\right) \\
&+ \frac{2\rho_n\,\sigma_n\,\eta_n}{a_n\,b_n}\left[\zeta_i - \frac{1 - e^{-a_n\zeta_i}}{a_n} - \frac{1 - e^{-b_n\zeta_i}}{b_n} + \frac{1 - e^{-(a_n+b_n)\zeta_i}}{a_n+b_n}\right] \\
&+ \frac{2\rho_r\,\sigma_r\,\eta_r}{a_r\,b_r\,(a_r+b_r)}\left(1 - e^{-a_r\zeta_i}\right)\left(1 - e^{-b_r\zeta_i}\right)\left(1 - e^{-(a_r+b_r)(T_{i-1}-t)}\right) \\
&+ \frac{2\rho_r\,\sigma_r\,\eta_r}{a_r\,b_r}\left[\zeta_i - \frac{1 - e^{-a_r\zeta_i}}{a_r} - \frac{1 - e^{-b_r\zeta_i}}{b_r} + \frac{1 - e^{-(a_r+b_r)\zeta_i}}{a_r+b_r}\right] \\
&- \frac{2\rho_{x_n,x_r}\,\sigma_n\,\sigma_r}{a_n\,a_r\,(a_n+a_r)}\left(1 - e^{-a_n\zeta_i}\right)\left(1 - e^{-a_r\zeta_i}\right)\left(1 - e^{-(a_n+a_r)(T_{i-1}-t)}\right) \\
&- \frac{2\rho_{x_n,x_r}\,\sigma_n\,\sigma_r}{a_n\,a_r}\left[\zeta_i - \frac{1 - e^{-a_n\zeta_i}}{a_n} - \frac{1 - e^{-a_r\zeta_i}}{a_r} + \frac{1 - e^{-(a_n+a_r)\zeta_i}}{a_n+a_r}\right] \\
&- \frac{2\rho_{x_n,y_r}\,\sigma_n\,\eta_r}{a_n\,b_r\,(a_n+b_r)}\left(1 - e^{-a_n\zeta_i}\right)\left(1 - e^{-b_r\zeta_i}\right)\left(1 - e^{-(a_n+b_r)(T_{i-1}-t)}\right) \\
&- \frac{2\rho_{x_n,y_r}\,\sigma_n\,\eta_r}{a_n\,b_r}\left[\zeta_i - \frac{1 - e^{-a_n\zeta_i}}{a_n} - \frac{1 - e^{-b_r\zeta_i}}{b_r} + \frac{1 - e^{-(a_n+b_r)\zeta_i}}{a_n+b_r}\right] \\
&- \frac{2\rho_{y_n,x_r}\,\eta_n\,\sigma_r}{b_n\,a_r\,(b_n+a_r)}\left(1 - e^{-b_n\zeta_i}\right)\left(1 - e^{-a_r\zeta_i}\right)\left(1 - e^{-(b_n+a_r)(T_{i-1}-t)}\right) \\
&- \frac{2\rho_{y_n,x_r}\,\eta_n\,\sigma_r}{b_n\,a_r}\left[\zeta_i - \frac{1 - e^{-b_n\zeta_i}}{b_n} - \frac{1 - e^{-a_r\zeta_i}}{a_r} + \frac{1 - e^{-(b_n+a_r)\zeta_i}}{b_n+a_r}\right] \\
&- \frac{2\rho_{y_n,y_r}\,\eta_n\,\eta_r}{b_n\,b_r\,(b_n+b_r)}\left(1 - e^{-b_n\zeta_i}\right)\left(1 - e^{-b_r\zeta_i}\right)\left(1 - e^{-(b_n+b_r)(T_{i-1}-t)}\right) \\
&- \frac{2\rho_{y_n,y_r}\,\eta_n\,\eta_r}{b_n\,b_r}\left[\zeta_i - \frac{1 - e^{-b_n\zeta_i}}{b_n} - \frac{1 - e^{-b_r\zeta_i}}{b_r} + \frac{1 - e^{-(b_n+b_r)\zeta_i}}{b_n+b_r}\right] \\
&+ \frac{2\rho_{x_n,I}\,\sigma_n\,\sigma_I}{a_n}\left[\zeta_i - \frac{1 - e^{-a_n\zeta_i}}{a_n}\right] + \frac{2\rho_{y_n,I}\,\eta_n\,\sigma_I}{b_n}\left[\zeta_i - \frac{1 - e^{-b_n\zeta_i}}{b_n}\right] \\
&- \frac{2\rho_{x_r,I}\,\sigma_r\,\sigma_I}{a_r}\left[\zeta_i - \frac{1 - e^{-a_r\zeta_i}}{a_r}\right] - \frac{2\rho_{y_r,I}\,\eta_r\,\sigma_I}{b_r}\left[\zeta_i - \frac{1 - e^{-b_r\zeta_i}}{b_r}\right]
\end{aligned}
$$

A.3 European stock-option pricing

In this appendix, we get the variance of the logarithm of the non-dividend-paying stock price, according to the model described in Section 3.1, which is required to solve the expected value of equation 3.12. It is important to remind that the variance is not affected by the change-of-numeraire technique, so according to equation 3.5 we have:

$$\mathrm{Var}_n^T\{\ln S(T) \mid \mathcal{F}_t\} = \mathrm{Var}_n\left\{\int_t^T n(u)\,du + \sigma_S\left(W_n^S(T) - W_n^S(t)\right) \mid \mathcal{F}_t\right\}$$

The solution of the latter equation is found to be to the sum of all its variance and covariance components. We point out that the variance-covariance component depending on the nominal state variables is given by equation 3.2. The variance component depending on the non-dividend-paying stock is given by:

$$\gamma_S(t,T) = \sigma_S^2\,(T-t)$$

Moreover, the covariance component depending on the first nominal state variable and the non-dividend-paying stock is given by:

$$\gamma_{x_n,S}(t,T) = \frac{\rho_{x_n,S}\,\sigma_n\,\sigma_S}{a_n}\left[T - t - \frac{1-e^{-a_n(T-t)}}{a_n}\right]$$

The covariance component depending on the second nominal state variable and the non-dividend-paying stock is analogous to the preceding one, and it can be found by replacing the parameters. Including all the terms, we finally obtain:

$$\mathrm{Var}_n^T\{\ln S(T) \mid \mathcal{F}_t\} = V_n(t,T) + \sigma_S^2\,(T-t) + \frac{2\rho_{x_n,S}\,\sigma_n\,\sigma_S}{a_n}\left[T - t - \frac{1-e^{-a_n(T-t)}}{a_n}\right]$$

$$+ \frac{2\rho_{y_n,S}\,\eta_n\,\sigma_S}{b_n}\left[T - t - \frac{1-e^{-b_n(T-t)}}{b_n}\right]$$

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