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Kazdan–Warner obstructions for a fourth-order boundary problem

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Abstract

We derive Kazdan–Warner type identities for the boundary problem of prescribing nonconstant interior Q curvature and boundary T curvature on the upper hemisphere $\mathbb{S}_+^4$ by a conformal change of the standard metric. Using the natural variational formulation and conformal variations generated by boundary-preserving conformal vector fields, we obtain nontrivial integral obstructions to solvability.

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1 | INTRODUCTION

Let (Σ, g) be a compact surface and let $\tilde{g} = e^{2u}g$ be a conformal metric. The classical *Kazdan–Warner problem* asks which functions K can arise as the Gaussian curvature of $\tilde{g}$. Analytically, u

solves the Liouville-type equation

$$-\Delta_g u + K_g = K e^{2u} \quad \text{in } \Sigma, \quad (1.1)$$

where K_g denotes the Gaussian curvature of Σ with respect to g . The Gauss–Bonnet theorem shows that the Equation (1.1) does not always admit a solution, since a topological constraint must hold, namely,

$$\int_{\Sigma} K e^{2u} dV_g = 2\pi\chi(\Sigma).$$

On the round sphere $(\mathbb{S}^2, g_0)$, this problem is known as the Nirenberg problem, whose difficulty is intimately related to the noncompactness of the group of conformal transformations of the sphere. One important feature of this setting is the existence of nontrivial integral obstructions to solvability, as observed by Kazdan and Warner in [13, 14] by differentiating the Euler–Lagrange functional along conformal flows. A particularly interesting case arises when taking the gradients of the first spherical harmonics as conformal vector fields, as highlighted by Chang and Yang in [5]. More precisely, if u solves $\Delta_{g_0} u + K e^{2u} = 1$ on $\mathbb{S}^2$, then one has the explicit obstruction

$$\int_{\mathbb{S}^2} \langle \nabla_{g_0} K, \nabla_{g_0} x_j \rangle e^{2u} dV_{g_0} = 0, \quad j = 1, 2, 3,$$

where x_j are the coordinate functions of the embedding $\mathbb{S}^2 \subset \mathbb{R}^3$. In particular, monotonicity of K along a conformal direction prevents existence of solutions.

A natural boundary analogue for (1.1) is obtained by prescribing, besides the interior curvature, the geodesic curvature of $\partial\Sigma$ to be equal to a given function h . If Σ has boundary $\partial\Sigma$ with geodesic curvature h_g and $\tilde{g} = e^{2u}g$, this problem reduces to solving the boundary problem

$$-\Delta_g u + K_g = K e^{2u} \quad \text{in } \Sigma, \quad \frac{\partial u}{\partial \nu} + h_g = h e^u \quad \text{on } \partial\Sigma,$$

where ν denotes the external unit normal vector. The case of the upper hemisphere $\Sigma = \mathbb{S}_+^2$ with the standard round metric g_0 is especially delicate, again due to the noncompact action of its conformal group. In this case, we arrive at the following Liouville-type boundary problem

$$\begin{cases} -\Delta u + 1 = K e^{2u} & \text{in } \mathbb{S}_+^2, \\ \frac{\partial u}{\partial \nu} = h e^u & \text{on } \mathbb{S}^1, \end{cases} \quad (1.2)$$

subject to the Gauss–Bonnet constraint

$$\int_{\mathbb{S}_+^2} K e^{2u} dV_{g_0} + \int_{\mathbb{S}^1} h e^u dS_{g_0} = 2\pi.$$

This problem with nonconstant K and h from the point of view of the disk has been studied in [8, 22, 23], and more recently in [16].

As in the case of the whole sphere, further obstructions to the existence of solutions in the form of Kazdan–Warner identities persist in the hemisphere. More precisely, Hamza [11] showed, via a careful integration by parts, that if u solves (1.2), then

$$\int_{\mathbb{S}_+^2} e^{2u} \langle \nabla K, \nabla F \rangle dV_{g_0} + 2 \int_{\mathbb{S}^1} e^u \langle \nabla h, \nabla F \rangle dS_{g_0} = 0. \quad (1.3)$$

for every F in the span of the restrictions of the coordinate functions. In particular, (1.3) yields a simple nonexistence criterion: if there exists such an F for which $\langle \nabla K, \nabla F \rangle$ and $\langle \nabla h, \nabla F \rangle$ have the same strict sign, then the problem admits no solution.

If (M, g) is a closed Riemannian manifold of dimension four, Branson's Q -curvature and the conformally covariant Paneitz operator P_g^4 provide a natural analogue of the Gaussian curvature in the context of conformal geometry. Under the change of the metric $\tilde{g} = e^{2u}g$, they satisfy the transformation law

$$P_g^4 u + 2Q_g = 2Q_{\tilde{g}} e^{4u} \quad \text{in } M; \quad (1.4)$$

see [1, 21]. In view of (1.4), one could address the existence of conformal metrics $\tilde{g}$ with prescribed Q -curvature equal to a given function Q (see [2, 6, 9]). In this regard, the case of the sphere $\mathbb{S}^4$ is again particularly challenging and so far only the partial results obtained in [15, 17, 24] for $Q > 0$ are available. Notably, this case displays the same obstruction mechanism as in the two-dimensional Nirenberg problem. Indeed, Chang and Yang pointed out in [6] that, if u solves

$$P_{g_0}^4 u + 2Q_{g_0} = 2Q e^{4u} \quad \text{on } \mathbb{S}^4,$$

then for the coordinate functions x_j on $\mathbb{S}^4 \subset \mathbb{R}^5$ one has

$$\int_{\mathbb{S}^4} \langle \nabla_{g_0} Q, \nabla_{g_0} x_j \rangle e^{4u} dV_{g_0} = 0, \quad j = 1, \dots, 5, \quad (1.5)$$

which again yields immediate nonexistence criteria under suitable monotonicity of Q along the Euclidean directions.

