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Dynamic portfolio optimization with stochastic investment opportunities and discontinuities

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Abstract

The present dissertation analyzes dynamic portfolio optimization problems with investment opportunities under different market scenarios. The portfolio allocation is based on decision-making to determine how many resources should allocated among various possible investments. It is well known that realistic portfolio choices must reflect the peculiarities of financial markets and economic agents; we use optimal control theory to replicate both economic investor behaviours and expected utility maximization. The novelty with respect to the existing literature lies in modelling precision instead of volatility, understood as the reciprocal of volatility, where both the stock and the state variables are subject to discontinuities. Such a non-affine structure guarantees a closed-form expression for the optimal weights for the financial assets involved in every different market setting. The optimal allocation result to be genuinely dynamic, consistent with Markowitz's well-known economic intuition that an inverse dependence of portfolio exposures on variance parameters should be observed. We consider several assumptions regarding investor preferences and market set-up. Empirical evidence shows that the investor is not only risk averse but also ambiguity averse. We attempt to provide a unifying approach to portfolio choice problems by studying a robust, dynamic, continuous-time optimal consumption and portfolio allocation problem for investors with recursive preferences who have access to both stock and derivatives markets.

Contents

Chapter 1

Introduction

The present work analyzes dynamic portfolio optimization problems with time-varying investment opportunities under different market scenarios. The portfolio allocation problem is based on the decision-making process to determine how resources should be allocated among various possible investments. Realistic portfolio choices should reflect the peculiarities of financial markets and economic agents; in particular, portfolio choices should strongly account for the investors' embedded qualities: investors are interested in earning as much as possible from their investments, but at the same time, they are concerned about the risks they face. Investors would like to maximize their returns without exceeding a given level of risk. Such a behaviour can be modelled mathematically using optimal control theory and expected utility maximization. Differently from the extant literature, in this work, we assume an economy allowing for discontinuities in both the stock price and the state variables. More precisely, we intend to model the avowed empirical property such that peaks in volatility levels correspond to drops in asset prices, see e.g. Eraker [2004], Todorov and Tauchen [2011], Bandi and Renò [2016]. Moreover, since the aforementioned changes in price and volatility occur simultaneously, we dub them *co-jumps,* and we introduce a unique counting process, typically a non-compensated Poisson, driving the non-diffusive component in the dynamics.
Although this approach is intuitively correct and well supported by empirical evidence, at a closer look, it turns out to be somewhat reductive due to the exclusion of the investor's *level of confidence* in the modelling used. For this reason, the possibility of uncertainty in the model and the possibility that investment strategies may be flawed must also be included in the setting, implying that the probability distribution underlying the model can be incorrect.
This work is organized as follows: Chapter 2 describes the mathematical model, with particular emphasis to the dynamics used to model the underlying and the volatility. We also review different market settings and investor preferences. In

Chapter 3, we solve a continuous-time dynamic optimal portfolio allocation problem in a complete market set up by considering non-affine dynamics for the state variables where both the stock and variance processes are subject to discontinuities. Such a non-affine framework ensures a closed-form expression for the optimal weights of all the financial assets involved, namely a riskless security, an equity and traded derivatives on such equity. The solution to the optimal allocation is genuinely dynamic, and an inverse dependence of the portfolio exposures with respect to the variability parameters should be observed. We assume that the investor has a classical utility function, namely *power utility*. In order to uncover the role of jumps in the optimal allocation, we numerically investigate some instances of the non-affine model proposed and analyze the corresponding risk profile. Furthermore, we perform a calibration procedure on real data, revealing that the performances of our investment strategy greatly benefit from the presence of both derivatives and co-jumps. In Chapter 4, we solve a dynamic optimization problem in an incomplete market, assuming the investor has access to one risk-free asset, one risky asset (univariate case) or n risky assets (multivariate case). To give more rationality to the economic agent, we consider different preferences for the investor. In particular, we exploit recursive preferences, where the investor's risk aversion is disentangled from the degree of the elasticity of intertemporal substitution of consumption. We further show the consequences of completing the market by adding a suitable number of derivative instruments. Similar considerations can be extended to the multivariate case. We derive an approximated closed-form solution to optimal rules by exploiting standard dynamic programming techniques. Our findings are manifold. First, we obtain dynamic optimal weights, inversely proportional to volatility. Second, we show that both co-jumps frequency and intensity play a crucial role, as they considerably limit potential losses in the investors' wealth. Third, we prove that jumps in precision reinforce the effect of jumps in price, further reducing optimal allocation. Finally, we highlight how co-jumps may influence investors' choices regarding intertemporal consumption. We also point out how the main evidence found for the incomplete market continues to apply when we complete the market by investing in derivative contracts. In the latter case, we show how ignoring the presence of co-jumps has an even more significant impact on the economic agent's wealth loss. In Chapter 5, we study a robust, dynamic, continuous-time optimal consumption and portfolio allocation problem for investors with recursive preferences who have access to both stock and derivatives markets. We obtain a closed-form approximate solution to the optimization problem for a non-unitary value of the elasticity of intertemporal substitution of consumption, being able to derive an exact solution as a particular case. Our theoretical findings show that the optimal policies are remarkably affected by the ambiguity-aversion parameters to diffusive and jump risks. The effectiveness of the theoretical results is confirmed by a detailed numerical analysis we carried out on real data. Finally, we prove that investors who do not believe in ambiguity may suffer considerable wealth losses.

1.1 The state-of-the-art

We first introduce a brief background on optimization problems over a finite time horizon in which terminal utility is maximized .
The first attempt to solve a portfolio allocation problem is due to Markowitz [1952], with the celebrated *mean-variance approach*. The latter continues to be widely applied in several financial frameworks, mainly because of its simplicity, since only the knowledge of the expectation and covariances of random variables is required.
The main drawback of this approach is the static nature of the optimal allocation. Indeed, the initial wealth is allocated between different assets at the beginning of the time horizon without allowing for changes in the allocation until maturity.
Moreover, such a behaviour ignores the presence of price volatility, so adopting this type of allocation model could result in large losses.
A solution widely adopted in the literature is considering continuous-time models for price dynamics. By allowing continuous trading, the investor can immediately react to possible changes in price volatility.

In this view, the pioneering work of Merton [1971] can be considered a starting point for continuous-time portfolio management. In this setting, the asset allocation problem is solved using the stochastic control method, and the optimal portfolio rules are expressed in terms of solutions to a second-order partial differential equation, the so-called *Hamilton-Jacobi-Bellman* (HJB) equation. In the seminal paper, an investor willing to allocate her wealth between stock and a riskless asset is taken into account, when the asset price follows a Geometric Brownian motion. Merton has proved that when investors select some special cases of HARA utility function, such as power utility, the maximization problem can be solved in a closed form.
Several authors generalized Merton's work in incomplete markets: we recall, among others, Liu et al. [2003], who used the event-risk framework of Duffie et al. [2000] to provide analytical solutions to optimal portfolio problems by assuming discontinuities in the state variable dynamics. In particular, the authors studied the implications of jumps in prices and volatility on investment strategies, providing analytical solutions to the optimal portfolio problem and finding that both jumps in prices and volatility significantly affect optimal allocation policy. Liu [2007] solved dynamic portfolio choice problems using affine models to face stochastic volatility. Three applications are presented: the first one is the bond portfolio selection problem when

bond returns are described by quadratic term structure models; the second one is the stock portfolio selection problem when volatility is stochastic as in the Heston model; the last application is a portfolio selection problem in incomplete markets when the interest rate is stochastic and stock returns have stochastic volatility.
A surprising and criticized feature of the optimal portfolio weights is their independence of the variance. One might be expected the agent to hold more stocks when the volatility is low and vice versa, but such a feature holds true if the risk premium is independent of the volatility: in this model, the risk premium is proportional to the conditional variance. Hence, when the variance is high, the risk premium is also high, and vice versa. We observe that the aforementioned findings are not related to a specific verification theorem, so we do not know under which conditions the candidate for the optimal portfolio strategy is the unique optimal solution to the allocation problem.
Kraft [2005] proved a verification result within the stochastic volatility framework, also deriving the parameters' conditions ensuring well-defined candidates for the solution of the problem.
Fleming and Hernandez-Hernandez [2003] derived explicit verifications results proving that their portfolio strategies are optimal. Buraschi et al. [2003] developed a new framework for multivariate intertemporal portfolio choices, where the correlations across asset classes and volatilities are stochastic. More precisely, in such a framework, volatilities and correlations are conditionally correlated with returns. To model stochastic variance-covariance risk, the authors specified the covariance matrix process as a Wishart diffusion process. The evidence highlighted in the aforementioned works shows once again that the optimal weights result to be independent of the instantaneous volatility when the dynamics of the risky asset are affine, implying a static allocation mechanism.
To overcome such a this drawback, Oliva and Renò [2018] study a continuous time optimal portfolio allocation with volatility and co-jump risk in a multi-asset framework. The authors deviate from affine models by specifying a Wishart jump-diffusion for the co-precision, namely the inverse of the covariance matrix, generalizing the dynamics proposed by Chacko and Viceira [2005] to a multivariate setting. They solved the optimal portfolio problem, providing an exact solution to this problem in the absence of jumps and an approximated solution in the presence of co-jumps. Furthermore the authors show that the optimal policies are dynamic, consistently with the well-known Markowitz economic intuition, along with the optimal weights should be proportional to the instantaneous co-precision.
So far we have assumed an incomplete market setting. To complete the market, we enrich the setting with a suitable number of financial securities susceptible to the whole range of risk components. The natural choice is to include derivative

contracts on the underlying portfolio equity. Among the numerous references, we first focus on Liu and Pan [2003], who studied the impact of options on wealth allocation in a stochastic volatility framework, considering only jumps in price. In this setting, the agent is interested in investing not only in stock and riskless bonds but also in derivatives. More specifically the derivatives involved must provide different exposures to the fundamental risk factors characterizing the economy. The optimal exposures to the risk factors, do not depend on instantaneous volatility. The authors showed as derivatives extend the risk and return tradeoffs associated with stochastic volatility and price jumps. In particular, they illustrated two significant examples. In the first one, they focused on the role of derivatives as a vehicle for volatility risk and, as a result the optimal portfolio weight in derivative security depends explicitly on the sensitivity of the derivative to volatility. The second example concerns the role of derivatives to disentangle jump risk from diffusive risk.
Branger et al. [2007] extended the Liu and Pan [2003] framework by adding discontinuities in the stochastic process that governs volatility. The authors showed that the demand for jump risk includes a hedging component which is not present in the models without volatility jump, being the main difference with the previous setting. Moreover, the authors showed how the volatility jump magnitude has a significant impact on the optimal portfolio. They analyzed the distribution of terminal wealth for an investor who uses the wrong model by ignoring volatility jumps or including wrong estimates of such jumps.
Within the aforementioned strand of the literature a CRRA-type utility function is assumed, with a specific focus on the *power utility* for the investor's preferences.
As an alternative, another strand of the extant literature has focused on the so called *Epstein-Zin preferences*, to include the effect of both risk aversion and separate EIS from the coefficient of relative risk aversion. As it is well known, power utility functions restrict risk aversion to be the reciprocal of the elasticity of intertemporal substitution, and such a constraint does not reflect the empirical evidence according to which the two parameters have very different effects on optimal consumption and portfolio choice, as highlighted in Chacko and Viceira [2005]. The latter examine the optimal consumption and portfolio-choice problem of long-horizon investors in an incomplete market setting, by first introducing the *precision* process, intended as the inverse of volatility, to obtain dynamic optimal portfolio rules. The authors consider recursive utility over consumption and derive an analytic expression for the optimal consumption and portfolio policies. The latter are exact when an investor has unit elasticity of intertemporal substitution of consumption, and approximate otherwise.
Since standard verification results are not applicable (Duffie and Epstein [1992]), due to the non-Lipschitzianity of Epstein-Zin preferences, Kraft et al. [2013] pro-

vided a suitable verification theorem for the associated HJB equation. This paper contributes to providing new explicit solutions to the HJB equation with recursive utility for a non-unit EIS. Those results represent the first explicit benchmark for the *Cambell-Shiller approximation*, used by Chacko and Viceira [2005] in their approximation. Kraft et al. [2016] extended their previous work and provided the existence and uniqueness of solutions of HJB equation by exploiting fixed point arguments, and developed a fast and accurate numerical method for computing both indirect utility and optimal strategies.

In a complete market setting, we refer to Hsuku [2007], where a recursive utility function defined on intermediate consumption is maximized, to reflect the realistic behavior of an investor who saves money for the future. The results obtained in Hsuku [2007] further show that optimal consumption–wealth ratio is a function of stochastic volatility when the elasticity of intertemporal substitution of consunption is different from 1. In particular, it is an increasing function for investors whose EIS is smaller than one, while it is a decreasing function for investors whose EIS is larger than one. Finally, the analyses confirm the conclusions of Liu and Pan [2003] in the case of the power utility, regarding the role of derivatives: derivative securities are a significant tool for expanding investors' dimension of risk-and-return tradeoffs, being a vehicle for the additional risk factor of stochastic volatility in the stock market.

On the other hand, realistic portfolio choices should reflect the peculiarities of financial markets and agents trading therein. First, portfolio choices should strongly account for the investors' embedded qualities, such as the corresponding risk aversion level. Although such an approach is intuitively correct and extremely well-supported by empirical evidence, at a deeper look it turns out to be rather curtailing, because of the exclusion of the investor's *level of confidence* in the modeling used. More precisely, the possibility of incorrect investment strategies has so far been ruled out, since the probability distribution underneath the model is incorrect. The role played by such uncertainty towards the model is widely documented in the literature, in terms of the so-called *Ellsberg paradox,* see e.g. Ellsberg [1961].

Thus, in the realm of optimal dynamic portfolio allocation, the challenge of including and managing both the risk component and the uncertainty factor has become popular.

The literature refers to two main ways of describing the behavior of the economic agents in the presence of model uncertainty, namely the *multiple priors* (MP) approach and the *robust control problem* (RC) approach, see e.g. Hansen and Sargent [2001], Hansen et al. [2006] and Epstein and Schneider [2003]. The MP approach has been launched for the first time in Gilboa and Schmeidler [1989], and is based on considering a max-min expected utility with multiple priors. On the contrary, the RC approach dates back to Anderson et al. [2003] and consists in identifying a *reference* model and a range of *alternative* models, the latter being difficult to distinguish from the reference one. The distance between reference and alternative models is measured in terms of relative entropy. Starting from the seminal paper by Anderson et al. [2003], the literature has produced tons of contributions: without claiming to be exhaustive, we mention Maenhout [2004], who apply robust control techniques to a Merton model for an investor with recursive preferences. Such a framework has been extended by Liu [2010], assuming mean-reverting dynamics for the expected return of the risky asset. Xu et al. [2011] investigate the implications of model uncertainty for the equity premium in a stochastic volatility model. Similarly, Faria and Correia-da Silva [2016] extend the Chacko and Viceira [2005] model by incorporating ambiguity with a different methodology of robust optimal control, considering either *constraints preferences,* where a constraint on the magnitude of perturbations is allowed for the reference model, or *multiplier preferences,* where deviations from the reference model are constructed by penalizing larger deviations more than smaller ones.

It is worth mentioning Aït-Sahalia and Matthys [2019] in the presence of discontinuities in all process dynamics, the author consider the consumption and optimal policies problem by allowing the incorrect specification for drift, jump intensity, and jump amplitude. Interestingly, the literature offers a large number of contributions dealing with robust control problems in complete markets, without including consumption. To name but a few, we mention Branger and Larsen [2013] and Escobar et al. [2015], where investors are ambiguous to both diffusive and jump risks, or diffusive and volatility risks, respectively. In particular, Escobar et al. [2015] determine the optimal portfolio for an ambiguity-averse investor when stock price follows a stochastic volatility jump-diffusion process, and when the investor can have different levels of uncertainty about diffusion parts of the stock and its volatility. The authors illustrate that the optimal exposures to stock and volatility risks are significantly affected by ambiguity aversion to the corresponding risk factor only. Moreover, they also show that volatility ambiguity has a smaller impact in incomplete markets. As an extension, Cheng and Escobar-Anel [2021] consider an optimal allocation problem with both risk and ambiguity aversion under a 4/2 model. The numerical analysis finds that the *wealth-equivalent loss* (WEL) deriving from ignoring uncertainty or market completeness is moderate, while the same metrics for investors who follow different models such as Heston or Merton (geometric Brownian Motion) is quite substantial.

In a multivariate setting, Bergen et al. [2018] solve a multivariate optimization problem in a complete market setting, for an investor ambiguous to both diffusive and covariance risk. In a similar context, Jin and Zhang [2012] examined a multi-asset model with constant volatility when the price is allowed to jump. Differently from other authors, such as Das and Uppal [2004] and Ascheberg et al. [2016], who focused on one type of jump and assumed constant coefficients in their dynamics, Jin and Zhang [2012] includes multiple types of jumps and a large number of assets and state variables. The authors also incorporate model uncertainty into the portfolio choice problem, tackling the problem by focusing on ambiguity aversion to inaccurate estimates of parameters associated with jumps. The theoretical results showed that ambiguity aversion increases the effect of jumps for a risk-averse investor, thus the exposure to jumps is smaller in the case of an investor not neutral to ambiguity. More recently Jin et al. [2021] formulates a portfolio choice problem in a multi-asset market characterized by ambiguous jumps and constant volatility. In the paper, therefore, a portfolio choice problem with ambiguity aversion in a continuous-time incomplete financial market is considered. The authors investigate the impact of tail risk on portfolio selection. The optimal portfolio is solved in closed form through a decomposition approach, with two different jump size distributions. They also show that underestimating tail risk might result in a sizeable wealth loss in the presence of jump ambiguity. Furthermore, they confirm that ambiguity-averse investors reduce more of their jump exposure if the jump distribution exhibits a fatter left tail. Very recently, Wei et al. [2023] examine the optimal control problem in a complete market setting by assuming a Bates model for the underlying risky asset, where investors are ambiguous about diffusive, volatility, and jump risks. Differently from the extant literature, the authors consider recursive preferences for investors.

Chapter 2

The model

This chapter describes the investment opportunity setting where the investor can allocate his wealth. Several crucial factors are involved in defining the investment choice setting for the investor. Throughout this work, we have made several assumptions about the investor's investment opportunities. We started with an incomplete market setting in which the investor can choose to invest his wealth between a risk-free investment and a risky asset. Then, we went on to complete the market by adding derivative contracts, i.e., complex instruments that give more realism to our model and carry many implications in the allocative choice of the economic agent, which we will explain in more detail in later sections. At the same time, we did not only focus on a *univariate* setting, so what happens for an investor who has a risky security at his disposal, but we considered, in some cases, a panel of risky securities, thus assuming the correlation between each asset that makes up the investor's portfolio. In addition, in this chapter, we also describe investor preferences: we started by considering classical preferences for the agent, such as non-separable preferences and the classic *power utility*. Then, we generalize the results obtained to more complex utility functions that, in a sense, better represent the rational behaviour of economic agents, as the *stochastic differential utility function*.
The chapter is organized as follows. Section 2.1 and 2.2 describe the model in the incomplete market, both in the univariate and multivariate case, respectively. Section 2.3 explains how the market is completed. Finally, Section 2.4 illustrates the different utility functions considered in this work to describe investor preferences.

2.1 Incomplete Market Setting

We consider three stochastic processes $M = \{M(t)\}_{t \in [0,T]}$, $S = \{S(t)\}_{t \in [0,T]}$, and $y = \{y(t)\}_{t \in [0,T]}$ on a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t \in [0,T]}, \mathbb{P}\right)$, representing the risk-free asset, the stock and the state variable, respectively. Regarding the latter, we employ the *precision* process y to describe the market diffusion feature. We recall that the precision, as introduced in Chacko and Viceira [2005], is defined as $y(t) = \frac{1}{v(t)}$, $t \in [0,T]$, where $v(t)$ specifies the variance process. Such a choice mainly indicates that herein we will refer to a *non-affine* model. The model is described as follows

$$\frac{dM(t)}{M(t)} = rdt \ , \tag{2.1}$$

$$\frac{dS(t)}{S(t)} = \mu dt + \sqrt{y(t)^{-1}}dB(t) + JdN(t) \ , \tag{2.2}$$

$$dy(t) = k\left(\theta - y(t)\right)dt + \sigma\sqrt{y(t)}d\bar{B}(t) + \xi\left(y(t)\right)dN(t), \tag{2.3}$$

where $B = \{B(t)\}_{t \in [0,T]}$, $\bar{B} = \{\bar{B}(t)\}_{t \in [0,T]}$ are two real-valued Wiener processes, while $N = \{N(t)\}_{t \in [0,T]}$ is a non-compensated Poisson process with intensity parameter λ, independent of B. The process $\bar{B}$ is correlated with the Wiener process B through

$$\bar{B}(t) = \rho B(t) + \sqrt{1-\rho^2}Z(t), t \in [0,T] \ , \tag{2.4}$$

being $\rho \in [-1,1]$ the correlation coefficient, and $Z = \{Z(t)\}_{t \in [0,T]}$ a Wiener process independent of B and N.
We expect a negative correlation between returns and volatility, which translates in a positive value for the coefficient ρ when we look at the precision. More precisely, since the shocks to precision are correlated with the instantaneous return on the risky asset through (2.4), then we have

$$\rho dt = Corr\left(dy(t), \frac{dS(t)}{S(t)}\right) = -Corr\left(dv(t), \frac{dS(t)}{S(t)}\right) = -\tilde{\rho}dt \ . \tag{2.5}$$

The stochastic process for precision implies a mean-reverting process for the instantaneous volatility: applying Ito's Lemma to Equation (2.3), we find that the volatility follow a mean-reverting, square-root process (namely 3/2 process).
Since

$$f(y(t)) = \frac{1}{y(t)}, \quad \frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial y(t)} = -\frac{1}{y(t)^2}, \quad \frac{\partial^2 f}{\partial y(t)^2} = \frac{2}{y(t)^3}$$

remember the Itô -Doeblin formula for one jump process [Shreve, 2004, p. 483], the dynamic of the volatility is given by:

$$\frac{dv(t)}{v(t)} = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial y(t)}dy(t)^C + \frac{1}{2}\frac{\partial^2 f}{\partial^2 y(t)} < dy^C(t), y^C(t) > + f\left(y(t) + \xi\left(y(t)\right)\left(y(t)\right)\right) - f\left(y(t)\right).$$

So we have:

$$\frac{dv(t)}{v(t)} = -\frac{1}{y(t)^2}\left[k(\theta - y(t))dt + \sigma\sqrt{y(t)}\left(\rho dB(t) + \sqrt{1-\rho^2}dZ(t)\right)\right] + \frac{\sigma^2}{y(t)^2}dt + \left(\frac{1}{y(t) + \xi\,(y(t))} - \frac{1}{y(t)}\right), \tag{2.6}$$

with simple algebraic adjustments we obtain

$$\frac{dv(t)}{v(t)} = k_v(\theta_v - v(t))dt - \sigma\sqrt{v(t)}\left(\rho dB(t) + \sqrt{1-\rho^2}dZ(t)\right) - \frac{\xi v(t)}{1 + \xi\,(v(t))\,v(t)}, \tag{2.7}$$

with $\theta_v = \left(k\theta - \sigma^2\right)^{-1}$ and $k_v = \frac{k}{\theta_v}$.

The risk-free interest rate r in (2.1) is assumed to be constant, while $\mu \in \mathbb{R}^+$ and J in (2.2) represent the investor's expected return and the price-jump amplitude, respectively.
The diffusive term in (2.2) depends on the precision $y(t)$, for $t \in [0, T]$. The latter is described in (2.3), and is characterized by the speed of reversion $k > 0$, the long-term mean level $\theta > 0$, and the vol-of-precision $\sigma > 0$. Moreover, we introduce a jump component in (2.3) driven by the same Poisson process N appearing in (2.2), with a precision-jump amplitude ξ. To ensure the positiveness of the precision process after a jump, we impose $y(t) + \xi > 0$, according to Ahn and Thompson [1988].
For more details about the existence of a solution of the jump-diffusion mean-reverting process (with negative jump amplitude), see Wu et al. [2008], where a condition is imposed on the parameters $(k + \lambda\xi > 0)$ to guarantee the existence of a solution. In general, ξ and J are random variables with suitable distributions. In all the thesis we will focus on deterministic jump-amplitudes. Absence of arbitrage in the market entails the possibility of moving from the physical measure $\mathbb{P}$ to the equivalent martingale measure $\mathbb{Q}$. In particular, the Girsanov theorem ensures that, for $T > 0$ and $t \in [0, T]$, there exists a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the processes $B(t)^{\mathbb{Q}}$, $Z(t)^{\mathbb{Q}}$, given by

$$dB(t)^{\mathbb{Q}} = dB(t) + \eta\sqrt{y(t)}dt, \tag{2.8a}$$

$$dZ(t)^{\mathbb{Q}} = dZ(t) + \nu\sqrt{y(t)}dt, \tag{2.8b}$$

are Wiener processes under $\mathbb{Q}$, where

$$\eta = \mu - r + \lambda^{\mathbb{Q}}J, \quad \nu = \frac{\nu^{\mathbb{Q}} - \rho\eta}{\sqrt{1-\rho^2}}, \tag{2.9}$$

with $\nu^{\mathbb{Q}} \in \mathbb{R}$ volatility risk premium. Therefore, the model under the risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$ reads as follows

$$\begin{cases} \dfrac{dS(t)}{S(t)_-} &= rdt + \sqrt{y(t)^{-1}}dB(t)^{\mathbb{Q}} + JdN(\lambda^{\mathbb{Q}})(t) - \lambda^{\mathbb{Q}}Jdt \\ dy(t) &= k^{\mathbb{Q}}(\theta^{\mathbb{Q}} - y(t)_-)dt + \sigma\sqrt{y(t)_-}\left[\rho dB(t)^{\mathbb{Q}} + \sqrt{1-\rho^2}dZ(t)^{\mathbb{Q}}\right] + \xi\left(y(t)\right)dN(\lambda^{\mathbb{Q}})(t) \end{cases}, \tag{2.10}$$

where $k^{\mathbb{Q}} = k + \sigma\nu^{\mathbb{Q}}$ and $\theta^{\mathbb{Q}} = \frac{k\theta^{\mathbb{Q}}}{k^{\mathbb{Q}}}$.
To determine the investor's optimal policies, we must resort, again the real-world probability measure $\mathbb{P}$, recalling the equations (2.8a) and (2.8b) we obtain

$$\begin{cases} \dfrac{dS(t)}{S(t)_-} &= (r+\eta)\,dt + J\left(\lambda^{\mathbb{P}} - \lambda^{\mathbb{Q}}\right)dt + \sqrt{y(t)^{-1}}dB(t) + +J(dN(t) - \lambda^{\mathbb{P}})dt \\ dy(t) &= k(\theta - y(t)_-)dt + \sigma\sqrt{y(t)_-}\left[\rho dB(t) + \sqrt{1-\rho^2}dZ(t)\right] + \xi\left(y(t)\right)dN(\lambda^{\mathbb{Q}})(t) \end{cases}. \tag{2.11}$$

This model for stock returns and precision (or volatility) can capture the main stylized empirical facts about the volatility of stock returns, particularly its mean reversion and negative correlation with stock returns, and is also empirically realistic in that changes in volatility are more pronounced in periods of high volatility than in periods of low volatility.

2.2 Multivariate Case

The previous model corresponds to the univariate case in which we have only one risky asset. Following the work of Oliva and Renò [2018], in a multivariate setting, let $\left(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\in[0,T]}, \mathbb{P}\right)$ a filtered probability space. Denoting by $GL_N(\mathbb{R})$ the set of invertible matrices in $\mathbb{R}^{N\times N}$ and by $S_N(\mathbb{R})$ (resp., $S_N^+(\mathbb{R})$) the set of the symmetric matrices (resp., the set of the symmetric and positive definite matrices) in $\mathbb{R}^{N\times N}$.

First consider the process $M = \{M(t)\}_{t\in[0,T]}$ representing the riskless asset, as in the univariate case (2.1), such that

$$\frac{dM(t)}{M(t)} = rdt,\ t\ \in\ [0,T]\,,$$

where $r \in \mathbb{R}$ is the instantaneous rate of return. We also consider N risky assets $S(t) = (S(t,1),\ldots,S(t,N))' \in \mathbb{R}^{N\times 1}$ and assume a multivariate stochastic volatility model, where we exploit the inverse of the instantaneous variance-covariance matrix through a Wishart process. Such a state variable is dubbed *co-precision.* The dynamics are given by

$$\begin{cases} dS(t) = & diag(S(t))\left[\alpha dt + \sqrt{Y(t)^{-1}}dZ(t) + JdN(\lambda)(t)\right] \\ dY(t) = & (\Omega\Omega' + KY(t) + Y(t)K')\,dt + \sqrt{Y(t)}d\bar{Z}(t)Q + Q'\left(d\bar{Z}(t)\right)'\sqrt{Y(t)} + \\ & \xi(Y(t))dN(\lambda)(t) \end{cases}, \tag{2.12}$$

for any $t \in [0,T]$. The dynamics of the risky assets include the square matrix $diag(S(t))$, with $S(t)$ in the diagonal and 0 on the off-diagonal elements, the drift $\alpha \in \mathbb{R}^{N\times 1}$ and the jump amplitude $J \in \mathbb{R}^{N\times 1}$. A vector of Wiener processes $Z(t) \in \mathbb{R}^{N\times 1}$ drives the diffusive part and a non-compensated Poisson process $N(\lambda)(t)$ with intensity $\lambda \in \mathbb{R}$ describes the discontinuous component.
The Wishart stochastic (co-)precision includes $K, Q \in \mathbb{R}^{N\times N}$, while $\Omega \in GL_N(\mathbb{R})$ is such that $\Omega\Omega' = aQQ'$, with $a \in \mathbb{R}$ and $a > N-1$, ensuring that $Y(t)$ is positive definite and mean-reverting at each time, see e.g. Bru [1991]. The matrix K represents the speed of the mean-reversion of the co-precision to its mean-reversion level $\bar{Y}$, which satisfies

$$aQQ' + K\bar{Y} + \bar{Y}K' = 0\,.$$

As highlighted in Oliva and Renò [2018], we expect the matrix K to be negative semi-definite. The diffusive part of $Y(t)$ is driven by a matrix of Wiener processes $\bar{Z}(t) \in \mathbb{R}^{N\times N}$. The Wiener processes are correlated according to

$$Z(t) = \bar{Z}(t)\rho + \sqrt{1-\rho'\rho}\tilde{Z}(t)\,, \tag{2.13}$$

where $\rho \in \mathbb{R}^{N\times 1}$, $\tilde{Z}(t)$ is a Wiener process in $\mathbb{R}^{N\times 1}$ and the elements of $\tilde{Z}(t)$ and $\bar{Z}(t)$ are all independent among them. Finally, we introduce a jump component for Y, driven by the same Poisson process appearing in the dynamics of the risky assets, with a precision-jump amplitude $\xi(Y(t)) \in S_N(\mathbb{R})$. A constant-precision jump dimension should be assumed, with

$$Y + \xi(Y) \in S_N^+(\mathbb{R})\,,$$

for all $Y \in S_N^+(\mathbb{R})$.

2.3 Complete Market Setting

The market described by (2.1) is incomplete and can be completed with a non-redundant set of instruments any of which is sensitive to the whole range of market risks. Following Liu and Pan [2003], by *non-redundant* we mean that the set of additional securities considering total price sensitivity, once all the risk factors are considered, is not the same for any two assets. Effectively, this makes possible to build portfolios that are sensitive to each of the individual risks across the whole holding period. In concrete, this is expressed in the framework presented as follows. We assume that N contingent claims maturing at time T are traded in the market, and defined in terms of their price at time $t \in [0, T]$, say $O(t)^{(i)}$, $i = 1, \ldots, N$. In general, the prices of these claims will be a function of both the stock price and the precision, namely

$$O(t)^{(i)} = g^{(i)}(S(t), y(t)),\ i = 1, \ldots, N, \tag{2.14}$$

with $g \in C^2(\mathbb{R}, \mathbb{R})$. As a minimal requirement, assuming the pair (J, ξ) takes value on a finite set of cardinality K, the number N of the additional assets selected must be equal to

$$N = N_D + N_J \cdot K - 1, \tag{2.15}$$

where $N_D \geq 1$ and $N_J \geq 1$ count the number of Wiener processes driving the diffusive terms in the dynamics and the number of Poisson processes and of the random variables describing the jump sizes, respectively. Therefore, together with the requirement that the spot price and precision discontinuities happen simultaneously, the assumption that the jumps take a finite number of values implicates that jumps can be perfectly hedged, and thus the market we consider can be completed by adding a finite number of traded derivatives. In our case we have $N_D = 2$ and $N_J = 1$ in (2.15), since we have two Brownian components and only one jump-driving Poisson process for both the asset returns and the stochastic volatility. In our framework we also assume $K = 1$, i.e. deterministic known jump size in both equity and precision.

Furthermore, the N assets selected must be non-redundant, that is, this set must span the whole risk space without linear dependency of their total risk profiles. The following definition formalizes this concept.

At any time $t \in [0, T]$, the derivatives represented by $O(t)^{(1)}$, $O(t)^{(2)}$ are said to be *non-redundant* if

$$\mathcal{D}(t) := \frac{g_y^{(2)}}{O(t)^{(2)}} \left(\frac{\Delta g^{(1)}}{J\, O(t)^{(1)}} - \frac{S(t) g_s^{(1)}}{O(t)^{(1)}} \right) - \frac{g_y^{(1)}}{O(t)^{(1)}} \left(\frac{\Delta g^{(2)}}{J\, O(t)^{(2)}} - \frac{S(t) g_s^{(2)}}{O(t)^{(2)}} \right) \neq 0\,, \tag{2.16}$$

where, for all $i = 1, 2$,

$$\Delta g^{(i)} := g^{(i)}((1 + J)S(t)_-, y(t)_- + \xi\,(y(t))) - g^{(i)}(S(t)_-, y(t)_-)\,, \tag{2.17a}$$

and $g_s^{(i)}$, $g_y^{(i)}$ are a shorthand to indicate the first-order partial derivatives w.r.t. S

and y. In the following, we will extend the same notation both to the first-order partial derivative of g w.r.t. t and to the the second-order derivatives w.r.t. S and y. Now using Ito's Lemma, the dynamics of the i-th derivative security $O_t^{(i)},\ i=1,2$, are

$$\begin{aligned} dO(t)^{(i)} = &\left[g(t) + rS(t)g_s^{(i)} - \lambda^{\mathbb{Q}} JS(t)g_s^{(i)} + g_y^{(i)}\kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - y(t)) + \frac{1}{2}\frac{g_{ss}^{(i)}S(t)^2}{y(t)} + \frac{1}{2}g_{yy}^{(i)}\sigma^2 y(t)\right. \\ &\left. + g_{sy}^{(i)}\sigma\rho S(t)\right]dt + \left[\frac{g_s^{(i)}S(t)}{\sqrt{y(t)}} + g_y^{(i)}\sigma\rho\sqrt{y(t)}\right]dB(t)^{\mathbb{Q}} + g_y^{(i)}\sigma\sqrt{y(t)}\sqrt{1-\rho^2}dZ(t)^{\mathbb{Q}} \\ &+ \Delta g^{(i)} dN(\lambda^{\mathbb{Q}})(t),\ t \in [0,T]. \end{aligned} \tag{2.18}$$

The drift in (2.18) is part of the following valuation equation for jump-diffusion models

$$\begin{aligned} &g_t + rS(t)g_s^{(i)} - \lambda^{\mathbb{Q}} JS(t)g_s^{(i)} + g_y^{(i)}\kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - y(t)) + \frac{1}{2}\left(\frac{g_{ss}^{(i)}S(t)^2}{y(t)} + g_{yy}^{(i)}S(t)^2 y(t) + 2g_{sy}^{(i)}\sigma\rho\right. \\ &\left. S(t)\right) + \Delta g^{(i)}\lambda^{\mathbb{Q}} = rg\,, \end{aligned} \tag{2.19}$$

so that we obtain the following martingale dynamics for $O^{(i)}(t),\ i=1,2$,

$$\begin{aligned} \frac{dO(t)^{(i)}}{O(t)^{(i)}} =& rdt + \left[\frac{g_s^{(i)}}{O(t)^{(i)}}S(t) + \sigma\rho\frac{g_y^{(i)}}{O(t)^{(i)}}y(t)\right]\sqrt{y(t)^{-1}}dB(t)^{\mathbb{Q}} + \sigma\sqrt{1-\rho^2}\frac{g_y^{(i)}}{O(t)^{(i)}}\sqrt{y(t)} \\ & dZ(t)^{\mathbb{Q}} + \frac{\Delta g^{(i)}}{O(t)^{(i)}}\left(dN(\lambda^{\mathbb{Q}})(t) - \lambda^{\mathbb{Q}}dt\right),\ t \in [0,T]\,. \end{aligned} \tag{2.20}$$

2.4 The investor preferences

An investor chooses among different strategies based on his/her preferences, and this is done in terms of *utility function,* according to the *expected utility criterion*, see e.g. Von Neuman and Morgenstern [1947].

If we assume that an investor compares random returns whose probability distributions are known on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we denote by $\succ$ the *preference order* that satisfies the *Von-Neumann Morgenstern criterion,* so that

$$X_1 \succ X_2 \iff \mathbb{E}[U(X_1)] > \mathbb{E}[U(X_2)], \tag{2.21}$$

where X_1, X_2 are random variables and U is an increasing real-valued function. The choice of the utility function allows defining the concept of *risk aversion* and *risk premium* of the investor. Among the properties that the utility function must have we recall the *concavity* of the utility function. If we have a utility function that satisfies *Jensen's inequality*, then an agent prefers to get the expectation $\mathbb{E}[X]$ of this return (with certainty), that is:

$$U(\mathbb{E}[X]) \geq \mathbb{E}[U(X)], \tag{2.22}$$

which holds only for concave functions. For a risk-averse agent with concave utility function U we can define the risk premium associated with a random return X he is willing to pay, to get a certain gain:

$$U(\mathbb{E}[X]-\pi)=\mathbb{E}[U(X)]. \tag{2.23}$$

The quantity $(\mathbb{E}[X]-\pi)$ is called *certainty equivalent* of X, which is smaller than the expectation of X.
It is easy to obtain the *absolute risk aversion* at level x, given by:

$$\alpha(x)=-\frac{U''(x)}{U'(x)}, \tag{2.24}$$

that is the *Arrow-Pratt* coefficient of absolute risk aversion of U at level x.
In the context of dynamic programming, the choice of the utility function is very relevant. The most preferred class of utility functions is given by HARA (*Hyperbolic Absolute Risk Aversion*) functions, where the inverse of absolute risk aversion is linear in consumption. We mention the quadratic, exponential, and isoelastic preferences. The most popular preference specification is either isoelastic preferences CRRA, or power utility see Hansen and Singleton [1982]. The power utility has the following specification:

$$U(x)=\begin{cases} \frac{1}{1-\gamma}x^{1-\gamma}, & if\ x>0 \\ -\infty, & if\ x\leq 0 \end{cases}, \tag{2.25}$$

here $\gamma>0$ and the second part of the utility specification impose a non-negative wealth constraint. The parameter γ plays a crucial role, indeed it is the risk aversion parameter and summarizes the behavior of the consumer toward risk, but at the same time its reciprocal is equal to the *constant elasticity of intertemporal substitution* (EIS) in consumption, and measures how consumption changes in time. So, only one parameter governs both investor risk aversion and EIS, and the latter is independent of the level of consumption.
To give rational support to observable investors' behavior decision-makers might take into account more complex utility functions, to use a nonlinear aggregator to bring together present and future utility thus dropping the hypothesis of expected utility.
In this perspective, the literature refers to the so-called *stochastic differential utilities* (SDU) introduced by Epstein and Zin [1989], who derive a parametrization of recursive utility in a discrete-time setting, and generalized by Duffie and Epstein [1992] and Fisher and Gilles [1999] in a continuous-time setting.
The Duffie and Epstein [1992] parametrization is

$$J=\mathbb{E}_t\left[\int_t^{\infty} f\left(C(s),J(s)\right)ds\right], \tag{2.26}$$

where $f(C(s), J(s))$ is a normalized aggregator of current consumption and utility with the following form:

$$f(C,J) = \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J\left[\left(\frac{C}{((1-\gamma)J)^{\frac{1}{1-\gamma}}}\right)^{1-\frac{1}{\psi}} - 1\right], \tag{2.27}$$

where $\beta > 0$ is the *rate of time preferences*, $\gamma > 0$ is the *coefficient of relative risk-aversion* and $\psi > 0$ is the *elasticity of intertemporal substitution* in consumption. In this setting there is a separation among preferences parameters, so the degree of risk aversion γ is disentangled from the elasticity of intertemporal substitution of consumption EIS. The Epstein-Zin preferences can be seen as a generalization of the standard time additive expected utility function, in that the former collapse to the power utility when the elasticity of intertemporal substitution of consumption EIS equals the reciprocal of the coefficient of relative risk aversion, so when we set $\psi = \frac{1}{\gamma}$.

When $\psi \to 1$ the normalized aggregator $f(C_s, J_s)$ takes the following form

$$f(C,V) = \beta(1-\gamma)V\left(\ln(C) - \frac{1}{1-\gamma}\ln((1-\gamma)V)\right). \tag{2.28}$$

Chapter 3

Dynamic portfolio allocation in complete markets with co-jumps and power utility

In the following chapter, we generalise the work of Oliva and Renò [2018] in the complete market, considering a univariate setting. In particular, we assume a scenario where the investor has separable preferences: we consider a classical utility function, i.e. *power utility*, as specified in Section 2.4. The reference market is complete, and the investor can access one risky asset, risk-free asset and derivative instruments.
We provide closed-form formulae for the optimal allocation of stocks and derivatives and the exposures to risk factors governing the market by using dynamic programming techniques and solving the corresponding Hamilton-Jacobi-Bellman equation. Our main theoretical result ensures that the optimal investment is genuinely dynamic. Besides, and at odds with the affine literature, we find that the optimal weights are state-dependent and inversely proportional to the instantaneous volatility, in line with the well-known Markowitz intuition.
We also show that derivatives represent a way to manage the additional sources of risk that intervene when a stochastic model with co-jumps is used. The optimal exposures to diffusive, volatility and jump risks for long-horizon investors consist of two terms. The first is a myopic component that guarantees the risk-and-return trade-off *à la* Merton. The second one is a hedging component, required by investors who want to exploit the time dependence of the investment. Such a hedging component is responsible for balancing allocation in the presence of sudden and unfavourable changes in volatility and returns. Consequently, derivatives allow the avoidance of any overestimation of diffusive risk. As expected, diffusive and volatility exposures are proportional to the corresponding risk premia and are inversely proportional to volatility, in contrast with the portfolio allocations provided by affine models. By calibrating the model on real data, we investigate two particular cases to highlight the main features of our findings. When assuming no jumps in the dynamics, we show that: *(i)* the demand for derivatives is mainly driven by the myopic demand; *(ii)* as the stock attractiveness increases, the corresponding allocation swells as well; *(iii)* when investors become more risk averse, they reduce their allocation to both stock and derivatives;

(iv) when the initial volatility raises, the position in derivatives and equities slackens. On the other hand, when assuming constant volatility, we obtain that jumps in price affect the overall optimal portfolio allocation since investors take a more significant position in stocks and derivatives in case of (absolute) jump amplitude reduction. Furthermore, we quantify the role played by derivatives in portfolio management in terms of *Certainty Equivalent Wealth (CEW).*

The chapter is organized as follows. Section 3.1 recalls the model we use and we explicitly solve the optimization problem. Section 3.2 explain in detail the economic aspects emerging from our theoretical results. The role of derivatives, both in the presence and absence of discontinuities in the state variables' dynamics, is detailed and empirically illustrated in Section 3.3.

