SHARP ASYMPTOTIC BEHAVIOR OF RADIAL SOLUTIONS OF SOME PLANAR SEMILINEAR ELLIPTIC PROBLEMS

ISABELLA IANNI AND ALBERTO SALDAÑA

Abstract. We consider the equation $-\Delta u = |x|^\alpha |u|^{p-1}u$ for any $\alpha \geq 0$, either in $\mathbb{R}^2$ or in the unit ball B of $\mathbb{R}^2$ centered at the origin with Dirichlet or Neumann boundary conditions. We give a sharp description of the asymptotic behavior as $p \to +\infty$ of all the radial solutions to these problems and we show that there is no uniform a priori bound for nodal solutions under Neumann or Dirichlet boundary conditions. This contrasts with the existence of uniform bounds for positive solutions, as shown in [32] for $\alpha = 0$ and Dirichlet boundary conditions.

1. Introduction

We consider the equation

$$-\Delta u = |x|^\alpha |u|^{p-1}u \qquad \text{in } \mathbb{R}^2, \tag{1.1}$$

where $\alpha \geq 0$ and $p > 1$, we also consider the problem in the unit ball B of $\mathbb{R}^2$ centered at the origin

$$-\Delta u = |x|^\alpha |u|^{p-1}u \qquad \text{in } B, \tag{1.2}$$

together with either Dirichlet boundary conditions

$$u = 0 \qquad \text{on } \partial B \tag{1.3}$$

or Neumann boundary conditions

$$\partial_\nu u = 0 \qquad \text{on } \partial B. \tag{1.4}$$

When $\alpha = 0$ these are commonly known as *Lane-Emden* problems, while when $\alpha > 0$ they were introduced first by M. Hénon in [30] and so are usually referred to as *Hénon* problems. As we will see the existence of radial solutions for all these planar problems can be easily proved for any $p > 1$.

In this work we give a sharp description of the asymptotic behavior of all the radial solutions to the problems (1.1), (1.2)-(1.3) and (1.2)-(1.4) as the exponent $p \to +\infty$, for any fixed $\alpha \geq 0$.

The interest in the study of the asymptotic behavior as $p \to +\infty$ for these 2-dimensional problems started from the seminal works by Ren and Wei ([37, 38]), which concern the study of the least energy solutions for the Lane-Emden equation ($\alpha = 0$) in general smooth bounded domains, under Dirichlet boundary conditions. In this case, the least energy solutions u_p (which are *positive*) exhibit a single point concentration and $u_p \to 0$ locally uniformly outside the concentration point as $p \to +\infty$. Moreover, differently from the almost critical higher dimensional case (which is much more studied, see for instance [5, 7, 29]), these solutions do not blow-up but stay uniformly bounded (in p) in L^∞ and (see [1]) $\|u_p\|_{L^\infty} \to \sqrt{e}$ as $p \to +\infty$.

Recently, an *a priori uniform bound* (in p) has been proven in [32] *for any positive solution* of the Dirichlet Lane-Emden problem, in any smooth bounded planar domain. More precisely it has been proved that, for any $p_0 > 1$, there exists a constant $C > 0$, which depends only on the domain and

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p_0, such that, for all $p \geq p_0$, any solution u_p of the Lane-Emden problem with Dirichlet condition on the boundary of the domain satisfies that

$$\|u_p\|_{L^\infty} \leq C. \tag{1.5}$$

Moreover in [17] a complete description of the asymptotic behavior of any positive solution of the Dirichlet Lane-Emden problem has been given, in any smooth bounded planar domain and under a uniform energy bound assumption, showing simple concentration at a finite number of distinct points, convergence to 0 locally uniformly outside the concentration set, convergence of the L^∞-norm to $\sqrt{e}$ and energy quantization to integer multiples of the value $8\pi e$ in the limit as $p \to +\infty$ (for the sharp quantization result and L^∞-norm limit behavior see [15] and also [42]). In particular, when the domain is the ball B centered at 0, then the Dirichlet Lane-Emden problem (i.e. (1.2)-(1.3) with $\alpha = 0$) admits a unique positive solution (which is the least energy) which is radial and strictly decreasing in the radial variable (by the symmetry result in [23]), so the unique concentration point is necessarily the origin, which is the maximum point, and one has that

$$u_p(0) = \|u_p\|_{L^\infty} \to \sqrt{e} \qquad \text{as } p \to +\infty, \tag{1.6}$$

$$pu_p(x) \to 4\sqrt{e}\log\frac{1}{|x|} \qquad \text{in } C^1_{loc}(\bar{B} \setminus \{0\}) \text{ as } p \to +\infty, \tag{1.7}$$

from which the following convergence of the derivative on the boundary directly follows

$$p|(u_p)'(1)| \to 4\sqrt{e} \qquad \text{as } p \to +\infty;$$

furthermore, the energy satisfies that

$$p\int_0^1 |(u_p)'(r)|^2 r\,dr \to 4e \qquad \text{as } p \to +\infty. \tag{1.8}$$

From [1] (see also [2, 21]) we also know that a suitable scaling of u_p converges to a radial solution Z_0 of the Liouville equation in the whole $\mathbb{R}^2$

$$\begin{cases} -\Delta Z_0 = e^{Z_0} & \text{in } \mathbb{R}^2, \\ Z_0(0) = 0, \\ \int_{\mathbb{R}^2} e^{Z_0} dx = 8\pi. \end{cases} \tag{1.9}$$

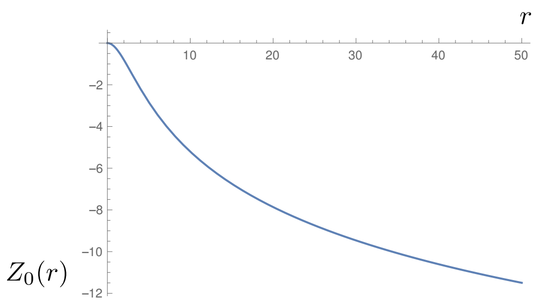

FIGURE 1. The solution of (1.9).

Concerning *sign-changing* solutions of the planar Dirichlet Lane-Emden problem, from [16] it is known that, under energy uniform bounds in p, nodal solutions are uniformly bounded in p, concentrate at a finite number of points and converge to zero locally uniformly outside the concentration set as $p \to +\infty$ (see also [26], where low energy sign-changing solutions have been studied).

In this paper we show that an a priori bound as in (1.5) does *not* hold in general for sign-changing solutions (see Theorem 2.8). Moreover, differently from the positive case, as $p \to +\infty$ the concentration may not be simple and a tower of bubbles may appear as shown in [27] in the ball B and later generalized to other symmetric domains in [16].

In particular, when the domain is the unit ball B centered at 0, it is known that the Dirichlet Lane-Emden problem (1.2)-(1.3) (with $\alpha = 0$) admits infinitely many radial solutions, one (up to sign) for each fixed number m of nodal regions, and they all concentrate only at the origin as $p \to +\infty$, where their absolute maximum (up to sign) is attained.
The asymptotic behavior of the *nodal radial solutions with* $m = 2$ *nodal regions* has been analyzed in full detail in [27]. The nodal radius r_p of these solutions shrinks to the origin as $p \to +\infty$ and there exists an explicit constant $\theta \sim 10.374$ (so that $t := \frac{4\sqrt{e}}{\theta-2}$ is the unique solution of the equation $2\sqrt{e}\log t + t = 0$) such that

$$|u_p(0)| = \|u_p\|_{L^\infty} \to \frac{\theta-2}{4} e^{\frac{2}{\theta+2}} (> \sqrt{e}) \qquad \text{as } p \to +\infty, \tag{1.10}$$

$$p|u_p(x)| \to (\theta+2)e^{\frac{2}{\theta+2}} \log\frac{1}{|x|} \qquad \text{in } C^1_{loc}(\bar{B}\setminus\{0\}) \text{ as } p \to +\infty. \tag{1.11}$$

From this convergence the following convergence of the derivative on the boundary directly follows

$$|p(u_p)'(1)| \to (\theta+2)e^{\frac{2}{\theta+2}} \qquad \text{as } p \to +\infty.$$

Moreover, in [27] it is proved that the nodal radius r_p satisfies

$$(r_p)^{\frac{2}{p-1}} \to \frac{4\sqrt{e}}{(\theta-2)} e^{-\frac{2}{\theta+2}} \qquad \text{as } p \to +\infty \tag{1.12}$$

and

$$|p(u_p)'(r_p)|r_p \to (\theta-2)e^{\frac{2}{\theta+2}} \qquad \text{as } p \to +\infty. \tag{1.13}$$

Furthermore, denoting by s_p the unique minimum (up to sign) of u_p, one has that it shrinks to 0,

$$(s_p)^{\frac{2}{p-1}} \to e^{-\frac{2}{\theta+2}} \qquad \text{as } p \to +\infty \tag{1.14}$$

and

$$|u_p(s_p)| \to e^{\frac{2}{\theta+2}} (< \sqrt{e}) \qquad \text{as } p \to +\infty. \tag{1.15}$$

Finally, the asymptotic value of the energy is explicitly determined

$$p\int_0^1 |(u_p)'(r)|^2 r\,dr \to \frac{(\theta+2)^2}{4} e^{\frac{4}{\theta+2}} \qquad \text{as } p \to +\infty. \tag{1.16}$$

The following is also known: a suitable scaling of u_p^+ converges to the solution Z_0 of the same limit problem (1.9) already involved in the asymptotic of the positive solution, but a suitable rescaling of u_p^- converges to a radial solution of a *different* limit problem, that is, the singular Liouville equation

$$\begin{cases} -\Delta Z = e^Z + 2\pi(2-\theta)\delta_0 & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^Z dx = 4\pi\theta, \end{cases} \tag{1.17}$$

where δ_0 is the Dirac measure centered at 0.

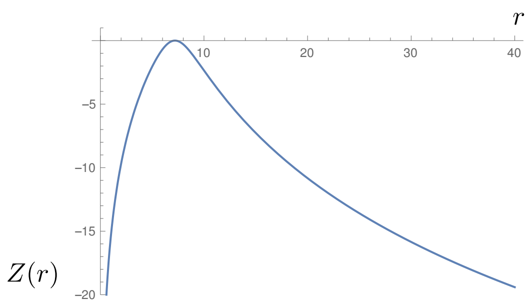

FIGURE 2. The solution of (1.17) with $\theta = 10.374$.

Observe that, in [18], this asymptotic information is the starting point to show that the Morse index of u_p is 12 when p is sufficiently large (see also [19] for the higher dimensional case).

In this paper, we extend and generalize to *any* radial solution u_p the asymptotic results described above. In particular (see the case $\alpha = 0$ in Theorems 2.4 and 2.5 below) we show similar convergence as in (1.6)-(1.7) and (1.10)-(1.11) and describe in a precise way the asymptotic behavior of all the critical points, all the roots and the values of u_p and u_p' at these points respectively as in (1.12)-(1.13)-(1.14)-(1.15), for *any* radial solution u_p. We also study the convergence of a suitable rescaling of the solution in each nodal region and explicitly determine the asymptotic value of the energy, generalizing (1.8) and (1.16).

Our asymptotic analysis is carried out also for the *Neumann Lane-Emden problem* (i.e. (1.2)-(1.4) with $\alpha = 0$), which is much less studied than the Dirichlet problem. These seem to be the first results concerning the asymptotic behavior of Neumann solutions in the 2-dimensional case. Observe that an easy application of the divergence theorem implies that nontrivial solutions u_p of the Neumann problem are necessarily sign-changing, since

$$0 = \int_{\partial B} \frac{\partial u_p}{\partial \nu} = -\int_B \Delta u_p = \int_B |u_p|^{p-1} u_p. \tag{1.18}$$

Moreover, when the domain is the ball B, radial solutions cannot be ground states (see [39, Corollary 1.4], see also [24]). Our Theorem 2.6 below (case $\alpha = 0$) provides a first description in dimension 2 of the asymptotic behavior of *all the radial solutions of the Neumann Lane-Emden problem in the ball* B as $p \to +\infty$. In particular, we obtain sharp constants for the asymptotic behavior of all the critical points, all the roots and the values of u_p and u_p' at these points respectively and explicitly determine the asymptotic value of the energy.

Our results, both for the Dirichlet and the Neumann Lane-Emden problems, complement the known results for the radial solutions of the same problems in higher dimension $N \geq 3$, as p approaches the critical Sobolev exponent $\frac{N+2}{N-2}$ from the left (see [6, 19] and in particular the recent paper [28] where explicit rates of blow-up and sharp constants are obtained, similarly to what we obtain in our results).

In this work we also consider the case $\alpha > 0$, namely the *Hénon problems*.

The first results on the asymptotic behavior of solutions to the Dirichlet Hénon problem are due to Cao e Peng ([13]) who study, in higher dimension $N \geq 3$ and as $p \to \frac{N+2}{N-2}$ from below, the ground states solutions (which are positive) showing a pointwise blow-up at a point approaching the boundary. As observed in [13] when the domain is a ball of $\mathbb{R}^N$, $N \geq 3$, this implies in particular that, differently with the case $\alpha = 0$, the least energy Dirichlet solution for the Hénon problem for any fixed $\alpha > 0$ is not radial for almost critical values p. This was already known from [40, Theorem 6.1], where it was also proved that, in any dimension $N \geq 2$, the breaking of symmetry also happens at any subcritical value of the exponent p, as soon as α is large enough.

When the domain is the ball B, radial solutions for the Dirichlet Hénon problem (i.e. (1.2)-(1.3) with $\alpha > 0$) exist for any $p \in (1, p_\alpha)$, where $p_\alpha = \frac{N+2+2\alpha}{N-2}$ if $N \geq 3$, $p_\alpha = +\infty$ if $N = 2$ (see [35]). A partial study of their asymptotic behavior, as $p \to p_\alpha$, may be found in [4] in dimension $N \geq 3$, and in [3] in dimension $N = 2$, where this analysis is used as the starting point to compute the Morse index of the solutions as in [18, 19]. In particular, [3] contains the sharp description, in dimension $N = 2$, of the asymptotic behavior, as $p \to +\infty$, of the nodal radii, critical points and values of the *least energy radial* solution (which is positive) and of the *least energy sign-changing radial* solution (which has 2 nodal regions). Also, the behavior of suitable rescalings of these solutions are studied.
In this paper, we derive the existence of a unique (up to a sign) radial sign-changing solution with m nodal regions for any $m \geq 2$, $p > 1$ and for $N = 2$, for the *Hénon problems* with both *Dirichlet*

and Neumann boundary conditions and we perform a sharp asymptotic analysis of solutions as $p \to +\infty$ (see Theorems 2.4, 2.5 and 2.6).
In the Dirichlet case, our results complement the ones about the asymptotic behavior of the solutions contained in [3, 4]. In particular, we extend the results in [3] to *any radial solution* and improve them, showing among other things that, similarly as for the Lane-Emden case, the solutions concentrate at the origin. Following [3, 4] and using our asymptotic analysis, one could then compute the Morse index for all the radial solutions also in dimension $N = 2$, this will be the object of future investigation (see the Appendix for some discussion and a conjecture).

The case of the *Neumann Hénon problem* is particularly interesting, since no results at all are available in the literature for this equation. The only papers considering Neumann Hénon type problems (where a linear term is added into the equation) are [22], where the existence of ground state solutions and the breaking of symmetry phenomenon is investigated, and [10] where the asymptotic behavior as $\alpha \to +\infty$ and for fixed p is studied for this solution. Our case is different not only because we are interested in radial solutions but above all because while [22, 10] deal with positive solutions, it is not difficult to see that any nontrivial solution of (1.2)-(1.4) is necessarily sign-changing (just apply the divergence theorem similarly as in (1.18)).

As a consequence of our results, we also obtain information on the asymptotic behavior of radial solutions for the equation in the whole plane (1.1), for any fixed $\alpha \geq 0$ (see Theorem 2.7).

Furthermore, from our sharp asymptotic analysis, we deduce the *lack of uniform (in p) a priori bounds for nodal solutions of the* 2*-dimensional Lane-Emden/Hénon equation* with either Dirichlet or Neumann boundary conditions, since we get that for radial solutions u_p with m nodal regions

$$\lim_{p\to+\infty} \|u_p\|_{L^\infty} \to +\infty \quad \text{as } m \to +\infty,$$

(see Theorem 2.8). This shows, in the Dirichlet Lane-Emden case, a difference with respect to the case of positive solutions, for which the uniform a priori bound (1.5) holds, as proved in [32].

We point out that we are interested in the asymptotic analysis as $p \to +\infty$ and for $\alpha \geq 0$ fixed, we refer to [9, 8, 10] for different results concerning the description of the asymptotic behavior as $\alpha \to +\infty$ and p is fixed ([9, 8] for Dirichlet ground states positive solutions and [10] for Neumann ground states positive solutions of an Hénon-type problem).

2. Main results and ideas

For a radial function $u : B \to \mathbb{R}$ defined on the unit ball B we freely vary between the notations $u(x), x \in B$ and $u(r)$, $0 \leq r \leq 1$.

In order to state our results, first we need to introduce the following definition:

Definition 2.1. *Let $(\theta_k)_{k\geq 0}$ be the sequence of real numbers uniquely defined by the following iteration:*

$$\begin{cases} \theta_0 = 2, \\ \theta_k = \dfrac{2}{\mathcal{L}\left[\dfrac{2}{2+\theta_{k-1}} e^{-\frac{2}{2+\theta_{k-1}}}\right]} + 2 \ (> 2), & \text{for } k \geq 1, \end{cases} \tag{2.1}$$

where $\mathcal{L}$ is the Lambert function, namely the inverse function of $f(L) = Le^L$.

The numbers θ_k are the building blocks to express the sharp constants involved in the asymptotic analysis of the solutions as $p \to \infty$. Below we define these constants (Definition 2.2) and explain their relationship with the concentration rates (Theorems 2.4, 2.6, and 2.7). Once these relationships have been established, a careful study of the behavior of these constants (Section 6) is used to show the lack of a universal uniform bound on nodal solutions for the Dirichlet and Neumann problems (see Theorem 2.8).

Definition 2.2. *Let*

$$\overset{k}{M}_{k-1} := e^{2/(2+\theta_{k-1})} \qquad \forall k \geq 1, \tag{2.2}$$

$$\overset{k}{R}_{k-1} := \frac{\overset{k-1}{M}_{k-2}\,(\theta_{k-2}+2)}{\overset{k}{M}_{k-1}(\theta_{k-1}-2)} \qquad \forall k \geq 2, \tag{2.3}$$

$$\overset{k}{D}_{k} := \overset{k}{M}_{k-1}(\theta_{k-1}+2) \qquad \forall k \geq 1, \tag{2.4}$$

$$\overset{k}{S}_{k-1} := \begin{cases} 0 & k=1, \\ (\overset{k}{M}_{k-1})^{-1} & k \geq 2, \end{cases} \tag{2.5}$$

$$\overset{m}{R}_{i} := \prod_{k=i+1}^{m} \overset{k}{R}_{k-1}, \qquad \overset{m}{D}_{i} := \frac{\overset{i}{D}_{i}}{\prod_{k=i+1}^{m} \overset{k}{R}_{k-1}}, \qquad i = 1, \dots, m-1, \tag{2.6}$$

$$\overset{m}{S}_{i} := \overset{i+1}{S}_{i} \prod_{k=i+2}^{m} \overset{k}{R}_{k-1}, \qquad \overset{m}{M}_{i} := \frac{\overset{i+1}{M}_{i}}{\prod_{k=i+2}^{m} \overset{k}{R}_{k-1}}, \qquad i = 0, \dots, m-2. \tag{2.7}$$

We have the following monotonicity properties.

Lemma 2.3. *It holds that*

$$0 = \overset{m}{S}_0 < \overset{m}{R}_1 < \overset{m}{S}_1 < \overset{m}{R}_2 < \overset{m}{S}_2 < \cdots < \overset{m}{S}_{m-2} < \overset{m}{R}_{m-1} < \overset{m}{S}_{m-1} < 1, \tag{2.8}$$

$$\overset{m}{M}_0 > \overset{m}{M}_1 > \ldots > \overset{m}{M}_{m-1} > 1. \tag{2.9}$$

The sequences $(\overset{m}{M}_j)_m$, $(\overset{m}{D}_j)_m$ are strictly increasing, while the sequences $(\overset{m}{S}_j)_m$, $(\overset{m}{R}_j)_m$ are strictly decreasing.

Next we state our asymptotic results for the Dirichlet problems (1.2)-(1.3). Recall the definition of $\overset{m}{M}_{m-1}$, θ_{m-1}, $\overset{m}{R}_i$, $\overset{m}{S}_i$, $\overset{m}{M}_i$, $\overset{m}{D}_i > 0$ given in Definitions 2.1-2.2 and let

$$G(x,0) = -\frac{1}{2\pi}\log|x|, \qquad x \in B,$$

denote the Green's function of $-\Delta$ in B with Dirichlet boundary condition, computed at the origin.

