Finite Fields and Their Applications 113 (2026) 102831

Contents lists available at ScienceDirect

Finite Fields and Their Applications

journal homepage: www.elsevier.com/locate/ffa

Maximal curves over finite fields and a modular isogeny

Valerio Dose[a], Guido Maria Lido[b], Pietro Mercuri[c,*], Claudio Stirpe[d]

[a] Department of Computer, Control and Management Engineering, Sapienza University of Rome, Roma, Italy
[b] Department of Mathematics, University of Rome "Tor Vergata", Roma, Italy
[c] Department of Engineering, Università di Palermo, Palermo, Italy
[d] Convitto Nazionale "R. Margherita", Anagni, Italy

ARTICLE INFO

Article history:
Received 10 September 2025
Accepted 24 March 2026
Available online xxxx
Communicated by Gary L. Mullen

MSC:
11G20
11G18
11T71
14G35

Keywords:
Many points
Finite fields
Modular curves
Chen's isogeny
Cartan subgroups
Hecke operators

ABSTRACT

We prove the existence of curves of genus 7 and 12 over the field with 11^5 elements, reaching the Hasse-Weil-Serre upper bound. These curves are quotients of modular curves and we give explicit equations. We compute the number of points of many quotient modular curves in the same family without providing equations. For various pairs (genus, finite field) we find new records for the largest known number of points. In other instances we find quotient modular curves that are maximal, matching already known results. To perform these computations, we provide a generalization of Chen's isogeny result.

* Corresponding author.
E-mail addresses: valerio.dose@uniroma1.it (V. Dose), guidomaria.lido@gmail.com (G.M. Lido), pietro.mercuri@unipa.it (P. Mercuri), clast@inwind.it (C. Stirpe).

https://doi.org/10.1016/j.ffa.2026.102831

1. Introduction

The Hasse-Weil-Serre bound prescribes that

$$|\#X(\mathbb{F}_q) - q - 1| \leq g\lfloor 2\sqrt{q} \rfloor \tag{1.1}$$

for every finite field $\mathbb{F}_q$ and every smooth, projective, absolutely irreducible algebraic curve X over $\mathbb{F}_q$ of genus g. We provide explicit equations for two smooth projective curves over $\mathbb{Q}$, of genus 7 and 12 respectively, which have good reduction at 11 and the largest possible number of points over the finite field with 11^5 elements: they have exactly 166666 and 170676 points over $\mathbb{F}_{11^5}$, respectively, achieving the Hasse-Weil-Serre bound.

The interest in explicit examples of curves with a number of points close to the Hasse-Weil-Serre bound is also motivated by their application to coding theory (see for example [45, Section 8.4]). The website [27] is devoted to collecting, for every choice of the genus $g \leq 50$ and $q = p^k$ with $p < 20$ a prime and $k \leq 5$, the current bounds about how large $\#X(\mathbb{F}_q)$ can be. Among the most recent efforts to populate this database, we mention [10], [29], [31].

Certain families of modular curves have already been used in [18] to produce many examples that improve the previously known lower bounds. The maximal curve of genus 12 over $\mathbb{F}_{11^5}$ described in Section 4 was found among those examples, even though explicit equations were not given.

In this work we extend the reach of the methods in [18] to a larger family of modular curves, finding new records and several examples of maximal curves, among which the new maximal curve of genus 7 over $\mathbb{F}_{11^5}$.

This instance and the one with genus 12 are particularly surprising, because they arise over an odd extension of a field with a prime number of elements. Indeed, modular curves are generally known to have many points over even extensions because of the presence of rational points associated to supersingular elliptic curves, which do not all show over an odd extension. We compute explicit equations for the two maximal curves by studying the behavior of the automorphisms on the modular forms involved.

In our study, we consider modular curves of Borel-Cartan type and their quotients by all subgroups made of Atkin-Lehner involutions or analogous involutions in the Cartan case. We refer to all these automorphisms as *Atkin-Lehner type involutions*. Curves in this family have already been the subject of a lot of recent studies because of their relation to open questions in the arithmetic of elliptic curves, see for example [4], [5], [16], [17], [20], [23], [24], [25], [33], [38]. Quotients, in particular, can result in better instances from the point of view of the number of points with respect to the genus, as observed for example in [15]. Concerning the explicit computation of modular forms and explicit equations for modular curves, we mention the recent works [1], [7], [8], [19], [36], [40], [47] as well as the more classical [26] and [44].

The paper is organized as follows. In Section 2 we recall the definition of modular curves of Borel-Cartan type, and we give an isogeny result of Chen type for the full family of those curves and their quotients by automorphisms of Atkin-Lehner type, generalizing [14], [13], [17], [18] and [43]. In Section 3 we recall how to compute the number of points over finite fields, using the new isogeny result, and we list all the improved lower bounds. In Section 4 we compute equations for the two new maximal curves of genus 7 and 12. In Section 5, we comment on all the maximal curves in the family in the range of our search. The results of our computations are summarized in the appendices: in Appendix B we report all the best results we found for each pair (genus, finite field), while in Appendix C we list all the maximal curves of genus larger than 1, in the range of our search.

As an extra, for X a curve with a subgroup $G \cong (\mathbb{Z}/2\mathbb{Z})^r$ of the automorphism group, we give in Appendix A some general results on the Jacobians of the quotient curves X/W, with W a subgroup of G. From this, we deduce formulas that relate the genera and the number of points over finite fields of quotients of X by different subgroups of G. These results can be relevant in the context of modular curves and their quotients, which in various cases have automorphism group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$ (see [3], [6], [11] and [17]).

2. Modular curves of Borel-Cartan type and their quotients

The curves we study are quotients of modular curves. Modular curves are smooth projective (algebraic) curves over $\mathbb{Q}$ that parametrize elliptic curves with a torsion structure and whose complex points are somewhat easy to describe by quotienting the complex upper half-plane. We refer to [22], [21] or to the small summary in the first section of [17], for more details.

Given $n_0, n_{\mathrm{ns}} \in \mathbb{Z}_{>0}$ coprime and $n = n_0 n_{\mathrm{ns}}$, let

$$X(n_0, n_{\mathrm{ns}}) = X_H,$$

be the modular curve associated to

$$H = B_0(n_0) \times C_{\mathrm{ns}}(n_{\mathrm{ns}}) \quad \text{inside} \quad \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}) = \mathrm{GL}_2(\mathbb{Z}/n_0\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}/n_{\mathrm{ns}}\mathbb{Z}),$$

where B_0 is a choice of a Borel subgroup and C_{ns} is a choice of a non-split Cartan subgroup. By the general theory of modular curves, X_H has good reduction over all primes $p \nmid n$ (see [21] for example). We say that these modular curves are of *Borel-Cartan (mixed) type*, see [18] for more details. The curves $X(n_0, n_{\mathrm{ns}})$ also appeared in [33] with the notation $X_{\mathrm{ns}}^{\vec{\varepsilon}}(n_{\mathrm{ns}}, n_0)$, where $\vec{\varepsilon}$ is a vector of non-square elements of $\mathbb{Z}/p\mathbb{Z}$ for each prime p dividing n_{ns}.

For every exact divisor k of $n = n_0 n_{\mathrm{ns}}$ (i.e., a divisor k such that k and $\frac{n}{k}$ are coprime), the curve $X(n_0, n_{\mathrm{ns}})$ has an involution W_k which we call *Atkin-Lehner type* involution, defined as follows:

- For k dividing n_0, we define W_k as a lift of the classical Atkin-Lehner involution on X_0 as follows: we represent a non-cuspidal point on $X(n_0, n_{\mathrm{ns}})$ as an elliptic curve E together with cyclic subgroups C_k of $E[k]$ and $C_{n_0/k}$ of $E[n_0/k]$ (they form the cyclic subgroup $C = C_k \oplus C_{n_0/k}$ of $E[n_0] = E[k] \oplus E[n_0/k]$) and together with an isomorphism $\phi\colon (\mathbb{Z}/n_{\mathrm{ns}}\mathbb{Z})^2 \cong E[n_{\mathrm{ns}}]$ (up to $C_{\mathrm{ns}}(n_{\mathrm{ns}})$). Then we define

$$W_k(E, C_k, C_{n_0/k}, \phi) = (E/C_k, E[k]/C_k, \pi(C_{n_0/k}), \pi \circ \phi)$$

  where π is the projection $E \to E/C_k$.
- For k a prime power dividing n_{ns}, we recall that under the Chinese Remainder Theorem we have $C_{\mathrm{ns}}(n_{\mathrm{ns}}) = C_{\mathrm{ns}}(k) \times C_{\mathrm{ns}}(n_{\mathrm{ns}}/k)$, and analogously for the normalizer $C_{\mathrm{ns}}^+(n_{\mathrm{ns}})$. Moreover, since k is a prime power, $C_{\mathrm{ns}}^+(k)/C_{\mathrm{ns}}(k)$ has two elements. Then we take the non-trivial element $w_k \in C_{\mathrm{ns}}^+(k)/C_{\mathrm{ns}}(k) < C_{\mathrm{ns}}^+(n_{\mathrm{ns}})/C_{\mathrm{ns}}(n_{\mathrm{ns}})$ and we define

$$W_k(E, C_{n_0}, \phi) = (E, C_{n_0}, \phi \circ w_k),$$

  for all non-cuspidal points (E, C_{n_0}, ϕ) on $X(n_0, n_{\mathrm{ns}})$. Moreover, in this case the quotient curve $X_H/\langle W_k \rangle$ is still of the form $X_{H'}$ for a suitable subgroup H' of $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ containing H.
- For a general k we extend the definition multiplicatively: each exact divisor k is uniquely the product of an exact divisor $k_0 \mid n_0$ and of pairwise coprime prime powers $q_i \mid n_{\mathrm{ns}}$ and we define $W_k = W_{k_0} \circ W_{q_1} \circ \cdots \circ W_{q_r}$ which does not depend on the order.

The above Atkin-Lehner type involutions have, indeed, order 2. They are defined over $\mathbb{Q}$ (this follows from the definition of X_H as coarse moduli space) and generate a subgroup of $\mathrm{Aut}(X(n_0, n_{\mathrm{ns}}))$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$ where r is the number of primes dividing $n_0 n_{\mathrm{ns}}$. One can consider the quotients

$$X(n_0, n_{\mathrm{ns}})/\langle W_{k_1}, \ldots, W_{k_t} \rangle$$

with $k_1, \ldots, k_t$ dividing exactly n. In the following theorem we compute an isogeny decomposition of the Jacobian of these curves, generalizing [18, Theorem 3.8].

Theorem 2.2. *Given a curve $Y = X(n_0, n_{\mathrm{ns}})$, let W be the group of its Atkin-Lehner type involutions. For each subgroup $K < W$ the Jacobian of the corresponding quotient satisfies the following isogeny relation over $\mathbb{Q}$*

$$\mathrm{Jac}(Y/K) \sim \textstyle\prod_f A_f^{m_f},$$

where f ranges among the newforms of weight 2 and level $d_0 d_{\mathrm{ns}}^2$ for divisors $d_0 | n_0$ and $d_{\mathrm{ns}} | n_{\mathrm{ns}}$ and for each given f the multiplicity m_f is

$$m_f := \sum_{\chi\in K^\perp}\left(\prod_{p^e||n_{\mathrm{ns}}}\left(\tfrac{1}{2}+\tfrac{1}{2}\varepsilon_{f,p^e}\chi(W_{p^e})\right)\cdot\prod_{p^e||n_0}\left(\tfrac{v+1}{2}+\varepsilon_{f,p^e}\chi(W_{p^e})\tfrac{1+(-1)^v}{4}\right)\right),$$

with $K^\perp$ being the set of characters of W that are trivial on K, the indexes p^e varying among all prime powers dividing exactly n_0 or n_{ns}, and ε_{f,p^e} being the eigenvalue of W_{p^e} applied to f, and $v := \mathrm{val}_p(n_0/d_0)$.

Proof. Let A be the Jacobian of Y. Then W acts on A and the Jacobian of Y/K is the connected component of 0 in the fixed locus of K in A, which we denote A^K. By [18, Theorem 3.8] A is isogenous to a product $\prod_f A_f^{n_f}$ with the same newforms f appearing in the statement and n_f suitably defined. Since the A_f's are all simple and non-isogenous over $\mathbb{Q}$, the action of W on A acts separately on each $A_f^{n_f}$. In particular the Jacobian of Y/K is isogenous over $\mathbb{Q}$ to the product of the fixed loci $(A_f^{n_f})^K$. Since the A_f's are simple, each $(A_f^{n_f})^K$ is isogenous to a power $A_f^{m_f}$. We are left to compute the multiplicities m_f.