3.1 The dynamic optimal allocation problem and the optimal policy

We consider three stochastic processes $M = \{M(t)\}_{t\in[0,T]}$, $S = \{S(t)\}_{t\in[0,T]}$, and $y = \{y(t)\}_{t\in[0,T]}$, whose dynamics are given by (2.1) and (2.10). The market described by (2.10) is incomplete, hence we include $N = 2$ contingent claims with underlying S and maturing at time T, whose price at time $t \in [0,T]$ is given by (2.14). The dynamics of the i-th derivative security $O^{(i)}(t),\ i = 1,2$, is given by (2.20) and the derivatives choosen must be satisfying the conditition of non-redundant given by (2.16). We want to maximize the expected utility from terminal wealth with respect to a power utility, (2.25) without considering intermediate consumption, as in Liu et al. [2003], Das and Uppal [2004], Oliva and Renò [2018]. We denote by $\epsilon(t)$, $\phi(t)$, $\psi(t)^{(i)}$, $t \in [0,T]$ the proportion of wealth invested in the riskless asset, the risky asset and the non-redundant i-th derivative, $i = 1,2$, respectively. Thus, by observing that $\epsilon(t) + \phi(t) + \sum_{i=1}^{2}\psi(t)^{(i)} = 1$, and using the self-financing property of the optimal portfolio, the dynamics of the wealth process are

$$
\begin{aligned}
\frac{dW(t)}{W(t)_-} &= \epsilon(t)\frac{dM(t)}{M(t)} + \phi(t)\frac{dS(t)}{S(t)} + \sum_{i=1}^{2}\psi(t)^{(i)}\frac{dO(t)^{(i)}}{O(t)^{(i)}} \\
&= rdt + \left[\phi(t) + \sum_{i=1}^{2}\psi(t)^{(i)}\left(S(t)\frac{g_s^{(i)}}{O(t)^{(i)}} + \sigma\rho\frac{g_y^{(i)}}{O(t)^{(i)}}y(t)\right)\right]\sqrt{y(t)^{-1}}dB_t^{\mathbb{Q}} \\
&\quad + \sigma\left[\sqrt{1-\rho^2}\sum_{i=1}^{2}\psi(t)^{(i)}\frac{g_y^{(i)}}{Oi}\right]\sqrt{y(t)}dZ(t)^{\mathbb{Q}} \\
&\quad + \left[\phi(t) + \sum_{i=1}^{2}\psi(t)^{(i)}\frac{\Delta g^{(i)}}{J\,O(t)^{(i)}}\right]\left(JdN(\lambda^{\mathbb{Q}})(t) - \lambda^{\mathbb{Q}}Jdt\right),\ t \in [0,T], \qquad (3.1)
\end{aligned}
$$

with initial wealth W_0. For any $t \in [0,T]$, we define the *risk exposures*

$$\theta^B(t) := \phi(t) + \sum_{i=1}^{2} \psi(t)^{(i)} \left(S(t)\frac{g_s^{(i)}}{O(t)^{(i)}} + \sigma\rho\frac{g_y^{(i)}}{O(t)^{(i)}}y(t) \right), \tag{3.2a}$$

$$\theta^Z(t) := \sigma\sqrt{1-\rho^2}\sum_{i=1}^{2} \psi(t)^{(i)} \frac{g_y^{(i)}}{O(t)^{(i)}}y(t), \tag{3.2b}$$

$$\theta^N(t) := \phi(t) + \sum_{i=1}^{2} \psi(t)^{(i)} \frac{\Delta g^{(i)}}{JO(t)^{(i)}}, \tag{3.2c}$$

so that, thanks to (3.2a), (3.2b) and (3.2c), the wealth $\mathbb{Q}$-dynamics can be written as follows

$$dW(t) = rW(t)dt + \theta^B(t)W(t)\sqrt{y(t)^{-1}}dB(t)^{\mathbb{Q}} + \theta^Z(t)W(t)\sqrt{y(t)^{-1}}dZ(t)^{\mathbb{Q}} + \theta^N(t)W(t)_-(JdN(\lambda^{\mathbb{Q}})(t) - \lambda^{\mathbb{Q}}Jdt),\ t \in [0,T].$$

Finally, by using (2.8a) and (2.8b), we obtain

$$dW(t) = W(t)\left[r + \theta^B(t)\eta + \theta^Z(t)\nu - \theta^N(t)\lambda^{\mathbb{Q}}J\right]dt + \theta^B(t)W(t)\sqrt{y(t)^{-1}}dB(t) + \theta^Z(t)W(t)\sqrt{y(t)^{-1}}dZ(t) + \theta^N(t)W(t)_-JdN(\lambda^{\mathbb{Q}})(t),\ t \in [0,T]. \tag{3.3}$$

The investment problem we intend to solve is

$$V(W,y,t) := \max_{u(t)\,\in\,\mathcal{U}} \mathbb{E}(t)\left[\frac{W(T)^{1-\gamma}}{1-\gamma}\right], \tag{3.4}$$

where $\mathbb{E}(t) = \mathbb{E}\left[\cdot|W(t)=W,\ y(t)=y\right]$ and $u(t) = \left(\theta^B(t), \theta^Z(t), \theta^N(t)\right) \in \mathcal{U}$ with $\mathcal{U}$ the set of *admissible strategies*. Eq. (3.4) can be solved by using standard dynamic programming techniques.
To determine the HJB equation associated to the investment problem (3.4), we set

$$\widetilde{V}(t,W,y,w) := \mathbb{E}(t)\left[U(t,W,y,w)\right], \tag{3.5}$$

where U depends on both the wealth and precision dynamics, as well as on the control process w. Thus, we write the optimal control problem as follows

$$V(W,y,t) := \max_{(u(t))_{t\,\in\,[0,T]}} \widetilde{V}(t,W,y,w). \tag{3.6}$$

By exploiting Ito's Lemma, we have

$$U(t+\Delta t, W+\Delta W, y+\Delta y, w) + U(t,W,y,w) = U_t dt + U_W dW_t + U_y dy(t) + \frac{1}{2}U_{WW}\langle dW, dW\rangle_t + \frac{1}{2}U_{yy}\langle dy, dy\rangle_t + +U_{Wy}\langle dW, dy\rangle_t + \Delta U dN(\lambda)(t), \tag{3.7}$$

where $\Delta U := U(t,W,y,w) - U(t-,W,y,w) = U(t, W(1+J\theta^N), y+\xi(y), w) - U(t,W,y,w)$, $U_t = \frac{\partial U}{\partial t}, U_W = \frac{\partial U}{\partial W}, U_y = \frac{\partial^2 U}{\partial y^2}$ and $U_{WW} = \frac{\partial^2 U}{\partial W^2}$. By replacing the budget constraint (3.3) and the $\mathbb{Q}$-precision dynamics (2.10) in (3.7), we get

$$\begin{aligned}U(t+\Delta t, W+\Delta W, y+\Delta y, w) - U(t, W, y, w) = U_t dt + U_W \left[W\left(r + \eta\theta^B(t) + \nu\theta^Z(t) - \lambda^{\mathbb{Q}}\theta^N(t)\right) dt\right. \\ \left. +\theta^B(t) W\sqrt{y^{-1}} dB^{\mathbb{Q}} + \theta^Z(t) W\sqrt{y^{-1}} dZ\right] \\ + U_y \left[k^{\mathbb{Q}}(\theta^{\mathbb{Q}} - y(t))dt + \sigma\sqrt{y}\left(\rho dB^{\mathbb{Q}} + \sqrt{1-\rho^2} dZ^{\mathbb{Q}}\right)\right] \\ + \frac{1}{2} U_{WW} W^2 \left(\left(\eta\theta^B(t)\right)^2 + \left(\theta^Z(t)\right)^2\right) y^{-1} dt \\ + \frac{1}{2} U_{yy}\sigma^2 y + +U_{Wy} W\sigma\left(\rho\theta^B(t) + \sqrt{1-\rho^2}\theta^Z(t)\right) + \Delta U dN(\lambda)(t)\ .\end{aligned} \tag{3.8}$$

Now we use (2.8a) and (2.8b), and evaluate the conditional expectation in (3.8), so that

$$\begin{aligned}\widetilde{V}(t+\Delta t, W+\Delta W, y+\Delta y, w) - \widetilde{V}(t, W, y, w) = \left[\widetilde{V}_t + \widetilde{V}_W W\left(r + \eta\theta^B(t) + \nu\theta^Z(t) - \lambda^{\mathbb{Q}}\theta^N(t)\right)\right. \\ +\widetilde{V}_y\left(k^{\mathbb{Q}}(\theta^{\mathbb{Q}} - y)\right) + \sigma\widetilde{V}_y\rho\eta y + \sigma\widetilde{V}_y\sqrt{1-\rho^2}\beta y \\ +\frac{1}{2}\widetilde{V}_{WW} W^2\left(\left(\theta^B(t)\right)^2 + \left(\theta^Z(t)\right)^2\right) \\ +\frac{1}{2}\widetilde{V}_{yy}\sigma^2 y + \widetilde{V}_{Wy} W\sigma\left(\rho\theta^B(t) + \sqrt{1-\rho^2}\theta^Z(t)\right) \\ \left.+\lambda\left[\widetilde{V}(t, W(1+J\theta^N(t)), y+\xi, w) - \widetilde{V}(t, W, y, w)\right]\right] dt\ .\end{aligned} \tag{3.9}$$

The associated Hamilton-Jacobi-Bellman equation (HJB) follows by applying the maximum over the control process w and evaluating the limit as Δt goes to zero, and so

$$\begin{aligned}0 = \max_{(u(t))_{t\,\in\,[0,T]}} \Big\{ V(t) + V_W W(t)\left(r + \theta^B(t)\eta + \theta^Z(t)\nu - \theta^N(t)\lambda^{\mathbb{Q}} J\right) + \\ V_y\left[k^{\mathbb{Q}}(\theta^{\mathbb{Q}} - y(t)) + \sigma\left(\rho\eta + \beta\sqrt{1-\rho^2}\right)\right] y(t) + \frac{1}{2} V_{WW} W(t)^2\left[\left(\theta^B(t)\right)^2 + \left(\theta^Z(t)\right)^2\right] y(t)^{-1} + \\ \frac{1}{2} V_{yy}\sigma^2 y(t) + V_{Wy} W(t)\sigma\left[\rho\theta^B(t) + \sqrt{1-\rho^2}\theta^Z(t)\right] \\ +\lambda\left[V\left((1+J\theta^N(t))W(t)_-, y(t)_- + \xi, t\right) - V(W(t)_-, y(t)_-, t)\right]\Big\}\ .\end{aligned} \tag{3.10}$$

In the proposed non-affine model with co-jumps the optimal portfolio weights can be expressed in closed form.

Proposition 3.1.1. *Consider the investment problem* (3.4) *and the associated HJB equation* (3.10)*. Then, for any $t \in [0,T]$, the value function is*

$$V(W, y, t) = \exp\{F(T-t)y(t) + G(T-t)\}\frac{W(t)^{1-\gamma}}{1-\gamma}\ , \tag{3.11}$$

where

$$F(T-t) = \frac{e^{k_2(T-t)}-1}{2k_2+(k_2+k_1)(e^{k_2(T-t)}-1)}\delta\,, \tag{3.12}$$

$$\begin{aligned} G(T-t) &= r(1-\gamma)(T-t) + 2\frac{k^{\mathbb{Q}}\theta^{\mathbb{Q}}}{\sigma^2}\ln\left(\frac{2k_2 e^{\frac{(k_1+k_2)(T-t)}{2}}}{2k_2+(k_1+k_2)(e^{k_2(T-t)}-1)}\right) \\ &\quad - \int_0^{(T-t)}\left(\lambda - \lambda e^{F(s)\xi}\left(1+J\theta^N(s)^*\right)^{1-\gamma} + \lambda^{\mathbb{Q}}J\theta^N(s)^*\right)ds\,, \end{aligned} \tag{3.13}$$

with

$$\delta := \frac{1-\gamma}{\gamma}\left(\eta^2+\beta^2\right),\; k_1 := k^{\mathbb{Q}} - \frac{\sigma}{\gamma}\left(\rho\eta+\nu\sqrt{1-\rho^2}\right),\; k_2 := \sqrt{k_1^2-\delta\sigma^2} \tag{3.14}$$

and $(T-t)=\tau$.
The optimal risk exposures to diffusive $B(t)$*, volatility* $Z(t)$ *and jump risk factors* $JN(\lambda)(t)$ *are*

$$\theta^B(t)^* = \left(\frac{\eta}{\gamma}+\frac{\sigma\rho F(\tau)}{\gamma}\right)y(t)\,, \tag{3.15a}$$

$$\theta^Z(t)^* = \left(\frac{\nu}{\gamma}+\frac{\sigma\sqrt{1-\rho^2}F(\tau)}{\gamma}\right)y(t)\,, \tag{3.15b}$$

$$\theta^N(t)^* = \frac{1}{J}\left[\left(\frac{\lambda e^{F(\tau)\xi}}{\lambda^{\mathbb{Q}}}\right)^{\frac{1}{\gamma}}-1\right]. \tag{3.15c}$$

Assume further that there exist non-redundant derivatives traded at time $t \le T$*. The optimal portfolio weights related to the risky asset and the derivative securities in* $t \le T$ *are*

$$\phi(t)^* = \theta^B(t)^* - \sum_{i=1}^{2}\psi(t)^{(i)*}\left(S(t)\frac{g_s^{(i)}}{O(t)^{(i)}}+\sigma\rho\frac{g_y^{(i)}}{O(t)^{(i)}}y(t)\right), \tag{3.16a}$$

$$\begin{aligned} \psi(t)^{(1)*} &= \frac{1}{\mathcal{D}(t)}\Bigg\{\left(-\theta^B(t)^*+\frac{\rho\theta^Z(t)^*}{\sqrt{1-\rho^2}}\right)\frac{g_y^{(2)}}{O(t)^{(2)}}+\Bigg[\theta^N(t)^*\frac{g_y^{(2)}}{O(t)^{(2)}}-\Bigg(\frac{\Delta g^{(2)}}{J\,O(t)^{(2)}}- \\ &\quad S(t)\frac{g_s^{(2)}}{O(t)^{(2)}}\Bigg)\frac{\theta^Z(t)^*}{\sigma\sqrt{1-\rho^2}y(t)}\Bigg]\Bigg\}, \end{aligned} \tag{3.16b}$$

$$\begin{aligned} \psi(t)^{(2)*} &= -\frac{1}{\mathcal{D}(t)}\Bigg\{\left(-\theta^B(t)^*+\frac{\rho\theta^Z(t)^*}{\sqrt{1-\rho^2}}\right)\frac{g_y^{(1)}}{O(t)^{(1)}}+\Bigg[\theta^N(t)^*\frac{g_y^{(1)}}{O(t)^{(1)}}-\Bigg(\frac{\Delta g^{(1)}}{J\,O(t)^{(1)}}- \\ &\quad S(t)\frac{g_s^{(1)}}{O(t)^{(1)}}\Bigg)\frac{\theta^Z(t)^*}{\sigma\sqrt{1-\rho^2}y(t)}\Bigg]\Bigg\}. \end{aligned} \tag{3.16c}$$

Proof. Assume that the value function is given in (3.11). Thus, setting $\tau = T-t$, the HJB equation (3.10) becomes

$$
\begin{aligned}
0 = \max_{\{u(t)\}_{t\in[0,T]}} & \Big\{ \left(-\dot{F}_\tau y(t) - \dot{G}_\tau\right) + (1-\gamma)\left(r + \theta^B(t)\eta + \theta^Z(t)\nu - \theta^N(t)\lambda^{\mathbb{Q}} J\right) \\
& + F(\tau)\left[k^{\mathbb{Q}}(\theta^{\mathbb{Q}} - y(t)) + \sigma\left(\rho\eta + \nu\sqrt{1-\rho^2}\right) y(t)\right] + \frac{1}{2}F(\tau)^2\sigma^2 y(t) \\
& - \frac{\gamma(1-\gamma)}{2}\left[\left(\theta^B(t)\right)^2 + \left(\theta^Z(t)\right)^2\right] y(t)^{-1} + (1-\gamma)F(\tau)\sigma\left[\rho\theta^B(t) + \sqrt{1-\rho^2}\theta^Z(t)\right] \\
& + \lambda\mathbb{E}\left[e^{F(\tau)\xi(y(t))}\left(1 + J\theta^N(t)\right)^{1-\gamma} - 1\right]\Big\} := \varphi(u(t)),
\end{aligned} \tag{3.17}
$$

with $\dot{F}_\tau = \frac{\partial F(\tau)}{\partial \tau}$ and $\dot{G}_\tau = \frac{\partial G(\tau)}{\partial \tau}$. Where $u(t) = \{\theta(t)^B, \theta(t)^Z, \theta(t)^N\}$. The first order conditions are given by

$$
\begin{aligned}
\frac{\partial\varphi}{\partial\theta^B} &= (1-\gamma)\eta - \gamma(1-\gamma)\theta^B(t)y(t)^{-1} + (1-\gamma)\sigma\rho F(\tau) = 0, \\
\frac{\partial\varphi}{\partial\theta^Z} &= (1-\gamma)\nu - \gamma(1-\gamma)\theta^Z(t)y(t)^{-1} + (1-\gamma)\sigma\sqrt{1-\rho^2}F(\tau) = 0, \\
\frac{\partial\varphi}{\partial\theta^N} &= -(1-\gamma)\lambda^{\mathbb{Q}}J + \lambda\mathbb{E}\left[(1-\gamma)e^{F(\tau)\xi(y(t))}J\left(1 + \theta^N(t)J\right)^{-\gamma}\right] = 0,
\end{aligned}
$$

which imply (3.15a), (3.15b), and (3.15c). Thanks to (3.2) we obtain the portfolio weights given by (3.16).

By replacing the optimal exposures in the HJB equation (3.17) and with some algebra we obtain an affine equation in y of the form

$$
\begin{aligned}
&\left(\dot{F}_\tau - F(\tau)\left[-k^{\mathbb{Q}} + \frac{\sigma}{\gamma}\left(\rho\eta + \nu\sqrt{1-\rho^2}\right)\right] - F(\tau)^2\frac{\sigma^2}{2\gamma} + \frac{1-\gamma}{2\gamma}(\eta^2 + \nu^2)\right) y(t) + \\
&\dot{G}_\tau - (1-\gamma)r - k^{\mathbb{Q}}\theta^{\mathbb{Q}}F(\tau) + (1-\gamma)\lambda^{\mathbb{Q}}J\theta^N(t) - \lambda\mathbb{E}\left[e^{F(\tau)\xi}\left(1 + J\theta^N(t)\right)^{1-\gamma} - 1\right] = 0,
\end{aligned} \tag{3.18}
$$

and so we have the following system of ODEs w.r.t. $F(\tau)$ and $G(\tau)$

$$
\begin{cases}
\dot{F}(\tau) - F(\tau)\left[-k^{\mathbb{Q}} + \frac{\sigma}{\gamma}\left(\rho\eta + \nu\sqrt{1-\rho^2}\right)\right] - F(\tau)^2\frac{\sigma^2}{2\gamma} + \frac{1-\gamma}{2\gamma}(\eta^2 + \nu^2) = 0, \\
\dot{G}(\tau) - (1-\gamma)r - k^{\mathbb{Q}}\theta^{\mathbb{Q}}F(\tau) + (1-\gamma)\lambda^{\mathbb{Q}}J\theta^N(t) - \lambda\mathbb{E}\left[e^{F(\tau)\xi}\left(1 + J\theta^N(t)\right)^{1-\gamma} - 1\right] = 0,
\end{cases} \tag{3.19}
$$

with initial conditions $F(0) = G(0) = 0$. We observe that the ODE for $F(\tau)$ in (3.19) is a Riccati-type equation of the form

$$
\dot{F}_\tau = +\frac{\sigma^2}{2\gamma}F(\tau)^2 + \tilde{k}_1 F(\tau) + \frac{\delta}{2}, \tag{3.20}
$$

where $\widetilde{k}_1 = -k_1$ and δ are defined in (3.14).

We set $y(\tau) := -\int_0^\tau F(s)ds$, so that $F(\tau) = -\frac{2}{\sigma^2}\frac{\dot{y}(\tau)}{y(\tau)}$, and Eq. (3.20) can be rewritten as

$$
\frac{2\gamma}{\sigma^2}\ddot{y}(\tau) - k_1\frac{2\gamma}{\sigma^2}\dot{y}(\tau) + \frac{\delta}{2} = 0. \tag{3.21}
$$

Eq. (3.21) is a second-order linear ODE, whose solution is of the form

$$y(\tau) = C_1 e^{x_1 \tau} + C_2 e^{x_2 \tau}\,, \tag{3.22}$$

with x_1, x_2 solutions to the corresponding characteristic equation

$$\frac{2\gamma}{\sigma^2}x^2 - \frac{2\gamma k_1}{\sigma^2}x + \frac{\delta}{2} = 0\,, \tag{3.23}$$

and C_1, C_2 constants to be determined. More precisely, solving Eq. (3.23) we obtain $x = \frac{k_1 \pm k_2}{2}$, with $k_2 = \sqrt{k_1^2 - \frac{\delta\sigma^2}{\gamma}}$. The boundary condition for F ensures that $C_2 = -C_1 \frac{x_1}{x_2}$. Replacing Eq. (3.22) in (3.21), with some algebra we get (3.12). The explicit expression for $G(\tau)$ is obtained by direct integration. □

3.1.1 The optimal policy in incomplete market

When derivatives are not included, the investor faces the same investment problem as in (3.4) and (3.10), with the additional constraint that she cannot invest in the derivatives market. In this case, to determine the associated HJB equation, we set

$$\widehat{V}(t, W, y, w) := \mathbb{E}(t)\left[U(t, W, y, w)\right], \tag{3.24}$$

where we want to find

$$V(t, W, y, w) = \max_{(\pi(t))_{t\in[0,T]}} \widehat{V}(t, W, y, w).$$

By following the same steps described in (3.8) and (3.9), and assuming that the value function is given in (3.11), the HJB equation is

$$\begin{aligned} 0 = \max_{\{\pi(t)\}_{t\,\in\,[0,T]}} \Bigg\{ & -\dot{\widehat{F}}_\tau y - \dot{\widehat{G}}_\tau + (1-\gamma)r + (1-\gamma)\left(\eta - \lambda^* J\right)\pi(t) + \widehat{F}(\tau)k^{\mathbb{Q}}\theta^{\mathbb{Q}} \\ & - \widehat{F}(\tau)k^{\mathbb{Q}}y + \widehat{F}(\tau)\sigma y\left(\eta\rho + \sqrt{1-\rho^2}\nu\right) - \frac{1}{2}\gamma(1-\gamma)\frac{1}{y}\pi(t)^2 + \\ & \frac{1}{2}\widehat{F}^2(\tau)\sigma^2 y + \widehat{F}(\tau)(1-\gamma)\sigma w_t + \lambda\mathbb{E}_t\left[e^{\widehat{F}(\tau)\xi(y(t))}\left(1 + J\pi(t)\right)^{1-\gamma} - 1\right]\Bigg\} \\ & := \widehat{\varphi}(\pi(t)). \end{aligned} \tag{3.25}$$

In this case, along the lines of Oliva and Renò [2018], we exploit the approximation

$$(1 + J\pi(t))^{1-\gamma} = 1 + (1-\gamma)\pi(t)J + o(\pi(t)J)^2\,.$$

Hence, the jump component in (3.25) can be written as $\lambda\mathbb{E}\left[e^{\widehat{F}(\tau)\xi(y(t))}\left(1 + (1-\gamma)\pi(t)J\right) - 1\right]$. The first order condition is given by

$$\frac{\partial\widehat{\varphi}(\pi(t))}{\partial\pi(t)} = (1-\gamma)(\eta - \lambda^* J) - (1-\gamma)\gamma\pi(t)\frac{1}{y(t)} + (1-\gamma)\rho\sigma\widehat{F}(\tau) + (1-\gamma)\lambda e^{\widehat{F}(\tau)\xi(y(t))}J = 0, \tag{3.26}$$

so that

$$\pi(t)^* = \left[\frac{\eta - \lambda^* J}{\gamma} + \frac{\sigma\rho\widehat{F}(\tau)}{\gamma} + \frac{\lambda e^{\widehat{F}(\tau)\xi}}{\gamma}\right] y(t) = \widehat{B}(t)y(t). \tag{3.27}$$

By replacing (3.27) in (3.25), and with some algebra, we obtain the following system of ODEs for $\widehat{F}(\tau)$ and $\widehat{G}(\tau)$

$$\begin{cases} \dot{\widehat{F}}_\tau - & (1-\gamma)\left(\eta - \lambda^* J\right)\widehat{B}(t) + \widehat{F}(\tau)k^{\mathbb{Q}} - \widehat{F}(\tau)\sigma\left(\eta\rho + \sqrt{1-\rho^2}\nu\right) + \frac{1}{2}\gamma(1-\gamma)\widehat{B}(t)^2 \\ & -\frac{1}{2}\widehat{F}(\tau)^2(\tau)\sigma^2 - (1-\gamma)\sigma\widehat{F}(\tau)\widehat{B}(t) - (1-\gamma)\lambda e^{\widehat{F}(\tau)\xi(y(t))}\widehat{B}(t)J = 0 \\ \dot{\widehat{G}}_\tau - & (1-\gamma)r - \widehat{F}(\tau)k^{\mathbb{Q}}\theta^{\mathbb{Q}} - \lambda\left[e^{\widehat{F}(\tau)\xi(y(t))} - 1\right] = 0 \end{cases}, \tag{3.28}$$

with initial conditions $\widehat{F}(0) = \widehat{G}(0) = 0$. The ODEs (3.28) are solved numerically.

3.2 Analysis of the optimal allocation

We first note that Proposition 3.1.1 can be both read in terms of (optimal) allocation of market components, i.e. stocks and derivatives, as well as with regards to the (optimal) exposure to the sources of risk supporting the wealth dynamics, as in Liu and Pan [2003] and Branger et al. [2008]. In order to provide additional economic insights to our result, we first look at the exposures. By inspecting (3.15a) and (3.15b), we distinguish two terms, which respectively represent the myopic and the inter-temporal hedging components for optimal allocation. As expected, for $\theta(t)^{B^*}$ the former is proportional to the premium η for diffusive risk, as it increases when the attractiveness of the stock raises. The same holds for $\theta(t)^{Z^*}$: a greater myopic risk exposure to the precision risk is due either to an increase of the premium demand ν or to a smaller risk aversion γ. Interestingly, our optimal portfolio exposures corresponding to the diffusive price stock B and the additional precision risk Z are inversely proportional to instantaneous volatility. This is the crucial difference with the affine models literature. Specifically, a myopic component proportional to the inverse of volatility reflects the static mean-variance allocation, reconciling the dynamic asset framework with Markowitz theory.
Now we investigate the exposure to jump risk given in (3.15c). In this view, we must impose realistic constraints on the jumps amplitudes. Hence, we require the jump-price magnitude to be negative, in line with the empirical evidence outlined in the literature, see e.g. Bandi and Renò [2016] and references therein.
The presence of co-jumps implies the existence of a further measure of potential losses deriving from such event risk, given by $\theta(t)^{N^*}$. A study of the latter as a function of the precision-jump amplitude $\xi(y(t))$ exhibits interesting features: first, we observe that $\theta(t)^{N^*}$ changes its sign for $\bar{\xi}(y(t)) := \frac{1}{F(\tau)} \ln\left(\frac{\lambda^{\mathbb{Q}}}{\lambda}\right)$, having imposed $J < 0$. Furthermore, $\bar{\xi}(y(t)) > 0$ for $F > 0$. The study of the derivative sign reveals that $\theta(t)^{N^*}$ is decreasing in ξ, if the function F turns out to be negative (i.e., for $\gamma < 1$). On the other hand, a financially significant choice is to take downward precision jumps, in order to guarantee

upward volatility jumps. Therefore, in our case the jump-risk exposure will always be positive, and decreases as the precision-jump amplitude increases.
We can rewrite the optimal jump exposure in a slight different form, i.e.

$$\theta(t)^{N^*} = \frac{1}{J}\left[\left(\frac{\lambda}{\lambda^{\mathbb{Q}}}\right)^{\frac{1}{\gamma}} - 1\right] + \frac{1}{J}\left(\frac{\lambda}{\lambda^{\mathbb{Q}}}\right)^{\frac{1}{\gamma}}\left[e^{\frac{\xi(y(t))F(\tau)}{\gamma}} - 1\right]. \tag{3.29}$$

Eq. (3.29) informs about both myopic and hedging components of the jump exposure. The former depends on the jump risk premium $\frac{\lambda}{\lambda^{\mathbb{Q}}}$. We first note that, since we consider negative jumps in price, the myopic component results to be positive (resp., negative) when $\lambda < \lambda^{\mathbb{Q}}$ (resp. $\lambda > \lambda^{\mathbb{Q}}$) when $\gamma > 1$. However, the empirical evidence shows that a suitable choice for the jump-risk premium is $\lambda^{\mathbb{Q}} = m\lambda$, with $m \in \mathbb{N}$, see e.g. Liu and Pan [2003] and references therein for further details. Thus, it makes sense to assume that $\lambda < \lambda^{\mathbb{Q}}$.
Moreover, the myopic component vanishes when $\lambda = \lambda^{\mathbb{Q}}$, namely when there is no risk premium for jump-timing uncertainty, see e.g. Pan [2002]. In this case, only the hedging component survives. Concerning the latter, we observe that the presence of co-jumps impacts on the second term in (3.29). Differently from $\theta(t)^{B^*}$ and $\theta(t)^{Z^*}$, the co-jump exposure does not depend on the volatility level.

3.3 The role of derivatives and co-jumps

To test our theoretical results we set the risk-free rate equal to $r = 0\%$, and we calibrate the model on real data. We perform different calibration procedures to obtain parameters' estimation under co-jumps, jumps-in-price, and no-jumps models. To obtain the $\mathbb{Q}$-parameters for the dynamics given in (2.10), we consider 78 CBOE quotes linked to the SPX index, downloaded from `https://www.cboe.com/delayed_quotes/spx/quote_table` on November 15, 2021. More precisely, we select 26 different strike prices for options expiring on January 21, 2022, 14 strike prices for options expiring on February 18, 2022, 24 strike prices for options expiring on March 18, 2022, and 14 strike prices for options expiring on May 20, 2022. We consider ITM and OTM options with moneyness smaller or equal to $\pm 4\%$, in order to have highly liquid options. As a further liquidity constraint, we also look at the so called *Open Interest* (INT), the number of long and short open positions. Such a metric reveals whether money is entering or leaving the market, and can be considered a measure of market activity. A low INT indicates few open positions. The higher the number, the easier it is to trade options at a reliable price. For this reason, we select options with $INT > 100$. Finally, we rule out the OTM options with shortest maturities, since the corresponding prices produced by the model are far from the average price (mid price), and therefore they could alter the estimation procedure, see e.g. Romo and Ortiz-Gracia [2021]. The quality of the calibration is assessed using the mean absolute error (MAE), i.e. the average of the absolute value of the difference between prices deriving from the model and market prices of the options. In terms of MAE there is no difference of the calibration performance between the model with jumps only in price and joint jumps. However, as we shall see this does not translate into similar behavior at the optimal allocation level, as jumps in volatility have an important impact on optimal allocation.

	$\theta^{\mathbb{Q}}$	$k^{\mathbb{Q}}$	σ	ρ	$\lambda^{\mathbb{Q}}$	J	ξ	MAE
Co-Jumps	205.0658	1.7184	10.4945	0.6207	0.4922	-0.1997	-0.1462	0.44%
Jumps-in-Price	228.4186	1.4183	11.9137	0.6184	0.4922	-0.1985	–	0.44%
No Jumps	98.2938	0.2129	9.7254	0.2196	–	–	–	4%

Table 3.1. Parameters estimates for the co-jumps, jump-in-price, and no-jumps models. The initial volatility (v_0) is equal to 9.6% for both the co-jumps and the jump-in-price models, and is equal to 13.16% for the no-jumps model. The jump intensity under the physical measure $\mathbb{P}$ is $\lambda = 0.2$.

By using the parameters obtained through the calibration procedure previously introduced, we first evaluate the optimal exposures for the model proposed, compared to the models with only jumps in price and without jumps, for two different values of the risk aversion parameter ($\gamma = 0.5$ to indicate low risk averse investor and $\gamma = 3$ to refer to risk averse investors). The results are shown in Fig. 3.1.

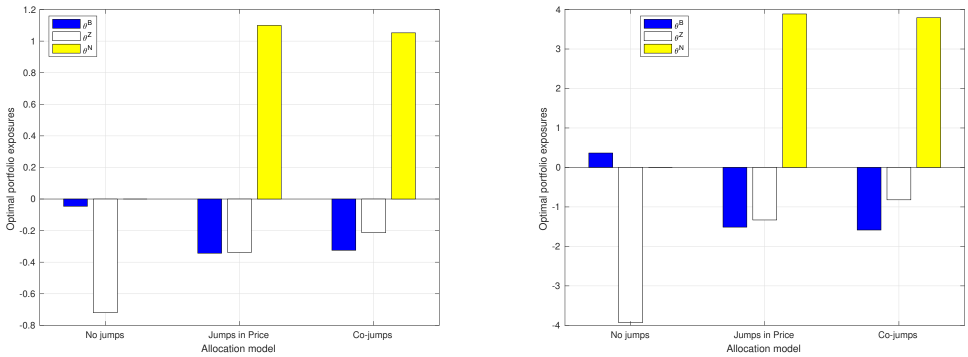

Figure 3.1. Optimal exposures for co-jumps, jumps-in price and no-jumps models for risk loving ($\gamma = 0.5$, left chart) and risk averse ($\gamma = 3$, right chart) investors. The investment maturity is 1 year, the diffusive and volatility risk premia are $\eta = -8.51\%$, $\nu = -5.05\%$ for the co-jumps model, $\eta = -8.45\%$, $\nu = -7.90\%$ for the jumps-in-price model, and $\eta = 0.84\%$, $\nu = -27.14\%$ for the no-jumps model, respectively. The remaining parameters' estimates are provided in Table 3.1.

When we exclude jumps, the primary source of uncertainty is the stochastic volatility, expressed in terms of its inverse. The latter decreases its impact if we allow stock prices to jump since the total uncertainty will be shared over the remaining components. Volatility risk reduces when the model includes co-jumps, exhibiting a subsequent increase in the (joint) jump risk exposure. We also observe how adding jumps in precision, without changing the number of risk sources, results in a reduction of aggregate risk if $\gamma > 1$ (and an increase if $\gamma < 1$). In addition, the exposure to jump risk is greater than zero, in line with the theoretical results provided in Section 3.2. In our case, θ^{N^*} is decreasing in $\xi(y(t))$, having assumed η, $\nu < 0$, hence the exposure to jump risk is greater in the model with only jumps in price, suggesting that volatility jumps are an important driver of event risk.

According to the discussion provided in Section 2, the market is complete with two non-redundant derivatives, so we include in the market two European options, namely a 3-month

5%-OTM call, and a 3-month 5%-ITM call.

According to Branger et al. [2008], for very short maturities, the exposures reflect predominantly myopic demand and they are highly sensitive to changes in risk premia. From (3.15a), (3.15b) and (3.15c) we deduce that the exposures inherit the analytical properties of the function F. In particular, for $\gamma > 1$, F is decreasing and convex, while F is increasing and concave for $0 < \gamma < 1$. Hence, by considering the parameter set given in Table 3.1 and assuming $\gamma > 1$, we conclude that the exposures are decreasing and convex as functions of T and have horizontal asymptotes when we consider large maturities. The asymptote is attained at $T = 2$. Such a behaviour is shown in the bottom-left chart of Fig. 3.2, which provides the optimal portfolio weights evaluated at $t = 0$, as functions of the time horizon T. To better understand the impact of the risk premia on the portfolio weights, we perform a sensitivity analysis w.r.t. η, ν, and $\frac{\lambda^{\mathbb{Q}}}{\lambda}$. The results are summarized in Fig. (3.2). The top-left chart shows the optimal portfolio allocation in a complete market as a function of the diffusive risk premium η. As expected, as the risk premium raises, the attractiveness of the underlying security increases as well, resulting in a larger allocation in the stock. The diffusive risk premium also affects the derivatives investment: the higher the shareholders' remuneration for the risk underneath, the larger the derivative investment. As a consequence, we obtain a long position in the 5%-ITM call and a short position in the 5%-OTM call option. When investors act non-myopically ($\eta = 0$), the total wealth invested in DER1 (i.e., the 5%-ITM call option) and in DER2 (i.e., the 5%-OTM call option) is 5% and -20%, respectively. For $\eta = 0.1$, the total wealth invested in DER1 (resp., DER2) is slightly more than 8% (resp., about -27%). For $\eta = -0.1$, the positions of the two derivatives do not change and the total wealth invested in DER1 (resp., DER2) is about 2% (resp., about -16%).

The top-right chart in Fig. 3.2 shows the optimal portfolio allocation in stock and derivatives as a function of the precision risk premium ν. We assume a risk-averse coefficient equal to $\gamma = 3$ for a 1-year investment. Consistently with Liu and Pan [2003], when the precision-risk premium is equal to zero, the non-myopic demand for DER1 (i.e., the 5%-ITM call option) and DER2 (i.e., the 5%-OTM call option) are less than 1% and -12% of the total wealth, respectively. When we consider more conservative values for the volatility-risk premium, namely we set $\nu = -0.5$, we witness an increasing investment in derivatives in absolute value, with the optimal allocation in DER1 (resp., DER2) equal to roughly 20% (resp., -67%) of the total wealth.

The bottom-right chart in Fig. 3.2 shows the optimal portfolio weights as a function of the jump risk premium λ^Q/λ, when λ is allowed to vary. We note that, while the ITM call option allocation remains almost unchanged (going from 2.1% when the jump frequency is 2 years to 2.8% when the jump frequency is 10 years), the OTM call option suffers more from the jump risk to the extent that the disinvestment is more than tripled. The greatest impact of the jump risk is associated with the investment in the stock, for which the change is about 200% when the jump frequency goes from 2 to 10 years.

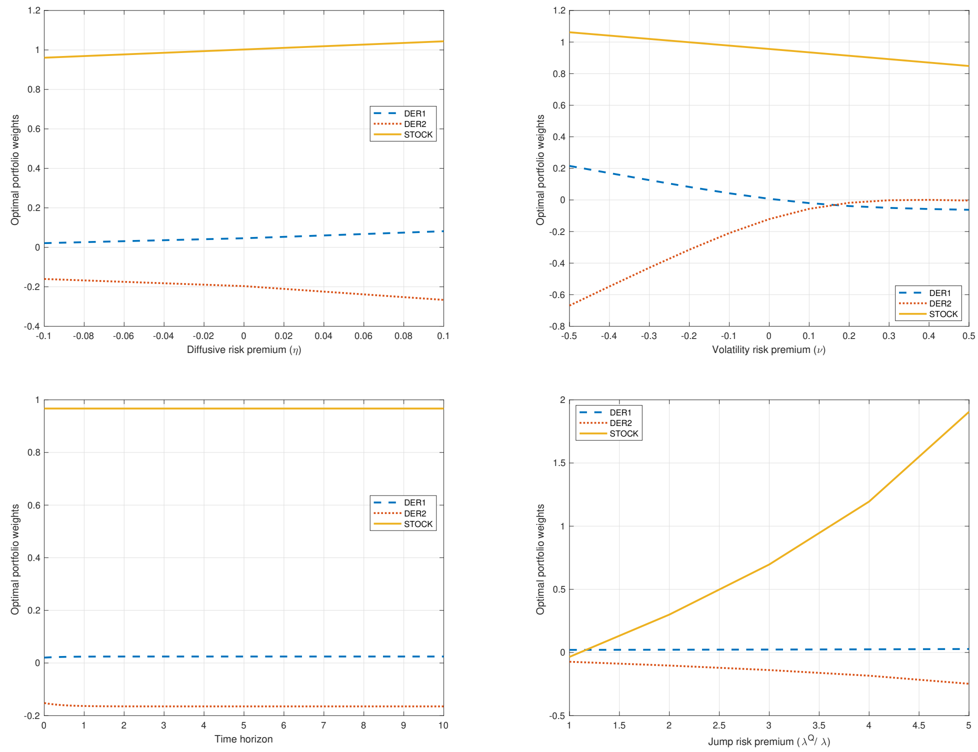

Figure 3.2. Optimal portfolio allocation as function of the diffusive risk-premium η, volatility risk premium ν, intensity of jump λ, and time horizon, respectively. The risk aversion coefficient is set equal to $\gamma = 3$, the risk-free interest rate is fix to $r = 0$. The investment maturity is 1 year. The remaining parameters estimates are provided in Table 3.1.

3.3.1 Optimal allocation with derivatives and no jumps

Turning to the optimal allocation given in (3.16), in order to better illustrate the formulae, we analyze a couple of special cases. First, we assume that no jumps occur, i.e. we set $\lambda = \xi = J = 0$ in (2.10). Hence, only one non-redundant derivative is needed. Setting $\tau = T - t$, the optimal allocation for $t \in [0, T]$ is

$$\psi(t)^* = \left(\frac{\nu}{\sigma\gamma\sqrt{1-\rho^2}} + \frac{F(\tau)}{\gamma} \right) \frac{O(t)}{g_y}\,, \tag{3.30a}$$

$$\phi(t)^* = \left(\frac{\eta}{\gamma} - \frac{\nu\rho}{\gamma\sqrt{1-\rho^2}} \right) y(t) - \psi(t)^* S(t) \frac{g_s}{O(t)}\,. \tag{3.30b}$$

Recalling that $y(t) = \frac{1}{v(t)}$ for all $t \in [0, T]$, we get $g_y = -\frac{g_v}{y(t)^2}$, so that the optimal derivative allocation becomes

$$\psi(t)^* = -\frac{1}{\sigma\sqrt{1-\rho^2}} \left(\frac{\nu}{\gamma} + \frac{F(\tau)}{\gamma} \right) \frac{O(t)}{g_v} y(t)^2\,. \tag{3.31}$$

Inspecting (3.31), we can detect some interesting features. First, we observe that the demand for derivatives is driven by two components, associated with two corresponding economic determinants of the investors behavior. The first one is the ratio between ν and γ, and defines the myopic demand of derivatives: investors are interested in taking positions in those contracts that ensure the hedging of volatility risk. The second one illustrates the surviving demand for derivatives when $\nu = 0$. Moreover, the percentage of wealth to be invested in the derivative is inversely proportional to $\frac{g_v}{O(t)}$. As pointed out in Liu and Pan [2003], such a quantity measures the volatility exposure for each unit of currency invested in the derivative. Our result confirms that the more volatility exposure provided, the smaller (in absolute value) the proportion of wealth to be invested. We observe two main differences with the extant affine literature: first, for investors with $\nu < 0$ and $\gamma > 1$, we witness an inversion in the derivatives holding, in that investors must either go long or short on derivatives with positive or negative volatility exposure g_v, respectively. More importantly, the optimal weight results to be inversely proportional to $v(t)^2$, i.e. the *quarticity*. As discussed in Carr and Wu [2009], Christensen et al. [2014], Barndorff-Nielsen and Veraart [2012], the latter captures the fluctuations of the returns' variance, making it possible, to some extent, to disentangle volatility and jumps contributions in price changes. As the quarticity increases, the volatility trade-off becomes less attractive for investors, as there is more uncertainty in the market; this leads to a reduction in the derivative position (in absolute value). Therefore, investors are more exposed to diffusive volatility risk under unstable market conditions, implying a lower derivative allocation.

By replacing (3.31) in (3.30a), the risky asset allocation becomes

$$\phi(t)^* = \left(\frac{\eta}{\gamma} - \frac{\nu\rho}{\gamma\sqrt{1-\rho^2}}\right) y(t) + \left(\frac{\nu}{\sigma\gamma\sqrt{1-\rho^2}} + \frac{F(\tau)}{\gamma}\right) \frac{S(t)g_s}{g_v} y(t)^2 \,. \tag{3.32}$$

When no jumps occur the market can be completed by introducing one non-redundant derivative in the market. For numerical illustration purposes, we select a 3-month 5%-OTM European call option. To illustrate some traits of optimal allocation in such a special case, we carry out a sensitivity analysis with respect to changes in the parameters involved in (3.30a) and (3.32), namely the risk premia, the investor's risk attitude, and the initial level of volatility.

In the top-left panel of Fig. 3.3 the optimal weights are represented as a function of the precision risk premium ν. We first highlight that when $\beta = 0$ the demand for the derivative is negligible, in particular equal to 0.0018% of the total amount of wealth for an investor with a risk aversion coefficient $\gamma = 3$ over a time horizon of 1 year. This means that the demand for derivatives is mainly driven by its myopic component; instead using $\nu = -0.4$ the optimal derivative weight increases up to 5.22% for the same investor. In the top-right panel we represent the optimal weights as a function of diffusion risk premium η. If we increase the diffusive risk premium in absolute value, the attractiveness of the stock increases as well, so the investment in the risky asset will raise. The bottom-left panel shows the optimal weights as a function of the initial volatility level; as the volatility increases the investor will allocate less (in absolute value) in the risky asset, since the optimal weight in the risky asset is inversely proportional to the instantaneous volatility. In stable markets the investor goes very short in equities and allocates little in the derivative. When the market becomes

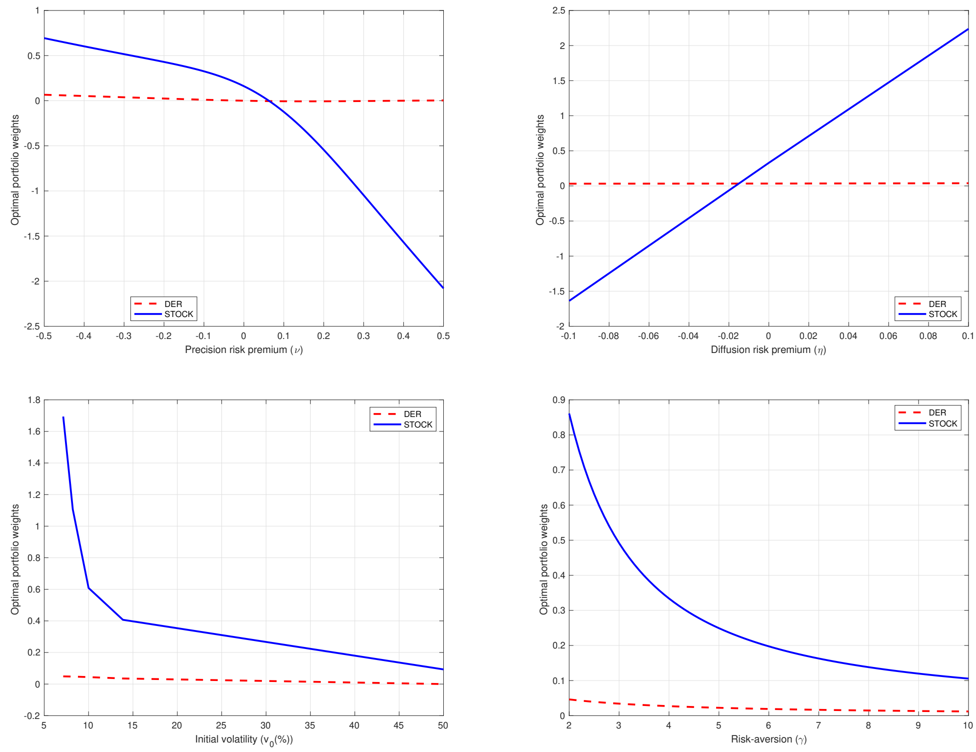

Figure 3.3. Optimal portfolio allocation for the no-jump case. The optimal weights ψ (derivative) and ϕ (stock) are represented on the y-axis. The risk aversion coefficient is set equal to $\gamma = 3$, the risk-free interest rate is fix to $r = 0$ and the time horizon is $T = 1$. The diffusive and volatility risk premia are $\eta = 0.84\%$, $\nu = -27.14\%$. The remaining parameters' estimates are provided in Table 3.1.

more turbulent, the situation changes with regard to the allocation in risky stocks, while we observe almost no impact on the demand for the derivative. The bottom-right panel shows the optimal weights as a function of the risk aversion coefficient: as expected, more risk-tolerant investors adopt riskier strategies. More precisely, we notice that the investor's risk aversion (when $\gamma > 1$) weakly affects the derivative demand, but increasingly pushes the investor towards the risky security, while maintaining a short position.

3.3.2 Optimal allocation with derivatives and constant volatility

In an economy with jump risk but no volatility risk, i.e. when assuming $\sigma = \xi = 0$ and $y(t) = \bar{y} \in \mathbb{R}^+$, for all $t \in [0, T]$, we also are in a circumstance where one derivative is enough to make the market complete. The optimal weights are given by

$$\psi(t)^* = \frac{1}{J} \frac{\left(\frac{\lambda}{\lambda^{\mathbb{Q}}}\right)^{\frac{1}{\gamma}} - \left(1 + \frac{\eta J}{\gamma}\bar{y}\right)}{\frac{\Delta g}{JO(t)} - \frac{S(t)g_s}{O_t}}, \tag{3.33a}$$

$$\phi(t)^* = \frac{\eta}{\gamma}\bar{y} - \psi(t)^* \frac{S(t)g_s}{O(t)} \,. \tag{3.33b}$$

Looking at (3.33a), we observe that the optimal allocation in the derivative is inversely proportional to $\frac{\Delta g}{JO(t)} - \frac{S(t)g_s}{O(t)}$. The latter represents the way to disentangle the diffusive price risk (represented by g_s) from the jump risk (described through Δg). Therefore, the more the derivative allows to separate the two risks mentioned above, the lower the portion of wealth to be invested in the derivative. On the other hand, such a financial tool also affects the numerator of (3.33a) in terms of risk premia. If the investor feels it is indifferent to focus on a specific risk factor, then

$$\frac{\lambda}{\lambda^{\mathbb{Q}}} = \left(1 + \frac{\eta J}{\gamma}\bar{y}\right)^{-\gamma} . \tag{3.34}$$

In this case, the allocation vanishes, i.e. $\psi(t)^* = 0$. However, the empirical evidence shows that $\frac{\lambda}{\lambda^{\mathbb{Q}}}$ is greater than the right-hand side of (3.34), see e.g. Liu and Pan [2003] and references therein. This implies that the jump risk is prevailing. Finally, it is worth highlighting that, unlike the affine case discussed in Liu and Pan [2003], the myopic component of both $\psi(t)^*$ and $\phi(t)^*$ depend on the (constant) level of volatility selected, according to an inverse-proportion relationship.