In the presence of a boundary, however, the integral quantity in (1.5) is no longer forced to vanish. More precisely, Gursky showed in [10, Theorem C] that the variation of the Q -curvature along a conformal vector field can be expressed as a boundary integral involving a symmetric fourth-order (0,2)-tensor introduced for this purpose and characterized by its structural properties.

If M has boundary, Chang and Qing [3, 4] introduced a third-order boundary operator P_g^3 and the corresponding boundary curvature T_g , so that

$$P_g^4 u + 2Q_g = 2Q_{\tilde{g}} e^{4u} \quad \text{in } M, \quad P_g^3 u + T_g = T_{\tilde{g}} e^{3u} \quad \text{on } \partial M. \quad (1.6)$$

Together with the Gauss–Bonnet–Chern formula, (1.6) suggests that the pair (Q_g, T_g) is the natural four-dimensional counterpart of (K_g, h_g) on surfaces.

If one aims to prescribe Q and T curvatures on M and ∂M , one needs to solve (1.6) with $Q_{\tilde{g}} = Q$ and $T_{\tilde{g}} = T$. For the sake of solvability, it is natural to impose an additional first-order boundary condition. In order to maintain the conformal nature of the problem, one natural condition to

work with is to impose that $(M, e^{2u}g)$ has minimal boundary, that is,

$$\frac{\partial u}{\partial \nu} + \frac{1}{3}H_g = 0 \quad \text{on } \partial M,$$

where ν denotes the unit normal vector pointing outwards.

In general terms, this problem has not been widely addressed, and the available results cover mostly the particular cases in which both curvatures are constant, and one of them is identically equal to zero, see [4, 18–20]. Moreover, all these results rule out the manifolds that are conformally equivalent to the hemisphere. The latter case, including nonconstant curvatures, was recently addressed in [7] for $Q, T \geq 0$; see also [12] for a T -curvature flow approach on $\mathbb{B}^4$.

The present work is precisely concerned with the model case of the upper hemisphere $\mathbb{S}_+^4 \subset \mathbb{R}^5$ endowed with the standard metric g , for which $\partial\mathbb{S}_+^4 = \mathbb{S}^3$ is totally geodesic. In this setting we have $P_g^4 = \Delta_g^2 - 2\Delta_g$ and $Q_g = 3$. Moreover, the boundary is totally geodesic, which means the second fundamental form $\mathbb{I}_g$ vanishes everywhere on $\mathbb{S}^3$. This gives $T_g = H_g = 0$ and, for functions satisfying $\partial_\nu u = 0$, one has $P_g^3 u = -\frac{1}{2}\partial_\nu(\Delta_g u)$. Therefore, prescribing smooth functions Q on $\mathbb{S}_+^4$ and T on $\mathbb{S}^3$ as the Q and T curvatures of $\tilde{g} = e^{2u}g$ leads to the boundary problem

$$\begin{cases} \Delta_g^2 u - 2\Delta_g u + 6 = 2Q e^{4u} & \text{in } \mathbb{S}_+^4, \\ -\frac{\partial \Delta_g u}{\partial \nu} = 2T e^{3u} & \text{on } \mathbb{S}^3, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \mathbb{S}^3, \end{cases} \quad (\text{QT})$$

where $\nu = -\nabla_g x_5$, together with the Gauss–Bonnet–Chern constraint

$$\int_{\mathbb{S}_+^4} Q e^{4u} dV_g + \int_{\mathbb{S}^3} T e^{3u} dS_g = 4\pi^2. \quad (1.7)$$

Throughout the paper, we assume that $Q \in C^\infty(\overline{\mathbb{S}_+^4})$ and $T \in C^\infty(\mathbb{S}^3)$.

In [7] we initiated the study of (QT) from the variational point of view, deriving existence results under symmetry assumptions by exploiting a mean-field type reformulation and sharp higher order Moser–Trudinger inequalities on $\mathbb{S}_+^4$.

