ON THE SEMISIMPLICITY OF THE CATEGORY KL_k FOR AFFINE LIE SUPERALGEBRAS

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ABSTRACT. We study the semisimplicity of the category KL_k for affine Lie superalgebras and provide a super analog of certain results from [8]. Let KL_k^{fin} be the subcategory of KL_k consisting of ordinary modules on which a Cartan subalgebra acts semisimply. We prove that KL_k^{fin} is semisimple when 1) k is a collapsing level, 2) $W_k(\mathfrak{g},\theta)$ is rational, 3) $W_k(\mathfrak{g},\theta)$ is semisimple in a certain category. The analysis of the semisimplicity of KL_k is subtler than in the Lie algebra case, since in super case KL_k can contain indecomposable modules. We are able to prove that in many cases when KL_k^{fin} is semisimple we indeed have $KL_k^{fin} = KL_k$, which therefore excludes indecomposable and logarithmic modules in KL_k. In these cases we are able to prove that there is a conformal embedding $W \hookrightarrow V_k(\mathfrak{g})$ with W semisimple (see Section 10). In particular, we prove the semisimplicity of KL_k for $\mathfrak{g} = sl(2|1)$ and $k = -\frac{m+1}{m+2}$, $m \in \mathbb{Z}_{\geq 0}$. For $\mathfrak{g} = sl(m|1)$, we prove that KL_k is semisimple for $k = -1$, but for k a positive integer we show that it is not semisimple by constructing indecomposable highest weight modules in KL_k^{fin}.

1. INTRODUCTION

The aim of this paper is to extend results from [8] to the super case. In [8] we proved semisimplicity of the category of ordinary modules KL_k for a simple affine vertex algebra $V_k(\mathfrak{g})$ when $\mathfrak{g}$ is a Lie algebra and one of the following assumptions holds:

(L-1) k is collapsing level, i.e., $W_k(\mathfrak{g},\theta)$ collapses to its affine subalgebra;
(L-2) $W_k(\mathfrak{g},\theta)$ is rational;
(L-3) $W_k(\mathfrak{g},\theta)$ is semisimple in a category of ordinary modules.

Here $W_k(\mathfrak{g},\theta)$ is the simple affine W-algebra attached to $\mathfrak{g}$ and a minimal nilpotent element [33]. It is natural to look for an extension of these results when $\mathfrak{g}$ is a Lie superalgebra.

Our paper [8] has attracted a lot of interest in various direction. Here we mention two interesting applications/generalizations:

- T. Creutzig and J. Yang showed in [19] that at all levels investigated in [8] there is a braided tensor category structure. It is interesting that they also use our previous work on the decomposition of conformal embeddings to prove rigidity of tensor categories.
- T. Arakawa, J. van Ekeren and A. Moreau in [15] have constructed a large new family of collapsing levels which are admissible.

In order to apply the results from [8], we need to extend certain structural resuls on self-extension of irreducible modules in KL_k. A first problem is that the category KL_k for Lie superalgebras is more complicated than in the case of Lie algebras. For instance, in the super case, KL_k can have non-semisimple and logarithmic modules. A nice illustration

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for these phenomena in given by the Lie superalgebra $\mathfrak{gl}(1|1)$ and its affine vertex algebra $V_1(\mathfrak{gl}(1|1))$, which admits highest weight modules whose top components are two dimensional indecomposable $\mathfrak{gl}(1|1)$–modules (cf. [12], [20]). One can also construct logarithmic $\mathfrak{gl}(1|1)$–modules in KL_k.

1.1. **Semisimplicity of KL_k^{fin}.** In order to extend directly our methods from [8], it seems that we have one natural choice. We can consider the smaller categories KL_k^{ss} (resp. KL_k^{fin}) consisting of modules from KL_k on which $L_{\mathfrak{g}}(0)$ acts semisimply (resp. a Cartan subalgebra of the affinization $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ acts semisimply), cf. Definition 2.1. We prove in Theorem 4.3 a super analog of [8, Theorem 5.5]:

- Let $\mathfrak{g}$ be a basic Lie superalgebra. If every highest weight $V_k(\mathfrak{g})$–module in KL_k^{fin} is irreducible, then the category KL_k^{fin} is semisimple.

With this modification we can prove the semisimplicity of KL_k^{fin} in the following cases

(S-2) $W_k(\mathfrak{g}, \theta)$ is rational.
(S-3) $W_k(\mathfrak{g}, \theta)$ is semisimple in the category of ordinary modules.

Regarding the super analogue of condition (L-1), i.e., k is a collapsing level, we introduce the notion of *collapsing chain* (cf. Definition 4.7) and in many cases we reduce to prove semisimplicity of KL_k^{fin} by looking at conditions (S-2) or (S-3) for explicitly determined subalgebras $\mathfrak{g}_n$ of $\mathfrak{g}$ (see Theorem 4.9). Collapsing levels for Lie superalgebras were classified in [8]. So our results immediately gives semisimplicity of KL_k^{fin} for these levels. A comprehensive list of all the cases covered by the above conditions is given in Corollary 4.12.

Let us mention some cases of rationality:

- $\mathfrak{g} = osp(1|2)$. Then the minimal W-algebra $W_k(\mathfrak{g}, \theta)$ is the $N = 1$ super-conformal algebra which is rational for certain k.
- $\mathfrak{g} = sl(2|1)$. Then for $k = -\frac{m+1}{m+2}$, $m \in \mathbb{Z}_{\geq 0}$, the algebra $W_k(\mathfrak{g}, \theta)$ is isomorphic to $N = 2$ superconformal vertex algebra at central charge $c = 3m/(m+2) = -3(2k+1)$, which is rational by [2].

Next we have interesting cases when $W_k(\mathfrak{g}, \theta)$ is semisimple in a certain category.

- $\mathfrak{g} = \mathfrak{psl}(2|2)$ and conformal level $k = 1/2$. Then $W_k(\mathfrak{g}, \theta)$ is the $N = 4$ superconformal vertex algebra with central charge $c = -9$ [4], which is semisimple in the category of ordinary modules (cf. Theorem 9.2).
- $\mathfrak{g} = D(2, 1, \alpha)$. Then for collapsing level k we can have that $W_k(\mathfrak{g}, \theta) = V_{k'}(sl(2))$. If k' is a positive integer, or admissible we conclude that $V_k(\mathfrak{g})$ is semisimple in KL_k. In [7] we described a conformal embedding $V_{k_1}(sl(2)) \otimes V_{k_2}(sl(2)) \hookrightarrow W_k(\mathfrak{g}, \theta)$. If both k_1, k_2 are admissible, we expect that then KL_k^{fin} is semisimple.

More results on the semisimplicity of KL_k, regarding $V_{-\frac{n+1}{2}}(C(n+1)), V_{-1}(psl(m|m)), m \geq 3$, are given in Theorems 6.3, 7.2 respectively.

1.2. **When $\mathbf{KL_k = KL_k^{fin}}$?** We have already observed that the category KL_k is the most natural choice of category of $V_k(\mathfrak{g})$–modules. The example of $V_1(\mathfrak{gl}(1|1))$ shows that in general $KL_k \neq KL_k^{fin}$. On the other hand, we can prove equality in some cases. In Proposition 5.1 we give two sufficient conditions for this equality to hold.

- Assume that $\mathfrak{g}_{\bar{0}}$ is a semisimple Lie algebra. Then $KL_k^{fin} = KL_k^{ss}$.
- There is a conformal embedding of $V_{k_1}(\mathfrak{g}_{\bar{0}}) \hookrightarrow V_k(\mathfrak{g})$ such that the category KL_{k_1} for $V_{k_1}(\mathfrak{g}_{\bar{0}})$ is semisimple.

These conformal embeddings were classified in [9].

We believe that when KL_k^{fin} is semisimple, we should have $KL_k = KL_k^{fin}$. Under the assumption that KL_k^{fin} is semisimple, we prove in Theorem 5.5 the following result:

- Assume that for any irreducible $V_k(\mathfrak{g})$–module M in KL_k we have

$$\mathrm{Ext}^1(M_{top}, M_{top}) = \{0\} \tag{1.1}$$

  in the category of finite-dimensional $\mathfrak{g}$–modules. Then KL_k is semisimple and $KL_k^{fin} = KL_k$.

We will prove that $KL_k = KL_k^{fin}$ in the following cases:

- $V_{-\frac{1}{2}}(C(n+1))$; the proof (see Theorem 6.2) relies on results from [9] and fusion rules arguments.
- $V_{-1}(sl(m|1))$ (see Theorem 7.7).
  Since $k = -1$ is a collapsing level, we get that KL_k^{fin} is semisimple. Next we refine the classification of irreducible modules in KL_k and prove that top components of irreducible modules in KL_k are atypical $\mathfrak{g}$–modules. Then the result of [24] on extensions of finite-dimensional $sl(m|1)$–modules implies that there are no self-extensions among irreducible modules in KL_k, so the condition (1.1) is satisfied. Then Theorem 5.5 gives that the larger category KL_k is semisimple.
- $V_{-(m+1)/(m+2)}(sl(2|1))$, $m \in \mathbb{Z}_{\geq 0}$. In this case we prove that the center of $\mathfrak{g}_{\bar{0}}$ belongs to a regular vertex operator algebra $D_{m+1,2}$ from [3]: see Section 10. Then the condition (1.1) is also satisfied, which gives that in this case the category KL_k is semisimple.

1.3. **Examples when KL_k is not semisimple.** M. Gorelik and V. Serganova in [26, Section 5.6.4] constructed examples of indecomposable weak $V_1(sl(2|1))$–modules on which the Sugawara operator $L(0)$ does not act semisimply. Their construction uses the theory of Zhu's algebras and a description of the maximal ideal in the universal affine vertex algebra $V^1(sl(2|1))$. In the present paper we use a different approach and apply free-field realisation. In Theorem 8.1 we construct a highest weight $V_1(sl(m|1))$–module $\widetilde{W} = V_1(sl(m|1)).(a^+)^{-m} \otimes |m>$ which has a proper submodule isomorphic to $V_1(sl(m|1))$. The module $\widetilde{W}$ belongs to KL_k^{fin}, and therefore KL_k is not semisimple for $k = 1$. We extend this example by showing that KL_k^{fin} is not semisimple for $k \in \mathbb{Z}_{>0}$. A complete analysis of indecomposable modules will appear in forthcoming papers.

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2. Setup

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a basic Lie superalgebra, i.e. a simple Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and there exists an even invariant supersymmetric bilinear form on it (see [28] for more details and the classification). Choose a Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{g}_{\bar{0}}$ and let $\Delta = \Delta_0 \cup \Delta_1$ be the set of roots. Fix a positive system Δ^+ in Δ and choose an even root

θ maximal in $\Delta_0^+ = \Delta^+ \cap \Delta_0$. We may choose root vectors e_θ and $e_{-\theta}$ such that

$$[e_\theta, e_{-\theta}] = x \in \mathfrak{h}, \qquad [x, e_{\pm\theta}] = \pm e_{\pm\theta}.$$

Due to the minimality of $-\theta$, the eigenspace decomposition of $ad\, x$ defines a *minimal* $\frac{1}{2}\mathbb{Z}$-grading ([33, (5.1)]):

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1, \tag{2.1}$$

where $\mathfrak{g}_{\pm 1} = \mathbb{C}e_{\pm\theta}$. Furthermore, one has

$$\mathfrak{g}_0 = \mathfrak{g}^\natural \oplus \mathbb{C}x, \quad \mathfrak{g}^\natural = \{a \in \mathfrak{g}_0 \mid (a|x) = 0\}. \tag{2.2}$$

Note that $\mathfrak{g}^\natural$ is the centralizer of the triple $\{f_\theta, x, e_\theta\}$. We can choose $\mathfrak{h}^\natural = \{h \in \mathfrak{h} \mid (h|x) = 0\}$, as a Cartan subalgebra of the Lie superalgebra $\mathfrak{g}^\natural$, so that $\mathfrak{h} = \mathfrak{h}^\natural \oplus \mathbb{C}x$.

For a given choice of a minimal root $-\theta$, we normalize the invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$ by the condition

$$(\theta|\theta) = 2. \tag{2.3}$$

The dual Coxeter number $h^\vee$ of the pair $(\mathfrak{g}, \theta)$ is defined to be half the eigenvalue of the Casimir operator of $\mathfrak{g}$ corresponding to $(\cdot|\cdot)$, normalized by (2.3). Let $\widehat{\mathfrak{g}}$ be the affinization of $\mathfrak{g}$, i.e.

$$\widehat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

where d acts as $t\frac{d}{dt}$, K is central and the bracket on $[\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]$ is defined by

$$[x_{(m)}, y_{(n)}] = [x, y]_{(m+n)} + m\delta_{m,-n}(x|y)K,$$

where $x_{(m)} = t^m \otimes x$, $x \in \mathfrak{g}$. Set $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$. Write $\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ where $\Lambda_0(K) = 1, \Lambda_0(\mathfrak{h}) = 0, \Lambda_0(d) = 0, \delta(K) = 0, \delta(\mathfrak{h}) = 0, \delta(d) = 1$.

By category $\mathcal{O}$ for $\widehat{\mathfrak{g}}$ we mean the set of $\widehat{\mathfrak{g}}$–modules which are $\widehat{\mathfrak{h}}$-diagonalizable with finite dimensional weight spaces and a finite number of maximal weights. Let $\mathcal{O}^k$ be the subcategory of $\widehat{\mathfrak{g}}$-modules in $\mathcal{O}$ of level k (i.e., K acts as $k\, Id$).

Let $\mathfrak{a}$ be a Lie superalgebra equipped with a nondegenerate invariant supersymmetric bilinear form B. The universal affine vertex algebra $V^B(\mathfrak{a})$ is the universal enveloping vertex algebra of the non–linear Lie conformal superalgebra $R = (\mathbb{C}[T] \otimes \mathfrak{a})$ with λ-bracket given by

$$[a_\lambda b] = [a, b] + \lambda B(a, b), \ a, b \in \mathfrak{a}.$$

In the following, we shall say that a vertex algebra V is an affine vertex algebra if it is a quotient of some $V^B(\mathfrak{a})$.

