A new connection protocol for multi-consensus of discrete-time systems

M. Mattioni[1], S. Monaco[1] and D. Normand-Cyrot[2]

***Abstract*— In this paper, a new connection protocol for consensus of multi-agent discrete-time systems under a general communication graph is proposed. In particular, the coupling is realized based on the outputs making each agent passive in the u-average sense so guaranteeing convergence to the agreement steady-state, with no need of mitigating the coupling gain, as typically done in concerned literature. The proposed connection rule is shown to apply for network dynamics under aperiodic sampling when the sampling sequence is known to all agents.**

***Index Terms*— Linear systems; Network analysis and control; Sampled-data control.**

I. Introduction

Consensus of multi-agent systems is nowadays fundamental as embedding different control problems spanning from several disciplines (e.g., [1]–[3]). It is well known that, in general, when dynamical units exchange information through a communication graph, clusters arise and, under particular coupling functions, the corresponding trajectories might asymptotically converge to common behaviors (that is, multiconsensus [4], [5]); such clusters and the number of consensuses are uniquely determined by the topology of the graph through the notion of almost equitable partitions.

In this setting, when considering scalar continuous-time integrator as agents of the network, the common neighbor-based coupling protocol [6], [7] ensures all agents converge to multi-consensus despite the strength of the connection. However, this does not hold for discrete-time systems in general, even when considering integrator-like units. As a matter of fact in this case, under the standard protocol, the coupling strength (i.e., the gain weighting the influence of the network on each agent) needs to be suitably set to ensure convergence to the agreement steady-state even when the network topology is fixed over time (e.g., [8], [9]). This results in conservative values for the gain which are generally inversely proportional to either the largest eigenvalue of the Laplacian (which must be known to all agents) or, alternatively, the number of agents involved in the network. Accordingly, the larger the network and the corresponding connectivity, the smallest is the necessary coupling gain which must be fixed for ensuring consensus so implying that the influence of the network over each agent must be significantly mitigated. This is not desirable in practice as discrete-time networks embed several classes of multi-agent systems as, to cite a few, communication systems and opinion dynamics [10], [11] for which the coupling strength of the network cannot be fixed small a priori, for both modeling and control reasons.

Motivated by these arguments, the objective of this paper is to provide a new coupling protocol for discrete-time multi-agent systems forcing all agents to a common multi-consensus (uniquely fixed by the communication topology) despite the value of the interconnection strength which might be then tuned to control the convergence rate. The underlying idea is that, when considering integrator continuous-time dynamics, consensus is generally guaranteed since the neighbor-based interconnection is realized via a passive map for each agent. Unfortunately, this is not true in the discrete-time context when exploiting the same connection rule. The following question naturally arises: how to realize the neighbor-based connection via new suitably defined mappings guaranteeing consensus?

In detail, the contribution of the paper is threefold. (i) Based on the notion of average passivity introduced in [12] for discrete-time systems at large, we construct a new neighbor-based coupling protocol by replacing the states of each agent with a new average passivating function which is directly depending on the coupling terms as well. The corresponding discrete-time network is proved to evolve with a new Laplacian-like matrix L_d which is structurally equivalent to the standard Laplacian L associated to the communication graph. (ii) Under the proposed coupling, all agents asymptotically converge to the multi-consensus uniquely dictated by the communication graph (via the Laplacian) and the corresponding initial condition despite the connection strength. This extends the results in [7] and overcomes the need of small coupling gains [6], [8]. (iii) The results are generalized to network agents under aperiodic (but synchronous) sampling under the assumption of known sampling sequences; i.e., when agents exchange information only at discrete-time instants sporadically spread over time. The proposed protocol allows large coupling strength and sampling periods, contrarily to the usual one (e.g., [13], [14]).

The remainder of the paper is organized as follows. In Section II, generalities on graph theories are recalled whereas the problem is motivated and formulated in Section III. The main result is proved in Section IV and revisited for sampled-data dynamics in Section V. In Section VI a simulated example is reported whereas conclusions and perspectives are in Section VII.

Notations: The symbol $|\cdot| \in \mathbb{R}$ denotes, depending on the argument, either the cardinality of a set $\mathcal{S}$ or the absolute

Supported by *Sapienza Università di Roma* (*Progetti di Ateneo 2018*-Piccoli progetti *RP11816436325B63*).

[1]Dipartimento di Ingegneria Informatica, Automatica e Gestionale *A. Ruberti* (Sapienza University of Rome); Via Ariosto 25, 00185 Rome, Italy {mattia.mattioni, salvatore.monaco}@uniroma1.it.