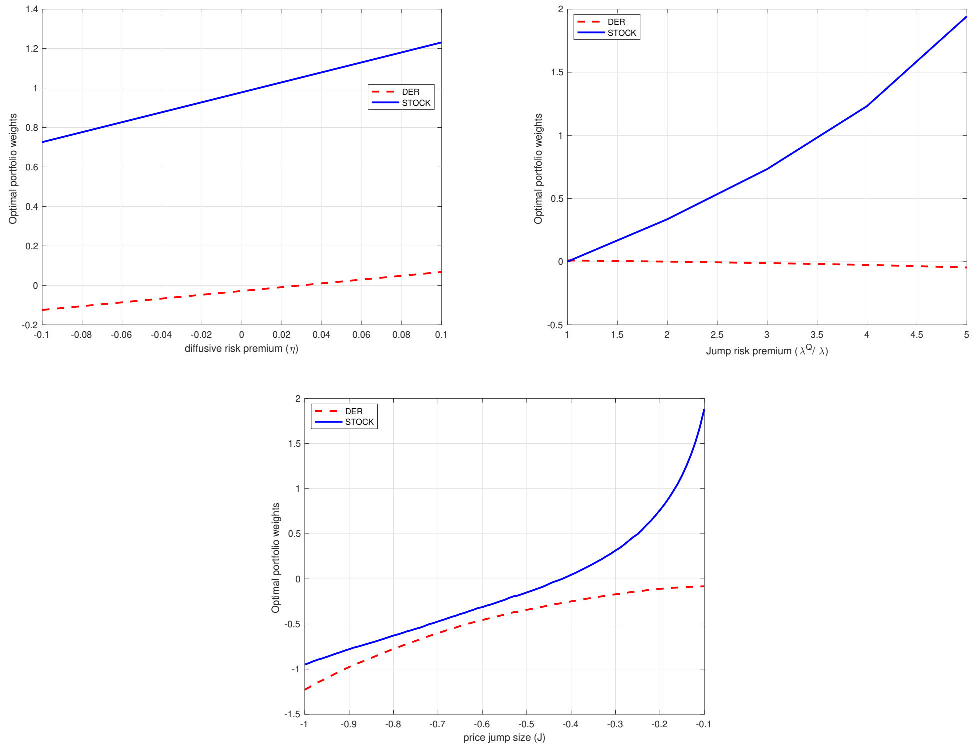

Figure 3.4. Optimal portfolio allocation as a function of the price-jump size, diffusive risk premium and jump intensity under the physical measure $\mathbb{P}$, respectively. The investment maturity is 1 year, the risk aversion parameter is $\gamma = 3$, the diffusive risk premium is $\eta = -8.45$. The remaining parameters' estimates are provided in Table 3.1.

The top-left chart in Fig. 3.4 shows the stock and derivative (5%-OTM European call option) allocation as a function of the diffusive risk premium η. As the latter increases, the

attractiveness of the stock intensifies. The allocation in the derivative is also influenced by changes in η, albeit to a lesser degree: for negative values of η a short position on the derivative is shown, as the investor is not remunerated for taking the risk on.
The top-right chart in Fig. 3.4 shows the optimal portfolio allocation as a function of the jump risk premium $\frac{\lambda^{\mathbb{Q}}}{\lambda}$. As λ decreases, the allocation in the stock grows, as a result of a larger stock premium. This also affects the call option, albeit in a not very significant way: the derivative takes a short position and the short position increases for higher values of the jump risk premium. If we kept the equity premium constant, we would obtain an increase in the jump risk premium, resulting in a larger allocation of derivatives (in absolute value) and a smaller attractiveness of the stock (as a result of the switch between the attractiveness of the stock and derivatives), according to Liu and Pan [2003].
The bottom chart in Fig. 3.4 better clarifies the impact of jumps in price in this special case. As expected, a jump in price reduces the optimal portfolio allocation: when the amplitude of the jump reduces (in absolute value), the investor tends to to increase their position in both stock and derivative (in absolute value), as a consequence of the reduction of the exposure to a jump risk-event.

3.3.3 The impact of derivatives in portfolio allocation

To assess the effect of derivatives in an optimal allocation strategy, we resort to the so called *certainty-equivalent wealth* $\mathcal{W}$, that is, a metric used to evaluate the portfolio improvement of an investor, as explained in Branger et al. [2008]. Given the initial wealth W_0 and the initial precision y_0, the investment problem (3.4) and the value function (3.11) ensure that there exists a value $\mathcal{W}$ of the investor's wealth such that

$$\frac{\mathcal{W}^{1-\gamma}}{1-\gamma} = \frac{W_0^{1-\gamma}}{1-\gamma} \exp\{F(T)y_0 + G(T)\} \,,$$

when derivatives are included in the market. This implies

$$\mathcal{W} = W_0 \exp\left\{ \frac{F(T)y_0 + G(T)}{1-\gamma} \right\} \,, \tag{3.35}$$

where $F(T)$, $G(T)$ are given in (3.12) and (3.13), respectively. At the same time, we exclude derivatives from the investment, mimicking the behaviour of investors who do not believe in the protecting role of such financial contracts. Hence, we evaluate

$$\widehat{\mathcal{W}} = \widehat{W}_0 \exp\left\{ \frac{\widehat{F}(T)y_0 + \widehat{G}(T)}{1-\gamma} \right\} \,, \tag{3.36}$$

where $\widehat{F}(T)$, $\widehat{G}(T)$ are the solutions to the ODEs for the investment problem without considering derivatives, see Section 3.1.1 for details. Finally, we define the portfolio improvement as

$$\mathcal{R}^{\mathcal{W}} := \log\left(\frac{\mathcal{W}}{\widehat{\mathcal{W}}}\right) = \frac{1}{1-\gamma}\left[(F(T) - \widehat{F}(T))y_0 + (G(T) + \widehat{G}(T))\right] \,. \tag{3.37}$$

The portfolio improvement obtained by introducing derivatives is evaluated as a function of the risk aversion parameter γ, in both cases of $\gamma < 1$ and $\gamma > 1$. By inspecting Fig. 3.5,

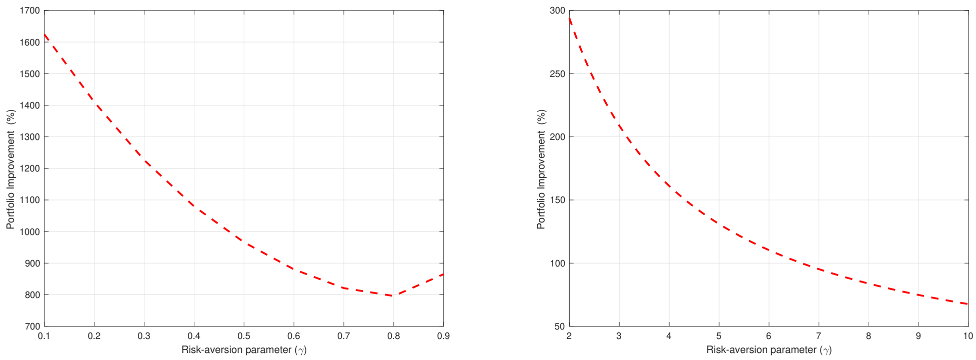

Figure 3.5. Portfolio improvement $\mathcal{R}^{\mathcal{W}}$ as a function of γ, for low risk averse (left chart) and risk averse (right chart) investors. The investment horizon is 1 year, the risk-free rate is fixed to $r = 0\%$. The diffusive and volatility risk premia are $\eta = -8.51\%$, $\nu = -5.05\%$. The remaining parameter estimates are provided in Table 3.1.

we note that the presence of derivatives always implies a portfolio improvement, regardless of the investor's aggressiveness.

Chapter 4

Dynamic portfolio allocation with co-jumps and recursive preferences in portfolio choices

In this chapter, we solve the optimal allocation problem by assuming that the investor has recursive preferences. First, we consider an incomplete market in which the investor has access to a risk-free asset and a panel of risky assets (multivariate case). Second, we focus on a particular case in which the investor can only invest in one risky asset (univariate case). Third, we complete the market by allowing economic agents to invest in derivative contracts. The main goal is to quantify the impact of jumps on optimal consumption, as well as on the proportion of wealth invested in the risky asset (and then in derivatives) within the non-separable preferences framework. Our theoretical results are obtained by exploiting classical dynamic programming techniques. We further analyse the economic implications of our theoretical results, taking into account the investors' characteristics: to do this, we perform a two-step calibration procedure, by using hedge fund data. We first show that both frequency and intensity of co-jumps play a crucial role, serving as a further hedging tool. In details for the incomplete market setting, we find out that: *(i)* infrequent jumps mainly influence the risky allocation, *(ii)* the impact of the precision-jump size is less relevant than the price ones, although it is not negligible, *(iii)* the most significant effect on optimal allocation is due to frequent and large jumps in price, instead of rare jumps of the same magnitude, and *(iv)* the presence of co-jumps increases the consumption-wealth ratio for investors who are extremely willing to substitute consumption over time, hence the larger the investor's propensity to save, the stronger this growth is. Finally, we measure the performances of the optimal rules through a suitable economic metric, namely the *Wealth Equivalent Loss* (WEL). Our findings show that investors who do not believe in the need to incorporate co-jumps can suffer significant losses in their investments. We highlight how the main evidence found for the incomplete market continues to apply when we complete the market by investing in derivative contracts. In the latter case, we show how ignoring the presence of co-jumps has an even more significant impact on the economic agent's wealth loss.

Furthermore, as the results proposed in this chapter provide analytical but approximate solutions, we also measure the effect of such an approximation on possible investment losses. We have performed a numerical study on real data in some particular cases where it is possible to solve the problem without using any approximation; our findings guarantee that the proposed strategy is comparable to the true one, making our proposal reliable even in the most general case.

The Chapter is organized as follows. In the Sections 4.1, 4.2 and 4.3, we describe our model, the investment problem we intend to solve and the main theoretical results respectively, for the multivariate, univariate and univariate cases in which we complete the market. In Section 4.4 we provide some numerical experiments on real data to show how the presence of co-jumps might affect the investment choice. In Section 4.5 we measure the reliability of approximate solutions of the optimal allocation and consumption problem.

4.1 Multivariate case

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ a filtered probability space.

In this chapter, consider the model defined in section (2.2). We assume that the investor's wealth $W = \{W(t)\}_{t\in[0,T]}$ is apportioned among the risk-free asset M and the stock S, also considering the related consumption $C = \{C(t)\}_{t\in[0,T]}$, so that we have

$$\frac{dW(t)}{W(t)} = \epsilon(t)'\frac{dM(t)}{M(t)} + \pi(t)'\frac{dS(t)}{S(t)} - \frac{C(t)}{W(t)}\,, \tag{4.1}$$

where $\pi(t) \in \mathbb{R}^{N\times 1}$ is the vector of wealth invested in the risky assets at time t. We define by $\mathbf{1}$ the $N \times 1$ vector of ones, so that the proportion of wealth invested in the riskless asset is given by $\epsilon(t)' = 1 - \pi(t)'\mathbf{1}$. Replacing the dynamics (2.12) in (4.1), we get the following budget constraint

$$\frac{dW(t)}{W(t)} = [r + \pi(t)'(\alpha - r\mathbf{1})W(t) - C(t)]\,dt + \pi(t)'\sqrt{Y(t)^{-1}}dZ(t) + \pi(t)'JdN(\lambda)(t)\,. \tag{4.2}$$

Following Chacko and Viceira [2005], we consider a SDU function given in (2.27), namely a Duffie-Epstein recursive utility function, see e.g. Duffie and Epstein [1992], to describe investors' preferences.

The investment problem to be solved is

$$V(t, W, y) = \max_{\{\pi(t), C(t)\}} \mathbb{E}\left[\int_t^{+\infty} f(C(s), V(s))ds \,\middle|\, \mathcal{F}_t\right]\,, \tag{4.3}$$

with (4.1) as intertemporal budget constraint. Eq. (2.27) depends on three parameters, namely the *rate of time preference* β, the *elasticity of intertemporal substitution of consumption* (EIS) ψ, and the *risk-aversion parameter* γ.

To find a solution to the intertemporal optimization problem (4.3) we use standard dynamic programming techniques, so that we consider the associated Hamilton-Jacobi-Bellman equation (HJB, from now on), given by

$$0 = \max_{\pi(t),C(t)} \Bigg\{ f(C,V) + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial W}\left(\pi(t)'(\alpha - r\mathbf{1}) + r\right)W(t) - C(t)\frac{\partial V}{\partial W} +$$
$$Tr\left[\left(\Omega\Omega' + KY(t) + Y(t)K'\right)\nabla V\right] + \frac{1}{2}W_t^2\pi(t)'Y(t)^{-1}\pi(t)\frac{\partial^2 V}{\partial W^2} + \left(2\pi(t)\nabla Q'\rho\frac{\partial V}{\partial W}\right)W(t) +$$
$$\frac{1}{2}Tr(4Y(t)\nabla Q'Q\nabla)V + \lambda\mathbb{E}\left[V(W(t) + \pi(t)'JW(t), Y(t) + \xi(Y(t))) - V(W(t), Y(t))\right]\Bigg\}, \tag{4.4}$$

where $\nabla := \left(\frac{\partial}{\partial Y_{i,j}}\right)_{i\leq 1, j\leq N}$, and the expectation in the last term of (4.4) is over the jump sizes. It is worth noting that, since we are considering constant values for J and ξ, in what follows the expectation in (4.4) will be unnecessary. The HJB equation is non-linear due to the aggregator f and the jump component. Besides, the non-linearity mentioned earlier implies the absence of an analytical solution, apart from some particular cases, see Chacko and Viceira [2005] for further details. Therefore, since we would like to carry our analyses on while remaining as general as possible, we propose a methodology for determining an analytical but approximate solution. To achieve this, we impose two restraints. Concerning the aggregator f, we proceed according to the approach proposed in Chacko and Viceira [2005], where the consumption-wealth ratio is approximated using an appropriate log-linear expansion around the unconditional mean $\overline{c-w} := \mathbb{E}\left[\frac{C(t)}{W(t)}\right]$, see Judd [1998] for further details. Concerning the jump component, we exploit a first-order Taylor expansion around $J\pi_t = 0$, so that

$$(1 + J\pi(t))^{1-\gamma} = 1 + (1-\gamma)J\pi(t) + o\left(J^2\pi(t)^2\right) \ , \tag{4.5}$$

and ignore the term $o\left(J^2\pi(t)^2\right)$, see Oliva and Renò [2018] for an in-depth analysis. We show the reliability of such approximations in Section 4.5. We set $H = \frac{1-\gamma}{1-\psi}$ and guess that the solution to (4.4) is of the form

$$V(t, W, y) = \exp\left\{-H\left(Tr(F(t)Y(t)) + G(t)\right)\right\}\frac{W(t)^{1-\gamma}}{1-\gamma} \ , \tag{4.6}$$

where the matrix function $F(t) \in S_N$ and the function $G \in \mathbb{R}$ have to be identified.

Now, we are ready to prove the following

Proposition 4.1.1. *Consider the investment problem* (4.3) *with* $\psi \neq 1$ *and the associated HJB equation* (4.4)*. Then, for* $t \in [0,T]$*, the value function is given in* (4.6)*, while the optimal consumption and portfolio rules are given by*

$$\frac{C(t)}{W(t)} = \beta^{\psi}\exp\left\{-\left(Tr(F(t)Y(t)) + G(t)\right)\right\} \ , \tag{4.7}$$

$$\pi(t) = Y(t)\left(\frac{(\alpha - r\mathbf{1})}{\gamma} + \frac{2HF(t)Q'(-\rho)}{\gamma} + \frac{\lambda J e^{-HTr(F(t)\xi(Y(t)))}}{\gamma}\right) =: Y(t)B(t) \ , \tag{4.8}$$

where $B(t) \in \mathbb{R}^{N\times 1}$ *and* $F \in S_N$, $G \in \mathbb{R}$ *satisfying the following system of ODEs*

$$\begin{cases} \dot{F}_t & = h_1 F(t) + (1-\psi)(\alpha - r\mathbf{1})B(t)' - F(t)K - F(t)K' - \frac{\gamma(1-\gamma)}{2H} B(t)B(t)' \\ & -2(1-\gamma)F(t)Q'\rho B(t)' - 2F(t)Q'QF(t) + \lambda \frac{(1-\gamma)}{H} e^{-HTr(F(t)\xi)} JB(t)', \quad F_T = 0 \\ \dot{G}_t & = -h_0 + \beta\psi - h_1\left(\psi \log(\beta) + G(t)\right) + (1-\psi)r + \frac{\lambda}{H}\left(e^{-HTr(F(t)\xi(Y(t))} - 1\right) \\ & +Tr(F(t)\Omega\Omega'), \quad G(T) = 0 \end{cases} \tag{4.9}$$

with $h_1 := \exp\left\{\overline{c - w}\right\}$ *and* $h_0 := h_1 - h_1 \log(h_1)$.

Proof. To determine the optimal policies we start from the HJB (4.4) equation associated to the investment problem, define $H := \frac{1-\gamma}{1-\psi}$, and guess

$$V(t, W, y) = \exp\{-H\left(Tr(F(t)Y(t)) + G(t)\right)\} \frac{W(t)^{1-\gamma}}{1-\gamma} . \tag{4.10}$$

Furthermore, we set $\dot{F}_t = \frac{\partial F(t)}{\partial t}$ and $\dot{G}_t = \frac{\partial G(t)}{\partial t}$, and evaluate the partial derivatives of the value function appearing in the HJB equation, namely

$$\frac{\partial V}{\partial t} = -H(Tr(\dot{F}_t Y(t)) + \dot{G}_t)V , \tag{4.11}$$

$$\frac{\partial V}{\partial W} = (1-\gamma)W(t)^{-1}V , \tag{4.12}$$

$$\nabla V = -HF(t)V \tag{4.13}$$

$$\frac{\partial^2 V}{\partial W^2} = -\gamma(1-\gamma)W(t)^{-2}V , \tag{4.14}$$

$$\nabla Q'\rho \frac{\partial V}{\partial W} = \nabla \frac{\partial V}{\partial W} Q'\rho = -HF(t)Q'\rho V(1-\gamma)W(t)^{-1} . \tag{4.15}$$

Moreover, we have

$$Tr(Y(t)\nabla Q'Q\nabla)V = -HTr(Y(t)F(t)Q'QF(t))V . \tag{4.16}$$

Finally, we use the linearization (4.5), so that the jump component in the HJB equation becomes

$$V(W(t) + \pi(t)'JW(t), Y(t) + \xi(Y(t))) - V(W(t), Y(t)) = V\Bigg(e^{-HTr(F(t)\xi(Y(t)))}(1+ \\ \pi(t)'J)^{1-\gamma} - 1\Bigg) . \tag{4.17}$$

Hence, we replace all the partial derivatives (4.11) –(4.15), the condition (4.16) and (4.17) in (4.4), so that we obtain

$$
\begin{aligned}
0 = \max_{\{\pi(t),C(t)\}} \Bigg\{ & \frac{f(C(t),V(t))}{V(t)} - HTr(\hat{F}_t Y(t)) - H\dot{G}_t + r(1-\gamma) + \pi(t)'(\alpha - r\mathbf{1})(1-\gamma) - \\
& \frac{C(t)}{W(t)}(1-\gamma) - HF(t)Tr(\Omega\Omega' + KY(t) + Y(t)K') - \frac{\gamma(1-\gamma)}{2}\pi(t)'Y(t)^{-1}\pi(t) \\
& -2H\pi(t)'F(t)Q'\rho(1-\gamma) - HTr(2Y(t)F(t)Q'QF(t)) + \lambda\Bigg[e^{-HTr(F(t)\xi(Y(t)))} \\
& (1+(1-\gamma)\pi(t)'J) - 1\Bigg]\Bigg\} =: \varphi(\pi(t),C(t)) \ .
\end{aligned} \tag{4.18}
$$

The first term in (4.18) can be written as

$$
\frac{f(C(t),V(t))}{V(t)} = H\beta\psi - \frac{H\beta\psi}{((1-\gamma)V)^{(\frac{1}{1-\gamma})(\frac{\psi-1}{\psi})}} C(t)^{\frac{\psi-1}{\psi}} \ . \tag{4.19}
$$

Hence, we are able to determine the first order condition for optimal consumption

$$
\frac{\partial\varphi}{\partial C(t)} = -\frac{H\beta\psi}{((1-\gamma)V)^{(\frac{1}{1-\gamma})(\frac{\psi-1}{\psi})}} \frac{\psi-1}{\psi} C(t)^{\frac{1}{\psi}} - \frac{(1-\gamma)}{W(t)} = 0 \ , \tag{4.20}
$$

and with some algebra we obtain (4.7). Similarly, we are able to determine the first order condition for optimal portfolio allocation

$$
\begin{aligned}
\frac{\partial\varphi}{\partial\pi(t)'} = & (\alpha - r\mathbf{1})(1-\gamma) - \gamma(1-\gamma)\pi(t)Y(t)^{-1} - 2HF(t)Q'\rho(1-\gamma) + \lambda(1-\gamma) \\
& e^{-HTr(F(t)\xi(Y(t)))} = 0 \ ,
\end{aligned} \tag{4.21}
$$

and (4.8) easily follows. Finally, we have to make explicit the expressions for F and G appearing in the value function. To do this, we replace the optimal consumption and portfolio allocation in (4.18) and we have

$$
\begin{aligned}
& \frac{A(t)}{H} - Tr(\dot{F}_t Y(t)) - \dot{G}_t + \frac{r(1-\gamma)}{H} + B_t' Y(t)\frac{(\alpha - r\mathbf{1})(1-\gamma)}{H} - F(t)Tr(\Omega\Omega' + KY(t) \\
& + Y(t)K') - \frac{\gamma(1-\gamma)}{2H}B(t)Y(t)B(t)' - 2F(t)Q'\rho Y(t)B(t)'(1-\gamma) - 2Tr(Y(t)F(t) \\
& Q'QF(t)) + \frac{\lambda}{H}\left[e^{-HTr(F(t)\xi)} - 1\right] + \frac{\lambda(1-\gamma)}{H}e^{-HTr(F(t)\xi(Y(t)))}JB(t)'Y(t) = 0 \ ,
\end{aligned} \tag{4.22}
$$

where we set

$$
B(t) := \frac{(\alpha - r\mathbf{1})}{\gamma} + \frac{2HF(t)Q'(-\rho)}{\gamma} + \frac{\lambda J e^{-HTr(F(t)\xi(Y(t)))}}{\gamma} \tag{4.23}
$$

and

$$
A(t) = \frac{f(C(t),V(t))}{V(t)} - \frac{C(t)}{W(t)}(1-\gamma) \ . \tag{4.24}
$$

With some algebra, we obtain $A(t) = H\beta\psi - H\beta^{\psi}\exp\{-(F(t)+G(t))\} = H\beta\psi - H\beta^{\psi}X^{-1}$.

The envelope condition ensures that

$$\beta^{\psi} X^{-1} = \frac{C_t}{W(t)} = \exp\{\log(C(t)) - \log(W(t))\} = \exp\{c(t) - w(t)\}, \tag{4.25}$$

so that

$$c(t) - w(t) = \log(\beta^{\psi} X^{-1}) = \psi \log(\beta) - Tr(F(t)Y(t)) - G(t). \tag{4.26}$$

Moreover, using the first-order taylor expansion of $\exp\{c(t)-w(t)\}$ around its unconditional mean $\overline{c-w}$, we have

$$A(t) \approx H\beta\psi - H\left[h_0 + h_1(c(t) - w(t))\right], \tag{4.27}$$

where $h_1 := \exp\{\overline{c-w}\}$ and $h_0 := h1 - h1\log(h1)$. By replacing (4.27) and (4.26) in (4.22), we have

$$\Bigg(- \dot{F}_t + h_1 F(t) + B(t)' \frac{(\alpha - r\mathbf{1})(1-\gamma)}{H} - F(t)K - F(t)K' - \frac{\gamma(1-\gamma)}{2H} B(t)B(t)' - 2F(t)$$
$$Q'\rho B(t)'(1-\gamma) - 2F(t)Q'QF(t) + \frac{\lambda}{H}(1-\gamma)e^{-HTr(F(t)\xi)} JB(t)' \Bigg) Tr(Y(t)) + \Bigg(- \dot{G}_t + \beta\psi$$
$$- h_0 - h_1 \left[\psi\log(\beta) - h_1 G(t)\right] + \frac{r(1-\gamma)}{H} + \frac{\lambda}{H}\left[e^{-HTr(F(t)\xi)} - 1\right] - Tr(\Omega\Omega' F(t)) \Bigg) = 0\,, \tag{4.28}$$

implying that (4.9) holds. □

An inspection of Proposition 4.1.1 shows that the optimal portfolio allocation, given by (4.8), comprises three terms: more precisely, $\frac{(\alpha - r\mathbf{1})}{\gamma}$ is the *myopic* component, $\frac{2HF(t)Q'(-\rho)}{\gamma}$ is the *intertemporal hedging* demand, and $\frac{\lambda J e^{-HTr(F(t)\xi(Y(t)))}}{\gamma}$ is the *jump hedging* demand. The first one depends on both the excess return and the risk-aversion parameter and describes the agents who completely ignore the multi-period nature of the investment. The latter is incorporated in the second term, relying on the investment opportunities σ, and the correlation between returns and volatility. The last term quantifies the hedging against tail risk and is related to the jump frequency λ and magnitudes J and ξ. While the absence of idiosyncratic risk ensures a positive myopic component, we can discuss the sign of the remaining terms. For the intertemporal hedging demand, we expect a negative correlation structure for correlated shocks to price returns and their variance-covariance matrix, since variances and covariances typically increase while prices decline. Assuming $\gamma > 1$, we distinguish two cases according to the value of EIS. For $\psi < 1$, we have $H < 0$, then the impact of the intertemporal hedging demand over the optimal consumption will be positive (resp., negative) when the function F is greater (resp., smaller) than zero. Vice versa, for $\psi > 1$, we have $H > 0$, thus the impact of the intertemporal hedging demand over the optimal consumption will be positive (resp., negative) when the function F is smaller (resp., greater) than zero. However, an in-depth analysis of Eq. (4.9) for F shows that $F(t) < 0$, for all $t \in [0, T]$, if $\psi < 1$, so that the intertemporal hedging demand curtails the optimal allocation

Remark 4.1.1. The properties of the function $F(t)$ are crucial to study the optimization problem. For this purpose, we make a change of variable by considering $\tau := T - t$, so that the ODE (4.9) admits an initial condition. We have

$$\begin{aligned}\dot{F}_\tau &= -h_1 F(\tau) - (1-\psi)(\alpha - r\mathbf{1})\left(\frac{(\alpha - r\mathbf{1})}{\gamma} + \frac{2HF(\tau)Q'(-\rho)}{\gamma} + \frac{\lambda J e^{-HTr(F(\tau)\xi(Y(t)))}}{\gamma}\right) \\ &+ F(\tau)K + K'F(\tau) + \frac{\gamma(1-\psi)}{2}\left(\frac{(\alpha - r\mathbf{1})}{\gamma} + \frac{2HF(\tau)Q'(-\rho)}{\gamma} + \frac{\lambda J e^{-HTr(F(\tau)\xi(Y(t)))}}{\gamma}\right) \\ &\left(\frac{(\alpha - r\mathbf{1})}{\gamma} + \frac{2HF(\tau)Q'(-\rho)}{\gamma} + \frac{\lambda J e^{-HTr(F(\tau)\xi(Y(t)))}}{\gamma}\right)' + 2(1-\gamma)F(\tau)Q'\rho \\ &\left(\frac{(\alpha - r\mathbf{1})}{\gamma} + \frac{2HF(\tau)Q'(-\rho)}{\gamma} + \frac{\lambda J e^{-HTr(F(\tau)\xi(Y(t)))}}{\gamma}\right)' + 2F(\tau)Q'QF(\tau) - \lambda(1-\psi)J \\ &e^{-HTr(F(\tau)\xi)}\left(\frac{(\alpha - r\mathbf{1})}{\gamma} + \frac{2HF(\tau)Q'(-\rho)}{\gamma} + \frac{\lambda J e^{-HTr(F(\tau)\xi(Y(t)))}}{\gamma}\right)', \end{aligned} \tag{4.29}$$

with $F_0 = 0$, $\dot{F}_\tau = \frac{\partial F}{\partial \tau}$ and we set

$$B(\tau) := \frac{(\alpha - r\mathbf{1})}{\gamma} + \frac{2HF(\tau)Q'(-\rho)}{\gamma} + \frac{\lambda J e^{-HTr(F(\tau)\xi(Y(t)))}}{\gamma}.$$

Thanks to the updated boundary condition, the ODE (4.29) evaluated at $\tau = 0$ can be written as

$$\begin{aligned}\dot{F}_0 &= -(1-\psi)(\alpha - r\mathbf{1})B(0) + \frac{\gamma(1-\psi)}{2}B(0)B(0)' - \lambda(1-\psi)JB(0)' \\ &= -\frac{\gamma(1-\psi)}{2}B(0)B(0)'.\end{aligned} \tag{4.30}$$

By assuming $\gamma > 1$ and $\psi < 1$, from (4.30) it is straightforward to verify that $\dot{F}(0) < 0$. Since F is continuous in γ, the previous discussion ensures that $F(\tau) < 0$. Analogously, we get $F(\tau) > 0$, for any τ, when $\gamma > 1$ and $\psi > 1$. Previous results are switched when $\gamma < 1$.

Concerning the jump hedging demand, we observe that its sign only depends on the price-jump amplitude, being positive (resp., negative) for $J > 0$ (resp., $J < 0$). On the other hand, the precision-jump component and the jump intensity affect the magnitude of the reduction/increase of the optimal allocation. More importantly, each component in (4.8) is proportional to the instantaneous precision, meaning that the allocation will reduce during volatile periods, reconciling the Markowitz intuition.

We would like to point out that, although it is taken for granted that modelling precision is equivalent to modelling variance (by assuming a 3/2-dynamics), remember the equation (2.7), this would still not be sufficient to overcome the problem of the non-dynamic nature of the investment. In the affine case, the optimal allocation depends neither on the long-run level nor the instantaneous volatility because of the choice to set the variance-risk premium proportional to volatility in the drift term. Therefore, using precision to model the state variable is not a trivial mathematical gimmick; instead, it is a requirement to obviate the imperative to impose a constraint on the drift architecture.

In addition, our investment strategy implies that, after a precision jump of size ξ, investors will change their allocation in the risky asset from $Y(t)S(t)$ to $(Y(t)+\xi(Y(t)))S(t)$. Since we impose a negative precision-jump size (to comply with the empirical evidence of positive peaks in volatility behind market crashes), we witness a reduction of risky investments after a market collapse, consistently with the current market practice. Such a circumstance is extensively discussed in Moreira and Muir [2017], where the authors argue that the mean-variance trade-off weakens in high-volatility regimes.

Finally, we provide some comments regarding the optimal consumption (4.7). As in Chacko and Viceira [2005], the log consumption-wealth ratio is an affine function of instantaneous precision. Furthermore, the role of jumps on optimal consumption cannot be directly inferred from (4.7), even though they affect $C(t)$ through functions $F(t)$ and $G(t)$, the latter being evaluated numerically.

From Proposition 4.1.1 we can obtain an approximated solution to the investment problem (4.3) when $\psi = 1$. In this case, the aggregator simplifies as (2.28). Hence, we have the following

Corollary 4.1.2. *Under the hypotheses of Proposition 4.1.1 and assuming that* (2.28) *holds true, the value function solving the investment problem* (4.3) *is*

$$V(t,W,y) = \exp\left\{Tr(\tilde{F}(t)Y(t)+\tilde{G}(t)\right\}\frac{W(t)^{1-\gamma}}{1-\gamma}\,, \tag{4.31}$$

where $\tilde{F}(t)$, $\tilde{G}(t)$ *are deterministic functions satisfying the following system of ODEs*

$$\begin{cases} \dot{\tilde{F}}_t &= \beta\tilde{F}(t) - (1-\gamma)(\alpha - r\mathbf{1})\tilde{B(t)}' - K\tilde{F}(t) - \tilde{F}(t)K' + \frac{1}{2}\gamma(1-\gamma)\tilde{B(t)}'\tilde{B(t)} \\ & -2(1-\gamma)\tilde{B(t)}'\tilde{F}(t)Q'\rho - 2Tr(\tilde{F}(t)'Q'Q\tilde{F}(t)) - \lambda(1-\gamma)e^{Tr(\tilde{F}(t)\xi(Y(t)))}J\tilde{B(t)},\ \tilde{F}(T)=0\,, \\ \dot{\tilde{G}}_t &= -\beta(1-\gamma)\log(\beta) + \beta\tilde{G}(t) - (1-\gamma)r + (1-\gamma)\beta - Tr(\tilde{\Omega}(t)\tilde{\Omega}(t)') \\ & +\lambda(1-e^{Tr(\tilde{F}(t)\xi(Y(t)))}),\ \tilde{G}(T)=0\,. \end{cases} \tag{4.32}$$

Moreover, the optimal consumption and portfolio rules are, for $t \in [0,T]$,

$$\frac{C(t)}{W(t)} = \beta\,, \tag{4.33}$$

$$\pi(t) = Y(t)\left(\frac{(\alpha - r\mathbf{1})}{\gamma} + \frac{2\tilde{F}(t)Q'\rho}{\gamma} + \frac{\lambda J e^{Tr(\tilde{F}(t)\xi(Y(t)))}}{\gamma}\right) =: Y(t)\tilde{B}(t)\,. \tag{4.34}$$

Proof. To determine the optimal policies in the special case in which $\psi = 1$ we start from the HJB equation (4.4). We guess

$$V(t,W(t),Y(t)) = \exp\{Tr(\tilde{F}(t)Y(t)) + \tilde{G}(t)\}\frac{W(t)^{1-\gamma}}{1-\gamma}. \tag{4.35}$$

We evaluate the partial derivatives of the value function and obtain

$$\frac{\partial V}{\partial t} = (Tr(\dot{\tilde{F}}(t)Y(t)) + \dot{\tilde{G}}(t))V \ , \tag{4.36}$$

$$\frac{\partial V}{\partial W} = VW(t)^{-1}(1-\gamma) \ , \tag{4.37}$$

$$\nabla V = \tilde{F}(t)V \tag{4.38}$$

$$\frac{\partial^2 V}{\partial W^2} = -V\gamma(1-\gamma)W(t)^{-2} \ , \tag{4.39}$$

$$\nabla Q'\rho\frac{\partial V}{\partial W} = \nabla\frac{\partial V}{\partial W}Q'\rho = \tilde{F}(t)Q'\rho V(1-\gamma)W(t)^{-1} \tag{4.40}$$

where $\dot{\tilde{F}}_t = \frac{\partial \tilde{F}(t)}{\partial t}$, $\dot{\tilde{G}}_t = \frac{\partial \tilde{G}(t)}{\partial t}$, and $Tr(Y(t)\nabla Q'Q\nabla)V = Tr(Y(t)\tilde{F}(t)Q'Q\tilde{F}(t))V$. Thanks to the linearization (4.5), the jump component in the HJB equation can still be written as

$$V(W(t)+\pi(t)'JW(t), Y(t)+\xi(Y(t))) - V(W(t), Y(t)) = V\left(e^{Tr(\tilde{F}(t)\xi(Y(t)))}(1+\pi(t)'J)^{1-\gamma} - 1\right) \ . \tag{4.41}$$

When $\psi = 1$ the normalized aggregator is given by equation (2.28), that is, it has the following form

$$f(C,V) = \beta(1-\gamma)V\left(\ln(C) - \frac{1}{1-\gamma}\ln((1-\gamma)V)\right) \ . \tag{4.42}$$

If we replace the partial derivatives (4.36) – (4.40) in (4.4) and the normalized aggregator (2.28) in (4.4), we obtain,

$$\begin{aligned} 0 = \max_{\{\pi(t),C(t)\}} \Bigg\{ & \frac{f(C(t),V(t))}{V} + Tr(\dot{\tilde{F}}_tY(t)) + \dot{\tilde{G}}_t + (1-\gamma)\pi(t)'(\alpha - r\mathbf{1}) + r(1-\gamma) \\ & - \frac{C(t)}{W(t)}(1-\gamma) + Tr(\Omega\Omega' + KY(t) + Y(t)K')\tilde{F}(t) - \frac{1}{2}\pi(t)'Y(t)^{-1}\pi(t)\gamma(1-\gamma) + \\ & 2\pi(t)'\tilde{F}(t)Q'\rho(1-\gamma) + 2Tr(Y(t)\tilde{F}(t)'Q'Q\tilde{F}(t))) + \lambda\left[e^{Tr(\tilde{F}(t)\xi(Y(t)))}(1+(1-\gamma)J\pi(t)') - 1\right]\Bigg\} \\ & =: \tilde{\varphi}(\pi(t), C(t)) \ . \end{aligned} \tag{4.43}$$

We determine the first order condition for optimal consumption

$$\frac{\partial\tilde{\varphi}}{\partial C(t)} = \frac{\beta(1-\gamma)}{C} - \frac{(1-\gamma)}{W(t)} = 0 \tag{4.44}$$

and (4.33) easily follows. Similarly, we are able to determine the first order condition for optimal portfolio allocation

$$\frac{\partial\tilde{\varphi}}{\partial\pi(t)} = (1-\gamma)(\alpha - r\mathbf{1}) - Y(t)^{-1}\pi(t)\gamma(1-\gamma) + 2\tilde{F}(t)Q'\rho(1-\gamma) + \lambda e^{Tr(\tilde{F}(t)\xi(Y(t)))}(1-\gamma)J = 0 \ , \tag{4.45}$$

so that we obtain the optimal portfolio rule (4.34). By using the optimal rules and with some algebra we obtain

$$(-\beta Tr(\tilde{F}(t)Y(t)) + Tr(\dot{\tilde{F}}_t Y(t)) + (1-\gamma)(\alpha - r\mathbf{1})\tilde{B}(t)'Y(t) + KTr(\tilde{F}(t)Y(t)) + Tr(\tilde{F}(t)Y(t))$$
$$K' - \frac{1}{2}\tilde{B}(t)Y(t)\tilde{B}(t)'\gamma(1-\gamma) + 2Q'\rho\tilde{B}(t)'Tr(\tilde{F}(t)Y(t))(1-\gamma) + 2Tr(Y(t)\tilde{F}(t)'Q'Q\tilde{F}(t)) +$$
$$\lambda e^{Tr(\tilde{F}(t)\xi(Y(t)))}(1-\gamma)J\tilde{B}(t)'Y(t) + \beta(1-\gamma)(\ln(\beta)-1) - \beta\tilde{G}(t) + \dot{\tilde{G}}_t + r(1-\gamma) + Tr(\Omega\Omega')$$
$$+ \lambda(e^{Tr(\tilde{F}(t)\xi(Y(t)))} - 1) = 0\,. \tag{4.46}$$

□

4.2 Univariate case

Now we want to analyze the model in the case where we have only one risky asset. We consider three stochastic processes $M = \{M(t)\}_{t\in[0,T]}$, $S = \{S(t)\}_{t\in[0,T]}$, and $y = \{y(t)\}_{t\in[0,T]}$ on a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\in[0,T]}, \mathbb{P}\right)$, see equation (2.1), (2.2) and (2.3), representing the risk-free asset, the stock and the state variable, respectively. The process $\bar{B}$ is correlated with the Wiener process B through equation (2.4).
We assume that the investor's wealth $W = \{W(t)\}_{t\in[0,T]}$ is apportioned among the risk-free asset M and the stock S, also considering the related consumption $C = \{C(t)\}_{t\in[0,T]}$, so that we have

$$\frac{dW(t)}{W(t)} = \epsilon(t)\frac{dM(t)}{M(t)} + \pi(t)\frac{dS(t)}{S(t)} - \frac{C(t)}{W(t)},\; t \in [0,T]\,, \tag{4.47}$$

where $\epsilon(t)$ and $\pi(t)$ represent the proportion of wealth invested at time t into the risk-free asset and the stock, respectively. Replacing the dynamics (2.1) and (2.2) in (4.47) and observing that $\epsilon(t) = 1 - \pi(t)$, we get the following budget constraint

$$dW(t) = [rW(t) + (\mu - r)\pi(t)W(t) - C(t)]dt + \pi(t)W(t)\sqrt{y(t)^{-1}}dB(t) + \pi(t)JdN(t),$$
$$t \in [0,T]\,. \tag{4.48}$$

The investment problem to be solved is given by (4.3) with (4.48) as intertemporal budget constraint.
As in the multivariate case, to find a solution to the intertemporal optimization problem we use standard dynamic programming techniques, so that we consider the associated Hamilton-Jacobi-Bellman equation (HJB, from now on), given by

$$0 = \max_{\{\pi(t),C(t)\}} \Bigg\{ f(C,V) + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial W}[rW(t) + (\mu - r)\pi(t)W(t) - C(t)] + \frac{\partial V}{\partial y}k(\theta - y(t))$$
$$+ \frac{1}{2}\frac{\partial^2 V}{\partial W^2}W_t^2\pi(t)^2\frac{1}{y(t)} + \frac{1}{2}\frac{\partial^2 V}{\partial y^2}\sigma^2 y(t) + \sigma\rho\frac{\partial^2 V}{\partial W\partial y}\pi(t)W(t) + \lambda\mathbb{E}\Bigg[V(t,(1+J\pi(t))W,$$
$$y + \xi(y(t))) - V(t,W,y)\Bigg]\Bigg\}\,. \tag{4.49}$$

Since HJB is non-linear, we impose the same constraint as in multivariate case: the consumption-wealth ratio is approximated using an appropriate log-linear expansion around the unconditional mean, as in Chacko and Viceira [2005] while for the jump component we exploit a first-order Taylor expansion around $J\pi_t = 0$, as in Oliva and Renò [2018], look at section (2.2) for more details. We guess that the solution to (4.49) is of the form

$$V(t, W, y) = \exp\{-H(F(t)y(t) + G(t))\} \frac{W(t)^{1-\gamma}}{1-\gamma}, \tag{4.50}$$

where $H = \frac{1-\gamma}{1-\psi}$ and F and G are deterministic function to be identified.

Proposition 4.2.1. *Consider the investment problem* (4.3) *with* $\psi \neq 1$ *and the associated HJB equation* (4.49)*. Then, for* $t \in [0, T]$*, the value function is given in* (4.50)*, while the optimal consumption and portfolio rules are given by*

$$\frac{C(t)}{W(t)} = \beta^{\psi} \exp\{-(F(t)y(t) + G(t))\}, \tag{4.51}$$

$$\pi_t = \left(\frac{\mu - r}{\gamma} + \frac{H\sigma(-\rho)F(t)}{\gamma} + \frac{\lambda J e^{-HF(t)\xi(y(t))}}{\gamma}\right) y(t) =: B(t)y(t), \tag{4.52}$$

where $H = \frac{1-\gamma}{1-\psi}$ *and* F, G *are deterministic functions satisfying the following system of ODEs*

$$\begin{cases} \dot{F}(t) & = h_1 F(t) + (1-\psi)(\mu - r)B(t) + kF(t) - \frac{1}{2}\gamma(1-\psi)B(t)^2 + \frac{1}{2}H\sigma^2 F(t)^2 \\ & -(1-\gamma)\sigma\rho F(t)B(t) + \lambda(1-\psi)Je^{-HF(t)\xi(y(t))}B(t), \\ & F(T) = 0 \\ \dot{G}(t) & = h_1 G(t) - h_1\psi\log\beta + \beta\psi - h_0 + (1-\psi)r - k\theta F(t) + \frac{\lambda}{H}\left(e^{-HF(t)\xi(y(t))} - 1\right), \\ & G(T) = 0 \end{cases}, \tag{4.53}$$

with $h_1 := \exp\{\overline{c - w}\}$ *and* $h_0 := h_1 - h_1\log(h_1)$.

Proof. To determine the optimal policies we start from the HJB equation (4.49), define $H := \frac{1-\gamma}{1-\psi}$ and guess

$$V(t, W, y) = \exp\{-H(F(t)y(t) + G(t))\} \frac{W(t)^{1-\gamma}}{1-\gamma}. \tag{4.54}$$

Furthermore, we set $\dot{F}_t = \frac{\partial F(t)}{\partial t}$ and $\dot{G}_t = \frac{\partial G(t)}{\partial t}$, and evaluate the partial derivatives of the value function appearing in the HJB equation, namely

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= -H(\dot{F}_t y(t) + \dot{G}_t)V \ , && (4.55)\\
\frac{\partial V}{\partial W} &= \exp\left\{-H(F(t)y(t) + G(t))\right\} W(t)^{-\gamma} \ , && (4.56)\\
\frac{\partial V}{\partial y} &= -HF(t)V && (4.57)\\
\frac{\partial^2 V}{\partial W^2} &= -\gamma \exp\left\{-H(F(t)y(t) + G(t))\right\} W(t)^{-\gamma-1} \ , && (4.58)\\
\frac{\partial^2 V}{\partial y^2} &= H^2 F(t)^2 V \ , && (4.59)\\
\frac{\partial^2 V}{\partial W \partial y} &= -HF(t) \exp\left\{-H(F(t)y(t) + G(t))\right\} W(t)^{-\gamma} \ . && (4.60)
\end{aligned}
$$

Finally, we use the linearization (4.5), so that the jump component in the HJB equation becomes

$$
V(t, (1 + J\pi)W, y + \xi) - V(t, W, y) = V \left(e^{-HF(t)\xi} (1 + \pi(t)J)^{1-\gamma} - 1 \right) \ . \qquad (4.61)
$$

Hence, we replace all the partial derivatives (4.55) – (4.60) and (4.61) in (4.49), so that we obtain

$$
\begin{aligned}
0 = \max_{\{\pi(t), C(t)\}} & \Big\{ f(C(t), V(t)) - H(\dot{F}_t y(t) + \dot{G}_t) V(t) + r \exp\{-H\left(F(t)y(t) + G(t)\right)\} W(t)^{1-\gamma} \\
& -HF(t)k(\theta - y(t))V(t) + (\mu - r) \exp\{-H\left(F(t)y(t) + G(t)\right)\} \pi(t) W(t)^{1-\gamma} - C(t) \\
& \exp\{-H\left(F(t)y(t) + G(t)\right)\} W(t)^{-\gamma} - \frac{\gamma}{2}\pi(t)^2 \frac{1}{y(t)} \exp\{-H\left(F(t)y(t) + G(t)\right)\} W(t)^{1-\gamma} + \\
& \frac{1}{2} H^2 F(t)^2 \sigma^2 y(t) V_t - \sigma\rho HF(t)\pi \exp\{(F(t)y(t) + G(t))\} W(t)^{1-\gamma} + \lambda V(t) \Big[e^{-HF(t)\xi(y(t))} \\
& -1 + (1-\gamma) e^{-HF(t)\xi(y(t))} J\pi \Big] \Big\} \qquad (4.62) \\
= \max_{\{\pi(t), C(t)\}} & \left\{ \frac{f(C(t), V(t))}{V(t)} - H(\dot{F}_t y(t) + \dot{G}_t) + r(1-\gamma) - HF(t)k(\theta - y(t)) + (\mu - r) \right. \\
& (1-\gamma)\pi(t) - \frac{C(t)}{W(t)}(1-\gamma) - \frac{\gamma(1-\gamma)}{2} \frac{\pi(t)^2}{y(t)} + \frac{1}{2} H^2 F(t)^2 \sigma^2 y(t) - \sigma\rho HF(t)\pi(1-\gamma) \\
& \left. + \lambda \left[e^{-HF(t)\xi} - 1 + (1-\gamma) e^{-HF(t)\xi(y(t))} J\pi \right] \right\} =: \varphi(C(t), \pi(t)) \ . \qquad (4.63)
\end{aligned}
$$

Moreover, the first term in (4.63) can be written as in (4.19), hence we are able to determine the first order condition for optimal consumption

$$
\frac{\partial \varphi}{\partial C(t)} = -\frac{H\beta\psi}{\left[(1-\gamma)V(t)\right]^{\frac{1}{1-\gamma}\left(1-\frac{1}{\psi}\right)}} C(t)^{-\frac{1}{\psi}} - \frac{1-\gamma}{W(t)} = 0 \ ,
$$

and with some algebra we obtain (4.51). Similarly, we are able to determine the first order condition for optimal portfolio allocation

$$\frac{\partial \varphi}{\partial \pi(t)} = (\mu - r)(1-\gamma) - \gamma(1-\gamma)\frac{\pi(t)}{y(t)} - \sigma\rho H F(t)(1-\gamma) + \lambda e^{-HF(t)\xi(y(t))} J(1-\gamma) = 0 \ ,$$

and (4.52) easily follows.