Theorem 2.4 (Dirichlet problem)**.** *Let $\alpha \geq 0$, $m \in \mathbb{N}$, $m \geq 1$ and $p > 1$. Then there exists a unique (up to a sign) radial solution of (1.2)-(1.3) with $m-1$ interior zeros. This solution does not vanish in the origin, between any two consecutive zeros it has exactly one critical point, which is either a minimum or a maximum. Let us denote by $\overset{m}{u}_{\alpha,p}$ the radial solution of (1.2)-(1.3) with $m-1$ interior zeros and such that $\overset{m}{u}_{\alpha,p}(0) > 0$. Then $\|\overset{m}{u}_{\alpha,p}\|_{L^\infty} = |\overset{m}{u}_{\alpha,p}(0)|$ and*

$$p\overset{m}{u}_{\alpha,p}(x) = 2\pi\gamma_{\alpha,m}G(x,0) + o(1), \text{ in } C^1_{loc}(B \setminus \{0\}) \qquad \text{as } p \to +\infty, \tag{2.10}$$

where

$$\gamma_{\alpha,m} := (-1)^{m-1}\frac{\alpha+2}{2}\overset{m}{M}_{m-1}(\theta_{m-1}+2). \tag{2.11}$$

Moreover,

$$p\int_0^1 |(\overset{m}{u}_{\alpha,p})'(r)|^2 r dr \;=\; p\int_0^1 |\overset{m}{u}_{\alpha,p}(r)|^{p+1} r^{\alpha+1} dr = \frac{\alpha+2}{8}(\overset{m}{M}_{m-1})^2(\theta_{m-1}+2)^2 + o(1) \tag{2.12}$$

as $p \to +\infty$. If $\overset{m}{r}_{i,\alpha,p}$ and $\overset{m}{s}_{i,\alpha,p}$ denote the zeros and the critical points of $\overset{m}{u}_{\alpha,p}$ respectively, so that

$$0 = \overset{m}{s}_{0,\alpha,p} < \overset{m}{r}_{1,\alpha,p} < \overset{m}{s}_{1,\alpha,p} < \overset{m}{r}_{2,\alpha,p} < \ldots < \overset{m}{r}_{m-1,\alpha,p} < \overset{m}{s}_{m-1,\alpha,p} < \overset{m}{r}_{m,\alpha,p} = 1,$$

then, as $p \to +\infty$,

$$(\overset{m}{r}_{i,\alpha,p})^{\frac{2}{p-1}} = \left(\overset{m}{R}_i\right)^{\frac{2}{\alpha+2}} + o(1), \qquad i = 1, \dots, m-1, \tag{2.13}$$

$$p|(\overset{m}{u}_{\alpha,p})'(\overset{m}{r}_{i,\alpha,p})|(\overset{m}{r}_{i,\alpha,p}) = \tfrac{\alpha+2}{2}\overset{m}{D}_i + o(1), \qquad i = 1, \dots, m, \tag{2.14}$$

$$(\overset{m}{s}_{i,\alpha,p})^{\frac{2}{p-1}} = \left(\overset{m}{S}_i\right)^{\frac{2}{\alpha+2}} + o(1), \qquad i = 1, \dots, m-1, \tag{2.15}$$

$$|\overset{m}{u}_{\alpha,p}(\overset{m}{s}_{i,\alpha,p})| = \overset{m}{M}_i + o(1), \qquad i = 0, \dots, m-1. \tag{2.16}$$

Observe that, from (2.10), it follows that the solution $\overset{m}{u}_{\alpha,p}$ *concentrates at the origin as* $p \to +\infty$ and that $\overset{m}{u}_{\alpha,p} \to 0$ uniformly on compact subsets of $B \setminus \{0\}$. Moreover from (2.16) we also know that $\overset{m}{u}_{\alpha,p}(0) \to \overset{m}{M}_0$ (> 1 from (2.9)). Notice also that Lemma 2.3 and (2.13), (2.15),(2.14) and (2.8) imply respectively that, as $p \to \infty$,

$$\overset{m}{r}_{i,\alpha,p} \sim (\overset{m}{R}_i)^{\frac{p-1}{\alpha+2}} \to 0, \qquad i = 1, \dots, m-1,$$
$$\overset{m}{s}_{i,\alpha,p} \sim (\overset{m}{S}_i)^{\frac{p-1}{\alpha+2}} \to 0, \qquad i = 1, \dots, m-1,$$
$$|(\overset{m}{u}_{\alpha,p})'(\overset{m}{r}_{i,\alpha,p})| \sim \begin{cases} \dfrac{\frac{\alpha+2}{2}\overset{m}{D}_i}{p(\overset{m}{R}_i)^{\frac{p-1}{\alpha+2}}} \to +\infty, & i = 1, \dots, m-1, \\[2ex] \dfrac{\frac{\alpha+2}{2}\overset{m}{D}_m}{p} \to 0, & i = m. \end{cases}$$

Next we investigate the asymptotic behavior as $p \to +\infty$ of suitable rescalings of the solution $\overset{m}{u}_{\alpha,p}$ in each nodal region, showing a *tower of bubbles* phenomenon.
To this aim, let us define the m parameters

$$\overset{m}{\varepsilon}_{i,\alpha,p} := \left(\frac{\alpha+2}{2}\right)^{\frac{2}{\alpha+2}} \left[p|\overset{m}{u}_{\alpha,p}(\overset{m}{s}_{i,\alpha,p})|^{p-1}\right]^{-\frac{1}{2+\alpha}}, \quad i = 0, \dots, m-1, \tag{2.17}$$

and the m rescaled functions

$$\overset{m}{\xi}_{i,\alpha,p}(r) := p\frac{(-1)^i\ \overset{m}{u}_{\alpha,p}(\overset{m}{\varepsilon}_{i,\alpha,p}r) - |\overset{m}{u}_{\alpha,p}(\overset{m}{s}_{i,\alpha,p})|}{|\overset{m}{u}_{\alpha,p}(\overset{m}{s}_{i,\alpha,p})|}, \quad i = 0, \dots, m-1, \tag{2.18}$$

for

$$r \in \begin{cases} [0, \frac{\overset{m}{r}_{1,\alpha,p}}{\overset{m}{\varepsilon}_{0,\alpha,p}}], & \text{if } i = 0, \\ \left[\frac{\overset{m}{r}_{i,\alpha,p}}{\overset{m}{\varepsilon}_{i,\alpha,p}}, \frac{\overset{m}{r}_{i+1,\alpha,p}}{\overset{m}{\varepsilon}_{i,\alpha,p}}\right], & \text{if } i \geq 1. \end{cases}$$

Let

$$Z_{i,\alpha}(r) = Z_{i,\alpha}(|x|) := \log \frac{2\theta_i^2 \beta_i^{\theta_i} |x|^{\frac{(\alpha+2)}{2}(\theta_i - 2)}}{(\beta_i^{\theta_i} + |x|^{\frac{(\alpha+2)}{2}\theta_i})^2}, \tag{2.19}$$

with $\beta_0 := 2\sqrt{2}$, $\beta_i := \frac{1}{\sqrt{2}}(\theta_i + 2)^{\frac{\theta_i+2}{2\theta_i}}(\theta_i - 2)^{\frac{\theta_i-2}{2\theta_i}}$ for $i = 1, \dots, m-1$, then $Z_{i,\alpha}$ is a radial solution of

$$\begin{cases} -\Delta Z = \left(\frac{\alpha+2}{2}\right)^2 |x|^\alpha e^Z + (\alpha+2)\pi(2-\theta_i)\delta_0 & \text{in } \mathbb{R}^2, \\ Z(\sigma_{i,\alpha}) = 0, \\ \int_{\mathbb{R}^2} e^Z |x|^\alpha dx = \frac{8\pi\theta_i}{\alpha+2}, \end{cases} \tag{2.20}$$

where δ_0 is the Dirac measure centered at 0 and θ_i (≥ 2) is given in (2.1). The limit profiles $Z_{i,\alpha}$'s are usually named *bubbles.*

Theorem 2.5 (Tower of bubbles for the Dirichlet problem)**.** *Let* $\alpha \geq 0$, $m \in \mathbb{N}$, $m \geq 1$, *then, as* $p \to +\infty$, $\overset{m}{\varepsilon}_{i,\alpha,p} = o(1)$ *and*

$$\frac{\overset{m}{r}_{i,\alpha,p}}{\overset{m}{\varepsilon}_{i,\alpha,p}} = o(1)\ (i \neq 0), \qquad \frac{\overset{m}{s}_{i,\alpha,p}}{\overset{m}{\varepsilon}_{i,\alpha,p}} = \sigma_{i,\alpha} + o(1), \qquad \frac{\overset{m}{\varepsilon}_{i,\alpha,p}}{\overset{m}{r}_{i+1,\alpha,p}} = o(1), \tag{2.21}$$

for all $i = 0, \dots, m-1$ *with* $\sigma_{i,\alpha} := \left(\frac{\theta_i^2 - 4}{2}\right)^{\frac{1}{2+\alpha}}$. *Moreover,*

$$\overset{m}{\xi}_{i,\alpha,p} = Z_{i,\alpha} + o(1) \quad \text{in } C^1_{loc}(0,+\infty) \qquad \text{for all } i = 0, \dots, m-1 \text{ as } p \to +\infty. \tag{2.22}$$

Observe that $\frac{\overset{m}{r}_{i,\alpha,p}}{\overset{m}{s}_{i,\alpha,p}} = o(1)$ and $\frac{\overset{m}{s}_{i,\alpha,p}}{\overset{m}{r}_{i+1,\alpha,p}} = o(1)$ so that

$$\frac{\overset{m}{\varepsilon}_{i,\alpha,p}}{\overset{m}{\varepsilon}_{i+1,\alpha,p}} = o(1) \quad \text{ as } p \to +\infty,$$

namely, in each nodal region the *bubble* appears at a *different scale*, for this reason we say that the solution has a *tower of bubbles* behavior.
Recall that $\theta_0 = 2$, hence $\sigma_0 = 0$, as a consequence the *first bubble* $Z_{0,\alpha}$ is a regular solution of the Liouville equation $-\Delta Z = \left(\frac{\alpha+2}{2}\right)^2 |x|^\alpha e^Z$ in $\mathbb{R}^2$ with $Z(0) = 0$ (see Figure 1).

Using a suitable change of variables, we can obtain from Theorem 2.4 a description of the radial solutions of the Neumann problem (1.2)-(1.4).

Theorem 2.6 (Neumann problem)**.** *Let* $\alpha \geq 0$, $m \in \mathbb{N}$, $m \geq 2$ *and* $p > 1$. *Then there exists a unique (up to a sign) radial solution of* (1.2)-(1.4) *with* $m-1$ *interior zeros. This solution does not vanish in the origin and between any two consecutive zeros it has exactly one critical point, which is either a minimum or a maximum.*
Let us denote by $\overset{m}{\bar{u}}_{\alpha,p}$ *the radial solution of* (1.2)-(1.4) *with* $m-1$ *interior zeros and such that* $\overset{m}{\bar{u}}_{\alpha,p}(0) > 0$. *Then*

$$\overset{m}{\bar{u}}_{\alpha,p}(r) = (\overset{m}{s}_{m-1,\alpha,p})^{\frac{\alpha+2}{p-1}} \overset{m}{u}_{\alpha,p}(\overset{m}{s}_{m-1,\alpha,p} r), \qquad r \in [0,1], \tag{2.23}$$

where $\overset{m}{u}_{\alpha,p}$ *and* $\overset{m}{s}_{m-1,\alpha,p}$ *are as in Theorem 2.4.*
Moreover,

$$p \int_0^1 |(\overset{m}{\bar{u}}_{\alpha,p})'(r)|^2 r dr = p \int_0^1 |\overset{m}{\bar{u}}_{\alpha,p}(r)|^{p+1} r^{1+\alpha} dr = \frac{\alpha+2}{8}(\theta_{m-1}+2)(\theta_{m-1}-2) + o(1) \tag{2.24}$$

as $p \to +\infty$ *and, letting* $\overset{m}{\bar{r}}_{i,\alpha,p}$ *and* $\overset{m}{\bar{s}}_{i,\alpha,p}$ *be the zeros and the critical points of* $\overset{m}{\bar{u}}_{\alpha,p}$ *respectively, so that*

$$0 = \overset{m}{\bar{s}}_{0,\alpha,p} < \overset{m}{\bar{r}}_{1,\alpha,p} < \overset{m}{\bar{s}}_{1,\alpha,p} < \overset{m}{\bar{r}}_{2,\alpha,p} < \ldots < \overset{m}{\bar{r}}_{m-1,\alpha,p} < \overset{m}{\bar{s}}_{m-1,\alpha,p} = 1,$$

then

$$(\overset{m}{\bar{r}}_{i,\alpha,p})^{\frac{2}{p-1}} = \left(\overset{m}{\bar{R}}_i\right)^{\frac{2}{\alpha+2}} + o(1), \qquad i = 1, \dots, m-1, \tag{2.25}$$

$$p|(\overset{m}{\bar{u}}_{\alpha,p})'(\overset{m}{\bar{r}}_{i,\alpha,p})|(\overset{m}{\bar{r}}_{i,\alpha,p}) = \tfrac{\alpha+2}{2}\overset{m}{\bar{D}}_i + o(1), \qquad i = 1, \dots, m-1, \tag{2.26}$$

$$(\overset{m}{\bar{s}}_{i,\alpha,p})^{\frac{2}{p-1}} = \left(\overset{m}{\bar{S}}_i\right)^{\frac{2}{\alpha+2}} + o(1), \qquad i = 1, \dots, m-2, \tag{2.27}$$

$$|\overset{m}{\bar{u}}_{\alpha,p}(\overset{m}{\bar{s}}_{i,\alpha,p})| = \overset{m}{\bar{M}}_i + o(1), \qquad i = 0, \dots, m-1, \tag{2.28}$$

as $p \to +\infty$, *where*

$$\overset{m}{\bar{R}}_i := \frac{\overset{m}{R}_i}{\overset{m}{S}_{m-1}}, \qquad \overset{m}{\bar{D}}_i := \overset{m}{S}_{m-1}\overset{m}{D}_i, \qquad \overset{m}{\bar{S}}_i := \frac{\overset{m}{S}_i}{\overset{m}{S}_{m-1}}, \qquad \overset{m}{\bar{M}}_i := \overset{m}{S}_{m-1}\overset{m}{M}_i, \tag{2.29}$$

and $\overset{m}{R}_i$, $\overset{m}{S}_i$, $\overset{m}{M}_i$ *and* $\overset{m}{D}_i$ *are the constants in Definitions 2.1-2.2.*

From Theorem 2.6 and Lemma 2.3 we deduce that

$$\begin{aligned}
&\overset{m}{\bar{r}}_{i,\alpha,p} \sim (\overset{m}{\bar{R}}_i)^{\frac{p-1}{\alpha+2}} \to 0, && i = 1, \ldots, m-1,\\
&\overset{m}{\bar{s}}_{i,\alpha,p} \sim (\overset{m}{\bar{S}}_i)^{\frac{p-1}{\alpha+2}} \to 0, && i = 1, \ldots, m-2,\\
&|(\overset{m}{\bar{u}}_{\alpha,p})'(\overset{m}{\bar{r}}_{i,\alpha,p})| \sim \frac{\frac{\alpha+2}{2}\overset{m}{\bar{D}}_i}{p(\overset{m}{\bar{R}}_i)^{\frac{p-1}{\alpha+2}}} \to +\infty, && i = 1, \ldots, m-1,\\
&|\overset{m}{\bar{u}}_{\alpha,p}(\overset{m}{\bar{s}}_{i,\alpha,p})| \sim \overset{m}{\bar{M}}_i > 1, && i = 0, \ldots, m-2,
\end{aligned}$$

and that $|\overset{m}{\bar{u}}_{\alpha,p}(1)| \to 1$ as $p \to +\infty$. We expect that $\overset{m}{\bar{u}}_{\alpha,p}(x) \to 1$ uniformly on compact subsets of $B \setminus \{0\}$, but this does not follow directly from Theorem 2.4 and would require further analysis.

Next we consider the radial solutions to (1.1). They oscillate infinitely many times and have a unique local maximum or minimum between any two consecutive zeros. It is not difficult to see that they are linked by a suitable change of variables to the radial solutions of the Dirichlet and Neumann problems, hence as a consequence of our previous results we deduce the following characterization.

Theorem 2.7 (Problem in $\mathbb{R}^2$)**.** *Let $\alpha \geq 0$ and $w_{\alpha,p}$ be the radial solution of (1.1) such that*

$$w_{\alpha,p}(0) = 1.$$

Then $w_{\alpha,p}$ has a sequence $(\rho_{m,\alpha,p})_{m\in\mathbb{N}}$ of zeros and a sequence $(\delta_{m,\alpha,p})_{m\in\mathbb{N}}$ of critical points such that

$$0 = \delta_{0,\alpha,p} < \rho_{1,\alpha,p} < \delta_{1,\alpha,p} < \rho_{2,\alpha,p} < \ldots < \delta_{m-1,\alpha,p} < \rho_{m,\alpha,p} < \delta_{m,\alpha,p} < \ldots;$$

and

$$\overset{m}{u}_{\alpha,p}(r) = (\rho_{m,\alpha,p})^{\frac{\alpha+2}{p-1}} w_{\alpha,p}(\rho_{m,\alpha,p} r), \qquad r \in [0,1], \tag{2.30}$$

where $\overset{m}{u}_{\alpha,p}$ is as in Theorem 2.4. Moreover,

$$\begin{aligned}
&(\rho_{m,\alpha,p})^{\frac{2}{p-1}} = \left(\overset{m}{M}_0\right)^{\frac{2}{\alpha+2}} + o(1),\\
&p|(w_{\alpha,p})'(\rho_{m,\alpha,p})|(\rho_{m,\alpha,p}) = \frac{\alpha+2}{2}\frac{\overset{m}{D}_m}{\overset{m}{M}_0} + o(1),\\
&(\delta_{m,\alpha,p})^{\frac{2}{p-1}} = \left(\overset{m+1}{M}_0 \overset{m+1}{S}_m\right)^{\frac{2}{\alpha+2}} + o(1),\\
&|w_{\alpha,p}(\delta_{m,\alpha,p})| = \frac{\overset{m+1}{M}_m}{\overset{m+1}{M}_0} + o(1),
\end{aligned}$$

as $p \to +\infty$, where $\overset{m}{M}_0$, $\overset{m+1}{M}_0$, $\overset{m+1}{M}_m$, $\overset{m}{D}_m$, $\overset{m+1}{S}_m$ are the constants in Definitions 2.1-2.2.

From the study of the constants performed in Section 6 (see Theorem 6.2), we deduce the following result which implies that there is no a priori bound for nodal solutions.

Theorem 2.8. *Let $\overset{m}{u}_{\alpha,p}$ and $\overset{m}{\bar{u}}_{\alpha,p}$ be as in Theorem 2.4 and Theorem 2.6 respectively and let $c_1 := \sqrt{\pi} \approx 1.77$ and $c_2 := 6\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \approx 2.02$. Then, for every $m \in \mathbb{N}$,*

$$c_1\ \frac{\Gamma(m)}{\Gamma(m-\frac{1}{2})} e^{\frac{1}{4m-1}} < \lim_{p\to\infty} \|\overset{m}{u}_{\alpha,p}\|_{L^\infty} < c_2\ \frac{\Gamma(m+\frac{1}{4})}{\Gamma(m-\frac{1}{4})} e^{\frac{1}{4m-2}}, \tag{2.31}$$

$$c_1\ \frac{\Gamma(m)}{\Gamma(m-\frac{1}{2})} e^{\frac{1}{4m-1}-\frac{1}{4m-2}} < \lim_{p\to\infty} \|\overset{m}{\bar{u}}_{\alpha,p}\|_{L^\infty} < c_2\ \frac{\Gamma(m+\frac{1}{4})}{\Gamma(m-\frac{1}{4})} e^{\frac{1}{4m-2}-\frac{1}{4m-1}}. \tag{2.32}$$

Furthermore,

$$c_1 \leq \liminf_{m\to\infty} \lim_{p\to\infty} \frac{\|\overset{m}{u}_{\alpha,p}\|_{L^\infty}}{\sqrt{m}} \leq \limsup_{m\to\infty} \lim_{p\to\infty} \frac{\|\overset{m}{u}_{\alpha,p}\|_{L^\infty}}{\sqrt{m}} \leq c_2, \tag{2.33}$$

$$c_1 \leq \liminf_{m\to\infty} \lim_{p\to\infty} \frac{\|\overset{m}{\bar{u}}_{\alpha,p}\|_{L^\infty}}{\sqrt{m}} \leq \limsup_{m\to\infty} \lim_{p\to\infty} \frac{\|\overset{m}{\bar{u}}_{\alpha,p}\|_{L^\infty}}{\sqrt{m}} \leq c_2. \tag{2.34}$$

An immediate consequence is that the i-th local maximum or minimum is also unbounded for any $i \in \mathbb{N}$ as $m, p \to \infty$ and with the same growth rate.

Corollary 2.9. *Using the notation of Theorems 2.4 and 2.6. Then*

$$0 < \liminf_{m\to\infty} \lim_{p\to\infty} \frac{\overset{m}{u}_{\alpha,p}(\overset{m}{s}_{i,\alpha,p})}{\sqrt{m}} \le \limsup_{m\to\infty} \lim_{p\to\infty} \frac{\overset{m}{u}_{\alpha,p}(\overset{m}{s}_{i,\alpha,p})}{\sqrt{m}} < \infty \qquad \text{for every } i \in \mathbb{N},$$

and the same result holds with $\overset{m}{\bar{u}}_{\alpha,p}(\overset{m}{\bar{s}}_{i,\alpha,p})$ *instead of* $\overset{m}{u}_{\alpha,p}(\overset{m}{s}_{i,\alpha,p})$.

Theorem 2.8 implies that, for the 2-dimensional Lane-Emden and Hénon problems with either Dirichlet or Neumann boundary conditions, *a priori uniform bounds do not hold true for nodal solutions in general*, and in fact the supremum norm grows as $\sqrt{m}$, where $m-1$ is the number of interior zeros of the solution (see Figure 4). We recall that positive solutions of Dirichlet Lane-Emden problems do satisfy an a priori uniform bound, see [32].

To close this section, we describe the organization and main ideas of the paper. Theorems 2.4 and 2.5 (Dirichlet Lane-Emden) are shown in Section 3 for $\alpha = 0$. Here, an inductive strategy is used together with the fact that $\overset{m-1}{u}_{0,p}$ and $\overset{m}{u}_{0,p}$ are related via a suitable *change of variables*. More details on the structure of our proof is in Section 3.2. In Section 4 we study the case $\alpha > 0$ in Theorems 2.4 and 2.5 (Dirichlet Hénon), which is deduced from the case $\alpha = 0$ using another change of variables which *works only in dimension* 2. Using (2.23) and (2.30), we show Theorems 2.6 (Neumann) and 2.7 (whole $\mathbb{R}^2$) in Section 5. In Section 6 we do a careful analysis of the constants and use this to show Theorem 2.8, Corollary 2.9, and Lemma 2.3. Finally, in the Appendix A we discuss a conjecture on a Morse index formula for the solutions.

3. The Dirichlet Lane-Emden problem in B

This section is devoted to the proof of the case $\alpha = 0$ in Theorem 2.4 and in Theorem 2.5. Hence, we consider the Dirichlet Lane-Emden problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases} \tag{3.1}$$

It is well known that, for any $p > 1$ and any $m \in \mathbb{N}$, $m \ge 1$, there exists a unique (up to a sign) radial solution to (3.1) with exactly $m-1$ interior zeros (see for instance [31, p. 263]).
This solution does not vanish in the origin so let us simply denote by $\overset{m}{u}_p$ the unique nodal radial solution of (3.1) having $m-1$ interior zeros and satisfying

$$\overset{m}{u}_p(0) > 0 \tag{3.2}$$

(namely $\overset{m}{u}_p = \overset{m}{u}_{p,0}$). With a slight abuse of notation, we often write $\overset{m}{u}_p(r) = \overset{m}{u}_p(|x|)$.

3.1. Preliminary properties.

Let us denote by $\overset{m}{r}_{i,p}$, $i = 1, \dots m$, the nodal radii of $\overset{m}{u}_p$, *i.e.*,

$$\begin{aligned} &0 < \overset{m}{r}_{1,p} < \overset{m}{r}_{2,p} < \cdots < \overset{m}{r}_{m-1,p} < \overset{m}{r}_{m,p} = 1, \\ &\overset{m}{u}_p(\overset{m}{r}_{i,p}) = 0 \qquad i = 1, \dots, m. \end{aligned} \tag{3.3}$$

Proposition 3.1. *Let* $p > 1$, *then*

(i) $\overset{m}{u}_p(0) = \|\overset{m}{u}_p\|_\infty$.

(ii) 0 *is the unique critical point of the map* $r \mapsto \overset{m}{u}_p(r)$ *in the interval* $[0, \overset{m}{r}_{1,p}]$; *in each interval* $[\overset{m}{r}_{i,p}, \overset{m}{r}_{i+1,p}]$ *the map* $r \mapsto \overset{m}{u}_p(r)$ *has exactly one critical point* $\overset{m}{s}_{i,p}$, *so*

$$\begin{aligned} &\overset{m}{s}_{0,p} := 0, \\ &\overset{m}{s}_{i,p} \in (\overset{m}{r}_{i,p}, \overset{m}{r}_{i+1,p}), \quad i = 1, \dots, m-1 \ (\textit{if } m \ge 2), \end{aligned} \tag{3.4}$$

$$(\overset{m}{u}_p)'(\overset{m}{s}_{i,p}) = 0, \quad i = 0, \dots, m-1,$$

$\overset{m}{s}_{2j,p}$ *is a local maximum and* $\overset{m}{s}_{2j+1,p}$ *is a local minimum point for* $j = 0, 1, 2, \dots$.

(iii)

$$|\overset{m}{u}_p(\overset{m}{s}_{0,p})| > |\overset{m}{u}_p(\overset{m}{s}_{1,p})| > \ldots > |\overset{m}{u}_p(\overset{m}{s}_{m-1,p})|. \tag{3.5}$$

Proof. The one variable function $r \mapsto \overset{m}{u}_p$ satisfies

$$(\overset{m}{u}_p)'' + \frac{1}{r}(\overset{m}{u}_p)' + |\overset{m}{u}_p|^{p-1}\overset{m}{u}_p = 0, \tag{3.6}$$

(ii) then follows immediately observing that for critical points $(\overset{m}{u}_p)'' = -|\overset{m}{u}_p|^{p-1}\overset{m}{u}_p$. Moreover, multiplying the equation (3.6) by $(\overset{m}{u}_p)'$ we have that $F'(r) = -\frac{1}{r}\left[(\overset{m}{u}_p)'(r)\right]^2 \le 0$, where

$$F(r) = \frac{1}{2}|(\overset{m}{u}_p)'|^2 + \frac{1}{p+1}|\overset{m}{u}_p|^{p+1}.$$

Thus F is non increasing, in particular $F(0) \ge F(r)$ for all $r \in (0,1)$, which implies (i), and $F(\overset{m}{s}_{i,p}) \ge F(\overset{m}{s}_{i+i,p})$, which implies (iii). □

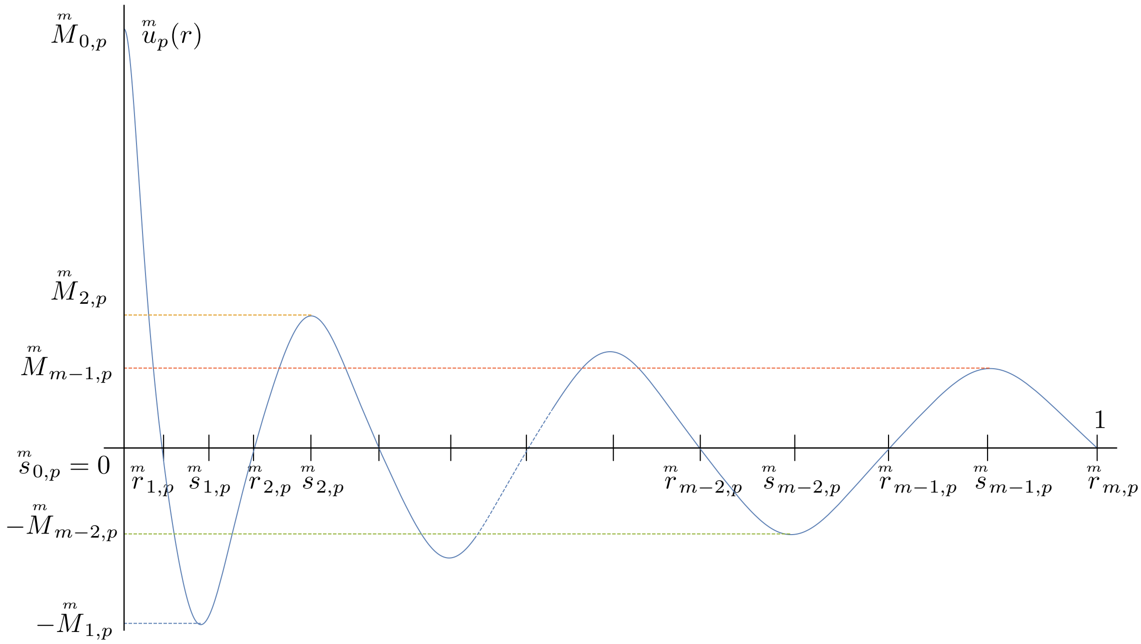

FIGURE 3. In our notation, $\overset{m}{s}_{i,p}$ denote the critical points, $\overset{m}{r}_{i,p}$ are roots, and $\overset{m}{M}_{i,p}$ are the alternating local minima and maxima in absolute value.