[2]Laboratoire de Signaux et Systèmes (L2S, CNRS and *Univ. Paris-Saclay*); 3, Rue Joliot Curie, 91192, Gif-sur-Yvette, France dorothee.normand-cyrot@centralesupelec.fr

value of a complex number $\lambda \in \mathbb{C}$. 0 denotes either the zero scalar or the zero matrix of suitable dimensions. $\mathbb{1}_c$ denotes the c-dimensional column vector whose elements are all ones while I is the identity matrix of suitable dimensions. Given a matrix $A \in \mathbb{R}^{n\times n}$, $\sigma\{A\} \subset \mathbb{C}$ is its spectrum. A is weakly Schur if all $\lambda \in \sigma\{A\}$ verify that: $|\lambda| < 1$ if the geometric and algebraic multiplicities are not the same; $|\lambda| \le 1$ otherwise. We denote by $\text{diag}\{\lambda_1, \dots, \lambda_N\} \in \mathbb{R}^{N\times N}$ the diagonal matrix with entries provided by the $\lambda_i \in \mathbb{R}$ ($i = 1, \dots, N$). Given a scalar real-valued function $H : \mathbb{R}^n \to \mathbb{R}$ we denote by $\Delta_k H(x) = H(x(k+1)) - H(x(k))$ the corresponding one step increment.

II. Recalls on graph-theory

We consider an unweighted directed graph (or digraph for short) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = N$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The set of neighbors to a node $\nu \in \mathcal{V}$ is defined as $\mathcal{N}(\nu) = \{\mu \in \mathcal{V} \text{ s.t. } (\mu, \nu) \in \mathcal{E}\}$. For all pairs of distinct nodes $\nu, \mu \in \mathcal{V}$, a directed path from ν to μ is defined as $\nu \rightsquigarrow \mu := \{(\nu_r, \nu_{r+1}) \in \mathcal{E} \text{ s.t. } \cup_{r=0}^{\ell-1} (\nu_r, \nu_{r+1}) \subseteq \mathcal{E} \text{ with } \nu_0 = \nu, \nu_\ell = \mu \text{ and } \ell > 0\}$. The reachable set from a node $\nu \in \mathcal{V}$ is defined as $R(\nu) := \{\nu\} \cup \{\mu \in \mathcal{V} \text{ s.t. } \nu \rightsquigarrow \mu\}$. A set $\mathcal{R}$ is called a reach if it is a maximal reachable set, that is, $\mathcal{R} = R(\nu)$ for some $\nu \in \mathcal{V}$ and there is no $\mu \in \mathcal{V}$ such that $R(\nu) \subset R(\mu)$. Since $\mathcal{G}$ possesses a finite number of vertices, such maximal sets exist and are uniquely determined by the graph itself. Denoting by $\mathcal{R}_i$ for $i = 1, \dots, \mu$, the reaches of $\mathcal{G}$, the exclusive part of $\mathcal{R}_i$ is defined as $\mathcal{H}_i = R_i \setminus \cup_{\ell=1, \ell\neq i}^{\mu} R_j$ with cardinality $h_i = |\mathcal{H}_i|$. Finally, the common part of $\mathcal{G}$ is given by $\mathcal{C} = \mathcal{V} \setminus \cup_{i=1}^{\mu} \mathcal{H}_i$ with cardinality $c = |\mathcal{C}|$.

The Laplacian matrix associated to $\mathcal{G}$ is given by $L = D - A$ with $D \in \mathbb{R}^{N\times N}$ and $A \in \mathbb{R}^{N\times N}$ being respectively the in-degree and the adjacency matrices. As proved in [15], [16], L possesses one eigenvalue $\lambda = 0$ with both algebraic and geometric multiplicities coinciding with μ, the number of reaches of $\mathcal{G}$. Hence, after a suitable re-labeling of nodes, the Laplacian admits the lower triangular form

$$L = \begin{pmatrix} L_1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & L_\mu & 0 \\ M_1 & \dots & M_\mu & M \end{pmatrix} \tag{1}$$

where: $L_i \in \mathbb{R}^{h_i\times h_i}$ ($i = 1, \dots, \mu$) is the Laplacian associated to the subgraph $\mathcal{H}_i$ and possessing one eigenvalue in zero with unitary algebraic and geometric multiplicities; $M \in \mathbb{R}^{c\times c}$ verifying $\sigma(M) \subset \mathbb{C}^+$ corresponds to the common component $\mathcal{C}$. Thus, the eigenspace associated to $\lambda = 0$ for L is spanned by the right eigenvectors given by $Z_0 = (z_1 \dots z_\mu)$ with

$$z_1 = \begin{pmatrix} \mathbb{1}_{h_1} \\ \vdots \\ 0 \\ \gamma^1 \end{pmatrix} \quad \dots \quad z_\mu = \begin{pmatrix} 0 \\ \vdots \\ \mathbb{1}_{h_\mu} \\ \gamma^\mu \end{pmatrix} \tag{2}$$

with $\sum_{i=1}^{\mu} \gamma^i = \mathbb{1}_c$ and $M_i \mathbb{1}_{h_i} + M\gamma^i = 0$ for all $i = 1, \dots, \mu$. In addition, the left eigenvectors associated to the zero eigenvalues are given by $V_0^\top = (\tilde{v}_1 \dots \tilde{v}_\mu)^\top$

$$\tilde{v}_1^\top = \begin{pmatrix} v_1^\top & \dots & 0 & 0 \end{pmatrix} \dots \tilde{v}_\mu^\top = \begin{pmatrix} 0 & \dots & v_\mu^\top & 0 \end{pmatrix} \tag{3}$$