If $k \in \mathbb{C}$, we will write simply $V^k(\mathfrak{g})$ for $V^{k(\cdot|\cdot)}(\mathfrak{g})$. We will always assume that k is non–critical, i.e. $k \neq -h^\vee$. With this assumption, it is known that $V^k(\mathfrak{g})$ has a unique simple quotient, denoted by $V_k(\mathfrak{g})$ (see [30, § 4.7 and Example 4.9b]). The vertex algebras $V^k(\mathfrak{g}), V_k(\mathfrak{g})$ are VOAs with Virasoro vector $L_\mathfrak{g}$ given by the Sugawara construction.

If M is a restricted module of level k for $\widehat{\mathfrak{g}}$ then it is a weak module for $V^k(\mathfrak{g})$; conversely, letting d act on weak modules by $-L_\mathfrak{g}(0)$ yields restricted modules for $\widehat{\mathfrak{g}}$.

Definition 2.1. *We denote by $KL^B(\mathfrak{g})$ the category of weak modules for $V^B(\mathfrak{g})$, which*

1. *are locally finite as $\mathfrak{g}$-modules;*
2. *admit a decomposition into generalized eigenspaces for $L_\mathfrak{g}(0)$ whose eigenvalues are bounded below.*

We denote by

- *$KL^B_{fin}(\mathfrak{g})$ the full subcategory of modules in $KL^B(\mathfrak{g})$ on which $\widehat{\mathfrak{h}}$ acts semisimply.*

- $KL^B_{ss}(\mathfrak{g})$ *the full subcategory of modules in* $KL^B(\mathfrak{g})$ *on which* $L_{\mathfrak{g}}(0)$ *acts semisimply.*

If $B = k(\cdot|\cdot)$ *we simply write* $KL^k(\mathfrak{g})$, $KL^k_{fin}(\mathfrak{g})$, $KL^k_{ss}(\mathfrak{g})$. *We also denote by* $KL_k(\mathfrak{g})$, $KL^{fin}_k(\mathfrak{g})$, $KL^{ss}_k(\mathfrak{g})$ *the full subcategories of* $KL^k(\mathfrak{g}), KL^k_{fin}(\mathfrak{g}), KL^k_{ss}(\mathfrak{g})$ *consisting of the* $V_k(\mathfrak{g})$*-modules. If* $\mathfrak{g}$ *is clear from the context we omit it in the notation.*

Let V be a conformal vertex algebra. Denote by L its conformal vector (with $Y(L,z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$). We say that a V–module is ordinary if $L(0)$ acts semisimply with finite dimensional eigenspaces. In our settings, let C_k be the category of ordinary modules.

Remark 2.2. *Note that (2) is exactly the definition of a logarithmic module for a conformal vertex algebra. Condition (1) says that each generalized eigenspace for* $L(0)$ *is a sum of finite-dimensional* $\mathfrak{g}$*–modules. When* $\mathfrak{g}$ *is a simple Lie algebra, we have that each generalized eigenspace for* $L(0)$ *is a direct sum of finite-dimensional modules. In particular, we have* $KL^{fin}_k = KL^{ss}_k$. *But when* $\mathfrak{g}$ *is a Lie superalgebra, we really have modules in* KL_k *and* KL^{ss}_k *with non–semisimple action of* $\mathfrak{g}$. *One such example is the vertex algebra* $V_1(\mathfrak{gl}(1|1))$ *and its modules considered in* [12]. *More examples are given in Section 8.*

3. W-algebras and collapsing levels

Denote by $W^k(\mathfrak{g},\theta)$ the affine W–algebra obtained from $V^k(\mathfrak{g})$ by Hamiltonian reduction relative to the minimal nilpotent element $e_{-\theta}$. More precisely, let M be a restricted $V^k(\mathfrak{g})$-module. Consider the complex

$$\mathcal{C}^M = M \otimes F(A_{ch}) \otimes F(A_{ne}), \tag{3.1}$$

where $F(A_{ch}), F(A_{ne})$ are fermionic vertex algebras, defined in [33], attached to the following superspaces. Denote by A_{ne} the vector superspace $\mathfrak{g}_{1/2}$ with the bilinear form

$$\langle a,b\rangle_{ne} = (e_{-\theta}|[a,b])\,. \tag{3.2}$$

Denote by A (resp. A^*) the vector superspace $\mathfrak{g}_{1/2}\oplus\mathfrak{g}_1$ (resp. $(\mathfrak{g}_{1/2}\oplus\mathfrak{g}_1)^*$) with the reversed parity, let $A_{ch} = A\oplus A^*$ and define an even skew-supersymmetric non-degenerate bilinear form $\langle .\,,.\rangle_{ch}$ on A_{ch} by

$$\begin{aligned} \langle A,A\rangle_{ch} &= 0 = \langle A^*,A^*\rangle_{ch}\,, \\ \langle a,b^*\rangle_{ch} &= -(-1)^{p(a)p(b^*)}\langle b^*,a\rangle_{ch} = b^*(a) \text{ for } a\in A, b^*\in A^*\,. \end{aligned} \tag{3.3}$$

Here and further, $p(a)$ stands for the parity of an (homogeneous) element of a vector superspace. Choose a basis $\{u_\alpha\}_{\alpha\in S_j}$ of each $\mathfrak{g}_j$ in (2.1), and let $S = \coprod_{j\in\frac12\mathbb{Z}} S_j$, $S_+ = \coprod_{j>0} S_j$. Let $p(\alpha)\in\mathbb{Z}/2\mathbb{Z}$ denote the parity of u_α, and let $m_\alpha = j$ if $\alpha\in S_j$. Define the structure constants $c^\gamma_{\alpha\beta}$ by $[u_\alpha,u_\beta] = \sum_\gamma c^\gamma_{\alpha\beta}u_\gamma$ $(\alpha,\beta,\gamma\in S)$. Denote by $\{\varphi_\alpha\}_{\alpha\in S_+}$ the basis of A corresponding to u_α in the identification $A = \mathfrak{g}_{1/2}\oplus\mathfrak{g}_1$, and by $\{\varphi^\alpha\}_{\alpha\in S_+}$ the basis of A^* such that $\langle\varphi_\alpha,\varphi^\beta\rangle_{ch} = \delta_{\alpha\beta}$. Denote by $\{\Phi_\alpha\}_{\alpha\in S_{1/2}}$ the corresponding basis of A_{ne}, and by $\{\Phi^\alpha\}_{\alpha\in S_{1/2}}$ the dual basis with respect to $\langle .\,,.\rangle_{ne}$, i.e., $\langle\Phi_\alpha,\Phi^\beta\rangle_{ne} = \delta_{\alpha\beta}$. Define

$$H(M) = H^0(\mathcal{C}^M, d_0)$$

where d_0 is defined in [33]. Then

$$W^k(\mathfrak{g},\theta) = H(V^k(\mathfrak{g})).$$

TABLE 1.

$\mathfrak{g}$	$p(k)$
$sl(m\|n)$, $n \neq m$	$(k+1)(k+(m-n)/2)$
$psl(m\|m)$	$k(k+1)$
$osp(m\|n)$	$(k+2)(k+(m-n-4)/2)$
$spo(n\|m)$	$(k+1/2)(k+(n-m+4)/4)$
$D(2,1;a)$	$(k-a)(k+1+a)$
$F(4)$, $\mathfrak{g}^\natural = so(7)$	$(k+2/3)(k-2/3)$
$F(4)$, $\mathfrak{g}^\natural = D(2,1;2)$	$(k+3/2)(k+1)$
$G(3)$, $\mathfrak{g}^\natural = G_2$	$(k-1/2)(k+3/4)$
$G(3)$, $\mathfrak{g}^\natural = osp(3\|2)$	$(k+2/3)(k+4/3)$

Note: when writing $osp(m|n) = spo(n|m)$ we adopt the conventions of [29, 2.1.2], so that n is even.

Denote by $W_k(\mathfrak{g},\theta)$ the unique simple quotient of $W^k(\mathfrak{g},\theta)$. It is proved in [33] that $W^k(\mathfrak{g},\theta)$ has a vertex subalgebra isomorphic to $V^{\beta_k}(\mathfrak{g}^\natural)$, where

$$\beta_k(a,b) = \tfrac{1}{4}\left((k+h^\vee/2)(a|b) - \tfrac{1}{4}\kappa_0(a,b)\right),\ a,b \in \mathfrak{g}^\natural.$$

and κ_0 is the Killing form of $\mathfrak{g}_0$. Let $\mathcal{V}_k(\mathfrak{g}^\natural)$ be the image of $V^{\beta_k}(\mathfrak{g}^\natural)$ in $W_k(\mathfrak{g},\theta)$.

Definition 3.1. *If $W_k(\mathfrak{g},\theta) = \mathcal{V}_k(\mathfrak{g}^\natural)$, we say that k is a collapsing level.*

Theorem 3.2. [6, Theorem 3.3] *Let $p(k)$ be the polynomial listed in Table 1 below. Then k is a collapsing level if and only if $p(k) = 0$.*

4. RESULTS IN KL_k^{fin}

For $\lambda \in \widehat{\mathfrak{h}}^*$, denote by $L(\lambda)$ the irreducible $\widehat{\mathfrak{g}}$-module of highest weight λ.

Lemma 4.1. *$Ext^1_{\mathcal{O}^k}(L(\lambda), L(\lambda)) = 0$.*

Proof. Suppose that there is an extension

$$0 \to L(\lambda) \xrightarrow{h} N \xrightarrow{f} L(\lambda) \to 0. \tag{4.1}$$

Since $\widehat{\mathfrak{h}}$ acts diagonally, this implies that

$$0 \to L(\lambda)_\lambda \to N_\lambda \to L(\lambda)_\lambda \to 0 \tag{4.2}$$

splits. We now prove that (4.1) splits too. Indeed, let $g_\lambda : L(\lambda)_\lambda \to N_\lambda$ be a section. Let $M(\lambda)$ be the Verma module and $\pi : M(\lambda) \to L(\lambda)$ be the canonical projection. Let $\eta : M(\lambda) \to N$ be the unique map such that $\eta(1) = g_\lambda(\pi(1))$. Remark that obviously, $f(\eta(Ker\,\pi)) = 0$, so $\eta(Ker\,\pi) \subset h(L(\lambda))$. Since $\eta(Ker\,\pi)_\lambda = 0$, we have $\eta(Ker\,\pi) = 0$. Define $g : L(\lambda) \to N$ by setting $g(\pi(v)) = \eta(v)$. It is easy to verify that g is a well defined section which splits (4.1). □

Proposition 4.2. *Suppose that $M \in KL_k^{fin}$ is finitely generated. Then $M \in \mathcal{O}^k$.*

Proof. By assumption $\widehat{\mathfrak{h}}$ acts semisimply. Let $\widehat{\mathfrak{g}}_+ = \mathfrak{g} \oplus t\mathbb{C}[t]\mathfrak{g}$. Let $\{m_1,\dots,m_k\}$ be a set of generators for M. By the finiteness of the $\mathfrak{g}$-action and since the conformal weights are bounded below, $M' = U(\widehat{\mathfrak{g}}_+)(\sum_{i=1}^k \mathbb{C}m_i)$ is finite dimensional, in particular, it has only a finite number of weights. Let $\widehat{\mathfrak{g}}_- = t^{-1}\mathbb{C}[t^{-1}]\mathfrak{g}$, so that $U(\widehat{\mathfrak{g}}) = U(\widehat{\mathfrak{g}}_-) \otimes U(\widehat{\mathfrak{g}}_+)$ and $M = U(\widehat{\mathfrak{g}}_-)M'$, hence M has a finite number of maximal weights. Since

$$M_\lambda = \sum_\nu U(\widehat{\mathfrak{g}}_-)_{\lambda-\nu} M'_\nu,$$

we see that the $\widehat{\mathfrak{h}}$-eigenspaces are finite dimensional. □

Theorem 4.3. *Let $\mathfrak{g}$ be a basic Lie superalgebra. If every highest weight $V_k(\mathfrak{g})$–module in KL_k^{fin} is irreducible, then the category KL_k^{fin} is semisimple.*

Proof. Let M be a module in KL_k^{fin}. If M is finitely generated, then, by Proposition 4.2, we have that $M \in \mathcal{O}^k$. Since Lemma 4.1 holds, we can then use the argument in [8, Theorem 5.5] to prove the complete reducibility of M. As in *loc. cit.*, the case when M is not finitely generated can be reduced to the finitely generated case. □

If V is a vertex algebra, we say that a V–module is ordinary if $L(0)$ acts semisimply with finite dimensional eigenspaces. Recall that we denote by C_k the category of ordinary modules in KL_k.

Corollary 4.4. *Assume that $\mathfrak{g}_{\bar{0}}$ is semisimple. If every highest weight $V_k(\mathfrak{g})$–module in C_k is irreducible, then C_k is semisimple.*

Proof. If M is a highest weight module in KL_k^{fin}, then, by Proposition 4.2, $M \in C_k$, thus M is irreducible. By Theorem 4.3, KL_k^{fin} is semisimple. Let M be a module in C_k. Since $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}}$ is a Cartan subalgebra of the semisimple Lie algebra $\mathfrak{g}_{\bar{0}}$, and $\mathfrak{h}, L(0)$ commute, then $\widehat{\mathfrak{h}}$ acts semisimply on M, thus $M \in KL_k^{fin}$ and therefore M is completely reducible. □

Recall from [8, Lemma 5.6] the following result.