Each subvariety $A_f^{n_f}$ is a module for the anemic Hecke algebra $\mathbb{T}$ (i.e., the algebra of Hecke operators whose indices are coprime with the level) of A_f and the action of $\mathbb{T}$ commutes with W. In particular the space of 1-forms on

$$V := H^0(A_f, \Omega^1)^{n_f}$$

is a module both for $\mathbb{T}$ and for W, and the multiplicity m_f is simply the rank, as $\mathbb{T}$-module, of the fixed locus V^K. Since the irreducible representations of W are characters, then V has a $\mathbb{T}$-basis $f_1,\dots,f_{n_f}$ with respect to which W acts diagonally (such f_i can be found explicitly extending [2, Equations (5.1) and (5.2)]): for each i, the group W acts on $\mathbb{T}\cdot f_i$ through a certain character $\chi\colon W\to\{\pm1\}$. For each character χ we denote d_χ the number of f_i's such that W acts on $\mathbb{T}\cdot f_i$ with the character χ. Then we have

$$m_f = \mathrm{rank}_{\mathbb{T}} V^K = \sum_{\chi\in K^\perp} d_\chi. \tag{2.3}$$

Therefore, to prove our theorem it is enough to prove

$$d_\chi = d'_\chi, \quad \text{with} \quad d'_\chi := \prod_{p^e||n_{\mathrm{ns}}}\left(\tfrac{1}{2}+\tfrac{1}{2}\varepsilon_{f,p^e}\chi(W_{p^e})\right)\cdot\prod_{p^e||n_0}\left(\tfrac{v+1}{2}+\varepsilon_{f,p^e}\chi(W_{p^e})\tfrac{1+(-1)^v}{4}\right). \tag{2.4}$$

We prove it with [18] and some linear algebra.

Let Q be the set of Atkin-Lehner type involutions W_{p^e} for $p^e||n_0n_{\mathrm{ns}}$: then Q generates W and

$$W \cong (\mathbb{Z}/2\mathbb{Z})^Q \cong (\mathbb{Z}/2\mathbb{Z})^{\#Q}.$$

Hence the characters of W are in bijection with $S := \{\pm1\}^{\#Q}$ since each character of W is determined by its value on each involution W_{p^e}. In particular the vectors $d := (d_\chi)_\chi$

and $d' := (d'_\chi)_\chi$ live in the vector space $\mathbb{R}^S \cong \mathbb{R}^{2^{\#Q}}$. Now for each subset $Z \subset Q$ consider $\tilde{Z} \in \mathbb{R}^S$ the vector only containing 0's or 1's as follows: for $s \in S$ we have $\tilde{Z}_s = 1$ if and only if s satisfies $s(q) = 1$ for each q in Z. Then, by induction on $\#Q$, the set $\{\tilde{Z} : Z \subset Q\}$ is a basis for $\mathbb{R}^S$. Hence, denoting by $\langle \cdot, \cdot \rangle$ the standard scalar product on $\mathbb{R}^S$, determining the vector d is equivalent to determining $\langle d, \tilde{Z} \rangle$ for each subset $Z \subset Q$. In particular to prove Equation (2.4) it is enough to prove $\langle d, \tilde{Z} \rangle = \langle d', \tilde{Z} \rangle$ for each Z. By definition of $\tilde{Z}$ we have

$$\langle d, \tilde{Z} \rangle = \sum_{\substack{\chi(W_q)=1 \\ \forall q \in Z}} d_\chi = \sum_{\langle W_q : q \in Z \rangle^\perp} d_\chi$$

which, by Equation (2.3), is the multiplicity of A_f in $Y/\langle W_q : q \in Z \rangle$. This multiplicity is computed in [18, Theorem 3.8] and it is exactly equal to $\langle d', \tilde{Z} \rangle$. $\square$

3. Computing the number of points

The curve $X = X(n_0, n_{\mathrm{ns}})/\langle W_{k_1}, \dots, W_{k_t} \rangle$ is defined over $\mathbb{Q}$ and has smooth reduction for all primes $p \nmid n_0 n_{\mathrm{ns}}$, in fact the full level curve $X(n)$ has good reduction by [21, Theorem IV.6.7], while [34, Corollary 4.10] or [46, Théorème 2.2] or [41, Lemma 5.1] take care of the quotient by H and by $\langle W_{k_1}, \dots, W_{k_t} \rangle$.

Theorem 2.2 allows one to compute its number of points over a finite extension of $\mathbb{F}_p$, starting from the knowledge of the newforms appearing in the decomposition of Theorem 2.2. In more detail, for q a power of p, one can compute $\#X(\mathbb{F}_q)$ applying [18, Algorithm 4.5], after substituting m_f with the multiplicities in Theorem 2.2.

Although we refer to [18] for the algorithm, we show the procedure in a couple of examples.

Example 3.5. Consider the curve $X = X(6, 7)/\langle W_6, W_7 \rangle$. We follow the notation of Theorem 2.2. Listing the possible positive divisors d_0 of 6 and d_{ns} of 7 we have $d_0 d_{\mathrm{ns}}^2 \in \{1, 2, 3, 6, 49, 98, 147, 294\}$. Since there are no cusp forms of weight 2 and level 14 or less than 10, we only consider $d_0 d_{\mathrm{ns}}^2 \in \{49, 98, 147, 294\}$ corresponding to 15 Galois conjugacy classes of forms. The set $K^\perp$ only contains $(1, 1, 1)$ and $(-1, -1, 1)$, where (k_1, k_2, k_3) corresponds to the values of the associated character on (W_2, W_3, W_7). Since $\chi(W_7) = 1$ for all $\chi \in K^\perp$, then we have $m_f = 0$ for each form f having negative eigenvalue with respect to W_7 (i.e., having $\varepsilon_{f,7} = -1$). Thus 6 Galois conjugacy classes of forms remain to be considered. For every f of level 294, we have $\varepsilon_{f,2}\chi(W_2) = \varepsilon_{f,3}\chi(W_3) = -1$, hence again $m_f = 0$. Thus only 4 Galois conjugacy classes of forms $\{f_1, f_2, f_3, f_4\}$ need to be considered, one of level 98 and three of level 147, and each has $m_f = 1$. The associated abelian varieties A_f have dimension 2 except one which has dimension 1, so the genus is 7. An alternative method to compute the genus is shown in Example A.8, in Appendix A.

For all forms f_i and their Galois conjugates (seven forms in total) the Hecke eigenvalues $a_{11}(f)$ are equal to -2. Hence the Frobenius Frob_{11} acting on X has two eigenvalues

α_1, α_2, each of multiplicity 7, which are the roots of the polynomial $x^2 - a_{11}(f)x + 11 = x^2 + 2x + 11$. We deduce that the number of points of X over $\mathbb{F}_{11}$ is

$$\#X(\mathbb{F}_{11}) = 11 + 1 - \sum_{i=1}^{7}(\alpha_1 + \alpha_2) = 12 + 14 = 26.$$

Similarly the number of points over $\mathbb{F}_{11^5}$ is

$$\#X(\mathbb{F}_{11^5}) = 11^5 + 1 - \sum_{i=1}^{7}(\alpha_1^5 + \alpha_2^5) = 161052 + 5614 = 166666.$$

This is our maximal curve of genus 7, further studied in Section 4.

Example 3.6. We consider another quotient $X(6,7)$, namely $X = X(6,7)/\langle W_3, W_7\rangle$. We follow the notation of Example 3.5. Again we only consider the 15 Galois conjugacy classes of level $d_0 d_{\mathrm{ns}}^2 \in \{49, 98, 147, 294\}$ but now $K^\perp = \{(1,1,1), (-1,1,1)\}$. Since $\chi(W_3) = \chi(W_7) = 1$ for all $\chi \in K^\perp$, then we have $m_f = 0$ for each form f having negative eigenvalue with respect to W_3 and W_7 (i.e., having $\varepsilon_{f,3} = -1$ or $\varepsilon_{f,7} = -1$). Thus 3 Galois conjugacy classes of forms are left to consider: f_1, f_2, f_3, of level 98, 147 and 294, respectively. Among these $m_{f_1} = m_{f_3} = 1$ and $m_{f_2} = 2$. The associated abelian varieties A_f have dimension 2 except the last one which has dimension 1, so the genus is 7.

For the forms f_1 and f_2 and their Galois conjugates (six forms in total) the Hecke eigenvalues $a_{11}(f)$ are equal to -2 but, for the last form, $a_{11}(f) = 5$. Hence the Frobenius Frob_{11} acting on X has the same eigenvalues α_1 and α_2 of Example 3.5 with multiplicity 6 and two new eigenvalues β_1 and β_2, which are the roots of the polynomial $x^2 - a_{11}(f)x + 11 = x^2 - 5x + 11$. We deduce that the number of points of X over $\mathbb{F}_{11}$ is

$$\#X(\mathbb{F}_{11}) = 11 + 1 - \beta_1 - \beta_2 - \sum_{i=1}^{6}(\alpha_1 + \alpha_2) = 12 - 5 + 12 = 19.$$

Similarly the number of points over $\mathbb{F}_{11^5}$ is

$$\#X(\mathbb{F}_{11^5}) = 11^5 + 1 - \beta_1^5 - \beta_2^5 - \sum_{i=1}^{6}(\alpha_1^5 + \alpha_2^5) = 161052 + 725 + 4812 = 166589.$$

This curve already belongs to the family studied in [18] and the small computation above gives the same result contained there.

We implemented the algorithm described in [18] (and sketched in the examples) for curves of type $X(n_0, n_{\mathrm{ns}})/\langle W_{k_1}, \dots, W_{k_t}\rangle$, and applied it to all such curves with $n = n_0 n_{\mathrm{ns}}^2 \leq 10000$. Following [27], we also restricted our analysis to genus $g \leq 50$ and field $\mathbb{F}_{p^k}$ with $k \leq 5$ and odd characteristic $p \leq 19$, respectively $k \leq 7$ for $p = 2$.

Table 3.1
Improved bounds for $\#X(\mathbb{F}_q)$ among curves of genus g, here obtained with X of the form $X(n_0, n_{\rm ns})/\langle W_{k_1}, \dots, W_{k_t}\rangle$ with $n_0 n_{\rm ns}^2 \leq 10000$ and $g \leq 50$.

g	q	$n_0, n_{\rm ns}$	$k_1, \dots, k_t$	$\#X(\mathbb{F}_q)$	g	q	$n_0, n_{\rm ns}$	$k_1, \dots, k_t$	$\#X(\mathbb{F}_q)$
7	11^5	6, 7	6, 7	166666	35	17^2	208, 3	16, 39	1152
10	17^5	208, 1	208	1436834	36	5^2	1271, 1	1271	238
10	19^2	319, 2	58, 319	703	36	17^2	208, 3	208, 39	1182
13	13^5	378, 1	54, 7	385275	38	2^2	29, 7	203	65
14	7^2	110, 3	3, 55	196	38	5^2	593, 2	1186	232
17	11^2	480, 1	5, 96	416	39	2^2	897, 1	299	68
19	2^2	545, 1	545	40	39	17^2	104, 3	39	1284
22	3^2	1102, 1	2, 551	82	40	5^2	814, 1	407	272
24	11^2	306, 1	34	530	40	19^2	1020, 1	17, 15	1440
27	5^2	574, 1	287	204	41	5^2	861, 1	287	260
28	5^2	23, 7	161	202	42	3^2	1001, 2	13, 154	134
29	5^2	1342, 1	2, 671	205	43	5^2	1343, 1	1343	276
30	5^2	1007, 1	1007	216	44	2^2	1169, 1	1169	70
32	5^2	1246, 1	2, 623	214	45	5^2	696, 1	87	280
33	5^2	924, 1	12, 308	226	47	2^2	51, 7	3, 119	75
34	3^2	638, 1	319	114	48	5^2	2014, 1	2, 1007	306
34	19^2	506, 1	253	1300	49	2^2	1023, 1	341	82

We got the eigenvalues of the Hecke operators on the relevant modular forms from [35]: the database contains the q-expansions of all the newforms of level at most 10000.

We found 36 curves that, according to [27], achieve a new (nice) record in terms of largest known number of points over $\mathbb{F}_q$ given the genus g. Following [27], a new record is considered *nice* only when it is at least $q + 1 + \frac{M_g(\mathbb{F}_q) - q - 1}{\sqrt{2}}$, where $M_g(\mathbb{F}_q)$ is the current (theoretical) upper bound for the number of points on a curve of genus g over $\mathbb{F}_q$ (in particular $M_q(\mathbb{F}_q)$ is less than or equal to the Hasse-Weil-Serre bound). We list these 36 record curves in Table 3.1.