with $v_i^\top = \begin{pmatrix} v_i^1 & \dots & v_i^{h_i} \end{pmatrix} \in \mathbb{R}^{1\times h_i}$, $v_i^s > 0$ if the corresponding node is root and zero otherwise. $Z_r \in \mathbb{R}^{N\times(N-\mu)}$ and $V_r^\top \in \mathbb{R}^{(N-\mu)\times N}$ denote the matrices composed of all other eigenvectors of the Laplacian. A partition (or, as an alternative, a cluster) $\pi = \{\rho_1, \dots, \rho_r\}$ of $\mathcal{V}$ is a collection of cells $\rho_i \subseteq \mathcal{V}$ verifying $\rho_i \cap \rho_j = \emptyset$ for all $i \neq j$ and $\cup_{i=1}^{r} \rho_i = \mathcal{V}$. Given two partitions π_1 and π_2, π_1 is said to be finer than π_2 ($\pi_1 \preceq \pi_2$) if all cells of π_1 are a subset of some cell of π_2; equivalently, we say that π_2 is coarser than π_1 ($\pi_2 \succeq \pi_1$). We name $\pi = \mathcal{V}$ the trivial partition as composed of a unique cell with all nodes. A partition $\pi = \{\rho_1, \dots, \rho_r\}$ of $\mathcal{V}$ is said to be an *almost equitable partition* (AEP, in short) if each node of ρ_i has the same number of neighbors in ρ_ℓ, for all $i, \ell \in \{1, \dots, r\}$ with $i \neq \ell$. More precisely, denote by $\mathcal{N}(\nu_i, \rho) = \{\nu \in \rho \text{ s.t } (\nu, \nu_i) \in \mathcal{E}\}$ the set of neighbors of ν_i in the cell ρ; π is an AEP of $\mathcal{G}$ if, for each $i, j \in \{1, 2, \dots, r\}$, with $i \neq j$, there exists an integer d_{ij} such that $|\mathcal{N}(\nu, \rho_j)| = d_{ij}$ for all $\nu \in \rho_i$. We say that a non trivial partition $\pi^\star$ is the coarsest AEP of $\mathcal{G}$ if for all non trivial π AEP of $\mathcal{G}$ then $\pi^\star \succeq \pi$.

III. Motivations and problem statement

Consider a multi-agent system exchanging information via a communication unweighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where each vertex $\nu_i \in \mathcal{V}$ is a dynamical unit of the form

$$x_i(k+1) = x_i(k) + u_i(k) \tag{4}$$

with $x_i \in \mathbb{R}$ the state and $u_i \in \mathbb{R}$ the coupling terms. As well-known [6], [8], the usual coupling rule

$$u_i = -\kappa \sum_{j: \nu_j \in \mathcal{N}(\nu_i)} (x_i - x_j) \tag{5}$$

does not ensure all agents converge to a consensus (or multi-consensus) steady-state for all $\kappa > 0$, the so-called coupling strength. As a matter of fact, denoting $x = (x_1 \ \dots \ x_N)^\top$, the multi-agent system

$$x(k+1) = (I - \kappa L)x(k)$$

does not converge to the multi-consensus dictated by $\mathcal{G}$ for all $\kappa > 0$. Despite the matrix $I - \kappa L$ possesses one eigenvalue in $\lambda_d = 1$ (with algebraic and geometric multiplicities equal to μ), it might not be weakly Schur: the remaining eigenvalues lie within the unit circle if κ verifies

$$\kappa < \frac{1}{2\lambda^\star}, \quad \lambda^\star = \max_{\lambda > 0} \{\lambda \in \sigma\{L\}\}.$$

If κ is not small enough consensus might be lost and, even worse, the multi-agent dynamics might diverge. In addition, even when consensus is preserved (under small κ), the exchange of information is significantly filtered by all agents so affecting the convergence rate. Finally, the information on $\lambda^\star$ might not be known to all agents and, even if upper bounds can be computed (e.g., $\kappa < \frac{1}{N}$), the transient performances might not be acceptable. Both these situations are not likely to happen in practice.

The question we address is hence the following one. Given the agent dynamics (4), is it possible to define a new discrete-time coupling rule (or consensus protocol)

$$u_i = -\kappa \sum_{j:\nu_j \in \mathcal{N}(\nu_i)} (y_i - y_j) \tag{6}$$

and a suitably defined output $y_i \in \mathbb{R}$, so that the corresponding multi-agent system converges to the (multi-)consensus associated to $\mathcal{G}$ for all $\kappa > 0$?

The answer we provide relies on a deeper understanding of the properties of the continuous-time consensus problem [17], [18] as reported in the following section.