Finally, we have to make explicit the expressions for F and G appearing in the value function. To do this, we replace the optimal consumption and portfolio allocation in (4.63) and we have

$$\begin{aligned} &-A(t) - \dot{F}_t y(t) - \dot{G}_t + \frac{1-\gamma}{H} r + \frac{1-\gamma}{H}(\mu - r)B(t)y(t) - k\theta F(t) + kF(t)y(t) - \frac{1}{2}\frac{\gamma(1-\gamma)}{H} \\ &B(t)^2 y(t) + \frac{1}{2}\sigma^2 H F(t) t^2 y(t) - (1-\gamma)\sigma\rho F(t)B(t)y(t) + \frac{1-\gamma}{H}\lambda e^{-HF(t)\xi} J B(t) y(t) + \frac{\lambda}{H} \\ &\left[e^{-HF(t)\xi} - 1\right] = 0 \ , \end{aligned} \tag{4.64}$$

where we set

$$B_t := \frac{\mu - r}{\gamma} + \frac{\sigma H(-\rho)F(t)}{\gamma} + \frac{\lambda \mathbb{E}\left[e^{-HF(t)\xi} J\right]}{\gamma} \ , \tag{4.65}$$

and

$$\begin{aligned} A(t) :=& \beta\psi \left[\left(\frac{\beta^{\psi} \exp\left\{-(F(t)y(t) + G(t))\right\} W(t)}{[(1-\gamma)V]^{\frac{1}{1-\gamma}}} \right)^{1-\frac{1}{\psi}} - 1 \right] + \frac{1-\gamma}{H}\beta^{\psi} \\ & \exp\left\{-(F(t)y(t) + G(t)))\right\} \ . \end{aligned} \tag{4.66}$$

With some algebra, we obtain $A(t) = \beta\psi - \beta^{\psi} \exp\left\{-(F(t)y(t) + G(t)))\right\} =: \beta\psi - \beta^{\psi} X^{-1}$. The envelope condition ensures that

$$\beta^{\psi} X^{-1} = \frac{C(t)}{W(t)} = \exp\left\{\log(C(t)) - \log(W(t))\right\} = \exp\{c(t) - w(t)\} \ , \tag{4.67}$$

so that

$$c(t) - w(t) = \log\left(\beta^{\psi} X^{-1}\right) = \psi \log(\beta) - F(t)y(t) - G(t) \ . \tag{4.68}$$

Moreover, using the first-order Taylor expansion of $\exp\{c(t) - w(t)\}$ around its unconditional mean $\overline{c - w}$, we have

$$A(t) \approx \beta\psi - h_0 - h_1(c(t) - w(t)) \ , \tag{4.69}$$

where $h_1 := \exp\{\overline{c - w}\}$ and $h_0 := h_1 - h_1 \log(h_1)$. By replacing (4.69) and (4.68) in (4.64), we have

$$\Bigg(- \dot{F}_t + h_1 F(t) + \frac{1-\gamma}{H}(\mu - r)B(t) + kF(t) - \frac{1}{2}\frac{\gamma(1-\gamma)}{H}B(t)^2 + \frac{1}{2}\sigma^2 H F(t)^2 -$$

$$(1-\gamma)\sigma\rho B(t)F(t) + \frac{1-\gamma}{H}\lambda J e^{-HF(t)\xi(y(t))}B(t)\Bigg) y(t) + \Bigg(- \dot{G}_t - h_0 + \beta\psi - h_1(\psi \log(\beta) -$$

$$G(t)) + \frac{1-\gamma}{H} r - k\theta F(t) + \frac{1}{H}\lambda\left[e^{-HF(t)\xi(y(t))} - 1\right]\Bigg) = 0 \ ,$$

implying that (4.53) holds true. □

Similar comments apply to the multivariate case concerning optimal weights and consumption-wealth ratio. From Proposition 4.2.1 we can obtain an approximated solution to the investment problem when $\psi = 1$. In this case, the aggregator simplifies as (2.28). Hence, we have the following

Corollary 4.2.2. *Under the hypotheses of Proposition 4.2.1 and assuming that* (2.28) *holds true, the value function solving the investment problem* (4.3) *is*

$$V(t, W, y) = \exp\left\{\tilde{F}(t)y(t) + \tilde{G}(t)\right\}\frac{W(t)^{1-\gamma}}{1-\gamma} \ , \tag{4.70}$$

where $\tilde{F}(t)$, $\tilde{G}(t)$ are deterministic functions satisfying the following system of ODEs

$$\begin{cases} \dot{\tilde{F}}_t = \beta\tilde{F}(t) - (1-\gamma)(\mu - r)\tilde{B(t)} + \tilde{F}(t)k + \frac{1}{2}\gamma(1-\gamma)\tilde{B(t)}^2 - \frac{1}{2}\tilde{F}(t)^2\sigma^2 \\ -(1-\gamma)\tilde{F}(t)\sigma\rho\tilde{B(t)} - \lambda(1-\gamma)e^{\tilde{F}(t)\xi(y(t))}J\tilde{B(t)}, \ \tilde{F}(T) = 0 \\ \dot{\tilde{G}}_t = -\beta(1-\gamma)\log(\beta) + \beta\tilde{G}(t) - (1-\gamma)r + (1-\gamma)\beta - \tilde{F}(t)k\theta + \lambda(1 - e^{\tilde{F}(t)\xi(y(t))}), \\ \tilde{G}(T) = 0 \end{cases} . \tag{4.71}$$

Moreover, the optimal consumption and portfolio rules are, for $t \in [0, T]$,

$$\frac{C(t)}{W(t)} = \beta \ , \tag{4.72}$$

$$\pi(t) = \left(\frac{\mu - r}{\gamma} + \frac{\sigma\rho\tilde{F}(t)}{\gamma} + \frac{\lambda J e^{\tilde{F}(t)\xi(y(t))}}{\gamma}\right) y(t) =: \tilde{B}(t)y(t) \ . \tag{4.73}$$

Proof. To determine the optimal policies in the special case in which $\psi = 1$ we start from the HJB equation (4.49). We guess

$$V(t, W(t), y) = \exp\{\tilde{F}(t)y + \tilde{G}(t)\}\frac{W(t)^{1-\gamma}}{1-\gamma}. \tag{4.74}$$

We evaluate the partial derivatives of the value function and obtain

$$\frac{\partial V}{\partial t} = (\dot{\tilde{F}}_t y(t) + \dot{\tilde{G}}_t)V \ , \tag{4.75}$$
$$\frac{\partial V}{\partial W} = \exp\left\{\tilde{F}(t)y(t) + \tilde{G}(t)\right\} W(t)^{-\gamma} \ , \tag{4.76}$$
$$\frac{\partial V}{\partial y} = \tilde{F}(t)V \tag{4.77}$$
$$\frac{\partial^2 V}{\partial W^2} = -\gamma \exp\left\{\tilde{F}(t)y(t) + \tilde{G}(t)\right\} W(t)^{-\gamma-1} \ , \tag{4.78}$$
$$\frac{\partial^2 V}{\partial y^2} = \tilde{F}(t)^2 V \ , \tag{4.79}$$
$$\frac{\partial^2 V}{\partial W \partial y} = \tilde{F}(t) \exp\left\{\tilde{F}(t)y(t) + \tilde{G}(t)\right\} W(t)^{-\gamma} \ , \tag{4.80}$$

with $\dot{\tilde{F}}_t = \frac{\partial \tilde{F(t)}}{\partial t}$ and $\dot{\tilde{G}}_t = \frac{\partial \tilde{G(t)}}{\partial t}$. Thanks to the linearization (4.5), the jump component in the HJB equation can still be written as

$$V(t, (1 + J\pi(t))W(t), y(t) + \xi(y(t))) - V(t, W(t), y(t)) = V\left(e^{\tilde{F}(t)\xi(y(t))}(1 + \pi(t)J)^{1-\gamma} - 1\right) \ . \tag{4.81}$$

When $\psi = 1$ the normalized aggregator (2.27) is simplified as in the equation,(2.28). If we replace the partial derivatives (4.75) – (4.80), (4.81) and the normalized aggregator (2.28) in (4.49), we obtain,

$$\begin{aligned} 0 = \max_{\pi(t), C(t)} \Bigg\{ & \frac{f(C(t), V(t))}{V(t)} + (\dot{\tilde{F}}_t y(t) + \dot{\tilde{G}}_t) + r(1-\gamma) + \pi(t)(\mu - r)(1-\gamma) - \frac{C(t)}{W(t)}(1-\gamma) \\ & + \tilde{F}(t)k(\theta - y(t)) - \frac{\gamma(1-\gamma)}{2}\frac{\pi(t)^2}{y(t)} + \frac{1}{2}\tilde{F}(t)^2\sigma^2 y(t) + (1-\gamma)\tilde{F}(t)\sigma\rho\pi(t) + \lambda\Bigg[e^{\tilde{F}(t)\xi(y(t))} - 1 \\ & + (1-\gamma)e^{\tilde{F}(t)\xi(y(t))}J\pi(t)\Bigg]\Bigg\} =: \tilde{\varphi}(C(t), \pi(t)) \ . \end{aligned} \tag{4.82}$$

We determine the first order condition for optimal consumption

$$\frac{\partial \tilde{\varphi}}{\partial C(t)} = \frac{\beta(1-\gamma)}{C(t)} - \frac{1-\gamma}{W(t)} = 0 \ ,$$

and (4.72) easily follows. Similarly, we are able to determine the first order condition for optimal portfolio allocation

$$\frac{\partial \varphi}{\partial \pi(t)} = (1-\gamma)(\mu - r) - \frac{\gamma(1-\gamma)}{y(t)}\pi(t) + (1-\gamma)\sigma\rho\tilde{F}(t) + \lambda(1-\gamma)Je^{\tilde{F}(t)\xi(y(t))} = 0,$$

so that we obtain the optimal portfolio rule (4.34).

By using the optimal rules in (4.82) and with some algebra we obtain

$$
\begin{aligned}
&\left(\dot{\tilde{F}}_t + \left[\frac{\sigma^2}{2\gamma} (\gamma + (1-\gamma)\rho^2) \right] \tilde{F}(t)^2 + \left(\frac{(\mu - r)(1-\gamma)\rho\sigma}{\gamma} - (\beta + k) \right) \tilde{F}(t) \right. \\
&\left. + \lambda e^{\tilde{F}(t)\xi(y(t))} J \left[\frac{(\mu - r)(1-\gamma)}{\gamma} + \frac{\rho\sigma(1-\gamma)\tilde{F}(t)}{\gamma} + \frac{1}{2} \frac{\lambda(1-\gamma) J e^{F(t)\xi(y(t))}}{\gamma} \right] + \frac{(1-\gamma)(\mu - r)^2}{2\gamma} \right) y(t) \\
&+ \dot{\tilde{G}}_t + (1-\gamma)(\beta \log(\beta) - \beta + r) - \beta \tilde{G} + \tilde{F}(t) k\theta + \lambda \left(e^{\tilde{F}(t)\xi(y(t))} - 1 \right) = 0. \qquad (4.83)
\end{aligned}
$$

□

4.3 Adding derivatives to complete the market

The model we have so far analyzed with recursive preferences assumes an incomplete market setting. Suppose we want to complete the market by allowing the investor to invest in derivatives. In this way, the investor will allocate his wealth between a risk-free investment, a risky asset, and derivative instruments. We start with the model (2.10) described in the section (2) and assume that to complete the market, we need only two derivatives, the dynamics of which are expressed by the equation (2.20). We always consider the investment problem given by the equation (4.3).
The investment problem is subject to the intertemporal budget constraint given by the wealth dynamic.
We denote by $\epsilon(t)$, $\phi(t)$, $\pi(t)^{(i)}$, $t \in [0,T]$ the proportion of wealth, $W = \{W(t)\}_{t\in[0,T]}$, invested in the riskless asset, the risky asset and the non-redundant i-th derivative, $i = 1, 2$, respectively, also considering the related consumption $C = \{C(t)\}_{t\in[0,T]}$. Thus, by observing that $\epsilon(t) + \phi(t) + \sum_{i=1}^{2} \pi(t)^{(i)} = 1$, the dynamic of the wealth process is

$$
\begin{aligned}
\frac{dW(t)}{W(t)} &= \epsilon(t)\frac{dM(t)}{M(t)} + \phi(t)\frac{dS(t)}{S(t)} + \sum_{i=1}^{2} \pi(t)^{(i)} \frac{dO(t)^{(i)}}{O(t)^{(i)}} - \frac{C(t)}{W(t)} \\
&= \epsilon(t)rdt + \phi(t)\left[rdt - \lambda^{\mathbb{Q}} Jdt + \sqrt{y(t)^{-1}} dB(t)^{\mathbb{Q}} + JdN(t)\right] + \\
&\sum_{i=1}^{2} \pi(t)^{(i)} \left[rdt + \sqrt{y(t)^{-1}} \left(S(t)\frac{g_s^{(i)}}{O^{(i)}} + \sigma\rho \frac{g_y^{(i)}}{O^{(i)}} y \right) dB(t)^{\mathbb{Q}} \right. \\
&\left. + \sigma\sqrt{1-\rho^2}\sqrt{y(t)}\frac{g_y^{(i)}}{O^i} dZ(t)^{\mathbb{Q}} + \frac{\Delta g^{(i)}}{O^{(i)}} J\left(dN(t) - \lambda^{\mathbb{Q}} dt\right) \right] - \frac{C(t)}{W(t)}dt \\
&= r\left(\epsilon(t) + \phi(t) + \sum_{i=1}^{2} \pi(t)^{(i)}\right) dt - \frac{C(t)}{W(t)}dt - \phi(t)\lambda^{\mathbb{Q}} Jdt + \phi(t)\sqrt{y(t)^{-1}} \\
&dB(t)^{\mathbb{Q}} + \phi(t)JdN(t) + \sqrt{y(t)^{-1}} \sum_{i=}^{2} \pi(t)^{(i)} \left(S(t)\frac{g_s^{(i)}}{O^{(i)}} + \sigma\rho y \frac{g_y^{(i)}}{O^{(i)}} \right) dB(t)^{\mathbb{Q}} + \\
&\sigma\sqrt{1-\rho^2}\sqrt{y(t)} \sum_{i=1}^{2} \pi(t)^{(i)} \frac{g_y^{(i)}}{O^{(i)}} dZ(t)^{\mathbb{Q}} + \sum_{i=1}^{2} \pi(t)^{(i)} \frac{\Delta g^{(i)}}{O^{(i)}} J\left(dN(t) - \lambda^{\mathbb{Q}} Jdt\right) \quad (4.84) \\
&= rdt - \frac{C}{W(t)}dt + \left[\phi(t) + \sum_{i=1}^{2} \pi(t)^{(i)} \left(S(t)\frac{g_s^{(i)}}{O^{(i)}} + \sigma\rho y \frac{g_y^{(i)}}{O^{(i)}} \right) \sqrt{y(t)^{-1}} dB(t)^{\mathbb{Q}} \right] \\
&+ \sigma\sqrt{1-\rho^2}\sqrt{y(t)} \sum_{i=1}^{2} \pi(t)^{(i)} \frac{g_y^{(i)}}{O^{(i)}} dZ(t)^{\mathbb{Q}} + \left[\phi(t) + \sum_{i=1}^{2} \pi(t)^{(i)} \frac{\Delta g^{(i)}}{O^{(i)}}\right] \left(JdN(t) - J\lambda^{\mathbb{Q}}\right).
\end{aligned}
$$

For any $t \in [0,T]$, we define the *risk exposures*

$$\theta^B(t) := \phi(t) + \sum_{i=1}^{2} \pi(t)^{(i)} \left(S(t) \frac{g_s^{(i)}}{O(t)^{(i)}} + \sigma\rho \frac{g_y^{(i)}}{O(t)^{(i)}} y(t) \right) , \tag{4.85a}$$

$$\theta^Z(t) := \sigma\sqrt{1-\rho^2} \sum_{i=1}^{2} \pi(t)^{(i)} \frac{g_y^{(i)}}{O(t)^{(i)}} y(t) , \tag{4.85b}$$

$$\theta^N(t) := \phi(t) + \sum_{i=1}^{2} \pi(t)^{(i)} \frac{\Delta g^{(i)}}{O(t)^{(i)}} , \tag{4.85c}$$

so that, thanks to (4.85a), (4.85b) and (4.85c), the wealth $\mathbb{Q}$-dynamics can be written as follows

$$dW(t) = rW(t)dt + \theta^B(t)W(t)\sqrt{y(t)^{-1}}dB(t)^{\mathbb{Q}} + \theta^Z(t)W(t)\sqrt{y(t)^{-1}}dZ(t)^{\mathbb{Q}} + \theta^N(t)W(t)(JdN(t)^{\mathbb{Q}} - \lambda^{\mathbb{Q}}Jdt) - C(t)dt, \; t \in [0,T] .$$

Finally, by using (2.8a) and (2.8b), we obtain

$$dW(t) = W(t)\left[r + \theta^B(t)\eta + \theta^Z(t)\nu - \theta^N(t)\lambda^{\mathbb{Q}}J - \frac{C(t)}{W(t)} \right] dt + \theta^B(t)W(t)\sqrt{y(t)^{-1}}dB(t) + \theta^Z(t)W(t)\sqrt{y(t)^{-1}}dZ(t) + \theta^N(t)W(t)JdN(t)^{\mathbb{Q}}, \; t \in [0,T] . \tag{4.86}$$

To find a solution to the intertemporal optimization problem (4.3) we use standard dynamic programming techniques. The associated Hamilton-Jacobi-Bellman equation (HJB) is

$$\begin{aligned} 0 = \max_{(u(t),C(t))_{t \in [0,T]}} & \left\{ f(C,V) + V(t) + V_W W(t) \left(r + \theta^B(t)\eta + \theta^Z(t)\nu - \theta^N(t)\lambda^{\mathbb{Q}}J - \frac{C(t)}{W(t)} \right) + \right. \\ & V_y \left[k^{\mathbb{Q}}(\theta^{\mathbb{Q}} - y(t)) + \sigma \left(\rho\eta + \nu\sqrt{1-\rho^2} \right) y(t) \right] + \frac{1}{2} V_{WW} W_t^2 \left[\left(\theta^B(t)\right)^2 + \left(\theta^Z(t)\right)^2 \right] y(t)^{-1} \\ & \left. + \frac{1}{2} V_{yy}\sigma^2 y(t) + V_{Wy}W_t\sigma \left[\rho\theta^B(t) + \sqrt{1-\rho^2}\theta^Z(t) \right] + \lambda\Delta V \right\} , \end{aligned} \tag{4.87}$$

with $u(t) = \left(\phi(t), \pi(t)^{(1)}, \pi(t)^{(2)}\right)$, and $V(t) = \frac{\partial V}{\partial t}, V_W = \frac{\partial V}{\partial W}, V_y = \frac{\partial V}{\partial y}, V_{WW} = \frac{\partial^2 V}{\partial W^2}, V_{yy} = \frac{\partial^2 V}{\partial y^2}, V_{Wy} = \frac{\partial^2 V}{\partial W \partial y}, \Delta V = \left[V\left((1 + J\theta^N(t))W(t-), y(t-) + \xi(y(t-)), t \right) - V(W(t-), y(t-), t) \right]$. The HJB equation is nonlinear due to the aggregator, f, and the jump component. Regarding the f aggregator, we proceed according to the approach presented in section (4.1), where the wealth-consumption ratio is approximated using an appropriate log-linear expansion around the unconditional mean, $\overline{c_t - w_t} := \mathbb{E}\left[\frac{C(t)}{W(t)}\right]$. In section (4.5) we show that the approximation error is negligible in an incomplete market setting: this continues to hold if we complete the market.

Proposition 4.3.1. *Consider the investment problem* (4.3) *and the associated HJB equation* (4.87)*. Then. for any* $t \in [0.T]$*, the value function, for* $\psi \neq 1$*, is*

$$V(t, W, y) = \exp\{-H\left(F(T-t)y_t + G(T-t)\right)\} \frac{W(t)^{1-\gamma}}{1-\gamma}, \tag{4.88}$$

where $H = \frac{1-\gamma}{1-\psi}$, *and*

$$F(\tau) = \frac{\delta}{H}\left(\frac{1-e^{k_2\tau}}{(k_2-k_1)(e^{k_2\tau}-1)+2k_2}\right) , \tag{4.89}$$

$$\begin{aligned} G(\tau) &= \psi\log(\beta) + \frac{h_0}{h_1} + -\frac{\beta\psi}{h_1} - \frac{r(1-\gamma)}{Hh_1} - \frac{\lambda^{\mathbb{Q}}(1-\gamma)}{h_1} + e^{-h_1 t}\int_0^\tau F(s)k^{\mathbb{Q}}\theta^{\mathbb{Q}}e^{h_1 s}ds \\ &+ e^{-h_1 t}\left\{\int_0^\tau \left[\lambda^{\mathbb{Q}}J(1-\gamma)\theta^N(t)(s) - \lambda e^{F(s)\xi(y(t))}(1+\theta^N(t)(s)J)^{1-\gamma}\right]e^{h_1 s}ds\right\} , \end{aligned} \tag{4.90}$$

with $\tau = T - t$ *and we can define*

$$\delta := \frac{1-\gamma}{\gamma}\left(\eta^2+\nu^2\right),\ k_1 := (k^{\mathbb{Q}}+h_1) + \frac{\sigma}{\gamma}\left(\rho\eta + \nu\sqrt{1-\rho^2}\right),\ k_2 := \sqrt{k_1^2 - \frac{\sigma^2\delta}{\gamma}} . \tag{4.91}$$

The optimal risk exposures to diffusive $B(t)$, *volatility* $Z(t)$ *and jump risk factors* $JN(\lambda)(t)$ *are*

$$\theta^B(t)^* = \left(\frac{\eta}{\gamma} - \frac{H\sigma\rho F(\tau)}{\gamma}\right)y(t) , \tag{4.92a}$$

$$\theta^Z(t)^* = \left(\frac{\nu}{\gamma} - \frac{H\sigma\sqrt{1-\rho^2}F(\tau)}{\gamma}\right)y(t) , \tag{4.92b}$$

$$\theta^N(t)^* = \frac{1}{J}\left[\left(\frac{\lambda e^{F(\tau)\xi(y(t))}}{\lambda^{\mathbb{Q}}}\right)^{\frac{1}{\gamma}} - 1\right] . \tag{4.92c}$$

While the optimal consumption-wealth ratio is given by

$$\frac{C(t)}{W(t)} = \beta^\psi \exp\{-(F(\tau)y(t) + G(\tau))\}. \tag{4.93}$$

Assume further that exist non-redundant derivatives traded at time $t \leq T$. *The optimal portfolio weights related to risky asset and derivative securities in* $t \leq T$ *are*

$$\phi(t)^* = \theta^B(t)^* - \sum_{i=1}^{2}\pi(t)^{(i)*}\left(S(t)\frac{g_s^{(i)}}{O(t)^{(i)}} + \sigma\rho\frac{g_y^{(i)}}{O(t)^{(i)}}y(t)\right) , \tag{4.94a}$$

$$\begin{aligned} \pi(t)^{(1)*} &= \frac{1}{\mathcal{D}(t)}\left\{\left(-\theta^B(t)^* + \frac{\rho\theta^Z(t)^*}{\sqrt{1-\rho^2}}\right)\frac{g_y^{(2)}}{O(t)^{(2)}} + \left[\theta^N(t)^*\frac{g_y^{(2)}}{O(t)^{(2)}}\right.\right. \\ &\left.\left. - \left(\frac{\Delta g^{(2)}}{O(t)^{(2)}} - S(t)\frac{g_s^{(2)}}{O(t)^{(2)}}\right)\frac{\theta^Z(t)^*}{\sigma\sqrt{1-\rho^2}y(t)}\right]\right\}, \end{aligned} \tag{4.94b}$$

$$\begin{aligned} \pi(t)^{(2)*} &= \frac{1}{\mathcal{D}(t)}\left\{\left[\left(\frac{\Delta g^{(1)}}{J\,O(t)^{(1)}} - S_t\frac{g_s^{(1)}}{O(t)^{(1)}}\right)\frac{\theta^Z(t)^*}{\sigma\sqrt{1-\rho^2}y_t} - \theta^N(t)^*\frac{g_y^{(1)}}{O(t)^{(1)}}\right] - \right. \\ &\left.\left(-\theta^B(t)^* + \frac{\rho\theta^Z(t)^*}{\sqrt{1-\rho^2}}\right)\frac{g_y^{(1)}}{O(t)^{(1)}}\right\} , \end{aligned} \tag{4.94c}$$

where $\mathcal{D}(t)$ *is given by* (2.16).

Proof. To determine the optimal policies we can start from the HJB (4.87) equation associated to the investment problem, define $H := \frac{1-\gamma}{1-\psi}$ and guess

$$V(t, W, y) = \exp\{-H\left(F(t)y_t + G(t)\right)\}\frac{W(t)^{1-\gamma}}{1-\gamma}. \tag{4.95}$$

We evaluate the partial derivatives of the value function appearing in the HJB equation, namely

$$\frac{\partial V}{\partial t} = -H(\dot{F}_t y(t) + \dot{G}_t V \ , \tag{4.96}$$
$$\frac{\partial V}{\partial W} = \exp\left\{-H(F(t)y(t) + G(t))\right\} W_t^{-\gamma} \ , \tag{4.97}$$
$$\frac{\partial V}{\partial y} = -HF(t)V \tag{4.98}$$
$$\frac{\partial^2 V}{\partial W^2} = -\gamma \exp\left\{-H(F(t)y(t) + G(t))\right\} W(t)^{-\gamma-1} \ , \tag{4.99}$$
$$\frac{\partial^2 V}{\partial y^2} = H^2 F(t)^2 V \ , \tag{4.100}$$
$$\frac{\partial^2 V}{\partial W \partial y} = -HF(t) \exp\left\{-H(F(t)y(t) + G(t))\right\} W(t)^{-\gamma} \ . \tag{4.101}$$
$$\begin{aligned}\Delta V &= \exp\{-H(F(t)(y + \xi(y(t))) + G(t))\}\frac{W^{1-\gamma}(1+\theta^N(t)J)^{1-\gamma}}{1-\gamma} - \exp\{-H(F(t)y(t) + G_t)\}\frac{W^{1-\gamma}}{1-\gamma} \\ &= \exp\{-H(F(t)y(t) + G(t))\}\frac{W(t)^{1-\gamma}}{1-\gamma}\left[e^{-HF(t)\xi(y(t))}(1+\theta^N(t)J)^{1-\gamma} - 1\right] \\ &= V\left[e^{-HF(t)\xi(y(t))}\left(1+\theta^N(t)J\right)^{1-\gamma} - 1\right].\end{aligned} \tag{4.102}$$

We replace all the derivatives (4.96) – (4.102) in (4.87), so that we obtain

$$\max_{\{u(t),C(t)\}}\Bigg\{f(C(t),V(t)) - H(\dot{F}(t)y(t)+\dot{G}(t))V + \exp\{-H(F(t)y(t)+G)\}W(t)^{-\gamma}W(t)$$
$$(r+\theta^B(t)\eta+\theta^Z(t)\nu-\theta^N(t)\lambda^{\mathbb{Q}}J-\frac{C}{W(t)}) - HF_tV[k^{\mathbb{Q}}(\theta^{\mathbb{Q}}-y_t)+\sigma\rho\eta y(t)+$$
$$\sigma\nu\sqrt{1-\rho^2}y(t)] - \gamma\frac{1}{2}\exp\{-H(F(t)y(t)+G(t))\}W(t)^{-\gamma-1}W^2[(\theta^B(t))^2y(t)^{-1}+$$
$$(\theta^Z(t))^2y(t)^{-1}] + \frac{1}{2}H^2F(t)^2V\sigma^2y(t) - HW(t)F(t)\exp\{-H(F(t)y(t)+G(t))\}W(t)^{-\gamma}$$
$$\left(\sigma\theta^B(t)\rho+\sigma\sqrt{1-\rho^2}\theta^Z(t)\right) + V\lambda\left[e^{F(t)\xi(y(t))}(1+\theta^N(t)J)^{1-\gamma}-1\right]\Bigg\} = \tag{4.103}$$

$$\max_{\{u(t),C(t)\}}\Bigg\{\frac{f(C(t),V(t))}{V(t)} - H(\dot{F}(t)y(t)+\dot{G}(t)) + r(1-\gamma) + \theta^B(t)\eta(1-\gamma) + \theta^Z(t)\nu(1-\gamma) -$$
$$\theta^N(t)\lambda^{\mathbb{Q}}J(1-\gamma) - \frac{C(t)}{W(t)}(1-\gamma) - HF(t)\left[k^{\mathbb{Q}}(\theta^{\mathbb{Q}}-y(t))+\sigma\rho\eta y(t)+\sigma\nu\sqrt{1-\rho^2}y(t)\right] -$$
$$\frac{\gamma}{2}(1-\gamma)\left[(\theta^B(t))^2y(t)^{-1}+(\theta^Z(t))^2y(t)^{-1}\right] + \frac{1}{2}H^2F(t)^2\sigma^2y_t - HF(t)(1-\gamma)$$
$$\left(\sigma\theta^B(t)\rho+\sigma\sqrt{1-\rho^2}\theta^Z(t)\right) + \lambda\left[e^{F(t)\xi(y(t))}(1+\theta^N(t)J)^{1-\gamma}-1\right]\Bigg\} =: \varphi(C(t),w(t)). \tag{4.104}$$

The first term in (4.104) can be written as

$$\begin{aligned}\frac{f(C(t),V(t))}{V(t)} &= \frac{\psi}{\psi-1}\beta(1-\gamma)\left[\left(\frac{C(t)}{((1-\gamma)V)^{\frac{1}{1-\gamma}}}\right)^{1-\frac{1}{\psi}}-1\right] = \\ &= -H\beta\psi\left[\frac{C(t)^{1-\frac{1}{\psi}}}{((1-\gamma)V)^{\left(\frac{1}{1-\gamma}\right)\left(\frac{\psi-1}{\psi}\right)}}-1\right] = \\ &= H\beta\psi - \frac{H\beta\psi}{((1-\gamma)V)^{\left(\frac{1}{1-\gamma}\right)\left(\frac{\psi-1}{\psi}\right)}}C(t)^{\frac{\psi-1}{\psi}}.\end{aligned} \tag{4.105}$$

We are able to determine the FOC's condition

$$\frac{\partial\varphi}{\partial C(t)} = -\frac{H\beta\psi}{((1-\gamma)V)^{\left(\frac{1}{1-\gamma}\right)\left(\frac{\psi-1}{\psi}\right)}}\frac{\psi-1}{\psi}C(t)^{\frac{1}{\psi}} - \frac{(1-\gamma)}{W(t)} = 0 \tag{4.106}$$

follows that

$$C(t)^{-\frac{1}{\psi}} = -\frac{(1-\gamma)}{W(t)}\frac{\psi}{\psi-1}\frac{\left((1-\gamma)V^{\frac{\psi-1}{(1-\gamma)\psi}}\right)}{H\beta\psi}, \tag{4.107}$$

substituting the value function, H and with some algebra we get

$$C(t)^{-\frac{1}{\psi}} = \frac{\left[(1-\gamma)\exp\{-H(F(t)y(t)+G(t))\}\frac{W(t)^{1-\gamma}}{1-\gamma}\right]^{\frac{\psi-1}{(1-\gamma)\psi}}}{\beta W(t)}$$
$$C(t) = \beta^{\psi}\exp\{-(F(t)y(t)+G(t))\}W(t). \tag{4.108}$$

Similarly we are able to determine the first order condition for optimal portfolio exposures

$$\frac{\partial\varphi}{\partial\theta^B(t)} = \eta(1-\gamma) - 2\frac{\gamma}{2}(1-\gamma)\theta^B(t)\frac{1}{y(t)} - HF(t)(1-\gamma)\sigma\rho = 0, \tag{4.109}$$
$$\frac{\partial\varphi}{\partial\theta^Z(t)} = \nu(1-\gamma) - 2\frac{\gamma}{2}(1-\gamma)\theta^Z(t)y(t)^{-1} - HF(t)(1-\gamma)\sigma\sqrt{1-\rho^2} = 0, \tag{4.110}$$
$$\frac{\partial\varphi}{\partial\theta^N(t)} = -\lambda^{\mathbb{Q}}J(1-\gamma) + \lambda J\left[e^{F(t)\xi(y(t))}(1-\gamma)(1+\theta^N(t)J)^{-\gamma}\right]; \tag{4.111}$$

from which it follows that

$$\theta^B(t) = \left[\frac{\eta - H\sigma F(t)\rho}{\gamma}\right]y(t), \tag{4.112}$$
$$\theta^Z(t) = \left[\frac{\nu - H\sigma F\sqrt{1-\rho^2}}{\gamma}\right]y(t), \tag{4.113}$$
$$\theta^N(t) = \frac{1}{J}\left[\left(\frac{\lambda}{\lambda^{\mathbb{Q}}}e^{F(t)\xi(y(t))}\right)^{\frac{1}{\gamma}} - 1\right]. \tag{4.114}$$

By replacing the optimal consumption and optimal exposures in the equation (4.104) we obtain

$$\frac{A(t)}{H} - \dot{F(t)}y(t) - \dot{G}(t) + \frac{r(1-\gamma)}{H} + \left(\frac{\eta - H\sigma F(t)\rho}{\gamma}\right) y(t)\eta(1-\gamma)\frac{1}{H} + \left(\frac{\nu - H\sigma F(t)\sqrt{1-\rho^2}}{\gamma}\right)$$
$$y(t)\nu(1-\gamma)\frac{1}{H} - \theta^N(t)\lambda^{\mathbb{Q}}J(1-\gamma) - F(t)\left[(k^{\mathbb{Q}}(\theta^{\mathbb{Q}} - y(t)) + \sigma\rho\eta y(t) + \sigma\nu\sqrt{1-\rho^2 y(t)}\right] -$$
$$\frac{\gamma}{2}(1-\gamma)\frac{1}{H}\left\{\left[\left(\frac{\eta - H\sigma F(t)\rho}{\gamma}\right)y(t)\right]^2 y(t)^{-1} + \left[\left(\frac{\nu - HF(t)\sigma\sqrt{1-\rho^2}}{\gamma}\right)y(t)\right]^2 y(t)^{-1}\right\} + \frac{1}{2}HF(t)^2$$
$$\sigma^2 y(t) - F(t)(1-\gamma)\left[\sigma\rho\left(\frac{\eta - H\sigma F(t)\rho}{\gamma}\right)y(t) + \sigma\sqrt{1-\rho^2}\left(\frac{\nu - HF(t)\sigma\sqrt{1-\rho^2}}{\gamma}\right)y(t)\right] +$$
$$\lambda\left[e^{F(t)\xi(y(t))}\left(1+\theta^N(t)J\right)^{1-\gamma} - 1\right] = 0 \tag{4.115}$$

$$\frac{A(t)}{H} - \dot{F(t)}y(t) - \dot{G}(t) + \frac{r(1-\gamma)}{H} + \frac{\eta^2}{\gamma}(1-\gamma)\frac{y(t)}{H} - \frac{\eta(1-\gamma)\sigma F(t)\rho}{\gamma}y(t) + \frac{\nu^2(1-\gamma)}{\gamma}\frac{y(t)}{H} -$$
$$\frac{\nu(1-\gamma)F(t)\sigma\sqrt{1-\rho^2}}{\gamma}y(t) - \theta^N(t)\lambda^{\mathbb{Q}}J(1-\gamma) - F(t)k^{\mathbb{Q}}\theta^{\mathbb{Q}} + F(t)k^{\mathbb{Q}}y(t) - F(t)\sigma\rho\eta y(t) - F(t)\sigma\nu$$
$$\sqrt{1-\rho^2}y(t) - \frac{\eta^2(1-\gamma)}{2\gamma}\frac{y(t)}{H} - \frac{F(t)^2 H\sigma^2\rho^2(1-\gamma)}{2\gamma}y(t) + \frac{(1-\gamma)\eta\sigma F(t)\rho}{\gamma}y(t) - \frac{\nu^2(1-\gamma)}{2\gamma}\frac{y(t)}{H} -$$
$$\frac{H\sigma^2F(t)^2(1-\gamma)}{2\gamma}y(t) + \frac{(1-\gamma)H\sigma^2\rho^2F(t)^2}{2\gamma}y(t) + \frac{(1-\gamma)\nu F(t)\sigma\sqrt{1-\rho^2}}{\gamma}y(t) + \frac{1}{2}HF(t)^2\sigma^2 y(t) -$$
$$\frac{F(t)(1-\gamma)}{\gamma}\sigma\rho\eta y(t) + \frac{F(t)t^2(1-\gamma)H\sigma^2\rho^2}{\gamma}y(t) - \frac{F(t)\nu(1-\gamma)y(t)\sigma\sqrt{1-\rho^2}}{\gamma} + \frac{F(t)^2H(1-\gamma)\sigma^2 y(t)}{\gamma}$$
$$-\frac{F(t)^2H(1-\gamma)\rho^2\sigma^2}{\gamma}y(t) + \lambda\left[e^{F(t)\xi(y(t))}(1+\theta^N(t)J)^{1-\gamma} - 1\right] = 0 \tag{4.116}$$

$$\frac{A(t)}{H} - \dot{F(t)}y(t) - \dot{G}(t) + \frac{r(1-\gamma)}{H} + \frac{\eta^2(1-\gamma)}{2\gamma}\frac{y(t)}{H} + \frac{\nu^2(1-\gamma)}{2\gamma}\frac{y(t)}{H} - \frac{F(t)(1-\gamma)\sigma\rho\eta}{\gamma}y(t) + \frac{1}{2}HF(t)^2$$
$$\sigma^2 y(t) - \frac{(t)\nu(1-\gamma)\sigma\sqrt{1-\rho^2}}{\gamma}y(t) + \frac{F(t)^2H(1-\gamma)\sigma^2 y(t)}{2\gamma} - \theta^N(t)\lambda^{\mathbb{Q}}J(1-\gamma) - F(t)k^{\mathbb{Q}}\theta^{\mathbb{Q}} +$$
$$F(t)k^{\mathbb{Q}}y(t) - F(t)\sigma\rho\eta y(t) - F(t)\sigma\nu\sqrt{1-\rho^2}y(t) + \lambda\left[e^{F(t)\xi(y(t))}(1+\theta^N(t)J)^{1-\gamma} - 1\right] = 0;$$

where $A(t)$ is given by

$$A(t) = \frac{f(C(t), V(t))}{V(t)} - \frac{C(t)}{W(t)}(1-\gamma), \tag{4.117}$$

substituting the equation (2.27) and the optimal consumption wealth ratio (4.106) we obtain

$$\begin{aligned}
A(t) &= H\beta\psi - \frac{H\beta\psi}{\left((1-\gamma)\exp\{-H(F(t)y(t)+G(t))\frac{W^{1-\gamma}}{1-\gamma}\}\right)^{\frac{\psi-1}{(1-\gamma)\psi}}}\left(W(t)\beta^{\psi}\exp\{-(F(t)y(t)+G(t))\}\right)^{\frac{\psi-1}{\psi}} \\
&- \frac{W(t)\beta^{\psi}\exp\{-(F(t)y(t)+G(t))\}}{W(t)}(1-\gamma) \\
&= H\beta\psi - \frac{H\beta\psi}{(\exp\{-H(F(t)y(t)+G(t))\}W(t)^{1-\gamma})^{\frac{\psi-1}{(1-\gamma)\psi}}}W(t)^{\frac{\psi-1}{\psi}}\beta^{\psi-1}\exp\{-(Fy(t)+G(t))\}^{\frac{\psi-1}{\psi}} \\
&- \beta\psi\exp\{-(F(t)y(t)+G(t))\}(1-\gamma) \\
&= H\beta\psi - \frac{H\beta\psi}{\exp\{\frac{1-\gamma}{1-\psi}\frac{1-\psi}{(1-\gamma)\psi}(F(t)y(t)+G(t))\}W(t)^{1-\gamma\frac{\psi-1}{(1-\gamma)\psi}}}W(t)^{\frac{\psi-1}{\psi}}\beta^{\psi-1}\exp\{-(F(t)+G(t))\}^{\frac{\psi-1}{\psi}} \\
&- \beta^{\psi}\exp\{-(F(t)+G(t))\}(1-\gamma) \\
&= H\beta\psi - H\beta^{\psi}\psi\exp\left\{-\frac{1}{\psi}(F(t)y(t)+G(t))\right\}\exp\left\{-\frac{\psi-1}{\psi}(F(t)y(t)+G(t))\right\} - \beta^{\psi}\exp\{-(F(t) \\
&y(t)+G(t))\}(1-\gamma) \\
&= H\beta\psi - H\beta^{\psi}\psi\exp\left\{\left(-\frac{1}{\psi}+\frac{1}{\psi}-1\right)(Fy(t)+G(t))\right\} - \beta^{\psi}\exp\{-(F(t)y(t)+G(t))\}(1-\gamma) \\
&= H\beta\psi - H\beta^{\psi}\psi\exp\{-(F(t)y(t)+G(t)\} - \beta^{\psi}\exp\{-(F(t)y(t)+G(t))\}(1-\gamma) \\
&= H\beta\psi - H\exp\{-(F(t)y(t)+G(t))\}\left(\beta^{\psi}\psi+\frac{(1-\gamma)}{H}\beta^{\psi}\right) \\
&= H\beta\psi - H\exp\{-(F(t)y(t)+G(t)\}\left(\beta^{\psi}\psi+\beta^{\psi}-\beta^{\psi}\psi\right) \\
&= H\beta\psi - H\beta^{\psi}\exp\{-(F(t)+G_(t))\} =: H\beta\psi - H\beta^{\psi}X^{-1}. \qquad (4.118)
\end{aligned}$$

Now using the envelope condition, as in Chacko and Viceira [2005], we have

$$\beta^{\psi}X^{-1} = \frac{C(t)}{W(t)} = \exp\{\log(C(t)) - \log(W(t))\} = \exp\{c(t)-w(t)\}, \qquad (4.119)$$

so that

$$c(t)-w(t) = \log(\beta^{\psi}X^{-1}) = \psi\log(\beta) - F(t)y(t) - G(t). \qquad (4.120)$$

Moreover, using the first-order taylor expansion of $\exp\{c(t)-w(t)\}$ around its unconditional mean $\overline{c-w}$, we have

$$A(t) \approx H\beta\psi - H\left[h_0 + h_1(c(t)-w(t))\right], \qquad (4.121)$$

where $h_1 := \exp\{\overline{c-w}\}$ and $h_0 := h1 - h1\log(h1)$. By replacing (4.121) and (4.120) in

(4.117), we have

$$\Bigg(- \dot{F}_t - \frac{F(t)(1-\gamma)\sigma\rho\eta}{\gamma} - \frac{F(t)\nu(1-\gamma)\sigma\sqrt{1-\rho^2}}{\gamma} + F(t)k^{\mathbb{Q}} - F(t)\sigma\rho\eta - F(t)\sigma\nu\sqrt{1-\rho^2} +$$

$$\frac{1}{2}HF(t)^2\sigma^2 + \frac{F(t)^2H(1-\gamma)\sigma^2}{2\gamma} + h_1F(t) + \frac{\eta^2(1-\gamma)}{2\gamma}\frac{1}{H} + \frac{\nu^2(1-\gamma)}{2\gamma}\frac{1}{H}\Bigg)y(t) + \Bigg(- \dot{G}(t)$$

$$- h_0 + \beta\psi - h_1(\psi\log(\beta) - G(t)) + \frac{r(1-\gamma)}{H} - \theta^N(t)\lambda^{\mathbb{Q}}J(1-\gamma) - F(t)k^{\mathbb{Q}}\theta^{\mathbb{Q}} + \lambda\Bigg[e^{F(t)\xi(y(t))}(1+$$

$$tNJ)^{1-\gamma} - 1\Bigg]\Bigg) = 0,$$

and so we have the following system of ODEs, where $F(t)$ and $G(t)$ have the following form

$$\begin{cases} -\dot{F}(t) - F(t)\left[\frac{(1-\gamma)}{\gamma}\sigma(\rho\eta - \nu\sqrt{1-\rho^2}) + \sigma(\rho\eta - \nu\sqrt{1-\rho^2}) + k^{\mathbb{Q}} + h_1\right] + \frac{1}{2\gamma}F(t)^2H\sigma^2 \\ (\gamma + (1-\gamma)) + \frac{(1-\gamma)}{2\gamma}\frac{1}{H}\left(\eta^2+\nu^2\right) = 0, \\ -\dot{G}_t - h_0 + \beta\psi - h_1\left(\psi\log(\beta) - G(t)\right) + \frac{r(1-\gamma)}{H} - \theta^N(t)\lambda^{\mathbb{Q}}J(1-\gamma) - F(t)k^{\mathbb{Q}}\theta^{\mathbb{Q}} \\ +\lambda\left[e^{F(t)\xi(y(t))}(1+\theta^N(t)J)^{1-\gamma} - 1\right] = 0, \end{cases} \tag{4.122}$$

$$\begin{cases} \dot{F}_t + F(t)\left[(k^{\mathbb{Q}} + h_1) + \frac{\sigma}{\gamma}\left(\rho\eta + \nu\sqrt{1-\rho^2}\right)\right] - \frac{1}{2\gamma}HF(t)^2\sigma^2 - \frac{(1-\gamma)}{2\gamma}\frac{1}{H}(\eta^2+\nu^2) = 0, \\ \dot{G}_t + h_0 - \beta\psi + h_1\left(\psi\log(\beta) - G(t)\right) - \frac{r(1-\gamma)}{H} + \theta^N(t)\lambda^{\mathbb{Q}}J(1-\gamma) + F(t)k^{\mathbb{Q}}\theta^{\mathbb{Q}} - \\ \lambda\left[e^{F(t)\xi}(1+\theta^N(t)J)^{1-\gamma} - 1\right] = 0, \end{cases} \tag{4.123}$$

with final condition $F(T) = G(T) = 0$.
If we set $\tau = T - t$, remembering that $\dot{F}_\tau = \frac{\partial F}{\partial \tau} = \frac{\partial F}{\partial t}\frac{\partial t}{\partial \tau} = -\frac{\partial F}{\partial t}$, the previous system becomes

$$\begin{cases} \dot{F}_\tau = F(\tau)\left[(k^{\mathbb{Q}} + h_1) + \frac{\sigma}{\gamma}\left(\rho\eta + \nu\sqrt{1-\rho^2}\right)\right] - \frac{1}{2\gamma}HF(\tau)^2\sigma^2 - \frac{(1-\gamma)}{2\gamma}\frac{1}{H}(\eta^2+\nu^2), \\ \dot{G}_\tau = h_0 - \beta\psi + h_1\left(\psi\log(\beta) - G(\tau)\right) - \frac{r(1-\gamma)}{H} + \theta^N(t)\lambda^{\mathbb{Q}}J(1-\gamma) + F(\tau)k^{\mathbb{Q}}\theta^{\mathbb{Q}} - \\ \lambda\left[e^{F(\tau)\xi(y(t))}(1+\theta^N(t)J)^{1-\gamma} - 1\right], \end{cases} \tag{4.124}$$

where F and G as function of τ have initial condition $F(0) = G(0) = 0$.
We observe that the ODE for $F(\tau)$ in (4.124) is a Riccati-type equation of the form

$$\dot{F}_\tau = -\frac{H\sigma^2}{2\gamma}F(\tau)^2 + k_1F(\tau) - \frac{\delta}{2H} \tag{4.125}$$

where $k_1 = k^{\mathbb{Q}} + h_1 + \frac{\sigma}{\gamma}\left(\rho\eta + \nu\sqrt{1-\rho^2}\right)$ and $\delta = \frac{1-\gamma}{\gamma}(\eta^2+\nu^2)$.