4. Explicit equations for two new maximal curves

Among the curves we analyzed, we found two curves that are the first known curves, for their genera, to reach the Hasse-Serre-Weil upper bound over $\mathbb{F}_{11^5}$. The two curves are $X(6,7)/\langle W_6, W_7\rangle$ and $X(156,1)/\langle W_{13}\rangle$: the first curve was unknown, to our knowledge, the second one already appeared in [18], but we have not computed explicit equations nor have we underlined the maximality. We give (singular) planar equations for both curves and we also give smooth equations for the first one, the one with lowest genus, using the canonical embedding.

4.1. Canonical model of $\boldsymbol{X(6,7)/\langle W_6,W_7\rangle}$

The following equations describe a canonical model of $X(6,7)/\langle W_6, W_7\rangle$ in $\mathbb{P}^6$ over $\mathbb{Q}$, which has genus 7, it is smooth over $\mathbb{Z}[\frac{1}{42}]$ and reaches Serre's bound, as computed in Table 3.1.

$$10x_1^2 + 22x_2x_5 - 18x_2x_6 + 5x_2x_7 - 9x_3^2 - 14x_3x_4 - x_3x_5 - 4x_3x_6 + 7x_3x_7 - 4x_4^2 +$$

$$
\begin{aligned}
&- 13x_4x_5 + 11x_4x_6 - 16x_4x_7 + 2x_5^2 - 17x_5x_6 - 7x_5x_7 + 10x_6^2 + x_6x_7 - x_7^2 = 0,\\
&- 4x_1^2 + 28x_1x_2 - 56x_2x_5 + 49x_2x_6 + 17x_3^2 + 23x_3x_4 + 21x_3x_5 - 6x_3x_7 - 8x_4^2 +\\
&\quad - 2x_4x_5 - 2x_4x_6 - x_4x_7 - 16x_5^2 + 56x_5x_6 + 3x_5x_7 - 23x_6^2 + 13x_6x_7 - 7x_7^2 = 0,\\
&- 6x_1^2 - 13x_1x_2 + x_1x_3 + 9x_2x_5 - 9x_2x_6 - 4x_2x_7 - x_3^2 - 9x_3x_5 + 3x_3x_6 - 3x_3x_7 + 7x_4^2 +\\
&\quad + 10x_4x_5 - 7x_4x_6 + 13x_4x_7 + 6x_5^2 - 13x_5x_6 + 4x_5x_7 + 3x_6^2 - 7x_6x_7 + 4x_7^2 = 0,\\
&- 6x_1^2 - 21x_1x_2 + x_1x_4 + 15x_2x_5 - 18x_2x_6 - 9x_2x_7 - 3x_3^2 - 2x_3x_4 - 14x_3x_5 + 4x_3x_6 +\\
&\quad - 2x_3x_7 + 10x_4^2 + 16x_4x_5 - 10x_4x_6 + 15x_4x_7 + 12x_5^2 - 24x_5x_6 + 5x_5x_7 + 5x_6^2\\
&\quad - 13x_6x_7 + 6x_7^2 = 0,\\
&- 3x_1^2 - 8x_1x_2 + x_1x_5 + 11x_2x_5 - 8x_2x_6 - x_2x_7 - 2x_3^2 - 2x_3x_4 - 6x_3x_5 + 2x_3x_6 +\\
&\quad - x_3x_7 + 4x_4^2 + 4x_4x_5 - 3x_4x_6 + 8x_4x_7 + 3x_5^2 - 11x_5x_6 + x_5x_7 + 4x_6^2 - 3x_6x_7\\
&\quad + 3x_7^2 = 0,\\
&- 8x_1^2 - 2x_1x_2 + x_1x_6 - 13x_2x_5 + 12x_2x_6 - 2x_2x_7 + 6x_3^2 + 10x_3x_4 + 3x_3x_6 - 5x_3x_7 +\\
&\quad + 4x_4^2 + 9x_4x_5 - 8x_4x_6 + 13x_4x_7 - 2x_5^2 + 11x_5x_6 + 5x_5x_7 - 6x_6^2 + x_7^2 = 0,\\
&- 6x_1^2 - 13x_1x_2 + x_1x_7 + 9x_2x_5 - 9x_2x_6 - 4x_2x_7 - x_3^2 - 9x_3x_5 + 3x_3x_6 - 3x_3x_7 + 7x_4^2 +\\
&\quad + 12x_4x_5 - 8x_4x_6 + 13x_4x_7 + 6x_5^2 - 13x_5x_6 + 4x_5x_7 + 3x_6^2 - 7x_6x_7 + 4x_7^2 = 0,\\
&- 4x_1^2 - 7x_1x_2 + x_2^2 + 4x_2x_5 - 3x_2x_6 - x_2x_7 + x_3x_4 - 5x_3x_5 + 2x_3x_6 - 2x_3x_7 + 4x_4^2 +\\
&\quad + 7x_4x_5 - 5x_4x_6 + 8x_4x_7 + 2x_5^2 - 5x_5x_6 + 3x_5x_7 + x_6^2 - 3x_6x_7 + 2x_7^2 = 0,\\
&x_2x_3 - 6x_2x_5 + 3x_2x_6 - x_2x_7 + x_3^2 + 2x_3x_4 + x_3x_5 + 2x_4x_5 - x_4x_6 - 2x_4x_7 +\\
&\quad + 4x_5x_6 + x_5x_7 - 2x_6^2 - x_6x_7 - x_7^2 = 0,\\
&- 4x_1^2 - 21x_1x_2 + x_2x_4 + 26x_2x_5 - 23x_2x_6 - 3x_2x_7 - 6x_3^2 - 7x_3x_4 - 15x_3x_5 + 3x_3x_6 +\\
&\quad - x_3x_7 + 9x_4^2 + 11x_4x_5 - 6x_4x_6 + 13x_4x_7 + 10x_5^2 - 29x_5x_6 + 3x_5x_7 + 10x_6^2 - 10x_6x_7\\
&\quad + 6x_7^2 = 0.
\end{aligned}
$$

In [37, file `Equations_curve_X(6,7)_mod_w6,w7.txt`] we have written the above equations in Magma language (see 12), which we computed using the Magma script `Computation_of_equations.txt` in the same repository.

In the remainder of this subsection we describe how we computed the equations above and prove the correctness of the computation.

Let $X = X(6,7)/\langle W_6, W_7\rangle$ be our maximal curve. To compute a canonical model it is enough to compute a basis of $\Omega^1(X)$, since the methods of [36] allow us to find quadrics defining the image of a canonical embedding. We have $X = Y/\langle W_6\rangle$, where $Y = X(6,7)/\langle W_7\rangle$, and the idea is to compute $\Omega^1(X)$ as the subspace of $\Omega^1(Y)$ which is W_6-invariant. Since Y is of the form X_H, the algorithm by David Zywina in [47] (see also [48, files `GL2GroupTheory.m` and `ModularCurves.m`]) computes a basis of $S_2(\Gamma_H) = \Omega^1(Y)$. In our algorithm we numerically compute a candidate S for $\Omega^1(X)$, then we compute the candidate equations (the ones above). Finally we prove, by geometric arguments, that the equations we found must be equations for X.

For the computation of S we use that W_6 acts on the upper half-plane by the matrix $m_6 = \left(\begin{smallmatrix} 6 & -1 \\ 6 & 0 \end{smallmatrix}\right)$. Given a basis $b_1, \ldots, b_{16}$ of $S_2(\Gamma_H) = \Omega^1(Y)$ we computed the values

$$b_r(\tau_s), \quad (b_r|_2 m_6)(\tau_s)$$

with precision 10^{-10} for $\tau_s = \frac{s}{10g} + i\sqrt{\frac{1}{6} - \frac{s^2}{100g^2}}$, with $s = -15, \ldots, 15$ (see [39, Section 2.1, Equation 2.1.5] for the definition of $f|_2 m$ for $m \in \mathrm{GL}_2^+(\mathbb{R})$). By solving a linear system we computed $b_r|_2 m_6$ as a linear combination of the b_i's with approximated coefficient, denote it $(b_r|_2 m_6)^\sim$. We know that $\Omega^1(X) = \Omega^1(Y)^{W_6}$ is generated by the forms $b_r + b_r|_2 m_6$ and we define S, our numerical candidate for $\Omega^1(X)$, as the span of $b_r + (b_r|_2 m_6)^\sim$; in the computation such forms $b_r + (b_r|_2 m_6)^\sim$ turned out to be linearly dependent and we extracted a basis $s_1, \ldots, s_7$. The values for τ are carefully chosen to have $\mathrm{Im}(\tau)$ as close as possible to $\mathrm{Im}(W_6\tau)$ and the coefficients of the linear combinations are rationals with small denominators (we look for denominators less than 40) up to a precision of 10^{-10} (using 5800 Fourier coefficients of the forms b_r).

A first clue for S being $\Omega^1(X)$ is that S has dimension 7. Using the method in [36] to compute canonical embeddings, we computed by [37, file `Computation_of_equations.txt`] quadratic relations which hold for a basis of forms $s_1, \ldots, s_7$ of S. Such quadratic relations are the equations above, which define a smooth curve over $\mathbb{Q}$ of genus 7 whose reduction modulo 11 is smooth and has the expected number of points over $\mathbb{F}_{11^5}$, which is the number of points of X computed independently in Section 3, giving further clues.

To turn clues into a proof, we notice that the differential forms $s_1, \ldots, s_7 \in \Omega^1(Y)$ define a map $Y \dashrightarrow X \subset \mathbb{P}^6$, which is defined over $\mathbb{Q}$ and has degree 2 by Riemann-Hurwitz Formula (indeed $s_1, \ldots, s_7$, despite being chosen with an approximation process, were chosen in $\Omega^1(Y)$, and the equations above are quadratic relations that are exactly satisfied by the s_i's). Hence X is a quotient by an automorphism u of Y defined over $\mathbb{Q}$. The automorphisms of a curve of genus at least 2 are determined by their action on the Jacobian. Hence, it is enough to prove that $u = W_6$ in the Jacobian of Y. Applying Theorem 2.2, we compute that the Jacobian J_Y of Y satisfies

$$J_Y \sim \prod_{i=1}^{6} A_{h_i}^{m_i}$$

for certain distinct newforms $h_1, \ldots, h_6$ with multiplicities $m_i \in \{1, 2\}$. Each A_{h_i} is simple and non-isomorphic to the other factors. By [32, Theorem 6], the ring of endomorphisms of J_Y defined over $\mathbb{Q}$, up to isogeny, is

$$\mathrm{End}^0_{\mathbb{Q}}(J_Y) = \mathrm{End}_{\mathbb{Q}}(J_Y) \otimes \mathbb{Q} \cong \prod_{i=1}^{6} \mathrm{End}^0_{\mathbb{Q}}(A_{h_i})^{m_i \otimes m_i} \cong \prod_{i=1}^{6} K_{h_i}^{m_i \times m_i}, \qquad (4.7)$$

where K_{h_i} is the field of coefficients of h_i and in this concrete case is either $\mathbb{Q}$ or a real quadratic extension of $\mathbb{Q}$. Using (4.7), we can write the action of u on J_Y as a collection of matrices $M_{u,i} \in K_{h_i}^{m_i \times m_i}$; analogously we have matrices $M_{W_2,i}$ and $M_{W_3,i}$ for the action of W_2 and W_3.

We now prove that u commutes with W_2 or, equivalently, that for each i the matrices $M_{u,i}$ and $M_{W_2,i}$ commute. Since u and W_2 are involutions, the matrices $M_{u,i}$ and $M_{W_2,i}$ have order at most 2, hence each matrix is either a multiple of the identity or a 2×2 matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$. Suppose by contradiction that for some i the matrices $M_{u,i}$ and $M_{W_2,i}$ do not commute. Then neither of them is a multiple of the identity, hence both of them have eigenvalues 1 and -1. In particular $\det(M_{u,i}) = \det(M_{W_2,i}) = -1$ and $\det(M_{u,i}M_{W_2,i}) = 1$. Now notice that the matrix $N_i := M_{u,i}M_{W_2,i}$ is a matrix of finite order, since, using (4.7), it describes the action of the automorphism uW_2 on a factor of the Jacobian of Y and the group $\mathrm{Aut}(Y)$ is finite. Moreover, the order of N_i is in $\{1, 2, 3, 4, 5, 6, 8, 10, 12\}$: indeed since N_i is a 2×2 matrix with entries in a quadratic extension of $\mathbb{Q}$, its eigenvalues have degree over $\mathbb{Q}$ dividing 4; the only roots of unity with degree dividing 4 are the ones whose order is in that list. If N_i has order 1, then $M_{u,i}$ and $M_{W_2,i}$ are inverse to each other, hence they commute. If N_i has order 2, using that $\det(N_i) = 1$, we deduce that $N_i = M_{u,i}M_{W_2,i} = -\mathrm{Id}$, again implying that $M_{u,i}$ and $M_{W_2,i}$ commute. In the other cases we deduce that uW_2 has order a multiple of 3, 4, or 5, contradicting the computations in [37, file `Order_3_4_5_automorphisms_test.txt`] that prove, using the test in [30], that Y has no automorphism of order 3, 4, or 5.