This paper goes in the opposite direction: we investigate *nonexistence phenomena* for (QT) by deriving Kazdan–Warner type identities in the spirit of (1.3) and (1.5). The proof follows the classical strategy of differentiating the energy along conformal orbits of u generated by boundary-preserving conformal vector fields, as done in [5]. More precisely, if $\{\phi_t\}$ is a conformal flow on $\mathbb{S}_+^4$ preserving the boundary and $\phi_t^* g = e^{2P_t} g$ for some smooth function P_t on $\mathbb{S}_+^4$, we test the Euler–Lagrange functional along the curve

$$u_t = u \circ \phi_t + P_t.$$

However, the fourth-order nature of (QT) requires establishing a cocycle identity for the quadratic part of the energy induced by P_g^4 and P_g^3 . Differentiating at $t = 0$ and using this identity together with conformal covariance yields an integral relation linking $X(Q)$ and $X(T)$, where X is the infinitesimal generator of ϕ_t . Our main result reads as follows:

Theorem 1.1. *Let $u \in \mathcal{H}$ be a solution to (QT). Then for every boundary-preserving conformal vector field X on $\mathbb{S}_+^4$,*

$$\int_{\mathbb{S}_+^4} X(Q)e^{4u} dV_g + \frac{4}{3} \int_{\mathbb{S}^3} X(T)e^{3u} dS_g = 0. \quad (1.8)$$

As a consequence, the problem admits no solutions if Q and T are simultaneously monotone along a conformal direction. In particular, we obtain the analogue of (1.3) by choosing X among the standard conformal vector fields associated to the coordinate functions.

The paper is organized as follows. In Section 2 we recall the direct variational formulation of (QT) and the basic conformal invariance properties needed in the argument. Section 3 is devoted to the derivation of the Kazdan–Warner identity on $\mathbb{S}_+^4$. Finally, in Section 4 we discuss several concrete nonexistence criteria and illustrative examples. We also include an appendix detailing the weak formulation of the problem.

2 | THE VARIATIONAL SETTING

Let

$$\mathcal{H}(M) = \left\{ u \in H^2(M) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial M \right\}.$$

We adopt the convention that, when we write $\mathcal{H}$, we are referring to $\mathcal{H}(\mathbb{S}_+^4)$.

Definition 2.1. We define the energy functional $I : \mathcal{H} \rightarrow \mathbb{R}$ by

$$\begin{aligned} I(u) = & \int_{\mathbb{S}_+^4} (\Delta_g u)^2 dV_g + 2 \int_{\mathbb{S}_+^4} |\nabla u|^2 dV_g + 12 \int_{\mathbb{S}_+^4} u dV_g \\ & - \int_{\mathbb{S}_+^4} Qe^{4u} dV_g - \frac{4}{3} \int_{\mathbb{S}^3} Te^{3u} dS_g. \end{aligned} \quad (2.1)$$

Integrating by parts and using the condition $\frac{\partial u}{\partial \nu} = 0$ on $\mathbb{S}^3$, it is easy to check that critical points of I solve (QT) in the usual weak sense. More precisely, if u is a critical point of I then

$$\begin{aligned} & \int_{\mathbb{S}_+^4} \Delta_g u \Delta_g v dV_g + 2 \int_{\mathbb{S}_+^4} \langle \nabla u, \nabla v \rangle dV_g + 6 \int_{\mathbb{S}_+^4} uv dV_g \\ & - \int_{\mathbb{S}^3} 2Te^{3u}v dS_g - \int_{\mathbb{S}_+^4} 2Qe^{4u}v dV_g = 0 \quad \forall v \in \mathcal{H}, \end{aligned}$$

which is the weak formulation of (QT). The details of this computation can be found in Appendix A.

Now, we decompose our functional in three parts that will be studied separately. The following definitions and result do not require us to be in the particular setting of the hemisphere, so we will place ourselves in a more general context.

Definition 2.2. We define the quadratic form $\mathcal{E}_g : \mathcal{H}(M) \times \mathcal{H}(M) \rightarrow \mathbb{R}$ as

$$\mathcal{E}_g(u, v) = \int_M (P_g^4 u) v \, dV_g + 2 \int_{\partial M} (P_g^3 u) v \, dS_g \quad u, v \in \mathcal{H}(M).$$

In the case of $(\mathbb{S}_+^4, g)$, we can integrate by parts twice, using the definitions of P_g^4 and P_g^3 introduced in Section 1, to obtain the simplified expression:

$$\begin{aligned} \mathcal{E}_g(u, v) &= \int_{\mathbb{S}_+^4} (\Delta^2 u - 2\Delta u) v \, dV_g - \int_{\mathbb{S}^3} \frac{\partial \Delta u}{\partial \nu} v \, dS_g \\ &= \int_{\mathbb{S}_+^4} \Delta_g u \Delta_g v \, dV_g + 2 \int_{\mathbb{S}_+^4} \nabla_g u \cdot \nabla_g v \, dV_g, \quad u, v \in \mathcal{H}. \end{aligned} \quad (2.2)$$

Definition 2.3. Let (M, g) be a four-dimensional Riemannian manifold with boundary ∂M , and let Q_g and T_g be its Q and boundary T curvatures. We define the functional $S : \mathcal{H}(M) \rightarrow \mathbb{R}$ by

$$S_g(u) = \mathcal{E}_g(u, u) + 4 \int_M Q_g u \, dV_g + 4 \int_{\partial M} T_g u \, dS_g, \quad u \in \mathcal{H}(M).$$

Lemma 2.4. Let $w, v \in \mathcal{H}(M)$ and set $\tilde{g} = e^{2w} g$. Then

$$S_g(w + v) = S_g(w) + S_{\tilde{g}}(v).$$

Proof. Expanding by linearity,

$$\begin{aligned} S_g(w + v) &= \mathcal{E}_g(w + v, w + v) + 4 \int_M Q_g(w + v) \, dV_g + 4 \int_{\partial M} T_g(w + v) \, dS_g \\ &= S_g(w) + \mathcal{E}_g(v, v) + 2\mathcal{E}_g(w, v) + 4 \int_M Q_g v \, dV_g + 4 \int_{\partial M} T_g v \, dS_g. \end{aligned}$$