Assume

$$F = \frac{2\gamma}{H\sigma^2}\left(\frac{y'}{y}\right) \tag{4.126}$$

$$\dot{F} = \frac{2\gamma}{H\sigma^2}\left[\frac{y''y-(y')^2}{y^2}\right] = \frac{2\gamma}{H\sigma^2 y}\left[y'' - \frac{(y')^2}{y}\right]. \tag{4.127}$$

Replacing the expression for F and $\dot{F}$ above in (4.125)

$$\begin{aligned}
\frac{2\gamma}{H\sigma^2 y}\left[y'' - \frac{(y')^2}{y}\right] &= -\frac{H\sigma^2}{2\gamma}\left(\frac{4\gamma^2}{H^2\sigma^4}\frac{(y')^2}{y^2}\right) + k_1\left(\frac{2\gamma}{H\sigma^2}\left(\frac{y'}{y}\right)\right) - \frac{\delta}{2H};\\
\frac{2\gamma}{H\sigma^2}\frac{y''}{y} - \frac{2\gamma}{H\sigma^2}\frac{(y')^2}{y^2} &= -\frac{2\gamma}{H\sigma}\frac{(y')^2}{y^2} + k_1\left(\frac{2\gamma}{H\sigma^2}\frac{y'}{y}\right) - \frac{\delta}{2H};\\
\frac{2\gamma}{H\sigma^2}y'' - k_1\left(\frac{2\gamma}{H\sigma^2}y'\right) + \frac{\delta y}{2H} &= 0.
\end{aligned} \tag{4.128}$$

If we put $y' = xe^{x\tau} = xy \iff y'' = x^2e^{x\tau} = x^2y$, we obtain

$$\begin{aligned}
\frac{2\gamma}{H\sigma^2}x^2y - \frac{2k_1\gamma}{H\sigma^2}xy + \frac{\delta}{2H}y &= 0\\
e^{x\tau}\left[\frac{2\gamma}{H\sigma^2}x^2 - \frac{2k_1\gamma}{H\sigma^2}x + \frac{\delta}{2H}\right] &= 0.
\end{aligned} \tag{4.129}$$

Therefore solving (4.125) is equivalent to solve the characteristic equation in (4.129)

$$\frac{2\gamma}{H\sigma^2}x^2 - \frac{2k_1\gamma}{H\sigma^2}x + \frac{\delta}{2H} = 0; \tag{4.130}$$

$$\begin{aligned}
x &= \frac{\frac{k_1\gamma}{H\sigma^2} \pm \sqrt{\frac{k_1^2\gamma^2}{\sigma^4H^2} - \frac{2\gamma}{H\sigma^2}\frac{\delta}{2H}}}{\frac{2\gamma}{H\sigma^2}} = \frac{\frac{k_1\gamma}{H\sigma^2} \pm \sqrt{\frac{k_1^2\gamma^2 - H\sigma^2\delta\gamma}{H^2\sigma^4}}}{\frac{2\gamma}{H\sigma^2}} = \frac{k_1\gamma \pm \sqrt{\gamma^2\left(k_1^2 - \frac{H\sigma^2\delta}{\gamma}\right)}}{2\gamma}\\
&= \frac{k_1 \pm \sqrt{\left(k_1^2 - \frac{H\sigma^2\delta}{\gamma}\right)}}{2}.
\end{aligned} \tag{4.131}$$

If we define $k_2 := \sqrt{k_1^2 - \frac{H\sigma^2\delta}{\gamma}}$, thus we have $x = \frac{1}{2}(k_1 \pm k_2)$. Standard arguments allow us to prove that a general solution to (4.129) is

$$y = c_1e^{x_1\tau} + c_2e^{x_2\tau}. \tag{4.132}$$

In any case, we have

$$y = c_1e^{\frac{1}{2}(k_1+k_2)\tau} + c_2e^{\frac{1}{2}(k_1-k_2)\tau} \tag{4.133}$$

$$y = \frac{1}{2}c_1(k_1+k_2)e^{\frac{1}{2}(k_1+k_2)\tau} + \frac{1}{2}c_2(k_1-k_2)e^{\frac{1}{2}(k_1-k_2)\tau}. \tag{4.134}$$

Replacing in (4.126), we obtain

$$
\begin{aligned}
F &= \frac{2\gamma}{H\sigma^2}\frac{\left[\frac{1}{2}c_1(k_1+k_2)e^{\frac{1}{2}(k_1+k_2)\tau}+\frac{1}{2}c_2(k_1-k_2)e^{\frac{1}{2}(k_1-k_2)\tau}\right]}{\left[c_1e^{\frac{1}{2}(k_1+k_2)\tau}+c_2e^{\frac{1}{2}(k_1-k_2)\tau}\right]} \\
&= \frac{\gamma c_1 e^{\frac{1}{2}(k_1+k_2)\tau}\left[(k_1+k_2)+\frac{c_2}{c_1}(k_1-k_2)e^{\frac{1}{2}(k_1-k_2)\tau-\frac{1}{2}(k_1+k_2)\tau}\right]}{H\sigma^2 c_1 e^{\frac{1}{2}(k_1+k_2)\tau}\left[1+\frac{c_2}{c_1}e^{\frac{1}{2}(k_1-k_2)\tau-\frac{1}{2}(k_1+k_2)\tau}\right]} \\
&= \frac{\gamma\left[(k_1+k_2)+C(k_1-k_2)e^{-k_2\tau}\right]}{H\sigma^2\left[1+Ce^{-k_2\tau}\right]} = \frac{\gamma\left[(k_1+k_2)e^{k_2\tau}+C(k_1-k_2)\right]}{H\sigma^2\left[e^{k_2\tau}+C\right]}
\end{aligned}
\tag{4.135}
$$

having set $C=\frac{c_2}{c_1}$.
Moreover we know that the F as a function of τ has as initial condition $F(0)=0$,

$$
\begin{aligned}
&\frac{\gamma\left[(k_1+k_2)+C(k_1-k_2)\right]}{H\sigma^2\left[1+C\right]} = 0; \\
&k_1+k_2+C(k_1-k_2)=0 \iff C=\frac{k_1+k_2}{k_2-k_1}.
\end{aligned}
\tag{4.136}
$$

Then by substituting C into the expression (4.135) we get

$$
\begin{aligned}
F(\tau) &= \frac{\gamma\left[(k_1+k_2)e^{k_2\tau}+\frac{k_1+k_2}{k_2-k_1}(k_1-k_2)\right]}{H\sigma^2\left[e^{k_2\tau}+\frac{k_1+k_2}{k_2-k_1}\right]} \\
&= \frac{\gamma\left[(k_1+k_2)(k_2-k_1)e^{k_2\tau}-(k_1+k_2)(k_2-k_1)\right]}{H\sigma^2\left[e^{k_2\tau}(k_2-k_1)+k_1+k_2\right]} \\
&= \frac{\gamma(k_2-k_1)\left[(k_1+k_2)e^{k_2\tau}-(k_1+k_2)\right]}{H\sigma^2\left[e^{k_2\tau}(k_2-k_1)+k_1+k_2\right]} \\
&= \frac{\gamma(k_2-k_1)(k_1+k_2)(e^{k_2\tau}-1)}{H\sigma^2\left(k_1+k_2+e^{k_2\tau}(k_2-k_1)\right)} \\
&= \frac{\gamma(k_2^2-k_1^2)(e^{k_2\tau}-1)}{H\sigma^2\left(k_1+k_2+k_2-k_2+(k_2-k_1)e^{k_2\tau}\right)} \\
&= -\frac{\delta}{H}\frac{(e^{k_2\tau}-1)}{H\sigma^2\left[(k_2-k_1)(e^{k_2\tau}-1)+2k_2\right]} \\
&= \frac{\delta}{H}\frac{1-e^{k_2\tau}}{(k_2-k_1)(e^{k_2\tau}-1)+2k_2},
\end{aligned}
\tag{4.137}
$$

which follows from the fact that $\frac{(k_2^2-k_1^2)\gamma}{\sigma^2}=-\delta$.
The explicit expression for $G(\tau)$ is obtained by observing that it is a first-order differential equation. More precisely,

$$
\dot{G}_\tau + h_1G(\tau) = h_1\psi\log(\beta)+h_0-\beta\psi+\theta^N(t)\lambda^{\mathbb{Q}}J(1-\gamma)+Fk^{\mathbb{Q}}\theta^{\mathbb{Q}}-\frac{r(1-\gamma)}{H}- \tag{4.138}
$$
$$
\lambda\left[e^{F(\tau)\xi(y(t))}(1+\theta^N(t)J)^{1-\gamma}-1\right],
$$

with initial condition $G(0) = 0$, is a first order differential equation of the type

$$\begin{cases} y'(t) + a_0(t)y(t) = g(t) \\ y(t_0) = y_0 \end{cases},$$

with solution of the form

$$y(t) = e^{-At}\left[y_0 + \int_{t_0}^{t} g(s)e^{A(s)}ds,\right]$$

with $A(t) = \int_0^t a_0(s)ds$.
From which it follows following the solution scheme for the differential equation

$$A(t) = \int_0^t h_1 ds = h_1 t, \tag{4.139}$$

$$\begin{aligned}
G(\tau) =& e^{-h1\tau}\left\{\int_0^\tau \left[h_1\psi\log(\beta) + h_0 - \beta\psi + \theta^N(t)(s)\lambda^{\mathbb{Q}}J(1-\gamma) + F(s)k^{\mathbb{Q}}\theta^{\mathbb{Q}} - \frac{r(1-\gamma)}{H} -\right.\right. \\
& \left.\left.\lambda e^{F(s)\xi(y(t))}(1+\theta^N(t)(s)J)^{1-\gamma} + \lambda\right]e^{h_1 s}ds\right\}; \\
& = e^{-h1\tau}\left\{\int_0^\tau h_1\psi\log\beta e^{h_1 s}ds + \int_0^\tau h_0 e^{h_1 s}ds - \beta\psi\int_0^\tau e^{h_1 s}ds + \int_0^\tau \theta^N(t)(s)\lambda^{\mathbb{Q}}(1-\gamma)\right. \\
& Je^{h_1 s}ds + \int_0^\tau F(s)k^{\mathbb{Q}}\theta^{\mathbb{Q}}e^{h_1 s}ds - \int_0^\tau \frac{r(1-\gamma)}{H}e^{h_1 s}ds - \int_0^\tau \lambda e^{F(s)\xi(y(t))}(1+\theta^N(t)(s)J)^{1-\gamma}e^{h_1 s}ds + \\
& \left.\int_0^\tau \lambda e^{h_1 s}ds\right\}; \\
& = e^{-h1\tau}\left\{\psi\log(\beta)e^{h_1\tau} + \frac{h_0}{h_1}e^{h_1\tau} - \frac{\beta\psi}{h_1}e^{h_1\tau} - \frac{r(1-\gamma)}{Hh_1}e^{h_1\tau} - \frac{\lambda^{\mathbb{Q}}(1-\gamma)}{h_1}e^{h_1\tau} + \int_0^\tau F(s)k^{\mathbb{Q}}\theta^{\mathbb{Q}}e^{h1s}\right. \\
& \left. ds + \int_0^\tau \theta^N(t)(s)\lambda^{\mathbb{Q}}(1-\gamma)Je^{h_1 s}ds - \int_0^\tau \lambda e^{F(s)\xi}(1+\theta^N(t)(s)J)^{1-\gamma}e^{h_1 s}ds\right\};
\end{aligned} \tag{4.140}$$

using some algebra we get

$$
\begin{aligned}
G(\tau) =&\psi \log(\beta) + \frac{h_0}{h_1} - \frac{\beta\psi}{h_1} - \frac{r(1-\gamma)}{Hh_1} - \frac{\lambda^{\mathbb{Q}}(1-\gamma)}{h_1} + \\
&\frac{k^{\mathbb{Q}}\theta^{Q}\delta}{(k_1+k_2)H}\left\{\frac{\mathcal{F}_1\left[1, \frac{h_1}{k_2}, \frac{h_1+k_2}{k_1+k_2}, \frac{e^{k_2\tau}(k_1-k_2)}{k_1+k_2}\right] - e^{-h_1\tau}\mathcal{F}_1\left[1, \frac{h_1}{k_2}, \frac{h_1+k_2}{k_2}, -1+\frac{2k_1}{k_1+k_2}\right]}{h_1} + \right. \\
&\left.\frac{-e^{k_2\tau}\mathcal{F}_1\left[1, \frac{h_1+k_2}{k_2}, 2+\frac{h_1}{k_2}, \frac{e^{k_2\tau}(k_1-k_2)}{k_1+k_2}\right] + e^{-h_1\tau}\mathcal{F}_1\left[1, \frac{h_1+k_2}{k_2}, 2+\frac{h_1}{k_2}, -1+\frac{2k_1}{k_1+k_2}\right]}{h_1+k_2}\right\} \\
&+ e^{-h_1\tau}\left\{\int_0^\tau \left[\lambda^{\mathbb{Q}}(1-\gamma)J\theta^N(t)(s) - \lambda e^{F(s)\xi(y(t))}(1+\theta^N(t)(s)J)^{1-\gamma}\right] e^{h_1 s} ds\right\};
\end{aligned}
$$

where $\mathcal{F}_1$ is the hypergeometric function. □

4.4 The impact of jumps and recursive preferences on optimal policies

In this Section, we show how the theoretical results provided in Section 4.1 work on real data. In particular, we focus on the general case $\psi \neq 1$, and spotlight the role played by preferences in optimal allocation.

4.4.1 Numerical example: the multivariate case:incomplete market setting

In the first numerical example, we consider a portfolio with $N = 12$ hedge funds from the Credit Suisse Hedge Fund index, namely Convertible Arbitrage (CA), Emerging Markets (EM), Equity Market Neutral (EMN), Event Driven (ED), Event Driven Distressed (EDD), Event Driven Multi-Strategy (EDMS), Event Driven Risk Arbitrage (EDRA), Fixed Income (FI), Global Macro (GM), Long/Short (L/S), Managed Futures (MF), and Multi-Strategy (MS).

The choice of considering hedge funds instead of the equity market is motivated by considering that the former are more exposed to the tail risk, as explained e.g. in Kelly and Jiang [2012]. We perform a calibration procedure to obtain the parameter estimates under three kinds of models, depending on the presence of the jump component. Hereinafter, we will refer to such models as *Co-Jumps, Jumps-in-Price,* and *No-Jumps.* Calibration consists of two parts: first, we estimate the precision and identify the parameters characterizing the jump component through a kernel estimator and a data-driven threshold, respectively, as in Corsi et al. [2010]. Then, we use a generalized (simulated) method of moments to detect the remaining parameters, as in Oliva and Renò [2018]. The parameter estimates of the three models are reported in Table 4.1. Furthermore, to be compliant with the empirical evidence, see e.g. Liu and Pan [2003] and Branger et al. [2008], we constrain the price-jump amplitudes to be smaller than zero. The results for the optimal allocation are provided in Table 4.2. We observe that the overall optimal allocation reduces as more sources of risk are included in the model. We also observe that the reduction in the overall allocation is minimal when only price jumps are considered (the variation is 0.4%). The presence of co-jumps, on the other hand, causes a sizeable decrease in the optimal investment in risky securities, equal to 10%.

Thus, we extend this analysis to different types of investors. As discussed in Section 4.1, a crucial role is played by the log-linearization coefficient appearing in (4.9). In what follows, we apply the scheme proposed in Chacko and Viceira [2005] and implement a recursive numerical procedure so that we fix an initial value for h_1, we use it to calculate (4.7) and update h_1 until the convergence is reached. Furthermore, we consider investors with coefficients of relative risk-aversion ranging in $[2, 40]$, and elasticity of intertemporal substitution spanning between $1/2$ and $1/40$.

Remark 4.4.1. It is worth stressing that, while for the risk-aversion parameter, the empirical evidence suggests $\gamma > 1$, the literature does not express a uniform agreement for the estimation of ψ, as highlighted in Kraft et al. [2013]. Hall [1988], Campbell and Viceira [1999, 2001, 2002], and Vissin-Jorgensen [2002] provide a value for EIS smaller than one and justify this result on extensive analyses of aggregated and disaggregated data. On the other hand, Bansal and Yaron [2004], Bansal [2007] and Hansen and Singleton [1982] argue that such an estimate of the parameter ψ is based on misspecification of the model, showing that ignoring the effects of time-varying consumption volatility and excluding fluctuating economic uncertainty leads to a severe downward bias in EIS estimates. However, while recognising the importance of an appropriate specification of the parameters characterising recursive preferences, it must be pointed out that this is beyond the scope of this work. This is why in the following we will not address the estimation of the risk aversion and EIS parameters, but will provide a detailed sensitivity analysis of our numerical results w.r.t. changes in the parameters. We will also set specific values for γ and ψ when this will be strictly necessary for our study.

The optimal allocation is highest when there are no jumps in the dynamics, implying that the optimal allocation is mainly affected by co-jumps. When we consider both jumps in price and precision (top panel in Table 4.3), the optimal allocation reduces, compared to the no-jump case (bottom panel). For example, for $\gamma = 2$ and $\psi = 1/2$, in the absence of

	Co-Jumps					**Jumps-in-Price**					**No-Jumps**				
	α	$diag(Q)$	$diag(K)$	J	$diag(\xi)$	α	$diag(Q)$	$diag(K)$	J	$diag(\xi)$	α	$diag(Q)$	$diag(K)$	J	$diag(\xi)$
CA	0.68	0.05	-0.10	-8.48	-0.33	0.68	0.05	-0.04	-8.48	–	0.57	0.13	-0.40	–	–
EM	0.72	0.06	-0.33	-2.47	-0.73	0.72	0.09	-0.26	-2.47	–	0.69	0.2	-1.01	–	–
EMN	0.51	0.1	-0.43	-7.48	-0.46	0.51	0.08	-0.09	-7.48	-	0.42	0.13	-0.40	–	–
ED	0.76	0.02	-0.02	-7.40	-0.46	0.76	0.08	-0.09	-7.40	–	0.67	0.11	-0.27	–	–
EDD	0.85	0.03	-0.03	-7.51	-0.44	0.85	0.06	-0.07	-7.51	–	0.76	0.1	-0.22	–	–
EDMS	0.69	0.02	-0.02	-4.99	-0.37	0.69	0.05	-0.05	-4.99	–	0.63	0.14	-0.43	–	–
EDRA	0.56	0.05	-0.11	-6.15	-0.10	0.56	0.09	-0.11	-6.15	–	0.48	0.08	-0.20	–	–
FI	0.55	0.05	-0.10	-8.25	-0.13	0.55	0.07	-0.07	-8.25	–	0.45	0.1	-0.24	–	–
GM	0.89	0.15	-1.47	-1.45	-1.18	0.89	0.06	-0.08	-1.45	–	0.87	0.14	-0.47	–	–
L/S	0.89	0.07	-0.26	-9.62	-0.77	0.89	0.08	-0.14	-9.62	–	0.77	0.18	-0.70	–	–
MF	0.41	0	-0.28	3.18	-0.01	0.41	0.08	-0.48	3.18	–	0.45	0.19	-1.17	–	–
MS	0.68	0.07	-0.22	-4.04	-0.42	0.68	0.07	-0.08	-4.04	–	0.63	0.13	-0.34	–	–

Table 4.1. Parameter estimates for the Co-Jumps model ($\psi = 146.9339$), the Jumps-in-Price model ($\psi = 48.3789$), and the No-Jumps model ($\psi = 62.0173$). The jump intensity λ is equal to 0.0120 when jumps occur, or is equal to 0 otherwise, $diag(Q)$, $diag(K)$, $diag(\xi)$ represent the main diagonal of matrices Q, K, ξ, respectively. Estimates are for returns expressed in percentages and in monthly units, see Oliva and Renò [2018] for further details.

Hedge Fund	No-Jumps (%)	Jumps-in-Price (%)	Co-Jumps (%)
CA	7.8328	7.9276	7.0144
EM	2.1976	2.1866	2.1372
EMN	9.7140	9.6576	8.2937
ED	7.9774	7.9344	7.3218
EDD	8.5607	8.5233	7.9016
EDMS	6.5082	6.5047	6.1320
EDRA	11.6174	11.6437	10.5577
FI	10.1383	10.0992	8.6735
GM	4.7001	4.7022	4.6266
L/S	3.8061	3.8118	3.4474
MF	1.4062	1.1577	1.0893
MS	10.9068	10.8845	10.3487
Total Allocation	85.3654	85.0334	77.5439

Table 4.2. Percentage of optimal portfolio allocation for the No-Jumps model (second column), the Jumps-in-Price model (third column), and the Co-Jumps model (fourth column). The maturity of the investment is 1 year, the rate of time preferences is $\beta = 6\%$, the risk aversion parameter is $\gamma = 3$, the elasticity of intertemporal substitution of consumption is $\psi = 0.8$, the jump intensity is equal to $\lambda = 0.0120$ for models with jumps, and 0 otherwise. The remaining parameters are given in Table 4.1.

jumps, the investment guarantees 7% more of the wealth allocated in the stock. Our findings confirm that price and volatility jumps can overshadow the skewness effect. Furthermore, the impact of EIS on the optimal portfolio weights is negligible. At the same time, the risk aversion parameter strongly affects the results. For example, if we quadruple γ (moving from $\gamma = 5$ to $\gamma = 20$), the optimal allocation is reduced by more than 75% of the previous value. The reduction proportion remains almost the same even when only price jumps are considered or if no jumps are assumed. We carry on a similar analysis for optimal consumption. More precisely, we evaluate

$$\frac{C(0)}{W(0)} = e^{\mathbb{E}[c(0)-x(0)]} \times 100 \,,$$

i.e., we consider the (exponentiated) optimal mean log consumption-wealth ratio for several values of EIS and risk-aversion parameter. The results are shown in Table 4.4, for a relative risk-aversion coefficient ranging in $[0.75, 10]$, and an elasticity of intertemporal substitution spanning between $1/0.75$ and $1/3$.

When the elasticity of intertemporal substitution ψ is small, investors are unwilling to substitute consumption in time. Instead, they prefer investing in portfolios with a higher consumption-wealth ratio. A bigger EIS corresponds to a lower value of the consumption-wealth ratio. We first focus on the last column of Table 4.4, namely, we comment on results for investors who are extremely reluctant to intertemporally substitute consumption. These investors wish to keep their expected consumption growth rate constant, regardless of the current investment opportunity set, by consuming the average return of their portfolios with a precautionary-savings adjustment for risk. Then, we study another borderline case,

Co-Jumps Model	Optimal Allocation weight (%)					
	E.I.S. (ψ)					
R.R.A (γ)	1/2	1/3	1/5	1/10	1/20	1/40
2	120.1778	120.5374	120.9167	121.2331	121.4216	121.5315
3	78.0377	78.4475	78.8628	79.2447	79.4717	79.5963
5	45.7806	46.1425	46.5233	46.8650	47.0663	47.1516
10	22.5196	22.7411	22.9836	23.1979	23.3192	23.3766
20	11.1586	11.2853	11.4210	11.5400	11.6092	11.6447
40	5.5588	5.6236	5.6931	5.7572	5.7924	5.8104
Jumps in Price Model	**Optimal Allocation weight (%)**					
	E.I.S. (ψ)					
R.R.A (γ)	1/2	1/3	1/5	1/10	1/20	1/40
2	127.9925	128.0254	128.0622	128.0924	128.1119	128.1215
3	85.0789	85.1217	85.1622	85.2003	85.2246	85.2364
5	50.9311	50.9680	51.0055	51.0402	51.0620	51.0724
10	25.4236	25.4478	25.4716	25.4963	25.5092	25.5161
20	12.7015	12.7151	12.7292	12.7427	12.7500	12.7534
40	6.3479	6.3554	6.3630	6.3698	6.3735	6.3760
No-Jumps Model	**Optimal Allocation weight (%)**					
	E.I.S. (ψ)					
R.R.A (γ)	1/2	1/3	1/5	1/10	1/20	1/40
2	128.3049	128.3287	128.3538	128.3768	128.3898	128.3967
3	85.3936	85.4191	85.4463	85.4716	85.4858	85.4934
5	51.1689	51.1900	51.2129	51.2341	51.2461	51.2525
10	25.5596	25.5728	25.5871	25.6005	12.8006	12.8028
20	12.7737	12.7810	12.7890	12.7964	12.7500	12.7534
40	6.3853	6.3892	6.3973	6.3698	6.3995	6.4007

Table 4.3. Percentage of Optimal Portfolio weights, for total funds, evaluated at $t = 0$ for Co-Jumps Model, Jumps-in-Price Model and No-Jumps Model, for different risk-aversion coefficients and elasticities of intertemporal substitution of consumption. The rate of time preferences β is set equal to 6% annually, the risk-free rate is set to $r = 0\%$. The remaining parameters are given in Table 4.1.

and we focus on extreme risk-averse parameters (last row in each panel of Table 4.4). In this case, investors are more likely to choose safer portfolios, thus achieving smaller returns. The polar opposite is represented by risk-tolerant investors, who are encouraged to allocate a more significant part of their wealth into risky assets so that an extreme risk-return profile will characterize the corresponding portfolios. This is why the consumption-wealth ratio is higher when investors are less risk-averse. Furthermore, we examine investors more willing to substitute consumption in time and assume $\psi > 1$. This implies that we concentrate on the first column in Table 4.4, so that, assuming $\beta > 0$, market players would rather face higher savings but lower consumption, instead of those who are reluctant to substitute consumption. In such a situation, an investor with low-risk aversion γ will result in a portfolio with higher expected returns: the higher EIS, the lower the consumption-wealth ratio. The reasoning above can be summarized by focusing on the percentage variation of the optimal consumption wealth ratio: if we halve EIS, e.g. going from 1/(1.25) to 1/(2.5), the optimal consumption experiences an increase by 78% if the investor is very risk-loving ($\gamma = 0.75$). Such an increment reduces to 76% if the investor's risk aversion is high enough

Co-Jumps Model	**Consumption-Wealth Ratio** (%)					
	E.I.S. (ψ)					
R.R.A (γ)	1/0.75	1/1.25	1/1.4	1/2	1/2.5	1/3
0.75	1.7669	12.6430	17.5656	39.9612	58.4263	76.8304
1.25	1.9554	12.0471	16.3623	36.4625	54.5553	72.0171
1.4	1.9841	11.9380	16.2277	35.9236	53.5491	71.3246
2	2.0584	11.7116	15.7729	34.6257	52.0020	68.9755
3	2.1128	11.5195	15.3992	33.6265	50.1415	67.1444
10	2.1870	11.2862	14.9703	31.9578	47.4859	63.3784
Jumps in Price Model	**Consumption-Wealth Ratio** (%)					
	E.I.S. (ψ)					
R.R.A (γ)	1/0.75	1/1.25	1/1.4	1/2	1/2.5	1/3
0.75	1.7677	16.0520	23.1559	57.7313	88.0512	116.6679
1.25	2.1074	14.7027	20.3604	47.1304	71.1238	95.0293
1.4	2.1643	14.4651	19.8900	45.6892	68.4882	91.0613
2	2.3133	13.8875	18.8697	42.2475	62.9560	83.7390
3	2.4343	13.5094	18.1111	39.4521	58.8529	78.2802
10	2.6166	12.9568	17.0442	36.3073	52.7644	70.0088
No-Jumps Model	**Consumption-Wealth Ratio** (%)					
	E.I.S. (ψ)					
R.R.A (γ)	1/0.75	1/1.25	1/1.4	1/2	1/2.5	1/3
0.75	1.2039	16.0147	24.5951	73.9475	124.5295	177.9226
1.25	1.5425	13.8906	20.1538	52.9806	85.4590	118.6873
1.4	1.6036	13.5660	19.4807	49.8798	80.1968	111.7709
2	1.7697	12.8252	17.9599	43.7385	68.3943	94.3549
3	1.9081	12.2476	16.8093	39.2259	60.3549	82.5620
10	2.1201	11.4995	15.3557	33.5010	50.2659	67.5364

Table 4.4. Percentage of consumption-wealth ratio for each models (Co-Jumps Model, Only Jumps in Price and No-Jumps Model) for different risk aversion coefficient and elasticities of intertemporal substitution of consumption. The rate of time preferences β is set equal to 6% annually, the risk-free rate is set to $r = 0\%$. The remaining parameters are given in Table 4.1.

($\gamma = 10$). Finally, by comparing the Co-Jumps, Jumps in price, and No Jumps models, we detect that the presence of jumps affects the optimal consumption when $\psi < 1$. The effect is even more intense the smaller the EIS value. By comparing the mid and bottom panels in Table 4.4, we find that jumps in price reduce the optimal consumption, regardless of the risk aversion level, because of the greater risk investors are exposed to. However, the impact of jumps in volatility (or precision) is significant, as stressed by the comparison between the top and mid panel in Table 4.4. Therefore, we can state that jumps in volatility further reduce optimal consumption in a very significant way. When $\psi > 1$, we notice a different behaviour: investors exhibit more significant savings and negligible consumption. Moreover, if jumps in price are included, the average portfolio return reduces since investors consider a smaller portion of the risky asset. This implies that investors choose portfolios

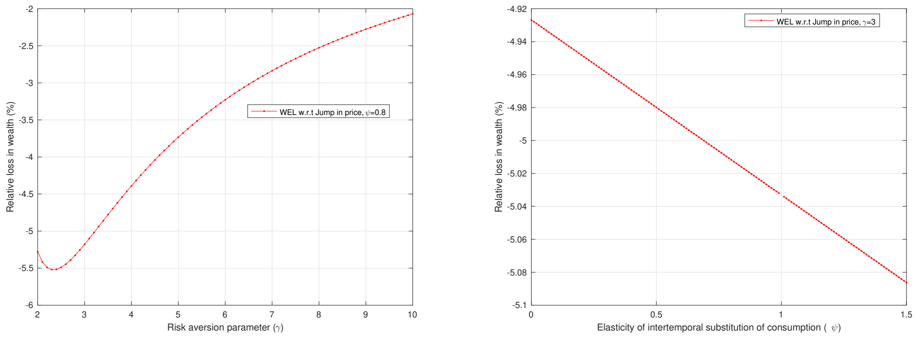

Figure 4.1. Percentage of relative loss in wealth as a function of the risk aversion parameter with $\psi = 0.8$. The maturity investment is 1 year, the rate of time preferences is $\beta = 6\%$, the risk-free rate is set to $r = 0\%$. The remaining parameters are given in Table 4.1.

with low expected returns, so a higher ψ corresponds to a higher consumption-wealth ratio.

To assess the effect of jumps in volatility, we resort to the *Wealth Equivalent Loss* (WEL), i.e. an economic metric that measures the possible loss suffered by an investor who assumes only jumps in price (or the absence of jumps) as a condition for determining the investment strategy. The WEL is defined as

$$WEL = 1 - \tilde{W}(t) \ ,$$

where $\tilde{W}$ is obtained by equating the value function $V(t, y, \tilde{W}) = \exp\{-H(Tr(F(t)Y(t)) + G(t))\}\frac{\tilde{W}(t)^{1-\gamma}}{1-\gamma}$ associated with the strategy involving co-jumps and the value function $\hat{V}(t, y, 1) = \exp\{-H(Tr(\hat{F}(t)Y(t)) + \hat{G}(t))\}\frac{1}{1-\gamma}$ for the sub-optimal strategy, where $H = \frac{1-\gamma}{1-\psi}$. More precisely,

$$\tilde{W}(t) = \left[\frac{\exp\{-H(\hat{F}(t)Y(t) + \hat{G}(t))\}}{\exp\{-H(F(t)Y(t) + G(t))\}}\right]^{\frac{1}{1-\gamma}} = \left[\exp\{(\hat{F}(t) - F(t))Y(t) + (\hat{G}(t) - G(t))\}\right]^{-\frac{1}{1-\psi}} \ , \tag{4.141}$$

where $F(t), G(t)$ (resp., $\hat{F}(t), \hat{G}(t)$) are the solutions to the ODEs provided in Proposition 4.1.1, suitably modified according to the presence (resp., the absence) of the jumps in volatility. Figure 4.1 shows the loss in wealth that an investor might suffer when she does not believe in jumps in volatility, as a function of the risk-aversion parameter γ and the elasticity of intertemporal substitution of consumption ψ. In particular, the figure shows that not including volatility jumps in the model would result in non-trivial losses in terms of wealth. For example, with a risk aversion parameter $\gamma = 3$, the investor would suffer a monetary loss of about 5%. This result reinforces the point that volatility jumps play a crucial role in determining the optimal behaviour of an investor in hedge funds portfolios.

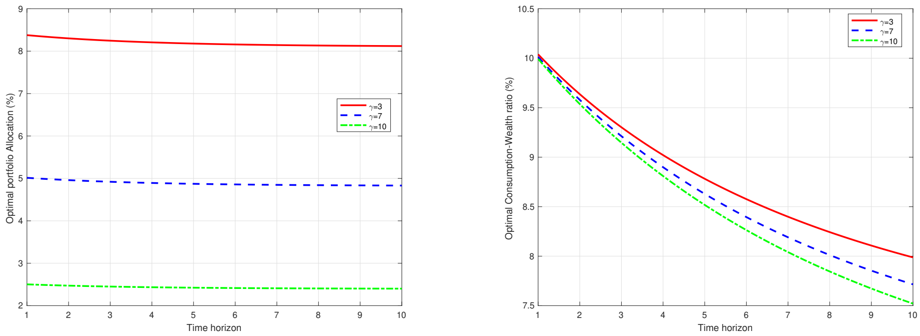

Figure 4.2. Optimal portfolio allocation (left chart) and consumption (right chart) as functions of time horizon, for several values of the risk-aversion parameter γ. The EIS parameter is fix to $\psi = 0.8$. The rate of time preferences β is set equal to 6% annually, the constant risk-free rate is equal to $r = 0\%$. The remaining parameters are given in Table 4.1.

4.4.2 Numerical example: the univariate case:incomplete market setting

To highlight the impact of jumps on the optimal allocation, in this section, we provide some analyses when a single fund (CA) is considered. The CA index displays a positive mean (6.89%) and low standard deviation (6.25%), making the fund attractive. However, it has negative skewness (-2.73) and high kurtosis (21.58), transforming the fund into a risky investment vehicle.

Our first analysis shows optimal portfolio allocation and consumption-wealth ratio, evaluated at $t = 0$, as functions of the investment time horizon, for different risk-aversion levels. The results are presented in Figure 4.2. As expected, the optimal allocation decreases over time due to the negative intertemporal hedging demand, see e.g. Liu and Pan [2003] and Branger et al. [2008]. Considering the parameters obtained through the calibration procedure and assuming $\gamma > 1$, the function F results to be negative, see Remark 4.1.1 for further details. Furthermore, as the maturity becomes larger, both the intertemporal and jump hedging demands rise (in absolute value) to compensate for the additional risk by increasing the investment time horizon. Consequently, the value to be subtracted from the myopic component is more significant as the maturity is pushed back in time, leading to a reduction of the overall optimal allocation. Concerning the optimal consumption, we note a shrinkage of the latter when the maturity becomes larger, assuming a fixed elasticity of intertemporal substitution of consumption equal to $\psi = 0.8$. This is not surprising, since an EIS smaller than one conveys that the agent is unwilling to base today's decision on events distant in time.

To further clarify to what extent jumps may affect the optimal policies, we study the optimal portfolio weights and consumption-wealth ratio as functions of the risk aversion parameter γ, for various price-jump sizes J, and keeping the precision-jump size fixed. The results are shown in Figure 4.3 (top charts). We note that when the price-jump size reduces (in absolute value), the investor tends to take a more sizeable position in the risky asset, especially when the investor's risk attitude goes to one, for a fixed value of EIS ($\psi = 0.8$).

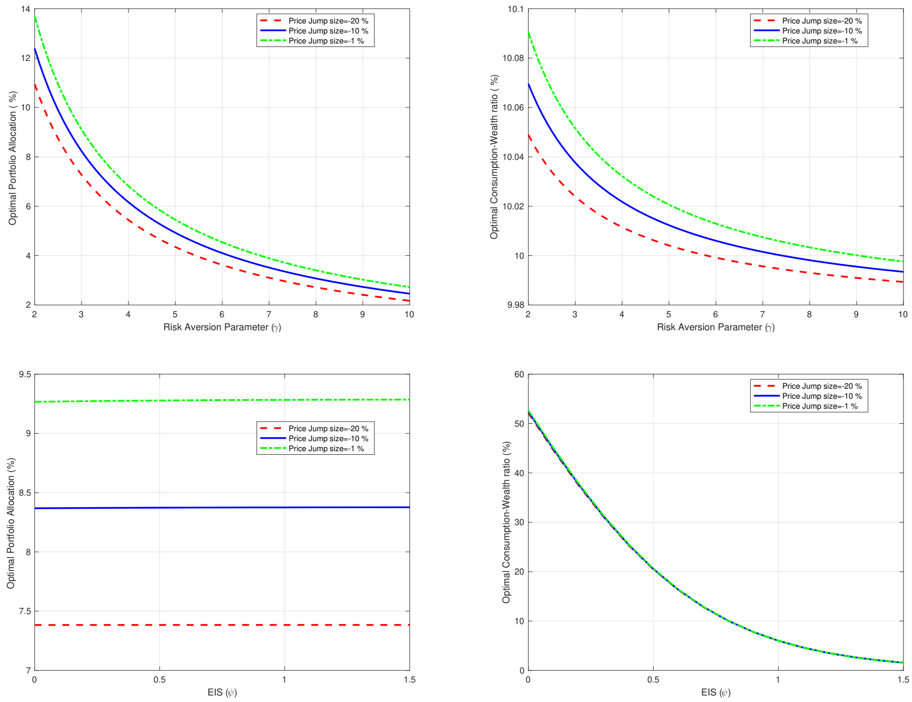

Figure 4.3. Optimal portfolio allocation (left charts) and consumption (right charts) as functions of γ (top) and ψ (bottom), for different price-jump sizes. In top charts the EIS is fixed to $\psi = 0.8$, for the bottom ones the risk-aversion parameter is fixed to $\gamma = 3$. The rate of time preferences β is set equal to 6% annually, the constant risk-free rate is $r = 0\%$.The remaining parameters are given in Table 4.1.

From an economic point of view, this means that assuming a model with less exposure to the jump risk event leads to considering a considerable position in the risky asset.

The optimal consumption is affected by changes in the price-jump amplitude, albeit to a lesser degree. By choosing again $\psi = 0.8$ and increasing (in absolute value) the price-jump size, the optimal consumption-wealth ratio reduces as a consequence of the lower expected portfolio returns. We further stress that the optimal consumption decreases as the investor's risk aversion increases, as she will choose safer portfolios with a less expected portfolio return.

A similar study can be accomplished by analysing optimal allocation and consumption-wealth ratio as functions of EIS for different price-jump amplitudes, as shown in Figure 4.3 (bottom charts). The numerical results confirm that the optimal allocation is not remarkably affected by EIS variations if we keep the risk-aversion parameter fixed. Moreover, in this case, the optimal allocation significantly increases if we consider a price-jump size equal to $J = -1\%$. Looking at the bottom-right chart of Figure 4.3, we note that for an investor extremely reluctant to substitute consumption over time, a greater price-jump size (in absolute value) results in smaller consumption. As the investor becomes more willing to substitute consumption over time, the impact of jumps in price becomes negligible since these investors have more significant savings and lower consumption.

To investigate the effect of jumps in precision on optimal policies, we study the latter as

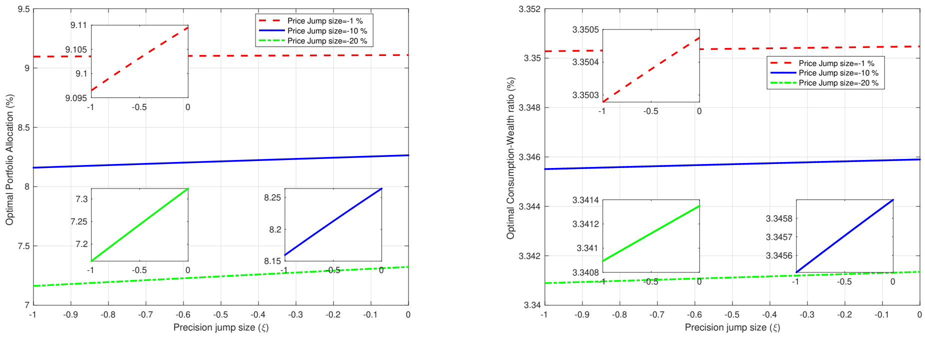

Figure 4.4. Optimal portfolio allocation (left chart) and Optimal consumption-wealth ratio (right chart) as a function of precision-jump size, for different price-jump sizes. The risk-aversion parameter is $\gamma = 3$, and the elasticity of intertemporal substitution is set equal to $\psi = 0.8$. The rate of time preferences β is set equal to 6% annually, the constant risk-free rate is $r = 0\%$. The remaining parameters are given in Table 4.1.

functions of the precision-jump magnitudes, assuming different price-jump sizes. We refer to Figure 4.4 for the corresponding results. We observe that introducing jumps in volatility produces a further effect for optimal weights: a more remarkable volatility jump translates into a smaller jump in precision, causing a reduction in the optimal portfolio allocation. However, such a curtailment has little impact and depends on the value of the price-jump size. Concerning the optimal consumption (right chart in Figure 4.4), as expected, we witness a reduction for wider price-jump magnitudes (in absolute value): as the amplitude of the jump in precision rises (in absolute value), the consumption-wealth ratio decreases as well due to the reduction of the expected portfolio return.

Finally, we investigate the role played by the jump-frequency over the optimal portfolio weights, see Table 4.5. We assume that $freq = 1/(\lambda \times 12)$, i.e., we consider the jump frequency expressed in years and intended as the reciprocal of the jump intensity, adjusted by considering that the parameters are in the form of monthly estimates. More precisely, optimal portfolio weights are obtained by considering four jump intensities, namely $\lambda = \{0.007,\ 0.0033,\ 0.0016, 0.0008\}$. Such a set corresponds to a jump frequency of 11.94, 25, 50 and 100 years on average, respectively. Our findings move along three directions. First, we recover that the optimal portfolio allocation is significantly affected by jumps in price. We further note that infrequent jumps mainly influence the risky allocation. Moreover, we confirm that jumps in precision have little impact on optimal policy, regardless of the jump frequency. More precisely, the impact of the precision-jump size is less relevant but not negligible, especially when considering variations in the jump intensity. The investor tends to take a more prominent position in optimal allocation when the jump size is small: the smaller the jump intensity, the higher the portfolio allocation. Third, we observe that frequent and large jumps in price impact optimal allocation more than rare jumps of the same magnitude. Assuming that a jump occurs every 12 years, the allocation decreases by approximately 6.5% if the price jump increases by 5 percentage points, namely, we move from $J = -20\%$ to $J = -15\%$. For rare jumps (e.g., one jump every 100 years), the variation of the optimal allocation is by 0.6%. The proportions remain almost unchanged as the jump

Optimal allocation weights					
Risk Aversion (γ)	Precision-Jump size (ξ)	Jump intensity (λ)	Price-Jump Size (J)		
			$J = -25\%$	$J = -20\%$	$J = -15\%$
3	-0.20	0.007	6.8173	7.2914	7.7669
		0.0033	8.0734	8.2985	8.5241
		0.0016	8.6541	8.7638	8.8735
		0.0008	8.9171	8.9743	9.0315
3	-0.50	0.007	6.7651	7.2435	7.7259
		0.0033	8.0384	8.2689	8.5006
		0.0016	8.6345	8.7476	8.8611
		0.0008	8.9062	8.9654	9.0248
5	-0.20	0.007	4.0804	4.3643	4.6493
		0.0033	4.8330	4.9681	5.1034
		0.0016	5.1815	5.2473	5.3132
		0.0008	5.3394	5.3737	5.4081
5	-0.50	0.007	4.0429	4.3299	4.6197
		0.0033	4.8078	4.9467	5.0865
		0.0016	5.1673	5.2357	5.3042
		0.0008	5.3315	5.3673	5.4032
10	-0.20	0.007	2.0364	2.1782	2.3206
		0.0033	2.4124	2.4799	2.5476
		0.0016	2.5867	2.6196	2.6525
		0.0008	2.6656	2.6828	2.7000
10	-0.50	0.007	2.0154	2.1589	2.3040
		0.0033	2.3982	2.4679	2.5380
		0.0016	2.5787	2.6130	2.6475
		0.0008	2.6612	2.6792	2.6973
20	-0.20	0.007	1.0173	1.0881	1.1593
		0.0033	1.2052	1.2389	1.2728
		0.0016	1.2923	1.3088	1.3253
		0.0008	1.3318	1.3404	1.3490
20	-0.50	0.007	1.0062	1.0779	1.1505
		0.0033	1.1977	1.2326	1.2677
		0.0016	1.2881	1.3053	1.3226
		0.0008	1.3295	1.3385	1.3476

Table 4.5. Percentage optimal portfolio allocation for different values of jump-intensity, price-jump size, risk-aversion parameter, and precision-jumps size. The EIS is fix equal to $\psi = 0.8$. The rate of time preferences β is set equal to 6% annually, and the constant risk-free rate is $r = 0\%$. The remaining parameters are given in Table 4.1.

amplitudes in precision and/or the risk aversion parameter vary.