Analogously u commutes with W_6, i.e., with the whole $\langle W_2, W_6\rangle$. Hence the elements of $\langle W_2, W_6\rangle$ descend to automorphisms of X defined over $\mathbb{Q}$. If u were not in $\langle W_2, W_6\rangle$, then all the elements of $\langle W_2, W_6\rangle$ would descend to distinct automorphisms of X defined over $\mathbb{Q}$, contradicting the Magma computations given by [37, file `Check_non-modular_automorphisms_mod_5.txt`], which show that $\#\mathrm{Aut}(X_{\mathbb{F}_5}) \leq 2$, hence $\#\mathrm{Aut}(X_{\mathbb{Q}}) \leq 2$ (indeed the composition $\mathrm{Aut}(X_{\mathbb{Q}}) \to \mathrm{Aut}(X_{\mathbb{F}_5}) \to \mathrm{End}(J_{\mathbb{F}_5})$ is equal to $\mathrm{Aut}(X_{\mathbb{Q}}) \to \mathrm{End}(J_{\mathbb{Q}}) \to \mathrm{End}(J_{\mathbb{F}_5})$ which is injective). Hence u lies in the group $\langle W_2, W_6\rangle$. Furthermore, by looking at the genus of $Y/\langle W_2\rangle$ we exclude the case $u = W_2$ and by looking at the number of points of the quotient curves $Y/\langle W_3\rangle$ and $Y/\langle W_6\rangle$ over some finite field (computed in Examples 3.5 and 3.6), we see that u must be equal to W_6, i.e., that the above equations actually describe the curve $X = Y/\langle W_6\rangle$.

4.2. A planar singular model for $X(6,7)/\langle W_6, W_7\rangle$

We can obtain a (singular) planar model of $X(6,7)/\langle W_6, W_7\rangle$ by projecting the canonical model onto a plane. In practice, we did it by a linear change of variables and eliminating all variables except the first three. Here is a planar equation for our curve, after dehomogenizing and reducing modulo 11:

$$(9x-x^2)y^9+(4x^3+4x^2+2-1)y^8+(7x^4-x^3+4x^2+3)y^7+(5x^4+8x^3+x^2+2x+1)y^6$$
$$+(8x^6+3x^5+x^4+9x^3+3x+5)y^5$$

$$+(9x^7+7x^6+2x^5+x^4+9x^3+4x+6)y^4+(8x^8+6x^7+8x^5-x^4+6x^3+8x^2+7x+3)y^3$$
$$+(3x^8+8x^6+3x^5+5x^4+2x^3-x^2$$
$$+x+1)y^2+(2x^{10}+9x^9+9x^8+5x^7+6x^6+7x^5+6x^4+5x^2+6x)y+7x^{11}+5x^{10}+2x^9-x^8+6x^7$$
$$-x^6+7x^5+4x^3=0.$$

As it can be computed using the dedicated Magma command, the above equation defines a curve whose normalization of the projective closure, which we denote $\widetilde{X}$, has genus 7. It is not so hard to (re)prove, with the help of Magma, that $\widetilde{X}$ has 166666 points over $\mathbb{F}_{11^5}$: the closure in $\mathbb{P}^2_{\mathbb{F}_{11^5}}$ of the above planar model has 166665 rational points of which three are singular: $(0:0:1)$, $(0:1:0)$ and $(9:7:1)$; the normalization $\widetilde{X}$ contains 4 rational points lying over these 3 singularities since all of them are nodal, the first two are split while the third is not.

The above equation can be found at [37, file `Equations_curve_X(6,7)_mod_w6,w7.txt`] in Magma readable format.

4.3. Equations for $\boldsymbol{X(156,1)/\langle W_{13}\rangle}$

In the family of curves we considered, there is another maximal curve, of genus 12, namely $X(156,1)/\langle W_{13}\rangle$, which has 170676 points over $\mathbb{F}_{11^5}$. We also computed equations for its canonical embedding which can be found in Magma readable format at [37, file `Equations_curve_X0(156)_mod_w13.txt`].

This curve is actually the same as $X_0(156)/\langle W_{13}\rangle$, thus to compute equations we used classical methods such as those in [26] and [36]:

- We computed the q-expansion of a basis of eigenforms $\{f_1,\dots,f_g\}$ of $\Omega^1(X_0(156))$ up to q^m, with $m=180$.
- Using Magma's exact computation of the Atkin-Lehner eigenvalues, we computed a basis $g_1,\dots,g_{12}$ of the space of invariants $\Omega^1(X_0(156))^{W_{13}}=\Omega^1(X)$ (again we computed the q-expansion of these forms up to q^{180}).
- We looked for quadratic polynomials F_j such that $F_j(g_1,\dots,g_{12})=O(q^{180})$: since $F_j(g_1,\dots,g_{12})$ is a section of the sheaf $(\Omega^1_X)^{\otimes 2}$ which has degree $4g-4=44$, then $F_j(g_1,\dots,g_{12})$ must be exactly 0, otherwise it would have more than 44 zeroes, which is absurd (see also [9, Section 2.1, Lemma 2.2, p. 1329]).
- After collecting enough F_j's we realized that they cut a curve $C\subset\mathbb{P}^{11}$ of genus 12 and that this curve C is isomorphic to X: indeed the image of the canonical map $X\to\mathbb{P}^{11}$ given by the sections g_i is contained in C and, by Riemann-Hurwitz, the induced map $X\to C$ is an isomorphism.

A planar singular model for the curve, when reduced modulo 11, is given by the following equation:

$$
\begin{aligned}
&x^{19}(6+2y)+x^{18}\left(-1+5y+5y^2\right)+x^{17}\left(8-y+7y^2+y^3\right)+x^{16}\left(-y^2+4y^3+y^4\right)+\\
&x^{15}\left(3y+2y^2+y^5\right)+x^{14}\left(4+6y+2y^2+9y^4-y^5+5y^6\right)+\\
&x^{13}\left(2+y+8y^2+2y^3+y^4+3y^5+y^6+3y^7\right)+x^{12}\left(8+4y+9y^2+4y^3+2y^4+5y^5+y^6+\right.\\
&\left.9y^7-y^8\right)+x^{11}\left(9-y+3y^2-y^3+3y^4+y^5+6y^6+8y^7+9y^8+4y^9\right)+x^{10}\left(6-y+6y^2-y^3+\right.\\
&\left.6y^5+5y^6-y^7+6y^8+6y^9+4y^{10}\right)+x^9\left(9+4y+8y^2+y^3-y^4+8y^5+9y^6+4y^7+8y^8+6y^{10}\right)+\\
&x^8\left(2+3y+9y^2+2y^3+4y^4+6y^5+2y^6+4y^7+9y^8+4y^9+6y^{10}+7y^{11}+4y^{12}\right)+x^7\left(5+8y+\right.\\
&\left.6y^2+8y^3+6y^4-y^5+8y^6+y^7+4y^8+8y^9+2y^{10}+y^{11}+7y^{13}\right)+x^6\left(1+9y+3y^2+4y^3+3y^4+\right.\\
&\left.y^5+6y^6+4y^7+7y^8+7y^9+4y^{10}+2y^{11}+4y^{12}+2y^{13}+5y^{14}\right)+x^5\left(4+7y+5y^2+9y^3-y^4+\right.\\
&\left.2y^5+2y^6+3y^7+3y^8+y^9+5y^{10}+8y^{12}+9y^{13}+3y^{14}+2y^{15}\right)+x^4\left(4-y+3y^2+5y^3+2y^4+\right.\\
&\left.6y^5+7y^7+y^8+5y^{12}+5y^{14}+7y^{15}+9y^{16}\right)+x^3\left(9-y+9y^2+6y^3+6y^4-y^6+2y^7+7y^8+8y^9+\right.\\
&\left.4y^{10}+y^{11}+7y^{12}+7y^{13}+y^{14}+y^{15}+3y^{16}+3y^{17}\right)+x^2\left(4+9y^2+y^3+6y^4-y^5+5y^6+\right.\\
&\left.5y^7-y^8+y^9-y^{10}+2y^{11}+3y^{13}+2y^{15}+9y^{16}+9y^{17}+8y^{18}\right)+x\left(3+7y-y^2+3y^3+4y^4+\right.\\
&\left.9y^5-y^6+8y^7+3y^8+3y^9+y^{10}+8y^{11}+4y^{12}+5y^{13}+3y^{16}+9y^{17}+3y^{18}+6y^{19}\right)+5+y+6y^2+\\
&6y^3+9y^4+2y^6-y^7+2y^8+3y^9+7y^{10}+7y^{11}+3y^{12}+3y^{13}+6y^{14}+9y^{15}+3y^{16}+2y^{17}+3y^{19}+\\
&y^{20}=0.
\end{aligned}
$$

It has been obtained from the canonical model by projection onto a plane, i.e., in practice by a linear change of variable plus variable elimination plus dehomogenization. The previous equation in Magma readable format can be found at [37, file `Equations_curve_X0(156)_mod_w13.txt`].

5. Other maximal quotient modular curves

Besides the two curves analyzed in Section 4, inside the set of curves studied in Section 3 we found several other examples of maximal curves. We did not compute explicit equations for all these examples, since, as tracked by [27], a maximal curve for that genus and field is already known. We collected these instances for genus larger than 1, and we computed the eigenvalues of Frobenius and the real Weil polynomial. The data is shown in Appendix C.

For some pairs (genus, finite field), we find many modular curves that are maximal and, in some cases, they have different real Weil polynomial. In particular this shows the existence of non-isomorphic maximal curves. This problem has also been studied by Rigato: in [42] she proved that for $q = 2$ and $g \leq 5$ the maximal curves are all isomorphic over $\mathbb{F}_2$ and she shows the existence of non-isomorphic maximal curves of genus 6 and 7. We also point out that all the maximal curves of genus 4 over $\mathbb{F}_{2^7}$ are isomorphic by the uniqueness remark made by Everett Howe in [27] (see [28] for a static version). In particular all curves in the corresponding line of Table C.12 are isomorphic over $\mathbb{F}_{2^7}$.

We now make more explicit a case of genus 4 where we observe more than one isomorphism class of maximal curves, and one case of genus 6 where explicit equations can be given.

Example 5.8. In genus 4 over the field $\mathbb{F}_{2^2}$ we observe at least four isomorphism classes of maximal curves, among the curves we analyzed (see Table 3.1). The following three curves have different real Weil polynomial (as in Table C.12):

$$X(1,11) = X_{\mathrm{ns}}(11) : \begin{cases} x^2 + xy + xz + xw + y^2 + yz + yw + z^2 + zw = 0 \\ x^2z + x^2w + xz^2 + xw^2 + y^3 + yzw + yw^2 + z^3 + zw^2 \\ \quad +w^3 = 0, \end{cases}$$

$$X(81,1) = X_0(81) : \begin{cases} xw + y^2 = 0 \\ x^3 + y^3 + z^3 + z^2w + zw^2 = 0, \end{cases}$$

$$X(251,1)/\langle W_{251}\rangle = X_0^+(251) : \begin{cases} x^2 + xy + yz + yw + zw = 0 \\ x^2z + xz^2 + y^2w + yw^2 = 0. \end{cases}$$

The equations of $X_{\mathrm{ns}}(11)$ can be found in [35], while equations (over $\mathbb{Q}$) for the other curves were computed with the code [37, file `Equations_other_maximal_curves.txt`]. Although we only found three distinct real Weil polynomials, the isomorphism classes are at least 4, in fact one can check with Magma that

$$X(177,1)/\langle W_{59}\rangle = X_0(177)/\langle W_{59}\rangle : \begin{cases} x^2 + xy + y^2 + yz + z^2 + zw = 0 \\ x^2z + xz^2 + xzw + y^2w + yzw + yw^2 + z^2w = 0 \end{cases}$$

is not isomorphic to $X_0(81)$ over $\mathbb{F}_{2^2}$ even if they have the same real Weil polynomial.

