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Evidence of fluctuation-induced first-order phase transition in active matter

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Abstract

We investigate the effects of density fluctuations on the near-ordering phase of a flock by studying the Malthusian Toner–Tu theory. Because of the birth/death process, characteristic of this Malthusian model, density fluctuations are partially suppressed. We show that unlike its incompressible counterpart, where the absence of the density fluctuations renders the ordering phase transition similar to a second-order phase transition, in the Malthusian theory density fluctuations may turn the phase from continuous to first-order. We study the model using a perturbative renormalization group approach. At one loop, we find that the renormalization group flow drives the system in an unstable region, suggesting a fluctuation-induced first-order phase transition.

Collective behaviours in active matter systems can be very different from their equilibrium siblings. A paradigmatic example of how being active can drastically change the emergent phenomenology of a system is represented by the Vicsek model (VM) [1]. Despite the similarities between the VM and its equilibrium counterpart, the Heisenberg model [2], their behaviour in both the low-temperature and the near-ordering phase are not very alike. In fact, the addition of particles' self-propulsion—that is what we call, in this context, activity—to the alignment interaction of the Heisenberg model gives rise to a richer and more complex phenomenology, such as the occurrence of collective order in two dimensions [3] prohibited in equilibrium systems by the Mermin–Wagner theorem.

The Heisenberg model is known to approach the ordered phase through a continuous, second-order, phase transition [4]. Similarly, it was initially believed that the Vicesk model phase transition, dividing the ordered phase from the disordered one, was second-order like [1]. Though, further numerical studies have shown that, for large enough systems, the polarization is discontinuous at the transition point. In this regime, the phase transition is more similar to a first-order phase transition rather than to a second-order one [5, 6]. A naive explanation of this fact is that the rewiring of the interaction network, caused by the particles' movement, implies positive feedback between density and velocity fluctuations that turns an equilibrium second-order phase transition into an off-equilibrium first-order phase transition [6–8].

From a theoretical perspective, the critical properties of the VM are captured by its corresponding hydrodynamic theory: the celebrated Toner–Tu (TT) theory [3]. Hence, to understand how the interplay between density and velocity fluctuations renders the phase transition first order, one would have to study the TT in its near ordering regime. However, precisely because of the velocity-density coupling, this is not a piece of cake; and, from a mere technical point of view, many things could go wrong with trying to study the TT in its critical phase.

An important stepping stone toward the solution of the problem is the seminal paper [9] by Chen, Toner and Lee. Here, the authors studied the critical dynamics of the TT theory in the incompressible limit (ITT). They found that the phase transition is second-order, provided that density fluctuations are *completly* suppressed; in this case, a novel active off-equilibrium fixed-point rules the critical dynamics [9, 10]. Although activity is relevant and changes the critical dynamics, it is not responsible *per se* for turning the phase transition from second to first order. A feedback mechanism between density and polarization fluctuations seems to be a *fundamental ingredient* to explain the first order phenomenology of the transition [11]. Hence, to understand why this ordering phase transition is first order, density fluctuations must also be taken into account.

In the present work, we study a modification of the TT theory, namely the Malthusian Toner–Tu (MTT) theory [12]. In the MTT theory, the presence of birth and death process *partially* suppresses density fluctuations. In this sense, MTT theory poses in between the ITT theory, where density fluctuations are absent, and the TT theory, where the conservation of the total density allows for hydrodynamic density fluctuations. The Malthusian Toner–Tu theory is easy enough to be tackled with perturbative renormalization group techniques; furthermore, recent numerical simulations [13] showed that a dynamical instability destroys the homogeneous flocking state, suggesting that the MTT is rich enough to give rise to a phenomenology qualitatively different from the one of the Incompressible Toner–Tu theory.

1. Malthusian Toner–Tu theory

The Malthusian Toner–Tu theory, introduced by Toner in [12], is a model for self-propelled agents that reproduce and die. In its hydrodynamic description the equations of motion (EOM) for the velocity field $\boldsymbol{v}$ and the density field ρ, as introduced in [12], are:

$$\partial_t \boldsymbol{v} + \gamma_1 \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\frac{dV[\boldsymbol{v}]}{d\boldsymbol{v}} + \Gamma_0\nabla^2\boldsymbol{v} - \nabla P(\rho) + \boldsymbol{f} + \dots \tag{1}$$

$$\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) = g(\rho) \qquad V[\boldsymbol{v}] = \int d^dx\,\frac{1}{2}m_0\boldsymbol{v}^2 + \frac{1}{4}J_0\boldsymbol{v}^4. \tag{2}$$