Lemma 4.5. *Let $k \in \mathbb{Q} \setminus \mathbb{Z}_{\geq 0}$. Assume that $H(U)$ is an irreducible, non-zero $W_k(\mathfrak{g}, \theta)$–module for every non-zero highest weight $V_k(\mathfrak{g})$–module U from the category KL_k^{fin}. Then every highest weight $V_k(\mathfrak{g})$–module in KL_k^{fin} is irreducible.*

4.1. **Rational case.**

Theorem 4.6. *Let $\mathfrak{g}$ be a basic Lie superalgebra and $k \notin \mathbb{Z}_{\geq 0}$. If $W_k(\mathfrak{g}, \theta)$ is rational then $KL_k^{fin}(\mathfrak{g})$ is semisimple.*

Proof. By Theorem 4.3 it suffices to prove that every highest weight $V_k(\mathfrak{g})$–module in KL_k^{fin} is irreducible. Let $U \neq 0$ be such a module. Then $H(U)$ is a highest weight module for $W_k(\mathfrak{g}, \theta)$, which is nonzero since $k \notin \mathbb{Z}_{\geq 0}$. If $W_k(\mathfrak{g}, \theta)$ is rational then $H(U)$ is irreducible, hence U is irreducible by Lemma 4.5. □

4.2. **Collapsing case.**

Definition 4.7. *Write $(\mathfrak{g}_1, k_1) \triangleright (\mathfrak{g}_2, k_2)$ if $H(V_{k_1}(\mathfrak{g}_1, \theta)) = V_{k_2}(\mathfrak{g}_2)$. We call a sequence $(\mathfrak{g}, k) \triangleright (\mathfrak{g}_1, k_1) \triangleright \ldots \triangleright (\mathfrak{g}_n, k_n)$ a collapsing chain for $(\mathfrak{g}, k)$.*

It follows from [13, Main Theorem] that if $(\mathfrak{g}_1, k_1) \triangleright (\mathfrak{g}_2, k_2)$, then $k_1 \notin \mathbb{Z}_{\geq 0}$.

Proposition 4.8. *Assume that $H(V_k(\mathfrak{g})) = V_{k'}(\mathfrak{g}^\natural)$. If M is a highest weight module in KL_k, then $H(M) \in KL_{k'}^{fin}$.*

Proof. By [33], we know that $H(M)$ is a highest weight module, in particular $\widehat{\mathfrak{h}}^\natural$ acts semisimply and the $L_{\mathfrak{g}^\natural}(0)$-eigenvalues are bounded below. It remains only to show that the action of $\mathfrak{g}^\natural$ on the complex $\mathcal{C}^M$ (3.1) is locally finite. If $v \in \mathfrak{g}^\natural$ the the action of v on the complex is given by $J_0^{\{v\}}$, where

$$J^{\{v\}} = v + \sum_{\alpha,\beta \in S_+} (-1)^{p(\alpha)} c_{\alpha\beta}(v) : \varphi_\alpha \varphi^\beta : + (-1)^{p(v)/2} \sum_{\alpha \in S_{1/2}} : \Phi^\alpha \Phi_{[u_\alpha, v]} : . \tag{4.3}$$

$J^{\{v\}}(0)$ acts on the tensor product $M \otimes F$, where $F = F(A_{ch}) \otimes F(A_{ne})$ is a Clifford vertex algebra. The action of $v(0)$ on M is locally finite, since the action of $\mathfrak{g}$ is locally finite. More precisely, M admits a $\mathbb{Z}_{\geq 0}$ gradation:

$$M = \bigoplus_{\ell \in \mathbb{Z}_{\geq 0}} M_\ell, \quad \mathfrak{g} \text{ acts locally finite on } M_\ell.$$

On the other hand F admits a Virasoro vector L_F and the corresponding eigenvalue decomposition

$$F = \bigoplus_{\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}} F_\ell,$$

has finite-dimensional eigenspaces. By (4.3), $J^{\{v\}} - v$ is primary of conformal weight 1, hence $J^{\{v\}}(0) - v(0)$ commutes with $L_F(0)$. In particular it stabilizes the eigenspaces of L_F, so that we have a gradation:

$$M \otimes F = \bigoplus_{\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}} (M \otimes F)_\ell,$$

where

$$(M \otimes F)_\ell = \bigoplus_{i=0}^{\ell} M_i \otimes F_{\ell - i}$$

and each $(M \otimes F)_\ell$ is a $\mathfrak{g}^\natural$–module. Since each M_i is $\mathfrak{g}$–locally finite and F_j is finite-dimensional, we conclude that $(M \otimes F)_\ell$ is $\mathfrak{g}^\natural$–locally finite. The claim follows. □

Theorem 4.9. *Let $(\mathfrak{g}, k) \triangleright (\mathfrak{g}_1, k_1) \triangleright \ldots \triangleright (\mathfrak{g}_n, k_n)$ be a collapsing chain. Assume that $KL_{k_n}^{fin}(\mathfrak{g}_n)$ is semisimple. Then KL_k^{fin} is semisimple. In particular this happens when $V_{k_n}(\mathfrak{g}_n)$ is rational or admissible.*

Proof. We proceed by induction on the length of the collapsing chain. The base case $n = 0$ is obvious. Now assume $n > 0$. First remark that every highest weight module U in KL_k^{fin} is irreducible. Indeed if U is such a module, then $H(U)$ is a $V_{k_1}(\mathfrak{g}_1)$–module in $KL_{k_1}^{fin}$ by Proposition 4.8 and non-zero and highest weight by [13], [33]. By induction $KL_{k_1}^{fin}$ is semisimple, hence $H(U)$ is irreducible. By Lemma 4.5 U is irreducible, so we are in the hypothesis of Theorem 4.3 and therefore we can conclude that KL_k^{fin} is semisimple. In particular, if $V_{k_n}(\mathfrak{g}_n)$ is rational or admissible, then $KL_{k_n}^{fin}(\mathfrak{g}_n)$ is semisimple (by definition in the rational case, by Arakawa's Main Theorem from [14] in the admissible case). □

Lemma 4.10. *There is a collapsing chain* $(so(m), \frac{4-m}{2}) \triangleright \ldots \triangleright (\mathfrak{g}', k')$ *with*

$$(\mathfrak{g}', k') = \begin{cases} \mathbb{C} & \textit{if } m \equiv 0,1 \mod 4, \\ M(1) & \textit{if } m \equiv 2 \mod 4, \\ (sl(2),1) & \textit{if } m \equiv 3 \mod 4. \end{cases} \tag{4.4}$$

Proof. Using Table 2 below, one sees that for $m \gg 0$

$$H(V_{\frac{4-m}{2}}(so(m))) = V_{\frac{8-m}{2}}(so(m-4)).$$

Since $\frac{8-m}{2}$ is again a collapsing level for $so(m-4)$, by induction on m we are reduced to the cases when $m = 4, 5, 6, 7$, where one concludes using Table 2 once again. □

Proposition 4.11. *The following are collapsing chains*

(4.5)
$(F(4), -1) \triangleright (D(2,1;2), 1/2) = (D(2,1;1/2), 1/2) \triangleright (sl(2), -4/3)$

(4.6)
$(F(4), 2/3) \triangleright (so(7), -2) \triangleright (sl(2), -1/2)$

(4.7)
$(G(3), 1/2) \triangleright (G_2, -5/3) \triangleright \mathbb{C}$

(4.8)
$(spo(2|3), -3/4) \triangleright (sl(2), 1)$

(4.9)
$(spo(2|1), -5/4) \triangleright \mathbb{C}$

(4.10)
$m \neq n+1, n \geq 2,\ (spo(n|m), -1/2) \triangleright \mathbb{C}$

(4.11)
$m \ odd:$
$(spo(n|m), \frac{m-n-4}{4}) \triangleright (spo(n-2|m), \frac{m-n-2}{4}) \triangleright \ldots (spo(2|m), \frac{m-6}{4}) \triangleright (so(m), \frac{4-m}{2}) \triangleright (4.4)$

(4.12)
$(osp(n+8|n), -2) \triangleright \mathbb{C}$

(4.13)
$m \neq n+4, n+8, m \geq 4:\ (osp(m|n), -2) \triangleright (sl(2), \frac{m-n-8}{2})$

(4.14)
$(osp(3|n), n+1) = (spo(n|3), -\frac{n+1}{4}) \triangleright \ldots \triangleright (spo(2|3), -\frac{3}{4}) \triangleright (sl(2), 1)$

(4.15)
$(osp(5|n), \frac{n-1}{2}) \triangleright (osp(1|n), \frac{-n-3}{4}) = (spo(n|1), \frac{-n-3}{4}) \triangleright \ldots \triangleright (spo(2|1), -\frac{5}{4}) \triangleright \mathbb{C}$

(4.16)
$(osp(7|n), \frac{n-3}{2}) \triangleright (osp(3|n), 1+n) \triangleright (4.14)$

(4.17)
m *odd*, $m \geq 8$: $(osp(m|n), \frac{n-m+4}{2}) \triangleright (osp(m-4|n), \frac{8-m+n}{2}) \triangleright \ldots \triangleright$ (4.15) *or* (4.16)

(4.18)
m *even*, $m > n+4, m \equiv n \mod 4$: $(osp(m|n), \frac{n-m+4}{2}) \triangleright (osp(m-4|n), \frac{8-m+n}{2}) \triangleright \ldots \triangleright$ (4.12)

(4.19)
m *even*, $m > n+4, m \equiv n+2 \mod 4$:

$(osp(m|n), \frac{n-m+4}{2}) \triangleright (osp(m-4|n), \frac{8-m+n}{2}) \triangleright \ldots \triangleright (osp(n+6|n), -1) \triangleright (osp(n+2|n), 1)$

(4.20)
$(psl(m|m), -1) \triangleright \mathbb{C}$

(4.21)
$m \neq n, n+1, n+2, m \geq 2, (sl(m|n), -1) \triangleright (M(1), 1)$

(4.22)
n *odd* $(sl(2|n), n/2-1) \triangleright (sl(n), -n/2) \triangleright \ldots \triangleright (sl(3), -3/2) \triangleright \mathbb{C}$

(4.23)
n *even*

$(sl(3|n), \frac{n-3}{2}) \triangleright (sl(1|n), \frac{1-n}{2}) = (sl(n|1), \frac{1-n}{2}) \triangleright (sl(n-2|1), \frac{3-n}{2}) \triangleright \cdots \triangleright (sl(2|1), -\frac{1}{2}) \triangleright \mathbb{C}$

(4.24)
$3 < m, n$, m *even*, n *odd*

$(sl(m|n), \frac{n-m}{2}) \triangleright (sl(m-2|n), \frac{n-m}{2}+1) \triangleright \ldots \triangleright (sl(2|n), \frac{n}{2}-1) \triangleright$ (4.22)

(4.25)
$3 < m, n$, m *odd*, n *even*

$(sl(m|n), \frac{n-m}{2}) \triangleright (sl(m-2|n), \frac{n-m}{2}+1) \triangleright \ldots \triangleright (sl(3|n), \frac{n-3}{2}) \triangleright$ (4.23)

(4.26)
$3 < n < m, \ m \equiv n \mod 2$

$(sl(m|n), \frac{n-m}{2}) \triangleright (sl(m-2|n), \frac{n-m}{2}+1) \triangleright \ldots \triangleright (sl(n+2|n), -1) = (sl(n|n+2), 1)$

(4.27)
$(D(2,1;a), a) \triangleright (sl(2), -\frac{1+2a}{1+a})$

(4.28)
$(D(2,1;a), -a-1) \triangleright (sl(2), -\frac{1+2a}{a})$

Proof. The proof of the proposition is based on the data in Table 2. This table is built up by using the data computed in [6, Tables 5,6,7]. Case (4.5) is proven by looking at lines 19 and 17 in Table 2. A similar direct analysis works in cases (4.6) (lines 21, 15), (4.7) (lines 23, 28), (4.8) (line 10), (4.9) (line 11), (4.10) (line 12), (4.12) (line 16), (4.13) (line 15), (4.16) (line 13′), (4.18) and (4.19) (line 13),(4.20) (line 7), (4.21) (line 6), (4.27) (line 17) , (4.28) (line 18).

Cases (4.11) Since n is even and m is odd, $(m-n-2)/4$ is never a integer, so $H(V_{(m-n-2)/4}) \neq 0$; use line 8 up to arriving to $(spo(2|m), \frac{m-6}{4})$. Using line 8 prevents to consider the case $(spo(m-2, m), -\frac{1}{2})$: on the other hand in this case we have collapsing to $\mathbb{C}$ by line 12. If $m \geq 5$ by line 9 we arrive at $(so(m), \frac{4-m}{2})$ which collapses according to Lemma 4.10. For $m = 3$ we have $(spo(2|3), -\frac{3}{4})$ which collapses to $(sl(2), 1)$ by line 10. For $m = 1$ we have $(spo(2|1), -\frac{5}{4})$ which collapses to $\mathbb{C}$ by line 11.

vskip5pt **Case** (4.14)**.** Use line 8 and (4.8).

Case (4.15)**.** Use lines 14, 8 and (4.9).

Case (4.17)**.** Use line 13 up to arriving to (4.15), (4.16).

Case (4.22)**.** Use lines 4, 1, 3. Since n is odd, all the levels involved are non-integral.

Case (4.23)**.** Use lines 2, 1, 5. Since n is even, all the levels involved are non-integral.

Cases (4.24)**,** (4.25)**.** These cases are reduced to (4.22), (4.23) by using (4.22), (4.23).

Cases (4.26) Use line 1 (note that the levels involved are negative, hence the functor H is nonzero).

□

Corollary 4.12. *In cases* (4.5)–(4.12), (4.13) *with* $m-n\geq 5$, (4.14)–(4.26) *the category* KL_k^{fin} *is semisimple. If* $a\notin\mathbb{Q}$ *or* $a=\frac{q}{p}-1, p,q\in\mathbb{Z}_{\geq 0}$, $p\geq 1$ *(resp.* $a=-\frac{q}{p}$*), the category* KL_a^{fin} *(resp.* KL_{-a-1}^{fin}*) for* $D(2,1;a)$ *is semisimple.*

Proof. We use Theorem 4.9. In cases (4.5), (4.6), the final level in the collapsing chain is admissible. In case (4.13) the final vertex algebra is rational if $m-n\geq 7$ is even and admissible if it is odd. The cases $m-n=5,6$ are covered by [19, Theorem 4.4.1]. In case (4.27), the vertex algebra $V_{k'}(sl(2))$ is

(1) rational or admissible if $a=\frac{q}{p}-1$, $p,q\in\mathbb{Z}_{\geq 0}$, $p\geq 2$, and $KL_{k'}$ is semi-simple by [11],
(2) isomorphic to $V_{-2+1/q}(sl(2))$ if $p=1$, and $KL_{k'}$ is semi-simple by [19, Theorem 4.4.1],
(3) generic if $a\notin\mathbb{Q}$, and $KL_{k'}$ is semi-simple by [36].