From [20, Proposition 2.1] we have a bound of the total energy.

Lemma 3.2. *There exist $p_m > 1$ and $E_m > 0$ such that*

$$p\int_0^1 |(\overset{m}{u}_p)'(r)|^2 r dr = p\int_0^1 |\overset{m}{u}_p(r)|^{p+1} r dr \le E_m \qquad \text{for } p \ge p_m. \tag{3.7}$$

By the classical Strauss inequality for radial functions in $H^1(\mathbb{R}^2)$ ([41]) and the energy estimate in (3.7) we deduce the following pointwise bound.

Lemma 3.3. *There exists $C_m > 0$ such that*

$$|\overset{m}{u}_p(r)| \le \frac{C_m}{\sqrt{r}} \qquad \text{for any } r \ne 0 \text{ and } \ p \ge p_m.$$

Moreover, integrating the equation (3.1) written in polar coordinates, we have that

$$-\big((\overset{m}{u}_p)'(r)r\big)' = |\overset{m}{u}_p(r)|^{p-1}\overset{m}{u}_p(r)r \qquad \text{in } (0,1).$$

As a consequence, we deduce the next identity.

Lemma 3.4. *Let $s, t \in (0,1)$, then*

$$(\overset{m}{u}_p)'(s)s - (\overset{m}{u}_p)'(t)t = \int_s^t |\overset{m}{u}_p(r)|^{p-1}\overset{m}{u}_p(r) r\, dr. \tag{3.8}$$

We then obtain the following estimate for the derivative.

Lemma 3.5. *There exists $C_m > 0$ and $p_m > 1$ such that*

$$p\big|(\overset{m}{u}_p)'(r)\big| \leq \frac{C_m}{r} \qquad \text{for all } r \in (0,1] \text{ and } p \geq p_m.$$

Proof. Let $r \in (0,1)$. Choosing $s = 0$ and $t = r$ in the identity (3.8) (recall that $(\overset{m}{u}_p)'(0) = 0$), by Hölder's inequality,

$$p\big|(\overset{m}{u}_p)'(r)\big|\, r \leq p \int_0^r |\overset{m}{u}_p(s)|^p s\, ds \leq p \left[\int_0^1 |\overset{m}{u}_p(s)|^{p+1} s\, ds\right]^{\frac{p}{p+1}}$$

and the conclusion follows from (3.7). The case $r = 1$ is obtained by continuity. □

Moreover, choosing $s = \overset{m}{s}_{m-1,p}$ and $t = 1$ into (3.8), we also have the following.

Lemma 3.6. *It holds that*

$$-\,p\,(\overset{m}{u}_p)'(1) = p \int_{\overset{m}{s}_{m-1,p}}^1 |\overset{m}{u}_p|^{p-1}\overset{m}{u}_p r\, dr = (-1)^{m-1} p \int_{\overset{m}{s}_{m-1,p}}^1 |\overset{m}{u}_p|^p r\, dr. \tag{3.9}$$

Lemma 3.7 (Pohozaev identity)**.** *It holds that*

$$p\int_0^1 |\overset{m}{u}_p|^{p+1} r\, dr = \left(1 + \frac{1}{p}\right)\frac{1}{4}\left[p(\overset{m}{u}_p)'(1)\right]^2.$$

Proof. The claim follows from the Pohozaev identity

$$\frac{2}{p+1}\int_B |\overset{m}{u}_p(x)|^{p+1} dx = \frac{1}{2}\int_{\partial B} (x \cdot \nu(x)) \left(\frac{\partial \overset{m}{u}_p(x)}{\partial \nu}\right)^2 d\sigma_x,$$

where ν denotes the exterior unit normal vector on ∂B. □

We conclude this paragraph recalling some known asymptotic results. Thanks to the energy bound (3.7), the general asymptotic analysis in [16] apply to the solution $\overset{m}{u}_p$ (see [16, Proposition 2.2] or also [17, Proposition 2.4, Corollary 2.6]). In particular, since the domain is now a ball and $\overset{m}{u}_p$ is radial, it follows that it concentrates at only 1 point, which is the origin (i.e. $k = 1$ and $\mathcal{S} = \{0\}$ in the notation of [16, 17]). In our notations:

Lemma 3.8. *There exists $C_m, \widetilde{C}_m > 0$ such that, for p large,*

$$1 \leq \|\overset{m}{u}_p\|_\infty = \overset{m}{u}_p(0) \leq C_m,$$

$$pr^2|\overset{m}{u}_p(r)|^{p-1} \leq \widetilde{C}_m, \quad \text{for all } r \in [0,1]. \tag{3.10}$$

Moreover, there exists a function $v \in C^1(B \setminus \{0\})$ such that

$$p\overset{m}{u}_p = v + o(1) \quad \text{in } C^1_{loc}(B \setminus \{0\}), \text{ as } p \to +\infty.$$

In particular,

$$p(\overset{m}{u}_p)^p = o(1) \text{ uniformly in } K \text{ for all compact sets } K \subset \mathcal{B} \setminus \{0\}. \tag{3.11}$$

Observe that from [16] we also know that a *first bubble* always appears. Indeed, [16, Proposition 2.2] also says that, when $\alpha = 0$, $\overset{m}{\varepsilon}_{0,\alpha,p} = o(1)$, where $\overset{m}{\varepsilon}_{0,\alpha,p}$ is the first scaling parameter as defined in (2.17), and that $\overset{m}{\eta}_{0,\alpha,p} = Z_{0,\alpha} + o(1)$ in $C^1_{loc}(0,+\infty)$ as $p \to +\infty$, where $\overset{m}{\eta}_{0,\alpha,p}$ is a first rescaling of $\overset{m}{u}_p$ defined formally as the first rescaling $\overset{m}{\xi}_{0,\alpha,p}$ in (2.18), but in a larger domain, namely for any $r \in [0, \frac{1}{\overset{m}{\varepsilon}_{0,\alpha,p}}]$. Next, we restrict to the first nodal region to prove the convergence of $\overset{m}{\xi}_{0,\alpha,p}$ stated in (2.22).

3.2. **Main partial results and an iterative strategy.** The proof for the case $\alpha = 0$ in Theorem 2.4 and in Theorem 2.5 (Dirichlet Lane-Emden) is postponed to the end of Section 3 (see Sections 3.5 and 3.6, respectively). It is a consequence of some preliminary results from Section 3.1 and of the following two partial results.

Theorem 3.9 (Sharp asymptotic constants)**.** *Let $m \in \mathbb{N}$, $m \geq 1$ and let $\overset{m}{u}_p$ be the radial solution of (3.1)-(3.2) with $m-1$ interior zeros and let $\overset{m}{r}_{i,p}$ and $\overset{m}{s}_{i,p}$ be its zeros and critical points (see (3.3),(3.4)). Then*

$$(\overset{m}{r}_{i,p})^{\frac{2}{p-1}} = \overset{m}{R}_i + o(1), \qquad i = 1, \dots, m-1, \tag{3.12}$$

$$p|(\overset{m}{u}_p)'(\overset{m}{r}_{i,p})|(\overset{m}{r}_{i,p}) = \overset{m}{D}_i + o(1), \qquad i = 1, \dots, m, \tag{3.13}$$

$$(\overset{m}{s}_{i,p})^{\frac{2}{p-1}} = \overset{m}{S}_i + o(1), \qquad i = 1, \dots, m-1, \tag{3.14}$$

$$|\overset{m}{u}_p(\overset{m}{s}_{i,p})| = \overset{m}{M}_i + o(1), \qquad i = 0, \dots, m-1, \tag{3.15}$$

as $p \to +\infty$, where $\overset{m}{R}_i, \overset{m}{S}_i, \overset{m}{M}_i, \overset{m}{D}_i > 0$ are the constants in Definition 2.2.

Theorem 3.10 (Last bubble)**.** *Let $m \in \mathbb{N}$, $m \geq 1$ and set*

$$\overset{m}{\varepsilon}_{m-1,p} := \left[p|\overset{m}{u}_p(\overset{m}{s}_{m-1,p})|^{p-1}\right]^{-\frac{1}{2}},$$

$$\overset{m}{z}_{m-1,p}(r) := \frac{p}{|\overset{m}{u}_p(\overset{m}{s}_{m-1,p})|}\left[(-1)^{m-1}\overset{m}{u}_p(\overset{m}{s}_{m-1,p} + \overset{m}{\varepsilon}_{m-1,p} r) - |\overset{m}{u}_p(\overset{m}{s}_{m-1,p})|\right],$$

for $r \in (\overset{m}{a}_{m-1,p}, \overset{m}{b}_{m-1,p})$ where

$$\overset{m}{a}_{m-1,p} := \begin{cases} 0, & \text{if } m = 1, \\ \frac{\overset{m}{r}_{m-1,p} - \overset{m}{s}_{m-1,p}}{\overset{m}{\varepsilon}_{m-1,p}} \ (<0), & \text{if } m \geq 2, \end{cases}$$

and

$$\overset{m}{b}_{m-1,p} := \frac{1 - \overset{m}{s}_{m-1,p}}{\overset{m}{\varepsilon}_{m-1,p}} \ (>0).$$

Then

$$\overset{m}{\varepsilon}_{m-1,p} = o(1), \tag{3.16}$$

$$\frac{\overset{m}{r}_{m-1,p}}{\overset{m}{\varepsilon}_{m-1,p}} = o(1) \quad (m \neq 1), \tag{3.17}$$

$$\frac{\overset{m}{s}_{m-1,p}}{\overset{m}{\varepsilon}_{m-1,p}} = \sigma_{m-1} + o(1), \tag{3.18}$$

and

$$\overset{m}{z}_{m-1,p} = Z_{m-1}(\cdot + \sigma_{m-1}) + o(1) \quad \text{in } C^1_{loc}(-\sigma_{m-1}, +\infty), \tag{3.19}$$

as $p \to +\infty$, where

$$\sigma_{m-1} := \sigma_{m-1,0} = \sqrt{\frac{\theta_{m-1}^2 - 4}{2}} \tag{3.20}$$

and

$$Z_{m-1} := Z_{m-1,0} = \log \frac{2(\theta_{m-1})^2(\beta_{m-1})^{\theta_{m-1}}|x|^{(\theta_{m-1}-2)}}{((\beta_{m-1})^{\theta_{m-1}} + |x|^{\theta_{m-1}})^2}, \tag{3.21}$$

(with $\beta_0 := 2\sqrt{2}$, $\beta_{m-1} := \frac{1}{\sqrt{2}}(\theta_{m-1}+2)^{\frac{\theta_{m-1}+2}{2\theta_{m-1}}}(\theta_{m-1}-2)^{\frac{\theta_{m-1}-2}{2\theta_{m-1}}}$, $m \geq 2$) are as in Theorem 2.5 with $\alpha = 0$.

The proof of Theorem 3.9 is done by an iterative argument on the number m. More precisely, it is obtained once the following *inductive basis* and *inductive step* are shown.

Proposition 3.11 (Inductive basis)**.** *Theorem 3.9 holds for $m = 1$.*

Proposition 3.12 (Inductive step)**.** *Let $m \geq 2$. If Theorem 3.9 holds for the solution $\overset{m-1}{u}_p$, then it holds for the solution $\overset{m}{u}_p$.*

Proposition 3.11 (*inductive basis*) was essentially already known from the general works [37, 38, 1] on the asymptotic behavior of the least energy solutions of the Dirichlet Lane-Emden problem. In Section 3.3 we will sketch a proof for completeness. As we will see it exploits the following:

Proposition 3.13. *Theorem 3.10 holds for $m = 1$.*

The proof of Proposition 3.12 (*inductive step*) is based on the key observation that the radial solution $\overset{m}{u}_p$ with $m-1$ interior zeros and the radial solution $\overset{m-1}{u}_p$ with $m-2$ interior zeros are related by the following change of variable

$$\overset{m-1}{u}_p(r) = (\overset{m}{r}_{m-1,p})^{\frac{2}{p-1}} \overset{m}{u}_p(\overset{m}{r}_{m-1,p} r), \quad r \in [0,1].$$

As a consequence, it can be proved that, if one knows that Theorem 3.9 holds for the solution $\overset{m-1}{u}_p$, then, in order to prove that it holds for the solution $\overset{m}{u}_p$, one has to show only the *last* 4 relations

$$\begin{aligned}
(\overset{m}{r}_{m-1,p})^{\frac{2}{p-1}} &= \overset{m}{R}_{m-1} + o(1),\\
p|(\overset{m}{u}_p)'(1)| &= \overset{m}{D}_m + o(1),\\
(\overset{m}{s}_{m-1,p})^{\frac{2}{p-1}} &= \overset{m}{S}_{m-1} + o(1),\\
|\overset{m}{u}_p(\overset{m}{s}_{m-1,p})| &= \overset{m}{M}_{m-1} + o(1),
\end{aligned}$$

as $p \to +\infty$. The proof of these 4 sharp limits is at the core of Section 3.4. It is mainly based on integral estimates and ODE techniques, and we exploit many arguments from [27], where the case $m = 2$ was investigated. One of the ingredients of the proof is the description of the bubble behavior for the *last* rescaled function $\overset{m}{z}_{m-1,p}$ of $\overset{m}{u}_p$ as $p \to +\infty$.

Proposition 3.14. *Let $m \geq 2$. If Theorem 3.9 holds for radial solutions $\overset{m-1}{u}_p$ with $m-2$ interior zeros, then Theorem 3.10 holds.*

Observe that, once Propositions 3.13 and 3.14 are proven, an iterative argument on m gives also the proof of Theorem 3.10, using the propositions as *inductive basis* and *inductive step* respectively.

In conclusion in order to show Theorems 3.9 and 3.10 it remains to show

- the *inductive basis*: Propositions 3.11 and 3.13 (see Section 3.3);
- the *inductive steps*: Propositions 3.12 and 3.14 (see Section 3.4).

3.3. **The case $m = 1$.** Here we prove Propositions 3.11 and 3.13.

Proof of Proposition 3.13. We have to show that

$$\overset{1}{\varepsilon}_{0,p} = o(1), \tag{3.22}$$

$$\overset{1}{z}_{0,p} = Z_0 + o(1). \tag{3.23}$$

as $p \to +\infty$ (indeed, since $\theta_0 = 2$, $\sigma_0 = \sqrt{\frac{\theta_0^2 - 4}{2}} = 0$). (3.22) was known from [37], we repeat the proof for completeness. By Poincare inequality,

$$\int_\Omega |\nabla u_p|^2 \, dx = \int_\Omega |u_p|^{p+1} \, dx \leq \|u_p\|_{L^\infty(\Omega)}^{p-1} \int_\Omega |u_p|^2 \, dx \leq \frac{\|u_p\|_{L^\infty(\Omega)}^{p-1}}{\lambda_1(\Omega)} \int_\Omega |\nabla u_p|^2 \, dx,$$

where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, hence

$$u_p(0)^{p-1} = \|u_p\|_{L^\infty(\Omega)}^{p-1} \geq \lambda_1(\Omega) > 0 \tag{3.24}$$

and, as a consequence, recalling that $\overset{1}{\varepsilon}_{0,p} := \left[p|\overset{1}{u}_p(0)|^{p-1}\right]^{-\frac{1}{2}}$, we derive (3.22).
The proof of (3.23) can be found in [1] (see also [2], where this scaling argument was introduced and [21] where the limit problem was firstly deduced). Observe that the functions $\overset{1}{z}_{0,p}$ satisfy

$$\begin{cases} (\overset{1}{z}_{0,p})'' + \dfrac{1}{r}(\overset{1}{z}_{0,p})' + \left|1 + \dfrac{\overset{1}{z}_{0,p}}{p}\right|^{p-1}\left(1 + \dfrac{\overset{1}{z}_{0,p}}{p}\right) = 0, \quad r \in (0, \overset{1}{b}_{0,p}), \\ \overset{1}{z}_{0,p}(0) = (\overset{1}{z}_{0,p})'(0) = 0, \\ \overset{1}{z}_{0,p}(\overset{1}{b}_{0,p}) = -p, \\ \overset{1}{z}_{0,p} < 0, \end{cases}$$

with $\left|1 + \frac{\overset{1}{z}_{0,p}}{p}\right| \leq 1$ and $\overset{1}{b}_{0,p} = (\overset{1}{\varepsilon}_{0,p})^{-1} \to +\infty$, as $p \to +\infty$. In [1, Theorem 1.1] it is shown that $\overset{1}{z}_{0,p}$ are locally uniformly bounded in $(0, +\infty)$. The rest of the proof is then standard: by elliptic estimates, we have that $\overset{1}{z}_{0,p}$ are uniformly bounded in $C^2_{loc}(0, +\infty)$, so they converge in $C^1_{loc}(0, +\infty)$ to a non-positive solution of $-z'' - \frac{z'}{r} = e^z$ with $z(0) = 0$. Observe that the radial function $\mathbb{R}^2 \ni x \mapsto z(|x|)$ has finite energy by Fatou's lemma so, by the classification results for the Liouville equation, it must necessarily be the function Z_0. □

Proof of Proposition 3.11. We have to show

$$\lim_{p\to+\infty} |\overset{1}{u}_p(0)| = \overset{1}{M}_0, \tag{3.25}$$

$$\lim_{p\to+\infty} p|(\overset{1}{u}_p)'(1)| = \overset{1}{D}_1. \tag{3.26}$$

From [37, 38] we know that the least energy solutions satisfy the following energy condition

$$p\int_0^1 |\overset{1}{u}_p|^{p+1} r dr = 4e + o(1), \quad \text{as } p \to +\infty. \tag{3.27}$$

Identity (3.25) was obtained in [1] by a contradiction argument which uses (3.27) and Proposition 3.13. Here we write a more direct proof which exploits also a pointwise estimate recently obtained in [17] for the rescaled function $\overset{1}{z}_{0,p}$ of the positive solution $\overset{1}{u}_p$. Indeed, in [17, Lemma 4.4 and Proposition 4.3] it is proven that, for any $\delta > 0$, there exist $R_\delta > 1$, $C_\delta > 0$ and $p_\delta > 1$ such that

$$\overset{1}{z}_{0,p}(y) \leq (4 - \delta)\log\frac{1}{|y|} + C_\delta, \quad 2R_\delta \leq |y| \leq \frac{r}{\overset{1}{\varepsilon}_{0,p}}, \tag{3.28}$$

provided $p \geq p_\delta$. From (3.28) one immediately derives (3.25) using the energy estimate (3.27), changing variable in the integral and passing to the limit by Lebesgue's theorem (thanks to (3.28)) and the results in Proposition 3.13:

$$\begin{aligned} 4e + o(1) &\overset{(3.27)}{=} p\int_0^1 |\overset{1}{u}_p(r)|^{p+1} r dr \\ &= |\overset{1}{u}_p(0)|^2 \int_0^{1/\overset{1}{\varepsilon}_{0,p}} \left(1 + \frac{\overset{1}{z}_{0,p}}{p}\right)^{p+1} r dr \\ &\overset{(3.28)+\text{Prop.}3.13}{=} |\overset{1}{u}_p(0)|^2 \left(\int_0^\infty e^{Z_0} r dr + o(1)\right) \\ &= |\overset{1}{u}_p(0)|^2 \left(2\theta_0 + o(1)\right) \end{aligned}$$

as $p \to +\infty$. Identity (3.25) follows recalling that $\theta_0 = 2$, and that $\overset{1}{M}_0 = \sqrt{e}$.
Similarly, recalling that $\overset{1}{D}_1 = (2 + \theta_0)\sqrt{e} = 4\sqrt{e}$, one has that (3.26):

$$p|(\overset{1}{u}_p)'(1)| \overset{\text{Lemma } 3.4}{=} p\int_0^1 |\overset{1}{u}_p(r)|^p r dr = \overset{1}{u}_p(0)\int_0^{1/\overset{1}{\varepsilon}_{0,p}} \left(1 + \frac{\overset{1}{z}_{0,p}}{p}\right)^p r dr$$

$$\overset{(3.25)}{=} \quad \overset{1}{M}_0 \int_0^{\infty} e^{Z_0} r dr + o(1) = (2+\theta_0)\overset{1}{M}_0 + o(1) = 4\sqrt{e} + o(1),$$

as $p \to +\infty$. □

3.4. **The inductive step.** In this section we prove Propositions 3.12 and 3.14. In order to shorten the notations let us set

$$\begin{aligned} \varepsilon_p &:= \overset{m}{\varepsilon}_{m-1,p}, \\ s_p &:= \overset{m}{s}_{m-1,p}, \\ r_p &:= \overset{m}{r}_{m-1,p}, \\ z_p &:= \overset{m}{z}_{m-1,p}, \\ a_p &:= \overset{m}{a}_{m-1,p}\ (<0), \\ b_p &:= \overset{m}{b}_{m-1,p}\ (>0). \end{aligned} \tag{3.29}$$

Observe that z_p solves

$$\begin{cases} (z_p)'' + \dfrac{1}{r+\frac{s_p}{\varepsilon_p}}(z_p)' + \left|1+\dfrac{z_p}{p}\right|^{p-1}\left(1+\dfrac{z_p}{p}\right) = 0, & r \in (a_p, b_p) \\ z_p(0) = (z_p)'(0) = 0 \\ z_p(a_p) = -p \\ z_p(b_p) = -p \\ z_p < 0. \end{cases} \tag{3.30}$$

Lemma 3.15 (Relation between $\overset{m}{u}_p$ and $\overset{m-1}{u}_p$)**.** *Let $m \in \mathbb{N}$, $m \geq 2$, then*

$$\overset{m-1}{u}_p(r) = (r_p)^{\frac{2}{p-1}} \overset{m}{u}_p(r_p r). \tag{3.31}$$

As a consequence,

$$\overset{\mathrm{m-1}}{r}_{j,p} = \frac{\overset{\mathrm{m}}{r}_{j,p}}{r_p}, \qquad 1 \leq j \leq m-1, \tag{3.32}$$

$$\overset{\mathrm{m-1}}{s}_{j,p} = \frac{\overset{\mathrm{m}}{s}_{j,p}}{r_p}, \qquad 0 \leq j < m-1, \tag{3.33}$$

$$\frac{1}{\overset{\mathrm{m-1}}{\varepsilon}_{j,p}} = \frac{r_p}{\overset{\mathrm{m}}{\varepsilon}_{j,p}}, \qquad 0 \leq j < m-1. \tag{3.34}$$

Proof. By a direct computation it is easy to verify that the function $w(r) := (r_p)^{\frac{2}{p-1}} \overset{m}{u}_p(r_p r)$, $r \in (0,1)$, solves (3.1) and (3.2). Moreover, it has $m-1$ nodal regions and so, by the uniqueness, $w = \overset{m-1}{u}_p$. □

Hence, we have the following.