Example 5.9. In genus 6 and over the field $\mathbb{F}_{2^2}$, we find only one curve. Since there are no explicit equations on website [27], we display below some equations for the curve we found, computed with the code in [37, file `Equations_other_maximal_curves.txt`].

$$X(299,1)/\langle W_{299}\rangle = X_0^+(299) : \begin{cases} x_1^2 + x_2^2 + x_2x_4 + x_2x_6 + x_3x_4 + x_5x_6 = 0 \\ x_1x_2 + x_2^2 + x_2x_3 + x_2x_6 + x_3^2 + x_3x_5 + x_4^2 + x_5x_6 = 0 \\ x_1x_3 + x_2x_6 + x_5x_6 = 0 \\ x_1x_4 + x_2x_3 + x_3x_5 + x_3x_6 + x_4^2 = 0 \\ x_1x_5 + x_2x_3 + x_3^2 + x_3x_6 + x_4x_6 + x_5x_6 = 0 \\ x_1x_6 + x_2x_5 + x_3x_4 + x_3x_5 + x_3x_6 = 0. \end{cases}$$

Acknowledgments

We thank Jeremy Rouse for the helpful discussion and the whole LMFDB team for their remarkable work, which we also use here.

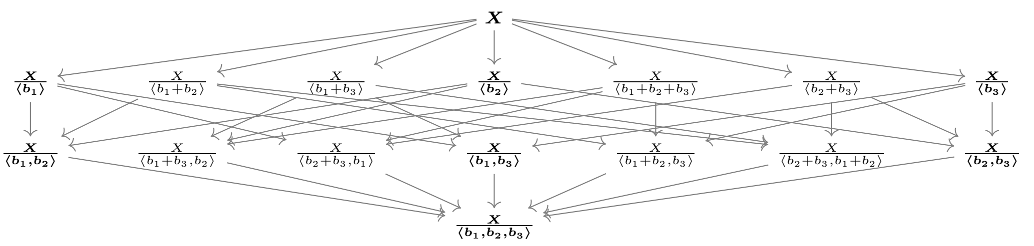
The second and the third authors are supported by the National Group for Algebraic and Geometric Structures and their Applications (GNSAGA - INdAM). The second author is also supported by the MUR Excellence Department Project MatMod@TOV (CUP

E83C23000330006) awarded to the Department of Mathematics, University of Rome Tor Vergata, by the “Programma Operativo (PON) “Ricerca e Innovazione” 2014-2020” and by Prin PNRR 2022 Mathematical Primitives for Post Quantum Digital Signatures (CUP D53D23018900001).

Appendix A. Jacobians of quotient curves

The Atkin-Lehner automorphisms on a curve $X_0(n)$ and, slightly more in general, the automorphisms that we consider for a curve $X(n_0, n_{\mathrm{ns}})$, form a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$. In the general setting where X is a smooth projective curve over a field with an action of $(\mathbb{Z}/2\mathbb{Z})^r$, this appendix gives some elementary results regarding the Jacobians of the quotient curves X/W for W a subgroup of $(\mathbb{Z}/2\mathbb{Z})^r$.

In this setting, all the possible curves X/W form a lattice, for example when $r = 3$ we get the following, where $\{b_1, b_2, b_3\}$ is a basis of $(\mathbb{Z}/2\mathbb{Z})^3$.



Given W a subgroup of $(\mathbb{Z}/2\mathbb{Z})^r$, we denote

$$J_W := \mathrm{Jac}(X/W). \tag{A.1}$$

In Theorem A.5 we prove that a few Jacobians J_{W_i} (in the above diagram they correspond to the bold curves) determine all the other J_W. In the case $r = 2$ we have the following statement.

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[d] \arrow[dr] & \\
X/\langle\sigma\rangle \arrow[dr] & X/\langle\tau\rangle \arrow[d] & X/\langle\sigma+\tau\rangle \arrow[dl] \\
 & X/\langle\sigma,\tau\rangle &
\end{tikzcd}$$

Lemma A.2. *Let X be a smooth projective curve which is acted upon by the group $(\mathbb{Z}/2\mathbb{Z})^2$, generated by elements σ and τ. Denote by J the Jacobian of X and let J_W be as in (A.1). Then, there is an isogeny*

$$J \times J_{\langle\sigma,\tau\rangle}^2 \sim J_{\langle\sigma\rangle} \times J_{\langle\tau\rangle} \times J_{\langle\sigma+\tau\rangle}.$$

Proof. Given two signs $\epsilon_\sigma, \epsilon_\tau \in \{+,-\}$, denote by $J^{\epsilon_\sigma\,\epsilon_\tau}$ the connected component of the identity of the subgroup scheme $\ker(\sigma^* - \epsilon_\sigma \operatorname{Id}) \cap \ker(\tau^* - \epsilon_\tau \operatorname{Id})$ in J. In other words, $J^{\epsilon_\sigma\,\epsilon_\tau}$ is the maximal abelian subvariety of J where σ acts as $\epsilon_\sigma \operatorname{Id}$ and τ acts as $\epsilon_\tau \operatorname{Id}$. The inclusions $J^{\epsilon_\sigma\,\epsilon_\tau} \subset J$ induce an isogeny

$$J \sim J^{++} \times J^{+-} \times J^{-+} \times J^{--}\,. \tag{A.3}$$

Finally recall that the Jacobian of X/W is isogenous, by pullback of divisors, to the connected component of the identity of the W-invariant part of J. Hence our statement follows from Equation (A.3) together with the identities

$$J_{\langle\sigma,\tau\rangle} \sim J^{++}, \qquad J_{\langle\sigma\rangle} \sim J^{++} \times J^{+-}, \qquad J_{\langle\tau\rangle} \sim J^{++} \times J^{-+}, \qquad J_{\langle\sigma+\tau\rangle} \sim J^{+-} \times J^{-+}.$$

□

The following lemma generalizes Lemma A.2. Given a subset I of $(\mathbb{Z}/2\mathbb{Z})^r$, we denote by $J_I = J_{\langle I\rangle}$ the Jacobian of $X/\langle I\rangle$, with $\langle I\rangle$ the subgroup generated by I. Moreover, the statement of the lemma contains a product $\prod A_i^{e_i}$ of abelian varieties A_i with possibly negative exponents e_i: we interpret this product as a formal object living in the Grothendieck group of the category of abelian varieties up to isogeny; *a posteriori*, the lemma itself implies that the product below is a real abelian variety, i.e., after decomposing every factor in irreducible abelian varieties, every irreducible factor appearing with negative exponents cancels out.

Lemma A.4. *Let X be a smooth projective curve with an action of $(\mathbb{Z}/2\mathbb{Z})^r$ and let $\{b_1, \dots, b_r\}$ be a basis of $(\mathbb{Z}/2\mathbb{Z})^r$. For all subsets $K \subset (\mathbb{Z}/2\mathbb{Z})^r$ and all non-negative integers $s \le r$, we have the following isogeny:*

$$J_{K\cup\{b_1+\cdots+b_s\}} \sim J_K^{\frac{1+(-1)^s}{2}} \times \prod_{j=1}^{s} \Bigg(\prod_{\substack{I \subseteq \{b_1,\dots,b_s\} \\ \#I = j}} J_{I\cup K} \Bigg)^{(-1)^{j+s}\cdot 2^{j-1}}.$$

Proof. To simplify formulas, we use the additive notation for the product of abelian varieties. The proof is by induction: the case $s = 0$ is immediate, so we assume that the equality holds for $s-1$. By Lemma A.2 applied to the curve $X/\langle K\rangle$, with $\sigma = b_1 + \cdots + b_{s-1}$ and $\tau = b_s$, we have

$$J_{\{b_1+\cdots+b_s\}\cup K} \sim J_K + 2J_{\{b_1+\cdots+b_{s-1}, b_s\}\cup K} - J_{\{b_1+\cdots+b_{s-1}\}\cup K} - J_{\{b_s\}\cup K}.$$

By inductive hypothesis on the second and the third summand, we obtain

$$
\begin{aligned}
J_{\{b_1+\dots+b_s\}\cup K} \sim J_K &+ 2\left(\tfrac{1-(-1)^s}{2} J_{\{b_s\}\cup K} + (-1)^s \sum_{j=1}^{s-1} (-2)^{j-1} \sum_{\substack{I\subseteq\{b_1,\dots,b_{s-1}\}\\ \#I=j}} J_{I\cup K\cup\{b_s\}}\right) + \\
&- \left(\tfrac{1-(-1)^s}{2} J_K + (-1)^s \sum_{j=1}^{s-1} (-2)^{j-1} \sum_{\substack{I\subseteq\{b_1,\dots,b_{s-1}\}\\ \#I=j}} J_{I\cup K}\right) - J_{\{b_s\}\cup K} \\
\sim \tfrac{1+(-1)^s}{2} J_K &+ (-1)^{s+1} \sum_{j=1}^{s-1} (-2)^{j-1} \sum_{\substack{I\subseteq\{b_1,\dots,b_{s-1}\}\\ \#I=j}} J_{I\cup K} \\
&+ (-1)^{s+1} \sum_{j=1}^{s-1} (-2)^{j} \sum_{\substack{I\subseteq\{b_1,\dots,b_{s-1}\}\\ \#I=j}} J_{I\cup K\cup\{b_s\}} + (-1)^{s+1} J_{\{b_s\}\cup K} \\
\sim \tfrac{1+(-1)^s}{2} J_K &+ (-1)^{s+1} \sum_{j=1}^{s-1} (-2)^{j-1} \sum_{\substack{I\subseteq\{b_1,\dots,b_{s}\}\\ \#I=j \text{ and } b_s\notin I}} J_{I\cup K} \\
&+ (-1)^{s+1} \sum_{j=2}^{s} (-2)^{j} \sum_{\substack{I\subseteq\{b_1,\dots,b_{s-1}\}\\ \#I=j-1}} J_{I\cup K\cup\{b_s\}} + (-1)^{s+1} J_{\{b_s\}\cup K} \\
\sim \tfrac{1+(-1)^s}{2} J_K &+ (-1)^{s+1} \sum_{j=1}^{s} (-2)^{j-1} \sum_{\substack{I\subseteq\{b_1,\dots,b_{s}\}\\ \#I=j \text{ and } b_s\notin I}} J_{I\cup K} + (-1)^{s+1} \sum_{j=1}^{s} (-2)^{j-1} \sum_{\substack{I\subseteq\{b_1,\dots,b_{s}\}\\ \#I=j \text{ and } b_s\in I}} J_{I\cup K} \\
\sim \tfrac{1+(-1)^s}{2} J_K &+ (-1)^{s+1} \sum_{j=1}^{s} (-2)^{j-1} \sum_{\substack{I\subseteq\{b_1,\dots,b_{s}\}\\ \#I=j}} J_{I\cup K}. \quad \square
\end{aligned}
$$

Theorem A.5. *Let X be a smooth projective curve with an action of $(\mathbb{Z}/2\mathbb{Z})^r$ and let B be a basis of $(\mathbb{Z}/2\mathbb{Z})^r$. For all subgroups W of $(\mathbb{Z}/2\mathbb{Z})^r$, the Jacobian J_W is isogenous to an integer combination of the Jacobians $J_{\langle S\rangle}$, with S varying among the subsets of B.*

Proof. Choose a set I of generators for W and apply Lemma A.4 repeatedly to each element of I, which can be written as sum of elements of B. $\square$

The above results imply analogous "numerical" corollaries. The next corollary, about genera, can also be proven directly, controlling the ramification of the quotient maps and applying Riemann-Hurwitz Formula.

Corollary A.6. *Let X be a smooth projective curve with an action by $(\mathbb{Z}/2\mathbb{Z})^r$ and let $B = \{b_1, \dots, b_r\}$ be a basis of $(\mathbb{Z}/2\mathbb{Z})^r$. For I a subset of $(\mathbb{Z}/2\mathbb{Z})^r$, we denote by g_I the genus of $X/\langle I \rangle$.*

For all subsets K of $(\mathbb{Z}/2\mathbb{Z})^r$ and all non-negative integers $s \le r$, we have

$$g_{\{b_1+\cdots+b_s\}\cup K} = \tfrac{1+(-1)^s}{2} g_K + (-1)^{s+1} \sum_{j=1}^{s} (-2)^{j-1} \sum_{\substack{I \subseteq \{b_1,\dots,b_s\} \\ \#I = j}} g_{I\cup K}.$$

Moreover, for each subset I of $(\mathbb{Z}/2\mathbb{Z})^r$, the genus g_I can be written as an integral linear combination of the genera g_S with S varying among all the subsets of B.