Similarly for (4.28). In all other cases except $V_1(osp(n+2|n), V_1(sl(n|n+2))$ the final vertex algebra in the collapsing chain is rational. In the two above special cases, semisimplicity of KL_k^{fin} is given by the following arguments:

- Recall from [9, Proposition 4.8] that the following decomposition holds:

$$V_1(osp(2m|n)) = V_1(so(2m))\otimes V_{-1/2}(sp(n)) + V_1(\omega_1)\otimes V_{-1/2}(\omega_1),$$

  and the embedding $V_1(so(2m))\otimes V_{-1/2}(sp(n))\hookrightarrow V_1(osp(2m|n))$ is conformal. Since $V_1(osp(2m))$ is rational and $V_{-1/2}(sp(2n))$ is admissible, we can apply Proposition 5.1 (2) below to obtain $KL_k^{fin}=KL_k$.

  Assume that W is any highest weight $V_1(osp(2m|n))$–module in KL_k. Then W is a completely reducible as $V_1(so(2m))\otimes V_{-1/2}(sp(n))$ and it contains an irreducible $V_1(so(2m))\otimes V_{-1/2}(sp(n))$–submodule isomorphic to the exactly one of the following modules

$$\begin{aligned} M_0 &\cong V_1(so(2m))\otimes V_{-1/2}(sp(n)),\\ M_1 &\cong V_1(\omega_1)\otimes V_{-1/2}(\omega_1),\\ M_2 &\cong V_1(\omega_1)\otimes V_{-1/2}(sp(n)),\\ M_3 &\cong V_1(so(2m))\otimes V_{-1/2}(\omega_1).\end{aligned}$$

  Using the fusion rules arguments as in the proof of [9, Proposition 4.8] we easily get that W is isomorphic to exactly one of the following two irreducible modules

$$W_0\cong M_0\oplus M_1,\ W_1=M_2\oplus M_3.$$

TABLE 2.

Values of k and k'. Assume that $k \notin \mathbb{Z}_{\geq 0}$.

	$\mathfrak{g}$	$V_{k'}(\mathfrak{g}^\natural)$	k	k'
1	$sl(m\|n),\ m \neq n, m > 3, m-2 \neq n$	$V_{k'}(sl(m-2\|n)))$	$\frac{n-m}{2}$	$\frac{n-m+2}{2}$
2	$sl(3\|n),\ n \neq 3, n \neq 1, n \neq 0$	$V_{k'}(sl(1\|n)))$	$\frac{n-3}{2}$	$\frac{1-n}{2}$
3	$sl(3)$	$\mathbb{C}$	$-\frac{3}{2}$	0
4	$sl(2\|n),\ n \neq 2, n \neq 1, n \neq 0$	$V_{k'}(sl(n)))$	$\frac{n-2}{2}$	$-\frac{n}{2}$
5	$sl(2\|1) = spo(2\|2)$	$\mathbb{C}$	$-\frac{1}{2}$	0
6	$sl(m\|n),\ m \neq n, n+1, n+2, m \geq 2$	$M(1)$	-1	1
7	$psl(m\|m),\ m \geq 2$	$\mathbb{C}$	-1	0
8	$spo(n\|m), m \neq n, n+2, n \geq 4$	$V_{k'}(spo(n-2\|m))$	$\frac{m-n-4}{4}$	$\frac{m-n-2}{4}$
9	$spo(2\|m), m \geq 5$	$V_{k'}(so(m))$	$\frac{m-6}{4}$	$\frac{4-m}{2}$
10	$spo(2\|3)$	$V_{k'}(sl(2))$	$-\frac{3}{4}$	1
11	$spo(2\|1)$	$\mathbb{C}$	$-\frac{5}{4}$	0
12	$spo(n\|m), m \neq n+1, n \geq 2$	$\mathbb{C}$	$-1/2$	0
13	$osp(m\|n), m \neq n, m \neq n+8, m \geq 8$	$V_{k'}(osp(m-4\|n))$	$\frac{n-m+4}{2}$	$\frac{8-m+n}{2}$
$13'$	$osp(7\|n)$	$V_{k'}(osp(3\|n))$	$\frac{n-3}{2}$	$1+n$
14	$osp(m\|n), n \neq m, 0; 4 \leq m \leq 6$	$V_{k'}(osp(m-4\|n))$	$\frac{n-m+4}{2}$	$\frac{m-n-8}{4}$
15	$osp(m\|n), m \neq n+4, n+8; m \geq 4$	$V_{k'}(sl(2))$	-2	$\frac{m-n-8}{2}$
16	$osp(n+8\|n), n \geq 0$	$\mathbb{C}$	-2	0
17	$D(2,1;a)$	$V_{k'}(sl(2))$	a	$-\frac{1+2a}{1+a}$
18	$D(2,1;a)$	$V_{k'}(sl(2))$	$-a-1$	$-\frac{1+2a}{a}$
19	$F(4)$	$V_{k'}(D(2,1;2))$	-1	$\frac{1}{2}$
20	$F(4)$	$\mathbb{C}$	$-3/2$	0
21	$F(4)$	$V_{k'}(so(7))$	$\frac{2}{3}$	-2
22	$F(4)$	$\mathbb{C}$	$-\frac{2}{3}$	0
23	$G(3)$	$V_{k'}(G_2)$	$\frac{1}{2}$	$-\frac{5}{3}$
24	$G(3)$	$\mathbb{C}$	$-\frac{3}{4}$	0
25	$G(3)$	$V_{k'}(osp(3\|2))$	$-\frac{2}{3}$	1
26	$G(3)$	$\mathbb{C}$	$-\frac{4}{3}$	0
27	G_2	$V_{k'}(sl(2))$	$-\frac{4}{3}$	1
28	G_2	$\mathbb{C}$	$-\frac{5}{3}$	0

Therefore $V_1(osp(2m|n))$ has two irreducible modules in KL_k and every highest weight module in KL_k is irreducible. Now using Theorem 4.3 we have that KL_k^{fin} is semisimple. In particular, for $V_1(osp(n+2|n))$, the category $KL_k^{fin} = KL_k$ is semisimple.

- $V_1(sl(n|n+2)) = V_{-1}(sl(n+2|n))$, and by results of Section 7.2 we have semisimplicity in KL_k^{fin}.

□

5. Category KL_k of $\mathfrak{g}$–locally finite $V_k(\mathfrak{g})$–modules

We first investigate some sufficient conditions to have either $KL_k^{ss} = KL_k^{fin}$ or $KL_k = KL_k^{fin}$.

Proposition 5.1. (1) *Assume that $\mathfrak{g}_{\bar{0}}$ is a semisimple Lie algebra. Then $KL_k^{fin} = KL_k^{ss}$.*
(2) *Assume that there is a conformal embedding of $V_{k_1}(\mathfrak{g}_{\bar{0}}) \hookrightarrow V_k(\mathfrak{g})$ and every module W from KL_k is semisimple as a $V_{k_1}(\mathfrak{g}_{\bar{0}})$–module. Then $KL_k^{fin} = KL_k$.*

Proof. Consider case (1). Assume that W is any module from KL_k^{ss}. We need to show that $\mathfrak{h}$ acts semisimply on W. Each $L(0)$–eigenspace of W is a sum of finite-dimensional $\mathfrak{g}$–modules, therefore W is a sum of finite-dimensional $\mathfrak{g}$–module. Since there is an embedding $V_{k_1}(\mathfrak{g}_{\bar{0}}) \hookrightarrow V_k(\mathfrak{g})$ and $\mathfrak{g}_{\bar{0}}$ is semisimple, we conclude that W is a direct sum of finite-dimensional $\mathfrak{g}_{\bar{0}}$–modules. Since the action of the Cartan subalgebra $\mathfrak{h}$ is obtained by the action of operators from $V_{k_1}(\mathfrak{g}_{\bar{0}})$, we conclude that these operators act semisimply and therefore W is in KL_k^{fin}.
Now we consider the case (2). Let W be a module from KL_k. We have directly that W is a semisimple as $V_{k_1}(\mathfrak{g}_{\bar{0}})$–module. So W is a direct sum of irreducible $V_{k_1}(\mathfrak{g}_{\bar{0}})$–modules in KL_{k_1}, which are highest weight modules, and therefore, since the embedding is conformal, we get that $\mathfrak{h}$ and $L(0)$ must act semisimply. The claim follows. □

We have the following consequence:

Corollary 5.2. *Assume that the condition (2) of Proposition 5.1 holds and that*
(3) *Any highest weight $V_k(\mathfrak{g})$–module in KL_k is irreducible.*
Then KL_k is semisimple.

Proof. The assumption (2) implies that the Cartan algebra and the Virasoro element $L(0)$ acts semisimply. This implies that $KL_k = KL_k^{fin}$. Then the result follows by applying Theorem 4.3. □

The conformal embeddings $\mathfrak{g}_{\bar{0}} \hookrightarrow \mathfrak{g}$ were classified in [9], and they include all collapsing levels for Lie superalgebras. Only in some cases when $\mathfrak{g}_{\bar{0}}$ is reductive, the semisimplicity of $V_k(\mathfrak{g}_{\bar{0}})$ is still an open problem.

Lemma 5.3. *Let M be a non-zero $V_k(\mathfrak{g})$–module from KL_k. Then there is a non-zero $V_k(\mathfrak{g})$–submodule $M^{fin} \subset M$ which belongs to KL_k^{fin}.*

Proof. Since M_{top} is a locally finite $\mathfrak{g}$–module, we get that $\mathfrak{h}$ and $L(0)$ acts locally finitely on M_{top}. So there is a common eigenvector w for the action of $\mathfrak{h}$ and $L(0)$. Therefore $M^{fin} = V_k(\mathfrak{g}).w$ is a $V_k(\mathfrak{g})$–submodule of M which is in the category KL_k^{fin}. □

Lemma 5.4.
(1) *Let $\mathcal{M}$ be a logarithmic $V_k(\mathfrak{g})$–module in KL_k. Then $L(0) - L_{ss}(0)$ is a $V_k(\mathfrak{g})$–homomorphism.*

(2) *For any module $\mathcal{M}$ in KL_k, the operator $h(0) - h_{ss}(0)$ is a $V_k(\mathfrak{g})$–homomorphism for any $h \in \mathfrak{h}$.*

Proof. The assertion (1) is already proved in [27, Remark 2.21]. For completeness, we present here a version of their proof.

We have

$$\mathcal{M} = \bigoplus_{\alpha \in \mathbb{C}} \mathcal{M}_\alpha, \ \mathcal{M}_\alpha = \{v \in \mathcal{M} \mid (L(0) - \alpha)^{N_\alpha} v = 0 \text{ for some } N_\alpha > 0\}.$$

Define $Q \in \mathrm{End}(\mathcal{M})$ by

$$Qv = (L(0) - \alpha)v, \quad v \in \mathcal{M}_\alpha.$$

Therefore $Q = L(0) - L_{ss}(0)$. Take $v \in \mathcal{M}_\alpha$ and $n \in \mathbb{Z}$. Then for each $N \in \mathbb{Z}_{>0}$ we have

$$(L(0) - (\alpha - n))^N x(n)v = x(n)(L(0) - \alpha)^N v$$

which gives that $x(n)v \in \mathcal{M}_{\alpha - n}$. Therefore

$$x(n)Qv = x(n)L(0)v - \alpha x(n)v = L(0)x(n)v - (\alpha - n)x(n)v = Qx(n)v,$$

which implies that Q is a $\widehat{\mathfrak{g}}$–homomorphism, hence a $V_k(\mathfrak{g})$–homomorphism. This proves (1). The proof of (2) is completely analogous. □

Theorem 5.5. *Assume that the category KL_k^{fin} is semisimple and that for any irreducible $V_k(\mathfrak{g})$–module M in KL_k we have*

$$Ext^1(M_{top}, M_{top}) = \{0\} \tag{5.1}$$

in the category of finite-dimensional $\mathfrak{g}$–modules. Then KL_k is semisimple and $KL_k^{fin} = KL_k$.

Proof. In a view of [25, Lemma 1.3.1] it suffices to show that

(1) $\mathrm{Ext}^1(M, N) = \{0\}$ for any two irreducible modules M, N in KL_k;
(2) Each module M in KL_k contains an irreducible submodule.

Assume that we have a non-split extension

$$0 \to M \to M^{ext} \to N \to 0 \tag{5.2}$$

for a certain $\mathbb{Z}_{\geq 0}$–gradable module M^{ext} in KL_k. If M^{ext} is in KL_k^{fin}, then $M^{ext} \cong M \oplus N$ because KL_k^{fin} is semisimple. This contradicts the assumption that M^{ext} is the non-split extension (5.2).

If M^{ext} does not belong to KL_k^{fin}, then $L(0)$ does not act semisimply or there is $h \in \mathfrak{h}$ such that $h(0)$ does not act semisimply. Define accordingly the operator Q as in Lemma 5.4, i.e. $Q = L(0) - L(0)_{ss}$ or $Q = h(0) - h(0)_{ss}$. Then Q is a non-zero $V_k(\mathfrak{g})$–homomorphism and therefore $N \cong QM^{ext} \cong M$. Indeed, since M is irreducible, we have that $QM = 0$. It follows that $QM^{ext} \cong M^{ext}/Ker\, Q \subset M^{ext}/M \cong N$. Since $N \neq 0$, we find $QM^{ext} \cong N$. If the extension does not split, we must have $QM^{ext} \cap M \neq 0$, so $QM^{ext} \cong M$, and at the end $M \cong N$. By applying the Zhu's functor to (5.2) we get a non-split extension

$$0 \to M_{top} \to (M^{ext})_{top} \to M_{top} \to 0$$

in the category of finite-dimensional $\mathfrak{g}$–modules. This contradicts (5.1). Thus (1) holds.