Remember that, by conformal covariance,

$$\begin{aligned} P_{\tilde{g}}^4(\varphi) &= e^{-4w} P_g^4(\varphi), \quad dV_{\tilde{g}} = e^{4w} dV_g, \\ P_{\tilde{g}}^3(\varphi) &= e^{-3w} P_g^3(\varphi), \quad dS_{\tilde{g}} = e^{3w} dS_g. \end{aligned} \quad (2.3)$$

These imply that

$$\mathcal{E}_g(v, v) = \int_M (e^{4w} P_{\tilde{g}}^4 v) v \, e^{-4w} dV_{\tilde{g}} + 2 \int_{\partial M} (e^{3w} P_{\tilde{g}}^3 v) v \, e^{-3w} dS_{\tilde{g}} = \mathcal{E}_{\tilde{g}}(v, v). \quad (2.4)$$

Now, multiply the transformation rules

$$P_g^4 w + 2Q_g = 2Q_{\tilde{g}} e^{4w}, \quad P_g^3 w + T_g = T_{\tilde{g}} e^{3w},$$

by v , integrate and use the conformal identities (2.3) to obtain

$$\int_M (P_g^4 w)v \, dV_g + 2 \int_M Q_g v \, dV_g = 2 \int_M Q_{\tilde{g}} e^{4w} v \, dV_g = 2 \int_M Q_{\tilde{g}} v \, dV_{\tilde{g}}, \quad (2.5)$$

$$\int_{\partial M} (P_g^3 w)v \, dS_g + \int_{\partial M} T_g v \, dS_g = \int_{\partial M} T_{\tilde{g}} e^{3w} v \, dS_g = \int_{\partial M} T_{\tilde{g}} v \, dS_{\tilde{g}}. \quad (2.6)$$

Combining (2.5) and (2.6) we get

$$\begin{aligned} & 2\mathcal{E}_g(w, v) + 4 \int_M Q_g v \, dV_g + 4 \int_{\partial M} T_g v \, dS_g \\ &= 2 \left(\int_M (P_g^4 w)v \, dV_g + 2 \int_M Q_g v \, dV_g + 2 \int_{\partial M} (P_g^3 w)v \, dS_g + 2 \int_{\partial M} T_g v \, dS_g \right) \\ &= 4 \left(\int_M Q_{\tilde{g}} v \, dV_{\tilde{g}} + \int_{\partial M} T_{\tilde{g}} v \, dS_{\tilde{g}} \right). \end{aligned} \quad (2.7)$$

Finally, by (2.4) and (2.7), we reach the desired conclusion:

$$S_g(w + v) = S_g(w) + S_{\tilde{g}}(v). \quad \square$$

In view of Definitions 2.2 and 2.3, the identity (2.2) and the fact that $Q_g = 3$ and $T_g = 0$ in $(\mathbb{S}_+^4, g)$, we can rewrite the energy functional (2.1) as

$$I = S - \mathcal{N}_Q - \frac{4}{3}B_T, \quad (2.8)$$

with

$$S(u) = \int_{\mathbb{S}_+^4} (\Delta_g u)^2 \, dV_g + 2 \int_{\mathbb{S}_+^4} |\nabla u|^2 \, dV_g + 12 \int_{\mathbb{S}_+^4} u \, dV_g$$

and

$$\mathcal{N}_Q(u) = \int_{\mathbb{S}_+^4} Q e^{4u} \, dV_g, \quad B_T(u) = \int_{\mathbb{S}^3} T e^{3u} \, dS_g. \quad (2.9)$$

3 | CONFORMAL VARIATIONS: PROOF OF THEOREM 1.1

Let X be a C^1 vector field on $\mathbb{S}_+^4$ that preserves the boundary, namely

$$\langle X, \nu \rangle = -\langle X, \nabla_g x_5 \rangle = 0 \quad \text{on } \mathbb{S}^3.$$

Let $\{\phi_t\}$ for $t \in (-\varepsilon, \varepsilon)$ be its flow:

$$\phi_0 = \text{Id}, \quad \frac{d}{dt} \phi_t(p) = X(\phi_t(p)).$$

We write

$$X(f) = \frac{d}{dt} \Big|_{t=0} f(\phi_t) \quad (3.1)$$

for the action of X on functions. Assume now that ϕ_t is conformal, that is, there exists a smooth function P_t on $\mathbb{S}_+^4$ such that

$$\phi_t^* g = e^{2P_t} g, \quad P_0 = 0. \quad (3.2)$$

To simplify the notation, we will refer to the above metric as g_t . Given a solution u to (QT), we consider the curve

$$u_t = u \circ \phi_t + P_t. \quad (3.3)$$

Any weak solution $u \in \mathcal{H}$ of (QT) is smooth by standard elliptic regularity for the corresponding fourth-order boundary value problem. Consequently, the curve u_t is a differentiable curve in $\mathcal{H}$. Since u is a critical point of I , it follows that

$$\left. \frac{d}{dt} \right|_{t=0} I(u_t) = 0. \quad (3.4)$$

Proposition 3.1. *For any boundary-preserving conformal flow (3.2) and any $u \in \mathcal{H}$,*

$$\left. \frac{d}{dt} \right|_{t=0} S(u_t) = 0.$$

Proof. First, we observe that $u_t \in \mathcal{H}$: since ϕ_t is a C^∞ diffeomorphism of $\mathbb{S}_+^4$, $u \circ \phi_t \in H^2(\mathbb{S}_+^4)$. Moreover $P_t \in C^\infty(\mathbb{S}_+^4) \subset H^2(\mathbb{S}_+^4)$, so $u_t \in H^2(\mathbb{S}_+^4)$.