A. A remark on continuous-time consensus

Consider a network of continuous-time agents

$$\dot{x}_i = u_i$$

which are passive with respect to the outputs $y_i = x_i$ and storage $H(x_i) = \frac{1}{2}x_i^2$. Then, the network coupling $u_i = -\kappa \sum_{j:\nu_j \in \mathcal{N}(\nu_i)} (y_i - y_j)$ induces a feedback interconnection of the agents via the passive outputs. Accordingly, the closed-loop system, (i.e., the network)

$$\dot{x} = -\kappa L x, \quad \kappa > 0 \tag{7}$$

possesses a stable center set and all agents converge to a suitably defined multi-consensus associated to the coarsest AEP of $\mathcal{G}$ (see [7] for a precise characterization). More in detail, setting the storage function $\hat{H}(x) = \frac{1}{2}x^\top x = \sum_{i=1}^N H(x_i)$ for the network dynamics one gets

$$\dot{\hat{H}}(x) = x^\top u = -x^\top L x = -x^\top V_r \Lambda_r V_r^\top x \leq 0$$

with $u = (u_1 \ \dots \ u_N)^\top$, $(V^\top)^{-1} = Z = \begin{pmatrix} Z_0 & Z_r \end{pmatrix}$,

$$V^\top L Z = \Lambda = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_r \end{pmatrix}, \ \Lambda_r = \text{diag}\{\lambda^1, \dots, \lambda^{N-\mu}\} \tag{8}$$

and $Re[\lambda^i] > 0$ for $i = 1, \dots, N - \mu$. By the inequality above, the trajectories of the network converge to the multi-consensus subspace

$$\mathcal{V}_s = ker\{V_r^\top\} = Im\{Z_0\} \tag{9}$$

being the eigenspace associated to the zero eigenvalue of L; i.e., $Z_0 = \begin{pmatrix} z_1 & \dots & z_\mu \end{pmatrix}$ and $z_i \in \mathbb{R}^N$ as in (2).
To summarize, the continuous-time network is realized via the feedback interconnection of agents through damping of the corresponding passive outputs so that stability of the closed-loop consensus subspace is preserved independently on the gain magnitude [17].

IV. The average discrete-time consensus protocol

The considerations above suggest that a passive connection in discrete time may work as well. Indeed, the scalar agents (4) are not passive (at least in the usual sense) with respect to the output $y_i = x_i$ because of the so-called direct throughput obstruction [12]; namely, a necessary condition for discrete-time systems to be passive is that the output explicitly depends on the input. Then, the usual coupling (5) does not realize a passive connection.

With this in mind, we propose to fix the interconnecting output in (6) based on the notion of average passivity [12].

Lemma 4.1: The scalar agents (4) are passive with respect to the (average) output[1]

$$y_i = x_i + \frac{1}{2}u_i \tag{10}$$

and storage functions $H(x_i) = \frac{1}{2}x_i^2$ with dissipation equality

$$\Delta_k H(x_i) = u_i(k)(x_i(k) + \frac{1}{2}u_i(k)) = u_i(k)y_i(k).$$

The next result proves that consensus is guaranteed for all $\kappa > 0$ when realizing the connection (10) via the passive outputs (10).

Theorem 4.1: Consider the multi-agent system with each agent of the form (4). Then, for all $\kappa > 0$, the coupling rule (6) with the passive outputs (10) ensures all agents converge to the consensus steady-states associated to the coarsest AEP of $\mathcal{G}$; namely, the consensus subspace (9) is attractive for the multi-agent dynamics

$$x(k+1) = L_d x(k) \tag{11}$$

with $L_d = \big(I + \frac{\kappa}{2}L\big)^{-1}(I - \frac{\kappa}{2}L\big)$.

Proof: Considering the agglomerate coupling vector $u = (u_1 \dots u_N)^\top$ with (6) and (10), one gets

$$u = -\kappa L(x + \frac{1}{2}u) \implies u = -\kappa(I + \frac{\kappa}{2}L)^{-1}Lx$$

with $(I + \frac{\kappa}{2}L)$ invertible for all $\kappa > 0$. Thus, the network dynamics gets the form (11). At this point, one must prove that the matrix L_d possesses: (i) $\lambda_d = 1$ as eigenvalue with geometric and algebraic multiplicities $\mu \geq 1$ (i.e., the number of reaches of $\mathcal{G}$) with corresponding eigenspace (9); (ii) the remaining eigenvalues within the strict unit circle. To this end, we prove that L_d admits the spectral decomposition

$$L_d = Z \Lambda_d V^\top$$

with Z and $V^\top$ as in (8), $\Lambda_d = \text{diag}\{I_\mu, \Lambda_{d.r}\}$, I_μ the μ-dimensional identity matrix and $\Lambda_{d,r} \in \mathbb{R}^{(N-\mu)\times(N-\mu)}$ a diagonal matrix with all elements within the open unit circle. For, exploiting (8), one rewrites

$$(I + \frac{\kappa}{2}L)^{-1} = (I + \frac{\kappa}{2}Z\Lambda V^\top)^{-1} = Z(I + \frac{\kappa}{2}\Lambda)^{-1}V^\top$$
$$I - \frac{\kappa}{2}L = Z(I - \frac{\kappa}{2}\Lambda)V^\top$$

[1] We refer to (10) as *average* as it is the average of the mapping $h(x_i) = x_i$ along the dynamics (4) and with respect to the input u_i; namely, it is $y_i = \frac{1}{u_i}\int_0^{u_i}(x_i + w)\mathrm{d}w$. We refer to [12] for a more general definition.