Let us prove (2). From Lemma 5.3 we get that M contains a non-zero submodule M^{fin} in KL_k^{fin}. Since KL_k^{fin} is semisimple, we conclude that M^{fin} contains an irreducible submodule. The claim follows. □

6. The case $\mathfrak{g} = C(n+1)$

6.1. Collapsing level $k = -\frac{1}{2}$.

Lemma 6.1. *Let $\mathfrak{g} = C(n+1)$ and $k = -\frac{1}{2}$. Then the unique irreducible modules in $KL_{-\frac{1}{2}}$ are $V_{-\frac{1}{2}}(\mathfrak{g})$ and $L(-\frac{1}{2}\Lambda_0 + \frac{1}{2}\theta)$, where θ is the highest root of $sp(2n)$.*

Proof. The discussion preceding [8, Lemma 4.1] applies: if $L(\Lambda)$ is irreducible in $KL_{-\frac{1}{2}}$ then $\Lambda = -\frac{1}{2}\Lambda_0 + \ell\theta$ and $\ell^2 - (k+1)\ell = 0$. □

Theorem 6.2. *Let $\mathfrak{g} = C(n+1)$ and $k = -\frac{1}{2}$. Then we have:*

- *Irreducible modules in KL_k have no self-extensions.*
- *The category KL_k^{fin} is semisimple and $KL_k = KL_k^{fin}$.*

Proof. It suffices to check that condition (2) of Proposition 5.1 and condition (3) of Corollary 5.2 hold.

Let us first check that:

(*) any $V_k(\mathfrak{g})$–module W in KL_k is completely reducible as $V_k(\mathfrak{g}_{\bar{0}}) = V_{-1/2}(sp(2n)) \otimes M(1)$–module, where $M(1)$ is the Heisenberg vertex algebra of rank one.

Let $V_D = M(1) \otimes \mathbb{C}[D]$ be the lattice vertex algebra associated to rank one lattice $D = \mathbb{Z}\alpha, \langle \alpha, \alpha \rangle = 4$. It has 4 non-isomorphic modules:

$$U_i = V_{D + \frac{i}{4}\alpha}, i = 0, 1, 2, 3,$$

and the following fusion rules:

$$U_i \times U_j = U_{(i+j) \mod 4}.$$

From [9, Proposition 4.15], we have that

$$V_{-\frac{1}{2}}(\mathfrak{g}) = (V_{-1/2}(sp(2n)) \otimes U_0) \bigoplus (L_{-1/2}(\omega_1) \otimes U_2)$$

(note that in *loc. cit.* a different normalization is used, so that the level 1 used there turns into level $-1/2$). The irreducible module $L(-\frac{1}{2}\Lambda_0 + \frac{1}{2}\theta)$ decomposes as

$$L(-\tfrac{1}{2}\Lambda_0 + \tfrac{1}{2}\theta) = V_{-1/2}(sp(2n)) \otimes U_1 \bigoplus L_{-1/2}(\omega_1) \otimes U_3.$$

By using the regularity of V_D (i.e., complete reducibility in the entire category of weak V_D–modules, cf. [21]) and the concept of Heisenberg coset (cf. [18]), we get that

$$W = M_0 \otimes U_0 \bigoplus M_1 \otimes U_1 \bigoplus M_2 \otimes U_2 \bigoplus M_3 \otimes U_3$$

for certain $V_{-1/2}(sp(2n))$–modules M_i in KL_k, $i = 0, 1, 2, 3$. By using complete reducibility for the admissible vertex algebra $V_{-1/2}(sp(2n))$ in the category $\mathcal{O}$ (cf. [1]), we get that the M_i are direct sum of copies of $V_{-1/2}(sp(2n))$ or $L_{-1/2}(\omega_1)$. This implies that W is a direct sum of irreducible $V_{-1/2}(sp(2n)) \otimes U_0$–modules. Since each irreducible U_0–module is a direct sum of irreducible modules for the Heisenberg vertex algebra $M(1)$, we get that W is a direct sum of $V_k(\mathfrak{g}_{\bar{0}})$–modules. So (*) holds.

It remains to prove that any highest weight $V_k(\mathfrak{g})$–module in KL_k is irreducible. By using the same arguments as above and fusion rules, we see that if W is a highest weight module in KL_k, it decomposes as

$$V_{-1/2}(sp(2n)) \otimes U_0 \bigoplus L_{-1/2}(\omega_1) \otimes U_2 \quad \text{or} \quad V_{-1/2}(sp(2n)) \otimes U_1 \bigoplus L_{-1/2}(\omega_1) \otimes U_3.$$

Therefore it is irreducible. The claim follows. □

6.2. **Collapsing level $k=-\frac{n+1}{2}$.**

Theorem 6.3. *Assume that $\mathfrak{g}=C(n+1)$, $k=-\frac{n+1}{2}$. Then KL_k^{fin} is semisimple.*

Proof. First we consider the case $n=1$, so $\mathfrak{g}=C(2)\cong sl(2|1)$. Then $k=-1$ is the critical level. Recently T. Creutzig and J. Yang in [19, Theorem 6.6] proved that in this case every module in KL_k^{fin} is completely reducible (to match Creutzig-Yang's result with our setting note that $V_{-1}(sl(m|n))=V_1(sl(n|m))$). Therefore we have the collapsing chain

$$(C(n+1),-\frac{n+1}{2})\triangleright(C(n),-\frac{n}{2})\triangleright\ldots\triangleright(C(2),-1)$$

implying that KL_k^{fin} is semisimple. □

7. Semisimplicity of KL_{-1} for $\mathfrak{g}=\mathfrak{sl}(m|n)$ and $\mathfrak{g}=\mathfrak{psl}(n|n)$.

In $sl(n|m), n\neq m$ set $\alpha_i^\vee=E_{ii}-E_{i+1i+1}$ for $i\neq n$ and $\alpha_n^\vee=E_{nn}+E_{n+1n+1}$ (E_{ij} are matrix units). Define $\omega_i\in\mathfrak{h}^*$ by setting $\omega_i(\alpha_j^\vee)=\delta_{ij}$ and $\omega_0=0$. When $n=m$, we work modulo the identity.

Recall that the defect def $\mathfrak{g}$ of a basic classical Lie superalgebra $\mathfrak{g}$ is the dimension of a maximal isotropic subspace in the real span of roots. When $\mathfrak{g}=sl(m|n)$ the defect is $\min\{m,n\}$. Also recall that the atypicality of a weight λ is the maximal number of linearly independent mutually orthogonal isotropic roots which are also orthogonal to λ. The atypicality of an irreducible finite dimensional $\mathfrak{g}$-module V of highest weight λ is the atypicality of $\lambda+\rho$ (here ρ is the half sum of positive even roots minus the half sum of positive odd roots).

In [32] it is shown that the atypicality does not depend on the choice of the set of positive roots. Let $L_{\mathfrak{g}_{\bar 0}}(\lambda)$ be the finite dimensional irreducible $\mathfrak{g}_{\bar 0}$–module of highest weight λ. The following conditions are equivalent [28]:

1. $\lambda+\rho$ is atypical.
2. The Kac module $Ind_{\mathfrak{g}_{\bar 0}}^{\mathfrak{g}}(L_{\mathfrak{g}_{\bar 0}}(\lambda))$ is not irreducible.

It turns out that the atypicality of the trivial module, i.e. that of ρ, is def $\mathfrak{g}$.

7.1. **The representation theory of $V_1(psl(m|m))$ via decomposition of conformal embedding.** Let us consider first the case $V=V_1(psl(m|m))$ for $m\geq 3$. For $s\in\mathbb{Z}_{\geq 0}$ consider the following $V_{-1}(sl(m))$-modules

$$\pi_s:=L_{-1}(s\omega_1),\quad \pi_{-s}:=L_{-1}(s\omega_{m-1}).$$

In order to prove that V is semisimple in KL_1, we need to prove the following:

Proposition 7.1. *Assume that M is any highest weight V–module in KL_1. Then $M\cong V$.*

Proof. Since $V_1(sl(m))$ is rational and $KL_{-1}(sl(m))$ is semisimple [8, Theorem 1.2] we conclude that M is completely reducible as $U=V_1(sl(m))\otimes V_{-1}(sl(m))$–module. This implies that M contains a U–submodule isomorphic to $U_{r_0,s_0}=L_1(\omega_{r_0})\otimes\pi_{-s_0}$, for some $0\leq r_0\leq n-1$, and $s_0\in\mathbb{Z}$. (For $r_0=0$, we set $L_1(\omega_0)=V_1(sl(m))$.

The lowest conformal weight of $U_{r,s}$ and $U_{r,-s}$ are

$$\begin{aligned}
h[r,s] &= \frac{(\omega_r,\omega_r+2\rho)}{2(m+1)}+\frac{(s\omega_{m-1},s\omega_{m-1}+2\rho)}{2(m-1)}\\
&= \frac{(m-r)r}{2m}+\frac{s^2+ms}{2m}=\frac{(m-r)r+s(s+m)}{2m}.\\
&= \frac{(\omega_r,\omega_r+2\rho)}{2(m+1)}+\frac{(s\omega_1,s\omega_1+2\rho)}{2(m-1)}=h[r,-s]
\end{aligned}$$

We can choose (r_0, s_0) so that the conformal weight of U_{r_0,s_0} coincides with the conformal weight of the highest weight vector of M.

For $s \in \mathbb{Z}$, let $\overline{s} \in \{0, 1, \dots, m-1\}$ be such that $s \equiv \overline{s} \mod (m-1)$. For $0 \le r_1, r_2 \le m-1$, we set $r_3 = \overline{r_1 + r_2}$. Recall from [9, Theorem 4.4] that

$$V = \bigoplus_{s \in \mathbb{Z}} U_{\overline{s},s},$$

and that V is generated by $V_1(sl(m)) \otimes V_{-1}(sl(m)) \oplus U_{1,1} \oplus U_{n-1,-1}$.

Using the fusion rules:

$$L_1(\omega_1) \times L_1(\omega_r) = L_1(\omega_{\overline{r+1}}),$$
$$L_1(\omega_{m-1}) \times L_1(\omega_r) = L_1(\omega_{\overline{m-1-r}}),$$
$$\pi_s \times \pi_{s'} = \pi_{s+s'},$$

we conclude that M also contains submodules:

$$U_{\overline{r_0+\ell},s_0+\ell} \quad (\ell \in \mathbb{Z}). \tag{7.1}$$

Since M is a highest weight module, it is generated by U_{r_0,s_0}

$$M = V.U_{r_0,s_0} = \bigoplus_{\ell \in \mathbb{Z}} U_{\overline{r_0+\ell},s_0+\ell}.$$

We conclude that M is $\mathbb{Z}_{\ge 0}$–graded. By a direct calculation of the lowest conformal weight of $U_{\overline{r+\ell},s+\ell}$ we get that the following statements are equivalent:

1. M is $\mathbb{Z}$–graded;
2. $h[r+1, s+1] - h[r, s] = \frac{m-r+s}{m} \in \mathbb{Z}$ for $r \in \{0, 1, \dots, m-2\}$, $s \in \mathbb{Z}$;
   $h[r-1, s-1] - h[r, s] = -\frac{m-r+s}{m} \in \mathbb{Z}$ for $r \in \{1, 2, \dots, m-1\}$, $s \in \mathbb{Z}$;
3. $r \equiv s \mod n$.

But then M contains a $V_1(sl(m)) \otimes V_{-1}(sl(m))$–submodule isomorphic to $U_{r,r}$, implying that $M = V$. The claim follows. $\square$

Since $\mathfrak{g}_{\bar{0}} = sl(m) \times sl(m)$ is semisimple, and the categories KL_1 and KL_{-1} are semisimple for $sl(m)$, using Proposition 7.1 we conclude:

Theorem 7.2. *Assume that $\mathfrak{g} = psl(m|m)$ for $m \ge 3$ and $k = -1$. The category KL_k is semisimple.*

7.2. **The case $V_{-1}(sl(m|1))$.** Let $\mathfrak{g} = sl(m|1)$. Recall [9] that there is a conformal embedding $V_{-1}(sl(m)) \otimes M_c(1) \hookrightarrow V_{-1}(\mathfrak{g})$ with the following decomposition:

$$V_{-1}(\mathfrak{g}) = \bigoplus_{q \in \mathbb{Z}} \pi_q \otimes M_c(1, -q\sqrt{\tfrac{m-1}{m}}).$$

where we set

$$c = \frac{1}{\sqrt{m(m-1)}} \begin{pmatrix} I_m & 0 \\ 0 & m \end{pmatrix}.$$

so that $[c_\lambda c] = \lambda$.

For $\ell \in \mathbb{Z}$ and $r \in \mathbb{C}$, we define the following $V_{-1}(sl(m)) \otimes M_c(1)$–module:

$$L[\ell, r] = \bigoplus_{q \in \mathbb{Z}} \pi_{q+\ell} \otimes M_c(1, -(q+r)\sqrt{\tfrac{m-1}{m}}).$$

Note that for each $s \in \mathbb{Z}$ we have $L[\ell + s, r + s] = L[\ell, r]$.

Recall that $V_{-1}(sl(m|n))$ is realized as a vertex subalgebra of $M_m \otimes F$, where M_m is the Weyl vertex algebra with generators $a_i^{\pm}$, $i = 1, \dots, m$, and F_n is the Clifford vertex algebra with generators $\Psi_i^{\pm}$ (cf. [31], [9]).

In this section we set $F = F_1$, $\Psi^{\pm} = \Psi_1^{\pm}$. We consider $V_{-1}(\mathfrak{g})$ as subalgebra of $M_m \otimes F$.

Proposition 7.3. *For every $\ell \in \mathbb{Z}$, $L[\ell, -\frac{\ell}{m-1}]$ has the structure of an irreducible $V_{-1}(\mathfrak{g})$–module. It is realized as*

$$L[\ell, -\tfrac{\ell}{m-1}] = V_{-1}(\mathfrak{g}) w_\ell$$

where $w_\ell =: (a_1^+)^{\ell-1} : \Psi^+$ for $\ell \in \mathbb{Z}_{>0}$ and $w_\ell =: (a_m^-)^{|\ell|} :$ for $\ell \in \mathbb{Z}_{\le 0}$.

Moreover, $L[\ell, -\frac{\ell}{m-1}]_{top} = U(\mathfrak{g}).w_\ell$ is an atypical irreducible, finite-dimensional $\mathfrak{g}$–module.

Proof. The results from [31] give that $M_m \otimes F$ is a completely reducible $V_{-1}(gl(m|1))$–module so that

$$M_m \otimes F = \bigoplus_{\ell \in \mathbb{Z}} V_{-1}(gl(m|1)).w_\ell.$$

This implies that $V_{-1}(\mathfrak{g}).w_\ell$ is an irreducible $V_{-1}(\mathfrak{g})$–module. Set $r = -\ell/(m-1)$.