It remains to check the boundary condition. Fix $p \in \mathbb{S}^3$. By the chain rule,

$$d(u \circ \phi_t)_p(\nu(p)) = du_{\phi_t(p)}(d\phi_t|_p(\nu(p))).$$

Since $\phi_t(\mathbb{S}^3) = \mathbb{S}^3$, we have $d\phi_t(T_p \mathbb{S}^3) = T_{\phi_t(p)} \mathbb{S}^3$. As ϕ_t is conformal, $d\phi_t$ preserves orthogonality, hence it maps the normal line $N_p(\mathbb{S}^3) = \text{span}\{\nu(p)\}$ onto $N_{\phi_t(p)}(\mathbb{S}^3) = \text{span}\{\nu(\phi_t(p))\}$. Therefore there exists $\lambda_t(p) \in \mathbb{R}$ such that

$$d\phi_t|_p(\nu(p)) = \lambda_t(p) \nu(\phi_t(p)).$$

Note that $\lambda_t(p) \neq 0$ since $d\phi_t|_p$ is invertible. Moreover $\lambda_0(p) = 1$ because $\phi_0 = \text{Id}$, and by continuity $\lambda_t(p) > 0$ for $|t|$ small.

Consequently,

$$\frac{\partial(u \circ \phi_t)}{\partial \nu}(p) = \lambda_t(p) \frac{\partial u}{\partial \nu}(\phi_t(p)) = 0,$$

because $u \in \mathcal{H}$ and $\phi_t(p) \in \mathbb{S}^3$.

Finally, we observe that the conformal factor P_t satisfies

$$\frac{\partial P_t}{\partial \nu} = 0 \quad \text{on } \mathbb{S}^3. \quad (3.5)$$

Indeed, since ϕ_t is a diffeomorphism with $\phi_t(\mathbb{S}^3) = \mathbb{S}^3$, then $\phi_t : (\mathbb{S}_+^4, g_t) \rightarrow (\mathbb{S}_+^4, g)$ is an isometry. In particular, the second fundamental form $\mathbb{I}_g$ is transported via ϕ_t , so $(\mathbb{S}_+^4, g_t)$ also has totally geodesic boundary.

It is known that, under the conformal change of metrics $g_t = e^{2P_t}g$, we have

$$e^{-P_t} \mathbb{I}_{g_t} = \mathbb{I}_g + \frac{\partial P_t}{\partial \nu} g|_{\mathbb{S}^3}$$

on $\mathbb{S}^3$, which gives (3.5).

Hence

$$\frac{\partial u_t}{\partial \nu} = \frac{\partial(u \circ \phi_t)}{\partial \nu} + \frac{\partial P_t}{\partial \nu} = 0 \quad \text{on } \mathbb{S}^3,$$

showing that $u_t \in \mathcal{H}$.

By Lemma 2.4 with $w = P_t$, $\tilde{g} = e^{2P_t}g = \phi_t^*g$, and $v = u \circ \phi_t$, and the invariance of S under diffeomorphisms,

$$\begin{aligned} S_g(u_t) &= S_g(u \circ \phi_t + P_t) = S_g(P_t) + S_{e^{2P_t}g}(u \circ \phi_t) \\ &= S_g(P_t) + S_{\phi_t^*g}(u \circ \phi_t) = S_g(P_t) + S_g(u). \end{aligned}$$

Hence, since $S_g(u)$ is independent of t ,

$$\left. \frac{d}{dt} \right|_{t=0} S_g(u_t) = \left. \frac{d}{dt} \right|_{t=0} S_g(P_t). \quad (3.6)$$

Let us prove that $\left. \frac{d}{dt} \right|_{t=0} S_g(P_t) = 0$. First, we see that