We conclude our study with a sensitivity analysis for WEL w.r.t. the parameters characterising the preferences involved, namely the EIS and the risk-aversion parameters. We consider a span for the risk aversion coefficient γ in $[1.1, 10]$. The results are illustrated in Figure 4.5. First, we observe that excluding jumps from the model always results in a potential loss for investors, regardless of their risk attitudes and consumption tendencies. Our qualitative analysis also shows that as the elasticity of substitution of consumption increases and the investor's risk aversion coefficient decreases, the loss suffered by the investor increases. Therefore, the aggressiveness of an investor who believes in the sub-optimal strategy (No-Jumps model) is rewarded through a maximum loss. As the investor becomes less aggressive, the potential loss she may incur decreases until a plateau is reached. When the investor's propensity to consume increases ($\psi \geq 0.5$), we see a WEL trend similar to the previous one: the very aggressive investor reaches a maximum loss, which is more significant

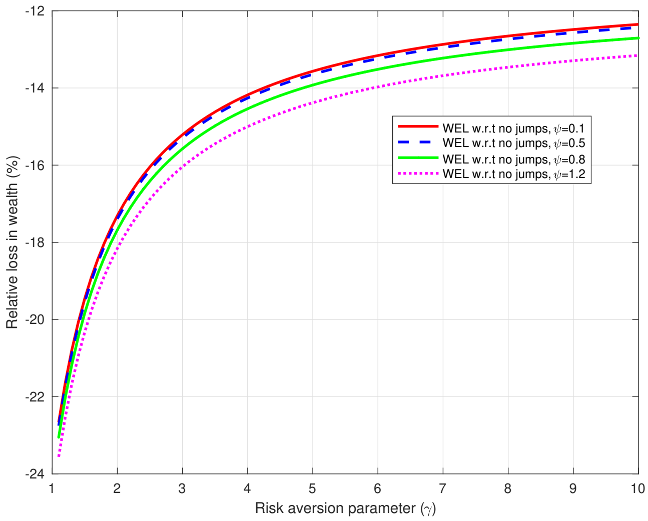

Figure 4.5. Percentage of relative loss in wealth for the sub-optimal strategy without jumps as a function of the risk aversion parameter with different values for the elasticity of intertemporal substitution of consumption (EIS). The maturity investment is one year, the rate of time preferences is $\beta = 6\%$ and the risk-free rate is set to $r = 0\%$. The remaining parameters are given in Table 4.1.

as the EIS grows.

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4.4.3 Numerical example: how do the numerical results change if we complete the market?

This section analyses the impact of derivative inclusion and the assumption of a complete market setting on optimal policy and consumption. We consider the parameterization given by the table (3.1).

We complete the market with two non-redundant derivatives, so we include two European options, a 3-month 5% -OTM call, and a 3-month 5% -ITM call. The first analysis, from Figure (4.6), concerns the optimal weights and consumption as a function of the risk aversion parameter. The results of the univariate case are confirmed: as risk aversion increases, the allocation in the risky asset decreases, as do the allocations in derivatives (albeit in absolute value) and optimal consumption. The allocations in the optimal weights depend on the derivatives chosen. Still, the impact of the risk aversion coefficient on the optimal allocation is the same even if we make different choices in terms of the derivatives chosen.

In Figure (4.7), we analyze the wealth loss that the investor suffers when not considering co-jumps or jumps at all in a complete market setting. Then we again resort the Wealth Equivalent loss, as defined in the subsection (4.4.1) in the multivariate case by the equation (4.141), which is an economic metric that shows us the loss suffered by the agent when considering as an optimal strategy the one in which co-jumps are included in the market and as a sub-optimal strategy the one in which the investor believes in a market without jumps (either price and volatility or only price). From the figure (4.7), we can see that the loss of wealth is huge when jumps are not included in the market, especially for very risk-averse agents.

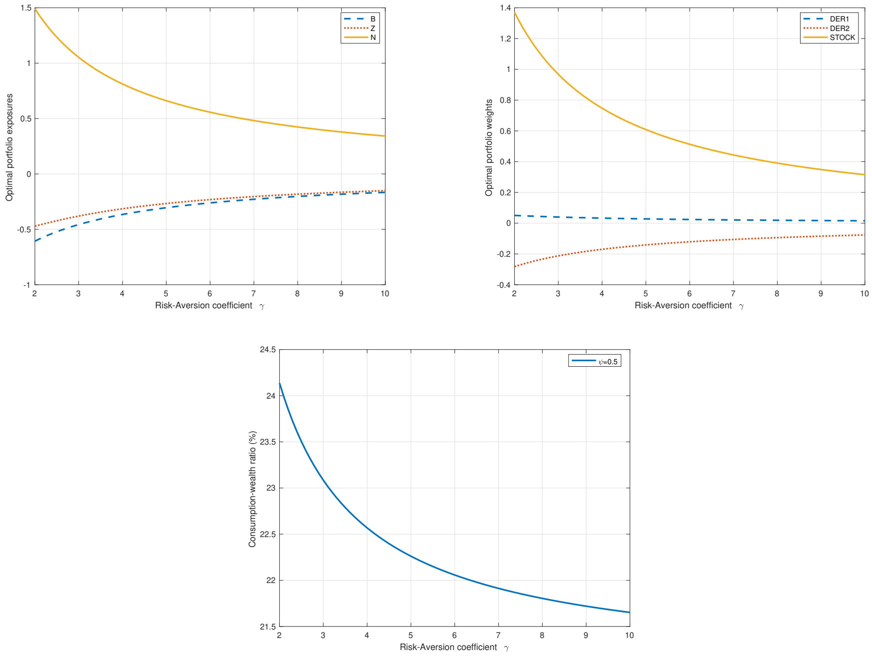

Figure 4.6. Optimal weights and consumption wealth ratio as a function of risk aversion parameter. The maturity investment is one year, the rate of time preferences is $\beta = 6\%$ and the risk-free rate is set to $r = 0\%$. The remaining parameters are given in Table 4.1.

4.5 The accuracy of the approximate solution

To understand whether the solutions proposed in Section 4.1 are accurate, we assess the reliability of the approximation assumptions. We recall that we are assuming two types of linearization procedures: one for the jump component and one for the logarithmic consumption wealth ratio.

4.5.1 Reliability of the jump-component linearization

To compare true and approximate solutions, we consider a simpler form of the model, where we consider only a single risky asset and by assuming that the precision $y(t)$ is constant over time. More precisely, the dynamics (2.12) simplifies to the well-known Merton's Jump-Diffusion model

$$\begin{cases} \frac{dS(t)}{S(t)} = \alpha dt + \sqrt{y(0)^{-1}} dB(t) + J dN(t) \\ dy(t) = 0 \end{cases} . \tag{4.142}$$

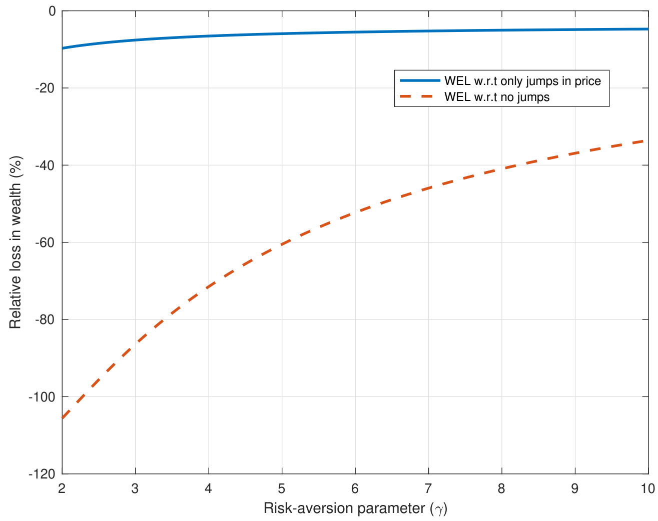

Figure 4.7. Percentage of relative loss in wealth for the sub-optimal strategy without jumps as a function of the risk aversion parameter with several values for the elasticity of intertemporal substitution of consumption (EIS). The maturity investment is one year, the rate of time preferences is $\beta = 6\%$, and the risk-free rate is set to $r = 0\%$. The remaining parameters are given in Table 4.1.

Assume first $\psi = 1$. Under such assumptions, we write the investor's indirect utility function as

$$V(t, W) = \frac{W(t)^{1-\gamma}}{1-\gamma} \exp\{F(t)\} , \tag{4.143}$$

for any $t \in [0, T]$.

If we do not linearize the jump component, the first order condition for the *true* optimal allocation weights is

$$0 = (\alpha - r) - \frac{\gamma \pi(t)}{y(0)} + \lambda(1 + \pi(t)J)^{-\gamma} J , \tag{4.144}$$

while the function F appearing in (4.143) is the solution to the following ODE

$$\begin{aligned}\dot{F}_t = & -\beta(1-\gamma)\log(\beta) + \beta F(t) - r(1-\gamma) - \pi_t(\alpha - r)(1-\gamma) + \beta(1-\gamma) \\ & + \frac{1}{2}\gamma(1-\gamma)\frac{\pi(t)^2}{y_0} - \lambda\left[(1 + J\pi(t))^{1-\gamma} - 1\right] .\end{aligned} \tag{4.145}$$

Both Eq. (4.144) and (4.145) can be easily solved by using *ad-hoc* numerical procedures.

Analogously, when we linearize the jump component, the first order condition for the *approximate* optimal allocation weights is

$$0 = (\alpha - r) - \frac{\gamma \pi(t)}{y(0)} + \lambda J , \tag{4.146}$$

	$J=-10\%$				$J=-15\%$				$J=-25\%$		
γ	$\tilde{\pi}(t)(\%)$	$\pi(t)(\%)$	$RWEL(\%)$	γ	$\tilde{\pi}(t)(\%)$	$\pi(t)(\%)$	$RWEL(\%)$	γ	$\tilde{\pi}(t)(\%)$	$\pi(t)(\%)$	$RWEL(\%)$
2	13.93	13.92	-0.0190	2	13.92	13.91	-0.0287	2	13.90	13.90	-0.0482
3	9.28	9.28	-0.0127	3	9.28	9.27	-0.0191	3	9.27	9.26	-0.0321
4	6.96	6.96	-0.0095	4	6.96	6.95	-0.0143	4	6.95	6.95	-0.0241
5	5.57	5.57	-0.0076	5	5.57	5.56	-0.0114	5	5.56	5.56	-0.0192
6	4.64	4.64	-0.0063	6	4.64	4.63	-0.0095	6	4.63	4.63	-0.0160
7	3.98	3.97	-0.0054	7	3.98	3.97	-0.0082	7	3.97	3.97	-0.0137
8	3.48	3.48	-0.0047	8	3.48	3.47	-0.0071	8	3.48	3.47	-0.0120
9	3.09	3.09	-0.0042	9	3.09	3.09	-0.0063	9	3.09	3.08	-0.0107
10	2.79	2.78	-0.0038	10	2.78	2.78	-0.0057	10	2.78	2.78	-0.0096

Table 4.6. Optimal approximate and true allocation weights and associated RWEL (expressed in percentage) for several risk aversions parameter γ and price-jump sizes. The time preference rate β is set at 6% per year, the risk-free rate is set at $r = 0\%$.

providing a closed-form expression for the optimal weights, while the function F appearing in (4.143) is the solution to the following ODE

$$\begin{aligned}\dot{F}_t = &-\beta(1-\gamma)\log(\beta) + \beta F(t) - r(1-\gamma) - \frac{1}{2}\frac{(\alpha-r)^2(1-\gamma)}{\gamma}y(0) + \beta(1-\gamma)\\ &-\frac{\lambda^2 J^2(1-\gamma)}{2\gamma}y(0) - \frac{J\lambda(1-\gamma)(\alpha-r)}{\gamma}y(0)\ . \end{aligned} \tag{4.147}$$

By indicating with V and π (resp., $\tilde{V}$ and $\tilde{\pi}$) the value function and the optimal weight for the true model (resp., the approximate model), we can evaluate the *Relative WEL* (RWEL), i.e. the percentage of initial wealth that can be sacrificed by an investor using the optimal strategy, in order to have the same indirect utility using the approximate one, see Ascheberg et al. [2016] for further details. The RWEL is obtained by equating $V(t, W(1-RWEL); \pi) = \tilde{V}(t, W; \tilde{\pi})$, hence, for $t \in [0, T]$, we get

$$RWEL = 1 - \frac{1}{1-\gamma}\exp\{\tilde{F}(t) - F(t)\}\ . \tag{4.148}$$

We execute a sanity check by comparing the *true* solution with the *approximate* one in terms of RWEL.

The results are depicted in Table 4.5.1, for several risk aversion parameter values and price-jump amplitudes. We note that the approximate optimal strategy overestimates the true one, providing a potential loss. However, such a loss is minimal, varying between 0.0038% (when $\gamma = 10$ and $J = -10\%$) to 0.0482% (when $\gamma = 2$ and $J = -25\%$).

Now assume $\psi \neq 1$. The investor's indirect utility function is

$$V(t, W) = \frac{W(t)^{1-\gamma}}{1-\gamma}\exp\{-HF(t)\}\ , \tag{4.149}$$

for any $t \in [0, T]$. As for the *true* strategy, the first order condition for the optimal allocation weights is given in (4.144), while the function F appearing in (4.149) is the solution to the following ODE

$$\begin{aligned}\dot{F}_t = &\, h_1 F(t) - h_1\psi\log\beta + \beta\psi - h_0 + r(1-\psi) + \pi(t)(\alpha-r)(1-\psi)\\ &-\frac{1}{2}\gamma(1-\psi)\frac{\pi(t)^2}{y_0} + \frac{\lambda}{H}\left[(1+J\pi(t))^{1-\gamma} - 1\right]\ . \end{aligned} \tag{4.150}$$

	$J=-10\%$				$J=-15\%$				$J=-25\%$		
γ	$\psi=0.2$	$\psi=0.5$	$\psi=1.5$	γ	$\psi=0.2$	$\psi=0.5$	$\psi=1.5$	γ	$\psi=0.2$	$\psi=0.5$	$\psi=1.5$
3	-0.0000741	-0.0000784	-0.0000867	3	-0.0001676	-0.0001773	-0.0001962	3	-0.0004705	-0.0004978	-0.0005508
5	-0.0000446	-0.0000469	-0.0000519	5	-0.0001008	-0.00010620	-0.0001175	5	-0.0002827	-0.0002977	-0.0003294
7	-0.00000318	-0.0000335	-0.0000371	7	-0.0000719	-0.0000758	-0.0000838	7	-0.0002016	-0.0002123	-0.0002349
10	-0.0000229	-0.0000234	-0.0000259	10	-0.0000503	-0.0000530	-0.0000586	10	-0.000141	-0.0001485	-0.0001643

Table 4.7. RWEL (expressed in percentage) for several values of risk aversions parameter γ, elasticity of intertemporal substitution of consumption ψ and price-jump sizes. The time preference rate β is set at 6% per year, the risk-free rate is $r=0\%$.

As for the *approximate* strategy, the first order condition for the optimal allocation weights is given in (4.146) and the function F appearing in (4.149) is the solution to the following ODE

$$\begin{aligned}\dot{F}_t &= h_1 F(t) - h_1\psi\log\beta + \beta\psi - h_0 + r(1-\psi) + \pi_t(\alpha - r)(1-\psi) \\ &\quad - \frac{1}{2}\gamma(1-\psi)\frac{\pi(t)^2}{y_0} + \lambda(1-\psi)J\pi(t)\,.\end{aligned}$$

The RWEL for $t\in[0,T]$ is

$$RWEL = 1 - \frac{1}{1-\gamma}\exp\{-H[\tilde{F}(t) - F(t)]\}\,. \tag{4.151}$$

We compare *true* and *approximate* solutions in terms of RWEL. The results are depicted in Table 4.7, for several values of risk aversion parameter γ, elasticity of intertemporal substitution of consumption ψ, and price-jump amplitudes. In this case, the potential loss is always lesser than 0.02%. As expected, the loss reduces for higher values of ψ and γ, while heightens for larger price-jump magnitudes.

4.5.2 The accuracy of the consumption-wealth linearization

We recall that the consumption-wealth ratio is approximated using an appropriate log-linear expansion around the unconditional mean $\overline{c-w} := \mathbb{E}\left[\frac{C(t)}{W(t)}\right]$. The reliability of such an approximation moves along the lines of Chacko and Viceira [2005] and can be measured by calculating the unconditional standard deviation of the optimal log consumption-wealth ratio, given by

$$\mid F_t \mid \sqrt{Var(y(t))} = \mid F(t) \mid \sqrt{\frac{\sigma^2}{2k}\left(\theta + \frac{\xi\lambda}{k}\right)}\,.$$

Figure 4.8 shows that the optimal consumption-wealth ratio exhibits high volatility only for investors with small values for γ and ψ. For example, when $\psi = 0.2$, the standard deviation is about 1.5%, while, for $\psi = 0.9$, it is about 0.16%, when $\gamma = 2$. Our results are corroborated by the discussion in Campbell and Koo [1997], where the authors explain that the approximation error can be considered bearable when the standard deviation of the log consumption-wealth ratio is smaller than 5%.

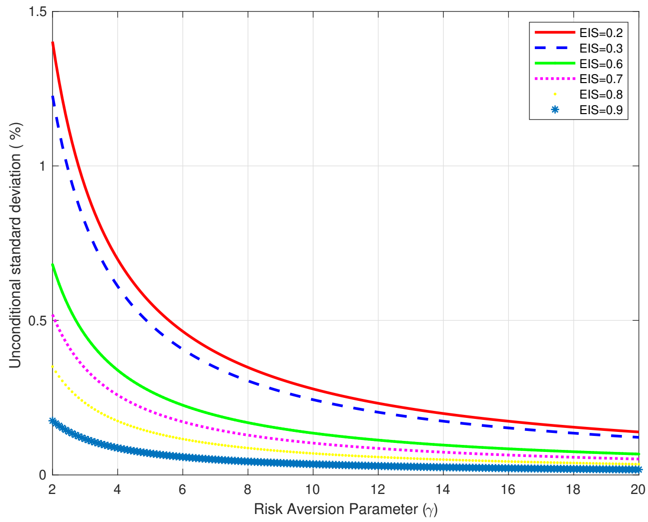

Figure 4.8. Percentage values of the unconditional standard deviation of the optimal log consumption–wealth ratio.

Chapter 5

Robust optimal consumption and portfolio choice in complete markets with co-jumps

This chapter attempts to provide a unifying approach for portfolio choice problems that simultaneously captures all the features discussed hitherto, i.e. *(i)* the agent's risk aversion and uncertainty about the market model, *(ii)* the intention to maximize profits and minimize losses, accounting for the investor's preference for either consumption or savings over time, *(iii)* the occurrence of sudden events that generate discontinuities in stock and volatility trends, and *(iv)* the access to the derivatives market.

We study a robust, dynamic, continuous-time optimal consumption and portfolio allocation problem, assuming investors have recursive preferences, as in Chapter (4.4), and may access both stock and derivative markets. Besides, we consider investors to be both risk and ambiguity-averse, focusing on uncertainty about all the risk factors that may affect the market. We allow for different levels of ambiguity aversion against diffusive and jump risks and treat the model uncertainty problem as a robust control problem along the lines of Anderson et al. [2003]. Therefore, we assume there exists both a reference model and a family of alternative models, statistically indistinguishable from each other, and we choose to weigh the distance between an alternative model and the reference one by assigning a penalty factor. Thus, our objective is to be able to select the best proportion of wealth to be invested and consumption to be produced in such a way as to maximize the investor's utility in the worst-case scenario. In such a complete market setting, we obtain analytical solutions to the optimal investment problem, depending on suitable deterministic functions that are quickly evaluated via numerical procedures. Our optimal strategies are exact for investors with unitary intertemporal elasticity of substitution), and approximate otherwise. Our results show that optimal exposures are driven by the corresponding ambiguity aversion parameters. As ambiguity risk aversion increases, the corresponding optimal exposures decrease due to the deterioration of the investors' opportunity set. Optimal consumption is also affected by both diffusive and jump ambiguity parameters: in particular, for more consumption-oriented investors, the consumption-wealth ratio decreases as a function of the ambiguity-aversion parameters due to deteriorating expectations about investment opportunities due to the negative income effect. A thorough sensitivity analysis concerning the main model parameters corroborates the theoretical results. Furthermore, to quantify the impact of model uncertainty on optimal wealth and consumption policies, we measure the potential loss of an investor who decides to adopt a sub-optimal (unambiguous) investment strategy. The numerical results, obtained in terms of *Wealth Equivalent Loss* (WEL), show that ignoring ambiguity might cause considerable losses that augment in case of higher ambiguity-aversion levels. The Chapter is organized as follows. Section 5.1 contains the technical specifications for the market model and we build the optimal investment and consumption problem for an ambiguity-averse investor with recursive preferences and derive the optimal solutions for the investor's utility maximization problem. In Section 5.2, we provide numerical examples on real data, to quantify the impact of co-jumps, ambiguity, and derivatives on investment choices. Section 5.3 estimates the degree of reliability of the approximations in a context where we consider uncertainty about the model.

5.1 The dynamic optimal allocation problem and the optimal policy

In this chapter all the stochastic processes involved in our successive analyses will be considered w.r.t. a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$, being $\mathbb{F} := \{\mathcal{F}(t),\, t \in [0,T]\}$ the natural filtration verifying the standard requirements and $\mathbb{Q}$ the risk-neutral measure equivalent to the real-world measure $\mathbb{P}$.

We consider a frictionless, continuous-time financial market consisting of a riskless asset $M = \{M(t)\}_{t\in[0,T]}$ and a stock $S = \{S(t)\}_{t\in[0,T]}$. The model and dynamics are defined in the (2.1) section. Specifically, the dynamics of the risk-free asset is given by Equation (2.1) while the dynamics of the risky asset is that of the equation (2.10). The market thus described is incomplete: we include $N = 2$ contingent claims with underlying S and maturity T whose price, dynamics, and non-redundancy condition are given in the section (2.3).

To determine the investor's optimal policies, we must resort to the real-world probability measure $\mathbb{P}$. Let us then consider the dynamics provided by the equation (2.11) in section (2.1) for the risky asset.

Eq. (2.11) is referred to as the *reference model,* i.e., the best representation of data available to the investor, see e.g. Wei et al. [2023]. On the other hand, the investor might be ambiguous about the probability distribution for the reference model; thus, it could be useful to look at plausible alternatives. If the investor is not sure about the *reference model*, she could choose among a family of models, that are difficult to distinguish from the reference one, Wei et al. [2023]. We incorporate ambiguity into our model and treat it as an additional source of uncertainty, along the lines of Anderson et al. [2003] and Hansen et al. [2006]. Thus, we must introduce a set $\mathcal{P} = \{\mathbb{P}^\epsilon : \epsilon \in \mathcal{E}\}$ of prior probability measures reflecting the investors' subjective beliefs, see e.g. Wei et al. [2023], where $\mathcal{E}$ is the set of admissible Markovian drift contaminations, see e.g. Faria and Correia-da Silva [2016].

We further consider two $\mathbb{P}^\epsilon$-Wiener processes $\tilde{B} = \{\tilde{B}(t)\}_{t \in [0,T]}$ and $\tilde{Z} = \{\tilde{Z}(t)\}_{t \in [0,T]}$, related to the corresponding $\mathbb{P}$-Wiener process in the reference model through

$$d\tilde{B}(t) = dB(t) + \epsilon^S(t)dt, \; t \in [0,T] \; , \tag{5.1}$$

$$d\tilde{Z}(t) = dZ(t) + \epsilon^y(t)dt, \; t \in [0,T] \; . \tag{5.2}$$

The same reasoning applies to the $\mathbb{P}^\epsilon$-Poisson process $\tilde{N}^\epsilon = \left\{\tilde{N}^\epsilon(t)\right\}_{t \in [0,T]}$, such that

$$d\tilde{N}^\epsilon(t) = dN(t) - \lambda^{\mathbb{P}}\epsilon^N(t)dt = d\tilde{N}(t) + \lambda^{\mathbb{P}}\left(1 - \epsilon^N(t)\right)dt \; , \tag{5.3}$$

being $\lambda^{\mathbb{P}}\epsilon^N(t)dt$ the compensator of the Poisson process. By exploiting (5.1), (5.2) and (5.3), we are able to write the $\mathbb{P}^\epsilon$-dynamics for the *alternative model* as follows

$$\begin{cases} \frac{dS(t)}{S(t)-} & = \left(r + \eta - \epsilon^S(t)\sqrt{y(t)^{-1}}\right)dt + \sqrt{y(t)^{-1}}d\tilde{B}(t) + Jd\tilde{N}(t)^\epsilon + J\left(\lambda^{\mathbb{P}}\epsilon^N(t) - \lambda^{\mathbb{Q}}\right)dt \\ dy(t) & = k(\theta - y(t))dt - \sigma\sqrt{y(t)}\rho\epsilon^S(t)dt - \sigma\sqrt{y(t)}\sqrt{1-\rho^2}\epsilon^y(t)dt \\ & +\sigma\sqrt{y(t)}\left(\rho d\tilde{B}(t) + \sqrt{1-\rho^2}d\tilde{Z}(t)\right) + \xi(y(t))d\tilde{N}^\epsilon(t) \end{cases} . \tag{5.4}$$

Using Ito's Lemma, the $\mathbb{P}^\epsilon$-dynamics of the i-th derivative security $O^{(i)}(t)$, $i = 1, 2$, are

$$\begin{aligned} \frac{dO^{(i)}(t)}{O^{(i)}(t)} = & \left[r + \left(\frac{g_s^i}{O_t^i}S(t) + \sigma\rho\frac{g_y^i}{O_t^i}y(t)\right)\eta - \left(\frac{g_s^i}{O_t^i}S(t) + \sigma\rho\frac{g_y^i}{O_t^i}y(t)\right)\epsilon^S(t)\sqrt{y(t)^{-1}}\right. \\ & \left. +\sigma\sqrt{1-\rho^2}\frac{g_y^{(i)}}{O^{(i)}(t)}\beta y(t) - \sigma\sqrt{1-\rho^2}\frac{g_y^i}{O_t^i}\sqrt{y(t)}\epsilon^y(t)\right]dt \\ & + \left[\frac{g_s^{(i)}}{O^{(i)}(t)}S(t) + \sigma\rho\frac{g_y^{(i)}}{O^{(i)}(t)}y(t)\right]\sqrt{y(t)^{-1}}d\tilde{B}(t) + \sigma\sqrt{1-\rho^2}\frac{g_y^{(i)}}{O^{(i)}(t)}\sqrt{y(t)}d\tilde{Z}(t) \\ & + \frac{\Delta g^{(i)}}{O^{(i)}(t)}\left[d\tilde{N}^\epsilon(t) + (\lambda^{\mathbb{P}}\epsilon^N(t) - \lambda^{\mathbb{Q}})dt\right], \; t \in [0,T]. \end{aligned} \tag{5.5}$$

Let $W = \{W(t)\}_{t \in [0,T]}$ and $C = \{C(t)\}_{t \in [0,T]}$ be the *wealth* and the *consumption* process, respectively. For any $t \in [0,T]$, we denote by $\epsilon(t)$, $\phi(t)$, $\pi^{(i)}(t)$ the proportion of wealth invested in the riskless asset, in the risky asset, and in the non-redundant i-th derivative, $i = 1, 2$, respectively, such that

$$\epsilon(t) + \phi(t) + \sum_{i=1}^{2}\pi^{(i)}(t) = 1.$$

Thanks to Eqs. (2.1), (5.4) and (5.5), the dynamics of the wealth process are

$$
\begin{aligned}
\frac{dW(t)}{W(t)} =& \epsilon(t)\frac{dM(t)}{M(t)} + \phi(t)\frac{dS(t)}{S(t)} + \sum_{i=1}^{2}\pi^{(i)}(t)\frac{dO(t)^{(i)}}{O(t)^{(i)}} - \frac{C(t)}{W(t)}dt \\
=& rdt - \frac{C(t)}{W(t)}dt + \eta dt\left[\phi(t) + \sum_{i=1}^{2}\pi^{(i)}(t)\left(\frac{g_s^{(i)}}{O^{(i)}(t)}S(t) + \sigma\rho\frac{g_y^{(i)}}{O^{(i)}(t)}y(t)\right)\right] \\
&- \sqrt{y(t)^{-1}}\epsilon^S(t)dt\left[\phi(t) + \sum_{i=1}^{2}\pi^{(i)}(t)\left(\frac{g_s^{(i)}}{O^{(i)}(t)}S(t) + \sigma\rho\frac{g_y^{(i)}}{O^{(i)}(t)}y(t)\right)\right] \\
&+ \sigma\sqrt{1-\rho^2}\sum_{i=1}^{2}\pi^{(i)}(t)\frac{g_y^{(i)}}{O^{(i)}(t)}\nu y(t)dt \\
&- \sigma\sqrt{1-\rho^2}\sum_{i=1}^{2}\pi^{(i)}(t)\frac{g_y^{(i)}}{O^{(i)}(t)}\sqrt{y(t)}\epsilon^y(t)dt \\
&+ \sqrt{y(t)^{-1}}d\tilde{B}(t)\left[\phi(t) + \sum_{i=1}^{2}\pi^{(i)}(t)\left(\frac{g_s^{(i)}}{O^{(i)}(t)}S(t) + \sigma\rho\frac{g_y^{(i)}}{O^{(i)}(t)}y(t)\right)\right] \\
&+ \sigma\sqrt{1-\rho^2}\sum_{i=1}^{2}\pi^{(i)}(t)\frac{g_y^{(i)}}{O^{(i)}(t)}\sqrt{y(t)}d\tilde{Z}(t) \\
&+ J\left[d\tilde{N}^{\epsilon}(t) + (\lambda^{\mathbb{P}}\epsilon^N(t) - \lambda^{\mathbb{Q}})dt\right]\left(\phi(t) + \sum_{i=1}^{2}\pi^{(i)}(t)\frac{\Delta g^{(i)}}{O(t)^{(i)}}\right) . \qquad (5.6)
\end{aligned}
$$

By setting the *risk exposures* to the risk factors $\tilde{B}$, $\tilde{Z}$, and $\tilde{N}$ as

$$
\theta^B(t) := \phi(t) + \sum_{i=1}^{2}\pi^{(i)}(t)\left(S(t)\frac{g_s^{(i)}}{O(t)^{(i)}} + \sigma\rho\frac{g_y^{(i)}}{O(t)^{(i)}}y(t)\right) , \qquad (5.7a)
$$

$$
\theta^Z(t) := \sigma\sqrt{1-\rho^2}\sum_{i=1}^{2}\pi^{(i)}(t)\frac{g_y^{(i)}}{O(t)^{(i)}}y(t) , \qquad (5.7b)
$$

$$
\theta^N(t) := \phi(t) + \sum_{i=1}^{2}\pi^{(i)}(t)\frac{\Delta g^{(i)}}{O(t)^{(i)}} , \qquad (5.7c)
$$

the dynamics of wealth simplifies as

$$
\begin{aligned}
\frac{dW(t)}{W(t)} =& \Bigg[r - \frac{C(t)}{W(t)} + \theta^B(t)(\eta - \sqrt{y(t)^{-1}}\epsilon^S(t)) + \theta^Z(t)(\nu - \sqrt{y(t)^{-1}}\epsilon^y(t)) + J\theta^N(t) \\
&\left(\lambda^P\epsilon^N(t) - \lambda^{\mathbb{Q}}\right)\Bigg]dt + \theta^B(t)\sqrt{y(t)^{-1}}d\tilde{B}(t) + \theta^Z(t)\sqrt{y(t)^{-1}}d\tilde{Z}(t) + J\theta^N(t) \\
&d\tilde{N}^{\epsilon}(t),\ t \in [0,T] . \qquad (5.8)
\end{aligned}
$$

We need to characterize the optimization problem. Hence, we denote by $\mathcal{U}(t,w,y)$ the set of all the *admissible strategies* for the initial data $(t,w,y) \in [0,T]\times\mathbb{R}^+\times\mathbb{R}^+$, and consider the set $u = \left\{\alpha(t),\, \phi(t),\, \pi^{(1)}(t),\, \pi^{(2)}(t), c(t)\right\}_{t\in[0,T]}$. We say that $u \in \mathcal{U}$ if

1. u is $\mathcal{F}(t)$-progressively measurable,

2. the following constraint holds true

$$\mathbb{E}\left\{\int_0^T (W(t))^2 \left[\left(\theta^B(t)\right)^2 y(t)^{-1} + \left(\theta^Z(t)\right)^2 y(t)^{-1} + J^2\left(\theta^N(t)\right)^2\right] dt\right\} < \infty\ ,$$

3. the SDE (5.8) has a unique strong solution for the initial data $(t, w, y) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+$.

In general, the investment problem to be solved, for an investor with a recursive utility function *à la* Duffie-Epstein, see e.g. Duffie and Epstein [1992], is the problem given by equation (4.3). Hence, we remember with the equation (2.27) the normalized aggregator of current consumption and continuation utility to describe to describe investors' preferences. As in Schroder and Skiadas [1999] and Wei et al. [2023], the necessary conditions to the existence of a solution to (4.3), are

$$\frac{\gamma - \frac{1}{\psi}}{1 - \frac{1}{\psi}} < 1, \text{ and } \frac{1}{\psi} > \max\left(0, \frac{\gamma - \frac{1}{\psi}}{\gamma - 1}\right) .$$

Consequently, we may face two alternative circumstances, namely, $\gamma > 1$ and $\psi < 1$, or $\gamma < 1$ and $\psi > 1$. In line with the literature, starting with the seminal paper by Maenhout [2004] and continuing up to the most recent contributions, see e.g. Wei et al. [2023], in the present paper we will concentrate on the most plausible and financially significant case, i.e. the case with $\gamma > 1$.

However, in the presence of ambiguity, the investor would like to determine the optimal investment policy and consumption-wealth ratio that maximize the expected utility *under the worst-case scenario.* Therefore, we have to introduce two penalty terms, say $\Psi(t, W, y)$ and $\Psi_N(t, W, y)$, representing the investor's ambiguity aversion to diffusion risk and jump risk, respectively, to point out the distance between the reference model and the alternative one, see e.g. Maenhout [2004] and Wei et al. [2023]. Hence, Eq. (4.3) can be generalized into the following *robust* investment problem, for any $t \in [0, T]$,

$$\begin{aligned} V(t, W, y) = \sup_{u \in \mathcal{U}} \inf_{\epsilon \in \mathcal{E}} \mathbb{E}_t^\epsilon \Bigg[\int_0^\infty \Bigg(& f(C, V) + e^{-\beta(s-t)} \frac{(\epsilon^S(t))^2 + (\epsilon^y(t))^2}{2\Psi(t, W, y)} \\ &+ e^{-\beta(s-t)} \frac{(\epsilon^N(t) \ln \epsilon^N(t) - \epsilon^N(t) + 1)\lambda^{\mathbb{Q}}}{\Psi_N(t, W, y)}\Bigg) ds\Bigg] , \end{aligned} \tag{5.9}$$

subject to the budget constraint (5.8). The second and third terms of (5.9) are the so-called *relative entropies* arising from diffusion and jump risks, respectively. According to Maenhout [2004], we assume that the investor's ambiguity aversion to diffusion and jump risks are state-dependent, meaning that the scaling factors depend on the value function.

In particular, we set

$$\Psi(t,W,y)=\frac{\Gamma}{(1-\gamma)V(t,W,y)}>0, \tag{5.10}$$

$$\Psi_N(t,W,y)=\frac{\Gamma_N}{(1-\gamma)V(t,W,y)}>0, \tag{5.11}$$

where Γ and Γ_N are non-negative constants, representing the ambiguity-aversion parameters.

The value function $V(t,W,y)$ satisfies the robust HJB equation, for any $t\in[0,T]$,

$$\sup_{u\in\mathcal{U}}\inf_{\epsilon\in\mathcal{E}}\mathbb{E}^{\epsilon}\left[f(C,V)+\mathcal{A}^{\epsilon}V(t,W,y)+\frac{(\epsilon^S(t))^2+(\epsilon^y(t))^2}{2\Psi(t,W,y)}+\frac{(\epsilon^N(t)\ln\epsilon^N(t)-\epsilon^N(t)+1)\lambda}{\Psi_N(t,W,y)}\right]=0\ , \tag{5.12}$$

where

$$\begin{aligned}\mathcal{A}^{\epsilon}V(t,W,y)=&\ V_t+V_wW(t)\left(r-\frac{C(t)}{W(t)}+\eta\theta^B(t)-\theta^B(t)\epsilon^S(t)\sqrt{y(t)^{-1}}+\theta^Z(t)\nu\right.\\&\left.-\theta^Z(t)\epsilon^y(t)\sqrt{y(t)^{-1}}-\theta^N(t)\lambda^{\mathbb{Q}}J\right)\\&+V_y\left[k^{\mathbb{Q}}(\theta^{\mathbb{Q}}-y(t))-\sigma\rho\epsilon^S(t)\sqrt{y(t)}-\sqrt{1-\rho^2}\sigma\sqrt{y(t)}\epsilon^y(t)\right]+\frac{1}{2}V_{yy}\sigma^2y(t)\\&+\frac{1}{2}V_{WW}W(t)^2\left((\theta^B(t))^2+(\theta^Z(t))^2\right)y(t)^{-1}+V_{wy}W(t)\sigma\left[\theta^B(t)\rho+\sqrt{1-\rho^2}\theta^Z(t)\right]\\&+\lambda^{\mathbb{P}}\Delta V\epsilon^N(t)\ ,\end{aligned} \tag{5.13}$$

where we indicate by V_t, V_W, V_y the first-order partial derivatives, by V_{WW}, V_{yy}, V_{Wy} the second-order partial derivatives, and by $\Delta V=V\left(t,(1+J\theta^N(t))W(t),y(t)+\xi(y(t))\right)$-$V(t,W(t-),y(t-))$ the jump-variation of the value function V, respectively.

We are ready to prove the following

Proposition 5.1.1. *Consider the robust investment problem* (5.9) *and the associated HJB equation* (5.12)*, and assume the normalized aggregator of current consumption and continuation utility* (2.27)*. Then, for any $t\in[0,T]$, the value function is*

$$V(t,W,y)=\exp\{-H(F(t)y(t)+G(t))\}\frac{W(t)^{1-\gamma}}{1-\gamma}\ , \tag{5.14}$$

where $H=\frac{1-\gamma}{1-\psi}$, while $F(t)$ and $G(t)$ are deterministic functions satisfying the following system of ODEs

$$\begin{cases}\dot F(t) &= h_1F(t)+Fk+\frac{\eta^2}{2(\gamma+\Gamma)}(1-\psi)-\frac{\eta F(t)\sigma\rho}{\gamma+\Gamma}(1-\gamma-\Gamma)+\frac{\nu^2}{2(\gamma+\Gamma)}(1-\psi)\\ &\quad -\frac{\nu F(t)\sigma\sqrt{1-\rho^2}}{\gamma+\Gamma}(1-\gamma-\Gamma)+\frac{F(t)^2H^2\sigma^2(1-\gamma-\Gamma)}{2(1-\gamma)}\\ \dot G(t) &= \beta\psi-h_0-h_1(\psi\log(\beta)+G(t))+r(1-\psi)-\theta^N(t)J\lambda^{\mathbb{Q}}(1-\psi)-F(t)K\theta\ ,\\ &\quad +\frac{\lambda^{\mathbb{P}}(1-\psi)}{\Gamma_N}\left[1-\exp\left\{\frac{\Gamma_Ne^{-HF(t)\xi(y(t))}\left[(1+\theta^N(t)J)^{1-\gamma}-1\right]}{1-\gamma}\right\}\right]\end{cases} \tag{5.15}$$

with $h_1 := \exp\{\overline{c-w}\}$ *and* $h_0 := h_1 - h_1 \log(h_1)$*. The optimal consumption-wealth ratio is*

$$\frac{C(t)}{W(t)} = \beta^{\psi} \exp\{-(F(t)y(t) + G(t))\} , \tag{5.16}$$

the optimal risk exposures to diffusive and volatility risk factors are

$$\theta^{\tilde{B},*}(t) = \left[\frac{\eta}{\gamma+\Gamma} - \frac{\sigma\rho(1-\gamma-\Gamma)F(t)}{(1-\psi)(\gamma+\Gamma)}\right] y(t) , \tag{5.17}$$

$$\theta^{\tilde{Z},*}(t) = \left[\frac{\nu}{\gamma+\Gamma} - \frac{\sigma\sqrt{1-\rho^2}(1-\gamma-\Gamma)F(t)}{(1-\psi)(\gamma+\Gamma)}\right] y(t) , \tag{5.18}$$

and the risk exposure to jump risk solves the non-linear equation

$$J\theta^{\tilde{N},*}(t) + 1 = \left[\frac{\lambda^{\mathbb{Q}}}{\lambda^{\mathbb{P}}} e^{HF(t)\xi(y(t))} \exp\left\{\frac{\Gamma_N e^{-HF(t)\xi(y(t))}\left[(1+\theta^N(t)J)^{1-\gamma} - 1\right]}{1-\gamma}\right\}\right]^{-\frac{1}{\gamma}} . \tag{5.19}$$

Moreover, the optimal distortion measures are

$$\epsilon^{s,*}(t) = \left(\frac{\eta}{\gamma+\Gamma} - \frac{\rho\sigma F(t)}{(1-\psi)(\gamma+\Gamma)}\right)\Gamma\sqrt{y(t)} , \tag{5.20}$$

$$\epsilon^{y,*}(t) = \left(\frac{\nu}{\gamma+\Gamma} - \frac{\sigma\sqrt{1-\rho^2}F(t)}{(1-\psi)(\gamma+\Gamma)}\right)\Gamma\sqrt{y(t)} , \tag{5.21}$$

$$\epsilon^{N,*}(t) = \exp\left\{\Gamma_N \frac{\left[1-(1+J\theta^{\tilde{N},*}(t))^{1-\gamma}\right]}{1-\gamma} e^{-HF(t)\times(y(t))i}\right\} . \tag{5.22}$$

Proof. To determine the optimal allocation and consumption policies we start from Eq. (5.12) associated with the investment problem. We first solve the infimization part by applying the first-order condition w.r.t. the distortion measures, namely

$$\begin{cases} \frac{\epsilon^S(t)}{\Psi} & -\theta^B(t)\sqrt{y(t)^{-1}}W(t)V_w - V_y\sigma\rho\sqrt{y(t)} = 0 \\ \frac{\epsilon^y(t)}{\Psi} & -\theta^Z(t)\sqrt{y(t)^{-1}}W(t)V_w - V_y\sigma\sqrt{1-\rho^2}\sqrt{y(t)} = 0 \\ \lambda\Delta V & +\frac{(\ln\epsilon^N(t)+\frac{1}{\epsilon^N(t)}\epsilon^N(t)-1)\lambda}{\Psi_N} = 0 \end{cases} , \tag{5.23}$$

and we obtain

$$\begin{cases} \epsilon^S(t)^* & = \sqrt{y(t)}\Psi\left(V_wW(t)\theta^B(t)\frac{1}{y(t)} + V_y\sigma\rho\right) \\ \epsilon^y(t)^* & = \sqrt{y(t)}\Psi\left(V_wW(t)\theta^Z(t)\frac{1}{y(t)} + V_y\sigma\sqrt{1-\rho^2}\right) \\ \epsilon^N(t)^* & = \exp\{-\Psi_N\Delta V\} \end{cases} . \tag{5.24}$$

Remember the HJB equation

$$\sup_{W(t),C(t)} \Bigg\{ f(C,V) + V_t + V_w W(t) \Bigg(r - \frac{C(t)}{W(t)} + \eta\theta^B(t) - \theta^B(t)\epsilon^S(t)\sqrt{y(t)^{-1}} + \theta^Z(t)\nu$$
$$- \theta^Z(t)\epsilon^y(t)\sqrt{y(t)^{-1}} - \theta^N(t)J\lambda^{\mathbb{Q}} \Bigg) + V_y \Big[k^{\mathbb{Q}}(\theta^{\mathbb{Q}} - y(t)) - \sigma\rho\epsilon^S(t)\sqrt{y(t)} - \sqrt{1-\rho^2}\sigma\sqrt{y(t)}\epsilon^y(t) \Big]$$
$$+ \frac{1}{2} V_{WW} W(t)^2 \left((\theta^B(t))^2 + (\theta^Z(t))^2 \right) y(t)^{-1} + \frac{1}{2} V_{yy}\sigma^2 y(t) + V_{wy} W(t)\sigma \Big[\theta^B(t)\rho + \sqrt{1-\rho^2}\theta^Z(t) \Big] +$$
$$+ \lambda^{\mathbb{P}}\Delta V \epsilon^N(t) \frac{(\epsilon^S(t))^2 + (\epsilon^y(t))^2}{2\Psi(t,W,y)} + \frac{(\epsilon^N(t)\ln\epsilon^N(t) - \epsilon^N(t) + 1)\lambda^{\mathbb{P}}}{\Psi_N(t,W,y)} \Bigg\} = 0 \tag{5.25}$$