The first equation is the same as in the TT theory, where P is a pressure term, $\boldsymbol{f}$ is Gaussian white noise with variance $2D_0$, and $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ is the material derivative accounting for self-advection [14]. The only difference with TT theory lies in the term $g(\rho)$ in the equation for the density, which accounts for the birth/death process [12, 15]. Near to ρ_0 the function $g(\rho)$ acts as a harmonic restoring force, $\partial_t\rho = g'(\rho_0)\delta\rho$, where $\delta\rho = \rho - \rho_0$, and

$$g(\rho_0) = 0 \qquad\qquad g'(\rho_0) < 0\,. \tag{3}$$

At first glance the birth/death mechanism seems to further complicate the problem, but in fact it is a significant simplification. Because of the term $g(\rho)$, density is not a conserved field [12], $\partial_t\rho(\boldsymbol{k}=0,t)\neq 0$, and in this sense ρ is not an hydrodynamic variable anymore [16]. Following [12, 17, 18], from the continuity equation (2) it is possible to derive a *linear* relation between density and velocity hydrodynamic fluctuations:

$$\delta\rho(\boldsymbol{x},t) = -\frac{\rho_0}{|g'(\rho_0)|}\,\nabla\cdot\boldsymbol{v}(\boldsymbol{x},t). \tag{4}$$

The force term $\boldsymbol{F}(\boldsymbol{v}) = -dV/d\boldsymbol{v}$ is responsible for the order/disorder phase transition, and it arises from the imitation/ferromagnetic interaction. The state of the system depends on the values of m_0 and J_0 [19]. While the coupling J_0 must be positive—otherwise the potential V would be unbounded—the sign of m_0 determines, at the mean-field level, whether or not the system is in a polarized phase $\langle\boldsymbol{v}\rangle\neq 0$. Precisely, if $m_0<0$ the average polarization of the system is non-zero,

$$\begin{aligned} m_0 > 0 &\implies \langle\boldsymbol{v}\rangle = 0\\ m_0 < 0 &\implies \langle\boldsymbol{v}\rangle = \sqrt{\frac{-m_0}{J_0}}. \end{aligned} \tag{5}$$

1.1. Equation of motion

Before proceeding with the explicit derivation of the EOM, a remark is in order. In general, the TT theory breaks Galilean invariance [14, 20], and the Malthusian TT theory is no exception. Galilean invariance is the symmetry under change of reference frame, and it is defined by the following transformation [14]:

$$\boldsymbol{x}' = \boldsymbol{x} + \bar{\boldsymbol{v}}t \qquad t' = t \qquad \boldsymbol{v}' = \boldsymbol{v} + \bar{\boldsymbol{v}} \tag{6}$$

implying that the derivatives in the new reference frame $(\nabla', \partial_{t'})$ are different from the derivates in the old reference frame (∇, ∂_t):

$$\nabla' = \nabla \qquad\qquad \partial_{t'} = \partial_t - \bar{\boldsymbol{v}}\cdot\nabla. \tag{7}$$

The force term $\boldsymbol{F}(\boldsymbol{v})$ breaks explicitly Galilean invariance,and the TT theory is not invariant under the transformation (6). In the hydrodynamic approach, symmetries determine which terms are allowed in the equation [2, 21], and particularly in Galilean-invariant theories the only allowed term, with one derivative and two fields, is $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$ [14]. Conversely, in a theory with broken Galilean invariance, as the TT theory, we must include all the possible terms with one derivative and two fields, namely $\boldsymbol{v}\nabla\cdot\boldsymbol{v}$ and $\nabla\boldsymbol{v}^2$ [20, 22].

The first of these two terms can be interpreted as a density-dependent alignment force [23]; assuming that the alignment force $m_0\boldsymbol{v}$ depends on density, and using the relation (4), leads to:

$$m(\rho) = m_0 + m'(\rho_0)\delta\rho \ , \tag{8}$$

$$m(\rho)\boldsymbol{v} = m(\rho_0)\boldsymbol{v} - m'(\rho_0)\frac{\rho_0}{|g'(\rho_0)|}\boldsymbol{v}\nabla\cdot\boldsymbol{v} \tag{9}$$

where the second term reproduces the expected structure $\boldsymbol{v}\nabla\cdot\boldsymbol{v}$.

The second non-Galilean-invariant term can be interpreted as a non-linear pressure term, of the kind $P\sim-\gamma_3\boldsymbol{v}^2$, which would lead to the following pressure force:

$$-\nabla P(\rho,\boldsymbol{v}) = -\nabla P(\rho) + \gamma_3\nabla\boldsymbol{v}^2 \tag{10}$$

which reproduces the structure of the second non-Galilean-invariant term, $\nabla\boldsymbol{v}^2$.