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} S_g(P_t) &= \left. \frac{d}{dt} \right|_{t=0} \left(\int_{\mathbb{S}_+^4} (\Delta_g P_t)^2 dV_g + 2 \int_{\mathbb{S}_+^4} |\nabla P_t|^2 dV_g + 12 \int_{\mathbb{S}_+^4} P_t dV_g \right) \\ &= 12 \int_{\mathbb{S}_+^4} \dot{P}_0 dV_g, \end{aligned} \quad (3.7)$$

since all the quadratic terms vanish because $P_0 = 0$. Now, we aim to differentiate the identity $\phi_t^*g = e^{2P_t}g$ at $t = 0$. Remember that, for any C^1 vector fields Y, Z on $\mathbb{S}_+^4$, we have

$$(\phi_t^*g)(Y, Z) = g((\phi_t)_*Y, (\phi_t)_*Z) \circ \phi_t,$$

that is,

$$(\phi_t^*g)_p(Y_p, Z_p) = g_{\phi_t(p)}(d\phi_t(p)[Y_p], d\phi_t(p)[Z_p]).$$

Therefore,

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} (\phi_t^*g)(Y, Z) &= X(g(Y, Z)) + g\left(\left. \frac{d}{dt} \right|_{t=0} (\phi_t)_*Y, Z\right) + g\left(Y, \left. \frac{d}{dt} \right|_{t=0} (\phi_t)_*Z\right) \\ &= X(g(Y, Z)) + g([X, Y], Z) + g(Y, [X, Z]) \\ &= (\mathcal{L}_X g)(Y, Z), \end{aligned}$$

where the right-hand side is the Lie derivative. Consequently, we have

$$\mathcal{L}_X g = 2\dot{P}_0 e^{2P_0} g = 2\dot{P}_0 g. \quad (3.8)$$

If we now take the trace of (3.8) we get $2\operatorname{div}_g X = 8\dot{P}_0$, which gives

$$\dot{P}_0 = \frac{1}{4} \operatorname{div}_g X.$$

Integrating on $\mathbb{S}_+^4$ and applying the divergence theorem:

$$\int_{\mathbb{S}_+^4} \dot{P}_0 dV_g = \frac{1}{4} \int_{\mathbb{S}_+^4} \operatorname{div}_g X dV_g = \frac{1}{4} \int_{\mathbb{S}^3} X \cdot \nu dS_g = 0, \quad (3.9)$$

since X preserves the boundary. The result follows from combining (3.9), (3.7) and (3.6). $\square$

We now compute the derivatives of the nonlinear terms $\mathcal{N}_Q$ and $\mathcal{B}_T$ defined in (2.9) along (3.3). For the interior term,

$$\left. \frac{d}{dt} \right|_{t=0} \int_{\mathbb{S}_+^4} Q e^{4u_t} dV_g = \left. \frac{d}{dt} \right|_{t=0} \int_{\mathbb{S}_+^4} Q(x) e^{4u(\phi_t(x))} e^{4P_t(x)} dV_g(x).$$

Performing the change of variables $y = \phi_t(x)$, so that $x = \phi_{-t}(y)$, and using the conformal relation (3.2) to relate the Jacobian to P_t , we obtain

$$\left. \frac{d}{dt} \right|_{t=0} \int_{\mathbb{S}_+^4} Q e^{4u_t} dV_g = \left. \frac{d}{dt} \right|_{t=0} \int_{\mathbb{S}_+^4} Q(\phi_{-t}(y)) e^{4u(y)} dV_g(y) = - \int_{\mathbb{S}_+^4} X(Q) e^{4u} dV_g. \quad (3.10)$$

Here we have used the group property of the flow, which implies that $\phi_{-t} = \phi_t^{-1}$, and (3.1). More precisely,

$$\left. \frac{d}{dt} \right|_{t=0} Q \circ \phi_t^{-1} = \left. \frac{d}{dt} \right|_{t=0} Q \circ \phi_{-t} = - \left. \frac{d}{ds} \right|_{s=0} Q \circ \phi_s = -X(Q).$$

Analogously, for the boundary term we obtain

$$\left. \frac{d}{dt} \right|_{t=0} \int_{\mathbb{S}^3} T e^{3u_t} dS_g = \left. \frac{d}{dt} \right|_{t=0} \int_{\mathbb{S}^3} T(\phi_{-t}(y)) e^{3u(y)} dS_g(y) = - \int_{\mathbb{S}^3} X(T) e^{3u} dS_g. \quad (3.11)$$

Recalling the splitting of the functional given in (2.8) and combining (3.4) with Proposition 3.1, (3.10) and (3.11), we obtain the Kazdan–Warner type identity from Theorem 1.1.

4 | NONEXISTENCE CRITERIA

We first point out that the Kazdan–Warner identity (1.8) does not require any sign assumption on the prescribed curvatures Q and T . Independently of this identity, every solution of (QT) must satisfy the Gauss–Bonnet–Chern constraint (1.7). Consequently, if

$$Q \leq 0 \quad \text{in } \mathbb{S}_+^4, \quad T \leq 0 \quad \text{on } \mathbb{S}^3,$$

then (QT) admits no solutions. This is a purely topological obstruction, whereas (1.8) detects the variation of the prescribed curvatures along conformal directions that preserve the boundary. As we will show, the two obstruction mechanisms are therefore different and complementary.

As a direct consequence of (1.8), we obtain the following obstruction to the existence of solutions to the prescribed curvature problem (QT).

Corollary 4.1. *Assume there exists a boundary-preserving conformal vector field X on $\mathbb{S}_+^4$ such that*

$$X(Q) \geq 0 \text{ on } \mathbb{S}_+^4, \quad X(T) \geq 0 \text{ on } \mathbb{S}^3,$$

and at least one of the two functions $X(Q)$, $X(T)$ is not identically zero. Then (QT) has no solutions in $\mathcal{H}$.

As a particular case, we obtain the classical Kazdan–Warner result, which establishes that there can be no solution for curvatures with the same monotonicity with respect to the same Euclidean direction.