We replace (5.24) in the HJB equation and with some algebra we have

$$
\begin{aligned}
&\sup_{W(t),C(t)}\Bigg\{ f(C,V)+V_t+V_wW(t)\Bigg(r-\frac{C(t)}{W(t)}+\eta\theta^B(t)+\theta^Z(t)\nu- \\
&\quad \theta^N(t)J\lambda^{\mathbb{Q}}\Bigg)++V_y\left[k^{\mathbb{Q}}(\theta^{\mathbb{Q}}-y(t))\right]+\frac{1}{2}V_{WW}W(t)^2\left((\theta^B(t))^2+(\theta^Z(t))^2\right)y(t)^{-1}+ \\
&\quad +\frac{1}{2}V_{yy}\sigma^2y(t)+V_{wy}W(t)\sigma\left[\theta^B(t)\rho+\sqrt{1-\rho^2}\theta^Z(t)\right]+\lambda^{\mathbb{P}}\Delta V\epsilon^N(t)-\epsilon^S(t)\Bigg(\sqrt{y(t)^{-1}} \\
&\quad \theta^B(t)V_wW(t)+\sqrt{y(t)}\sigma\rho V_y\Bigg)-\epsilon^y(t)\left(\sqrt{y(t)^{-1}}\theta^Z(t)V_wW(t)+\sqrt{y(t)}\sigma\sqrt{1-\rho^2}V_y\right)+ \\
&\quad \frac{(\epsilon^S(t))^2}{2\Psi}+\frac{(\epsilon^y(t))^2}{2\Psi(t,W,y)}+\frac{(\epsilon^N(t)\ln\epsilon^N(t)-\epsilon^N(t)+1)\lambda^{\mathbb{P}}}{\Psi_N(t,W,y)}\Bigg\}=0; \\
&\sup_{W(t),C(t)}\Bigg\{ f(C,V)+V_t+V_wW(t)\Bigg(r-\frac{C(t)}{W(t)}+\eta\theta^B(t)+\theta^Z(t)\nu- \\
&\quad \theta^N(t)J\lambda^{\mathbb{Q}}\Bigg)+V_y\left[k^{\mathbb{Q}}(\theta^{\mathbb{Q}}-y(t))\right]+\frac{1}{2}V_{WW}W(t)^2\left((\theta^B(t))^2+(\theta^Z(t))^2\right)y(t)^{-1}+\frac{1}{2}V_{yy} \\
&\quad \sigma^2y(t)+V_{wy}W(t)\sigma\left[\theta^B(t)\rho+\sqrt{1-\rho^2}\theta^Z(t)\right]+\lambda^{\mathbb{P}}\Delta V\exp\{-\Psi_N\Delta V\}-\sqrt{y(t)}\Psi \\
&\quad \left(V_WW(t)\frac{\theta^B(t)}{y(t)}+V_y\sigma\rho\right)\sqrt{y(t)}\left(V_WW(t)\frac{\theta^B(t)}{y(t)}+V_y\sigma\rho\right)-\sqrt{y(t)}\Psi\Bigg(V_WW(t)\frac{\theta^Z(t)}{y(t)}+V_y \\
&\quad \sigma\sqrt{1-\rho^2}\Bigg)\sqrt{y(t)}\left(V_WW(t)\frac{\theta^Z(t)}{y(t)}+V_y\sigma\sqrt{1-\rho^2}\right)+\frac{1}{2\Psi}\Bigg(\sqrt{y(t)}\Psi\Bigg(V_WW(t)\frac{\theta^B(t)}{y(t)}+ \\
&\quad V_y\sigma\rho\Bigg)\Bigg)^2+\frac{1}{2\Psi}\left(\sqrt{y(t)}\Psi\left(V_WW(t)\frac{\theta^Z(t)}{y(t)}+V_y\sigma\sqrt{1-\rho^2}\right)\right)^2+ \\
&\quad \frac{\left(\exp\{-\Psi_N\Delta V\}\ln\exp\{-\Psi_N\Delta V\}-\exp\{-\Psi_N\Delta V\}\right)\lambda^{\mathbb{P}}}{\Psi_N}\Bigg\}=0
\end{aligned}
\tag{5.26}
$$

$$
\begin{aligned}
&\sup_{W(t),C(t)}\Bigg\{ f(C,V)+V_t+V_wW(t)\left(r-\frac{C(t)}{W(t)}+\eta\theta^B(t)+\theta^Z(t)\nu-\theta^N(t)J\lambda^{\mathbb{Q}}\right) \\
&\quad +V_y\left[k^{\mathbb{Q}}(\theta^{\mathbb{Q}}-y(t))\right]+\frac{1}{2}V_{WW}W(t)^2\left((\theta^B(t))^2+(\theta^Z(t))^2\right)y(t)^{-1} \\
&\quad +\frac{1}{2}V_{yy}\sigma^2y(t)+V_{wy}W(t)\sigma\left[\theta^B(t)\rho+\sqrt{1-\rho^2}\theta^Z(t)\right]-\frac{1}{2}\Psi y(t)\left(V_WW(t)\frac{\theta^B(t)}{y(t)}+V_y\sigma\rho\right)^2 \\
&\quad -\frac{1}{2}\Psi y(t)\left(V_WW(t)\frac{\theta^Z(t)}{y(t)}+V_y\sigma\sqrt{1-\rho^2}\right)^2+\frac{\lambda^{\mathbb{P}}}{\Psi_N}\left(1-\exp\{-\Psi_N\Delta V\}\right)\Bigg\}=0\,.
\end{aligned}
\tag{5.27}
$$

We rearrange the first term in (5.27) so that

$$
\begin{aligned}
f(C,V) &= V_t \frac{\psi}{\psi-1}\beta(1-\gamma)\left[\left(\frac{C(t)}{((1-\gamma)V)^{\frac{1}{1-\gamma}}}\right)^{1-\frac{1}{\psi}} - 1\right] = \\
&= -V_t H\beta\psi \left[\frac{C(t)^{1-\frac{1}{\psi}}}{((1-\gamma)V)^{\left(\frac{1}{1-\gamma}\right)\left(\frac{\psi-1}{\psi}\right)}} - 1\right] = \\
&= VH\beta\psi - V\frac{H\beta\psi}{((1-\gamma)V)^{\left(\frac{1}{1-\gamma}\right)\left(\frac{\psi-1}{\psi}\right)}} C^{\frac{\psi-1}{\psi}} \; .
\end{aligned} \tag{5.28}
$$

By applying the first-order condition w.r.t. the consumption, we have

$$
\frac{V_t H\beta\psi}{((1-\gamma)V)^{\left(\frac{1}{1-\gamma}\right)\left(\frac{\psi-1}{\psi}\right)}} \frac{\psi-1}{\psi} C(t)^{\frac{1}{\psi}} + \frac{V_w W(t)}{W(t)} = 0 \; , \tag{5.29}
$$

so that the optimal consumption is

$$
C(t)^{-\frac{1}{\psi}} = V_W \frac{\psi}{\psi-1} \frac{\left((1-\gamma)V^{\frac{\psi-1}{(1-\gamma)\psi}}\right)}{-H\beta\psi V} \; . \tag{5.30}
$$

By applying the first-order condition w.r.t. the portfolio exposures, we have

$$
\begin{cases}
\eta V_W W(t) + \frac{1}{2} V_{WW} W(t)^2 2\theta^B(t) y(t)^{-1} + V_{Wy} W(t)\sigma\rho \\
-y(t)\Psi \left(V_W W(t)\frac{\theta^B(t)}{y(t)} + V_y \sigma\rho\right) V_W W(t)\frac{1}{y(t)} = 0 \\
\nu V_W W(t) + \frac{1}{2} V_{WW} W(t)^2 2\theta^Z(t) y(t)^{-1} + V_{Wy} W(t)\sigma\sqrt{1-\rho^2} \\
-2y(t)\Psi \left(V_W W(t)\frac{\theta^Z(t)}{y(t)} + V_y\sigma\sqrt{1-\rho^2}\right) V_W W(t)\frac{1}{y(t)} = 0 \\
V_W W(t)\lambda^{\mathbb{Q}} J + \frac{\lambda^{\mathbb{P}}}{\Psi_N}(\exp\{-\Psi_N \Delta V\})(-)\Psi_N \frac{\partial \Delta V}{\partial \theta^N(t)} = 0
\end{cases} \; . \tag{5.31}
$$

Setting $H := \frac{1-\gamma}{1-\psi}$, we guess

$$
V(t,W,y) = \exp\{-H(F(t)y(t) + G(t))\}\frac{W(t)^{1-\gamma}}{1-\gamma}, \; t \in [0,T] \; . \tag{5.32}
$$

The partial derivatives of the value function are

$$V_t = -H(\dot{F}_t y(t) + \dot{G}_t)V \ , \tag{5.33}$$

$$V_W = \exp\left\{-H(F(t)y(t) + G(t))\right\} W(t)^{-\gamma} = \frac{V}{W(t)}(1-\gamma) \ , \tag{5.34}$$

$$V_y = -HF(t)V \tag{5.35}$$

$$V_{WW} = -\gamma \exp\left\{-H(F(t)y(t) + G(t))\right\} W(t)^{-\gamma-1} = -\gamma V(1-\gamma)W^{-2} \ , \tag{5.36}$$

$$V_{yy} = H^2 F(t)^2 V \ , \tag{5.37}$$

$$V_{Wy} = -HF(t)\exp\left\{-H(F(t)y(t) + G(t))\right\} W(t)^{-\gamma} = \frac{-HVF(t)}{W(t)}(1-\gamma) \ . \tag{5.38}$$

$$\begin{aligned}\Delta V &= \exp\{-H(F(t)(y+\xi) + G(t))\}\frac{W^{1-\gamma}(1+\theta^N(t)J)^{1-\gamma}}{1-\gamma} \\ &\quad - \exp\{-H(F(t)y(t) + G(t))\}\frac{W^{1-\gamma}}{1-\gamma} = V\left[e^{-HF(t)\xi}\left(1+\theta^N(t)J\right)^{1-\gamma} - 1\right] \ .\end{aligned} \tag{5.39}$$

From (5.31) we recover the optimal risk exposures (5.17), (5.18) and (5.19). We replace the partial derivatives in (5.30) and (5.31) and we get

$$\frac{C(t)}{W(t)} = \beta^{\psi}\exp\{-(F(t)y(t) + G(t))\} \ ,$$

$$\theta^B(t)^* = \left[\frac{\eta}{\gamma+\Gamma} - \frac{\sigma\rho(1-\gamma-\Gamma)\frac{F(t)}{1-\psi}}{(\gamma+\Gamma)}\right] y(t) \ ,$$

$$\theta^Z(t)^* = \left[\frac{\nu}{\gamma+\Gamma} - \frac{\sigma\sqrt{1-\rho^2}(1-\gamma-\Gamma)\frac{F(t)}{1-\psi}}{(\gamma+\Gamma)}\right] y(t) \ ,$$

$$\theta^N(t)^* = \frac{\left[\frac{\lambda^{\mathbb{Q}}}{\lambda^{\mathbb{P}}} e^{HF(t)\xi(y(t))}\exp\left\{\frac{\Gamma_N e^{-HF(t)\xi(y(t))}\left[(1+\theta^N(t)J)^{1-\gamma}-1\right]}{1-\gamma}\right\}\right]^{-\frac{1}{\gamma}} - 1}{J} .$$

To solve the supremization part, we replace the optimal risk exposures, the partial derivatives of the value function, and the explicit expression of the penalty terms in (5.24).

Concerning the diffusive distortion measure, we have

$$
\begin{aligned}
\epsilon^S(t)^* &= \sqrt{y(t)}\frac{\Gamma}{(1-\gamma)V}\left[\frac{V(1-\gamma)}{y(t)}y(t)\left(\frac{\eta}{\gamma+\Gamma}-\frac{\sigma\rho(1-\gamma-\Gamma)\frac{F(t)}{1-\psi}}{(\gamma+\Gamma)}\right)-HF(t)V\sigma\rho\right]\\
&= \sqrt{y(t)}\Gamma\left(\frac{\eta}{\gamma+\Gamma}-\frac{F(t)\sigma\rho(1-\gamma-\Gamma)}{(\gamma+\Gamma)(1-\psi)}\right)-\Gamma\sqrt{y(t)}\frac{F(t)\sigma\rho H}{(1-\gamma)}\\
&= \frac{\eta(1-\gamma)\sqrt{y}_t\Gamma-\rho\sigma(1-\gamma-\Gamma)\frac{F}{1-\psi}(1-\gamma)\sqrt{y(t)}\Gamma-(\gamma+\Gamma)F\sigma\rho H\Gamma\sqrt{y(t)}}{(1-\gamma)(\gamma+\Gamma)}\\
&= \frac{\sqrt{y(t)}\Gamma\left[\eta(1-\gamma)-\rho\sigma HF(1-\gamma-\Gamma)-F\sigma\rho H(\gamma+\Gamma)\right]}{(1-\gamma)(\gamma+\Gamma)}\\
&= \frac{\sqrt{y(t)}\Gamma\left[\eta(1-\gamma)-\rho\sigma HF+\rho\sigma HF\gamma+\rho\sigma HF\Gamma-F\sigma\rho H\gamma-F\sigma\rho H\Gamma\right]}{(1-\gamma)(\gamma+\Gamma)}\\
&= \frac{\sqrt{y(t)}\,\Gamma\left(\eta-\rho\sigma\frac{F}{1-\psi}\right)}{(\gamma+\Gamma)}\,.
\end{aligned}
$$

Using the same algebra, we obtain the remaining optimal distortion measures, namely

$$
\begin{aligned}
\epsilon^y(t)^* &= \frac{\Gamma\sqrt{y(t)}\left(\nu-\sigma\sqrt{1-\rho^2}\frac{F(t)}{1-\psi}\right)}{(\gamma+\Gamma)}\,,\\
\epsilon^N(t)^* &= \exp\left\{\frac{\Gamma_N e^{-HF\xi(y(t))}\left[1-(1+\theta^N J)^{1-\gamma}\right]}{(1-\gamma)}\right\}.
\end{aligned}
$$

Finally, we replace the partial derivatives, the optimal policies, and the optimal distortion measures in the HJB equation

$$
\begin{aligned}
&f(C,V) - H(\dot{F}_t y + \dot{G}_t) + \frac{V}{W(t)}(1-\gamma)W(t)\Bigg[r - \frac{C(t)}{W(t)} + \left(\frac{\eta}{\gamma+\Gamma} - \frac{\sigma\rho(1-\gamma-\Gamma)\frac{F(t)}{1-\psi}}{\gamma+\Gamma}\right)\eta y(t) \\
&+ \left(\frac{\nu}{\gamma+\Gamma} - \frac{\sigma\sqrt{1-\rho^2}(1-\gamma-\Gamma)\frac{F(t)}{1-\psi}}{\gamma+\Gamma}\right)\nu y(t) - \theta^N(t)J\lambda^{\mathbb{Q}}\Bigg] - HF(t)VK^{\mathbb{Q}}(\theta^{\mathbb{Q}} - y(t)) - \gamma\frac{1}{2} \\
&V(1-\gamma)W(t)^{-2}W(t)^2\Bigg[\left(\frac{\eta}{\gamma+\Gamma} - \frac{\sigma\rho(1-\gamma-\Gamma)\frac{F(t)}{1-\psi}}{\gamma+\Gamma}\right)^2 y(t)^2 + \left(\frac{\nu}{\gamma+\Gamma} - \frac{\sigma\sqrt{1-\rho^2}(1-\gamma-\Gamma)\frac{F(t)}{1-\psi}}{\gamma+\Gamma}\right)^2 \\
&y(t)^2\Bigg]\frac{1}{y(t)} + \frac{1}{2}H^2F^2V\sigma^2 y(t) - W(t)\frac{HVF(t)}{W(t)}(1-\gamma)\sigma\Bigg[\rho\left(\frac{\eta}{\gamma+\Gamma} - \frac{\sigma\rho(1-\gamma-\Gamma)\frac{F(t)}{1-\psi}}{\gamma+\Gamma}\right)y(t) + \sqrt{1-\rho^2} \\
&\left(\frac{\nu}{\gamma+\Gamma} - \frac{\sigma\sqrt{1-\rho^2}(1-\gamma-\Gamma)\frac{F(t)}{1-\psi}}{\gamma+\Gamma}\right)y(t)\Bigg] - \frac{1}{2}y(t)\frac{\Gamma}{(1-\gamma)V}\Bigg[\frac{V}{W(t)}(1-\gamma)\frac{W(t)}{y(t)}y(t)\Bigg(\frac{\eta}{\gamma+\Gamma} - \\
&\frac{\sigma\rho(1-\gamma-\Gamma)\frac{F(t)}{1-\psi}}{\gamma+\Gamma}\Bigg) - HF(t)V\sigma\rho\Bigg]^2 - \frac{1}{2}\frac{\Gamma}{(1-\gamma)V_t}\Bigg[\frac{V}{W(t)}(1-\gamma)\frac{W(t)}{y(t)}y(t)\Bigg(\frac{\nu}{\gamma+\Gamma} - \\
&\frac{\sigma\sqrt{1-\rho^2}(1-\gamma-\Gamma)\frac{F(t)}{1-\psi}}{\gamma+\Gamma}\Bigg) - HF(t)V\sigma\sqrt{1-\rho^2}\Bigg]^2 + \frac{(1-\gamma)V\lambda^{\mathbb{P}}}{\Gamma_N}\Bigg[1 - \exp\Bigg\{\frac{\Gamma_N}{(1-\gamma)V}V \\
&\left[e^{-HF(t)\xi(y(t))}(1+\theta^N(t)J)^{1-\gamma} - 1\right]\Bigg\}\Bigg] = 0.
\end{aligned}
$$

At this point we can divide the previous expression by V and with some algebraic manipulation, we obtain

$$
\begin{aligned}
&\frac{f(C(t),V_t)}{V_t} - \frac{C(t)}{W(t)}(1-\gamma) - H\dot{F}_t y(t) - H\dot{G}_t + r(1-\gamma) - (1-\gamma)\theta^N(t)J\lambda^{\mathbb{Q}} - HF(t)k\theta^{\mathbb{Q}} \\
&\quad + HF(t)k^{\mathbb{Q}}y(t) + \frac{\eta^2}{2(\gamma+\Gamma)}(1-\gamma)y(t) - \frac{H\eta F(t)\sigma\rho}{(\gamma+\Gamma)}y(t)(1-\gamma-\Gamma) + \frac{\nu^2}{2(\gamma+\Gamma)} \\
&\quad (1-\gamma)y(t) - \frac{H\nu F(t)\sigma\sqrt{1-\rho^2}}{(\gamma+\Gamma)}y(t)(1-\gamma-\Gamma) + \frac{1}{2}\frac{H^2F(t)^2\sigma^2}{(1-\gamma)}(1-\gamma-\Gamma)y(t) + \\
&\quad \frac{(1-\gamma)\lambda^{\mathbb{P}}}{\Gamma_N}\left[1 - \exp\left\{\frac{\Gamma_N}{(1-\gamma)}\left[e^{-HF(t)\xi(y(t))}(1+\theta^N(t)J)^{1-\gamma} - 1\right]\right\}\right] = 0.
\end{aligned}
\tag{5.40}
$$

We replace the guess (5.32) in the normalized aggregator, as well as the optimal consumption-wealth ratio (5.30), so that the first two terms of (5.40) can be rewritten as

$$
\begin{aligned}
&\frac{f(C(t),V_t)}{V_t}-\frac{C(t)}{W(t)}(1-\gamma)=\\
&=H\beta\psi-\frac{H\beta\psi}{\left((1-\gamma)\exp\{-H(F(t)y(t)+G(t))\frac{W^{1-\gamma}}{1-\gamma}\}\right)^{\frac{\psi-1}{(1-\gamma)\psi}}}\left(W(t)\beta^{\psi}\exp\{-(F(t)y(t)+G(t))\}\right)^{\frac{\psi-1}{\psi}}\\
&-\frac{W(t)\beta^{\psi}\exp\{-(F(t)y(t)+G(t))\}}{W(t)}(1-\gamma)\\
&=H\beta\psi-\frac{H\beta\psi}{(\exp\{-H(F(t)y(t)+G(t))\}W(t)^{1-\gamma})^{\frac{\psi-1}{(1-\gamma)\psi}}}W(t)^{\frac{\psi-1}{\psi}}\beta^{\psi-1}\exp\{-(Fy(t)+G(t))\}^{\frac{\psi-1}{\psi}}\\
&-\beta\psi\exp\{-(F(t)y(t)+G(t))\}(1-\gamma)\\
&=H\beta\psi-\frac{H\beta\psi}{\exp\{\frac{1-\gamma}{1-\psi}\frac{1-\psi}{(1-\gamma)\psi}(F(t)y(t)+G(t))\}W(t)^{1-\gamma\frac{\psi-1}{(1-\gamma)\psi}}}W(t)^{\frac{\psi-1}{\psi}}\beta^{\psi-1}\exp\{-(F(t)+G(t))\}^{\frac{\psi-1}{\psi}}\\
&-\beta^{\psi}\exp\{-(F(t)+G(t))\}(1-\gamma)\\
&=H\beta\psi-H\beta^{\psi}\psi\exp\left\{-\frac{1}{\psi}(F(t)y(t)+G(t))\right\}\exp\left\{-\frac{\psi-1}{\psi}(F(t)y(t)+G(t))\right\}-\beta^{\psi}\exp\{-(F(t)\\
&y(t)+G(t))\}(1-\gamma)\\
&=H\beta\psi-H\beta^{\psi}\psi\exp\left\{\left(-\frac{1}{\psi}+\frac{1}{\psi}-1\right)(Fy(t)+G(t))\right\}-\beta^{\psi}\exp\{-(F(t)y(t)+G(t))\}(1-\gamma)\\
&=H\beta\psi-H\beta^{\psi}\psi\exp\{-(F(t)y(t)+G(t)\}-\beta^{\psi}\exp\{-(F_y y(t)+G(t))\}(1-\gamma)\\
&=H\beta\psi-H\exp\{-(F(t)y(t)+G(t))\}\left(\beta^{\psi}\psi+\frac{(1-\gamma)}{H}\beta^{\psi}\right)\\
&=H\beta\psi-H\exp\{-(F(t)y(t)+G(t))\}\left(\beta^{\psi}\psi+\beta^{\psi}-\beta^{\psi}\psi\right)\\
&=H\beta\psi-H\beta^{\psi}\exp\{-(F(t)+G(t))\}\ .
\end{aligned}
\tag{5.41}
$$

Recalling the consumption-wealth ratio formula and using the envelope condition as in Chacko and Viceira [2005], we have

$$
\beta^{\psi}\exp\{-(F(t)+G(t))\}=\frac{C(t)}{W(t)}=\exp\{\log(C(t))-\log(W(t))\}=\exp\{C(t)-W(t)\}\ ,
\tag{5.42}
$$

so that

$$
C(t)-W(t)=\psi\log(\beta)-F(t)y(t)-G(t)\ .
\tag{5.43}
$$

The first-order Taylor expansion of $\exp\{C(t)-W(t)\}$ around its unconditional mean $\overline{c-w}$ ensures that

$$
\frac{f(C(t),V_t)}{V_t}-\frac{C(t)}{W(t)}(1-\gamma)\approx H\beta\psi-H\left[h_0+h_1(C(t)-W(t))\right],
\tag{5.44}
$$

where $h_1:=\exp\{\overline{c-w}\}$ and $h_0:=h_1-h_1\log(h_1)$. We replace (5.44) in (5.40) and obtain the system of ODEs

$$
\begin{cases}
-\dot{F}_t + F(t)k^{\mathbb{Q}} + h_1 F(t) + \frac{\eta^2}{2(\gamma+\Gamma)}(1-\psi) - \frac{\eta F(t)\sigma\rho}{(\gamma+\Gamma)}(1-\gamma-\Gamma) + \frac{\nu^2}{2(\gamma+\Gamma)}(1-\psi) \\
\frac{\nu F(t)\sigma\sqrt{1-\rho^2}}{(\gamma+\Gamma)}(1-\gamma-\Gamma) + \frac{1}{2}\frac{HF(t)^2\sigma^2}{(1-\gamma)}(1-\gamma-\Gamma) = 0 \\
\beta\psi - h_0 - h_1\psi\log(\beta) - h_1 G(t) - \dot{G}_t + r(1-\psi) - \theta^N(t)J\lambda^{\mathbb{Q}}(1-\psi) - F(t)k^{\mathbb{Q}}\theta^{\mathbb{Q}} \quad \cdot \\
+\frac{(1-\psi)\lambda^{\mathbb{P}}}{\Gamma_N}\left[1 - \exp\left\{\frac{\Gamma_N}{(1-\gamma}\left[e^{-HF(t)\xi(y(t))}(1+\theta^N(t)J)^{1-\gamma} - 1\right]\right\}\right] = 0
\end{cases}
\tag{5.45}
$$

$\square$

Prop. 5.1.1 provides some interesting results, which are worth commenting on. We first note that the log consumption-wealth ratio is affine w.r.t. the instantaneous precision. More precisely, an increase in volatility implies a deterioration in investment opportunities, creating a negative income effect and a positive substitution effect. Moreover, such a ratio depends on both the risk aversion level and the model ambiguity degree, through the terms constituting the exponential component. We also observe that the consumption-wealth ratio is explicitly related to the elasticity of intertemporal substitution of consumption (EIS), indicated by ψ. An investor with $\psi < 1$ will have an income effect that dominates the substitution effect, hence she will consume less wealth as volatility increases, consistent with the findings in Chacko and Viceira [2005]. Being β the time preference, the consumption-wealth ratio is always positive, whatever the value of the intertemporal substitution parameter. Moreover, the ratio will be greater the lower the elasticity of intertemporal substitution of consumption ($\psi < 1$), as the investor prefers to consume today rather than save for future consumption. In Eqs. (5.17) and (5.18) we provide the optimal exposures to diffusive and volatility risks. In line with the literature, such exposures consist of two terms, namely, a myopic and an intertemporal hedging component. The former depends on the diffusive risk premia and is inversely proportional to the exogenous parameters characterizing the investment (i.e. risk aversion and ambiguity). The latter relies on the continuous-part model parameters, apart from the long-run mean. Remarkably, the risk exposures depend on the value of instantaneous volatility, through an inverse proportionality relationship. Such a condition confirms the results in Chacko and Viceira [2005] within a jump-free stochastic volatility framework, as well as the findings provided in Oliva and Renò [2018] in an SVCJ model with power utilities. To conclude, we argue that the optimal exposures turn out to be truly dynamic, persuading agents to reduce and adjust their investment in case of turbulent market scenarios. The value of the jump-risk exposure is obtained via numerical techniques. By inspecting (5.19) we find that a positive exposure to jump risk occurs when we assume a negative jumps in price (i.e., when we assume $J < 1$). Such a condition depends on the choice of the jump risk premia, that is, we assume $\lambda^{\mathbb{Q}} > \lambda^{\mathbb{P}}$, as in Liu et al. [2003]. Such a similar behavior is exhibited by the optimal distortion measures, provided in (5.20) - (5.22). More precisely, the penalty terms associated with the diffusive components consist of two terms, playing the same role as the myopic and the intertemporal hedging components in the exposures, respectively. This is in line with well-established literature results, see e.g. Wei et al. [2023] and references therein. Interestingly, the model proposed in the present work shows the presence of a propagation factor inversely proportional to instantaneous volatility: such a feature represents the novelty of our proposal compared with the extant affine volatility literature: the inverse proportionality of optimal weights w.r.t. volatility entails that a larger variability in turbulent markets follows a reduction in the optimal allocation. At the same time, also the ambiguity risk premia are inversely proportional to volatility. Hence, in case of market crashes, a reduction of the optimal allocation also diminishes the distance between the reference model and the alternative ones, being the latter statistically close to the reference model. It is possible to obtain an analytical solution to the investment problem (5.9) in the special case with $\psi = 1$. Here, the aggregator $f(C, V)$ assumes the simplified form of equation (2.28). We are able to prove the following

Proposition 5.1.2. *Consider the robust investment problem* (5.9) *and the associated HJB equation* (5.12)*, and assume the normalized aggregator of current consumption and continuation utility* (2.28)*. Then, for* $t \in [0, T]$*, the value function is*

$$V(t, W, y) = \exp\{\bar{F}(t)y(t) + \bar{G}(t)\}\frac{W(t)^{1-\gamma}}{1-\gamma} \ , \tag{5.46}$$

where $\bar{F}(t)$ *and* $\bar{G}(t)$ *are deterministic functions satisfying the following system of ODEs*

$$\begin{cases} \dot{\bar{F}}_t & = \beta\bar{F}(t) - \left(\frac{(1-\gamma-\Gamma)}{(\gamma+\Gamma)}\right)\bar{F}(t)\sigma\left(\rho\eta + \nu\sqrt{1-\rho^2}\right) + \frac{1}{2}\frac{(1-\gamma-\Gamma)}{(\gamma+\Gamma)}\sigma^2\bar{F}^2(\tau) \\ & -\frac{1}{2}\frac{(1-\gamma)}{(\gamma+\Gamma)}(\eta^2+\nu^2) \ , \\ \dot{\bar{G}}_t & = -\beta(1-\gamma)\log(\beta) + \beta\bar{G}(t) - r(1-\gamma) + \beta(1-\gamma) + \theta^N(t)J(1-\gamma) - \frac{\lambda^{\mathbb{P}}}{\Gamma_N}(1-\gamma) \\ & \times\exp\left\{\frac{e^{\bar{F}(t)\xi(y(t))}\Gamma_N(1-(1+\theta^N(t)J)^{1-\gamma})}{1-\gamma}\right\} . \end{cases} \tag{5.47}$$

The optimal consumption-wealth ratio is

$$\frac{C(t)}{W(t)} = \beta \ , \tag{5.48}$$

the optimal risk exposures to diffusive and volatility risk factors are

$$\theta^{\tilde{B},*}(t) = \left(\frac{\eta}{\gamma+\Gamma} + \frac{\bar{F}(t)\sigma\rho(1-\gamma-\Gamma)}{(\gamma+\Gamma)(1-\gamma)}\right)y(t) \ , \tag{5.49}$$

$$\theta^{\tilde{Z},*}(t) = \left(\frac{\nu}{\gamma+\Gamma} + \frac{\bar{F}(t)\sigma\sqrt{1-\rho^2}(1-\gamma-\Gamma)}{(\gamma+\Gamma)(1-\gamma)}\right)y(t) \ , \tag{5.50}$$

and the risk exposure to jump risk solves the non-linear equation

$$J\theta^{\tilde{N},*}(t) + 1 = \left(\frac{\lambda^{\mathbb{Q}}}{\lambda^{\mathbb{P}}}e^{-\bar{F}(t)\xi(y(t))}\exp\left\{\frac{\Gamma_N e^{\bar{F}(t)\xi(y(t))}\left[(1+J\theta^{\tilde{N},*}(t))^{1-\gamma}-1\right]}{1-\gamma}\right\}\right)^{-\frac{1}{\gamma}} . \tag{5.51}$$

Moreover, the optimal distortion measures are

$$\epsilon^{s,*}(t) = \left(\frac{\eta}{\gamma+\Gamma} - \frac{\rho\sigma\bar{F}(t)}{(\gamma+\Gamma)(1-\gamma)}\right)\Gamma\sqrt{y(t)}\ , \tag{5.52}$$

$$\epsilon^{y,*}(t) = \left(\frac{\nu}{\gamma+\Gamma} - \frac{\sigma\sqrt{1-\rho^2}\bar{F}(t)}{(\gamma+\Gamma)(1-\gamma)}\right)\Gamma\sqrt{y(t)}\ , \tag{5.53}$$

$$\epsilon^{N,*}(t) = \exp\left\{\Gamma_N \frac{\left[1-(1+J\theta^{\tilde{N},*}(t))^{1-\gamma}\right]}{1-\gamma} e^{\bar{F}(t)\xi}\right\}. \tag{5.54}$$

Proof. To determine the optimal policies we can start from the (5.12) equation associated to the investment problem, and guess

$$V(t,W,y) = \exp\{\bar{F}(t)y(t) + \bar{G}(t)\}\frac{W_t^{1-\gamma}}{1-\gamma}. \tag{5.55}$$

We evaluate the partial derivative of the value function appearing in HJB equation

$$\frac{\partial V}{\partial t} = (\dot{\bar{F}}_t y_t + \dot{\bar{G}}_t)V\ , \tag{5.56}$$

$$\frac{\partial V}{\partial W} = \exp\left\{(F(t)y(t)+G(t))\right\}W_t^{-\gamma} = \frac{V}{W_t}(1-\gamma)\ , \tag{5.57}$$

$$\frac{\partial V}{\partial y} = F(t)V \tag{5.58}$$

$$\frac{\partial^2 V}{\partial W^2} = -\gamma\exp\left\{(F(t)y(t)+G(t))\right\}W(t)^{-\gamma-1} = -\gamma V(1-\gamma)W^{-2}\ , \tag{5.59}$$

$$\frac{\partial^2 V}{\partial y^2} = F(t)^2 V\ , \tag{5.60}$$

$$\frac{\partial^2 V}{\partial W \partial y} = F(t)\exp\left\{(F(t)y(t)+G(t))\right\}W(t)^{-\gamma} = \frac{VF(t)}{W(t)}(1-\gamma)\ . \tag{5.61}$$

$$\begin{aligned}\Delta V &= \exp\{(F(t)(y(t)+\xi)+G(t))\}\frac{W(t)^{1-\gamma}(1+\theta^N(t)J)^{1-\gamma}}{1-\gamma} \\ &- \exp\{(F(t)y(t)+G(t))\}\frac{W(t)^{1-\gamma}}{1-\gamma} = \exp\{(F(t)y(t)+G(t))\}\frac{W(t)^{1-\gamma}}{1-\gamma} \\ &\left[e^{F(t)\xi}(1+\theta^N(t)J)^{1-\gamma}-1\right] = V\left[e^{F(t)\xi}\left(1+\theta^N(t)J\right)^{1-\gamma}-1\right].\end{aligned} \tag{5.62}$$

We first solve the infimization part by applying the first-order condition w.r.t. the distortion measures, namely

$$\frac{\epsilon^s(t)}{\Psi} - \theta^B(t)\sqrt{y(t)^{-1}}W(t)V_w - V_y\sigma\rho\sqrt{y(t)} = 0 \tag{5.63}$$

$$\frac{\epsilon^y(t)}{\Psi} - \theta^Z(t)\sqrt{y(t)^{-1}}W(t)V_w - V_y\sigma\sqrt{1-\rho^2}\sqrt{y_t} = 0 \tag{5.64}$$

$$\lambda^{\mathbb{P}}\Delta V + \frac{(\ln\epsilon^N(t) + \frac{1}{\epsilon^N(t)}\epsilon^N(t) - 1)\lambda^{\mathbb{P}}}{\Psi_N} = 0, \tag{5.65}$$

and so we obtain

$$\begin{aligned}
\epsilon^s(t) &= \sqrt{y(t)}\Psi\left(V_w W(t)\theta^B(t)\frac{1}{y(t)} + V_y\sigma\rho\right)\\
\epsilon^y(t) &= \sqrt{y(t)}\Psi\left(V_w W(t)\theta^Z(t)\frac{1}{y(t)} + V_y\sigma\sqrt{1-\rho^2}\right)\\
\epsilon^N(t) &= \exp\{-\Psi_N \Delta V\}.
\end{aligned} \tag{5.66}$$

Remember the HJB equation

$$\sup_{W(t),C(t)}\Bigg\{f(C,V)+V_t+V_wW(t)\Bigg(r-\frac{C(t)}{W(t)}+\eta\theta^B(t)-\theta^B(t)\epsilon(t)^s\sqrt{y(t)^{-1}}+\theta^Z(t)\nu-$$
$$\theta^Z(t)\epsilon(t)^y\sqrt{y(t)^{-1}}-\theta^N(t)J\lambda^{\mathbb{Q}}\Bigg)+V_y\left[k^{\mathbb{Q}}(\theta^{\mathbb{Q}}-y(t))-\sigma\rho\epsilon(t)^s\sqrt{y(t)}-\sqrt{1-\rho^2}\sigma\sqrt{y(t)}\epsilon(t)^y\right]+$$
$$\frac{1}{2}V_{WW}W(t)^2\left((\theta(t)^B)^2+(\theta(t)^Z)^2\right)y(t)^{-1}+\frac{1}{2}V_{yy}\sigma^2y(t)+V_{wy}W(t)\sigma\left[\theta^B(t)\rho+\sqrt{1-\rho^2}\theta^Z(t)\right]+$$
$$\lambda^{\mathbb{P}}\Delta V\epsilon(t)^N+\frac{(\epsilon(t)^s)^2+(\epsilon(t)^y)^2}{2\Psi(W,y)}+\frac{(\epsilon(t)^N\ln\epsilon(t)^N-\epsilon(t)^N+1)\lambda^{\mathbb{P}}}{\Psi_N(W,y)}\Bigg\}=0, \tag{5.67}$$

we replace (5.66) in (5.25) and so we obtain

$$\sup_{W(t),C(t)} \Bigg\{ f(C,V) + V(t) + V_w W(t)\left(r - \frac{C(t)}{W(t)} + \eta\theta^B(t) + \theta^Z(t)\nu - \theta^N(t)J\lambda^{\mathbb{Q}}\right) +$$
$$+ V_y\left[k^{\mathbb{Q}}(\theta^{\mathbb{Q}} - y(t))\right] + \frac{1}{2}V_{WW}W(t)^2\left((\theta^B(t))^2 + (\theta^Z(t))^2\right)y(t)^{-1} + \frac{1}{2}V_{yy}\sigma^2 y(t) + V_{wy}W(t)\sigma\Bigg[\theta^B(t)\rho +$$
$$\sqrt{1-\rho^2}\theta^Z(t)\Bigg] + \lambda^{\mathbb{P}}\Delta V\epsilon(t)^N - \epsilon(t)^s\left(\sqrt{y(t)^{-1}}\theta^B(t)V_w W(t) + \sqrt{y(t)}\sigma\rho V_y\right) - \epsilon(t)^y\Bigg(\sqrt{y(t)^{-1}}\theta^Z(t)V_w W(t)$$
$$+ \sqrt{y(t)}\sigma\sqrt{1-\rho^2}V_y\Bigg) + \frac{(\epsilon(t)^s)^2}{2\Psi} + \frac{(\epsilon(t)^y)^2}{2\Psi(W,y)} + \frac{(\epsilon(t)^N\ln\epsilon(t)^N - \epsilon(t)^N + 1)\lambda^{\mathbb{P}}}{\Psi_N(W,y)}\Bigg\} = 0$$

$$\sup_{W(t),C(t)} \Bigg\{ f(C,V) + V_t + V_w W(t)\left(r - \frac{C(t)}{W(t)} + \eta\theta^B(t) + \theta^Z(t)\nu - \theta^N(t)J\lambda^{\mathbb{Q}}\right) +$$
$$+ V_y\left[k^{\mathbb{Q}}(\theta^{\mathbb{Q}} - y(t))\right] + \frac{1}{2}V_{WW}W(t)^2\left((\theta^B(t))^2 + (\theta^Z(t))^2\right)y(t)^{-1} + \frac{1}{2}V_{yy}\sigma^2 y(t) + V_{wy}W(t)\sigma$$
$$\left[\theta^B(t)\rho + \sqrt{1-\rho^2}\theta^Z(t)\right] + \lambda^{\mathbb{P}}\Delta V\exp\{-\Psi_N\Delta V\} - \sqrt{y(t)}\Psi\left(V_W W(t)\frac{\theta(t)^B}{y(t)} + V_y\sigma\rho\right)\sqrt{y(t)}$$
$$\left(V_W W(t)\frac{\theta^B(t)}{y(t)} + V_y\sigma\rho\right) - \sqrt{y(t)}\Psi\left(V_W W(t)\frac{\theta^Z(t)}{y(t)} + V_y\sigma\sqrt{1-\rho^2}\right)\sqrt{y(t)}\Bigg(V_W W(t)\frac{\theta^Z(t)}{y(t)} +$$
$$V_y\sigma\sqrt{1-\rho^2}\Bigg) + \frac{1}{2\Psi}\left(\sqrt{y(t)}\Psi\left(V_W W(t)\frac{\theta^B(t)}{y(t)} + V_y\sigma\rho\right)\right)^2 + \frac{1}{2\Psi}\Bigg(\sqrt{y(t)}\Psi\Bigg(V_W W(t)\frac{\theta^Z(t)}{y(t)} +$$
$$V_y\sigma\sqrt{1-\rho^2}\Bigg)\Bigg)^2 + \frac{\left(\exp\{-\Psi_N\Delta V\}\ln\exp\{-\Psi_N\Delta V\} - \exp\{-\Psi_N\Delta V\}\right)\lambda^{\mathbb{P}}}{\Psi_N}\Bigg\} = 0 \quad (5.68)$$

$$\mathcal{F} := \sup_{W(t),C(t)} \Bigg\{ f(C,V) + V_t + V_w W(t)\left(r - \frac{C(t)}{W(t)} + \eta\theta^B(t) + \theta^Z(t)\nu - \theta^N(t)J\lambda^{\mathbb{Q}}\right) +$$
$$+ V_y\left[k^{\mathbb{Q}}(\theta^{\mathbb{Q}} - y(t))\right] + \frac{1}{2}V_{WW}W(t)^2\left((\theta^B(t))^2 + (\theta^Z(t))^2\right)y(t)^{-1} +$$
$$+ \frac{1}{2}V_{yy}\sigma^2 y(t) + V_{wy}W(t)\sigma\left[\theta^B(t)\rho + \sqrt{1-\rho^2}\theta^Z(t)\right] - \frac{1}{2}\Psi y(t)\left(V_W W(t)\frac{\theta^B(t)}{y(t)} + V_y\sigma\rho\right)^2$$
$$- \frac{1}{2}\Psi y(t)\left(V_W W(t)\frac{\theta^Z(t)}{y(t)} + V_y\sigma\sqrt{1-\rho^2}\right)^2 + \frac{\lambda^{\mathbb{P}}}{\Psi_N}\left(1 - \exp\{-\Psi_N\Delta V\}\right)\Bigg\} = 0. \quad (5.69)$$

We replace all the derivatives (5.56) – (5.62) in the first order condition and we obtain

$$\begin{aligned}
\frac{C(t)}{W(t)} &= \beta \\
\theta^B(t) &= \left[\frac{\eta}{\gamma+\Gamma} + \frac{\bar{F}(t)\sigma\rho(1-\gamma-\Gamma)}{(\gamma+\Gamma)(1-\gamma)}\right] y(t) \\
\theta^Z(t) &= \left[\frac{\nu}{\gamma+\Gamma} + \frac{\bar{F}(t)\sigma\sqrt{1-\rho^2}(1-\gamma-\Gamma)}{(\gamma+\Gamma)(1-\gamma)}\right] y(t) \\
\theta^N(t) &= \frac{\left[\frac{\lambda^{\mathbb{Q}}}{\lambda^{\mathbb{P}}} e^{-\bar{F}(t)\xi(y(t))} \exp\left\{\frac{\Gamma_N e^{\bar{F}(t)\xi(y(t))}\left[(1+\theta^N(t)J)^{1-\gamma}-1\right]}{1-\gamma}\right\}\right]^{-\frac{1}{\gamma}} - 1}{J},
\end{aligned} \tag{5.70}$$

Then we substitute value function, all the derivatives (5.56) – (5.62) and optimal exposures (5.70), remembering that $\psi(W,y) = \frac{\Gamma}{(1-\gamma)V(W,y)} > 0$, $\psi_N(W,y) = \frac{\Gamma_N}{(1-\gamma)V(W,y)} > 0$, we get

$$\begin{aligned}
\epsilon(t)^s =& \sqrt{y(t)}\psi\left(V_W W(t)\theta^B(t)\frac{1}{y(t)} + V_y\sigma\rho\right) = \sqrt{y(t)}\frac{\Gamma}{(1-\gamma)V}\left(\frac{V}{W(t)}(1-\gamma)W(t)\frac{\theta^B(t)}{y(t)} + F(t)V\sigma\rho\right) = \\
=& \sqrt{y(t)}\frac{\Gamma}{(1-\gamma)V}\left[\frac{V(1-\gamma)}{y(t)}y(t)\left(\frac{\eta}{\gamma+\Gamma} + \frac{\sigma\rho(1-\gamma-\Gamma)F(t)}{(\gamma+\Gamma)(1-\gamma)}\right) + F(t)V\sigma\rho\right] = \\
& \sqrt{y(t)}\Gamma\left(\frac{\eta}{\gamma+\Gamma} + \frac{F(t)\sigma\rho(1-\gamma-\Gamma)}{(\gamma+\Gamma)(1-\psi)}\right) + \Gamma\sqrt{y(t)}\frac{F(t)\sigma\rho}{(1-\gamma)} = \\
& \frac{\eta(1-\gamma)\sqrt{y(t)}\Gamma + \rho\sigma(1-\gamma-\Gamma)F(t)\sqrt{y(t)}\Gamma + (\gamma+\Gamma)F\sigma\rho\Gamma\sqrt{y(t)}}{(1-\gamma)(\gamma+\Gamma)} = \\
& \frac{\sqrt{y(t)}\Gamma\left[\eta(1-\gamma) + \rho\sigma F(t)(1-\gamma-\Gamma) + F(t)\sigma\rho(\gamma+\Gamma)\right]}{(1-\gamma)(\gamma+\Gamma)} = \\
& \frac{\sqrt{y(t)}\Gamma\left[\eta(1-\gamma) + \rho\sigma F(t) - \rho\sigma F(t)\gamma - \rho\sigma F(t)\Gamma + (t)F\sigma\rho\gamma + F(t)\sigma\rho\Gamma\right]}{(1-\gamma)(\gamma+\Gamma)} = \\
& \frac{\sqrt{y(t)}\Gamma\left(\eta(1-\gamma) + \rho\sigma F(t)\right)}{(\gamma+\Gamma)(1-\rho)}.
\end{aligned}$$

$$\begin{aligned}
\epsilon(t)^y =& \sqrt{y(t)}\psi\left(V_W W(t)\theta^Z(t)\frac{1}{y(t)} + V_y\sigma\sqrt{1-\rho^2}\right) = \sqrt{y(t)}\frac{\Gamma}{(1-\gamma)V}\left(\frac{V}{W(t)}(1-\gamma)W(t)\right. \\
& \left.\frac{\theta^Z(t)}{y(t)} + F(t)V\sigma\sqrt{1-\rho^2}\right) = \sqrt{y(t)}\Gamma\left(\frac{\nu}{\gamma+\Gamma} + \frac{\sigma\sqrt{1-\rho^2}(1-\gamma-\Gamma)F(t)}{(\gamma+\Gamma)(1-\gamma)}\right) + \\
& \frac{\Gamma\sqrt{y(t)}F(t)\sigma\sqrt{1-\rho^2}}{(1-\gamma)} = \frac{\Gamma\sqrt{y(t)}\left(\nu(1-\gamma) + \sigma F(t)\sqrt{1-\rho^2}\right)}{(1-\gamma)(\gamma+\Gamma)}
\end{aligned}$$

$$
\begin{aligned}
\epsilon(t)^N =& \exp\left(\frac{-\Gamma_N V\left[e^{F\xi(y(t))}(1+\theta^N(t)J)^{1-\gamma}-1\right]}{(1-\gamma)V}\right)= \\
&= \exp\left(\frac{-\Gamma_N \left[e^{F(t)\xi(y(t))}(1+\theta^N(t)J)^{1-\gamma}-1\right]}{(1-\gamma)}\right)= \\
&= \exp\left(\frac{\Gamma_N e^{F(t)\xi(y(t))}\left[1-(1+\theta^N(t)J)^{1-\gamma}\right]}{(1-\gamma)}\right).
\end{aligned}
$$

Finally we replace all the derivatives (5.56) – (5.62), the optimal policy, the optimal consumption and the normalized aggregator (2.28) in (5.69), so that we obtain

$$
\begin{aligned}
&\beta(1-\gamma)V\log(\beta W(t)) - \beta V\log((1-\gamma)V)) + \dot{\bar{F}}_t V y(t) + \dot{\bar{G}}_t V + rV(1-\gamma) + \eta V(1-\gamma)y(t) \\
&\left(\frac{\eta}{\gamma+\Gamma} + \frac{\bar{F}(t)\sigma\rho(1-\gamma-\Gamma)}{(\gamma+\Gamma)(1-\gamma)}\right) + \nu V(1-\gamma)y(t)\left(\frac{\nu}{\gamma+\Gamma} + \frac{\bar{F}(t)(t)\sigma\sqrt{1-\rho^2}(1-\gamma-\Gamma)}{(\gamma+\Gamma)(1-\gamma)}\right) \\
&-\theta_t^N J\lambda^{\mathbb{Q}}V(1-\gamma) - \beta V(1-\gamma) - \frac{1}{2}\gamma V(1-\gamma)W(t)^{-2}W(t)^2\frac{1}{y(t)}\left[y(t)^2\left(\frac{\eta}{\gamma+\Gamma} + \frac{\bar{F}(t)\sigma\rho(1-\gamma-\Gamma)}{(\gamma+\Gamma)(1-\gamma)}\right)^2 + \right. \\
&\left. y(t)^2\left(\frac{\nu}{\gamma+\Gamma} + \frac{\bar{F}(t)\sigma\sqrt{1-\rho^2}(1-\gamma-\Gamma)}{(\gamma+\Gamma)(1-\gamma)}\right)^2\right] + \frac{1}{2}\bar{F}(t)^2 V_t y(t) + V\bar{F}\sigma(1-\gamma)\left[y(t)\rho\right. \\
&\left.\left(\frac{\eta}{\gamma+\Gamma} + \frac{\bar{F}(t)\sigma\rho(1-\gamma-\Gamma)}{(\gamma+\Gamma)(1-\gamma)}\right)\sqrt{1-\rho^2}y(t)\left(\frac{\nu}{\gamma+\Gamma} + \frac{\bar{F}(t)\sigma\sqrt{1-\rho^2}(1-\gamma-\Gamma)}{(\gamma+\Gamma)(1-\gamma)}\right)\right] \\
&-\frac{1}{2}\frac{\Gamma y(t)}{(1-\gamma)V}\left[V(1-\gamma)\frac{1}{y(t)}\left(\frac{\eta}{\gamma+\Gamma} + \frac{\bar{F}(t)\sigma\rho(1-\gamma-\Gamma)}{(\gamma+\Gamma)(1-\gamma)}\right) + \bar{F}V\sigma\rho\right]^2 - \frac{1}{2}\frac{\Gamma y(t)}{(1-\gamma)V} \\
&\left[V(1-\gamma)\frac{1}{y(t)}\left(\frac{\nu}{\gamma+\Gamma} + \frac{\bar{F}(t)\sigma\sqrt{1-\rho^2}(1-\gamma-\Gamma)}{(\gamma+\Gamma)(1-\gamma)}\right) + \bar{F}(t)V\sigma\sqrt{1-\rho^2}\right]^2 + \frac{\lambda^{\mathbb{P}}(1-\gamma)V}{\Gamma_N} \\
&\left[1-\exp\left(-\frac{\Gamma_N}{(1-\gamma)V}Ve^{\bar{F}(t)\xi(y(t))}[(1+\theta_t^N J)^{1-\gamma}-1]\right)\right].
\end{aligned}
\tag{5.71}
$$

Proceeding dividing by V, and adjusting the terms by simple algebraic manipulation we obtain

$$
\begin{aligned}
&\left(\dot{\bar{F}}_t - \beta\bar{F}(t) + \left(\frac{(1-\gamma-\Gamma)}{(\gamma+\Gamma)}\right)\bar{F}(t)\sigma\left(\rho\eta + \nu\sqrt{1-\rho^2}\right) - \frac{1}{2}\frac{(1-\gamma-\Gamma)}{(\gamma+\Gamma)}\bar{F}(t)^2\sigma^2\right. \\
&\left. + \frac{1}{2}\frac{(1-\gamma)}{(\gamma+\Gamma)}(\eta^2+\nu^2)\right)y(t) + \dot{\bar{G}}_t + \beta(1-\gamma)\log(\beta) - \beta\bar{G}(t) + r(1-\gamma) - \beta(1-\gamma) \\
&-\theta_t^N J(1-\gamma) - \frac{\lambda^{\mathbb{P}}}{\Gamma_N}(1-\gamma)\exp\left\{\frac{e^{\bar{F}(t)\xi(y(t))}\Gamma_N(1+(1+\theta_t^N J)^{1-\gamma})}{1-\gamma}\right\}.
\end{aligned}
$$

□

From Proposition 5.1.2 we learn that when $\psi = 1$ the structure of optimal exposures and optimal distortion measures are the same as the general case. Hence, we can reiterate the same comments as before. The unique difference relates to the optimal consumption-wealth ratio, which results to be constant and equal to the rate of time preferences β, and does

not depend on the level of volatility, nor on the ambiguity aversion parameters. Thus, we argue that investors consume a fixed level of wealth at each time $t \in [0, T]$.
Finally, we focus on the optimal allocation policies.