To derive the EOM of the model we need an explicit expression for the pressure force ∇P. Following the literature on active matter [19], the pressure term P can be expressed as a series in powers of density fluctuations:

$$P(\rho) = \sum_{n=1}\sigma_n\delta\rho^n \ . \tag{11}$$

Using equation (4), at the leading order in the fluctuations the pressure force reduces to

$$P\simeq\sigma_1\delta\rho = -\sigma_1\frac{\rho_0}{|g'(\rho_0)|}\nabla\cdot\boldsymbol{v}(\boldsymbol{x},t) \ . \tag{12}$$

This leads to the following EOM for the velocity field,

$$\partial_t\boldsymbol{v} + \gamma_1(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -(m_0 + J_0\boldsymbol{v}^2)\boldsymbol{v} + \Gamma_0\nabla^2\boldsymbol{v} + \gamma_2(\nabla\cdot\boldsymbol{v})\boldsymbol{v} + \gamma_3\nabla v^2 + \sigma_1\frac{\rho_0}{|g'(\rho_0)|}\nabla(\nabla\cdot\boldsymbol{v}) + \boldsymbol{F} \ , \tag{13}$$

$$\langle f_\alpha(\boldsymbol{x},t)f_\beta(\boldsymbol{x}',t')\rangle = 2D_0\delta^d(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t') \ . \tag{14}$$

We obtained an EOM for the sole velocity field; however, a relic of the coupling with density is hidden in the parameters γ_2 and σ_1. A remark is in order: the real technical difference between equation (13) and the equation of motions of the incompressible theory [9] resides in the fact that here we do not constraint the velocity field to be divergence-free. Nevertheless, density fluctuations are still non zero [18]; they can always be computed through equation (4).

1.2. Linearized theory

The starting point to build the perturbative expansion is the free (linear) theory, which is obtained by setting to zero all the non-linear terms, $\gamma_i=0$, $J=0$, in the equation of motion (13). In Fourier space, hence using the variables $\boldsymbol{k}$ and t, the linearized equation of motion is:

$$\partial_t v_\alpha(\boldsymbol{k},t) = -m_0 v_\alpha(\boldsymbol{k},t) - \Gamma_0\boldsymbol{k}^2 v_\alpha(\boldsymbol{k},t) - \sigma_1\frac{\rho_0}{|g'(\rho_0)|}k_\alpha k_\beta v_\beta(\boldsymbol{k},t) + f_\alpha(\boldsymbol{k},t). \tag{15}$$

In this equation of motion we can distinguish two different types of terms: relaxational terms that are in the same direction of the field $v_\alpha(\boldsymbol{k})$, and pressure terms that are always in the direction of the wave vector k_α. Because of that the dynamics of the Fourier transformed fields $\boldsymbol{v}^{\|}(\boldsymbol{k},t)$ and $\boldsymbol{v}^{\perp}(\boldsymbol{k},t)$ are different, where $\boldsymbol{v}^{\|}$ and $\boldsymbol{v}^{\perp}$ are respectively the component of $\boldsymbol{v}(\boldsymbol{k},t)$ parallel and orthogonal to the wave vector $\boldsymbol{k}$. For this reason it is practical to introduce the transverse $P^{\perp}_{\alpha\beta}(k)$ and longitudinal $P^{\|}_{\alpha\beta}(k)$ projection operators,

$$P^{\perp}_{\alpha\beta}(\boldsymbol{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta/k^2, \qquad P^{\|}_{\alpha\beta}(\boldsymbol{k}) = k_\alpha k_\beta/k^2 \ , \tag{16}$$

and the relative longitudinal $\Gamma_0^{\parallel}$ and transverse $\Gamma_0^{\perp}$ kinetic coefficients,

$$\Gamma_0^{\perp} = \Gamma_0 \qquad \Gamma_0^{\parallel} = \Gamma_0 + \sigma_1 \frac{\rho_0}{|g'(\rho_0)|} . \tag{17}$$

Using (16) and (17) the equation of motion (15) takes the form,

$$\partial_t v_\alpha = -(m_0 + \Gamma_0^{\perp} k^2) P_{\alpha\beta}^{\perp} v_\beta - (m_0 + \Gamma_0^{\parallel} k^2) P_{\alpha\beta}^{\parallel} v_\beta + f_\alpha , \tag{18}$$

which makes more evident that longitudinal and transverse fluctuations behaves differently. Intuitively, since longitudinal fluctuations are coupled to density fluctuations, as shown in equation (4), they relax faster than transverse fluctuations and $\Gamma_0^{\parallel} > \Gamma_0^{\perp}$. From the equation of motion (15) it is possible to derive the linear response and correlation functions $G_{\alpha\beta}^0(\boldsymbol{k},\omega)$ and $C_{\alpha\beta}^0(\boldsymbol{k},\omega)$,

$$G_{\alpha\beta}^0(\boldsymbol{k},\omega) = G_0^{\perp}(\boldsymbol{k},\omega) P_{\alpha\beta}^{\perp}(\boldsymbol{k}) + G_0^{\parallel}(\boldsymbol{k},\omega) P_{\alpha\beta}^{\parallel}(\boldsymbol{k}) \tag{19}$$