Proof. It follows by Lemma A.4 and Theorem A.5 recalling that the genus of a curve is the dimension of the Jacobian as an abelian variety. □

Corollary A.7. *Let X be a smooth projective curve over a finite field $\mathbb{F}_q$, with an action of $(\mathbb{Z}/2\mathbb{Z})^r$ and let $B = \{b_1, \dots, b_r\}$ be a basis of $(\mathbb{Z}/2\mathbb{Z})^r$.*

For all subsets K of $(\mathbb{Z}/2\mathbb{Z})^r$ and all non-negative integer $s \le r$, we have:

$$\begin{aligned} &\#(X/\langle \{b_1+\cdots+b_s\}\cup K\rangle)(\mathbb{F}_q) \\ =&\tfrac{1+(-1)^s}{2} \#(X/\langle K\rangle)(\mathbb{F}_q) + (-1)^{s+1} \sum_{j=1}^{s} (-2)^{j-1} \sum_{\substack{I \subseteq \{b_1,\dots,b_s\} \\ \#I = j}} \#(X/\langle I\cup K\rangle)(\mathbb{F}_q). \end{aligned}$$

Moreover, given I a subset of $(\mathbb{Z}/2\mathbb{Z})^r$, then $\#(X/\langle I\rangle)(\mathbb{F}_q)$ can be written as integral linear combination of $\#(X/\langle S\rangle)(\mathbb{F}_q)$ with S varying among all the subsets of B.

Proof. It follows by Lemma A.4 and Theorem A.5 observing that if an abelian variety J is isogenous to $J_1 \times J_2$, then the trace of Frobenius satisfies $\mathrm{tr}(\mathrm{Frob}) = \mathrm{tr}(\mathrm{Frob}_1) + \mathrm{tr}(\mathrm{Frob}_2)$, where, for a prime power q, we denote by Frob, Frob_1 and Frob_2 the q-th Frobenius on J, J_1 and J_2 respectively. □

Example A.8. Suppose that X is (a quotient of) a curve $X(n_0, n_{\mathrm{ns}})$ and let $\{q_1, \dots, q_s\}$ be a set of coprime prime powers dividing exactly the level $n_0 n_{\mathrm{ns}}$ of X. If we denote by $g(\cdot)$ the genus, we have

$$g(X/\langle W_{q_1\cdots q_s}\rangle) = \tfrac{1-(-1)^{s+1}}{2} g(X) + (-1)^{s+1} \sum_{j=1}^{s} (-2)^{j-1} \sum_{\substack{I \subseteq \{W_{q_1},\dots,W_{q_s}\} \\ \#I = j}} g(X/\langle I\rangle).$$

Using this formula we compute once again the genus of the record curve $X/\langle W_6\rangle$, with $X = X(6,7)/\langle W_7\rangle$. From the algorithm in [18] we get: $g(X) = 16$, $g(X/\langle W_2\rangle) = 6$, $g(X/\langle W_3\rangle) = 7$ and $g(X/\langle W_2, W_3\rangle) = 2$. The above formula gives

$$g(X/\langle W_6\rangle) = g(X) - g(X/\langle W_2\rangle) - g(X/\langle W_3\rangle) + 2g(X/\langle W_2, W_3\rangle) = 16 - 6 - 7 + 4 = 7,$$

in accordance with Example 3.5 and Section 4.

Appendix B. Tables for the number of points

In the following tables (B.2, B.3, B.4, B.5, B.6, B.7, B.8, B.9, B.10) we list the quotient curves $X(n_0, n_{\mathrm{ns}})/\langle W_{d_1}, \ldots, W_{d_k}\rangle$ achieving the largest number of points over each pair (genus, finite field). All items improving the actual best known bounds in [27] are in bold.

We explain the notation of the tables which was used to save space. Let

$$n_0 n_{\mathrm{ns}} = \prod_i p_i^{v_i} = \prod_i q_i$$

be the prime factorization of $n_0 n_{\mathrm{ns}}$, with $p_1 < p_2 < \ldots$ and $q_i = p_i^{v_i}$. Then each exact divisor d_h is uniquely the product of certain q_j's and, in the tables, we denote $X(n_0, n_{\mathrm{ns}})/\langle W_{d_1}, \ldots, W_{d_k}\rangle$ as

$$(n_0, n_{\mathrm{ns}})\,\{i_{1,1}, \ldots, i_{1,m_1}; \ldots\ldots\ldots; i_{k,1}, \ldots, i_{k,m_k}\}$$

where for each $h = 1, \ldots, k$,

$$d_h = \prod_{j=1}^{m_h} q_{i_{h,j}}.$$

As an example the entry "$(945, 1)\{2; 1, 3\}, 49$", in the table below corresponding to row $g = 27$ and column $\mathbb{F}_{2^2}$, means that the quotient of $X(945, 1)$ by $\langle W_5, W_{189}\rangle$ has 49 points over $\mathbb{F}_{2^2}$, where $d_1 = 5 = p_2$ and $d_2 = 189 = p_1^3 p_3$ since, in this case, $p_1 = 3$, $p_2 = 5$ and $p_3 = 7$.

Another example is the entry "$(418, 3)\{2; 1, 3; 3, 4\}, 4261$", in Table B.5, row $g = 41$ and column $\mathbb{F}_{5^5}$, that means that the quotient of $X(418, 3)$ by $\langle W_3, W_{22}, W_{209}\rangle$ has 4261 points over $\mathbb{F}_{5^5}$, where $d_1 = 3 = p_2$ and $d_2 = 22 = p_1 p_3$ and $d_3 = 209 = p_3 p_4$ since, in this case, $p_1 = 2$, $p_2 = 3$, $p_3 = 11$ and $p_4 = 19$.

Table B.2
Table for $\max\{|X(n_0, n_{\mathrm{ns}})/\langle W_{d_1}, \ldots, W_{d_t}\rangle(\mathbb{F}_q)|\}$ with $q = 2^k$, $k = 1, 2, 3$ and $n_0 n_{\mathrm{ns}}^2 \le 10^4$.

g	$\mathbb{F}_2$	$\mathbb{F}_{2^2}$	$\mathbb{F}_{2^3}$
1	$(11, 1)\{\}, 5$	$(1, 11)\{1\}, 9$	$(1, 7)\{\}, 14$
2	$(67, 1)\{1\}, 6$	$(23, 1)\{\}, 10$	$(35, 1)\{2\}, 16$
3	$(97, 1)\{1\}, 7$	$(17, 5)\{1; 2\}, 14$	$(3, 7)\{1, 2\}, 16$
4	$(47, 3)\{1; 2\}, 7$	$(1, 11)\{\}, 15$	$(119, 1)\{2\}, 16$
5	$(145, 1)\{1, 2\}, 8$	$(209, 1)\{1, 2\}, 17$	$(3, 7)\{2\}, 17$
6	$(223, 1)\{1\}, 9$	$(299, 1)\{1, 2\}, 20$	$(9, 7)\{1; 2\}, 17$
7	$(187, 1)\{1, 2\}, 10$	$(189, 1)\{1, 2\}, 21$	$(35, 3)\{1, 2, 3\}, 16$

(continued on next page)

Table B.2 (*continued*)

g	$\mathbb{F}_2$	$\mathbb{F}_{2^2}$	$\mathbb{F}_{2^3}$
8	$(427,1)\{1;2\},10$	$(545,1)\{1;2\},22$	$(175,1)\{2\},25$
9	$(273,1)\{2;3\},10$	$(341,1)\{1,2\},25$	$(3,7)\{\},26$
10	$(307,1)\{1\},10$	$(13,15)\{1;3;2\},27$	$(343,1)\{1\},17$
11	$(253,1)\{1,2\},11$	$(285,1)\{2,3\},26$	$(35,3)\{3\},24$
12	$(219,1)\{2\},11$	$(15,7)\{1;2,3\},29$	$(9,7)\{1,2\},17$
13	$(399,1)\{2;3\},12$	$(135,1)\{\},30$	$(621,1)\{1;2\},22$
14	$(115,3)\{2;1,3\},10$	$(155,3)\{2;1,3\},32$	$(115,3)\{2;1,3\},16$
15	$(433,1)\{1\},13$	$(879,1)\{1;2\},33$	$(9,7)\{1\},26$
16	$(291,1)\{2\},13$	$(569,1)\{1\},34$	$(561,1)\{1;2,3\},18$
17	$(309,1)\{2\},12$	$(5,11)\{1,2\},36$	$(39,5)\{1;2,3\},21$
18	$(427,1)\{1,2\},13$	$(897,1)\{1;2,3\},36$	$(1155,1)\{4;1,3;2,3\},18$
19	$(493,1)\{1,2\},14$	$\mathbf{(545,1)\{1,2\},40}$	$(13,7)\{1,2\},22$
20	$(805,1)\{2;1,3\},13$	$(861,1)\{3;1,2\},38$	$(805,1)\{2;1,3\},19$
21	$(517,1)\{1,2\},13$	$(689,1)\{1,2\},44$	$(245,1)\{\},28$
22	$(577,1)\{1\},16$	$\mathbf{(121,3)\{1,2\},43}$	$(175,3)\{3;1,2\},25$
23	$(345,1)\{2,3\},14$	$(483,1)\{2,3\},44$	$(219,1)\{\},18$
24	$(643,1)\{1\},16$	$(1469,1)\{1;2\},44$	$(765,1)\{2;1,3\},21$
25	$(795,1)\{1;2,3\},15$	$(13,15)\{2;1,3\},48$	$(175,3)\{2;3\},24$
26	$(203,3)\{1;2,3\},15$	$(259,3)\{1;2,3\},49$	$(1029,1)\{1;2\},21$
27	$(733,1)\{1\},16$	$(945,1)\{2;1,3\},49$	$(1191,1)\{1;2\},21$
28	$(787,1)\{1\},15$	$(941,1)\{1\},52$	$(175,3)\{1;3\},25$
29	$(697,1)\{1,2\},16$	$\mathbf{(1091,1)\{1\},55}$	$(95,3)\{2\},26$
30	$(1263,1)\{1;2\},16$	$(319,3)\{1;2,3\},53$	$(1035,1)\{2;1,3\},23$
31	$(517,3)\{1;3;2\},14$	$(1,57)\{1;2\},54$	$(621,1)\{2\},32$
32	$(57,5)\{2;3\},16$	$(245,3)\{2;1,3\},55$	$(235,3)\{2;1,3\},26$
33	$(817,1)\{1,2\},19$	$(349,3)\{1;2\},56$	$(665,1)\{1,2,3\},22$
34	$(943,1)\{1,2\},18$	$(1109,1)\{1\},60$	$(943,1)\{1,2\},21$
35	$(561,1)\{2,3\},20$	$(1139,1)\{1,2\},61$	$(7,15)\{3\},22$
36	$(1065,1)\{1,3;2,3\},16$	$(305,3)\{1;3\},59$	$(245,3)\{1;2,3\},23$
37	$(669,1)\{2\},18$	$(675,1)\{2\},63$	$(15,7)\{2\},26$
38	$(1479,1)\{1;2,3\},17$	$\mathbf{(29,7)\{1,2\},65}$	$(997,1)\{1\},21$
39	$(913,1)\{1,2\},17$	$\mathbf{(897,1)\{2,3\},68}$	$(665,1)\{2,3\},26$
40	$(353,3)\{1;2\},20$	$(1559,1)\{1\},66$	$(353,3)\{1;2\},23$
41	$(1281,1)\{2;3\},20$	$(1391,1)\{1,2\},65$	$(715,1)\{2\},28$
42	$(1123,1)\{1\},21$	$(1361,1)\{1\},73$	$(1123,1)\{1\},24$
43	$(973,1)\{1,2\},19$	$(1635,1)\{1;2,3\},68$	$(13,7)\{\},26$
44	$(1147,1)\{1,2\},21$	$\mathbf{(1169,1)\{1,2\},70}$	$(1731,1)\{1;2\},23$
45	$(1207,1)\{1,2\},20$	$(451,3)\{1;2,3\},74$	$(31,7)\{1,2\},24$
46	$(2257,1)\{1;2\},18$	$(1349,1)\{1,2\},76$	$(1863,1)\{1;2\},25$
47	$(377,3)\{1;2,3\},18$	$\mathbf{(51,7)\{1;2,3\},75}$	$(1155,1)\{1,4;2,4\},26$
48	$(1237,1)\{1\},21$	$(1679,1)\{1,2\},76$	$(1237,1)\{1\},27$
49	$(1683,1)\{1;2,3\},20$	$\mathbf{(1023,1)\{2,3\},82}$	$(735,1)\{1,2\},28$
50	$(1929,1)\{1;2\},21$	$(2067,1)\{2;3\},78$	$(1929,1)\{1;2\},24$

Table B.3
Table for $\max\{|X(n_0,n_{\mathrm{ns}})/\langle W_{d_1},\dots,W_{d_t}\rangle(\mathbb{F}_q)|\}$ with $q=2^k$, $k=4,5,6,7$ and $n_0n_{\mathrm{ns}}^2\le 10^4$.