Corollary 5.1.3. *Consider the investment problem* (5.1.1) *and the associated HJB equation* (5.12)*. Assume further that there exist non-redundant derivatives traded at time* $t \leq T$*. The optimal portfolio weights related to the risky asset and the derivative securities in* $t \leq T$ *are*

$$\phi^*(t) = \theta^{\tilde{B},*}(t) - \sum_{i=1}^{2} \pi^{(i),*}(t) \left(S(t) \frac{g_s^{(i)}}{O(t)^{(i)}} + \sigma\rho \frac{g_y^{(i)}}{O(t)^{(i)}} y(t) \right) , \tag{5.72a}$$

$$\begin{aligned} \pi^{(1),*}(t) = \frac{1}{\mathcal{D}(t)} \Bigg\{ & \left(-\theta^{\tilde{B},*}(t) + \frac{\rho\theta^{\tilde{Z},*}(t)}{\sqrt{1-\rho^2}} \right) \frac{g_y^{(2)}}{O(t)^{(2)}} \\ & + \left[\theta^{\tilde{N},*}(t) \frac{g_y^{(2)}}{O(t)^{(2)}} - \left(\frac{\Delta g^{(2)}}{O(t)^{(2)}} - S(t) \frac{g_s^{(2)}}{O(t)^{(2)}} \right) \frac{\theta^{\tilde{Z},*}(t)}{\sigma\sqrt{1-\rho^2} y(t)} \right] \Bigg\} , \end{aligned} \tag{5.72b}$$

$$\begin{aligned} \pi^{(1),*}(t) = \frac{1}{\mathcal{D}(t)} \Bigg\{ & \left[\left(\frac{\Delta g^{(1)}}{J\, O(t)^{(1)}} - S(t) \frac{g_s^{(1)}}{O(t)^{(1)}} \right) \frac{\theta^{\tilde{Z},*}(t)}{\sigma\sqrt{1-\rho^2} y(t)} - \theta^{\tilde{N},*}(t) \frac{g_y^{(1)}}{O(t)^{(1)}} \right] \\ & - \left(-\theta^{\tilde{B},*}(t) + \frac{\rho\theta^{\tilde{Z},*}(t)}{\sqrt{1-\rho^2}} \right) \frac{g_y^{(1)}}{O(t)^{(1)}} \Bigg\} , \end{aligned} \tag{5.72c}$$

where $\theta^{\tilde{B},*}(t)$, $\theta^{\tilde{Z},*}(t)$, *and* $\theta^{\tilde{N},*}(t)$ *are given in* (5.7)*, and represent the optimal portfolio exposures on the risk factors* $\tilde{B}$, $\tilde{Z}$, $\tilde{N}$, *respectively.*

5.2 Numerical experiments

In this Section, we would like to measure the effect of the model uncertainty both for optimal policies and consumption. Hence, we first calibrate our model on real data, considering the closing price of European options with the SPX index as underlying. See the calibration procedure provided by Section 3.3 of this thesis. The estimates are provided in Table 5.1.

Stock		**Precision**	
Parameter	Value	Parameter	Value
$\lambda^{\mathbb{Q}}$	0.4922	θ	205.0658
ρ	0.6207	k	1.7184
J	-0.1997	ξ	-0.1462
		σ	10.4945

Table 5.1. Parameters estimate for stock and precision processes.

For the preferences parameters, we set $\beta = 0.06$, $\gamma = 3$, and $\psi = 0.7$ for the baseline case, letting such parameters vary when studying the sensitivity of our solutions. Moreover, we fix the risk premia as $\eta = 6\%$, $\nu = -7\%$, $\lambda^{\mathbb{P}} = 0.2$. The risk-free interest rate is set $r = 0\%$.

The market is complete with two non-redundant derivatives, namely, we consider a 3-month 5%-OTM European call and a 3-month 5%-ITM European call. It is worth stressing that such a choice is arbitrary and might have an impact on the functional form of the optimal weights. In this chapter we will thoroughly discuss the role of derivatives in optimal policies in Section 5.2.3.

5.2.1 Optimal policies and ambiguity

We study the sensitivity of the optimal policies to the ambiguity parameters. Our findings are displayed in Figure 5.1.

First, we analyze the optimal risk exposures to the diffusive, volatility, and jump risks, as a function of the ambiguity aversion parameter to diffusive and jump risks. Apparently, we expect that the exposures are decreasing (in absolute value) in the respective ambiguity parameters since the investor is more pessimistic about the investment opportunities. Such an economic intuition is confirmed by the results depicted in the top charts of Figure 5.1. The top-left panel shows that the diffusion risk exposures to B and Z are decreasing in absolute terms, as the uncertainty against the (diffusive) model misspecification increases. As expected, the jump risk exposure is unaffected by variations of Γ. The pattern is reversed in the top-right panel, where the exposure to jump risk is decreasing w.r.t. the ambiguity aversion to jump risk (Γ_N), but the optimal exposures to diffusive and volatility risks are insensitive to variations of Γ_N. Our findings suggest that the optimal exposures are primarily driven by the corresponding ambiguity aversion parameters.

Then, we investigate the impact of uncertainty on the investors' optimal consumption policy, under different levels of the elasticity of the intertemporal substitution of consumption parameter (EIS), as a function of ambiguity aversion parameters. The corresponding results are displayed in the middle charts of Figure 5.1. Heuristically, we could envision that an enhancement of the elasticity of intertemporal substitution of consumption would induce the investor to save wealth today while awaiting favorable investment periods. To confirm such an economic intuition, we first notice that, for $\psi < 1$, the investor's consumption dwindles as the ambiguity aversion to diffusive risk increases. Such behavior might be ascribed to the deterioration of expectations regarding investment opportunities: under the robust model, the investor becomes more pessimistic about investment opportunities. In general, the pessimistic outlook against investment opportunities within the alternative model generates a negative income fallout, as the investor expects both a worsening of future wealth and a positive intertemporal substitution effect. For investors with $\psi < 1$, the income effect dominates the substitution one, and consequently, produces a reduction in the investor's consumption when the ambiguity aversion parameter becomes larger. We may provide similar comments, also when the independent variable is Γ_N. The results are shown in the right panel of Figure 5.1. In this case, the magnitude of consumption reduction is smaller w.r.t. the ambiguity aversion parameter to jump risk.

The impact of ambiguity on the optimal distortion measures is reported in the bottom charts of Figure 5.1. More precisely, we show the distortion measures as functions of the ambiguity parameter to diffusive risk (bottom-left panel) and jump risk (bottom-right panel). We recall that the distortion measures represent the ambiguity risk premia, evaluating the

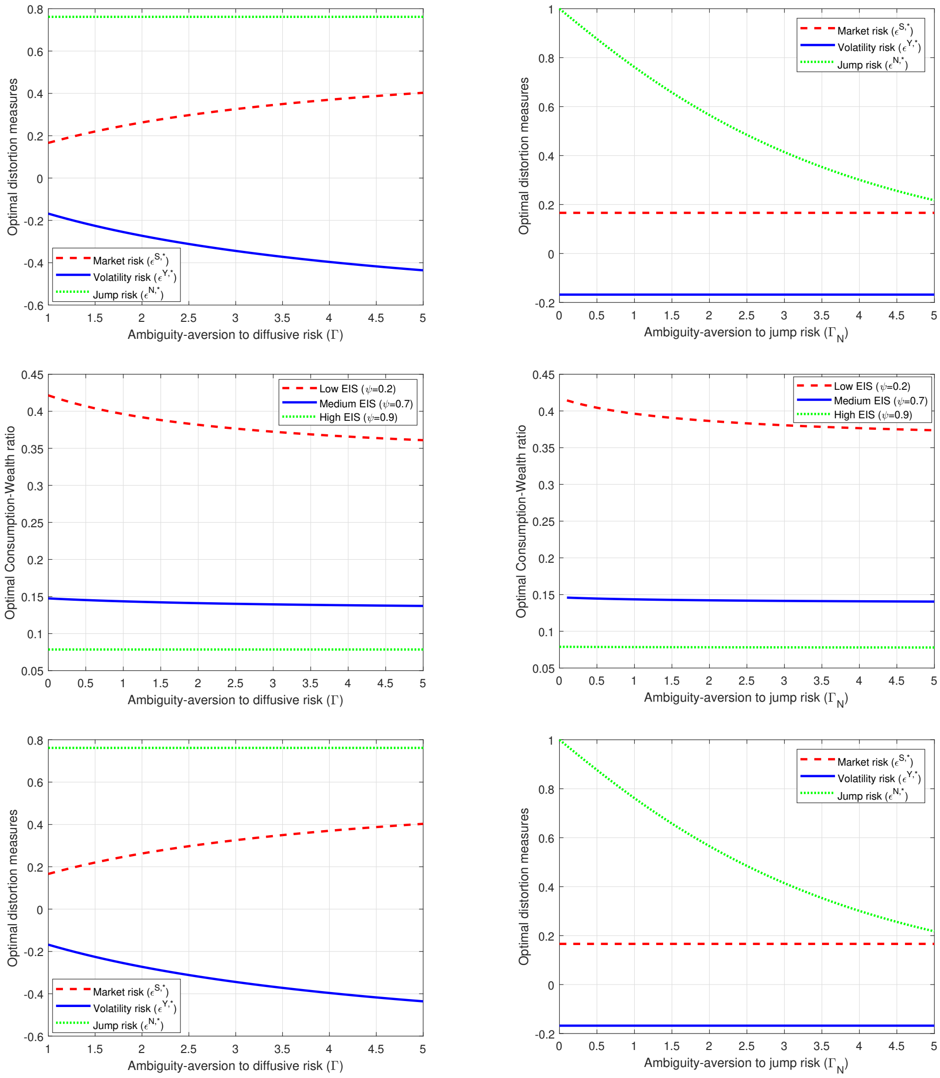

Figure 5.1. Optimal policies as a function of ambiguity aversion parameters to diffusive risk (left charts, setting $\Gamma_N = 1$) and jump risk (right charts, setting $\Gamma = 1$). The top, middle, and bottom graphs refer to the optimal exposures, the optimal consumption-wealth ratio, and the optimal distortion measures, respectively. The baseline scenario considers the EIS parameter fixed to $\psi = 0.7$, while the risk-aversion parameter is set to $\gamma = 3$. The rate of time preferences is $\beta = 6\%$ annually, the constant risk-free rate is equal to $r = 0\%$. The remaining parameters are provided in Table 5.1.

distance between the reference model and the alternative ones. The chart shows that a greater ambiguity aversion parameter to diffusive risk leads to an increase in the distortion measures to diffusive risks since the investor is less confident in the reference model, and the worst-case scenario can greatly deviate from the benchmark. As expected, the jump risk distortion measure is unaffected by changes in the ambiguity aversion parameter of diffusive risks. The bottom-right chart exhibits a different behavior: the distortion measure to diffusive risk is insensitive to changes in the ambiguity parameter to jump risk, but an increase in the ambiguity parameter related to the jump intensity misspecification leads to a reduction in the distortion measure to jump risk. The reason for such a result relies on the fact that the exposure to jump risk reduces since the reference model greatly deviates from the alternative one.

Finally, we examine the loss of wealth suffered by the investor who ignores model uncertainty. To quantify such a loss, we consider the *Wealth Equivalent Loss* (WEL). To obtain such an indicator, we consider that the investor pursuing the *optimal strategy* has an unknown wealth equal to $\tilde{W}$ and a value function given by

$$V(t,y,\tilde{W}) = \exp\{-H(F(t)y(t)+G(t))\}\frac{\tilde{W}(t)^{1-\gamma}}{1-\gamma},\; t\,\in\,[0,T]\;, \tag{5.73}$$

where $H = \frac{1-\gamma}{1-\psi}$ and the functions F and G are solutions to the system of ODEs given in (5.15). At the same time, we consider the *sub-optimal* strategy with wealth equal to one, such that the value function is

$$\hat{V}(t,y,1) = \exp\{-H(\hat{F}(t)y(t)+\hat{G}(t))\}\frac{1}{1-\gamma}\;, \tag{5.74}$$

being $\hat{F}(t)$, $\hat{G}(t)$ the solutions to the ODEs provided in (5.15) by setting $\Gamma = 0$ and $\Gamma_N = 0$ in the optimal policies and consumption. We have

$$\tilde{W}(t) = \left[\frac{\exp\{-H(\hat{F}(t)y(t)+\hat{G}(t))\}}{\exp\{-H(Tr(F(t)y(t))+G(t))\}}\right]^{\frac{1}{1-\gamma}}\;. \tag{5.75}$$

Finally, we define

$$WEL = 1-\tilde{W}(t),\; t\,\in\,[0,T]\;,$$

and we get

$$WEL = 1-\exp\left\{\frac{1}{1-\psi}\left[(\hat{F}(t)-F(t))y(t)+(\hat{G}(t)-G(t))\right]\right\}. \tag{5.76}$$

We report the numerical analysis in Figure 5.2. Our findings prove that ignoring ambiguity against both diffusion and jump risks induces considerable losses, and such a loss is increasing in the ambiguity aversion parameters.

5.2.2 Optimal policies and co-jumps

We would like to measure the impact of co-jumps on optimal policies, for different ambiguity-aversion levels. To reach such a goal, we exploit again the WEL metrics. The results are shown in Figure 5.3.

We notice that ignoring uncertainty for diffusive risk results in a slightly more significant loss than ignoring uncertainty for jump risk. Differently from the extant literature, see e.g. Wei et al. [2023], we recover that missing ambiguity to jump risk no longer has a negligible effect on the loss of investor wealth, due to the inclusion of jumps in volatility

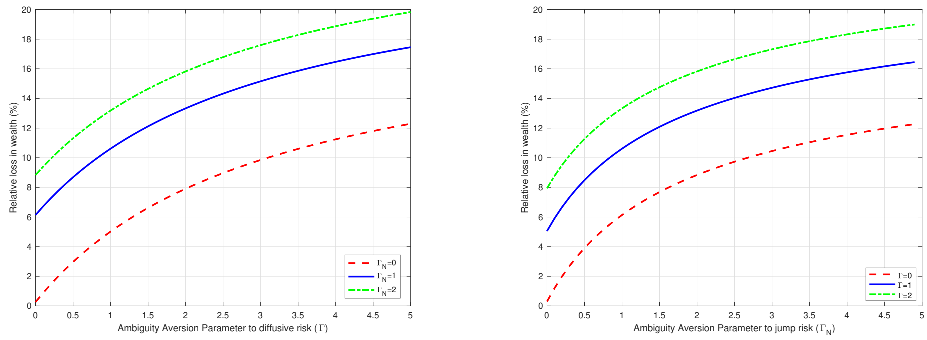

Figure 5.2. Wealth Equivalent Loss as a function of ambiguity aversion parameters to diffusive risk (left chart, for fixed $\Gamma_N = 1$) and jump risk (right chart, for fixed $\Gamma = 1$).The EIS parameter is $\psi = 0.7$ and the risk-aversion parameter is set to $\gamma = 3$. The rate of time preferences is $\beta = 6\%$ annually, and the constant risk-free rate is $r = 0\%$. The remaining parameters are provided in Table 5.1.

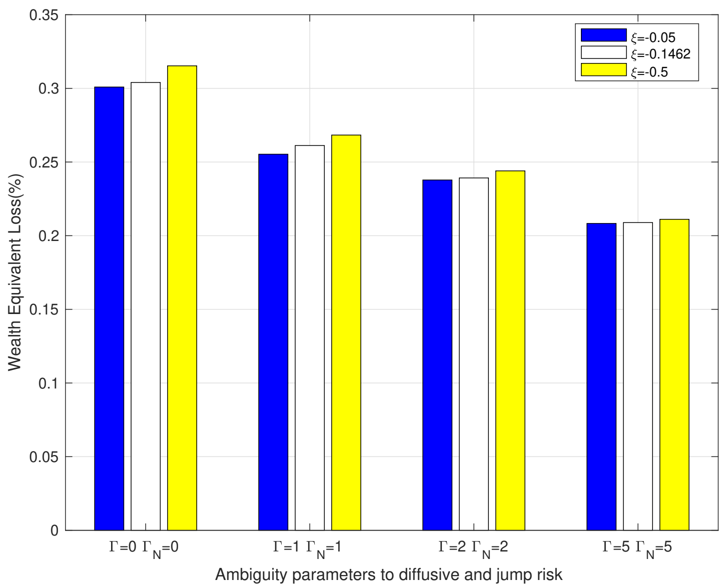

Figure 5.3. Wealth Equivalent Loss for ambiguity parameters equal to $\Gamma = 0, \Gamma_N = 0, \Gamma = 1, \Gamma_N = 1, \Gamma = 2, \Gamma_N = 2, \Gamma = 5, \Gamma_N = 5$ and volatility jump size equal to $\xi = 5\%, \xi = 14.62\%, \xi = 50\%$ respectively. The loss is calculated by considering the optimal strategy with volatility jumps w.r.t the sub-optimal strategy without jumps in volatility (i.e., $\xi = 0$). The EIS parameter is fixed to $\psi = 0.7$ and the risk-aversion parameter is set to $\gamma = 3$. The rate of time preferences β is set equal to 6% annually, the constant risk-free rate is equal to $r = 0\%$. The remaining parameters are provided in Table 5.1.

in the model. The addition of jumps in volatility implies that the loss is sensitive to jump intensity misspecification. In Figure 5.3 we depict the potential loss the investor might suffer when ignoring the presence of jumps in the state variable, for three different values of the precision-jump amplitude, for several ambiguity-aversion levels. As expected, numerical tests reveal that the potential loss suffered by investors decreases as they deviate from the reference model. The reason why such a potential loss is all the higher the lower the value of the ambiguity aversion parameters may be sought in the connection between ambiguity and risk aversion: a robust investor is, by construction, also relatively heavily risk-averse, as emphasized in Maenhout [2004]. Moreover, the loss is more pronounced the more marked the impact of the jump (in absolute value) in the state variable. The difference in terms of loss is attenuated when ambiguity aversion becomes more prominent.

5.2.3 Optimal policies and derivatives

We investigate the impact of derivatives on optimal robust policies. In Table 5.2 we report the percentage of optimal wealth invested in the market components, by assuming different figures for non-redundant derivatives. In particular, we consider four combinations of European options, with different maturities and moneyness.

	$\Gamma = \Gamma_N = 1$			$\Gamma = \Gamma_N = 2$			$\Gamma = \Gamma_N = 5$		
PTF of derivatives	$\phi(\%)$	$\pi^{(1)}(\%)$	$\pi^{(2)}(\%)$	$\phi(\%)$	$\pi^{(1)}(\%)$	$\pi^{(2)}(\%)$	$\phi(\%)$	$\pi^{(1)}(\%)$	$\pi^{(2)}(\%)$
3-month 5%-OTM + 3-month 5%- ITM European Call Options	95.62	20.93	-48.20	76.89	17.03	-39.48	49.09	10.93	-25.64
3-month 10%-OTM + 3-month 10%- ITM European Call Options	84.92	5.23	-2.799	68.27	4.25	-24.62	43.65	2.72	-16.25
3-month 10%-OTM European Call + 3-month 5%- OTM European Put Options	99.38	4.62	16.17	80.22	3.75	13.36	51.54	2.40	8.82
3-month 10%-OTM European Call + 3-month 10%- OTM European Put Options	95.28	5.09	25.13	76.83	4.14	20.77	49.30	2.65	13.71

Table 5.2. Optimal allocation with several choices for non-redundant derivatives. The ambiguity parameters to diffusive and jump risk are set equal to $\Gamma = \Gamma_N = 1$, $\Gamma = \Gamma_N = 2$, and $\Gamma = \Gamma_N = 5$, respectively. The EIS parameter is fixed to $\psi = 0.7$ and the risk-aversion parameter is set to $\gamma = 3$. The rate of time preferences β is set equal to 6% annually, the constant risk-free rate is equal to $r = 0\%$. The remaining parameters are provided in Table 5.1

We notice that the allocation in both stock and derivatives decreases for higher ambiguity-aversion levels, as a consequence of the investors' deteriorating investment opportunities. Moreover, for greater values of ITM or OTM call options, the allocation in both stock and derivatives reduces, because of the higher riskiness and the lower liquidity of such options. Hence, we argue that even considering different portfolio compositions or different OTM call/put combinations, the effect of ambiguity on the optimal portfolio remains the same. To substantiate such an intuition, we perform further analysis, to prove that the choice of derivatives does not affect the value function in our model. For the sake of simplicity, inspired by Zeng et al. [2018], we investigate the special case where no jumps occur, i.e. we set $J = 0, \xi = 0$, $\lambda^{\mathbb{P}} = 0$, and $\lambda^{\mathbb{Q}} = 0$.
Within such a particular framework, only one non-redundant derivative is needed. By exploiting Corollary 3.1.1 we find out the following functional form of the optimal strategy:

$$\phi(t) = \theta^B(t) - \frac{\theta^Z(t)S(t)}{y(t)\sigma\sqrt{1-\rho^2}}\frac{g_s}{g_y} - \frac{\theta^Z(t)\rho}{\sqrt{1-\rho^2}}\ , \tag{5.77}$$

$$\pi(t) = -\frac{\theta^Z(t)}{\sigma\sqrt{1-\rho^2}}\frac{O(t)}{g_v}y(t) \tag{5.78}$$

being $g_y = -\frac{g_v}{(y(t))^2}$. From (5.78), we deduce that the proportion of wealth invested in the derivative is inversely proportional to $\frac{g_v}{O(t)}$, that measures the volatility exposure for each unit of currency invested in the derivative, as pointed out in Liu and Pan [2003]. Therefore, we conclude that the greater the exposure to volatility, the lower (in absolute value) the share of wealth to be invested in the derivative.

Moreover, we notice that (5.77) and (5.78) depend on the derivatives via the terms $\frac{g_s}{g_y}$ and $\frac{O(t)}{g_y}$. By assuming the parameterization provided in Table 5.1, we obtain that the allocation in the risky stock, $\phi(t)$, is increasing w.r.t. the ratio $\frac{g_s}{g_y}$, since $-\frac{\theta^Z(t)S(t)}{y(t)\sigma\sqrt{1-\rho^2}} > 0$, while the allocation in the derivative, $\pi(t)$, is decreasing w.r.t. the ratio $\frac{O(t)}{g_y}$, since $\frac{\theta^Z(t)}{y(t)\sigma\sqrt{1-\rho^2}} < 0$. Table 5.3 exhibits the numerical experiments we carried out for three different types of derivatives, namely a European call option, a European put option, and a straddle option.

	$-\frac{\theta^Z(t)S(t)}{y(t)\sigma\sqrt{1-\rho^2}}$	$\frac{g_s}{g_y}$	$\theta^B(t)$	$\theta^Z(t)$	$\phi(t)$	$\frac{\theta^Z(t)(t)}{y(t)\sigma\sqrt{1-\rho^2}}$	$\frac{O(t)}{g_y}$	$\pi(t)$
5%-OTM Call Option	0.2115	−2.1215	1.0001	−1.8701	2.0318	−0.0021	−4.1780	0.0088
5%- OTM Put Option	0.2115	12.6938	1.0001	-1.8701	0.0847	-0.0021	-78.3613	0.1470
Straddle Option	0.2115	5.3402	1.0001	-1.8701	0.0879	-0.0021	-41.5404	0.0879

Table 5.3. Comparison of optimal allocation for the three types of derivatives considered. The ambiguity-aversion parameters are $\Gamma = \Gamma_N = 2$. The EIS parameter is fixed to $\psi = 0.7$ and the risk-aversion parameter is set to $\gamma = 3$. The rate of time preferences β is set equal to 6% annually, the constant risk-free rate is equal to $r = 0\%$. The remaining parameters are provided in Table 5.1

The substantial difference among the derivatives lies in the values reached by the ratio g_s/g_y, representing the sensitivity of the delta to changes in the vega. Such a ratio assumes a negative (resp., positive) value for the call option (resp., for the put option and the straddle). Contrariwise, the ratio $\frac{O(t)}{g_y}$ shows a negative sign for all the derivatives involved in our analysis. However, it is worth highlighting that there exists a difference in the magnitude of such a ratio when the put option provides the highest value.

Now, we study the optimal allocation policies dynamically in a general setting, where co-jumps are allowed. Figure 5.4 describes the simulated path for the optimal weights over the time horizon of two years. We consider two non-redundant derivatives, namely we have a 3-month 5%-OTM call option (DER1) and a 3-month 5%-ITM call option (DER2).

To better clarify the behavior of the risky assets over time, we consider three different scenarios, namely the worst case (5th quantile), the median case (50th quantile), and the best case (95th quantile). We observe that the risky asset (top chart) shows an optimal allocation with a downward trend over time. Besides, the middle graph reveals an almost unchanged allocation for the first derivative, whereby the investor takes a long position over the entire examined time frame. Conversely, the investment in the second derivative entails a short position at every time point. However, such a short position shrinks with time elapsing. In general, we would like to point out that the investment policy is first aggressive, in that at the beginning the investor allocates a larger amount of wealth into the risky asset and derivatives, regardless of the financial position. As the investment horizon widens, the strategy the investor deploys becomes more conservative.

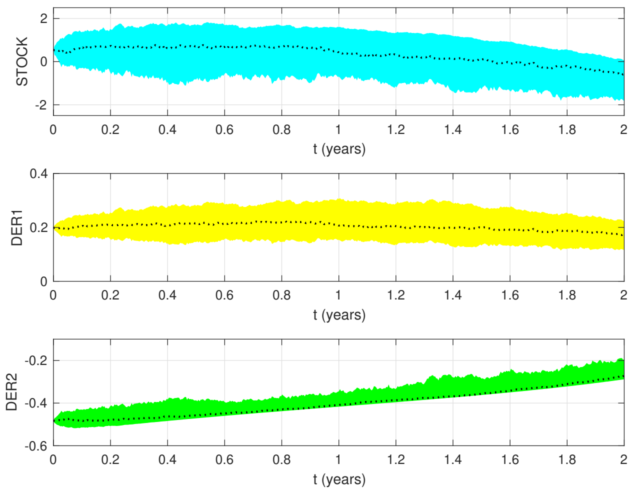

Figure 5.4. Median and extreme scenarios for optimal weights as a function of time (expressed in years). The top, middle, and bottom graphs refer to stock, the first derivative, and the second derivative, respectively. The dotted line in each chart represents the 50th quantile (median), while the area of each graph is bounded by the 5th and the 95th quantiles. The ambiguity aversion parameters are set equal to $\Gamma = 2$ and $\Gamma_N = 2$. The EIS parameter is fixed to $\psi = 0.7$ and the risk-aversion parameter is set to $\gamma = 3$. The rate of time preferences β is set equal to 6% annually, the constant risk-free rate is equal to $r = 0\%$. The remaining parameters are provided in Table 5.1.

5.3 The accuracy of log-linearization of consumption-wealth ratio around the unconditional mean

The optimal strategies proposed in Section 5.1 provide approximate solutions when $\psi \neq 1$. In such a case, the solution is based on a log-linearization procedure for the consumption process, around the unconditional mean, as in Section (4.5). Hence, it is worth investigating the reliability of such an approximation in terms of the unconditional standard deviation of optimal log consumption-wealth ratio, in a context of model uncertainty,

$$\hat{\varsigma} =\mid F(t) \mid \sqrt{var(y(t))} =\mid F(t) \mid \sqrt{\frac{\sigma^2}{2k}\left(\theta + \frac{\xi\lambda}{k}\right)}\,, \tag{5.79}$$

where $k^{\mathbb{Q}}$, $\theta^{\mathbb{Q}}$, and $\sigma > 0$ represent the speed of reversion, the long-term mean level, and the volatility of precision, respectively.

Table 5.3 shows that the percentage unconditional standard deviation ($\hat{\varsigma}(\%)$) is very low in most cases. The lowest values, close to zero, are obtained for large values of γ, but $\psi \simeq 1$. A quick comparison among the three panels composing Table 5.3 reveals that the unconditional standard deviation decreases significantly when the ambiguity aversion parameters become larger, consistently with Maenhout [2004], where the author argues that ambiguity induces a kind of increase in investor risk aversion without changing the EIS.

The aforementioned results suggest that the log-linearization of the consumption-wealth ratio around the unconditional mean works well in this context.

Panel A: $\Gamma = \Gamma_N = 0$					
	$\psi = 1$	$\psi = 0.8$	$\psi = 0.5$	$\psi = 0.2$	$\psi = 0.1$
$\gamma = 2$	0	1.80	4.25	6.71	7.53
$\gamma = 5$	0	0	2.22	3.14	3.46
$\gamma = 8$	0	0	0	2.41	2.57
Panel B: $\Gamma = \Gamma_N = 2$					
	$\psi = 1$	$\psi = 0.8$	$\psi = 0.5$	$\psi = 0.2$	$\psi = 0.1$
$\gamma = 2$	0	0.97	2.20	3.45	3.87
$\gamma = 5$	0	0	1.64	2.29	2.51
$\gamma = 8$	0	0	0	1.96	2.09
Panel C: $\Gamma = \Gamma_N = 5$					
	$\psi = 1$	$\psi = 0.8$	$\psi = 0.5$	$\psi = 0.2$	$\psi = 0.1$
$\gamma = 2$	0	0.57	1.28	2.00	2.24
$\gamma = 5$	0	0	1.17	1.62	1.78
$\gamma = 8$	0	0	0	1.53	1.63

Table 5.4. Unconditional standard deviation ($\hat{\varsigma}$, expressed in percentage) of optimal log consumption-wealth ratio for several values of elasticity of intertemporal substitution of consumption and risk aversion parameter. Panel A represents values in the absence of uncertainty in the model (i.e. $\Gamma = 0, \Gamma_N = 0$), Panel B and Panel C have gradually increasing values for ambiguity aversion parameters.

Chapter 6

Conclusion

In this thesis work we studied a dynamic optimal portfolio allocation problem in a non-affine jump-diffusion model in several market settings, where the investor can access not only a bond and equity but also derivatives written on such equity, and we assumed the underlying price is subject to simultaneous jumps in price and volatility. We modelled the instantaneous precision, the reciprocal of the volatility, as a familiar square-root diffusion process and solved the problem in a closed form. By analysing the risk exposures and optimal portfolio allocations, we showed that the optimal allocation depends on the stochastic determinants of the jump process. As a consequence of the non-affine framework, we assumed the optimal allocation is strongly affected by the precision process, in contrast to the existing literature and consistent with what occurs empirically in the markets. Our findings suggest that investors must rebalance their portfolio after a market crash, reviewing their optimal strategy and decreasing the allocation of risky assets. Moreover, we experienced a significant portfolio improvement in wealth for the investor when she has access to traded derivatives. Finally, we demonstrated a substantial reduction of wealth when investors do not include jumps in their model while supporting the opportunity to include jumps when the wealth dynamics are based on an inverse-square root diffusion model. Moreover, for the first time in the literature, we extend the model by considering both market characteristics (presence of co-jumps in process dynamics, possibility of completing the market by trading an adequate number of derivatives) and investor characteristics, i.e., risk aversion and ambiguity regarding the selected probability measure, by considering a utility that disentangles investor risk aversion from the elasticity of intertemporal substitution of consumption. Investors wary about ambiguity incur substantial losses during the allocation phase. The choice of considering co-jumps in the dynamics reveals that, even in the presence of model uncertainty, exposures to model risk factors decrease in more turbulent markets, implying smaller investments in risky securities, as well as a portfolio rebalancing after a market shock. Then, the type of derivative used to complete the market has no significant impact on allocation and consumption.

Bibliography

C.M Ahn and H.E Thompson. Jump-diffusion processes and the term structure of interest rates. *The Journal of Finance*, 43(1):155–174, 1988.

E. Anderson, L.P. Hansen, and T. Sargent. A quartet of semigroups for model specification, robustness, prices of risk, and model detection. *Journal of the European Economic Association*, 1(1):68–123, 2003.

M. Ascheberg, N. Branger, H. Kraft, and F. T. Seifried. When do jumps matter for portfolio optimization? *Quantitative Finance*, 16(8):1297–1311, 2016.

Y. Aït-Sahalia and F. Matthys. Robust consumption and portfolio policies when asset prices can jump. *Journal of Economic Theory*, 179:1–56, 2019.

F. Bandi and R. Renò. Price and volatility co-jumps. *Journal of Financial Economics*, 119 (1):107–146, 2016.

R. Bansal. Long-run risks and financial markets. *Fed. Reserve Bank St. Louis Rev.*, 89: 1–17, 2007.

R. Bansal and A. Yaron. Risks for the long run: A potential resolution of asset pricing puzzles. *The Journal of Finance*, 59:1481–1509, 2004.

O. E. Barndorff-Nielsen and A. E. D. Veraart. Stochastic volatility of volatility and variance risk premia. *Journal of Financial Econometrics*, 11(1):1–46, 2012.

V. Bergen, M. Escobar, A. Rubtsov, and R. Zagst. Robust multivariate portfolio choice with stochastic covariance in the presence of ambiguity. *Quantitative Finance*, 18(8): 1265–1294, 2018.

N. Branger and L.S. Larsen. Robust portfolio choice with uncertainty about jump and diffusion risk. *Journal of Banking and Finance*, 37(12):5036–5047, 2013.

N. Branger, C. Schlag, and E. Schneider. Optimal portfolios when volatility can jump. *Journal of Banking and Finance*, 32(6):1087–1097, 2007.

N. Branger, C. Schlag, and E. Schrneider. Optimal portfolios when volatility can jump. *Journal of Banking and Finance*, 32(6):1087–1097, 2008.

M.F. Bru. Wishart processes. *Journal of Theoretical Probability*, 4:725–751, 1991.

A. Buraschi, P. Porchia, and F. Trojani. Correlation risk and optimal portfolio choice. *The Journal of Finance*, 65(1):393–420, 2003.

J.Y. Campbell and H.K. Koo. A comparison of numerical and analytic approximate solutions to an intertemporal consumption choice problem. *Journal of Economic Dynamics and Control*, 21:273–295, 1997.

J.Y. Campbell and L.M. Viceira. Consumption and portfolio decisions when expected returns are time varying. *Quarterly Journal of Economics*, 114:433–495, 1999.

J.Y. Campbell and L.M. Viceira. Who should buy long-term bonds? *American Economic Review*, 91:99–127, 2001.

J.Y. Campbell and L.M. Viceira. Strategic asset allocation: Portfolio choice for long-tem investors. *Oxford University Press, Oxford U.K*, 2002.

P. Carr and L. Wu. Variance risk premiums. *Review of Financial Studies*, 22(3):1311–1341, 2009.

G. Chacko and L.M. Viceira. Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets. *Review of Financial Studies*, 18(4):1369–1402, 2005.

Y. Cheng and M. Escobar-Anel. Robust portfolio choice under the 4/2 stochastic volatility model. *IMA Journal of Management Mathematics*, 10, 2021.

K. Christensen, R.C.A. Oomen, and M. Podolskij. Fact or friction: Jumps at ultra high frequency. *Journal of Financial Economics*, 114(3):576–599, 2014.

F. Corsi, D. Pirino, and R. Renò. Threshold bipower variation and the impact of jumps on volatility forecasting. *J. Econom.*, 159:276–288, 2010.

S.R. Das and R. Uppal. Systemic Risk and International Portfolio Choice. *The Journal of Finance*, 59(6):2809–2834, 2004.

D. Duffie and L.G. Epstein. Stochastic differential utility. *Econometrica*, pages 353–394, 1992.

D. Duffie, J. Pan, and K. Singleton. Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 16(68):1343–1376, 2000.

D. Ellsberg. Risk, Ambiguity, and the Savage Axioms*. *The Quarterly Journal of Economics*, 75(4):643–669, 1961.

L.G. Epstein and M. Schneider. Recursive multiple-priors. *Journal of Economic Theory*, 113(1):1–31, 2003.

L.G. Epstein and S.E. Zin. Substitution, risk aversion, and temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica*, 3(3):973–969, 1989.

B. Eraker. Do stock prices and volatility jump? reconciling evidence from spot and option prices. *The Journal of Finance*, 59:1367–1404, 2004.

M. Escobar, S. Ferrando, and A. Rubtsov. Robust portfolio choice with derivative trading under stochastic volatility. *Journal of Banking and Finance*, 61:142–157, 2015.

G. Faria and J. Correia-da Silva. Is stochastic volatility relevant for dynamic portfolio choice under ambiguity? *The European Journal of Finance*, 22(7):601–626, 2016.

M. Fisher and C. Gilles. Consumption and asset prices with homothetic recursive preferences. *Federal Reserve Bank of Atlanta*, 22(7):99–17, 1999.

W.H. Fleming and D. Hernandez-Hernandez. An optimal consumption model with stochastic volatility. *Finance and Stochastics*, 7:245–262, 2003.

I. Gilboa and D. Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18(2):141–153, 1989.

R.E Hall. Intertemporal substitution in consumption. *Journal of Political Economy*, 96: 339–357, 1988.

L. P. Hansen and K. J. Singleton. Generalized instrumental variables estimation of nonlinear rational expectations models. *Econometrica*, 50(5):1269–1286, 1982.

L.P. Hansen and T.J. Sargent. Acknowledging misspecification in macroeconomic theory. *Review of Economic Dynamics*, 4(3):519–535, 2001.

L.P. Hansen, T. J. Sargent, G. Turmuhambetova, and N. Williams. Robust control and model misspecification. *Journal of Economic Theory*, 128(1):45–90, 2006.

Y.H. Hsuku. Dynamic consumption and asset allocation with derivative securities. *Quantitative Finance*, 7(2):137–149, 2007.

X. Jin and A.X. Zhang. Decomposition of optimal portfolio weight in a jump-diffusion model and its applications. *The Review of Financial Studies*, 25(9):2877–2919, 2012.

X. Jin, D. Luo, and X. Zeng. Tail risk and robust portfolio decisions. *Management Science*, 67(5):3254–3275, 2021.

K. L. Judd. *Numerical Methods in Economics.* MIT Press, Cambridge, MA, 1998.

B. Kelly and H. Jiang. Tail risk and hedge fund returns. *Working Paper, University of Chicago*, 2012.

H. Kraft. Optimal portfolios and heston's stochastic volatility model: an explicit solution for power utility. *Quantitative Finance*, 5(3):301–313, 2005.

H. Kraft, F.T. Seifried, and M. Steffensen. Consumption-portfolio optimization with recursive utility in incomplete markets. *Quantitative Finance*, 17:161–196, 2013.

H. Kraft, F.T. Seifried, and M. Steffensen. Optimal consumption and investment with epstein-zin recursive utility. *Finance and Stochastics*, 21:187–226, 2016.

H. Liu. Robust consumption and portfolio choice for time varying investment opportunities. *Annals of Finance*, 6(4):435–454, 2010.

J. Liu. Portfolio selection in stochastic environments. *The Review of Financial Studies*, 20 (1):1–39, 2007.

J. Liu and J. Pan. Dynamic derivative strategies. *Journal of Financial Economics*, 69: 401–430, 2003. doi: 10.1016/S0304-405X(03)00118-1.

J. Liu, F.A. Longstaff, and J. Pan. Dynamic asset allocation with event risk. *The Journal of Finance*, 58(1):231–259, Feb 2003.

P.J. Maenhout. Robust portfolio rules and asset pricing. *Review of Financial Studies*, 17 (4):951–983, 2004.

H. Markowitz. Portfolio selection. *Journal of Finance*, 7:77–91, 1952.

R.C. Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3(4):373–413, 1971.

A. Moreira and T. Muir. Volatility-managed portfolios. *The Journal of Finance*, LXXII, 2017.

I. Oliva and R. Renò. Optimal portfolio allocation with volatility and co-jump risk that markowitz would like. *Journal of Econimc Dynamics and Control*, 94:242–256, Sept 2018.

J. Pan. The jump-risk premia implicit in options: evidence from an integrated time-series study. *Journal of Financial Economics*, 63(1):3–50, 2002.

E. Romo and L. Ortiz-Gracia. SWIFT calibration of the Heston model. *Mathematics*, 9(5): 1–20, 2021.

M. Schroder and C. Skiadas. Optimal consumption and portfolio selection with stochastic differential utility. *Journal of Economic Theory*, 89(1):68–126, 1999.

S.E. Shreve. *Stochastic Calculus for Finance II, Continuous-Time Models*. Springer, 2004.

V. Todorov and G. Tauchen. Volatility jumps. *Journal of Business and Economic Statistics*, 3(29):356–371, 2011.

A. Vissin-Jorgensen. Limited asset market participation and the elasticity of intertemporal substitution. *Journal of Political Economy*, 110:825–853, 2002.

J. Von Neuman and O. Morgenstern. Theory of games and economic behavior. *Princeton University Press*, 1947.

P. Wei, C. Yang, and Y. Zhuang. Robust consumption and portfolio choice with derivatives trading. *European Journal of Operational Research*, 304(2):832–850, 2023.

F. Wu, X. Mao, and K. Chen. Strong convergence of monte carlo simulations of the mean-reverting square root process with jump. *Applied Mathematics and Computation*, 206(1): 494–505, 2008.

W. Xu, H. Li, and C. Wu. A robust general equilibrium stochastic volatility model with recursive preference investors. *Annals of Economics and Finance*, 12(2):217–231, 2011.

Y. Zeng, D. Li, Z. Chen, and Z. Yang. Ambiguity aversion and optimal derivative-based pension investment with stochastic income and volatility. *Journal of Economic Dynamics and Control*, 88:70–103, 2018.