$$C_{\alpha\beta}^0(\boldsymbol{k},\omega) = C_0^{\perp}(\boldsymbol{k},\omega) P_{\alpha\beta}^{\perp}(\boldsymbol{k}) + C_0^{\parallel}(\boldsymbol{k},\omega) P_{\alpha\beta}^{\parallel}(\boldsymbol{k}) \tag{20}$$

where the functions $G_0^{\parallel,\perp}(\boldsymbol{k},\omega)$ and $C_0^{\parallel,\perp}(\boldsymbol{k},\omega)$ are defined as follows

$$G_0^{\perp,\parallel}(\boldsymbol{k},\omega) = \frac{1}{-i\omega + \Gamma_0^{\perp,\parallel}\boldsymbol{k}^2 + m_0} \tag{21}$$

$$C_0^{\perp,\parallel}(\boldsymbol{k},\omega) = \frac{2D_0}{\omega^2 + (\Gamma_0^{\perp,\parallel}\boldsymbol{k}^2 + m_0)^2} . \tag{22}$$

These expressions, (21) and (22), can be obtained with standard procedures [21], and a detailed derivation can be found in the SM. Using equation (4) we can explicitly derive and expression for the density correlation function,

$$C_{\rho\rho,0}(\boldsymbol{k},\omega) = \left(\frac{\rho_0}{|g'(\rho_0)|}\right)^2 \frac{2D_0\boldsymbol{k}^2}{\omega^2 + (\Gamma_0^{\parallel}\boldsymbol{k}^2 + m_0)^2} . \tag{23}$$

As we pointed out, even if we removed density from the equation of motion of velocity, density is still a fluctuating field. An important parameter, ruling the relevance of density fluctuations, is the ratio between the transverse and longitudinal kinetic coefficients,

$$\mu = \frac{\Gamma_0^{\parallel}}{\Gamma_0^{\perp}}. \tag{24}$$

At the linear level—and also at the non-linear level as we shall see later—the MTT theory becomes equivalent to the ITT when $\mu \to \infty$, namely when $\Gamma_0^{\parallel} \gg \Gamma_0^{\perp}$. In this regime, both $G_0^{\parallel}$ and $C_0^{\parallel}$ become negligible, and the bare correlation and response functions are proportional to $P_{\alpha\beta}^{\perp}(\boldsymbol{k})$, which means that as in incompressible theories the field fluctuates only in the transverse direction [9, 24]. We anticipate that from the renormalization group perspective there are two possible scenarios. The first possibility is that the parameter μ is a relevant parameter in the renormalization group sense, namely that its scaling dimension is positive [2, 21]. This would mean that when looking at larger and larger length scales the effective value of μ grows and eventually the large scale behaviour of the system is described by a theory with $\mu \to \infty$; in this case, the behaviour of the system in the vicinity of the transition point would be well captured by the incompressible theory [9]. The second possibility, as opposed to the latter, is that density fluctuations are important in determining the large scale behaviour of the system.

1.3. Gaussian scaling equations

A first possible attempt to determine the relevance of μ is given by Gaussian, or naive, scaling analysis. The Gaussian scaling dimensions can be obtained by applying a rescaling transformation of the field and of the spatio-temporal scales,

$$x \to xb \qquad t \to tb^z \qquad \boldsymbol{v}(\boldsymbol{x},t) \to \boldsymbol{v}(\boldsymbol{x}b, tb^z) b^{d_v - d - z} , \tag{25}$$

or equivalently in Fourier space,

$$k \to k/b \qquad \omega \to \omega/b^z \qquad \boldsymbol{v}(\boldsymbol{k},\omega) \to \boldsymbol{v}(\boldsymbol{k}/b, \omega/b^z) b^{d_v} . \tag{26}$$

Here, z is the *dynamical critical exponent* and its Gaussian value is $z = 2$, and $d_v = \frac{d+3z}{2}$ is the naive field scaling dimension of the field [2, 25]. Plugging equation (25) in the equation of motion gives the following Gaussian scaling relations for the parameters,

$$\begin{aligned} &m_0 \to m_0 b^z \qquad J_0 \to J_0 b^{4-d} \qquad \gamma_{i,0} \to \gamma_{i,0} b^{\frac{4-d}{2}} \\ &D_0 \to D_0 b^{z-2} \qquad \Gamma_0^{\perp} \to \Gamma_0^{\perp} b^{z-2} \qquad \Gamma_0^{\perp} \to \Gamma_0^{\perp} b^{z-2}. \end{aligned} \tag{27}$$

The scaling dimensions of the non-linear terms are proportional to $\epsilon = d - 4$, revealing that the non-linearities are relevant in dimension $d < 4$.