g	$\mathbb{F}_{2^4}$	$\mathbb{F}_{2^5}$	$\mathbb{F}_{2^6}$	$\mathbb{F}_{2^7}$
1	$(3,5)\{1\},25$	$(11,3)\{1;2\},44$	$(1,11)\{1\},81$	$(11,1)\{\},145$
2	$(1,9)\{\},31$	$(13,3)\{2\},51$	$(3,7)\{1;2\},91$	$(23,1)\{\},172$
3	$(43,1)\{\},33$	$(11,3)\{2\},63$	$(105,1)\{1;2\},103$	$(33,1)\{\},174$
4	$(69,1)\{1\},32$	$(61,1)\{\},53$	$(1,15)\{1\},119$	$(29,3)\{1,2\},215$
5	$(555,1)\{1;3;2\},33$	$(13,3)\{\},62$	$(35,3)\{1,3;2,3\},129$	$(213,1)\{2\},198$
6	$(111,1)\{2\},45$	$(297,1)\{1;2\},66$	$(117,1)\{2\},143$	$(153,1)\{2\},199$
7	$(135,1)\{1\},39$	$(423,1)\{1;2\},56$	$(1,15)\{\},110$	$(69,1)\{\},228$
8	$(37,3)\{2\},43$	$(155,1)\{1\},64$	$(5,9)\{1\},113$	$(273,1)\{1;2\},201$
9	$(171,1)\{2\},62$	$(195,1)\{1,3\},64$	$(255,1)\{2;3\},153$	$(483,1)\{1,3;2,3\},285$
10	$(127,1)\{\},40$	$(65,3)\{1,3;2,3\},67$	$(175,3)\{1;3;2\},126$	$(483,1)\{1;2,3\},215$
11	$(43,3)\{1\},43$	$(247,1)\{2\},65$	$(35,3)\{1,3\},136$	$(207,1)\{1\},228$
12	$(513,1)\{1;2\},56$	$(47,3)\{2\},59$	$(85,3)\{1,3;2,3\},143$	$(321,1)\{2\},269$
13	$(9,5)\{\},60$	$(265,1)\{2\},76$	$(195,1)\{3\},142$	$(1105,1)\{1;3;2\},211$
14	$(15,7)\{3;1,2\},51$	$(297,1)\{2\},93$	$(3,11)\{2\},135$	$(645,1)\{2;3\},231$
15	$(555,1)\{2;3\},61$	$(79,3)\{2\},72$	$(279,1)\{2\},170$	$(483,1)\{1;2\},229$
16	$(23,5)\{2\},50$	$(445,1)\{2\},73$	$(9,7)\{2\},237$	$(375,1)\{2\},268$
17	$(65,3)\{1\},52$	$(427,1)\{2\},67$	$(309,1)\{2\},170$	$(273,1)\{2\},232$
18	$(333,1)\{2\},83$	$(423,1)\{2\},97$	$(365,1)\{2\},153$	$(145,3)\{1;3\},283$

Table B.3 (*continued*)

g	$\mathbb{F}_{2^4}$	$\mathbb{F}_{2^5}$	$\mathbb{F}_{2^6}$	$\mathbb{F}_{2^7}$
19	$(627, 1)\{1; 3\}, 55$	$(65, 3)\{2, 3\}, 84$	$(351, 1)\{2\}, 207$	$(285, 1)\{3\}, 284$
20	$(777, 1)\{1; 3\}, 65$	$(437, 1)\{2\}, 64$	$(1, 33)\{2\}, 151$	$(861, 1)\{1, 3; 2, 3\}, 227$
21	$(39, 5)\{1; 3\}, 65$	$(209, 3)\{2; 1, 3\}, 65$	$(441, 1)\{2\}, 188$	$(963, 1)\{1; 2\}, 258$
22	$(741, 1)\{1; 3\}, 67$	$(1365, 1)\{3; 2, 4; 1, 2\}, 67$	$(511, 1)\{2\}, 162$	$(1255, 1)\{1; 2\}, 232$
23	$(209, 3)\{3; 1, 2\}, 62$	$(435, 1)\{3\}, 84$	$(385, 1)\{1\}, 152$	$(483, 1)\{2, 3\}, 312$
24	$(185, 3)\{3; 1, 2\}, 67$	$(5, 21)\{3; 1, 2\}, 77$	$(1339, 1)\{1; 2\}, 138$	$(527, 1)\{2\}, 240$
25	$(85, 3)\{3\}, 72$	$(95, 3)\{3\}, 80$	$(539, 1)\{2\}, 220$	$(855, 1)\{1; 3\}, 284$
26	$(69, 5)\{1; 2, 3\}, 64$	$(1365, 1)\{2; 1; 4\}, 72$	$(765, 1)\{2; 3\}, 199$	$(545, 1)\{1\}, 229$
27	$(91, 3)\{1\}, 80$	$(247, 3)\{3; 1, 2\}, 77$	$(707, 1)\{2\}, 157$	$(107, 3)\{2\}, 271$
28	$(513, 1)\{2\}, 108$	$(1325, 1)\{1; 2\}, 91$	$(175, 3)\{1; 3\}, 167$	$(639, 1)\{2\}, 387$
29	$(1295, 1)\{3; 1, 2\}, 80$	$(465, 1)\{1, 2\}, 66$	$(1085, 1)\{1; 3\}, 164$	$(435, 1)\{1, 3\}, 262$
30	$(2553, 1)\{1; 3; 2\}, 63$	$(209, 3)\{1; 2\}, 86$	$(77, 5)\{3; 1, 2\}, 155$	$(203, 3)\{1; 3\}, 241$
31	$(975, 1)\{1; 3\}, 73$	$(247, 3)\{2; 3\}, 80$	$(9, 7)\{\}, 186$	$(483, 1)\{2\}, 232$
32	$(1505, 1)\{2; 1, 3\}, 71$	$(7, 11)\{1\}, 107$	$(579, 1)\{2\}, 161$	$(1311, 1)\{1, 3; 2, 3\}, 221$
33	$(363, 1)\{\}, 72$	$(19, 7)\{1\}, 65$	$(603, 1)\{2\}, 182$	$(43, 5)\{2\}, 311$
34	$(2145, 1)\{2; 3; 1, 4\}, 77$	$(253, 3)\{1; 2\}, 77$	$(737, 1)\{2\}, 170$	$(685, 1)\{2\}, 205$
35	$(15, 7)\{1, 2\}, 94$	$(585, 1)\{2, 3\}, 124$	$(775, 1)\{2\}, 270$	$(955, 1)\{2\}, 310$
36	$(1547, 1)\{1; 2, 3\}, 71$	$(1305, 1)\{3; 1, 2\}, 87$	$(157, 3)\{2\}, 171$	$(749, 1)\{2\}, 297$
37	$(555, 1)\{3\}, 98$	$(7, 15)\{2\}, 82$	$(23, 7)\{2\}, 196$	$(145, 3)\{1, 3\}, 310$
38	$(301, 3)\{1; 2, 3\}, 79$	$(847, 1)\{1\}, 69$	$(2145, 1)\{4; 1, 3; 2, 3\}, 163$	$(1435, 1)\{2; 3\}, 286$
39	$(627, 1)\{3\}, 92$	$(143, 3)\{3\}, 80$	$(585, 1)\{3\}, 242$	$(1449, 1)\{1; 2, 3\}, 288$
40	$(391, 3)\{2; 1, 3\}, 72$	$(1305, 1)\{2; 3\}, 67$	$(1, 45)\{1\}, 197$	$(1305, 1)\{1, 3; 2, 3\}, 231$
41	$(375, 1)\{\}, 78$	$(53, 5)\{2\}, 112$	$(715, 1)\{3\}, 174$	$(861, 1)\{2, 3\}, 312$
42	$(2103, 1)\{1; 2\}, 80$	$(1495, 1)\{1; 3\}, 63$	$(1505, 1)\{2; 3\}, 174$	$(319, 3)\{1; 3\}, 284$
43	$(39, 5)\{1, 3\}, 96$	$(1365, 1)\{1, 4; 2, 3, 4\}, 84$	$(1155, 1)\{2; 1, 3\}, 170$	$(39, 5)\{2\}, 308$
44	$(1169, 1)\{1, 2\}, 78$	$(1653, 1)\{3; 1, 2\}, 68$	$(1111, 1)\{2\}, 168$	$(2415, 1)\{1; 2, 3; 3, 4\}, 234$
45	$(741, 1)\{3\}, 100$	$(693, 1)\{3\}, 124$	$(1545, 1)\{2; 3\}, 202$	$(925, 1)\{2\}, 302$
46	$(4403, 1)\{1; 3; 2\}, 82$	$(47, 9)\{1; 2\}, 94$	$(9, 11)\{2\}, 259$	$(341, 3)\{1, 3; 2, 3\}, 289$
47	$(1073, 1)\{2\}, 92$	$(891, 1)\{2\}, 93$	$(211, 3)\{2\}, 216$	$(963, 1)\{2\}, 465$
48	$(1887, 1)\{2; 3\}, 96$	$(803, 3)\{1; 3; 2\}, 57$	$(965, 1)\{2\}, 201$	$(1177, 1)\{2\}, 285$
49	$(777, 1)\{3\}, 120$	$(1079, 1)\{2\}, 77$	$(1545, 1)\{3; 1, 2\}, 225$	$(1255, 1)\{2\}, 380$
50	$(2639, 1)\{1; 2, 3\}, 92$	$(1885, 1)\{1; 2\}, 73$	$(93, 5)\{1; 3\}, 206$	$(355, 3)\{2; 3\}, 287$

Table B.4
Table for $\max\{|X(n_0, n_{\mathrm{ns}})/\langle W_{d_1}, \ldots, W_{d_t}\rangle(\mathbb{F}_q)|\}$ with $q = 3^k$, $k = 1, 2, 3, 4, 5$ and $n_0 n_{\mathrm{ns}}^2 \leq 10^4$.