Lemma 3.16. *Let $m \in \mathbb{N}$, $m \geq 2$, then*

$$\frac{\overset{\mathrm{m-1}}{s}_{j,p}}{\overset{\mathrm{m-1}}{\varepsilon}_{j,p}} = \frac{\overset{\mathrm{m}}{s}_{j,p}}{\overset{\mathrm{m}}{\varepsilon}_{j,p}}, \qquad 0 \leq j < m-1, \tag{3.35}$$

$$\frac{\overset{\mathrm{m-1}}{r}_{j,p}}{\overset{\mathrm{m-1}}{\varepsilon}_{j,p}} = \frac{\overset{\mathrm{m}}{r}_{j,p}}{\overset{\mathrm{m}}{\varepsilon}_{j,p}}, \qquad 1 \leq j < m-1, \tag{3.36}$$

$$r_p = \frac{\overset{\mathrm{m}}{\varepsilon}_{j,p}}{\overset{\mathrm{m-1}}{\varepsilon}_{j,p}} \overset{(*)}{=} \frac{\overset{\mathrm{m}}{s}_{j,p}}{\overset{\mathrm{m-1}}{s}_{j,p}} \overset{(*)}{=} \frac{\overset{\mathrm{m}}{r}_{j,p}}{\overset{\mathrm{m-1}}{r}_{j,p}}, \qquad 0 \leq j < m-1, j \neq 0 \ \textit{in} \ (*). \tag{3.37}$$

As a consequence,

$$|\overset{m-1}{u}_p(\overset{m-1}{s}_{j,p})| = (r_p)^{\frac{2}{p-1}} |\overset{m}{u}_p(\overset{m}{s}_{j,p})| < |\overset{m}{u}_p(\overset{m}{s}_{j,p})|, \qquad 0 \leq j < m-1. \tag{3.38}$$

Lemma 3.17. *Let $m \geq 2$. If Theorem 3.9 holds for radial solutions $\overset{m-1}{u}_p$ with $m-2$ interior zeros, then*

$$\lim_{p\to+\infty}(r_p)^{-\frac{2}{p-1}}(\overset{m}{r}_{i,p})^{\frac{2}{p-1}} = \overset{m-1}{R}_i, \qquad i = 1,\dots,m-2 \ (\text{if } m\geq 3),$$

$$\lim_{p\to+\infty}(r_p)^{\frac{2}{p-1}}p|(\overset{m}{u}_p)'(\overset{m}{r}_{i,p})|(\overset{m}{r}_{i,p}) = \overset{m-1}{D}_i, \qquad i = 1,\dots,m-1,$$

$$\lim_{p\to+\infty}(r_p)^{-\frac{2}{p-1}}(\overset{m}{s}_{i,p})^{\frac{2}{p-1}} = \overset{m-1}{S}_i, \qquad i = 0,\dots,m-2,$$

$$\lim_{p\to+\infty}(r_p)^{\frac{2}{p-1}}|\overset{m}{u}_p(\overset{m}{s}_{i,p})| = \overset{m-1}{M}_i, \qquad i = 0,\dots,m-2,$$

where $\overset{m-1}{R}_i$, $\overset{m-1}{D}_i$, $\overset{m-1}{S}_i$, $\overset{m-1}{M}_i$ are the constants in Theorem 3.9.

Proof. By (3.31), (3.32), (3.33), and the inductive assumption,

$$\lim_{p\to+\infty}(r_p)^{-\frac{2}{p-1}}(\overset{m}{r}_{i,p})^{\frac{2}{p-1}} \overset{(3.32)}{=} \lim_{p\to+\infty}(\overset{m-1}{r}_{i,p})^{\frac{2}{p-1}} = \overset{m-1}{R}_i,$$

$$\lim_{p\to+\infty}(r_p)^{\frac{2}{p-1}}p|(\overset{m}{u}_p)'(\overset{m}{r}_{i,p})|(\overset{m}{r}_{i,p}) \overset{(3.31)}{\underset{(3.32)}{=}} \lim_{p\to+\infty}p|(\overset{m-1}{u}_p)'(\overset{m-1}{r}_{i,p})|(\overset{m-1}{r}_{i,p}) = \overset{m-1}{D}_i,$$

$$\lim_{p\to+\infty}(r_p)^{-\frac{2}{p-1}}(\overset{m}{s}_{i,p})^{\frac{2}{p-1}} \overset{(3.33)}{=} \lim_{p\to+\infty}(\overset{m-1}{s}_{i,p})^{\frac{2}{p-1}} = \overset{m-1}{S}_i,$$

$$\lim_{p\to+\infty}(r_p)^{\frac{2}{p-1}}|\overset{m}{u}_p(\overset{m}{s}_{i,p})| \overset{(3.31)}{\underset{(3.33)}{=}} \lim_{p\to+\infty}|\overset{m-1}{u}_p(\overset{m-1}{s}_{i,p})| = \overset{m-1}{M}_i.$$

□

The previous result implies that, in order to prove Proposition 3.12, it is enough to study the asymptotic behavior of the 4 quantities

$$(r_p)^{\frac{2}{p-1}}, \quad p|(\overset{m}{u}_p)'(1)|, \quad (s_p)^{\frac{2}{p-1}}, \quad |\overset{m}{u}_p(s_p)|.$$

We start with a preliminary result.

Lemma 3.18. *Let $m \in \mathbb{N}$, $m \geq 2$. For any $p > 1$,*

$$|\overset{m}{u}_p(s_p)|^{p-1} \geq \lambda_1 > 0, \tag{3.39}$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H^1_0(\Omega)$, as a consequence

$$\varepsilon_p = \left[p|\overset{m}{u}_p(s_p)|^{p-1}\right]^{-\frac{1}{2}} = o(1) \qquad \text{as } p\to+\infty. \tag{3.40}$$

Proof. (3.40) clearly follows from (3.39). The proof of (3.39) is similar to the one of (3.24). Indeed let $\Omega_p := \{r_p < |x| < 1\}$, then by the monotonicity property of the first eigenvalue $\lambda_1(\Omega) \leq \lambda_1(\Omega_p)$, moreover $|u_p(s_p)| = \|u_p\|_{L^\infty(\Omega_p)}$ and so, by Poincare inequality:

$$\int_{\Omega_p}|\nabla u_p|^2\,dx = \int_{\Omega_p}|u_p|^{p+1}\,dx \leq |u_p(s_p)|^{p-1}\int_{\Omega_p}|u_p|^2\,dx \leq \frac{|u_p(s_p)|^{p-1}}{\lambda_1(\Omega)}\int_{\Omega_p}|\nabla u_p|^2\,dx.$$

□

From this we easily deduce the following.

Lemma 3.19 (The nodal lines shrink to $(0,0)$)**.** *Let $m \in \mathbb{N}^+$, $m \geq 2$. We have*

$$s_p = o(1) \ \textit{(and so also } r_p = o(1)\textit{)} \qquad \textit{as} \ \ p\to+\infty.$$

Proof. If, by contradiction, $s_{p_n} \geq \alpha > 0$ for a sequence $p_n \to +\infty$ as $n \to +\infty$, then, by Lemma 3.3,

$$\sqrt{p_n}|\overset{m}{u}_p(s_{p_n})| \leq \frac{C}{|\alpha|^{\frac{1}{2}}} \quad \text{for } n \text{ large}.$$

So, the sequence $(\sqrt{p_n}|\overset{m}{u}_p(s_{p_n})|)_n$ would be bounded in contradiction with (3.39). □

Lemma 3.20. *Let $m \in \mathbb{N}$, $m \geq 2$. If Theorem 3.9 holds for the solution $\overset{m-1}{u}_p$ with $m-2$ interior zeros, then there exist $p_m > 1$, $K_m, C_m > 0$ such that, for $p \geq p_m$,*

$$0 < K_m \leq (r_p)^{\frac{2}{p-1}} (< 1), \tag{3.41}$$

$$(0 < C \leq)\ |\overset{m}{u}_p(s_p)| \leq C_m. \tag{3.42}$$

Proof. The upper bound in (3.41) is immediate, since $r_p < 1$, moreover

$$\begin{aligned} p\int_0^1 |\overset{m-1}{u}_p(r)|^{p+1} r dr &\overset{\text{Lemma 3.7}}{=} \left(1+\frac{1}{p}\right)\frac{1}{4}\left[p(\overset{m-1}{u}_p)'(1)\right]^2 \\ &\overset{\text{Thm 3.9 for } \overset{m-1}{u}_p}{=} \frac{1}{4}\left[\overset{m-1}{D}_{m-1}\right]^2 + o(1), \end{aligned} \tag{3.43}$$

as $p \to +\infty$, where for the last equality we have used the assumption that Theorem 3.9 holds for $\overset{m-1}{u}_p$. Moreover,

$$\begin{aligned} p\int_0^1 |\overset{m-1}{u}_p(r)|^{p+1} r dr &\overset{(3.31)}{=} p(r_p)^{\frac{2(p+1)}{p-1}} \int_0^1 |\overset{m}{u}_p(r_p r)|^{p+1} r dr \\ &= p(r_p)^{\frac{4}{p-1}} \int_0^{r_p} |\overset{m}{u}_p(s)|^{p+1} s ds \\ &\leq (r_p)^{\frac{4}{p-1}} p \int_0^1 |\overset{m}{u}_p(s)|^{p+1} s ds \\ &\overset{(3.7)}{\leq} (r_p)^{\frac{4}{p-1}} E_m \end{aligned} \tag{3.44}$$

for $p \geq p_m$. The lower bound in (3.41) then follows combining (3.43) with (3.44).
The lower bound in (3.42) is due to (3.39), moreover from (3.41) and from Lemma 3.17 (for which again we need the assumption on the validity of Theorem 3.9 for $\overset{m-1}{u}_p$) we also obtain the upper bound in (3.42):

$$\overset{m-1}{M}_{m-2} + o(1) \overset{\text{Lemma 3.17}}{=} (r_p)^{\frac{2}{p-1}} |\overset{m}{u}_p(\overset{m}{s}_{m-2,p})| \overset{(3.41)}{\geq} K_m |\overset{m}{u}_p(\overset{m}{s}_{m-2,p})| \overset{(3.5)}{>} K_m |\overset{m}{u}_p(s_p)|.$$

□

Definition 3.21. *Let $m \in \mathbb{N}$, $m \geq 2$. If Theorem 3.9 holds for the solution $\overset{m-1}{u}_p$ with $m-2$ interior zeros then, by (3.41) it is well defined, up to a subsequence,*

$$R := \lim_{p\to+\infty} (r_p)^{\frac{2}{p-1}} \ (\in (0,1]). \tag{3.45}$$

Moreover, by (3.42) we can also define, up to a subsequence,

$$M := \lim_{p\to+\infty} |\overset{m}{u}_p(s_p)| \ (> 0). \tag{3.46}$$

Next we prove that $R = \overset{m}{R}_{m-1}$ and $M = \overset{m}{M}_{m-1}$.

Proposition 3.22. *Let $m \in \mathbb{N}^+$, $m \geq 2$. If Theorem 3.9 holds for the solution $\overset{m-1}{u}_p$ with $m-2$ interior zeros, then there exists $\sigma > 0$ such that*

$$\lim_{p\to+\infty} \frac{s_p}{\varepsilon_p} = \sigma > 0. \tag{3.47}$$

Proof. By (3.10), there exists $C > 0$ independent of p such that

$$\frac{s_p}{\varepsilon_p} \leq C. \tag{3.48}$$

Assume, by contradiction, that, up to a subsequence, $\frac{s_p}{\varepsilon_p} \to 0$ as $p \to +\infty$. Then, since $0 < r_p < s_p$, necessarily also $a_p = \frac{r_p - s_p}{\varepsilon_p} \to 0$ and, by the change of variables, $r = s_p + \varepsilon_p s$, since $|\overset{m}{u}_p(s_p)|$ is bounded (see (3.42)), we deduce that

$$\begin{aligned} p\int_{r_p}^{s_p} |\overset{m}{u}_p(r)|^p r\,dr &\leq p|\overset{m}{u}_p(s_p)|^p \int_{r_p}^{s_p} r\,dr \\ &= p|\overset{m}{u}_p(s_p)|^p(\varepsilon_p)^2 \int_{a_p}^{0} \left(\frac{s_p}{\varepsilon_p}+s\right) ds \\ &= |\overset{m}{u}_p(s_p)| \left(\frac{s_p}{\varepsilon_p}|a_p| - \frac{a_p^2}{2}\right) \longrightarrow 0 \end{aligned}$$

as $p \to +\infty$. But choosing $s = r_p$ and $t = s_p$ into (3.8) one obtains, from Lemma 3.17 (which holds true by the assumption on the validity of Theorem 3.9 for $\overset{m-1}{u}_p$) and (3.45), that

$$p\int_{r_p}^{s_p} |\overset{m}{u}_p(r)|^p r\,dr \overset{(3.8)}{=} p\left|(\overset{m}{u}_p)'(r_p)\right| r_p \longrightarrow \frac{\overset{m-1}{D}_{m-1}}{R} > 0 \quad \text{as } p \to +\infty,$$

which gives a contradiction. □

Proposition 3.23. *Let $m \in \mathbb{N}^+$, $m \geq 2$. If Theorem 3.9 holds for the solution $\overset{m-1}{u}_p$ with $m-2$ interior zeros, then*

$$\frac{r_p}{\varepsilon_p} = o(1) \qquad \text{as } p \to +\infty. \tag{3.49}$$

Proof. Let $\sigma > 0$ be the constant in Proposition 3.22 and let $\rho := \lim_{p\to+\infty} \frac{r_p}{\varepsilon_p}$, then $\rho \in [0, \sigma]$, since $\frac{s_p}{\varepsilon_p} \to \sigma$ by Proposition 3.22 and $0 < r_p < s_p$. As a consequence $a_p = \frac{r_p - s_p}{\varepsilon_p} \to h := \rho - \sigma \in [-\sigma, 0]$. Proving $\rho = 0$ is then equivalent with showing that $h = -\sigma$.
Assume by contradiction that $h > -\sigma$. As $z_p(0) = 0$ and $z_p(a_p) = -p$, by the mean value theorem in $(a_p, 0)$ we have the existence of $t_p \in (a_p, 0)$ such that

$$|(z_p)'(t_p)| = \frac{|z_p(a_p) - z_p(0)|}{|a_p|} = \frac{p}{|a_p|} = \frac{p}{h + o(1)} \to +\infty, \tag{3.50}$$

as $p \to +\infty$. Moreover, since $h > -\sigma$ by assumption, there exists $C_1 > 0$ such that

$$t_p \geq -\sigma + C_1 \tag{3.51}$$

for p large. On the other hand, from the equation (3.30),

$$\left[(z_p)'(r)\left(r + \frac{s_p}{\varepsilon_p}\right)\right]' = -\left(r + \frac{s_p}{\varepsilon_p}\right)\left|1 + \frac{z_p}{p}\right|^{p-1}\left(1 + \frac{z_p}{p}\right), \text{ for } r \in (a_p, b_p).$$

Recall that $(z_p)'(0) = 0$ and that, by definition, $\left|1 + \frac{z_p}{p}\right| \leq 1$. Then, integrating on $(t_p, 0)$, we have for p large that

$$|(z_p)'(t_p)|\left|t_p + \frac{s_p}{\varepsilon_p}\right| \leq \int_{t_p}^{0} \left|s + \frac{s_p}{\varepsilon_p}\right| ds,$$

from which we obtain

$$|(z_p)'(t_p)|\,|t_p + \sigma + o(1)| \leq \int_{-\sigma}^{0} (|s| + |\sigma| + o(1))\, ds \leq C_2. \tag{3.52}$$

But (3.51) implies that $|t_p + \sigma + o(1)| \geq C_3 > 0$ for p large and so, by (3.52),

$$|(z_p)'(t_p)| \leq C_3$$

uniformly in p, for p large, reaching a contradiction with (3.50). □

Using Proposition 3.22 and Proposition 3.23 we can prove the convergence of the last rescaling z_p of $\overset{m}{u}_p$.

Proposition 3.24. *Let $m \in \mathbb{N}^+$, $m \geq 2$. Assume that Theorem 3.9 holds for the solution $\overset{m-1}{u}_p$ with $m-2$ interior zeros. Let $\sigma > 0$ be the constant in Proposition 3.22, then*

$$z_p \longrightarrow Z(\cdot + \sigma) \quad \textit{in } C^1_{loc}(-\sigma, +\infty), \quad \textit{as } p \to +\infty, \tag{3.53}$$

where

$$Z(r) := \log\left(\frac{2\theta^2 \beta^\theta |x|^{\theta-2}}{(\beta^\theta + |x|^\theta)^2}\right), \tag{3.54}$$

with

$$\beta = \frac{1}{\sqrt{2}}(\theta+2)^{\frac{\theta+2}{2\theta}}(\theta-2)^{\frac{\theta-2}{2\theta}} \tag{3.55}$$

and

$$\theta := \sqrt{2(\sigma^2+2)}\ (>2) \tag{3.56}$$

is a singular radial solution of

$$\begin{cases} -\Delta Z = e^Z + 2\pi(2-\theta)\delta_0 & \textit{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^Z dx = 4\pi\theta, \end{cases}$$

where δ_0 is the Dirac measure centered at 0.

Proof. The rescaled function z_p satisfies (3.30) in (a_p, b_p) with $\left|1 + \frac{z_p}{p}\right| \leq 1$. Observe that, by Propositions 3.22 and 3.23,

$$a_p \to -\sigma < 0 \quad \text{ and } \quad b_p \to +\infty \qquad \text{as } p \to +\infty.$$

Following the proof of [1, Theorem 1.1] one can then show that z_p is locally uniformly bounded in $(-\sigma, +\infty)$. By standard elliptic estimates, we have that z_p is uniformly bounded in $C^2_{loc}(-\sigma, +\infty)$, so $z_p \to z$ in $C^1_{loc}(-\sigma, +\infty)$, as $p \to +\infty$, where z solves

$$\begin{cases} z'' + \dfrac{1}{r+\sigma} z' + e^z = 0, & r \in (-\sigma, +\infty), \\ z(0) = z'(0) = 0, \\ z \leq 0. \end{cases} \tag{3.57}$$

Let us set $Z(r) := z(r - \sigma)$, so Z satisfies

$$\begin{cases} Z'' + \dfrac{1}{r} Z' + e^Z = 0, & r \in (0, +\infty), \\ Z(\sigma) = Z'(\sigma) = 0, \\ Z \leq 0. \end{cases} \tag{3.58}$$

Hence Z must be the function in (3.54) with (3.55) and (3.56), which satisfies

$$-\Delta Z = e^Z + 2\pi H \delta_0 \quad \text{ in } \mathbb{R}^2,$$

where δ_0 is the Dirac measure centered at 0 and $H := -\int_0^\sigma e^{Z(s)} s ds$ (see [27, Proof of Proposition 3.1]).

Using (3.54) with (3.55) and (3.56) we have that

$$-H = \int_0^\sigma e^{Z(s)} s ds \overset{(3.54)}{=} 2\theta^2\beta^\theta \int_0^\sigma \frac{s^{\theta-1}}{(\beta^\theta + s^\theta)^2} ds = \frac{2\theta\sigma^\theta}{\beta^\theta + \sigma^\theta} \overset{(3.55)\&(3.56)}{=} \theta - 2 \tag{3.59}$$

and similarly that

$$\int_{\mathbb{R}^2} e^Z dx = 4\pi\theta.$$

□

Now we aim to show that $\sigma = \sigma_{m-1}$. Once this equality has been established, Proposition 3.14 follows by combining (3.40) and Propositions 3.22, 3.23 and 3.24. Observe that, by (3.56) and (3.20), the equality $\sigma = \sigma_{m-1}$ is equivalent to $\theta = \theta_{m-1}$. We begin with some auxiliary lemmas.

Lemma 3.25. *Let $m \in \mathbb{N}^+$, $m \geq 2$. Assume that Theorem 3.9 holds for the solution $\overset{m-1}{u}_p$ with $m-2$ interior zeros. Then*

$$p \int_{s_p}^{1} |\overset{m}{u}_p(r)|^p r\, dr = (M + o(1))\, I_p$$

as $p \to +\infty$, where $M > 0$ is the constant in (3.46) and

$$I_p := \int_0^{b_p} \left|1 + \frac{z_p(r)}{p}\right|^p \left(\frac{s_p}{\varepsilon_p} + r\right) dr. \tag{3.60}$$

Proof. The claim follows from the change of variable $s = s_p + \varepsilon_p r$ and the definitions of z_p and ε_p, because

$$\begin{aligned} p \int_{s_p}^{1} |\overset{m}{u}_p(s)|^p s\, ds &= p\varepsilon_p \int_0^{b_p} |\overset{m}{u}_p(s_p + \varepsilon_p r)|^p (s_p + \varepsilon_p r)\, dr \\ &= p\,(\varepsilon_p)^2 \int_0^{b_p} |\overset{m}{u}_p(s_p + \varepsilon_p r)|^p \left(\frac{s_p}{\varepsilon_p} + r\right) dr \\ &\overset{(3.103)}{=} |\overset{m}{u}_p(s_p)| \int_0^{b_p} \left|1 + \frac{z_p(r)}{p}\right|^p \left(\frac{s_p}{\varepsilon_p} + r\right) dr \\ &\overset{(3.46)}{=} (M + o(1)) \int_0^{b_p} \left|1 + \frac{z_p(r)}{p}\right|^p \left(\frac{s_p}{\varepsilon_p} + r\right) dr. \end{aligned} \tag{3.61}$$

□

Using Proposition 3.24 we can now integrate $|\overset{m}{u}_p(r)|^{p+1} r$.

Lemma 3.26. *Let $m \in \mathbb{N}^+$, $m \geq 2$. Assume that Theorem 3.9 holds for the solution $\overset{m-1}{u}_p$ with $m-2$ interior zeros. Then*

$$p \int_{r_p}^{s_p} |\overset{m}{u}_p|^{p+1} r\, dr = M^2(\theta - 2) + o(1), \tag{3.62}$$

as $p \to +\infty$, where $M > 0$ is the constant in (3.46) and θ (> 2) is the constant in (3.56).

Proof. To compute (3.62) we use the change of variable $r = s_p + \varepsilon_p s$, exploit the definition of z_p and ε_p, use that $\frac{s_p}{\varepsilon_p} = \sigma + o(1)$, $a_p = -\sigma + o(1)$ (by Propositions 3.23 and 3.22), use the convergence of z_p to Z given by Proposition 3.24 and the fact that $|\overset{m}{u}_p(s_p)|^2 = M^2 + o(1)$ by (3.46) (observe that, by assumption, Theorem 3.9 holds for the solution $\overset{m-1}{u}_p$). Namely, we have that

$$\begin{aligned} p \int_{r_p}^{s_p} |\overset{m}{u}_p|^{p+1} r\, dr &= p(\varepsilon_p)^2 \int_{a_p}^{0} |\overset{m}{u}_p(s_p + \varepsilon_p s)|^{p+1} \left(\frac{s_p}{\varepsilon_p} + s\right) ds \\ &= |\overset{m}{u}_p(s_p)|^2 \int_{a_p}^{0} \left|1 + \frac{z_p}{p}\right|^{p+1} \left(\frac{s_p}{\varepsilon_p} + s\right) ds \\ &\overset{(*)}{=} M^2 \int_{-\sigma}^{0} e^{Z(\sigma+s)} (\sigma + s) ds + o(1) \\ &= M^2 \int_0^{\sigma} e^{Z(s)} s ds + o(1) \\ &= M^2(\theta - 2) + o(1), \end{aligned}$$

as $p \to +\infty$, where the passage to the limit in $(*)$ is due to Lebesgue's theorem ($(1 + \frac{z_p}{p}) \leq 1$ and the integration is on a bounded interval), while the last equality is a consequence of (3.54) with (3.55) and (3.56) in Proposition 3.24 (as already observed in (3.59)). □

Lemma 3.27. *Let $m \in \mathbb{N}^+$, $m \geq 2$. Assume that Theorem 3.9 holds for the solution $\overset{m-1}{u}_p$ with $m-2$ interior zeros. Then*

$$p\int_{s_p}^1 |\overset{m}{u}_p|^{p+1} r\,dr \leq \left(M^2 + o(1)\right) I_p,$$

as $p \to +\infty$, where $M > 0$ is the constant in (3.46) and I_p is defined in (3.60).