because $V^\top Z = ZV^\top = I$, so getting

$$L_d = Z\Lambda_d V^\top,\ \Lambda_d = (I + \frac{\kappa}{2}\Lambda)^{-1}(I - \frac{\kappa}{2}\Lambda).$$

The so-computed Λ_d, by (8), is diagonal and given by

$$\Lambda_{d,r} = (I + \frac{\kappa}{2}\Lambda_r)^{-1}(I - \frac{\kappa}{2}\Lambda_r) = diag(\lambda_d^1, \dots, \lambda_d^{N-\mu})$$

with $\lambda_d^i = \frac{1-\frac{\kappa}{2}\lambda^i}{1+\frac{\kappa}{2}\lambda^i}$ for all $\lambda^i \in \sigma\{L\}\backslash\{0\}$ and $i = 1, \dots, N-\mu$. This concludes the proof. ■

From now on, we refer to (6) as the *average coupling* since it is deduced starting from the average output (10).

The proof of Theorem 4.1 highlights that there is a one-to-one correspondence among the eigenvalues of L and L_d: if $\lambda \in \sigma\{L\}$ then $\lambda_d = \frac{1-\frac{\kappa}{2}\lambda}{1+\frac{\kappa}{2}\lambda} \in \sigma\{L_d\}$. In addition, the dynamical matrix L_d governing the dynamics of the network (11) under the coupling (6) shares the same invariant structure decomposition of the Laplacian of $\mathcal{G}$: the same eigenspace is associated to $\lambda \in \sigma\{L\}$ and $\lambda_d \in \sigma\{L_d\}$. By these properties, L_d is defined as the *discrete-time Laplacian*.

Remark 4.1: The coupling rule (6) is implicitly defined and, in general, cannot be computed in a fully decentralized manner (i.e., locally to each agent and based on partial information from the neighbors).

A. A discrete-time passivity interpretation

The result in Theorem 4.1 comes with a passivity interpretation that is the counterpart of the continuous-time one in Section III-A: the network coupling (6), realized via the average passive output (10), induces a feedback interconnection that guarantees consensus for all $\kappa > 0$.

Starting from Lemma 4.1 and Theorem 4.1, we set the function $\hat{H}(x) = \frac{1}{2}x^\top x$ which verifies, along (11)

$$\begin{aligned}\Delta_k \hat{H}(x) &= u^\top(k)y(k)\\ &= -x^\top(k)(I + \frac{\kappa}{2}L)^{-\top}L(I + \frac{\kappa}{2}L)^{-1}x(k)\end{aligned}$$

with $u = (u_1\ \dots\ u_N)^\top$, $y = (y_1\ \dots\ y_N)^\top$, u_i and y_i as in (6)-(10) for $i = 1, \dots, N$. From (8), one gets

$$\begin{aligned}&\Delta_k \hat{H}(x) =\\ &- x^\top(k)V_r(I + \frac{\kappa}{2}\Lambda_r)^{-\top}\Lambda_r(I + \frac{\kappa}{2}\Lambda_r)^{-1}V_r^\top x(k) \leq 0\end{aligned}$$

with $\Lambda_r \succ 0$ by definition and $V_r^\top \in \mathbb{R}^{(N-\mu)\times N}$ the matrix of the left eigenvectors associated to the non-zero eigenvalues of L. Thus, the trajectories of the discrete-time network converge to the consensus-subspace (9).

Also, we highlight that the discrete network dynamics (11) gets the form of a discrete-time Hamiltonian dynamics [19]

$$x^+(u) = x + \bar{\nabla}H\big|_x^{x^+} + u = x + \frac{1}{2}(x^+ + x) + u$$

with $\bar{\nabla}H\big|_x^{x^+} = \frac{1}{2}(x^+ + x)$ the discrete-gradient associated to $H(x)$, $x = x(k)$, $u = u(k)$, $x^+(u) = x^+(u(k)) = x(k+1)$ and $x^+ = x^+(0)$. The coupling term gets the form of a damping injection

$$u = -\kappa\bar{\nabla}H\big|_{x^+}^{x^+(u)} = -\kappa\frac{\kappa}{2}L(x^+(u) + x^+)$$

ensuring

$$\Delta_k H(x) = -\kappa\bar{\nabla}^\top H\big|_{x^+}^{x^+(u)} L\bar{\nabla}H\big|_{x^+}^{x^+(u)} \leq 0.$$

Also, the discrete Laplacian L_d corresponds to the biliner transformation of the continuous-time $-L$, typically used in approximate discretization of Hamiltonian dynamics [20]. Those arguments provide a different proof of Theorem 4.1.