While it is clear, even at the Gaussian level, that J and γ_i are relevant parameters, at this stage, it is not possible to address the relevance of density fluctuations. The naive scaling dimension of μ is zero, $\mu_0 \to \mu_0 b^0$; the leading term of its scaling dimension is given by non-linear corrections and we cannot say much about it at this level. To understand whether density fluctuations are relevant in the vicinity of the transition point or not, we must study the full non-linear equation of motion.

2. Non-linear effects and renormalization group equations

We study the effects of the non-linear terms using the momentum shell renormalization group technique. The momentum-shell RG approach [21] unfolds through two stages: (a) integrating out small length-scale details, on the so-called momentum shell $\Lambda/b < k < \Lambda$, where $b \simeq 1$; (b) rescaling of momenta, $\boldsymbol{k}' = \boldsymbol{k}/b$, frequencies, $\omega' = \omega/b^z$, and fields, $\boldsymbol{v}'(\boldsymbol{k}, \omega) = b^{d_v}\boldsymbol{v}(b^{-1}\boldsymbol{k}, b^{-z}\omega)$. The effect of step (b) is to rescale each parameter by its corresponding naive dimension. The effect of step (a) is to provide each parameter with a perturbative scaling dimension, which is computed using the Feynman diagram technique (the technical details can be found in the SM). Combining the perturbative and the naive scaling dimensions leads to the following set of recursion relations, which determine how the parameters of the model change when we observe the system at larger and larger length scales [21]:

$$m_b = m_0\, b^z(1 + \delta m \ln b) \tag{28}$$

$$\Gamma_b^{\parallel,\perp} = \Gamma_0^{\parallel,\perp}\, b^{z-2-\eta}(1 + \delta\Gamma^{\parallel,\perp} \ln b) \tag{29}$$

$$\gamma_{i,b} = \gamma_{i,0}\, b^{d_v-d-1-\eta}(1 + \delta\gamma_i \ln b) \tag{30}$$

$$J_b = J_0\, b^{2d_v-z-2d-\eta}(1 + \delta J \ln b) \tag{31}$$

$$D_b = D_0\, b^{-2d_v+d+z-2\eta}(1 + \delta D \ln b). \tag{32}$$

Here, δX represents the perturbative correction to the parameter X, and η is the perturbative anomalous dimension of the field. A set of 15 Feynman diagrams determines the perturbative contributions—their explicit expressions can be found in the SM. Iterating the RG transformation results in a flow in the space of theories: the *renormalization group flow*.

The large scale behavior of the system is captured by a set of effective parameters,

$$\begin{aligned} &g_{1,0} = \frac{\gamma_{1,0}}{\Gamma_0^{\perp}}\sqrt{\frac{D_0}{\Gamma_0^{\perp}}} \qquad g_{2,0} = \frac{\gamma_{2,0}}{\Gamma_0^{\perp}}\sqrt{\frac{D_0}{\Gamma_0^{\perp}}} \qquad g_{3,0} = \frac{\gamma_{3,0}}{\Gamma_0^{\perp}}\sqrt{\frac{D_0}{\Gamma_0^{\perp}}} \\ &u_0 = \frac{J_0}{\Gamma_0^{\perp}}\frac{D_0}{\Gamma_0^{\perp}} \qquad \mu_0 = \frac{\Gamma_0^{\parallel}}{\Gamma_0^{\perp}} \qquad r_0 = \frac{m_0}{\Gamma_0^{\perp}} \end{aligned} \tag{33}$$

where r_0 and u_0 are the effective couplings relative to the force term $m_0\boldsymbol{v} + J_0\boldsymbol{v}^2\boldsymbol{v}$, and $g_{i,0}$ are the effective couplings relative to the three non-linearities, $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$, $\boldsymbol{v}\nabla\cdot\boldsymbol{v}$ and $\nabla\boldsymbol{v}^2$. It is not surprising that also the parameter μ plays an important role in the renormalization group equations, as anticipated it rules the relevance of density fluctuations.

The recursion relations for the effective parameters can be derived by the relations (28)–(32):

$$r_b = r_0 b^2\left[1 + (\delta m_0 - \delta\Gamma^{\perp})\ln b\right] \tag{34}$$

$$\mu_b = \mu_0 \left[1 + (\delta\Gamma^{\parallel} - \delta\Gamma^{\perp}) \ln b\right] \tag{35}$$

$$u_b = u_0 b^{\epsilon} \left[1 + (\delta J + \delta D - 2\delta\Gamma^{\perp} - \eta) \ln b\right] \tag{36}$$

$$g_{i,b} = g_{i,0} b^{\frac{\epsilon}{2}} \left[1 + \left(\delta\gamma_i + \frac{1}{2}\delta D - \frac{3}{2}\delta\Gamma^{\perp} - \eta\right) \ln b\right]. \tag{37}$$

The naive scaling dimensions of the effective parameters are:

$$d_u^{(n)} = \epsilon \qquad d_g^{(n)} = \epsilon/2 \qquad d_\mu^{(n)} = 0\,, \tag{38}$$

where $\epsilon = 4 - d$. This means that in dimension $d < 4$ both u and the constants g_i are relevant in the renormalization group sense.