Proof. We make the change of variable $r = s_p + \varepsilon_p s$ in the left hand side and use the definition of z_p and of ε_p, then we observe that the inequality $(1 + \frac{z_p}{p}) \leq 1$ implies $(1 + \frac{z_p}{p})^{p+1} \leq (1 + \frac{z_p}{p})^p$, which allows to estimate the integral with I_p, that is,

$$\begin{aligned} p\int_{s_p}^1 |\overset{m}{u}_p(r)|^{p+1} r\,dr &= |\overset{m}{u}_p(s_p)|^2 \int_0^{b_p} \left|1 + \frac{z_p}{p}\right|^{p+1} \left(\frac{s_p}{\varepsilon_p} + s\right) ds \\ &\leq |\overset{m}{u}_p(s_p)|^2 I_p \\ &\overset{(3.46)}{=} \left(M^2 + o(1)\right) I_p. \end{aligned}$$

□

We are also able to compute the value of the derivative in the last zero of a solution.

Lemma 3.28. *Let $m \in \mathbb{N}$, $m \geq 2$. If Theorem 3.9 holds for radial solutions $\overset{m-1}{u}_p$ with $m-2$ interior zeros, then*

$$\overset{m-1}{M}_{m-2}\left(\theta_{m-2} + 2\right) = RM\left(\theta - 2\right), \tag{3.63}$$

where $R \in (0, 1]$ is the constant in (3.45), $M > 0$ is the constant in (3.46), θ (> 2) is the constant in (3.56) and $\overset{m-1}{M}_{m-2}$ and θ_{m-2} are given in Theorem 3.9.

Proof. Choosing $s = r_p$ and $t = s_p$ into (3.8) one has that

$$p|(\overset{m}{u}_p)'(r_p)|(r_p) = p\int_{r_p}^{s_p} |\overset{m}{u}_p(r)|^p r\,dr. \tag{3.64}$$

We can now compute this integral similarly as we did in the proof of Lemma 3.26, making the change of variable $r = s_p + \varepsilon_p s$ and passing to the limit as $p \to +\infty$. We use that $\frac{s_p}{\varepsilon_p} = \sigma + o(1)$, $a_p = -\sigma + o(1)$ (by Proposition 3.23 and Proposition 3.22), the convergence of z_p to Z in Proposition 3.24 and also that $|\overset{m}{u}_p(s_p)| = M + o(1)$ by (3.46) (by assumption Theorem 3.9 holds for the solution $\overset{m-1}{u}_p$, so all these results hold), hence we obtain that

$$\begin{aligned} p\int_{r_p}^{s_p} |\overset{m}{u}_p(r)|^p r\,dr &= p(\varepsilon_p)^2 \int_{a_p}^0 |\overset{m}{u}_p(s_p + s\varepsilon_p)|^p \left(\frac{s_p}{\varepsilon_p} + s\right) ds \\ &= |\overset{m}{u}_p(s_p)| \int_{a_p}^0 \left|1 + \frac{z_p}{p}\right|^p \left(\frac{s_p}{\varepsilon_p} + s\right) ds \\ &\overset{(*)}{=} M\int_{-\sigma}^0 e^{Z(\sigma+s)}(\sigma + s)ds + o(1) \\ &= M\int_0^{\sigma} e^{Z(s)} s ds + o(1) \\ &\overset{(**)}{=} M(\theta - 2) + o(1), \end{aligned} \tag{3.65}$$

as $p \to +\infty$, where the passage to the limit in $(*)$ is again due to Lebesgue's theorem ($(1 + \frac{z_p}{p}) \leq 1$ and the integration is on a bounded interval), while the equality in $(**)$ is again a consequence of (3.54) with (3.55) and (3.56) in Proposition 3.24 (as already observed in (3.59)). Putting together (3.64) with (3.65) one has

$$p|(\overset{m}{u}_p)'(r_p)|(r_p) = M(\theta - 2) + o(1) \tag{3.66}$$

as $p\to+\infty$. On the other side, since by assumption Theorem 3.9 holds for the solution $\overset{m-1}{u}_p$, one can use Lemma 3.17, the convergence in (3.45) and exploit the expression of the known constant $\overset{m-1}{D}_{m-1}$ (see (2.4)) to obtain

$$p|(\overset{m}{u}_p)'(r_p)|(r_p) \overset{\text{Lemma 3.17}}{\underset{(3.45)}{=}} R^{-1}\overset{m-1}{D}_{m-1}+o(1), \overset{(2.4)}{=} R^{-1}\overset{m-1}{M}_{m-2}\left(\theta_{m-2}+2\right)+o(1), \tag{3.67}$$

as $p\to+\infty$. Identity (3.63) then follows from (3.67) and (3.66). ☐

Lemma 3.29. *Let $m\in\mathbb{N}$, $m\geq 2$. Assume that Theorem 3.9 holds for radial solutions $\overset{m-1}{u}_p$ with $m-2$ interior zeros. Then*

$$I_p=\theta+2+o(1), \tag{3.68}$$

as $p\to+\infty$, where I_p is defined in (3.60) and θ (>2) is the constant in Proposition 3.24.

Proof. Observe that, differently from the previous proofs, we cannot pass to the limit into the integral that defines I_p, since now Lebesgue's Dominated Convergence Theorem does not apply. We follow similar ideas as in [27].

STEP 1. We show that I_p is bounded.

Recalling the definition of I_p and repeating the same computations as in (3.61) one has that

$$(0\leq)\ I_p \overset{(3.61)}{=} \frac{p\int_{s_p}^1 |\overset{m}{u}_p(r)|^p r\,dr}{|\overset{m}{u}(s_p)|} \overset{(3.42)}{\leq} Cp\int_0^1 |\overset{m}{u}_p|^p r dr \overset{\text{Hölder}}{\leq} Cp\left(\int_0^1 |\overset{m}{u}_p|^{p+1} r dr\right)^{\frac{p}{p+1}} \overset{(3.7)}{\leq} C.$$

STEP 2. We show that

$$\liminf_{p\to+\infty} I_p\geq\theta+2.$$

By the convergence of z_p to Z in Proposition 3.24 and the one of $\frac{s_p}{\varepsilon_p}$ to σ in Proposition 3.22, using Fatou's lemma we obtain

$$\begin{aligned}
\liminf_{p\to+\infty} I_p &= \liminf_{p\to+\infty}\int_0^{b_p}\left|1+\frac{z_p}{p}\right|^p\left(\frac{s_p}{\varepsilon_p}+r\right)dr \\
&\overset{Fatou}{\geq} \int_0^{+\infty} e^{Z(\sigma+r)}(\sigma+r)dr \\
&= \int_\sigma^{+\infty} e^{Z(s)}s ds \\
&\overset{(3.54)}{=} 2\theta^2\beta^\theta\int_\sigma^{+\infty}\frac{s^{\theta-1}}{(\beta^\theta+s^\theta)^2}ds \\
&= \frac{2\theta\beta^\theta}{\beta^\theta+\sigma^\theta} \\
&\overset{(3.55),(3.56)}{=} \theta+2.
\end{aligned}$$

STEP 3. We prove that

$$\frac{1}{4}\left[I_p\right]^2-I_p\leq\frac{(\theta-2)^2}{4}+(\theta-2)+o(1), \tag{3.69}$$

as $p\to+\infty$.

From (3.9) and Lemma 3.25

$$M^2\left[I_p\right]^2+o(1) \overset{\text{Lemma 3.25}}{=} \left[p\int_{s_p}^1 |\overset{m}{u}_p(r)|^p r\,dr\right]^2 \overset{(3.9)}{=} p^2\left[(\overset{m}{u}_p)'(1)\right]^2,$$

as $p \to +\infty$, moreover, by Lemma 3.7,

$$\left[(\overset{m}{u}_p)'(1)\right]^2 = \frac{4}{(p+1)} \int_0^1 |\overset{m}{u}_p(r)|^{p+1} r\, dr$$

and so

$$\frac{1}{4}\left[I_p\right]^2 = \frac{1}{M^2}\frac{p}{(p+1)}p \int_0^1 |\overset{m}{u}_p(r)|^{p+1} r\, dr + o(1) \tag{3.70}$$

as $p \to +\infty$. We decompose the integral in (3.70) into the sum of three terms:

$$p\int_0^1 |\overset{m}{u}_p(r)|^{p+1} r\, dr \;=\; p\int_0^{r_p} |\overset{m}{u}_p|^{p+1} r\, dr + p\int_{r_p}^{s_p} |\overset{m}{u}_p|^{p+1} r\, dr + p\int_{s_p}^{1} |\overset{m}{u}_p|^{p+1} r\, dr \tag{3.71}$$

For the first term we make the change of variable $r = sr_p$ and reduce it to the computation of the total energy for the solution $\overset{m-1}{u}_p$, which can be done using Lemma 3.7 and the inductive assumption that Theorem 3.9 holds for the solution $\overset{m-1}{u}_p$. Finally, we have the dependence on θ and M by exploiting Lemma 3.28, namely,

$$\begin{aligned}
p\int_0^{r_p} |\overset{m}{u}_p|^{p+1} r\, dr \quad &= \quad (r_p)^2 p\int_0^1 |\overset{m}{u}_p(sr_p)|^{p+1} s\, ds \\
&\overset{(3.31)}{=} \quad (r_p)^{-\frac{4}{p-1}} p\int_0^1 |\overset{m-1}{u}_p(s)|^{p+1} s\, ds \\
&\overset{(3.45)}{=} \quad R^{-2} p\int_0^1 |\overset{m-1}{u}_p(s)|^{p+1} s\, ds + o(1) \\
&\overset{\text{Lemma } 3.7}{=} \quad R^{-2}\frac{1}{4}\left[p(\overset{m-1}{u}_p)'(1)\right]^2 + o(1) \\
&\overset{\text{Theorem } 3.9 \text{ for } \overset{m-1}{u}_p}{=} \quad \frac{1}{4}\frac{(\overset{m-1}{D}_{m-1})^2}{R^2} + o(1) \\
&\overset{(2.4)}{=} \quad \frac{1}{4}\frac{(\overset{m-1}{M}_{m-2})^2}{R^2}(\theta_{m-2}+2)^2 + o(1) \\
&\overset{\text{Lemma } 3.28}{=} \quad M^2\frac{(\theta-2)^2}{4} + o(1),
\end{aligned} \tag{3.72}$$

as $p \to +\infty$. We use Lemma 3.26 and Lemma 3.27 to estimate the last two terms in (3.71), hence

$$p\int_0^1 |\overset{m}{u}_p(r)|^{p+1} r\, dr \le M^2\left[\frac{(\theta-2)^2}{4} + (\theta-2) + I_p\right] + o(1), \tag{3.73}$$

as $p \to +\infty$. Substituting (3.73) into (3.70) we deduce (3.69).

STEP 4. We show (3.68). Note that

$$\begin{aligned}
\frac{1}{4}\left[I_p\right]^2 - I_p \quad &\overset{\text{STEP 3}}{\le} \quad \frac{(\theta-2)^2}{4} + (\theta-2) + o(1), \\
&= \quad \frac{(\theta+2)^2}{4} - (\theta+2) + o(1),
\end{aligned} \tag{3.74}$$

as $p \to +\infty$. Since I_p is bounded by *STEP 1*, then, up to a subsequence, it converges to I_∞, as $p \to +\infty$. From (3.74) we then have that

$$F(I_\infty) \le F(\theta+2),$$

where $F : \mathbb{R} \to \mathbb{R}$ is the function $F(x) := \frac{x^2}{4} - x$. Since F is increasing for $x \ge 2$ and $I_\infty \ge \theta+2 (\ge 2)$ by *STEP 2*, it then follows that

$$I_\infty = \theta + 2.$$

□

Definition 3.30. *Let $m \in \mathbb{N}$, $m \geq 2$. If Theorem 3.9 holds for the solution $\overset{m-1}{u}_p$ with $m-2$ interior zeros, then we can define*

$$t := RM \ (>0), \tag{3.75}$$

where $R \in (0,1]$ is the constant in (3.45) and $M > 0$ is the constant in (3.46).
Then (3.63) can be written as

$$\theta = \frac{\overset{m-1}{M}_{m-2}(\theta_{m-2}+2)}{t} + 2, \tag{3.76}$$

where θ (>2) is the constant in (3.56).

Proposition 3.31. *Let $m \in \mathbb{N}$, $m \geq 2$. If Theorem 3.9 holds for the solution $\overset{m-1}{u}_p$ with $m-2$ interior zeros. Then the constant t defined as in (3.75) is the unique root of the equation*

$$\frac{\overset{m-1}{M}_{m-2}(\theta_{m-2}+2)}{2} \log x + x = 0,$$

namely

$$t = \frac{\overset{m-1}{M}_{m-2}\ (\theta_{m-2}+2)}{2} \mathcal{L}\left[\frac{2}{\overset{m-1}{M}_{m-2}\ (\theta_{m-2}+2)}\right],$$

where $\mathcal{L}$ is the Lambert function (i.e. the inverse function of $f(L) = Le^L$).

Proof. It is enough to show the following equality

$$1 = -\frac{(\theta-2)}{2} \log t, \tag{3.77}$$

the conclusion then follows by combining it with (3.76). To prove (3.77), observe that, for $s > \max\{-\frac{s_p}{\varepsilon_p}, -\sigma\}$,

$$\begin{aligned}
\log(s_p + \varepsilon_p s) - \log r_p &\overset{\text{Prop.3.22}}{=} \log(\sigma\varepsilon_p + \varepsilon_p s + o(1)\varepsilon_p) - \log r_p \\
&= \log(\sigma + s) + \log\frac{\varepsilon_p}{r_p} + o(1) \\
&= \log(\sigma+s) - \frac{p-1}{2}\log\left(p^{\frac{1}{(p-1)}}|\overset{m}{u}_p(s_p)|(r_p)^{\frac{2}{(p-1)}}\right) + o(1) \\
&\overset{(3.45)}{\underset{(3.46)}{=}} \log(\sigma+s) - \frac{p-1}{2}(\log t + o(1)) + o(1),
\end{aligned} \tag{3.78}$$

as $p \to +\infty$. Moreover, by multiplying both sides of the equation $-\left[(\overset{m}{u}_p)'(r)r\right]' = |\overset{m}{u}_p(r)|^{p-1}\overset{m}{u}_p(r)r$ by $(\log r - \log r_p)$ and integrating by parts, we have that

$$\begin{aligned}
\int_{r_p}^{s_p} |\overset{m}{u}_p(r)|^p r(\log r - \log r_p)dr &= (-1)^{m-1}\int_{r_p}^{s_p} |\overset{m}{u}_p(r)|^{p-1}\overset{m}{u}_p(r)r(\log r - \log r_p)dr \\
&= (-1)^m \int_{r_p}^{s_p}\left[(\overset{m}{u}_p)'(r)r\right]'(\log r - \log r_p)dr \\
&= (-1)^m(\overset{m}{u}_p)'(s_p)s_p(\log s_p - \log r_p) + (-1)^{m-1}\int_{r_p}^{s_p}(\overset{m}{u}_p)'(r)r\frac{1}{r}dr \\
&= (-1)^{m-1}\left(\overset{m}{u}_p(s_p) - \overset{m}{u}_p(r_p)\right) = |\overset{m}{u}_p(s_p)|.
\end{aligned} \tag{3.79}$$

In order to compute the integral on the left hand side of (3.79) we make the change of variable $r = s_p + \varepsilon_p t$ and recall the definition of z_p, to obtain that

$$\int_{r_p}^{s_p} |\overset{m}{u}_p(r)|^p r(\log r - \log r_p)dr = (\varepsilon_p)^2 \int_{a_p}^{0} |\overset{m}{u}_p(s_p+\varepsilon_p t)|^p \left(\frac{s_p}{\varepsilon_p} + t\right)(\log(s_p+\varepsilon_p t) - \log r_p)\, dt$$

$$= \frac{|\overset{m}{u}_p(s_p)|}{p}\int_{a_p}^{0}\left|1+\frac{z_p(t)}{p}\right|^p\left(\frac{s_p}{\varepsilon_p}+t\right)(\log(s_p+\varepsilon_p t)-\log r_p)\,dt.$$

We then pass to the limit as $p\to+\infty$, using the convergence of z_p to Z (Proposition 3.24), the computation already obtained in (3.78) and Lebesgue's convergence theorem:

$$\begin{aligned}
1 &\overset{(3.79)}{=} \frac{1}{p}\int_{a_p}^{0}\left|1+\frac{z_p(t)}{p}\right|^p\left(\frac{s_p}{\varepsilon_p}+t\right)(\log(s_p+\varepsilon_p t)-\log r_p)\,dt\\
&= \frac{1}{p}\int_0^{\sigma} e^{Z(s)}s\log s\,ds-\frac{p-1}{2p}\int_{-\sigma}^{0}e^{Z(\sigma+t)}(\sigma+t)\log\left(t+o(1)\right)dt+o(1)\\
&= -\frac{1}{2}\log t\int_0^{\sigma}e^{Z(s)}s ds+o(1)\\
&= -\frac{(\theta-2)}{2}\log t+o(1),
\end{aligned}$$

as $p\to+\infty$, where the last equality is a consequence of (3.54) with (3.55) and (3.56) in Proposition 3.24 (as already observed in (3.59)). □

Proposition 3.32. *Let $m\in\mathbb{N}^+$, $m\geq 2$. Assume that Theorem 3.9 holds for the solution $\overset{m-1}{u}_p$ with $m-2$ interior zeros and let θ (>2) be the constant in (3.56). Then*

$$\theta=\theta_{m-1}, \tag{3.80}$$

where θ_{m-1} is the constant in (2.1).

Proof. Substituting the value of t obtained in Proposition 3.31 in (3.76), we have that

$$\begin{aligned}
\theta &= \frac{\overset{m-1}{M}_{m-2}(\theta_{m-2}+2)}{t}+2=\frac{\overset{m-1}{M}_{m-2}(\theta_{m-2}+2)}{\frac{\overset{m-1}{M}_{m-2}\,(\theta_{m-2}+2)}{2}\mathcal{L}\left[\frac{2}{\overset{m-1}{M}_{m-2}\,(\theta_{m-2}+2)}\right]}+2\\
&= \frac{2}{\mathcal{L}\left[\frac{2}{(\theta_{m-2}+2)}e^{-2/(2+\theta_{m-2})}\right]}+2=\theta_{m-1}.
\end{aligned}$$

□

Proof of Proposition 3.14. Let $m\geq 2$ and we assume that Theorem 3.9 holds for the radial solution $\overset{m-1}{u}_p$ with $m-2$ interior zeros. We want to show that Theorem 3.10 holds, namely that (3.16), (3.17), (3.18) and (3.19) are satisfied. Observe that (3.16) is (3.40), while (3.17) follows from Proposition 3.23. By (3.80) in Proposition 3.32 and (3.56), $\sigma=\sigma_{m-1}$. Then, by Proposition 3.22, (3.18) holds true and therefore, by (3.54) and (3.55), we have that $Z=Z_{m-1}$. But then the convergence in (3.53) implies (3.19). □

Proof of Proposition 3.12. By assumption, Theorem 3.9 holds for the solution $\overset{m-1}{u}_p$ with $m-2$ interior zeros. We want to show that it holds for the solution $\overset{m}{u}_p$ with $m-1$ interior zeros, $m\geq 2$.

STEP 1. We prove the last convergence in (3.12), (3.13), (3.14) and (3.15). Namely we show that:

$$\lim_{p\to+\infty}(r_p)^{\frac{2}{p-1}}=\overset{m}{R}_{m-1}, \tag{3.81}$$

$$\lim_{p\to+\infty}p|(\overset{m}{u}_p)'(1)|=\overset{m}{D}_m, \tag{3.82}$$

$$\lim_{p\to+\infty}(s_p)^{\frac{2}{p-1}}=\overset{m}{S}_{m-1}, \tag{3.83}$$

$$\lim_{p\to+\infty}|\overset{m}{u}_p(s_p)|=\overset{m}{M}_{m-1}, \tag{3.84}$$

where $\overset{m}{M}_{m-1}$, $\overset{m}{R}_{m-1}$, $\overset{m}{D}_m$ *and* $\overset{m}{S}_{m-1}$ *are the constants in Theorem 3.9.*

Proof of STEP 1. Thanks to Lemma 3.5, one can define

$$D := \lim_{p\to+\infty} p|(\overset{m}{u}_p)'(1)|\ (\geq 0); \tag{3.85}$$

moreover, since by (3.41) $0 < K_m \leq (s_p)^{\frac{2}{p-1}} (< 1)$, also

$$S := \lim_{p\to+\infty} (s_p)^{\frac{2}{p-1}}\ (\in (0,1]). \tag{3.86}$$

Hence (3.81), (3.82), (3.83), (3.84) are equivalent to prove that

$$M = \overset{m}{M}_{m-1}, \tag{3.87}$$

$$R = \overset{m}{R}_{m-1}, \tag{3.88}$$

$$D = \overset{m}{D}_m, \tag{3.89}$$

$$S = \overset{m}{S}_{m-1}, \tag{3.90}$$

where M and R have been defined in (3.46) and (3.45). Following similar ideas as in the proof of (3.78) we obtain, for $s > 0$,

$$\begin{aligned}\log(s_p + \varepsilon_p s) &\overset{(3.18)}{=} \log(\sigma_{m-1}\varepsilon_p + \varepsilon_p s + o(1)\varepsilon_p)\\ &= \log(\sigma_{m-1} + s) + \frac{1}{2}\log(\varepsilon_p)^2 + o(1)\\ &\overset{(3.103)}{=} \log(\sigma_{m-1} + s) + \frac{1}{2}\log\frac{1}{p} - \frac{p-1}{2}\log|\overset{m}{u}_p(s_p)| + o(1).\end{aligned} \tag{3.91}$$

Then, as in the proof of (3.79), we multiply the equation $-\left[(\overset{m}{u}_p)'(r)r\right]' = |\overset{m}{u}_p(r)|^{p-1}\overset{m}{u}_p(r)r$ by $\log r$ and integrate by parts on the interval $(s_p, 1)$ to obtain that

$$\begin{aligned}\int_{s_p}^1 |\overset{m}{u}_p(r)|^p r\log r\,dr &= (-1)^{m-1}\int_{s_p}^1 |\overset{m}{u}_p(r)|^{p-1}\overset{m}{u}_p(r) r\log r\,dr\\ &= (-1)^m \int_{s_p}^1 \left[(\overset{m}{u}_p)'(r)r\right]'\log r\,dr\\ &= (-1)^{m-1}(\overset{m}{u}_p)'(s_p)s_p\log s_p + (-1)^{m-1}\int_{s_p}^1 (\overset{m}{u}_p)'(r)r\frac{1}{r}\,dr\\ &= (-1)^m\overset{m}{u}_p(s_p) = -|\overset{m}{u}_p(s_p)|.\end{aligned} \tag{3.92}$$

By the change of variable $r = s_p + \varepsilon_p t$ and recalling the definition of z_p it follows that

$$\begin{aligned}1 &\overset{(3.92)}{=} -\frac{1}{|\overset{m}{u}_p(s_p)|}\int_{s_p}^1 |\overset{m}{u}_p(r)|^p r\log r\,dr\\ &= -\frac{(\varepsilon_p)^2}{|\overset{m}{u}_p(s_p)|}\int_0^{b_p} |\overset{m}{u}_p(s_p + \varepsilon_p t)|^p\left(\frac{s_p}{\varepsilon_p} + t\right)\log(s_p + \varepsilon_p t)\,dt\\ &= -\frac{1}{p}\int_0^{b_p}\left|1 + \frac{z_p(t)}{p}\right|^p\left(\frac{s_p}{\varepsilon_p} + t\right)\log(s_p + \varepsilon_p t)\,dt\\ &\overset{(3.91)}{=} \frac{p-1}{2p} I_p\log|\overset{m}{u}_p(s_p)| + I_p o(1) - \overset{m}{J}_p,\end{aligned} \tag{3.93}$$

where

$$\overset{m}{J}_p := \frac{1}{p}\int_0^{b_p}\left|1 + \frac{z_p(t)}{p}\right|^p\left(\frac{s_p}{\varepsilon_p} + t\right)\log(\sigma_{m-1} + t)\,dt.$$

Following [27, Lemma 4.8] it is possible to show that $\overset{m}{J}_p = o(1)$ as $p \to +\infty$; moreover, $I_p = \theta_{m-1} + 2 + o(1)$ as $p \to +\infty$ by Lemma 3.29 and (3.80). Then, passing to the limit into (3.93),

$$1 = \frac{\theta_{m-1} + 2}{2} \log M,$$

namely,

$$M = e^{\frac{2}{2+\theta_{m-1}}} \overset{(2.2)}{=} \overset{m}{M}_{m-1},$$

which ends the proof of (3.87). Identity (3.87) implies (3.88), because

$$R \overset{\substack{(3.75)\\(3.87)}}{=} \frac{t}{\overset{m}{M}_{m-1}} \overset{\substack{(3.76)\\(3.80)}}{=} \frac{\overset{m-1}{M}_{m-2}(\theta_{m-2} + 2)}{\overset{m}{M}_{m-1}(\theta_{m-1} - 2)} \overset{(2.3)}{=} \overset{m}{R}_{m-1}.$$

Next we prove (3.89). Observe that

$$\begin{aligned} D &\overset{(3.85)}{=} \lim_{p\to+\infty} p|(\overset{m}{u}_p)'(1)| \\ &\overset{(3.9)}{=} \lim_{p\to+\infty} p \int_{s_p}^{1} |\overset{m}{u}_p(r)|^p r\, dr \\ &\overset{\substack{\text{Lemma 3.25}\\(3.87)}}{=} [\overset{m}{M}_{m-1} + o(1)] \lim_{p\to+\infty} I_p \\ &\overset{\substack{\text{Lemma 3.29}\\(3.80)}}{=} \overset{m}{M}_{m-1}(\theta_{m-1} + 2) \overset{(2.4)}{=} \overset{m}{D}_m. \end{aligned}$$

The proof of (3.90) also follows, since

$$S(\overset{m}{M}_{m-1}) \overset{(3.87)}{=} \lim_{p\to+\infty} (s_p)^{\frac{2}{p-1}} |\overset{m}{u}_p(s_p)| = \lim_{p\to+\infty} \left(\frac{s_p}{\varepsilon_p}\right)^{\frac{2}{p-1}} \overset{(3.18)}{=} \lim_{p\to+\infty} (\sigma_{m-1} + o(1))^{\frac{2}{p-1}} = 1,$$

so

$$S = (\overset{m}{M}_{m-1})^{-1} \overset{(2.5)}{=} \overset{m}{S}_{m-1}.$$

STEP 2. We prove all the other convergences in (3.12), (3.13), (3.14), (3.15).