Corollary 4.2. *For $i \in \{1, \dots, 4\}$ let $x_i : \mathbb{S}_+^4 \rightarrow \mathbb{R}$ be the i th Euclidean coordinate in $\mathbb{R}^5$ and set*

$$X_i = \nabla_g x_i.$$

Then X_i is a boundary-preserving conformal vector field on $\mathbb{S}_+^4$ and any solution $u \in \mathcal{H}$ to (QT) satisfies

$$\int_{\mathbb{S}_+^4} \langle \nabla_g x_i, \nabla_g Q \rangle e^{4u} dV_g + \frac{4}{3} \int_{\mathbb{S}^3} \langle \nabla_g x_i, \nabla_g T \rangle e^{3u} dS_g = 0.$$

In particular, if for some i one has

$$\langle \nabla_g x_i, \nabla_g Q \rangle \geq 0 \text{ in } \mathbb{S}_+^4, \quad \langle \nabla_g x_i, \nabla_g T \rangle \geq 0 \text{ on } \mathbb{S}^3,$$

and at least one of the two functions above is not identically zero, then (QT) admits no solutions in $\mathcal{H}$.

On $\mathbb{S}^4 \subset \mathbb{R}^5$ one has the explicit formula

$$X_i(x) = e_i - x_i x,$$

hence $X_i(x_i) = 1 - x_i^2$ and therefore, for curvature functions $Q(x) = f(x_i)$ and $T(y) = g(y_i)$,

$$X_i(Q) = f'(x_i)(1 - x_i^2), \quad X_i(T) = g'(y_i)(1 - y_i^2).$$

In particular, if $f' \geq 0$ and $g' \geq 0$ and at least one is not identically zero, then (QT) has no solutions in $\mathcal{H}$.

The preceding corollary can be extended to any conformal direction in the following sense:

Corollary 4.3. *Let $\Psi : \mathbb{S}_+^4 \rightarrow \mathbb{S}_+^4$ be a conformal diffeomorphism such that $\Psi(\mathbb{S}^3) = \mathbb{S}^3$. Fix $i \in \{1, \dots, 4\}$ and set $X_i = \nabla_g x_i$ on $\mathbb{S}_+^4$. Assume that*

$$X_i(Q \circ \Psi) \geq 0 \text{ in } \mathbb{S}_+^4, \quad X_i(T \circ \Psi) \geq 0 \text{ on } \mathbb{S}^3, \quad (4.1)$$

and that at least one of the two functions in (4.1) is not identically zero. Then the boundary problem (QT) admits no solutions in $\mathcal{H}$.

Proof. Since Ψ is conformal and preserves the boundary, the vector field $X = \Psi_* X_i$ is also a boundary-preserving conformal vector field on $\mathbb{S}_+^4$. Moreover, for any smooth function F on $\mathbb{S}_+^4$ one has

$$X(F) \circ \Psi = X_i(F \circ \Psi),$$

hence $X(Q) \geq 0$ in $\mathbb{S}_+^4$ and $X(T) \geq 0$ on $\mathbb{S}^3$ are equivalent to (4.1). Therefore, the claim follows from Corollary 4.1 applied to X . $\square$

We can construct easy explicit examples for Corollary 4.3 in the following way: given two distinct points $p, q \in \mathbb{S}^3$, one may choose Ψ mapping $\pm e_i \in \mathbb{S}^3$ to p, q , so that $X = \Psi_* X_i$ generates the boundary Möbius flow with fixed points p and q . Then (4.1) is equivalent to saying that Q and T are nondecreasing along that flow.

Let us illustrate this procedure.

Identify $\mathbb{S}_+^4 \subset \mathbb{R}^5$ as

$$\begin{aligned}\mathbb{S}_+^4 &= \{(\bar{x}, x_5) \in \mathbb{R}^4 \times \mathbb{R} : |\bar{x}|^2 + x_5^2 = 1, x_5 \geq 0\}, \\ \partial \mathbb{S}_+^4 &= \{(\bar{x}, 0) \in \mathbb{R}^4 \times \{0\} : |\bar{x}| = 1\} \cong \mathbb{S}^3 \subset \mathbb{R}^4.\end{aligned}$$

Consider the restriction to $\mathbb{S}_+^4$ of the stereographic projection from the south pole $(0, \dots, 0, -1)$,

$$\Pi(\bar{x}, x_5) = \frac{\bar{x}}{1 + x_5} \in B^4, \quad \Pi^{-1}(y) = \left(\frac{2y}{1 + |y|^2}, \frac{1 - |y|^2}{1 + |y|^2} \right),$$

which maps $\mathbb{S}_+^4$ conformally onto the unit ball $B^4 \subset \mathbb{R}^4$ and restricts to the identity on the boundary $\mathbb{S}^3$ (since $\Pi(\bar{x}, 0) = \bar{x}$ for $|\bar{x}| = 1$). For $a \in B^4$, we define the following Möbius automorphism of B^4 :

$$\Phi_a(y) = \frac{(1 - |a|^2)y + (1 + 2a \cdot y + |y|^2)a}{1 + 2a \cdot y + |a|^2|y|^2}, \quad y \in B^4,$$