2.1. Equilibrium case

There are two natural physical limits of the MTT model: the equilibrium limit, in this regime the model should reduce to a simple Heisemberg ferromagnet; and the incompressible limit, in this regime the model should reduce to an incompressible theory, and we expect to find the same results of the ITT theory [9].

First, we discuss the equilibrium limit, which is obtained by setting all the parameters related to activity to zero, namely $\gamma_i = 0$, $\sigma_1 = 0$ and $\mu = 1$.

In this limit the recursion relations of (35)–(37) reduce to

$$r_b = b^2 r_0 \left[1 + \frac{6u_0}{r_0(r_0 + \Lambda^2)} \ln b\right] \qquad \mu_b = 1 \tag{39}$$

$$u_b = b^{\epsilon} u_0 \left(1 - 12\, u_0 \ln b\right) \qquad g_{i,b} = 0. \tag{40}$$

This recursion relations reproduce correctly the one-loop fixed-point and critical exponents of an Heisenberg model with a four-dimensional order parameter [2],

$$r^* = -\frac{1}{4}\epsilon\Lambda^2 \qquad u^* = \frac{1}{12}\epsilon \tag{41}$$

$$\nu = \frac{1}{2} + \frac{9}{68}\epsilon \qquad \eta = 0 \qquad z = 2. \tag{42}$$

As it is reasonable to expect , a stability analysis shows that this fixed point is infrared-unstable. In particular, it is infrared-unstable on the off-equilibrium direction, corresponding to the introduction of the γ_i terms; the eigenvalue corresponding to this instability is $\lambda_e = \frac{\sqrt{151}+5}{48}$.

2.2. Incompressible case

The incompressible case corresponds to $\mu \to \infty$, namely to $\Gamma^{\parallel} \gg \Gamma^{\perp}$. In this limit, the bare correlation functions of the MTT theory coincide with the ones of an incompressible theory [9, 24]. Therefore, in this limit it is reasonable to expect that the transition is ruled by the exponents of incompressible active matter found in [9]. For $\Gamma^{\parallel} \gg \Gamma^{\perp}$, which implies $\mu \to \infty$, the recursion relations (21) and (22) reduce to the ones of [9] (see SM). The large scale behavior for $\mu \to \infty$ is thus ruled by the critical exponents,

$$z = 2 - \frac{31}{113}\epsilon \qquad \beta = \frac{1}{2} - \frac{6}{113}\epsilon \qquad \nu = \frac{1}{2} + \frac{29}{226}\epsilon\,, \tag{43}$$

which coincide with the ones of incompressible active matter [9]. However, for large but finite values of μ, namely in the neighborhood of the incompressible fixed point, the recursion relation of μ takes the form,

$$\mu_b = \mu_0 \left[1 - \frac{g_1^2}{4} \ln b\right] = \mu_0 b^{-g_1^2/4} \qquad \text{as} \quad \mu \gg 1\,, \tag{44}$$

revealing that the incompressible fixed point is unstable under a small deviation from incompressibility, its scaling dimension $-g_1^2/4$ is negative. This means that, for large but finite μ, the RG flow escapes this fixed point. Moreover, the larger μ, the longer the RG flow lingers near the incompressible active matter fixed point; this structure gives rise to a crossover [21]. To determine whether or not the critical dynamics is ruled by the incompressible fixed point we must consider the stopping condition of the RG flow. The RG flow stops when the correlation length, which trivially scales as $\xi_b = \xi/b$, is of the order of the inverse cutoff Λ^{-1}, namely when $\xi_{\rm stop} = \Lambda^{-1}$, where $l_{\rm stop} = \log_b(\Lambda\xi)$ is the number of RG iterations before stopping. Briefly, if at the end of the RG flow $\mu_{\rm stop}$ is still large, the incompressible active matter fixed point rules the critical dynamics. Therefore, for small enough correlation length,

$$\xi \ll \mu_0^{\phi}, \qquad \phi = \frac{31}{113}\epsilon \quad , \tag{45}$$

the critical dynamics is ruled by the incompressible active matter fixed point (43) - where ϕ is the so called crossover exponent [21].