Proof of STEP 2.
The proof is obtained combining Lemma 3.17 with the convergence of $(r_p)^{\frac{2}{p-1}}$ proved in (3.81) in STEP 1 and using also the explicit values of the constants (in (2.6) and (2.7)) related to the solution $\overset{m-1}{u}_p$ with $m-2$ interior zeros, which are known since Theorem 3.9 holds for it by assumption. We have that

$$\begin{aligned} \lim_{p\to+\infty} (\overset{m}{r}_{i,p})^{\frac{2}{p-1}} &\overset{\text{Lemma 3.17}}{=} \lim_{p\to+\infty} (r_p)^{\frac{2}{p-1}} \overset{m-1}{R}_i \overset{(3.81)}{=} \overset{m}{R}_{m-1} \overset{m-1}{R}_i \overset{(2.6)}{=} \overset{m}{R}_{m-1} \prod_{k=i+1}^{m-1} \overset{k}{R}_{k-1} \\ &= \prod_{k=i+1}^{m} \overset{k}{R}_{k-1} \overset{(2.6)}{=} \overset{m}{R}_i, \qquad i = 1, \dots, m-2. \end{aligned}$$

$$\begin{aligned} \lim_{p\to+\infty} p|(\overset{m}{u}_p)'(\overset{m}{r}_{i,p})|(\overset{m}{r}_{i,p}) &\overset{\text{Lemma 3.17}}{=} \frac{\overset{m-1}{D}_i}{\lim_{p\to+\infty} (r_p)^{\frac{2}{p-1}}} \overset{(3.81)}{=} \frac{\overset{m-1}{D}_i}{\overset{m}{R}_{m-1}} \overset{(2.6)}{=} \frac{\overset{i}{D}_i}{\overset{m}{R}_{m-1} \prod_{k=i+1}^{m-1} \overset{k}{R}_{k-1}} \\ &= \frac{\overset{i}{D}_i}{\prod_{k=i+1}^{m} \overset{k}{R}_{k-1}} \overset{(2.6)}{=} \overset{m}{D}_i, \qquad i = 1, \dots, m-1. \end{aligned}$$

$$\lim_{p\to+\infty}(\overset{m}{s}_{i,p})^{\frac{2}{p-1}} \overset{\text{Lemma 3.17}}{=} \lim_{p\to+\infty}(r_p)^{\frac{2}{p-1}}\overset{m-1}{S}_i \overset{(3.81)}{=} \overset{m}{R}_{m-1}\overset{m-1}{S}_i \overset{(2.7)}{=} \overset{m}{R}_{m-1}\overset{i+1}{S}_i\prod_{k=i+2}^{m-1}\overset{k}{R}_{k-1}$$
$$= \overset{i+1}{S}_i\prod_{k=i+2}^{m}\overset{k}{R}_{k-1} \overset{(2.7)}{=} \overset{m}{S}_i, \qquad i=0,\dots,m-2.$$

$$\lim_{p\to+\infty}|\overset{m}{u}_p(\overset{m}{s}_{i,p})| \overset{\text{Lemma 3.17}}{=} \frac{\overset{m-1}{M}_i}{\lim_{p\to+\infty}(r_p)^{\frac{2}{p-1}}} \overset{(3.81)}{=} \frac{\overset{m-1}{M}_i}{\overset{m}{R}_{m-1}} \overset{(2.7)}{=} \frac{\overset{i+1}{M}_i}{\overset{m}{R}_{m-1}\prod_{k=i+2}^{m-1}\overset{k}{R}_{k-1}}$$
$$= \frac{\overset{i+1}{M}_i}{\prod_{k=i+2}^{m}\overset{k}{R}_{k-1}} \overset{(2.7)}{=} \overset{m}{M}_i, \qquad i=0,\dots,m-2.$$

□

3.5. **Proof of Theorem 2.4 - case $\alpha = 0$.** The proof is a consequence of Theorem 3.9, Theorem 3.10 and some of the preliminary results in Section 3.1.

We keep the notation of Section 3, namely $\overset{m}{u}_p = \overset{m}{u}_{0,p}$, $\overset{m}{r}_{i,p} = \overset{m}{r}_{i,0,p}$, $\overset{m}{s}_{i,p} = \overset{m}{s}_{i,0,p}$.
(3.12), (3.13), (3.14) and (3.15) in Theorem 3.9 are exactly the case $\alpha = 0$ of the sharp limits in (2.13), (2.14), (2.15) and (2.16) in Theorem 2.4. The proof of the case $\alpha = 0$ in (2.12) follows instead from (3.13) in Theorem 3.9 combined with the preliminary Lemma 3.7, indeed

$$p\int_0^1 |\overset{m}{u}_p(r)|^{p+1} r\,dr \overset{\text{Lemma 3.7}}{=} \frac{1}{4}\left[p(\overset{m}{u}_p)'(1)\right]^2 + o(1) \overset{(3.13)}{=} \frac{(\overset{m}{D}_m)^2}{4} + o(1).$$

In order to conclude the proof of Theorem 2.4, we need to show the case $\alpha = 0$ in the convergence in (2.10), namely that

$$p\overset{m}{u}_p(x) = 2\pi(-1)^{m-1}\overset{m}{M}_{m-1}(\theta_{m-1}+2)G(x,0) + o(1), \text{ in } C^1_{loc}(B\setminus\{0\}), \tag{3.94}$$

as $p\to+\infty$. The proof of (3.94) exploits not only Theorem 3.9 but also Theorem 3.10 and some preliminary results in Section 3.1 (in particular the asymptotic results in Lemma 3.8). We start from the representation formula and splitting the integral into 2 terms:

$$\begin{aligned} p\overset{m}{u}_p(x) &= p\int_B G(x,y)|\overset{m}{u}_p(y)|^{p-1}\overset{m}{u}_p(y)dy \\ &= p\int_{\{|y|<\overset{m}{s}_{m-1,p}\}} G(x,y)|\overset{m}{u}_p(y)|^{p-1}\overset{m}{u}_p(y)dy + p\int_{\{\overset{m}{s}_{m-1,p}<|y|<1\}} G(x,y)|\overset{m}{u}_p(y)|^{p-1}\overset{m}{u}_p(y)dy. \end{aligned}$$

Let $\delta\in(0,1)$. For the first term we use that for any x,y such that $0<\delta\le|x|\le1$ and $|y|<\overset{m}{s}_{m-1,p}$ ($<\frac{\delta}{2}$ since $\overset{m}{s}_{m-1,p}\to0$ as $p\to+\infty$),

$$|G(x,y)-G(x,0)| \le |y|\sup_{\substack{\delta\le|x|\le1\\ |y|\le\frac{\delta}{2}}}|\nabla G(x,y)| = C|y| \le C\overset{m}{s}_{m-1,p} = o(1), \tag{3.95}$$

(where the constant C depends on δ); moreover, exploiting Lemma 3.4 we also have that

$$p\int_0^{\overset{m}{s}_{m-1,p}} |\overset{m}{u}_p(r)|^{p-1}\overset{m}{u}_p(r)rdr \overset{\text{Lemma 3.4}}{=} p(\overset{m}{u}_p)'(\overset{m}{s}_{m-1,p})\overset{m}{s}_{m-1,p} = 0. \tag{3.96}$$

Then

$$\begin{aligned} p\int_{\{|y|<\overset{m}{s}_{m-1,p}\}} G(x,y)|\overset{m}{u}_p(y)|^{p-1}\overset{m}{u}_p(y)dy &\overset{(3.95)}{=} (G(x,0)+o(1))p\int_{\{|y|<\overset{m}{s}_{m-1,p}\}}|\overset{m}{u}_p(y)|^{p-1}\overset{m}{u}_p(y)dy \\ &= 2\pi(G(x,0)+o(1))p\int_0^{\overset{m}{s}_{m-1,p}}|\overset{m}{u}_p(r)|^{p-1}\overset{m}{u}_p(r)rdr \end{aligned}$$

$$\overset{(3.96)}{=} \quad 0. \tag{3.97}$$

For the second term, we consider $\tau_p \in (\overset{m}{s}_{m-1,p}, 1)$ such that

$$\tau_p = o(1) \qquad \text{and} \qquad \frac{\tau_p - \overset{m}{s}_{m-1,p}}{\overset{m}{\varepsilon}_{m-1,p}} \to +\infty,$$

as $p \to +\infty$, and decompose the integral in the following way:

$$\begin{aligned} p\int_{\{\overset{m}{s}_{m-1,p}<|y|<1\}} G(x,y)|\overset{m}{u}_p(y)|^{p-1}\overset{m}{u}_p(y)dy &= p\int_{\{\overset{m}{s}_{m-1,p}<|y|<\tau_p\}} G(x,y)|\overset{m}{u}_p(y)|^{p-1}\overset{m}{u}_p(y)dy \\ &\quad + p\int_{\{\tau_p<|y|<1\}} G(x,y)|\overset{m}{u}_p(y)|^{p-1}\overset{m}{u}_p(y)dy. \end{aligned}$$

We show that

$$p\int_{\{\overset{m}{s}_{m-1,p}<|y|<\tau_p\}} G(x,y)|\overset{m}{u}_p(y)|^{p-1}\overset{m}{u}_p(y)dy = (-1)^{m-1}2\pi G(x,0)\overset{m}{M}_{m-1}(\theta_{m-1}+2) + o(1) \tag{3.98}$$

and that

$$p\int_{\{\tau_p<|y|<1\}} G(x,y)|\overset{m}{u}_p(y)|^{p-1}\overset{m}{u}_p(y)dy = o(1), \tag{3.99}$$

uniformly in $\{\delta < |x| < 1\}$, as $p \to +\infty$. The local uniform convergence of pu_p in (3.94) then follows from (3.97), (3.98) and (3.99) (the uniform convergence of the derivative follows in a similar way, we omit it) and this will conclude the proof.

Proof of (3.98):
observe that for any x, y such that $0 < \delta \le |x| \le 1$ and $|y| < \tau_p$ ($< \frac{\delta}{2}$ since $\tau_p \to 0$, as $p \to +\infty$),

$$|G(x,y) - G(x,0)| \le |y| \sup_{\substack{\delta\le|x|\le1 \\ |y|\le\frac{\delta}{2}}} |\nabla G(x,y)| = C|y| \le C\tau_p = o(1)$$

as $p \to +\infty$, hence

$$\begin{aligned} p\int_{\{\overset{m}{s}_{m-1,p}<|y|<\tau_p\}} G(x,y)|\overset{m}{u}_p(y)|^{p-1}\overset{m}{u}_p(y)dy &= (G(x,0)+o(1))p\int_{\{\overset{m}{s}_{m-1,p}<|y|<\tau_p\}} |\overset{m}{u}_p(y)|^{p-1}\overset{m}{u}_p(y)dy \\ &= 2\pi(G(x,0)+o(1))p\int_{\overset{m}{s}_{m-1,p}}^{\tau_p} |\overset{m}{u}_p(r)|^{p-1}\overset{m}{u}_p(r)rdr. \end{aligned}$$

We prove that

$$p\int_{\overset{m}{s}_{m-1,p}}^{\tau_p} |\overset{m}{u}_p(r)|^{p-1}\overset{m}{u}_p(r)rdr = (-1)^{m-1}\overset{m}{M}_{m-1}(\theta_{m-1}+2) + o(1), \tag{3.100}$$

as $p \to +\infty$. On the one hand, since $\tau_p < 1$,

$$\begin{aligned} \limsup_p p\int_{\overset{m}{s}_{m-1,p}}^{\tau_p} |\overset{m}{u}_p(r)|^p rdr &\le \limsup_p p\int_{\overset{m}{s}_{m-1,p}}^{1} |\overset{m}{u}_p(r)|^p rdr \\ &\overset{\text{Lemma 3.4}}{=} \limsup_p p|(\overset{m}{u}_p)'(1)| \\ &\overset{\text{Thm 3.9}}{=} \overset{m}{D}_m = \overset{m}{M}_{m-1}(\theta_{m-1}+2). \end{aligned}$$

On the other hand, since $\frac{\tau_p - \overset{m}{s}_{m-1,p}}{\overset{m}{\varepsilon}_{m-1,p}} \to +\infty$ as $p \to +\infty$, by making a change of variable $r = \overset{m}{\varepsilon}_{m-1,p}s + \overset{m}{s}_{m-1,p}$, recalling the definition of z_p, exploiting Theorem 3.10 and using Fatou's lemma (similarly as in the proof of STEP 2 in Lemma 3.29), we obtain that

$$\liminf_p p\int_{\overset{m}{s}_{m-1,p}}^{\tau_p} |\overset{m}{u}_p(r)|^p rdr \quad = \quad \liminf_p |\overset{m}{u}_p(\overset{m}{s}_{m-1,p})| \int_0^{\frac{\tau_p - \overset{m}{s}_{m-1,p}}{\overset{m}{\varepsilon}_{m-1,p}}} \left|1 + \frac{z_p}{p}\right|^p \left(s + \frac{\overset{m}{s}_{m-1,p}}{\overset{m}{\varepsilon}_{m-1,p}}\right) ds$$

$$\begin{aligned}
&\overset{\text{Thm } 3.10}{\geq} \overset{m}{M}_{m-1} \int_0^{+\infty} e^{Z_{m-1}(s+\sigma_{m-1})}(s+\sigma_{m-1})ds \\
&= \overset{m}{M}_{m-1} \int_{\sigma_{m-1}}^{+\infty} e^{Z_{m-1}(s)} s ds \\
&\overset{(3.21)}{=} \overset{m}{M}_{m-1} (\theta_{m-1})^2 (\beta_{m-1})^{\theta_{m-1}} \int_{\sigma_{m-1}}^{+\infty} \frac{s^{\theta_{m-1}-1}}{((\beta_{m-1})^{\theta_{m-1}} + s^{\theta_{m-1}})^2} ds \\
&= \overset{m}{M}_{m-1} \frac{2\theta_{m-1}(\beta_{m-1})^{\theta_{m-1}}}{(\beta_{m-1})^{\theta_{m-1}} + (\sigma_{m-1})^{\theta_{m-1}}} \\
&= \overset{m}{M}_{m-1} (\theta_{m-1}+2),
\end{aligned}$$

which ends the proof of (3.100) and, therefore, of (3.98).

Proof of (3.99):

$$\begin{aligned}
p\int_0^1 |\overset{m}{u}_p(r)|^{p-1}\overset{m}{u}_p(r) r dr &\overset{\text{Lemma } 3.4}{=} -p(\overset{m}{u}_p)'(1) \overset{\text{Thm } 3.9}{=} (-1)^{m-1}\overset{m}{D}_m + o(1) \\
&= (-1)^{m-1}\overset{m}{M}_{m-1}(\theta_{m-1}+2) + o(1)
\end{aligned}$$

as $p \to +\infty$ and so, by (3.96) and (3.100),

$$\begin{aligned}
p\int_{\tau_p}^1 |\overset{m}{u}_p(r)|^{p-1}\overset{m}{u}_p(r) r dr &= p\int_0^1 |\overset{m}{u}_p(r)|^{p-1}\overset{m}{u}_p(r) r dr + \\
&\quad - p\int_0^{\overset{m}{s}_{m-1,p}} |\overset{m}{u}_p(r)|^{p-1}\overset{m}{u}_p(r) r dr - p\int_{\overset{m}{s}_{m-1,p}}^{\tau_p} |\overset{m}{u}_p(r)|^{p-1}\overset{m}{u}_p(r) r dr \\
&\overset{(3.96),(3.100)}{=} o(1),
\end{aligned}$$

as $p \to +\infty$. Moreover, $|G(x,y)| \leq C$ for any x, y such that $0 < \delta \leq |x| \leq 1$ and $|y| < \frac{\delta}{2}$. As a consequence,

$$\begin{aligned}
\left| p\int_{\{\tau_p<|y|<\frac{\delta}{2}\}} G(x,y)|\overset{m}{u}_p(y)|^{p-1}\overset{m}{u}_p(y)dy \right| &\leq p\int_{\{\tau_p<|y|<\frac{\delta}{2}\}} |G(x,y)||\overset{m}{u}_p(y)|^p dy \\
&\leq Cp\int_{\{\tau_p<|y|<\frac{\delta}{2}\}} |\overset{m}{u}_p(y)|^p dy \\
&\leq Cp\int_{\{\tau_p<|y|<1\}} |\overset{m}{u}_p(y)|^p dy \\
&= 2\pi(-1)^{m-1}Cp\int_{\tau_p}^1 |\overset{m}{u}_p(r)|^{p-1}\overset{m}{u}_p(r) r dr \\
&= o(1). \qquad (3.101)
\end{aligned}$$

Finally, for the last term, we use (3.11) in Lemma 3.8, namely,

$$\begin{aligned}
\left| p\int_{\{\frac{\delta}{2}<|y|<1\}} G(x,y)|\overset{m}{u}_p(y)|^{p-1}\overset{m}{u}_p(y)dy \right| &\leq p\int_{\{\frac{\delta}{2}<|y|<1\}} |G(x,y)||\overset{m}{u}_p(y)|^p dy \\
&\leq \|p|\overset{m}{u}_p|^p\|_{L^\infty(\{\frac{\delta}{2}\leq|y|\leq1\})} \int_B |G(x,y)|dy \\
&= C\|p|\overset{m}{u}_p|^p\|_{L^\infty(\{\frac{\delta}{2}\leq|y|\leq1\})} \\
&\overset{(3.11)}{=} o(1) \qquad (3.102)
\end{aligned}$$

as $p \to +\infty$. Identity (3.99) follows from (3.101) and (3.102).

3.6. **Proof of Theorem 2.5 - case $\alpha = 0$.** We keep the notation of Section 3, namely $\overset{m}{u}_p = \overset{m}{u}_{0,p}$, $\overset{m}{r}_{i,p} = \overset{m}{r}_{i,0,p}$, $\overset{m}{s}_{i,p} = \overset{m}{s}_{i,0,p}$; similarly we set

$$\overset{m}{\varepsilon}_{i,p} := \overset{m}{\varepsilon}_{i,0,p}, \quad i = 0, \dots, m-1, \tag{3.103}$$

where $\overset{m}{\varepsilon}_{i,0,p}$ are the parameters defined in (2.17) in the case $\alpha = 0$. Let us define the rescaled functions

$$\overset{m}{z}_{i,p}(r) := \frac{p}{|\overset{m}{u}_p(\overset{m}{s}_{i,p})|}\left[(-1)^i \overset{m}{u}_p(\overset{m}{s}_{i,p} + \overset{m}{\varepsilon}_{i,p} r) - |\overset{m}{u}_p(\overset{m}{s}_{i,p})|\right], \quad r \in (\overset{m}{a}_{i,p}, \overset{m}{b}_{i,p}), \tag{3.104}$$

for $i = 0, \dots, m-1$, where

$$\overset{m}{a}_{i,p} := \begin{cases} 0, & \text{if } i = 0, \\ \frac{\overset{m}{r}_{i,p} - \overset{m}{s}_{i,p}}{\overset{m}{\varepsilon}_{i,p}} \ (< 0), & \text{if } i = 1, \dots, m-1, \end{cases} \tag{3.105}$$

and

$$\overset{m}{b}_{i,p} := \frac{\overset{m}{r}_{i+1,p} - \overset{m}{s}_{i,p}}{\overset{m}{\varepsilon}_{i,p}} \ (> 0), \qquad i = 0, \dots, m-1. \tag{3.106}$$

Theorem 3.33. *Let $m \in \mathbb{N}$, $m \geq 1$ then $\overset{m}{\varepsilon}_{i,p} = o(1)$,*

$$\frac{\overset{m}{r}_{i,p}}{\overset{m}{\varepsilon}_{i,p}} = o(1) \ (i \neq 0), \qquad \frac{\overset{m}{s}_{i,p}}{\overset{m}{\varepsilon}_{i,p}} = \sigma_{i,0} + o(1), \qquad \frac{\overset{m}{\varepsilon}_{i,p}}{\overset{m}{r}_{i+1,p}} = o(1), \tag{3.107}$$

and

$$\overset{m}{z}_{i,p} = Z_{i,0}(\cdot + \sigma_{i,0}) + o(1) \quad \text{in } C^1_{loc}(-\sigma_i, +\infty) \quad \text{as } p \to +\infty \tag{3.108}$$

for all $i = 0, \dots, m-1$, where $\sigma_{i,0}$ and $Z_{i,0}$ are the constants and functions in the case $\alpha = 0$ in Theorem 2.5.

Before proving Theorem 3.33, we show how the case $\alpha = 0$ in Theorem 2.5 is a consequence of it.