B. The discrete-time multi-consensus

As proved in [7, Theorem 1] for the continuous-time case, the multi-consensus is associated with the coarsest AEP of $\mathcal{G}$ which, assuming L of the form (1), is given by

$$\pi^\star = \{\mathcal{H}_1, \dots, \mathcal{H}_\mu, \mathcal{C}_{\mu+1}, \dots, \mathcal{C}_{\mu+p}\} \tag{12}$$

with $\mathcal{C} = \cup_{i=1}^p \mathcal{C}_{\mu+i}$ and nodes $\nu_{h_1+\dots+h_\mu+c_1+\dots+c_{\mu+i-1}+s}$ belonging to the same cell $\mathcal{C}_{\mu+j}$ if and only if they share the same component γ_s^i of γ^i in (2) for $c_{\mu+i} = |\mathcal{C}_{\mu+i}|$, $s = 1, \dots, c_\mu + i$ and $i = 1, \dots, p$.

Let us regroup nodes according to the cell of $\pi^\star$ they belong to; i.e., for $c_\ell = |\mathcal{C}_{\mu+\ell}|$ for $\ell = 1, \dots, p$, $i = 1, \dots, \mu$ and, for simplicity, $h_0 = 0f$ and $c_{p+1} = c_p$

$$\begin{aligned}\mathbf{x}_i &= \begin{pmatrix} x_{h_1+\dots+h_{i-1}+1} & \dots & x_{h_1+\dots+h_i}\end{pmatrix}^\top\\ \mathbf{x}_{\delta,\ell} &= \begin{pmatrix} x_{N-c_\ell-\dots-c_p+1} & \dots & x_{N-c_{\ell+1}-\dots-c_p}\end{pmatrix}^\top.\end{aligned} \tag{13}$$

Also, by such sorting, one can rewrite γ^i in (2) as

$$\gamma^i = \begin{pmatrix}\gamma_1^i \mathbb{1}_{c_1}^\top & \dots & \gamma_p^i \mathbb{1}_{c_p}^\top\end{pmatrix}^\top, \quad i = 1, \dots, \mu$$

with $\gamma_\ell^i \in \mathbb{R}$ and $\sum_{i=1}^\mu \gamma_\ell^i = 1$ for $\ell = 1, \dots, p$. The following result can be then given to explicitly characterize the multi-consensus of the discrete-time network (11).

Corollary 4.1: Consider a network of N discrete-time multi-agent systems of the form (4) evolving over a communication unweighted digraph $\mathcal{G}$ with Laplacian L in the form (1). Then, for all $\kappa > 0$, the coupling (6) makes the network (11) converge to the multi-consensus associated to $\mathcal{G}$. More in detail, as $\kappa \to \infty$, all nodes in the same cell of $\pi^\star$ in (12) converge to the same consensus given by

$$\mathbf{x}_i(k) \to x_s^i = v_i^\top \mathbf{x}_i(0), \quad i = 1, \dots, \mu \tag{14a}$$

$$\mathbf{x}_{\delta,\ell}(k) \to x_s^{\delta,\ell} = \sum_{i=1}^{\mu} \gamma_\ell^i x_s^i, \quad \ell = 1, \dots, p \tag{14b}$$

with $v_i^\top \in \mathbb{R}^{1\times h_i}$ as in (3).

Proof: Assume, with no loss of generality, the Laplacian of the form (1) with, by Theorem 4.1, left and right eigenvectors associated to $\lambda_d = 1$ as in (2)-(3). Then, the discrete-time Laplacian gets the form

$$L_d = \begin{pmatrix} L_{d,1} & \dots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & L_{d,\mu} & 0\\ M_{d,1} & \dots & M_{d,\mu} & M_d\end{pmatrix}$$

with

$$L_{d,i} =(I + \frac{\kappa}{2}L_i)^{-1}(I - \frac{\kappa}{2}L_i)$$
$$M_{d,\imath} = - \frac{k}{2}(I + \frac{\kappa}{2}M)^{-1}M_i(I + \frac{k}{2}L_i)^{-1}$$
$$M_d =(I + \frac{\kappa}{2}M)^{-1}(I - \frac{\kappa}{2}M)$$

verifying, since $M_i\mathbb{1}_{h_i} + M\gamma^i = 0$ and $L_i\mathbb{1}_{h_i} = 0$

$$L_{d,i}\mathbb{1}_{h_i} = \mathbb{1}_{h_i}, \quad v_i^\top L_{d,i} = 0, \quad M_{d,i}\mathbb{1}_{h_i} + M_d\gamma^i = \gamma^i.$$

By this structure, the proof follows the lines of the continuous time one in [7, Theorem 1], showing that, $k \to \infty$

$$x(k) \to \begin{pmatrix} \mathbb{1}_{h_1} \\ \vdots \\ 0 \\ \gamma^1 \end{pmatrix} x_s^1 + \cdots + \begin{pmatrix} 0 \\ \vdots \\ \mathbb{1}_{h_\mu} \\ \gamma^\mu \end{pmatrix} x_s^\mu$$

and, in particular, component-wise, (14). ■

By the result below, the consensus of the discrete-time network (11) exactly coincides with the one of the continuous-time counterpart (7), independently on the specific value of the gain $\kappa > 0$ (which is only modulating the convergence rate). This implies that, when the coupling rule is suitably set, the behavior of the network does not depend on the type of agents composing it, but only on the underlying communication graph. This fact is strengthened in the following section with reference to sampled-data networks.