In finite systems, the correlation length is always bounded by the system size L, and a relation akin to (45) can be derived. If $L \ll \mu^{\phi}$ the critical dynamics is ruled by the incompressible fixed point, meaning that the results of Chen *et al* [9] also hold in sufficiently small systems with mild density fluctuations $\mu \gg 1$, as it has been observed numerically in [10]. Conversely, if the system size is large enough, $L \gg \mu^{\phi}$, its properties at the transition point are not described by the incompressible active matter fixed point. In that case, we must study the recursion relations for finite μ. We remark that μ can be measured by looking at the system's transverse and longitudinal correlation functions, and, at least in principle, it could be measured both in numerical simulations and real experiments.

3. Fluctuation-induced first-order phase transition

The properties of the Malthusian TT theory in the vicinity of the transition are determined by the recursion relation (35)–(37), see SM. These recursion relations are complicated, and it is not possible to study them analytically. For this reason, we choose a set of initial values of the parameters and simulate the recursion relations (35)–(37) numerically. Usually, this procedure leads the system's parameters to an infrared-stable fixed point; however, things are more complicated in this model. In figure 1 we show the renormalization group flow obtained by numerical integration; we chose a large initial value of μ, meaning that the RG flow starts in a neighbourhood of the incompressible active matter fixed point. As soon as the flow escapes this fixed point—which means that the longitudinal fluctuations, along with density fluctuations, become relevant—the RG flow enters an unstable region, where the ferromagnetic coupling u_b becomes negative. Moreover, after that, the RG flow is characterized by run-away trajectories, and the couplings blow up to larger and larger values. The condition $u > 0$ ensures that the pseudo-potential $V(\boldsymbol{v})$ is bounded, and as shown in figure 1 after the potential enters the $u_b < 0$ region, the potential becomes unbounded.

To have an idea of what is happening, we consider how the pseudo-potential $V(\boldsymbol{v})$ changes along the RG flow. At the beginning of the RG flow, the pseudo-potential $V(\boldsymbol{v})$ takes a double-well shape, characteristic of $\lambda\varphi^4$ theories [21], guaranteeing that the order parameter is small. However, if the flow enters the $u_b < 0$ region, the pseudo-potential becomes unbounded. In this regime, the order parameter is clearly not fluctuating around zero anymore, suggesting a breakdown of the perturbative expansion. The situation may seem puzzling: all the theories along a RG trajectory describe the same system, which means that if the system is stable at the beginning of the flow, it must remain stable along the whole RG trajectory, but figure 1 clearly shows the contrary.

We can solve this apparent contradiction considering that the RG transformation may generate additional couplings that grant the stability of the theory also in the $u_b < 0$ region. In figure 2 (right) we show two Feynman diagrams that generate a $\boldsymbol{v}^4\boldsymbol{v}$ term in the equation of motion, which may stabilize the system in the $u_b < 0$ region. Even if the coefficient J_b becomes negative [26], a force term $-\mathcal{K}\boldsymbol{v}^4\boldsymbol{v}$ could still guarantee the stability of the equation of motion, provided that $\mathcal{K} > 0$. Adding this novel force term in the equation of motion is equivalent to modifying the pseudo-potential $V_{\rm new}(\boldsymbol{v})$ as follows:

$$V_{\rm new}(\boldsymbol{v}) = \frac{1}{2}m_0\boldsymbol{v}^2 + \frac{1}{4}J_0\boldsymbol{v}^4 + \frac{1}{6}\mathcal{K}_0\boldsymbol{v}^6. \tag{46}$$

If $\mathcal{K}_0$ is zero, the potential $V(\boldsymbol{v})$ develops a non-zero minimum only for $m_0 < 0$; moreover, these minima are arbitrarily close to zero provided that m_0 is small enough. This scenario gives rise to the phenomenology of a second-order phase transition. Conversely, if $\mathcal{K}_0 > 0$, the potential V may develop a minimum which is not close to zero when J_0 becomes negative. As shown in figure 2, when the constant J_0 becomes negative the potential may develop a global minimum far from $\boldsymbol{v} = 0$, provided that both m_0 and $\mathcal{K}_0$ remain positive; this would result in a discontinuous, first-order, phase transition.

In the literature, this scenario, where one or more couplings become negative and then run away, is often, but not always [27, 28], related to a fluctuation-induced first-order transition. The Heisenberg model with cubic anisotropy and the scalar electrodynamics [29] are two paradigmatic examples in which run-away trajectories result in a first-order phase transition. Non-perturbative techniques are often needed to prove