so that $\Phi_a(B^4) = B^4$ and $\Phi_a(\mathbb{S}^3) = \mathbb{S}^3$. Set $\Psi = \Pi^{-1} \circ \Phi_a \circ \Pi : \mathbb{S}_+^4 \rightarrow \mathbb{S}_+^4$, which is a conformal diffeomorphism with $\Psi(\mathbb{S}^3) = \mathbb{S}^3$. Now choose, for instance,

$$a = \left(0, \frac{1}{2}, 0, 0\right) \in B^4.$$

Since $\Pi|_{\mathbb{S}^3} = \text{Id}$, on the boundary we have $\Psi|_{\mathbb{S}^3} = \Phi_a|_{\mathbb{S}^3}$, and a direct computation gives

$$p = \Psi(e_1) = \Phi_a(e_1) = \left(\frac{3}{5}, \frac{4}{5}, 0, 0\right), \quad q = \Psi(-e_1) = \Phi_a(-e_1) = \left(-\frac{3}{5}, \frac{4}{5}, 0, 0\right).$$

Let $X_1 = \nabla_g x_1$ on $\mathbb{S}_+^4$ and define $X = \Psi_* X_1$, which generates a boundary Möbius flow with fixed points p and q on $\mathbb{S}^3$. Consider the data

$$Q(x) = 3 + \varepsilon(x_1 \circ \Psi^{-1})(x) \quad \text{in } \mathbb{S}_+^4, \quad T \equiv 1 \quad \text{on } \mathbb{S}^3,$$

with $\varepsilon > 0$. Then

$$(Q \circ \Psi)(x) = 3 + \varepsilon x_1, \quad (T \circ \Psi) \equiv 1,$$

and therefore

$$X_1(Q \circ \Psi) = \varepsilon X_1(x_1) = \varepsilon |\nabla_g x_1|^2 = \varepsilon(1 - x_1^2) \geq 0, \quad X_1(T \circ \Psi) = 0.$$

Since $X_1(Q \circ \Psi) \not\equiv 0$, the assumptions of Corollary 4.3 are satisfied, hence the boundary problem (QT) admits no solutions in $\mathcal{H}$ for this choice of (Q, T) . Note that, if $0 < \varepsilon < 3$, then $Q \geq 3 - \varepsilon > 0$ on $\mathbb{S}_+^4$ and $T > 0$ on $\mathbb{S}^3$. Therefore, the Gauss–Bonnet–Chern constraint alone does not prevent solvability; the obstruction is purely of Kazdan–Warner type.

APPENDIX A: WEAK FORMULATION

Let $\mathcal{H}$ be the functional space defined in Section 2 and fix $u \in \mathcal{H}$. For any $v \in \mathcal{H}$, multiplying the first equation in (QT) by v and integrating over $\mathbb{S}_+^4$ gives

$$\int_{\mathbb{S}_+^4} (\Delta_g^2 u - 2\Delta_g u + 6)v \, dV_g = \int_{\mathbb{S}_+^4} 2Qe^{4u}v \, dV_g. \quad (\text{A.1})$$

We integrate by parts in the term with $\Delta_g^2 u$:

$$\begin{aligned} \int_{\mathbb{S}_+^4} \Delta_g^2 u \, v \, dV_g &= \int_{\mathbb{S}_+^4} \Delta_g u \, \Delta_g v \, dV_g - \int_{\mathbb{S}^3} \Delta_g u \, \partial_\nu v \, dS_g + \int_{\mathbb{S}^3} \partial_\nu (\Delta_g u) v \, dS_g \\ &= \int_{\mathbb{S}_+^4} \Delta_g u \, \Delta_g v \, dV_g + \int_{\mathbb{S}^3} \partial_\nu (\Delta_g u) v \, dS_g, \end{aligned}$$

where we used $\partial_\nu v = 0$ on $\mathbb{S}^3$. Using the boundary condition $-\partial_\nu \Delta_g u = 2Te^{3u}$, we obtain

$$\int_{\mathbb{S}_+^4} \Delta_g^2 u \, v \, dV_g = \int_{\mathbb{S}_+^4} \Delta_g u \, \Delta_g v \, dV_g - \int_{\mathbb{S}^3} 2Te^{3u}v \, dS_g. \quad (\text{A.2})$$

Similarly,

$$-2 \int_{\mathbb{S}_+^4} \Delta_g u \, v \, dV_g = 2 \int_{\mathbb{S}_+^4} \langle \nabla u, \nabla v \rangle \, dV_g - 2 \int_{\mathbb{S}^3} \partial_\nu u \, v \, dS_g = 2 \int_{\mathbb{S}_+^4} \langle \nabla u, \nabla v \rangle \, dV_g. \quad (\text{A.3})$$

Plugging (A.2) and (A.3) into (A.1) yields the weak formulation

$$\int_{\mathbb{S}_+^4} \Delta_g u \, \Delta_g v \, dV_g + 2 \int_{\mathbb{S}_+^4} \langle \nabla u, \nabla v \rangle \, dV_g + 6 \int_{\mathbb{S}_+^4} v \, dV_g - \int_{\mathbb{S}^3} 2Te^{3u}v \, dS_g - \int_{\mathbb{S}_+^4} 2Qe^{4u}v \, dV_g = 0.$$

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