Proof of Theorem 2.5 - case $\alpha = 0$. If $i = 0$ then $\sigma_{i,0} = 0$ and $\overset{m}{\xi}_{i,0,p} = \overset{m}{z}_{i,p}$ so there is nothing to prove. Let $i \geq 1$. We show the local uniform convergence, the convergence of the derivatives may be proved in a similar way. Let $K \subset (0, +\infty)$ be a compact set, we want to show that

$$\sup_{r \in K} |\overset{m}{\xi}_{i,0,p}(r) - Z_{i,0}(r)| = o(1), \tag{3.109}$$

as $p \to +\infty$. For any $c > 0$ let

$$K - c := \{s \in \mathbb{R} \ : \ s + c \in K\} \qquad \text{and} \qquad K_c := \{s \in \mathbb{R} \ : \ \min_{y \in K} |s - y| \leq c\},$$

clearly $K - c$ and K_c are compact sets. Observe that, by the second equality in (3.107), for $\delta > 0$ there exists $p_\delta > 1$ such that

$$K - \frac{\overset{m}{s}_{i,p}}{\overset{m}{\varepsilon}_{i,p}} \subseteq \widetilde{K} := (K - \sigma_{i,0})_\delta, \qquad \text{for } p \geq p_\delta.$$

Moreover, note that

$$\overset{m}{\xi}_{i,0,p}(r) = \overset{m}{z}_{i,p}\Big(r - \frac{\overset{m}{s}_{i,p}}{\overset{m}{\varepsilon}_{i,p}}\Big).$$

Hence,

$$\begin{aligned}
\sup_{r \in K} |\overset{m}{\xi}_{i,0,p}(r) - Z_{i,0}(r)| &= \sup_{r \in K} \left| \overset{m}{z}_{i,p}\Big(r - \frac{\overset{m}{s}_{i,p}}{\overset{m}{\varepsilon}_{i,p}}\Big) - Z_{i,0}(r) \right| \\
&= \sup_{t \in K - \frac{\overset{m}{s}_{i,p}}{\overset{m}{\varepsilon}_{i,p}}} \left| \overset{m}{z}_{i,p}(t) - Z_{i,0}\Big(t + \frac{\overset{m}{s}_{i,p}}{\overset{m}{\varepsilon}_{i,p}}\Big) \right| \\
&\leq \sup_{t \in \widetilde{K}} \left| \overset{m}{z}_{i,p}(t) - Z_{i,0}\Big(t + \frac{\overset{m}{s}_{i,p}}{\overset{m}{\varepsilon}_{i,p}}\Big) \right|
\end{aligned}$$

$$\leq \sup_{t\in\widetilde{K}}\left|\overset{m}{z}_{i,p}(t) - Z_{i,0}(t+\sigma_{i,0})\right| + \sup_{t\in\widetilde{K}}\left|Z_{i,0}(t+\sigma_{i,0}) - Z_{i,0}(t+\frac{\overset{m}{s}_{i,p}}{\overset{m}{\varepsilon}_{i,p}})\right|$$

$$\overset{(3.108)}{\underset{(3.107)}{=}} o(1)$$

as $p \to +\infty$. □

Proof of Theorem 3.33. If $i = m-1$ then we directly apply Theorem 3.10. Hence, let us consider the case $i < m-1$. Recall that the solutions $\overset{m}{u}_p$ and $\overset{m-1}{u}_p$ are linked by the following change of variable

$$\overset{m-1}{u}_p(r) = (\overset{m}{r}_{m-1,p})^{\frac{2}{p-1}}\overset{m}{u}_p(\overset{m}{r}_{m-1,p}r), \qquad r \in [0,1]. \tag{3.110}$$

As a consequence,

$$\frac{\overset{m}{r}_{i,p}}{\overset{m}{\varepsilon}_{i,p}} \overset{(3.36)}{=} \frac{\overset{m-1}{r}_{i,p}}{\overset{m-1}{\varepsilon}_{i,p}}, \qquad \frac{\overset{m}{s}_{i,p}}{\overset{m}{\varepsilon}_{i,p}} \overset{(3.35)}{=} \frac{\overset{m-1}{s}_{i,p}}{\overset{m-1}{\varepsilon}_{i,p}}, \qquad \frac{\overset{m}{\varepsilon}_{i,p}}{\overset{m}{r}_{i+1,p}} = \frac{\overset{m-1}{\varepsilon}_{i,p}}{\overset{m-1}{r}_{i+1,p}}.$$

So also

$$\overset{m}{a}_{i,p} = \overset{m-1}{a}_{i,p}, \qquad \overset{m}{b}_{i,p} = \overset{m-1}{b}_{i,p},$$

and the i-th rescaled function $\overset{m}{z}_{i,p}$ of the solution $\overset{m}{u}_p$ coincides with the i-th rescaled function $\overset{m-1}{z}_{i,p}$ of the solution $\overset{m-1}{u}_p$, because

$$\begin{aligned}
\overset{m}{z}_{i,p}(r) &= p\left[(-1)^i\frac{\overset{m}{u}_p(\overset{m}{s}_{i,p}+\overset{m}{\varepsilon}_{i,p}r)}{|\overset{m}{u}_p(\overset{m}{s}_{i,p})|} - 1\right] \\
&\overset{(3.110)}{=} p\left[(-1)^i\frac{\overset{m-1}{u}_p\left(\frac{\overset{m}{s}_{i,p}}{\overset{m}{r}_{m-1,p}} + \frac{\overset{m}{\varepsilon}_{i,p}}{\overset{m}{r}_{m-1,p}}r\right)}{(\overset{m}{r}_{m-1,p})^{2/(p-1)}|\overset{m}{u}_p(\overset{m}{s}_{i,p})|} - 1\right] \\
&= p\left[(-1)^i\frac{\overset{m-1}{u}_p(\overset{m-1}{s}_{i,p}+\overset{m-1}{\varepsilon}_{i,p}r)}{|\overset{m-1}{u}_p(\overset{m-1}{s}_{i,p})|} - 1\right] = \overset{m-1}{z}_{i,p}(r).
\end{aligned}$$

Iterating this procedure $m-i-1$ times, we have that the *last* rescaled function of the solution $\overset{i+1}{u}_p$, namely,

$$\overset{m}{z}_{i,p} \underbrace{= \overset{m-1}{z}_{i,p} = \ldots =}_{m-i-1 \text{ times}} \overset{i+1}{z}_{i,p}$$

with

$$\overset{m}{a}_{i,p} = \overset{m-1}{a}_{i,p} = \ldots = \overset{i+1}{a}_{i,p}; \qquad \overset{m}{b}_{i,p} = \overset{m-1}{b}_{i,p} = \ldots = \overset{i+1}{b}_{i,p};$$

and similarly the relations

$$\begin{aligned}
\frac{\overset{m}{r}_{i,p}}{\overset{m}{\varepsilon}_{i,p}} &= \frac{\overset{m-1}{r}_{i,p}}{\overset{m-1}{\varepsilon}_{i,p}} = \ldots = \frac{\overset{i+1}{r}_{i,p}}{\overset{i+1}{\varepsilon}_{i,p}}, \\
\frac{\overset{m}{s}_{i,p}}{\overset{m}{\varepsilon}_{i,p}} &= \frac{\overset{m-1}{s}_{i,p}}{\overset{m-1}{\varepsilon}_{i,p}} = \ldots = \frac{\overset{i+1}{s}_{i,p}}{\overset{i+1}{\varepsilon}_{i,p}}, \\
\frac{\overset{m}{\varepsilon}_{i,p}}{\overset{m}{r}_{i+1,p}} &= \frac{\overset{m-1}{\varepsilon}_{i,p}}{\overset{m-1}{r}_{i+1,p}} = \ldots = \overset{i+1}{\varepsilon}_{i,p}.
\end{aligned}$$

The claim now follows from Theorem 3.10 (with $m = i+1$). □

4. The Dirichlet Hénon problem in B

In this section we prove the case $\alpha > 0$ in Theorems 2.4 and 2.5.

The case $\alpha = 0$ (Dirichlet Lane-Emden) has been proved in Section 3, the idea is to use a suitable *change of variable* in order to derive the case $\alpha > 0$ from the case $\alpha = 0$.

To this aim let us observe that the Lane-Emden equation (case $\alpha = 0$) and the Hénon equation (case $\alpha > 0$) are linked - *in the radial setting and in dimension* 2 - by a change of variable first considered in [12] and then also used in [14, 25, 40] (see also [34, 40] where it is exploited also in a non-radial setting). Indeed, setting

$$v(r) := \left[\frac{2}{\alpha+2}\right]^{-\frac{2}{p-1}} u(r^{\frac{\alpha+2}{2}}), \tag{4.1}$$

then it is easy to see that u is a solution of the Lane-Emden equation

$$-\Delta u = |u|^{p-1}u \qquad \text{in } B$$

if and only if v is a solution of the Hénon equation

$$-\Delta v = |x|^{\alpha}|v|^{p-1}v \qquad \text{or in } B;$$

moreover also the boundary conditions are preserved:

$$u = 0 \quad \text{on } \partial B \quad \text{iff} \quad v = 0 \quad \text{on } \partial B.$$

As a consequence, using the change of variable (4.1), one immediately obtains the existence and uniqueness of the radial solutions with $m-1$ interior zeros for the Dirichlet Hénon problems, since

$$\overset{m}{u}_{\alpha,p}(r) = \left[\frac{2}{\alpha+2}\right]^{-\frac{2}{p-1}} \overset{m}{u}_{0,p}(r^{\frac{\alpha+2}{2}}). \tag{4.2}$$

We remark that *this strategy cannot be used in higher dimension*. From (4.2) it follows that

$$\overset{m}{r}_{j,\alpha,p} = (\overset{m}{r}_{j,0,p})^{\frac{2}{\alpha+2}}, \qquad \overset{m}{s}_{j,\alpha,p} = (\overset{m}{s}_{j,0,p})^{\frac{2}{\alpha+2}}, \tag{4.3}$$

and then

$$|\overset{m}{u}_{\alpha,p}(\overset{m}{s}_{j,\alpha,p})| = \left[\frac{2}{\alpha+2}\right]^{-\frac{2}{p-1}} |\overset{m}{u}_{0,p}(\overset{m}{s}_{j,0,p})|.$$

Taking the derivative on both sides of (4.2),

$$|(\overset{m}{u}_{\alpha,p})'(\overset{m}{r}_{j,\alpha,p})|\overset{m}{r}_{j,\alpha,p} = \frac{\alpha+2}{2}\left[\frac{2}{\alpha+2}\right]^{-\frac{2}{p-1}} |(\overset{m}{u}_{0,p})'(\overset{m}{r}_{j,0,p})|\overset{m}{r}_{j,0,p}.$$

4.1. **Proof of Theorem 2.4 - case** $\alpha > 0$**.** (2.13)-(2.14)-(2.15)-(2.16) follows passing to the limit as $p \to +\infty$ into the previous 4 equalities and using the corresponding convergences in the case $\alpha = 0$.

Similarly one shows the convergence of the energy in (2.12), again using the change of variable (4.2) and passing into the limit by exploiting the convergence already proved for the energy in the case $\alpha = 0$, namely,

$$\begin{aligned} p\int_0^1 |\overset{m}{u}_{\alpha,p}(r)|^{p+1} r^{\alpha+1} dr &\overset{(4.2)}{=} \left[\frac{2}{\alpha+2}\right]^{-\frac{2(p+1)}{p-1}} p\int_0^1 |\overset{m}{u}_{0,p}(r^{\frac{\alpha+2}{2}})|^{p+1} r^{\alpha+1}\, dr \\ &= \left[\frac{2}{\alpha+2}\right]^{-\frac{4}{p-1}} \frac{\alpha+2}{2} p\int_0^1 |\overset{m}{u}_{0,p}(s)|^{p+1} s\, ds \\ &\overset{case\ \alpha=0}{=} \frac{\alpha+2}{8}(\overset{m}{M}_{m-1})^2(\theta_{m-1}+2)^2 + o(1). \end{aligned}$$

The convergence in (2.10) also follows from (4.2) and the corresponding $C^1_{loc}(B\setminus\{0\})$ convergence already proved for $\overset{m}{u}_{0,p}$ as $p\to+\infty$, because

$$\begin{aligned} p\overset{m}{u}_{\alpha,p}(|x|) &\overset{(4.2)}{=} \left[\frac{2}{\alpha+2}\right]^{-\frac{2}{p-1}} p\overset{m}{u}_{0,p}(|x|^{\frac{\alpha+2}{2}}) \\ &\overset{case\ \alpha=0}{=} 2\pi\gamma_{0,m}G(|x|^{\frac{\alpha+2}{2}},0)+o(1) \\ &= \gamma_{0,m}\log|x|^{-\frac{\alpha+2}{2}}+o(1) \\ &= \frac{\alpha+2}{2}\gamma_{0,m}\log|x|^{-1}+o(1) \\ &= 2\pi\gamma_{\alpha,m}G(x,0)+o(1), \qquad \text{in } C^1_{loc}(B\setminus\{0\}), \end{aligned}$$

with $\gamma_{\alpha,m}$ as in (2.11).

4.2. **Proof of Theorem 2.5 - case $\alpha>0$.** An easy computation from (4.2) gives

$$\overset{m}{\varepsilon}_{i,\alpha,p}=(\overset{m}{\varepsilon}_{i,0,p})^{\frac{2}{\alpha+2}}; \tag{4.4}$$

and hence,

$$\overset{m}{\xi}_{i,\alpha,p}(r)=\overset{m}{\xi}_{i,0,p}(r^{\frac{\alpha+2}{2}}), \tag{4.5}$$

where $\overset{m}{\varepsilon}_{i,\alpha,p}$ and $\overset{m}{\xi}_{i,\alpha,p}$ are the scaling parameters and the rescaled solutions as defined in (2.17), (2.18). Using the result already proven in the case $\alpha=0$, from (4.4) and (4.3) we then obtain (2.21), and from (4.5) we obtain

$$\overset{m}{\xi}_{i,\alpha,p}(r)=Z_{i,0}(r^{\frac{\alpha+2}{2}})+o(1) \qquad \text{in } C^1_{loc}(B\setminus\{0\}) \quad \text{as } p\to+\infty.$$

The conclusion follows observing that $Z_{i,\alpha}(r)=Z_{i,0}(r^{\frac{\alpha+2}{2}})$. Recalling that $Z_{i,0}$ solves

$$\begin{cases} -\Delta Z=e^Z+2\pi(2-\theta_i)\delta_0 & \text{in } \mathbb{R}^2,\\ Z(\sigma_{i,0})=0,\\ \int_{\mathbb{R}^2}e^Z dx=4\pi\theta_i, \end{cases}$$

it is not difficult to check that $Z_{i,\alpha}$ solves (2.20).

5. The Neumann problems and the equation in the whole $\mathbb{R}^2$

In this section we prove Theorems 2.6 and 2.7.

5.1. **The proof of Theorem 2.6.** Let us fix $m\in\mathbb{N}$, $m\geq 2$. The existence of a unique (up to a sign) radial solution $\overset{m}{\bar u}_{\alpha,p}$ for the Neumann problem (1.2)-(1.4) having m nodal regions follows observing that the solutions $\overset{m}{\bar u}_{\alpha,p}$ of the Neumann problem (1.2)-(1.4) and the solution $\overset{m}{u}_{\alpha,p}$ of the Dirichlet problem (1.2)-(1.3) are linked via the following change of variable:

$$\overset{m}{\bar u}_{\alpha,p}(r)=(s_{\alpha,p})^{\frac{\alpha+2}{p-1}}\overset{m}{u}_{\alpha,p}(s_{\alpha,p}r), \qquad r\in[0,1], \tag{5.1}$$

where we have simplified the notation of the last critical point of $\overset{m}{u}_{\alpha,p}$ setting

$$s_{\alpha,p}:=\overset{m}{s}_{m-1,\alpha,p}.$$

The proof of (2.25)-(2.26)-(2.27)-(2.28), with the sharp constants in (2.29), immediately follows from (5.1). Indeed, we deduce the relations among the zeros and critical points of $\overset{m}{\bar u}_{\alpha,p}$ and $\overset{m}{u}_{\alpha,p}$ given by

$$\bar r_{i,\alpha,p}=\frac{r_{i,\alpha,p}}{s_{\alpha,p}} \qquad \text{and} \qquad \bar s_{i,\alpha,p}=\frac{s_{i,\alpha,p}}{s_{\alpha,p}}.$$

Here, in order to simplify the notation, we have dropped the dependence on m. Then

$$|\overset{m}{\bar u}_{\alpha,p}(\bar s_{i,\alpha,p})|=(s_{\alpha,p})^{\frac{\alpha+2}{p-1}}|\overset{m}{u}_{\alpha,p}(s_{i,\alpha,p})|.$$

Moreover, taking the derivative on both sides in (5.1),

$$p|(\overset{m}{\bar{u}}_{\alpha,p})'(\bar{r}_{i,\alpha,p})|\bar{r}_{i,\alpha,p} = (s_{\alpha,p})^{\frac{\alpha+2}{p-1}} p|(\overset{m}{u}_{\alpha,p})'(r_{i,\alpha,p})|r_{i,\alpha,p}.$$

The claim now follows passing to the limit as $p \to +\infty$ into the previous equalities and applying (2.13)-(2.14)-(2.15)-(2.16).
In order to compute the energy and obtain (2.24) we use the relation (5.1), perform the change of variable $t = s_{\alpha,p} r$ and reduce it to the computation of a partial energy of the Dirichlet solution $\overset{m}{u}_{\alpha,p}$, that is,

$$\begin{aligned} p\int_0^1 |(\overset{m}{\bar{u}}_{\alpha,p})'(r)|^2 r\,dr &\overset{(1.2)-(1.4)}{=} p\int_0^1 |\overset{m}{\bar{u}}_{\alpha,p}(r)|^{p+1} r^{1+\alpha}\,dr \\ &\overset{(5.1)}{=} (s_{\alpha,p})^{\frac{(\alpha+2)(p+1)}{p-1}} p\int_0^1 |\overset{m}{u}_{\alpha,p}(s_{\alpha,p}r)|^{p+1} r^{1+\alpha}\,dr \\ &= (s_{\alpha,p})^{\frac{2(\alpha+2)}{p-1}} p\int_0^{s_{\alpha,p}} |\overset{m}{u}_{\alpha,p}(t)|^{p+1} t^{1+\alpha}\,dt. \end{aligned} \tag{5.2}$$

Using the change of variable (4.2) we can then reduce this last integral for the solution $\overset{m}{u}_{\alpha,p}$ of the Dirichlet Hénon problem ($\alpha > 0$) to an integral for the solution $\overset{m}{u}_{0,p}$ of the Dirichlet Lane-Emden problem ($\alpha = 0$), namely,

$$\begin{aligned} p\int_0^{s_{\alpha,p}} |\overset{m}{u}_{\alpha,p}(t)|^{p+1} t^{1+\alpha}\,dt &\overset{(4.2)}{=} \left(\frac{\alpha+2}{2}\right)^{\frac{2(p+1)}{p-1}} p\int_0^{s_{\alpha,p}} |\overset{m}{u}_{0,p}(t^{\frac{\alpha+2}{2}})|^{p+1} t^{1+\alpha}\,dt \\ &= \left(\frac{\alpha+2}{2}\right)^{\frac{p+3}{p-1}} p\int_0^{s_p} |\overset{m}{u}_{0,p}(r)|^{p+1} r\,dr, \end{aligned}$$

where $s_p := s_{0,p}$. Finally, we compute this integral using the results of Section 3. More precisely, by (3.72), (3.80) and (3.87),

$$p\int_0^{r_p} |\overset{m}{u}_{0,p}|^{p+1} r\,dr = (\overset{m}{M}_{m-1})^2 \frac{(\theta_{m-1}-2)^2}{4} + o(1),$$

and by Lemma 3.26, (3.80) and (3.87) we obtain that

$$p\int_{r_p}^{s_p} |\overset{m}{u}_{0,p}|^{p+1} r\,dr = (\overset{m}{M}_{m-1})^2 (\theta_{m-1}-2) + o(1),$$

so that

$$p\int_0^{s_{\alpha,p}} |\overset{m}{u}_{\alpha,p}|^{p+1} t^{1+\alpha}\,dt = \frac{\alpha+2}{8} (\overset{m}{M}_{m-1})^2 (\theta_{m-1}-2)(\theta_{m-1}+2) + o(1). \tag{5.3}$$

Collecting (5.2) and (5.3), and recalling that, by (2.15), $(s_{\alpha,p})^{\frac{2(\alpha+2)}{p-1}} = (\overset{m}{S}_{m-1})^2 + o(1)$ as $p \to +\infty$, we obtain (2.24).

5.2. **The proof of Theorem 2.7.** A radial solution $w_{\alpha,p}$ of (1.1) such that $w_{\alpha,p}(0) = 1$ exists by ODE considerations. Moreover it oscillates infinitely many times and has a unique local maximum or minimum $\delta_{m,\alpha,p}$ between any two consecutive zeros $\rho_{m,\alpha,p}$ and $\rho_{m+1,\alpha,p}$ (see [36, p. 294] for the case $\alpha = 0$ and use the change of variable (4.1) to obtain the same result for any $\alpha > 0$).
It is easy to see that

$$\overset{m}{u}_{\alpha,p}(r) = (\rho_{m,\alpha,p})^{\frac{\alpha+2}{p-1}} w_{\alpha,p}(\rho_{m,\alpha,p} r), \qquad r \in [0,1], \tag{5.4}$$

is a radial solution of the Dirichlet problem (1.2)-(1.3) with $m-1$ interior zeros. The conclusion then follows as a corollary of (2.13)-(2.14)-(2.15)-(2.16) in Theorem 2.4, observing that, from (5.4),

$$(\rho_{m,\alpha,p})^{\frac{\alpha+2}{p-1}} = \overset{m}{u}_{\alpha,p}(0) \quad \text{and} \quad \delta_{m-1,\alpha,p} = \rho_{m,\alpha,p} \overset{m}{s}_{m-1,\alpha,p},$$

and then

$$w_{\alpha,p}(\delta_{m-1,\alpha,p}) = (\rho_{m,\alpha,p})^{-\frac{\alpha+2}{p-1}} \overset{m}{u}_{\alpha,p}(\overset{m}{s}_{m-1,\alpha,p}).$$

Taking the derivative on both sides of (5.4), it follows that

$$p|(w_{\alpha,p})'(\rho_{m,\alpha,p})|\rho_{m,\alpha,p} = (\rho_{m,\alpha,p})^{-\frac{\alpha+2}{p-1}} p|(\overset{m}{u}_{\alpha,p})'(1)|.$$

6. Analysis of the constants

In this section we do a careful study of the constants in Definition 2.2 and show Theorem 2.8. The first result in this section characterizes the growth of θ_k.

Theorem 6.1 (Linear growth of θ_k)**.** *Let θ_k be the numbers in Definition 2.1. It holds that*

$$2+8k < \theta_k < 2 + \frac{2}{\mathcal{L}\left(\frac{1}{4k}\right)} < 4+8k \qquad \text{for all } k \in \mathbb{N}.$$

In particular, the sequence $(\theta_k)_k$ is strictly increasing and

$$\left[\frac{\theta_k}{2}\right] = 4k+1 \qquad \text{for all } k \in \mathbb{N}, \tag{6.1}$$

where $[x]$ denotes the integer part of x.

Using Theorem 6.1 we can deduce information on all the constants in Definition 2.2.