By identifying the highest weights, we get

$$V_{-1}(sl(m)) \otimes M_c(1).w_\ell = \pi_\ell \otimes M_c(1, -r\sqrt{\tfrac{m-1}{m}}).$$

By using fusion rules for $V_{-1}(sl(m)) \otimes M_c(1)$–modules we get:

$$\begin{aligned} V_{-1}(\mathfrak{g}).w_\ell &= V_{-1}(\mathfrak{g}).\pi_\ell \otimes M_c(1, -r\sqrt{\tfrac{m-1}{m}}) \\ &= \bigoplus_{q \in \mathbb{Z}} (\pi_q \otimes M_c(1, -q\sqrt{\tfrac{m-1}{m}})).\pi_\ell \otimes M_c(1, -r\sqrt{\tfrac{m-1}{m}}) \\ &= \bigoplus_{q \in \mathbb{Z}} \pi_{q+\ell} \otimes M_c(1, -(q+r)\sqrt{\tfrac{m-1}{m}}) = L[\ell, r]. \end{aligned}$$

The top component is then an irreducible $\mathfrak{g}$–module $U(\mathfrak{g}).w_\ell$ which has all 1–dimensional weight spaces. Therefore $U(\mathfrak{g}).w_\ell$ can not isomorphic to the Kac module obtained from the corresponding $gl(m)$–module. Therefore $U(\mathfrak{g}).w_\ell$ is atypical. (In Remark 7.5 below we check the atypicality by computing explicitly the highest weights). The claim follows. □

The same argument of Proposition 7.1 yields:

Proposition 7.4. *Let $m \ge 3$. Assume that M is any highest weight $V_{-1}(\mathfrak{g})$–module in KL_{-1}. Then M is irreducible and there is $\ell \in \mathbb{Z}$ such that*

$$M \cong L[\ell, -\tfrac{\ell}{m-1}].$$

In particular, M_{top} is an atypical $\mathfrak{g}$–module.

Proof. Since M is a highest weight $V_{-1}(\mathfrak{g})$–module in KL_k, its top component M_{top} must contain a singular vector w such that

$$V_{-1}(sl(m)) \otimes M_c(1).w \cong \pi_\ell \otimes M_c(1, -r\sqrt{\tfrac{m-1}{m}})$$

for certain $\ell \in \mathbb{Z}$ and $r \in \mathbb{C}$. By using fusion rules for $V_{-1}(sl(m)) \otimes M_c(1)$–modules we get:

$$\begin{aligned} M &= V_{-1}(\mathfrak{g}).\pi_\ell \otimes M_c(1, -r\sqrt{\tfrac{m-1}{m}}) \\ &= \bigoplus_{q \in \mathbb{Z}} \pi_{q+\ell} \otimes M_c(1, -(q+r)\sqrt{\tfrac{m-1}{m}}) = L[\ell, r]. \end{aligned}$$

Let $h[\ell, r]$ denotes the conformal weight of the top component of $\pi_\ell \otimes M_c(1, -r\sqrt{\frac{m-1}{m}})$. It is given by the formula

$$h[\ell, r] = \frac{\ell^2 + |\ell| m}{2m} + r^2 \frac{m-1}{2m}.$$

Since the embedding $V_{-1}(sl(m) \otimes M_c(1) \hookrightarrow V_{-1}(sl(m|1))$ is conformal, the conformal weight of $\pi_{q+\ell} \otimes M_c(1, -(q+r)\sqrt{\frac{m-1}{m}})$ must differ form $h[\ell, r]$ by a positive integer. In particular, we must have the following conditions:

- $h[\ell - 1, r - 1] - h[\ell, r] \in \mathbb{Z}_{>0}$,
- $h[\ell + 1, r + 1] - h[\ell, r] \in \mathbb{Z}_{\geq 0}$.

These relations have the solutions $r = -\frac{\ell+m}{m-1}$ for $\ell \geq 0$ and $r = -\frac{\ell}{m-1}$ for $\ell \leq 0$ (the solution is indeed unique if $\ell \neq 0$). Therefore

$$M \cong L[\ell, -\frac{\ell + m}{m-1}] = L[\ell + 1, -\frac{\ell + 1}{m-1}] \quad (\ell \geq 0),$$
$$M \cong L[\ell, -\frac{\ell}{m-1}] \quad (\ell \leq 0).$$

The atypicality of M_{top} follows from Proposition 7.3.

□

Remark 7.5. *Less conceptually, we can check the atypicality by computing explicitly the highest weights. Identify $\mathfrak{h}^*$ with $\{r_0\delta_1 + \sum_{i=1}^m r_i\varepsilon_i \mid r_0 + \sum_{i=1}^m r_i = 0\}$. Then the highest weight of* $:(a_1^+)^\ell \Psi^+:$ *($\ell > 0$) is*

$$\lambda_\ell^+ := \ell\omega_1 + (m+\ell)\left(-\frac{1}{m(m-1)}(\varepsilon_1 + \cdots + \varepsilon_m) + \frac{1}{m-1}\delta_1\right)$$
$$= \ell(\varepsilon_1 - \frac{1}{m}\sum_{i=1}^m \varepsilon_i) - \frac{m+\ell}{m(m-1)}\sum_{i=1}^m \varepsilon_i + \frac{m+\ell}{m-1}\delta_1$$
$$= (\ell - \frac{\ell}{m} + \frac{\ell + m}{m - m^2})\epsilon_1 + \sum_{i=2}^m \frac{1+\ell}{1-m}\varepsilon_i + \frac{m+\ell}{m-1}\delta_1.$$

Since $\rho = \sum_{i=1}^m (\frac{m}{2} - i + 1)\varepsilon_i - \frac{m}{2}\delta_1$, we have

$$\lambda_\ell^+ + \rho = \frac{m\ell - 2\ell - 1}{m-1}\varepsilon_1 + \sum_{i=1}^m (1 - i + \frac{1+\ell}{1-m} + \frac{m}{2})\varepsilon_i + (\frac{m+\ell}{m-1} - \frac{m}{2})\delta_1$$
$$= \frac{m\ell - 2\ell - 1}{m-1}\varepsilon_1 + (\frac{m}{2} - \frac{m+\ell}{m-1})\varepsilon_2 + \sum_{i=3}^m (1 - i + \frac{1+\ell}{1-m} + \frac{m}{2})\varepsilon_i + (\frac{m+\ell}{m-1} - \frac{m}{2})\delta_1,$$

and $(\lambda_\ell^+ + \rho | \delta_1 - \varepsilon_2) = 0$.

The highest weight of $:(a_m^-)^{-\ell}:$ *is*

$$\begin{aligned}\lambda_\ell^- &:= -\ell\omega_{m-1} + \ell\left(-\frac{1}{m(m-1)}(\varepsilon_1+\cdots+\varepsilon_m) + \frac{1}{m-1}\delta_1\right)\\ &= \ell\left(-\sum_{i=1}^{m-1}\varepsilon_i + \frac{m-2}{m-1}\sum_{i=1}^{m}\varepsilon_i + \frac{1}{m-1}\delta_1\right)\\ &= \ell\left(\frac{1}{1-m}\sum_{i=1}^{m-1}\varepsilon_i + \frac{m-2}{m-1}\varepsilon_m + \frac{1}{m-1}\delta_1\right).\end{aligned}$$

Hence

$$\begin{aligned}\lambda_\ell^- + \rho &= \ell\left(\frac{1}{1-m}\sum_{i=1}^{m-1}\varepsilon_i + \frac{m-2}{m-1}\varepsilon_m + \frac{1}{m-1}\delta_1\right) + \rho\\ &= \sum_{i=1}^{m-1}\left(\frac{m}{2} - i + 1 - \frac{\ell}{m-1}\right)\varepsilon_i + \left(1 + \frac{(m-2)\ell}{m-1} - \frac{m}{2}\right)\varepsilon_m + \left(\frac{\ell}{m-1} - \frac{m}{2}\right)\delta_1\end{aligned}$$

and $(\lambda_\ell^- + \rho|\delta_1 - \varepsilon_1) = 0$.

In the following theorem we need to use results from [24] and [42] (see also [26]), which we recall in our setting. Let $\mathcal{L}^{(k)}$ be the category of finite dimensional $sl(m|1)$-modules on which the center acts with Jordan blocks of size at most k. Let λ, μ be dominant weights, and let ρ_1 be the half sum of the positive odd roots. Combining [24, Proposition 6.1.2. (iii)] and [42, Lemma 6.6] one has

Proposition 7.6. *If* λ *has atypicality* 1*, then*

$$Ext_{\mathcal{L}^{(k)}}(L(\lambda), L(\mu)) = \begin{cases}\mathbb{C} & \text{if } \lambda = \mu \pm 2\rho_1,\\ 0 & \text{otherwise.}\end{cases}$$

Note that the non-trivial extensions above were explicitly realized inside a Kac module, which is a weight module. Since our category KL_k^{fin} is semi-simple, such extensions cannot appear for $V_k(\mathfrak{g})$–modules. We shall now see that there are no non-trivial extensions in the larger category KL_k.

Theorem 7.7. *Assume that* $\mathfrak{g} = sl(m|1)$ *for* $m \geq 2$ *and* $k = -1$*. Let* M *be an irreducible* $V_{-1}(\mathfrak{g})$*–module in* KL_{-1}*. Then*

1. $Ext^1(M_{top}, M_{top}) = \{0\}$ *in the category of finite-dimensional* $\mathfrak{g}$*–modules.*
2. $Ext^1(M, M) = \{0\}$ *in the category* KL_{-1}*.*
3. *The category* KL_k *is semisimple.*

Proof. The classification of irreducible modules in KL_k implies that the top component M_{top} of an irreducible module in KL_k is an irreducible highest weight $\mathfrak{g}$–module. Since def $sl(m|1) = 1$, the atypicality of M_{top} is at most 1. Then assertion (1) follows from Proposition 7.6. The assertions (2) and (3) follow from (1) and semisimplicity of KL_k^{fin} by using Theorem 5.5. □

Remark 7.8. *Theorem 7.7 generalizes to the whole category* $KL_{-1}(sl(m|1))$ *the semisimplicity result of Creutzig and Yang* [19]*, who deal with modules of finite length with semisimple* $\mathfrak{h}$*-action in* KL_{-1} *for* $V_{-1}(sl(m|n)) = V_1(sl(n|m))$ *in the more general case* $m \geq 2$*,* $n \geq 1$*. Both results rely on the classification of irreducible modules in* KL_{-1}*, that we construct using* *the conformal embedding* $V_{-1}(\mathfrak{g}) \hookrightarrow M_m \otimes F$, *whereas Creutzig and Yang use tensor categories and induced modules: see* [19, Corollary 6.11].

8. The category KL_k is not semisimple for $\mathfrak{g} = sl(m|1)$ and $k \in \mathbb{Z}_{>0}$.

Theorem 7.7 shows that indecomposable non-irreducible modules in KL_k do not exist for $k = -1$.

Using Zhu's algebra theory in [26], the authors construct indecomposable weak $V_k(sl(m|1))$–modules for $k = 1$ on which the element $L(0)$ of the Virasoro algebra does not act semisimply (these modules are also called logarithmic modules). Note that the level $k = 1$ is neither conformal nor collapsing for $\mathfrak{g} = sl(m|1)$.

In this section we shall first refine the example presented in [26] and show that even smaller category KL_k^{fin} is not semisimple for $k = 1$. Next we shall extend this result for $k \in \mathbb{Z}_{>0}$.

Recall that the vertex algebra $V_1(\mathfrak{g})$ is realized as a subalgebra of $M \otimes F_m$, where $M = M_1$ is the Weyl vertex algebra generated by $a^{\pm} = a_1^{\pm}$, and F_m the Clifford vertex algebra generated by $\Psi_i^{\pm}$, $i = 1, \dots, m$ (cf. [31]). Even generators of $V_1(\mathfrak{g})$ are realized by

$$E_{i,j} :=: \Psi_i^+ \Psi_j^- :, \quad i, j = 1, \dots, m,$$

and odd generators by

$$E_{1,j+1} :=: a^+ \Psi_j^- :, \ E_{j+1,1} :=: a^- \Psi_j^+ :, \quad j = 1, \dots, m.$$

Define $|m\rangle =: \Psi_1^+ \cdots \Psi_m^+ :\in F_m$. We know from [31] that

$$: (a^+)^{\ell} : \otimes |m > \tag{8.1}$$

is a singular vector for $V_1(\mathfrak{g})$ for each $\ell \in \mathbb{Z}_{\geq 0}$, and it generates an irreducible, highest weight $V_1(\mathfrak{g})$–module. In order to construct indecomposable, highest weight modules, we need to allow that ℓ in the formula (8.1) is a negative integer in certain sense. In order to achieve this we shall consider a larger vertex algebra containing $M \otimes F_m$ such that formula (8.1) makes sense for ℓ negative. Fortunately, there is a nice construction of the vertex algebra $\Pi(0)$ obtained using a localisation of the Weyl vertex algebra M. The vertex algebra $\Pi(0)$ was orginally constructed in [16], and has appeared recently in realisation of certain vertex algebras and their modules [22], [5], [12], [10].

Let $L = \mathbb{Z}c + \mathbb{Z}d$ be the rank two lattice such that

$$\langle c, d\rangle = 2, \ \langle c, c\rangle = \langle d, d\rangle = 0.$$

Let $V_L = M(1) \otimes \mathbb{C}[L]$ be the associated lattice vertex algebra (cf. [30]). Then $\Pi(0)$ is realized as the following subalgebra of V_L:

$$\Pi(0) = M(1) \otimes \mathbb{C}[\mathbb{Z}c].$$

There is an embedding of M into $\Pi(0)$ such that

$$a^+ = e^c, \ a^- = -\frac{c(-1) + d(-1)}{2} e^{-c}.$$

Then e^{-c} plays the role of the inverse $(a^+)^{-1}$ of a^+.

Theorem 8.1. *Define*

$$\widetilde{w} := (a^+)^{-m} \otimes |m\rangle = e^{-mc} \otimes |m\rangle \in \Pi(0) \otimes F_m.$$

Then we have:

- $\widetilde{W} = V_1(\mathfrak{g})\widetilde{w}$ *is a highest weight* $V_1(\mathfrak{g})$*–module in the category* KL_k^{fin}.