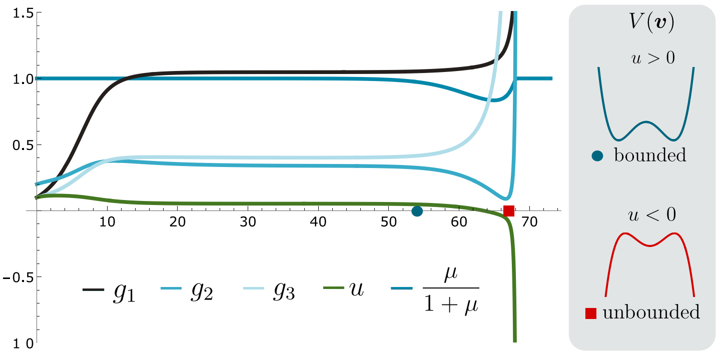

Figure 1. Renormalization group flow of the Malthusian Toner–Tu theory. In this example, the flow starts in the nearly incompressible regime, $\mu \gg 1$ and $\mu/(1+\mu) \simeq 1$; as soon as the flow escapes from the incompressible regime, the ferromagnetic coupling u_b (green curve) becomes negative, and then the flow is characterized by run-away trajectories. In the $u_b < 0$ regime the system is unstable. In the inserts, we show qualitatively how the pseudo-potential $V(\boldsymbol{v})$ changes before (blue dot) and after (red square) the flow enters the unstable region; when u_b becomes negative the pseudo-potential is unbounded.

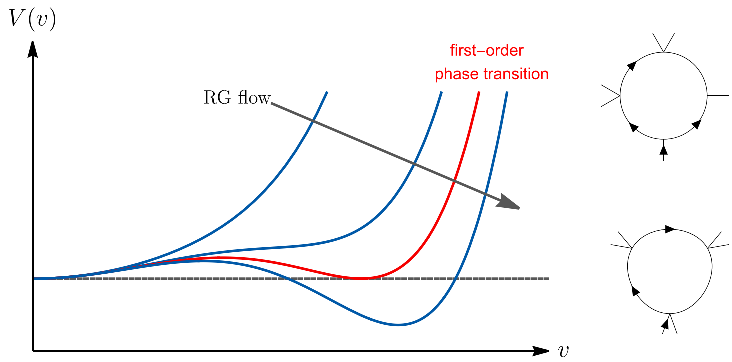

Figure 2. (Left) An idea of how the inclusion of the $\boldsymbol{v}^6$ term in the potential, $V_b(\boldsymbol{v}) = \frac{1}{2}m_b\boldsymbol{v}^2 + \frac{1}{4}J_b\boldsymbol{v}^4 + \frac{1}{6}\mathcal{K}_b\boldsymbol{v}^6$, may affect the renormalization group flow of the Malthusian Toner–Tu theory. J_b becoming negative, while both m_b and $\mathcal{K}_b$ staying positive, could lead to a first-order phase transition phenomenology. When the RG flow enters the $u_b < 0$ region (red curve), the perturbative expansion breaks down. (Right) Two of the 39 possible Feynman diagrams contributing to the generation of the coupling $\mathcal{K}_0\boldsymbol{v}^4\boldsymbol{v}$.

that a run-away RG trajectory results in a first-order phase transition [29, 30], but this is beyond the purposes of this perturbative RG calculation.

The stopping condition of the RG equations is determined by the correlation length ξ, or by the system size L; therefore, ξ determines if RG flow enters or not the unstable region. If the correlation length, or the system size, is small the RG flow does not enter the unstable region, and the system displays the phenomenology of a second-order phase transition, ruled by the incompressible active matter fixed point [9]. Conversely, if the correlation length is large enough, the RG flow enters the $u_b < 0$ region; when it happens because of the strong fluctuations, a phase transition, that should be second-order, becomes first-order [29]. In this sense, we call this phase transition a fluctuation-induced first-order phase transition [29].

4. Conclusions

We studied the properties of the Malthusian Toner–Tu theory in the vicinity of the transition point. The birth-death process characteristic of Malthusian systems suppresses density fluctuations; however, as opposed to incompressible systems, where density fluctuations are completely suppressed, in the MTT theory density fluctuations are only partially suppressed.

In the nearly incompressible regime, the MTT theory reproduces the results found in strictly incompressible active matter [9]. Thus, in agreement with the numerical evidence in [10], the critical exponents found in [9] hold also for systems with mild density fluctuations, provided that the correlation length (or the system size) is not too large. Furthermore, we found a relation between two *measurable* parameters, μ and the correlation length ξ (or the size of the system L) that determines the regime of validity of the incompressible approximation. While if the system size L is small enough the properties of the system are similar to the ones of an incompressible system, if L is large enough, an instability in the renormalization group equations suggests that a fluctuation-induced first-order phase transition occurs; corroborating the idea that fluctuations and renormalization of parameters play a crucial role in determining the order of the phase transition [23]. We believe that even though, in MTT theory, density fluctuations are partially suppressed, they are still strong enough to turn the second-order phase transition into a first-order phase transition. Further studies are needed to clarify the situation; two strategies are including other couplings in the equation of motion or using non-perturbative techniques. This calculation provides one of the first RG evidence that the phase transition in active matter is first-order for large enough systems.

Data availability statement

No new data were created or analysed in this study.

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