V. The case of aperiodic sampling

Consider a network of N dynamical agents

$$\dot{x}_i(t) = u_i(t_k), \quad t \in [t_k, t_{k+1}) \tag{15}$$

with unweighted digraph $\mathcal{G}$. Let us assume that all agents exchange information at discrete-time instants $t_k \in \Delta$ and $\Delta = \{t_0, t_1, \dots\}$ the sampling sequence with bounded sampling periods $\delta_k := t_{k+1} - t_k \in [\delta_m, \delta_M]$ for some $\delta_M \geq \delta_m > 0$ and $k \geq 0$. Then, denoting $u(k) := u(t_k)$ and $x(k) := x(t_k)$ for $t_k \in \Delta$, the multi-agent system under the discrete coupling rule with the average output

$$y_i(k) = x_i(k) + \frac{1}{2}\delta_k u_i(k) \tag{16}$$

reads

$$\dot{x}(t) = -\kappa(I + \frac{\kappa}{2}\delta_k L)^{-1}Lx(k), \quad t \in [t_k, t_{k+1}). \tag{17}$$

The following result can be proved.

Theorem 5.1: Consider the multi-agent sampled-data system (17) with sampling sequence Δ and $\delta_k := t_{k+1} - t_k \in [\delta_m, \delta_M]$ and communication unweighted digraph $\mathcal{G}$ with Laplacian of the form (1). Then, the discrete-coupling rule (6) with average outputs (16) ensures convergence of (17) to multi-consensus; namely, for $t \to \infty$, regrouping the states as in (13) and for $v_i^\top \in \mathbb{R}^{h_i}$ as in (3), one gets

$$\mathbf{x}_i(t) \to x_s^i = v_i^\top \mathbf{x}_i(0), \quad i = 1, \dots, \mu$$
$$\mathbf{x}_{\delta,\ell}(t) \to x_s^{\delta,\ell} = \sum_{i=1}^{\mu} \gamma_\ell^i x_s^i, \quad \ell = 1, \dots, p.$$

Proof: We adopt the hybrid formalism developed in [21, Chapter 3] by defining the hybrid model

$$\dot{\zeta} =f(\zeta), \quad \zeta \in C; \qquad z^+ = g(\zeta), \quad \zeta \in D$$

over the extended state $\zeta = (x^\top \ u^\top \tau)^\top \in \mathbb{R}^{2N+1}$ with

$$C =\{(x^\top \ u^\top \tau)^\top \in \mathbb{R}^{2N+1} \text{ s.t } \tau \in [0, \delta_M]\}$$
$$D =\{(x^\top \ u^\top \tau)^\top \in \mathbb{R}^{2N+1} \text{ s.t } \tau = 0\}$$
$$f(\zeta) = \begin{pmatrix} u \\ 0 \\ -1 \end{pmatrix}, \quad g(\zeta) = \begin{pmatrix} x \\ -\kappa(I + \frac{\kappa}{2}\delta_{k+1}L)^{-1}Lx \\ 0 \end{pmatrix}$$

where $\tau = t_{k+1} - t \geq 0$ is a decreasing timer to the next sampling instant to occur. At this point, for proving convergence to consensus, one must prove asymptotic stability of the hybrid dynamics with respect to the consensus set (see [14] for further detail on its definition)

$$\mathcal{A} = \{(x, u) \in \mathbb{R}^N \times \mathbb{R}^N \text{ s.t } V_r^\top x = 0\}$$

with $V_r^\top \in \mathbb{R}^{(N-\mu)\times N}$ the matrix of the left eigenvectors associated to the non-zero eigenvalue of L. The hybrid Lyapunov function

$$\mathcal{H}(\zeta) = \frac{1}{2}\left(x^\top u^\top\right) A_\tau^\top A_\tau \begin{pmatrix} x \\ u \end{pmatrix}, \quad A_\tau = \begin{pmatrix} I & \tau I \\ 0 & I \end{pmatrix}$$

verifies $\dot{\mathcal{H}}(\zeta) = 0$ for all $\zeta \in C$ and, for $\zeta \in D$, that

$$\mathcal{H}(g(\zeta)) - \mathcal{H}(\zeta) \leq$$
$$- x^\top V_r(I + \frac{\kappa}{2}\delta_{k+1}\Lambda_r)^{-\top}\Lambda_r(I + \frac{\kappa}{2}\delta_{k+1}\Lambda_r)^{-1}V_r^\top x \leq 0.$$

Thus it is strictly decreasing with respect to $\mathcal{A}$. ■

Remark 5.1: As $\delta_M \to \delta_m \to 0$, the coupling (6) naturally recovers the standard continuous-time counterpart.

Remark 5.2: When sampling is periodic (i.e., $\delta = \delta_k$ for all $k \geq 0$), then the proof of the result above notably simplifies as one can easily use only the sampled-data equivalent model of the time-invariant network dynamics.