Theorem 6.2 (Sublinear growth of $\overset{m}{M}_0$)**.** *Let $c_1 := \sqrt{\pi} \approx 1.77245$ and $c_2 := 6\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \approx 2.028$. Then, for every $m \in \mathbb{N}$,*

$$c_1 \frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})} e^{\frac{1}{3+4m}} < \overset{m+1}{M}_0 < c_2 \frac{\Gamma(m+\frac{5}{4})}{\Gamma(m+\frac{3}{4})} e^{\frac{1}{2+4m}}, \tag{6.2}$$

where Γ is the usual Gamma function. Furthermore,

$$c_1 \le \liminf_{m\to\infty} \frac{\overset{m+1}{M}_0}{\sqrt{m}} \le \limsup_{m\to\infty} \frac{\overset{m+1}{M}_0}{\sqrt{m}} \le c_2. \tag{6.3}$$

Corollary 6.3. *For every $i \in \mathbb{N}$,*

$$0 < \liminf_{m\to\infty} \frac{\overset{m}{M}_i}{\sqrt{m}} \le \limsup_{m\to\infty} \frac{\overset{m}{M}_i}{\sqrt{m}} < \infty, \qquad 0 < \liminf_{m\to\infty} \frac{\overset{m}{D}_i}{\sqrt{m}} \le \limsup_{m\to\infty} \frac{\overset{m}{D}_i}{\sqrt{m}} < \infty, \tag{6.4}$$

$$0 < \liminf_{m\to\infty} \overset{m}{S}_i\sqrt{m} \le \limsup_{m\to\infty} \overset{m}{S}_i\sqrt{m} < \infty, \qquad 0 < \liminf_{m\to\infty} \overset{m}{R}_i\sqrt{m} \le \limsup_{m\to\infty} \overset{m}{R}_i\sqrt{m} < \infty. \tag{6.5}$$

To show Theorem 6.1, we begin with an alternative formula for θ_k. Let

$$\phi : (0,\infty) \to (0,\infty) \qquad \text{be given by} \qquad \phi(s) := \frac{s}{1+2s} e^{-\frac{s}{1+2s}}$$

and let

$$A_1 := \mathcal{L}(\frac{1}{2}e^{-\frac{1}{2}}), \qquad A_{k+1} := \mathcal{L}(\phi(A_k)) \quad \text{for } k \in \mathbb{N}.$$

Lemma 6.4. *For every $m \in \mathbb{N}$,*

$$\theta_m = 2 + \frac{2}{A_m} \qquad . \tag{6.6}$$

Proof. We argue by induction. The inductive base is clear since $\theta_1 = 2 + \frac{2}{A_1}$ by definition. Let $k \in \mathbb{N}$ and assume that $\theta_k = 2 + \frac{2}{A_k}$. Then

$$\frac{2}{e^{2/(2+\theta_k)}\,(\theta_k+2)} = \frac{2}{e^{2/(4+\frac{2}{A_k})}\,(4+\frac{2}{A_k})} = \frac{A_k}{2A_k+1} e^{-\frac{A_k}{2A_k+1}},$$

and therefore

$$\theta_{k+1} = 2 + \frac{2}{\mathcal{L}\left[\frac{2}{e^{2/(2+\theta_k)}\,(\theta_k+2)}\right]} = 2 + \frac{2}{\mathcal{L}\left[\frac{A_k}{2A_k+1} e^{-\frac{A_k}{2A_k+1}}\right]} = 2 + \frac{2}{A_{k+1}}.$$

□

Next we deduce some bounds for A_k. We need first an auxiliary lemma. Recall the following well-known properties of the $\mathcal{L}$ function: For every $s>0$, $s\mapsto \mathcal{L}(s)$ is monotone increasing, $\mathcal{L}(0)=0$, $\mathcal{L}'(s)=\frac{\mathcal{L}(s)}{s(\mathcal{L}(s)+1)}$, and

$$s-s^2<\mathcal{L}(s)<s \qquad \text{for } s>0. \tag{6.7}$$

Lemma 6.5. *For every $k\in\mathbb{N}$,*

$$\phi\Big(\frac{1}{4k}\Big)=\frac{1}{4k+2}e^{-\frac{1}{4k+2}}<\frac{1}{4k+4}e^{\frac{1}{4k+4}}=\mathcal{L}^{-1}\Big(\frac{1}{4(k+1)}\Big), \tag{6.8}$$

$$\phi\Big(\mathcal{L}\Big(\frac{1}{4k}\Big)\Big)>\frac{1}{4(k+1)}. \tag{6.9}$$

Proof. Let $f:(0,\infty)\to(0,\infty)$ be given by $f(s)=\frac{s+\frac12}{s+1}-\ln\left(\left(1+\frac1s\right)^s\right)$. Observe that f is convex (because $f''(s)=\frac{1}{s(1+s)^3}>0$) and that $\lim_{s\to\infty}f(s)=0$; as a consequence, $f>0$ in $(0,\infty)$, and therefore $\ln\left(1+\frac1s\right)<\frac{s+\frac12}{s(s+1)}$, or, equivalently,

$$1+\frac1s<e^{\frac{s+\frac12}{s(s+1)}}=e^{\frac12\left(\frac{1}{1+s}+\frac1s\right)}.$$

If $k\in\mathbb{N}$ and $s=2k+1$, then

$$\frac{4k+4}{4k+2}=1+\frac{1}{2k+1}<e^{\frac12\left(\frac{1}{2k+2}+\frac{1}{2k+1}\right)},$$

and (6.8) follows. Next we show inequality (6.9). For $s>0$, let

$$h(s)=\frac{\phi(\mathcal{L}(s))}{\frac{s}{1+4s}},\qquad \text{then}\qquad h'(s)=\frac{4e^{-\frac{\mathcal{L}(s)}{1+2\mathcal{L}(s)}}\mathcal{L}(s)\left(s-\mathcal{L}(s)-\mathcal{L}(s)^2\right)}{s^2(2\mathcal{L}(s)+1)^3}.$$

From (6.7) we deduce that $h(0)=1$ and $h'(s)>0$ for $s\in(0,1)$, therefore $h>1$ in $(0,1)$. Indeed, let $G(s):=\frac{s}{\mathcal{L}(s)+\mathcal{L}(s)^2}$, then $G(0)=1$ (by (6.7)) and $G'(s)=\frac{\mathcal{L}(s)}{(\mathcal{L}(s)+1)^3}>0$ for $s\in(0,1)$, as a consequence $G>1$ in $(0,1)$ and thus $s-\mathcal{L}(s)-\mathcal{L}(s)^2>0$ for $s\in(0,1)$. Then, $h(\frac{1}{4k})>1$ for $k\in\mathbb{N}$ and thus $\phi(\mathcal{L}(\frac{1}{4k}))>\frac{1}{4(k+1)}$ as claimed in (6.9). □

Proposition 6.6. *It holds that*

$$\frac{1}{4k+1}<\mathcal{L}\Big(\frac{1}{4k}\Big)<A_k<\frac{1}{4k}\qquad \textit{for all } k\in\mathbb{N}.$$

Proof. We show first that $A_k<\frac{1}{4k}$ for all $k\in\mathbb{N}$. The inductive base is immediate, since $0.2388\approx A_1<\frac14$. Let $k\in\mathbb{N}$ and assume that $A_k<\frac{1}{4k}$. Using that $t\mapsto\mathcal{L}(\phi(t))$ is a monotone increasing function, it follows, by Lemma 6.5, that

$$A_{k+1}=\mathcal{L}(\phi(A_k))<\mathcal{L}\left(\phi\Big(\frac{1}{4k}\Big)\right)<\frac{1}{4(k+1)}.$$

Next, we show that $A_k>\mathcal{L}(\frac{1}{4k})$ for all $k\in\mathbb{N}$. Note that $A_1>\mathcal{L}(\frac14)\approx 0.2038$ and if $A_k>\mathcal{L}(\frac{1}{4k})$ then, using that $t\mapsto\mathcal{L}(\phi(t))$ and $t\mapsto\mathcal{L}(t)$ are monotone increasing functions and Lemma 6.5,

$$A_{k+1}=\mathcal{L}(\phi(A_k))>\mathcal{L}\left(\phi\Big(\mathcal{L}\Big(\frac{1}{4k}\Big)\Big)\right)>\mathcal{L}\Big(\frac{1}{4(k+1)}\Big).$$

Finally, for $s\in(0,1)$, let $g(s):=\frac{\mathcal{L}(s)}{\frac{s}{s+1}}$; then, by (6.7),

$$g'(s)=\frac{(s-\mathcal{L}(s))\mathcal{L}(s)}{s^2(\mathcal{L}(s)+1)}>0\quad \text{in } (0,1)\qquad \text{and}\qquad g(0)=1.$$

Therefore, $g>1$ in $(0,1)$ and, using $s=\frac{1}{4k}$ with $k\in\mathbb{N}$, we deduce that $\mathcal{L}(\frac{1}{4k})>\frac{1}{4k+1}$. □

We are now ready to prove Theorem 6.1.

Proof of Theorem 6.1. Let $k \in \mathbb{N}$. By Proposition 6.6 we know that $4k < \frac{1}{A_k} < 4k+1$, then, by Lemma 6.4, $2 + 8k < \theta_k < 2 + \frac{2}{\mathcal{L}(\frac{1}{4k})} < 4 + 8k$ and $[\frac{\theta_k}{2}] = 1 + [\frac{1}{A_k}] = 1 + 4k$. □

The proof of Theorem 6.2 relies on a formula for $\overset{m+1}{M}_0$ in terms of θ_k's (see Lemma 6.7 below), on the estimates for θ_k (see Theorem 6.1), and on some well-known properties of the Γ-function (see (6.10)–(6.12)).

Lemma 6.7. *Let $m \in \mathbb{N}$, then*

$$\overset{m+1}{M}_0 = \frac{(\theta_m - 2)}{4} e^{\frac{2}{2+\theta_m}} \prod_{k=1}^{m-1} \frac{\theta_k - 2}{\theta_k + 2}$$

Proof. By (2.7),

$$\begin{aligned}
\overset{m+1}{M}_0 &= \frac{\overset{1}{M}_0}{\prod_{k=2}^{m+1} \overset{k}{R}_{k-1}} = \frac{e^{2/(2+\theta_0)}}{\prod_{k=2}^{m+1} \frac{\overset{k-1}{M}_{k-2}\,(\theta_{k-2}+2)}{\overset{k}{M}_{k-1}(\theta_{k-1}-2)}} = e^{2/(2+\theta_0)} \prod_{k=2}^{m+1} \frac{\overset{k}{M}_{k-1}(\theta_{k-1}-2)}{\overset{k-1}{M}_{k-2}\,(\theta_{k-2}+2)} \\
&= e^{\frac{1}{2}} \prod_{k=2}^{m+1} \frac{\theta_{k-1}-2}{\theta_{k-2}+2} \prod_{k=2}^{m+1} \frac{\overset{k}{M}_{k-1}}{\overset{k-1}{M}_{k-2}} = e^{\frac{1}{2}} \prod_{k=2}^{m+1} \frac{\theta_{k-1}-2}{\theta_{k-2}+2} \prod_{k=2}^{m+1} \frac{e^{2/(2+\theta_{k-1})}}{e^{2/(2+\theta_{k-2})}} \\
&= e^{\frac{1}{2} + \sum_{k=2}^{m+1} \frac{2}{2+\theta_{k-1}} - \frac{2}{2+\theta_{k-2}}} \frac{\prod_{k=2}^{m+1} (\theta_{k-1}-2)}{\prod_{k=2}^{m+1} (\theta_{k-2}+2)} = \frac{(\theta_m - 2)}{4} e^{\frac{2}{2+\theta_m}} \prod_{k=1}^{m-1} \frac{\theta_k - 2}{\theta_k + 2}.
\end{aligned}$$

□

Recall that the Gamma function is the unique function that simultaneously satisfies that $\Gamma(1) = 1$,

$$z\Gamma(z) = \Gamma(z+1), \tag{6.10}$$

$$\lim_{n\to\infty} \frac{\Gamma(n+z)}{\Gamma(n)n^z} = 1 \qquad \text{for every } z \in \mathbb{C}. \tag{6.11}$$

In particular, by (6.10), we have, for every $\alpha \geq 0$ and $\beta \geq 0$, that

$$\prod_{k=1}^{m-1} \frac{k+\beta}{k+\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)} \frac{\Gamma(m+\beta)}{\Gamma(m+\alpha)}. \tag{6.12}$$

We are now ready to show Theorem 6.2.

Proof of Theorem 6.2. By Theorem 6.1,

$$e^{\frac{1}{3+4m}} < e^{\frac{2}{2+\theta_m}} < e^{\frac{1}{2+4m}} \qquad \text{and} \qquad 2m < \frac{\theta_m - 2}{4} < 2m + \frac{1}{2} \qquad \text{for } m \in \mathbb{N}$$

and, using that $s \mapsto \frac{s-2}{s+2}$ is a monotone increasing function,

$$\frac{k}{k+\frac{1}{2}} < \frac{\theta_k - 2}{\theta_k + 2} < \frac{k+\frac{1}{4}}{k+\frac{3}{4}} \qquad \text{for } k \in \mathbb{N}.$$

Then, by Lemma 6.7,

$$2m\, e^{\frac{1}{3+4m}} \prod_{k=1}^{m-1} \frac{k}{k+\frac{1}{2}} < \overset{m+1}{M}_0 = \frac{(\theta_m - 2)}{4} e^{\frac{2}{2+\theta_m}} \prod_{k=1}^{m-1} \frac{\theta_k - 2}{\theta_k + 2} < (2m + \frac{1}{2})\, e^{\frac{1}{2+4m}} \prod_{k=1}^{m-1} \frac{k+\frac{1}{4}}{k+\frac{3}{4}},$$

and, by (6.12),

$$2\frac{m\Gamma(m)}{\Gamma(m+\frac{1}{2})}\Gamma(1+\frac{1}{2})e^{\frac{1}{3+4m}} < \overset{m+1}{M}_0 < (2m+\frac{1}{2})\frac{\Gamma(m+\frac{1}{4})}{\Gamma(m+\frac{3}{4})}\frac{\Gamma(1+\frac{3}{4})}{\Gamma(1+\frac{1}{4})}e^{\frac{1}{2+4m}}. \tag{6.13}$$

Using (6.10) and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

$$2\frac{m\Gamma(m)}{\Gamma(m+\frac{1}{2})}\Gamma(1+\frac{1}{2}) = \sqrt{\pi}\,\frac{m\Gamma(m)}{\Gamma(m+\frac{1}{2})} = \sqrt{\pi}\,\frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})}, \tag{6.14}$$

$$(2m+\frac{1}{2})\frac{\Gamma(m+\frac{1}{4})}{\Gamma(m+\frac{3}{4})}\frac{\Gamma(1+\frac{3}{4})}{\Gamma(1+\frac{1}{4})} = 3\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\,(2m+\frac{1}{2})\frac{\Gamma(m+\frac{1}{4})}{\Gamma(m+\frac{3}{4})} = 6\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\frac{\Gamma(m+\frac{5}{4})}{\Gamma(m+\frac{3}{4})}. \tag{6.15}$$

Substituting (6.14) and (6.15) into (6.13), we obtain (6.2). Finally, from (6.2) and (6.11) one has (6.3), because

$$\liminf_{m\to\infty}\frac{\overset{m+1}{M}_0}{m^{\frac{1}{2}}} \geq \lim_{m\to\infty}\sqrt{\pi}\,\frac{\Gamma(m+1)}{m^{\frac{1}{2}}\Gamma(m+\frac{1}{2})}e^{\frac{1}{3+4m}} = \sqrt{\pi},$$

$$\limsup_{m\to\infty}\frac{\overset{m+1}{M}_0}{m^{\frac{1}{2}}} \leq \lim_{m\to\infty}6\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\frac{\Gamma(m+\frac{5}{4})}{\Gamma(m+\frac{3}{4})}e^{\frac{1}{2+4m}} = 6\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}.$$

□

Proof of Corollary 6.3. This is a direct consequence of Theorem 6.2 and the fact that, for every $i \in \mathbb{N}$,

$$\frac{\overset{m+1}{M}_0}{\overset{m+1}{M}_i} = \frac{\overset{1}{M}_0}{\overset{i+1}{M}_i}\prod_{k=2}^{i+1}\overset{k}{R}_{k-1}, \qquad \frac{\overset{m+1}{M}_0}{\overset{m+1}{D}_i} = \frac{\overset{1}{M}_0}{\overset{i}{D}_i}\prod_{k=2}^{i}\overset{k}{R}_{k-1}, \qquad \overset{m+1}{M}_0\overset{m+1}{S}_i = \overset{1}{M}_0\overset{i+1}{S}_i(\prod_{k=2}^{i+1}\overset{k}{R}_{k-1})^{-1},$$

$$\overset{m+1}{M}_0\overset{m+1}{R}_i = \overset{1}{M}_0(\prod_{k=2}^{i}\overset{k}{R}_{k-1})^{-1},$$

where all the right-hand sides are positive constants independent of m. □

We conclude the section with the proofs of Theorem 2.8 and Corollary 2.9.

Proof of Theorem 2.8. By Theorem 2.4, we have that $|\overset{m}{u}_{\alpha,p}(\overset{m}{s}_{0,\alpha,p})| = \overset{m}{M}_0 + o(1)$ as $p \to \infty$, and then (2.31), (2.33) follow from Theorem 6.2. Furthermore, by Theorem 2.6, $|\overset{m}{\bar{u}}_{\alpha,p}(\overset{m}{\bar{s}}_{0,\alpha,p})| = \overset{m}{\bar{M}}_0 + o(1) = \overset{m}{S}_{m-1}\overset{m}{M}_0 + o(1)$. Then (2.32) follows from Theorem 6.2 and the fact that

$$e^{-\frac{1}{4m-2}} < \overset{m}{S}_{m-1} = e^{-\frac{A_{m-1}}{2A_{m-1}+1}} < e^{-\frac{1}{4m-1}}.$$

Finally, (2.34) holds by Theorem 6.2 and because

$$\lim_{m\to\infty}\overset{m}{S}_{m-1} = \lim_{m\to\infty}e^{-\frac{2}{2+\theta_{m-1}}} = 1. \tag{6.16}$$

□

Proof of Corollary 2.9. By Theorems 2.4 and 2.6, $|\overset{m}{u}_{\alpha,p}(\overset{m}{s}_{i,\alpha,p})| = \overset{m}{M}_i + o(1)$ and $|\overset{m}{\bar{u}}_{\alpha,p}(\overset{m}{\bar{s}}_{i,\alpha,p})| = \overset{m}{\bar{M}}_i + o(1) = \overset{m}{S}_{m-1}\overset{m}{M}_i + o(1)$ as $p \to \infty$ and the claim follows from (6.16) and (6.4). □

Remark 6.8. *An efficient way of computing numerically the values of the θ_k's is via a for-cycle; for instance, in the computing software Mathematica, for a given number $N1$, the code*

```
T[0] = N[2]
psi[t_] := 2/(2 + t)*Exp[-2/(2 + t)]
For[k = 1, k < N1, k++, T[k] = 2 + 2 / ProductLog[psi[T[k - 1]]]]
```

computes N1 values of $\theta_k = T[k]$ (c.f. (2.1)). Using $N1 = 10^6$ and Lemma 6.7, we see that $\lim_{m\to\infty} \frac{\overset{m+1}{M}_0}{\sqrt{m}} \approx 1.82774$. *Indeed, numerically it seems that the sequence* $(\frac{\overset{m+1}{M}_0}{\sqrt{m}})_{m\in\mathbb{N}}$ *is monotone, and in particular it has a limit as $m \to \infty$. This would imply that Theorem 2.8 and Corollary 2.9 can be written with* lim *instead of* lim sup *and* lim inf. *However, a rigorous proof of this monotonicity seems to require sharper bounds that the ones given by Theorem 6.2.*

m	θ_m	$\overset{m}{M}_0$	$\overset{m}{M}_0/\sqrt{m}$
1	10.374	1.6487	1.6487
2	18.4277	2.46075	1.74001
3	26.4493	3.06521	1.7697
4	34.4609	3.56876	1.78438
5	42.4682	4.00957	1.79313
6	50.4731	4.40651	1.79895
7	58.4767	4.77053	1.80309
8	66.4795	5.10867	1.80619
9	74.4816	5.42579	1.8086
10	82.4833	5.72537	1.81052
11	90.4848	6.01003	1.81209
12	98.486	6.28181	1.8134
13	106.487	6.5423	1.81451
14	114.488	6.79282	1.81546
15	122.489	7.03442	1.81628
16	130.489	7.26799	1.817
17	138.49	7.49429	1.81763
18	146.49	7.71395	1.81819
19	154.491	7.92752	1.8187
20	162.491	8.13549	1.81915
21	170.492	8.33828	1.81956
22	178.492	8.53625	1.81993
23	186.492	8.72973	1.82027
24	194.493	8.91901	1.82059
25	202.493	9.10436	1.82087

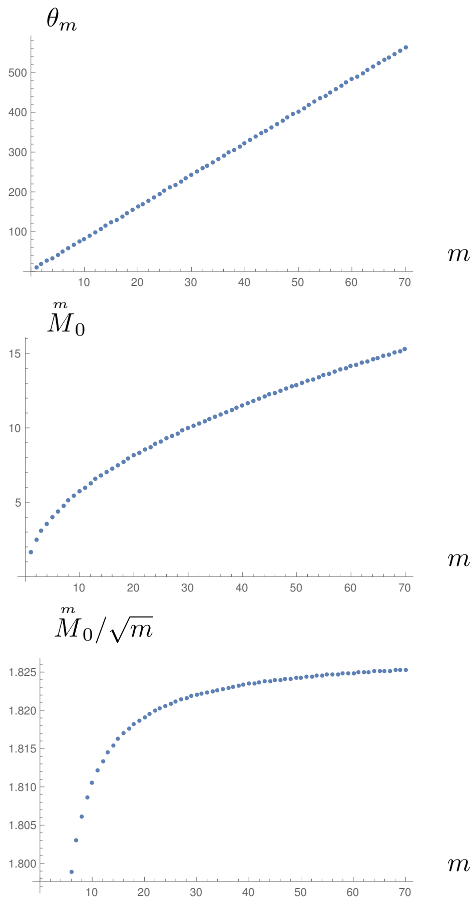

FIGURE 4. Some numerical values and plots of θ_m, $\overset{m}{M}_0$, and $\overset{m}{M}_0/\sqrt{m}$.

We close this section with the proof of Lemma 2.3.

Proof of Lemma 2.3. To see (2.8) and (2.9), observe that $\overset{m}{M}_{m-1} = e^{\frac{2}{2+\theta_{m-1}}} > 1$, and as a consequence, $\overset{m}{S}_{m-1} = (\overset{m}{M}_{m-1})^{-1} < 1$. Then, $\frac{\overset{m}{S}_i}{\overset{m}{R}_{i+1}} = \overset{i+1}{S}_i < 1$. Hence, $\overset{m}{S}_i < \overset{m}{R}_{i+1}$ for $i = 1, \ldots, m-2$. Furthermore, since $\mathcal{L}\left(\frac{y}{e^y}\right) < \frac{y}{e^y}$ we have (setting $y := \frac{2}{\theta_{m-2}+2}$) that $e^{\frac{2}{\theta_{m-2}+2}} \frac{(\theta_{m-2}+2)}{2} < \frac{\theta_{m-1}-2}{2}$, and then, by (2.3) and (2.5), we deduce that $\overset{m}{R}_{m-1} < \overset{m}{S}_{m-1}$. As a consequence, $\overset{m}{R}_i < \overset{m}{S}_i$ for $i = 1, \ldots, m-1$ and (2.8), (2.9) follow. The strict monotonicity of the sequence $(\overset{m}{M}_j)_m$ follows from (2.7), since $\overset{m-1}{M}_j = \overset{m}{R}_{m-1}\overset{m}{M}_j < \overset{m}{M}_j$. Similarly, one can prove the monotonicity for the sequences $(\overset{m}{S}_j)_m$, $(\overset{m}{R}_j)_m$ and the monotonicity of $(\overset{m}{D}_j)_m$ can be deduced from (2.6) and the monotonicity of $(\overset{m}{R}_j)_m$. □

Appendix A. Morse index conjecture for the Dirichlet Lane-Emden case

Let $\overset{m}{u}_p := \overset{m}{u}_{0,p}$ be the radial solution of the Lane-Emden equation (1.2) with $\alpha = 0$ and with Dirichlet boundary conditions (1.3), having $m-1$ interior zeros. We conjecture that for p sufficiently large the Morse index $\mathfrak{m}$ satisfies

$$\mathfrak{m}(\overset{m}{u}_p) = \sum_{k=0}^{m-1} \mathfrak{m}(Z_k), \tag{A.1}$$

where $Z_k := Z_{k,0}$ are the limit profiles in (2.19) with $\alpha = 0$, whose Morse indexes are known (see [11]), namely,

$$\mathfrak{m}(Z_0) = 1, \qquad \mathfrak{m}(Z_k) = 1 + 2\left[\frac{\theta_k}{2}\right] \qquad \text{for } k \geq 1,$$

where $[c]$ is the integer part of $c \in \mathbb{R}$. Then, by (6.1),

$$\mathfrak{m}(Z_k) = 1 + 2\left[\frac{\theta_k}{2}\right] = 8k + 3 \qquad \text{for all } k \in \mathbb{N},\ k \geq 1. \tag{A.2}$$

As a consequence from conjecture (A.1) one would obtain the following *Morse index formula.*

Conjecture A.1. *For any $m \geq 1$ there exists $p_m > 1$ such that*

$$\mathfrak{m}(\overset{m}{u}_p) = 4m^2 - m - 2, \qquad \forall p \geq p_m.$$

Indeed, simple computations would show that

$$\mathfrak{m}(\overset{m}{u}_p) \overset{(A.1)}{=} \sum_{k=0}^{m-1} \mathfrak{m}(Z_k) \overset{(A.2)}{=} 1 + 3(m-1) + 8\sum_{k=1}^{m-1} k = 3m - 2 + 4m(m-1) = 4m^2 - m - 2.$$

Conjecture A.1 holds in the case $m = 1$, since it can be proved that the least energy solution has Morse index 1 (see [33]). Moreover it has been proved in the case $m = 2$ in [18], where it is shown that $\mathfrak{m}(\overset{2}{u}_p) = 12$. We remark that in higher dimension $N \geq 3$, the Morse index is smaller, since, in this case, the Morse index grows linearly; to be more precise, in [19] it is shown that for $N \geq 3$ and $m \geq 1$ there exists $\delta_m > 0$ such that

$$\mathfrak{m}(\overset{m}{u}_p) = m + N(m-1) \qquad \text{for } \frac{N+2}{N-2} - \delta_m \leq p < \frac{N+2}{N-2}.$$

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