- $\widetilde{W}$ *is reducible and it contains a proper submodule isomorphic to* $V_1(\mathfrak{g})$.

In particular, the category KL_k^{fin} *is not semisimple for* $k=1$.

Proof. The proof that $\widetilde{w}$ is a singular vector is the same as in the case of the singular vector (8.1). Therefore $\widetilde{W}$ is an highest weight $V_k(\mathfrak{g})$–module. By using the action of $E_{1,j+1}(0)$, $j=1,\dots,m$ we see that

$$\widetilde{W}_{top} = \mathrm{span}_{\mathbb{C}}\{e^{-sc}\otimes :\Psi^+_{j_1}\dots\Psi^+_{j_s}:\},$$

for $0\le s\le m$, $1\le j_1<\dots<j_s\le m$. So $\dim \widetilde{W}_{top} = 2^m$. This implies that $\widetilde{W}$ is in the category KL_k^{fin}. Moreover $\widetilde{w}$ is a highest weight vector for $\widetilde{W}$, hence $\widetilde{W}$ is indecomposable. Since $\mathbf{1}_{M\otimes F_m} = e^0\otimes \mathbf{1}_{F_m}\in \widetilde{W}_{top}$, we conclude that $V_1(\mathfrak{g})\cong V_1(\mathfrak{g}).\mathbf{1}_{M\otimes F_m}$ is a proper submodule of $\widetilde{W}$. Therefore, $\widetilde{W}$ is reducible and indecomposable $V_k(\mathfrak{g})$–module. □

Remark 8.2. *Note that the building block for a construction of indecomposable* $V_k(\mathfrak{g})$*–module in* KL_k *is the indecomposable* M*–module* $\Pi(0)$*. By using the singular vectors* $(a_1^+)^{-m}\otimes|m\rangle$*, we get indecomposable, weight* $V_1(sl(m|n))$*–modules for every* $n\in\mathbb{Z}_{>0}$*. But one can show that these modules are in the category* KL_k *if and only if* $n=1$*. A more detailed analysis of these modules will appear elsewhere.*

Let now $k\in\mathbb{Z}_{>0}$ is arbitrary. In [26, Corollary 5.4.3], the authors proved that $V_k(\mathfrak{g}) = V^k(\mathfrak{g})/I$, where I is the ideal in $V^k(\mathfrak{g})$ generated by the singular vector $e_\theta(-1)^{k+1}\mathbf{1}$. Now we can combine this result with Theorem 8.1 and show that there exist indecomposable $V_k(\mathfrak{g})$–modules.

Corollary 8.3. *The category* KL_k^{fin} *is not semisimple for any* $k\in\mathbb{Z}_{>0}$.

Proof. It is clear that there is a diagonal action of $V^k(\mathfrak{g})$ on $V_1(\mathfrak{g})^{\otimes k}$. Using [26], one gets that

$$V_k(\mathfrak{g})\cong V^k(\mathfrak{g}).(\underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{k \text{ times}})\subset V_1(\mathfrak{g})^{\otimes k}.$$

As a consequence, we have that $\widetilde{W}\otimes V_1(\mathfrak{g})^{\otimes(k-1)}$ is a $V_k(\mathfrak{g})$–module. Define

$$\begin{aligned}\widetilde{w}^{(k)} &= \widetilde{w}\otimes\underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{(k-1)\text{ times}},\\ \widetilde{W}^{(k)} &= V_k(\mathfrak{g}).\widetilde{w}^{(k)}\subset \widetilde{W}\otimes V_1(\mathfrak{g})^{\otimes(k-1)}.\end{aligned}$$

One easily sees that:

$$\widetilde{W}^{(k)}_{top} = \mathrm{span}_{\mathbb{C}}\{(e^{-sc}\otimes :\Psi^+_{j_1}\dots\Psi^+_{j_s}:)\otimes\underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{(k-1)\text{ times}}\},$$

for $0\le s\le m$, $1\le j_1<\dots<j_s\le m$. We can then argue as in Theorem 8.1 and conclude that $\widetilde{W}^{(k)}$ is indecomposable with $V_k(\mathfrak{g})\cdot(\underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{k\text{ times}})$ a proper submodule. □

Remark 8.4. *Assume that for certain non-integral level* $k\in\mathbb{C}$*, we have*

$$V_k(\mathfrak{g})\hookrightarrow V_{k-1}(\mathfrak{g})\otimes V_1(\mathfrak{g}). \tag{8.2}$$

Then the same proof as above implies that KL_k^{fin} *is not semisimple. Beyond integral levels* $k\ge 2$*, we believe that (8.2) holds for principal admissible levels (cf.* [34], [35]*). As far as we* *know, this is not proved yet. So, one expects that $V_k(\mathfrak{g})$ will be semisimple only for principal admissible levels k such that $k-1$ is not admissible. In the case $\mathfrak{g} = \mathfrak{sl}(2|1)$, such levels will be described in Section 10.*

9. Semisimplicity of $V_{1/2}(psl(2|2))$

In this section we will present an example where KL_k^{fin} is semisimple with k non-collapsing and $W_k(\mathfrak{g}, \theta)$ irrational.

In this section we let $k = 1/2$, $k_0 = -3/2$. Set $V = W_k(psl(2|2), \theta)$. By [33, §8.4], V is isomorphic to the simple $N = 4$ superconformal vertex algebra at central charge $c = -9$. A free-field realization of the vertex algebra V was presented in [4], where it was proved that there is a conformal embedding $V_{k_0}(sl(2)) \hookrightarrow V$. Next, we consider the category of V–modules which belong to $KL_{k_0}(sl(2))$, which we denote by $KL_{N=4}$. Since $KL_{k_0}(sl(2))$ is semi-simple, $KL_{N=4}$ coincides with the category of ordinary V–modules.

Using the same methods as in Section 4 and the fact that for $\mathfrak{g} = sl(2)$ we have $KL_{k_0}(sl(2)) = KL_{k_0}^{fin}(sl(2))$ we get:

Lemma 9.1. *In $KL_{N=4}$ we have:*

$$Ext^1_{KL_{N=4}}(V, V) = \{0\}.$$

Theorem 9.2.

(1) *The category $KL_{N=4}$ is semisimple. In particular, every highest weight V–module in $KL_{N=4}$ is irreducible and isomorphic to V.*
(2) *The category KL_k^{fin} for $V_{1/2}(psl(2|2))$ is semisimple.*

Proof. By [4], V is the unique irreducible V–module in $KL_{N=4}$, Assume that M is a highest weight module in $KL_{N=4}$. By using the relation

$$[e]([\omega] + 1/2) = 0$$

in the Zhu's algebra $A(V)$ from [4, Proposition 3] (with notation used there), one easily sees that M is irreducible and isomorphic V. Now applying Lemma 9.1 we get that $KL_{N=4}$ is semisimple.

Assume now that W is a non-zero highest weight $V_{1/2}(psl(2|2))$–module in KL_k. Then $H(W)$ is a non-zero highest weight V–module in $KL_{N=4}$ and therefore $H(W) \cong V$. Now, using Lemma 4.5, we get that $KL_k^{fin}(psl(2|2))$ is semisimple. □

10. The category KL_k for $sl(2|1)$

In this section let $\mathfrak{g} = sl(2|1)$ and $k = -\frac{m+1}{m+2}$, $m \in \mathbb{Z}_{\geq 0}$. Recall that $W_k(\mathfrak{g}, \theta)$ is isomorphic to the $N = 2$ superconformal vertex algebra, which is rational by [2]. Therefore KL_k^{fin} is semisimple by Theorem 4.6. Let us see that in this case $KL_k = KL_k^{fin}$.

Theorem 10.1. *Let $\mathfrak{g} = sl(2|1)$ and $k = -\frac{m+1}{m+2}$, $m \in \mathbb{Z}_{\geq 0}$. Then KL_k is semisimple.*

Proof. Note that $V_k(\mathfrak{g})$ has a subalgebra isomorphic to the affine vertex algebra $\widetilde{V}_k(sl(2))$, which is a certain quotient of $V^k(sl(2))$. Theorem 10.6 below implies that there is a conformal embedding $U = \widetilde{V}_k(sl(2)) \otimes W \hookrightarrow V_k(\mathfrak{g})$, where W is isomorphic to the regular vertex algebra $D_{m+1,2}$ from [3], also investigated recently in [40, 41]. In the case $m = 0$, the assertion is already proved in Theorem 6.2 and the vertex algebra $D_{1,2}$ is the lattice vertex algebra V_D. Since $\mathfrak{g}_{\bar{0}} \cong sl(2) \oplus \mathbb{C}H^+$, and the center $\mathbb{C}H^+$ belongs to W (see (10.2)), we have that

$V_k(\mathfrak{g}_{\bar 0}) \subset U$. Note that the $sl(2)$ subalgebra of $\mathfrak{g}_{\bar 0}$ acts semisimply on any module M from KL_k^{ss}. Since $H^+ \in W$ and W is regular, we get that M is completely reducible as a module for the Heisenberg vertex algebra generated by H^+, which we denote by $M_{H^+}(1)$. Therefore, M is a sum of $\widetilde{V}_k(sl(2)) \otimes M_{H^+}(1)$–modules in KL_k^{ss} with semisimple action of $sl(2)$. This implies that the Cartan subalgebra of $\mathfrak{g}$ acts semisimply on any module in Kl_k^{ss}. Hence $KL_k^{ss} = KL_k^{fin}$.

Assume M and N are irreducible $V_k(\mathfrak{g})$–module in KL_k and that we have an extension

$$0 \to M \to M^{ext} \to N \to 0 \tag{10.1}$$

for a certain $\mathbb{Z}_{\geq 0}$–gradable module M^{ext} in KL_k. We have the following cases:

- If M^{ext} is in $KL_k^{ss} = KL_k^{fin}$, then $M^{ext} \cong M \oplus N$ because KL_k^{fin} is semisimple.
- If M^{ext} is logarithmic, then $Q = L(0) - L_{ss}(0)$ is a $V_k(\mathfrak{g})$–homomorphism and therefore $N \cong QM \cong M$. So we can assume that $M = N$. Then applying Zhu's functor we get an extension
$$0 \to M_{top} \to (M^{ext})_{top} \to M_{top} \to 0.$$
We proved above that $H^+(0)$ is semisimple. Note that $(M^{ext})_{top}$ is finite-dimensional and therefore the action of $sl(2)$ on $(M^{ext})_{top}$ is semi-simple. Thus the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ acts diagonally on $(M^{ext})_{top}$. Moreover, the action of $L(0)$ on $(M^{ext})_{top}$ is given by
$$L(0) = L^{sl(2)}(0) + L^W(0)$$
where $L^{sl(2)}$ is Sugawara Virasoro vector in $\widetilde{V}_k(sl(2))$, and L^W is Virasoro vector in W. Since W is regular, we have that the action of $L^W(0)$ is diagonal. Since the action of $L^{sl(2)}(0)$ on $(M^{ext})_{top}$ is proportional to the action of Casimir element of $sl(2)$, we conclude that $L^{sl(2)}(0)$ also acts diagonally on $(M^{ext})_{top}$. Therefore $L(0)$ acts diagonally on $(M^{ext})_{top}$. This implies that M^{ext} is a module from KL_k^{fin}, and therefore (10.1) splits.

□

Remark 10.2. *Using the language of tensor categories and concepts from* [19]*, Theorem 10.1 implies that KL_k is a semisimple braided tensor category.*

Remark 10.3. *Although $KL_k(sl(2))$ is semi-simple for k admissible, since $V^k(sl(2))$ is not simple, the category $KL^k(sl(2))$ is not semi-simple. Moreover, it is expected that $KL^k(sl(2))$ contains logarithmic modules. The paper* [38] *presents some conjectural logarithmic modules at admissible level, which to belong $KL^k(sl(2))$.* [1]

We believe that the subalgebra $\widetilde{V}_k(sl(2))$ of $V_k(\mathfrak{g})$ is simple for $k = -\frac{m+1}{m+2}$. But we don't need this information for proving semisimplicity of KL_k.

Based on Theorem 10.1 and the arguments presented in Remark 8.4 we expect that the following conjecture holds.

Conjecture 10.4. *The category KL_k is semisimple if and only if $k \in \{-1, -\frac{m+1}{m+2} \mid m \in \mathbb{Z}_{\geq 0}\}$.*

[1] We thank the referee for this information.

10.1. **The vertex algebra $D_{m+1,2}$.** The vertex algebras $D_{m+1,k}$ are defined in [3] for arbitrary $m \in \mathbb{C}$ and $k \in \mathbb{Z}_{>0}$. We shall here assume that $m \in \mathbb{Z}_{\geq 0}$ and $k = 2$.