VI. A simulated example

Consider a network of $N = 8$ agents under the communication graph $\mathcal{G}$ with Laplacian

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 3 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & -1 & 4 \end{pmatrix}$$

possessing two reaches $\mathcal{H}_1 = \{\nu_1, \nu_2, \nu_3\}$, $\mathcal{H}_2 = \{\nu_4, \nu_5\}$ and the common $\mathcal{C} = \{\nu_6, \nu_7, \nu_8\}$. Through simulations (Figure 1), we compare the cases in which agents are: continuous-time integrators (for a benchmark behavior); sampled-data integrators under aperiodic sampling with $\delta_m = 0.1$ and $\delta_M = 2$ seconds; discrete-time integrators. In the latter case, two situations are reported: the network under

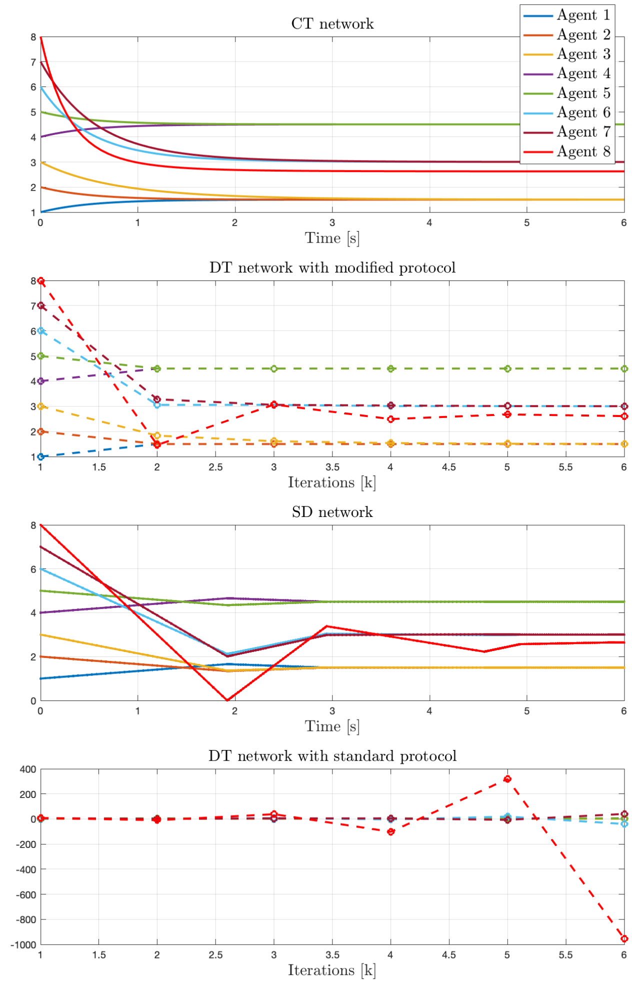

Fig. 1. $\kappa = 1$, $x_i(0) = i$ for $i = 1, \dots, 8$.

the standard coupling (5); the network under the proposed coupling (6). In all scenarios, the clusters dictated by the network are given by the AEP (12) with $p = 2$, $\rho_3 = \{\nu_6, \nu_7\}$ and $\rho_4 = \{\nu_8\}$. In particular, computing (2)-(3) with $\mu = 2$

$$v_1^\top = \frac{1}{2}\begin{pmatrix}\mathbb{1}_2^\top & 0\end{pmatrix}, \quad v_2^\top = \frac{1}{2}\mathbb{1}_2^\top$$
$$\gamma^1 = \begin{pmatrix}\frac{1}{2}\mathbb{1}_2^\top & \frac{5}{8}\end{pmatrix}^\top, \quad \gamma^2 = \begin{pmatrix}\frac{1}{2}\mathbb{1}_2^\top & \frac{3}{8}\end{pmatrix}^\top$$

four consensuses are induced by the network and given by

$$x_s^1 = \frac{1}{2}(x_1(0) + x_2(0)), \quad x_s^2 = \frac{1}{2}(x_4(0) + x_5(0))$$
$$x_s^{\delta,1} = \frac{1}{2}(x_s^1 + x_s^2), \quad x_s^{\delta,2} = \frac{1}{8}(5x_s^1 + 3x_s^2).$$

The results of the simulations are reported in Figure 1 when setting in all cases $\kappa = 1$ and initial conditions $x_i(0) = i$, for $i = 1, \dots, 8$ so getting $x_s^1 = \frac{3}{2}$, $x_s^2 = \frac{9}{2}$, $x_s^{\delta,1} = 3$ and $x_s^{\delta,2} = \frac{21}{8}$. When using standard coupling, consensus is not guaranteed for $\kappa \geq \frac{1}{2}$ contrarily to the case in which the proposed connection is introduced.

VII. Conclusions and perspectives

A new coupling rule for multi-agent discrete-time dynamics has been proposed by recurring to the notion of average passivity. In particular, it is realized by interconnecting all agents with respect to the (average) passivating outputs. Typical issues arising under the standard coupling rule are naturally overcome. The proposed method is shown to hold also for sampled-data systems under aperiodic sampling. Current work is devoted to making the new coupling computable distributedly and the case of sampled-data agents with no knowledge of the sampling instants.

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