For $p \in \mathbb{Z}$, let $F_p = M_\delta(1) \otimes \mathbb{C}[\mathbb{Z}\delta]$ be the rank one lattice vertex algebra associated to the lattice $\mathbb{Z}\delta^{(p)}$, $\langle \delta^{(p)}, \delta^{(p)} \rangle = p$. Here $M_{\delta^{(p)}}(1)$ is the Heisenberg vertex algebra generated by $\delta^{(p)}(z) = \sum_{n \in \mathbb{Z}} \delta^{(p)}(z) z^{-n-1}$. Following [3], let $D_{m+1,2}$ be the vertex subalgebra of $V_{m+1}(sl(2)) \otimes F_2$ generated by

$$\overline{X} = e(-1)\mathbf{1} \otimes e^{\delta^{(2)}}, \quad \overline{Y} = f(-1)\mathbf{1} \otimes e^{-\delta^{(2)}}.$$

It was proved in [3] that $D_{m+1,2}$ is a regular vertex operator algebra (i.e., every weak $D_{m+1,2}$–module is completely reducible) and that

$$D_{m+1,2} \otimes F_{-2} = V_{m+1}(sl(2)) \otimes F_{-2(m+2)}.$$

Consider now the vertex algebra $V_k(\mathfrak{g})$ for $k = -\frac{m+1}{m+2}$ and $\mathfrak{g} = sl(2|1)$. It is generated by four even fields $E^{1,2}, H^-, F^{1,2}, H^+$, such that $E^{1,2}, 2H^-, F^{1,2}$ define the homomorphism $\Phi_1 : V^k(sl(2)) \to V_k(\mathfrak{g})$ commuting with the Heisenberg subalgebra generated by H^+, and four odd fields E^1, E^2, F^1, F^2 (cf. [17]). Denote by

$$X = E^1(-1)F^2(-1)\mathbf{1}, \quad Y = F^1(-1)E^2(-1)\mathbf{1}.$$

Set $X(n) = X_{n+1}$, $Y(n) = Y_{n+1}$. Let W be the vertex subalgebra of $V_k(\mathfrak{g})$ generated by vectors X and Y. A direct computation, which uses the relations displayed in [17, Appendix D], shows that

$$X(1)Y = k\mathbf{1} + 2H^+. \tag{10.2}$$

Lemma 10.5. *In $V_k(\mathfrak{g})$, we have:*

$$X(-2m-4)X(-2m-2)\cdots X(-2)\mathbf{1} = 0, \tag{10.3}$$

$$X(-2m-5)X(-2m-2)\cdots X(-2)\mathbf{1} = 0. \tag{10.4}$$

Proof. The proof the relation (10.3) follows from the fact that there is $\nu \neq 0$ such that

$$X(-2m-4)X(-2m-2)\cdots X(-4)X(-2)\mathbf{1} = \nu F^2(-m-2)\cdots F^2(-1)E^1(-m-2)\cdots E^1(-1)\mathbf{1},$$

and

$$F^2(-m-2)\cdots F^2(-1)E^1(-m-2)\cdots E^1(-1)\mathbf{1}$$

is a charged singular vector in $V^k(\mathfrak{g})$ (cf. [39]). The relation (10.4) follows by applying derivation on (10.3). □

Consider the vertex algebra $V_k(\mathfrak{g}) \otimes F_{-2}$. Set $\delta = \delta^{(-2)}$. As shown in [17, (4.2)], there exists another vertex algebra homomorphism $\Phi_2 : V^{m+1}(sl(2)) \to W \otimes F_{-2} \subset V_k(\mathfrak{g}) \otimes F_{-2}$ uniquely determined by

$$e \mapsto \frac{1}{k+1} X \otimes e^{\delta}, \ f \mapsto \frac{1}{k+1} Y \otimes e^{-\delta}, \quad h \mapsto (m+1)\delta + 2(m+2)H^+.$$

Using Lemma 10.5 we get that $e(-1)^{m+2}\mathbf{1} = 0$ in $W \otimes F_{-2}$, implying that

$$V_{m+1}(sl(2)) \hookrightarrow W \otimes F_{-2}.$$

Theorem 10.6. *We have $W \cong D_{m+1,2}$.*

Proof. Let $\beta = (m+2)(\delta + 2H^+)$. Then $\beta \in \mathrm{Com}(V_{m+1}(sl(2)), W \otimes F_{-2})$ and $\beta(1)\beta = -2(m+2)$. Let $M_\beta(1)$ be the Heisenberg vertex algebra generated by β. Let

$$u^+ = X(-2m-2)\cdots X(-2)\mathbf{1} \otimes e^{(m+2)\delta}, \quad u^- = Y(-2m-2)\cdots Y(-2)\mathbf{1} \otimes e^{-(m+2)\delta}$$

Then

$$\beta(0)u^+ = (2(m+2)(m+1) - 2(m+2)^2)u^+ = -2(m+2)u^+, \; \beta(0)u^- = 2(m+2)u^-.$$

We have:

$$\begin{aligned}(k+1)e(0)u^+ &= X(-2m-6)X(-2m-2)\cdots X(-2)\mathbf{1} \otimes e^{\delta}_{2m+4}e^{(m+2)\delta}. \\ &\quad + X(-2m-5)X(-2m-2)\cdots X(-2)\mathbf{1} \otimes e^{\delta}_{2m+3}e^{(m+2)\delta} = 0\end{aligned}$$

Since $Y(2m+3+j)X(-2m-2)\cdots X(-2)\mathbf{1} = 0$ for $j \geq 0$, we conclude

$$(k+1)f(0)u^+ = Y(2m+3)X(-2m-2)\cdots X(-2)\mathbf{1} \otimes e^{-\delta}_{-2m-5}e^{(m+2)\delta} = 0$$

We prove analogous relations for u^-, which implies that

$$u^{\pm} \in \mathrm{Com}(V_{m+1}(sl(2)), W \otimes F_{-2}). \tag{10.5}$$

Let Z be the subalgebra of $W \otimes F_{-2}$ generated by $u^{\pm}$. So we have that $V_{m+1}(sl(2)) \otimes Z \subset W \otimes F_{-2}$. By applying the action of $f(n)$ on u^+ (resp. $e(n)$ of u^-), one gets that $e^{\pm\delta} \in V_{m+1}(sl(2)) \otimes Z$. From this one gets $X, Y \in V_{m+1}(sl(2)) \otimes Z$. Therefore:

$$W \otimes F_{-2} = V_{m+1}(sl(2)) \otimes Z.$$

We conclude that there is a conformal embedding $M_\beta(1) \hookrightarrow Z$, so that Z is generated by singular vectors $u^{\pm}$. Using results on the uniqueness of the lattice vertex algebras (cf. [37]), we get that $Z \cong F_{-2(m+2)}$. The isomorphism is determined by

$$e^{\delta^{(-2(m+2))}} \mapsto u^+, \quad e^{-\delta^{(-2(m+2))}} \mapsto au^-,$$

for a certain $a \neq 0$. Therefore we get an isomorphism

$$W \otimes F_{-2} \cong V_{m+1}(sl(2)) \otimes F_{-2(m+2)}.$$

On the other hand, using construction from [3, Section 6] we get an isomorphism

$$V_{m+1}(sl(2)) \otimes F_{-2(m+2)} \cong D_{m+1,2} \otimes F_{-2} \tag{10.6}$$

such that the subalgebra F_{-2} is generated by

$$f(-1)^{m+1}\mathbf{1} \otimes e^{\delta^{(-2(m+2))}}, \quad e(-1)^{m+1}\mathbf{1} \otimes e^{-\delta^{(-2(m+2))}}. \tag{10.7}$$

(This argument is essentially based on the fact that $f(-1)^{m+1}\mathbf{1}, e(-1)^{m+1}\mathbf{1}$ generate a subalgebra of $V_{m+1}(sl(2))$ isomorphic to F_{2m+2}.) The isomorphism (10.6) maps the generators (10.7) to elements of $W \otimes F_{-2}$:

$$f(-1)^{m+1}u^+, \; ae(-1)^{m+1}u^-,$$

which are proportional to e^{δ}, $e^{-\delta}$. Thus, we get an isomorphism $W \otimes F_{-2} \cong D_{m+1,2} \otimes F_{-2}$ which preserves F_{-2}. This implies that $W \cong D_{m+1,2}$.

□

References

[1] D. Adamović, Some rational vertex algebras, Glas. Mat. Ser. III **29** (1994), 25-40.

[2] D. Adamović, Vertex algebra approach to fusion rules for N=2 superconformal minimal models, Journal of Algebra **239** (2001) 549–572

[3] D. Adamović, A family of regular vertex operator algebras with two generators, Central European J. Math. **5** (2007), 1-18.

[4] D. Adamović, A realization of certain modules for the $N = 4$ superconformal algebra and the affine Lie algebra $A_2^{(1)}$, Transformation Groups **21**, n. 2 (2016) 299-327

[5] D. Adamović, Realizations of simple affine vertex algebras and their modules: the cases $\widehat{sl(2)}$ and $\widehat{osp(1,2)}$, Comm. Math. Phys. **366** (2019) 1025–1067, arXiv:1711.11342 [math.QA].

[6] D. Adamović, V. G. Kac, P. Möseneder Frajria, P. Papi, O. Perše, Conformal embeddings of affine vertex algebras in minimal W-algebras I: Structural results Journal of Algebra, **500**, (2018), 117–152

[7] D. Adamović, V. G. Kac, P. Möseneder Frajria, P. Papi, O. Perše, Conformal embeddings of affine vertex algebras in minimal W-algebras II: decompositions, *Japanese Journal of Mathematics*, **12**, 2, 261–315

[8] D. Adamović, V. G. Kac, P. Möseneder Frajria, P. Papi, O. Perše, An application of collapsing levels to the representation theory of affine vertex algebras, Int. Math. Res. Not., Volume 2020, **13**, 2020, 4103–4143

[9] D. Adamović, P. Möseneder Frajria, P. Papi, O. Perše, Conformal embeddings in affine vertex superalgebras , Advances in Mathematics, **360** (2020) 106918

[10] D. Adamović, K. Kawasetsu, D. Ridout, A realisation of the Bershadsky-Polyakov algebras and their relaxed modules, Letters in Math. Physics 111, 38 (2021)

[11] D. Adamović and A. Milas, Vertex operator algebras associated to modular invariant representations for $A_1^{(1)}$, *Math. Res. Lett.* **2** (1995), 563–575.

[12] D. Adamović, V. Pedić, On fusion rules and intertwining operators for the Weyl vertex algebra, Journal of Mathematical Physics **60** (2019), 081701, 18 pp., arXiv:1903.10248 [math.QA].

[13] T. Arakawa, Representation theory of superconformal algebras and the Kac-Roan-Wakimoto conjecture, *Duke Math. J.* **130** (3) (2005), 435–478.

[14] T. Arakawa, Rationality of admissible affine vertex algebras in the category O, Duke Math. J. **165** (2016), no. 1, 67–93.

[15] T. Arakawa, J. van Ekeren, A. Moreau, Singularities of nilpotent Slodowy slices and collapsing levels of W-algebras, arXiv:2102.13462

[16] S. Berman, C. Dong, and S. Tan, Representations of a class of lattice type vertex algebras, J. Pure Appl. Algebra, **176**, 27–47, 2002.

[17] P. Bowcock, B. L. Feigin, A. M. Semikhatov, A. Taormina, Affine $sl(2|1)$ and affine $D(2|1:\alpha)$ as vertex operator extensions of dual affine $sl(2)$ algebras. Comm. Math. Phys. **214**, 495-545 (2000)

[18] T. Creutzig, S. Kanade, A. Linshaw and D. Ridout, Schur-Weyl duality for Heisenberg cosets. Transformation Groups **24** (2019), 301–354.

[19] T. Creutzig, J. Yang, Tensor categories of affine Lie algebras beyond admissible level. Math. Ann. **380** (2021), no. 3-4, 1991–2040

[20] T. Creutzig, R. McRae, J.Yang, Tensor structure on the Kazhdan–Lusztig category for affine $\mathfrak{gl}(1|1)$, International Mathematics Research Notices, 2021; rnab080, https://doi.org/10.1093/imrn/rnab080

[21] C. Dong, H. Li, G. Mason, Regularity of rational vertex operator algebras, Adv. Math. **132**, 1997, 148–166

[22] E. Frenkel, Lectures on Wakimoto modules, opers and the center at the critical level, Adv. Math. 195 (2005) 297-404.

[23] I. Frenkel, Y. Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, Mem. Am. Math. Soc. 104 (1993)

[24] J. Germoni, Indecomposable representations of general linear Lie superalgebras, J. of Alg., 209, (1998), 367–401.

[25] M. Gorelik, V. Kac, On complete reducibility for infinite-dimensional Lie algebras, Advances in Mathematics 226 (2011) 1911–1972

[26] M. Gorelik, V. Serganova, Integrable Modules Over Affine Lie Superalgebras $\mathfrak{sl}(1|n)^{(1)}$, Comm. Math. Phys. 364, 635–654 (2018).

[27] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory for gen- eralized modules for a conformal vertex algebra, I: Introduction and strongly graded algebras and their generalized modules, Conformal Field Theories and Tensor Cate- gories, 169–248, Math. Lect. Peking Univ., Springer, Heidelberg, 2014.

[28] V. G. Kac. *Representation of classical Lie superalgebras*, in *Differential geometrical methods in Mathematical Physics, II (Proc. Conf. Univ. Bonn, Bonn 1977)*, LNM 676, Springer, Berlin, 1978, 597–626

[29] V. G. Kac. *Lie superalgebras*, Adv. Math. **26**, 8–96, (1977).

[30] V. G. Kac, Vertex Algebras for Beginners, University Lecture Series, Second Edition, AMS, Vol. 10 (1998).

[31] V. G. Kac, M. Wakimoto, Integrable highest weight modules over affine superalgebras and Appell's function, Comm. Math. Phys. 215 (2001), 631–682

[32] V. G. Kac, M. Wakimoto, Integrable highest weight modules over affine superalgebras and number theory. Lie theory and geometry, 415–456, Progr. Math., 123, Birkhäuser Boston, Boston, MA, 1994.

[33] V. G. Kac, M. Wakimoto, Quantum reduction and representation theory of superconformal algebras, Adv. Math. **185** (2004), 400–458.

[34] V. G. Kac, M. Wakimoto, Representations of affine superalgebras and mock theta functions, Trans. Groups **19**, no. 2 (2014): 383–455.

[35] V. G. Kac, M. Wakimoto, Representations of superconformal algebras and mock theta functions *Trans. Moscow Math. Soc.* 2017, 9–74.

[36] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras. I, II, *J. Amer. Math. Soc.*, 6 (1993), 905–947; 949–1011.

[37] H. Li, X. Xu, A characterization of vertex algebras associated to even lattices, J. Algebra 173 (1995) 253–270

[38] J. Rasmussen, Staggered and Kac modules over $A_1^{(1)}$, Nuclear Phys. B 950(2020), 114865, 46 pp.

[39] A. M. Semikhatov, A. Taormina, Twists and singular vectors in $sl(2|1)$ representations. Teor. Mat. Fiz. 128, 474-491 (2001); [Theor. Math. Phys. 128, 1236-251 (2001)]

[40] H. Yamada, H. Yamauchi, Simple Current Extensions of Tensor Products of Vertex Operator Algebras, Int. Math. Res. Not. 2021, no. 16, 12778–12807.

[41] H. Yamada, H. Yamauchi, $\mathbb{Z}_{2k}$–code vertex operator algebras, J. of Alg. **573**, 451–475 (2021)

[42] J. Van der Jeugt, J. W. B. Hughes, R. C. King, and J. Thierry-Mieg, A character formula for singly atypical modules of the Lie superalgebra $sl(m,n)$., Comm. Alg. **18**1990., 3454 –3480.

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