g	$\mathbb{F}_3$	$\mathbb{F}_{3^2}$	$\mathbb{F}_{3^3}$	$\mathbb{F}_{3^4}$	$\mathbb{F}_{3^5}$
1	$(23, 2)\{1, 2\}, 7$	$(1, 7)\{\}, 16$	$(1, 10)\{2\}, 38$	$(1, 10)\{2\}, 96$	$(1, 11)\{1\}, 275$
2	$(28, 1)\{\}, 8$	$(2, 5)\{\}, 20$	$(11, 4)\{2\}, 44$	$(23, 1)\{\}, 116$	$(7, 5)\{1, 2\}, 306$
3	$(52, 1)\{2\}, 10$	$(64, 1)\{\}, 28$	$(29, 2)\{2\}, 46$	$(25, 2)\{\}, 124$	$(4, 5)\{2\}, 304$
4	$(74, 1)\{2\}, 11$	$(14, 5)\{1; 2, 3\}, 30$	$(41, 2)\{2\}, 44$	$(80, 1)\{2\}, 138$	$(85, 1)\{2\}, 346$
5	$(106, 1)\{2\}, 12$	$(2, 7)\{\}, 32$	$(110, 1)\{3\}, 52$	$(46, 1)\{\}, 132$	$(14, 5)\{2; 3\}, 366$
6	$(170, 1)\{2; 3\}, 13$	$\mathbf{(398, 1)\{1; 2\}, 37}$	$(8, 5)\{1\}, 56$	$(95, 2)\{2, 3\}, 142$	$(121, 1)\{\}, 364$
7	$(266, 1)\{1; 2, 3\}, 13$	$(238, 1)\{2, 3\}, 40$	$(62, 1)\{\}, 54$	$(122, 1)\{1\}, 137$	$(380, 1)\{2; 3\}, 366$
8	$(127, 2)\{1, 2\}, 12$	$(91, 2)\{3\}, 42$	$(23, 4)\{2\}, 44$	$(101, 1)\{\}, 152$	$(205, 2)\{3; 1, 2\}, 396$
9	$(148, 1)\{2\}, 16$	$(128, 1)\{\}, 48$	$(7, 8)\{2\}, 52$	$(160, 1)\{2\}, 176$	$\mathbf{(85, 2)\{3\}, 464}$
10	$(277, 1)\{1\}, 16$	$(220, 1)\{2, 3\}, 54$	$(170, 1)\{1, 3\}, 52$	$(260, 1)\{2; 3\}, 180$	$(191, 2)\{2\}, 383$
11	$(170, 1)\{2, 3\}, 16$	$(439, 1)\{1\}, 53$	$(104, 1)\{\}, 56$	$(115, 1)\{\}, 164$	$(14, 5)\{2, 3\}, 366$
12	$(410, 1)\{1; 2, 3\}, 16$	$\mathbf{(28, 5)\{2; 3\}, 58}$	$(119, 2)\{1, 2\}, 54$	$(37, 4)\{2\}, 166$	$(28, 5)\{2; 3\}, 484$
13	$(212, 1)\{2\}, 18$	$(632, 1)\{1; 2\}, 56$	$(220, 1)\{3\}, 72$	$(224, 1)\{1\}, 168$	$(632, 1)\{1; 2\}, 448$
14	$(554, 1)\{1; 2\}, 18$	$\mathbf{(734, 1)\{1; 2\}, 59}$	$(124, 1)\{\}, 56$	$(122, 1)\{\}, 160$	$(595, 2)\{3; 2, 4; 1, 2\}, 370$
15	$(413, 1)\{1, 2\}, 18$	$(1055, 1)\{1; 2\}, 63$	$(238, 1)\{1, 3\}, 52$	$(35, 4)\{3\}, 198$	$(5, 14)\{1, 3\}, 404$
16	$(506, 1)\{1, 3; 2, 3\}, 18$	$(398, 1)\{2\}, 70$	$(368, 1)\{2\}, 52$	$(272, 1)\{2\}, 210$	$(305, 2)\{2; 3\}, 414$
17	$(634, 1)\{1; 2\}, 19$	$(782, 1)\{1; 2, 3\}, 72$	$(440, 1)\{1; 3\}, 64$	$(665, 2)\{2; 4; 1, 3\}, 174$	$(14, 5)\{2\}, 448$
18	$(266, 1)\{2, 3\}, 22$	$\mathbf{(878, 1)\{1; 2\}, 73}$	$(175, 2)\{2\}, 52$	$(296, 1)\{2\}, 170$	$(574, 1)\{1; 3\}, 374$
19	$(746, 1)\{1; 2\}, 21$	$(476, 1)\{2, 3\}, 78$	$(11, 10)\{2\}, 60$	$(185, 2)\{2\}, 196$	$(221, 2)\{1, 2\}, 402$
20	$(778, 1)\{1; 2\}, 19$	$\mathbf{(622, 1)\{2\}, 80}$	$(493, 2)\{3; 1, 2\}, 56$	$(520, 1)\{2; 3\}, 198$	$(425, 1)\{2\}, 526$
21	$(533, 1)\{1, 2\}, 20$	$(256, 1)\{\}, 80$	$(253, 2)\{1, 2\}, 56$	$(125, 2)\{\}, 208$	$(805, 1)\{1; 3\}, 446$
22	$(308, 1)\{2, 3\}, 24$	$\mathbf{(1102, 1)\{1; 2, 3\}, 82}$	$(247, 2)\{3\}, 64$	$(215, 2)\{1, 3\}, 194$	$(560, 1)\{2; 3\}, 484$
23	$(613, 1)\{1\}, 22$	$(542, 1)\{2\}, 92$	$(208, 1)\{\}, 56$	$(385, 2)\{2; 3\}, 214$	$(680, 1)\{3; 1, 2\}, 450$
24	$(653, 1)\{1\}, 23$	$(496, 1)\{2\}, 82$	$(11, 16)\{1; 2\}, 54$	$(385, 2)\{4; 3\}, 228$	$(680, 1)\{1; 3\}, 464$
25	$(340, 1)\{2, 3\}, 24$	$(440, 1)\{2, 3\}, 96$	$(245, 2)\{3\}, 56$	$(31, 5)\{2\}, 222$	$(340, 1)\{3\}, 552$
26	$(424, 1)\{2\}, 24$	$(28, 5)\{2, 3\}, 98$	$(10, 7)\{1, 2\}, 68$	$(812, 1)\{3; 1, 2\}, 188$	$(28, 5)\{2, 3\}, 482$
27	$(436, 1)\{2\}, 22$	$(604, 1)\{2\}, 100$	$(232, 1)\{\}, 56$	$(329, 2)\{3\}, 198$	$(329, 2)\{3\}, 472$
28	$(410, 1)\{2, 3\}, 24$	$(871, 1)\{1, 2\}, 104$	$(287, 2)\{3\}, 62$	$(1, 70)\{3; 1, 2\}, 218$	$(850, 1)\{3; 1, 2\}, 508$
29	$(757, 1)\{1\}, 25$	$(56, 5)\{2; 3\}, 104$	$(323, 2)\{3\}, 62$	$(77, 4)\{1, 2\}, 204$	$(632, 1)\{2\}, 604$
30	$(1114, 1)\{1; 2\}, 25$	$\mathbf{(1375, 1)\{1; 2\}, 99}$	$(496, 1)\{1\}, 56$	$(1, 35)\{2\}, 248$	$(1144, 1)\{1, 3; 2, 3\}, 508$
31	$(1265, 1)\{1; 2, 3\}, 23$	$(1, 46)\{1, 2\}, 103$	$(341, 2)\{2\}, 60$	$(1309, 2)\{1, 3; 2, 4; 3, 4\}, 222$	$(305, 2)\{3\}, 522$
32	$(986, 1)\{2; 3\}, 25$	$\mathbf{(1039, 1)\{1\}, 112}$	$(880, 1)\{3; 1, 2\}, 68$	$(946, 1)\{1; 2\}, 215$	$(1264, 1)\{1; 2\}, 490$
33	$(554, 1)\{2\}, 31$	$(103, 5)\{1; 2\}, 111$	$(440, 1)\{3\}, 80$	$(544, 1)\{2\}, 256$	$(43, 5)\{2\}, 524$
34	$(506, 1)\{2, 3\}, 28$	$\mathbf{(638, 1)\{2, 3\}, 114}$	$(490, 1)\{1, 2\}, 72$	$(335, 2)\{3\}, 246$	$(47, 5)\{2\}, 440$
35	$(989, 1)\{1, 2\}, 24$	$\mathbf{(958, 1)\{2\}, 122}$	$(595, 1)\{1\}, 72$	$(775, 1)\{2\}, 242$	$(28, 5)\{2\}, 608$
36	$(1178, 1)\{1; 2, 3\}, 26$	$\mathbf{(796, 1)\{2\}, 130}$	$(371, 2)\{3\}, 72$	$(592, 1)\{2\}, 274$	$(752, 1)\{2\}, 444$
37	$(634, 1)\{2\}, 28$	$(956, 1)\{2\}, 123$	$(413, 2)\{3\}, 62$	$(185, 4)\{1; 2\}, 222$	$(580, 1)\{1, 3\}, 446$
38	$(532, 1)\{2, 3\}, 32$	$\mathbf{(1231, 1)\{1\}, 131}$	$(1007, 2)\{3; 1, 2\}, 59$	$(1505, 2)\{3; 2, 4; 1, 2\}, 245$	$(1106, 1)\{2; 3\}, 362$
39	$(628, 1)\{2\}, 28$	$(782, 1)\{2, 3\}, 140$	$(23, 10)\{2\}, 66$	$(665, 2)\{2; 4\}, 268$	$(652, 1)\{2\}, 454$
40	$(1514, 1)\{1; 2\}, 29$	$\mathbf{(878, 1)\{2\}, 142}$	$(550, 1)\{1\}, 54$	$(34, 5)\{3\}, 260$	$(550, 1)\{1, 3\}, 548$
41	$(1037, 1)\{1, 2\}, 28$	$(1564, 1)\{1, 3; 2, 3\}, 132$	$(1003, 2)\{3; 1, 2\}, 57$	$(640, 1)\{2\}, 240$	$(290, 1)\{\}, 488$
42	$(109, 5)\{1; 2\}, 27$	$\mathbf{(1001, 2)\{4; 1, 2, 3\}, 134}$	$(451, 2)\{3\}, 64$	$(25, 7)\{2\}, 260$	$(1495, 1)\{1; 3\}, 436$
43	$(1117, 1)\{1\}, 32$	$(700, 1)\{2, 3\}, 144$	$(1190, 1)\{2, 3; 1, 4\}, 56$	$(385, 4)\{1; 3; 2\}, 252$	$(425, 2)\{3\}, 624$
44	$(746, 1)\{2\}, 35$	$\mathbf{(1966, 1)\{1; 2\}, 136}$	$(358, 1)\{\}, 54$	$(469, 2)\{3\}, 270$	$(247, 4)\{1; 2\}, 466$
45	$(616, 1)\{2, 3\}, 32$	$\mathbf{(1399, 1)\{1\}, 145}$	$(493, 2)\{3\}, 68$	$(322, 1)\{\}, 264$	$(1120, 1)\{2; 3\}, 604$
46	$(1642, 1)\{1; 2\}, 27$	$(2455, 1)\{1; 2\}, 139$	$(752, 1)\{1\}, 54$	$(74, 5)\{1; 3\}, 284$	$(1580, 1)\{3; 1, 2\}, 448$
47	$(1576, 1)\{1; 2\}, 30$	$(1886, 1)\{1; 2, 3\}, 146$	$(1190, 1)\{1, 4; 2, 3, 4\}, 68$	$(385, 2)\{3\}, 296$	$(805, 2)\{2; 4\}, 462$
48	$(1213, 1)\{1\}, 33$	$(1102, 1)\{2, 3\}, 154$	$(1189, 2)\{2; 1, 3\}, 54$	$(385, 4)\{1; 3; 4\}, 259$	$(805, 2)\{2; 3, 4\}, 412$
49	$(788, 1)\{2\}, 36$	$(1244, 1)\{2\}, 159$	$(29, 10)\{2\}, 72$	$(485, 2)\{3\}, 284$	$(790, 1)\{3\}, 584$
50	$(2285, 1)\{1; 2\}, 28$	$(2126, 1)\{1; 2\}, 154$	$(551, 2)\{2\}, 74$	$(2992, 1)\{1; 3; 2\}, 258$	$(826, 1)\{3\}, 514$

Table B.5
Table for $\max\{|X(n_0, n_{\mathrm{ns}})/\langle W_{d_1}, \ldots, W_{d_t}\rangle(\mathbb{F}_q)|\}$ with $q = 5^k$, $k = 1, 2, 3, 4, 5$ and $n_0 n_{\mathrm{ns}}^2 \leq 10^4$.

g	$\mathbb{F}_5$	$\mathbb{F}_{5^2}$	$\mathbb{F}_{5^3}$	$\mathbb{F}_{5^4}$	$\mathbb{F}_{5^5}$
1	$(7, 4)\{1; 2\}, 10$	$(1, 6)\{2\}, 36$	$(1, 8)\{\}, 148$	$(1, 11)\{1\}, 675$	$(13, 4)\{1; 2\}, 3227$
2	$(11, 6)\{3; 1, 2\}, 12$	$(1, 12)\{2\}, 46$	$(14, 3)\{1; 3\}, 170$	$(57, 1)\{1, 2\}, 724$	$(17, 6)\{3; 1, 2\}, 3328$
3	$(76, 1)\{2\}, 15$	$(19, 12)\{1; 3; 2\}, 56$	$(14, 3)\{3\}, 192$	$(29, 2)\{2\}, 738$	$(13, 4)\{1\}, 3404$
4	$(86, 1)\{2\}, 17$	$(16, 3)\{2\}, 66$	$(132, 1)\{1, 3; 2, 3\}, 188$	$(67, 2)\{1, 2\}, 774$	$(62, 3)\{1; 2, 3\}, 3530$

(*continued on next page*)

Table B.5 (*continued*)

g	$\mathbb{F}_5$	$\mathbb{F}_{5^2}$	$\mathbb{F}_{5^3}$	$\mathbb{F}_{5^4}$	$\mathbb{F}_{5^5}$
5	(21, 4){1; 3}, 16	**(209, 2){3; 1, 2}, 71**	(186, 1){1, 3; 2, 3}, 184	(133, 1){1, 2}, 823	(209, 2){3; 1, 2}, 3631
6	(163, 1){1}, 19	(84, 1){1, 3}, 74	(112, 1){1}, 178	(117, 1){2}, 902	(84, 1){3}, 3612
7	(44, 3){2; 3}, 21	(16, 3){}, 84	(28, 3){3}, 240	(232, 1){1; 2}, 854	(231, 1){1; 3}, 3644
8	(403, 1){1; 2}, 21	(174, 1){2, 3}, 90	(264, 1){1; 2, 3}, 188	(234, 1){3; 1, 2}, 885	(77, 2){1, 3}, 3690
9	(172, 1){2}, 24	(321, 2){2; 1, 3}, 94	(93, 1){}, 200	(19, 3){}, 898	(87, 1){}, 3768
10	(182, 1){2, 3}, 24	(286, 1){2, 3}, 108	(132, 1){2, 3}, 246	(164, 1){2}, 942	**(334, 1){2}, 4092**
11	(201, 1){2}, 24	(413, 2){2; 1, 3}, 109	(186, 1){1, 3}, 232	(201, 2){1, 3; 2, 3}, 924	(299, 1){2}, 3912
12	(203, 3){1; 3; 2}, 23	**(208, 1){2}, 122**	(11, 12){1; 3}, 204	(133, 2){2, 3}, 984	(231, 2){3; 1, 4}, 3804
13	(278, 1){2}, 26	**(572, 1){2; 3}, 126**	(129, 1){}, 200	(117, 2){3}, 936	(174, 1){1, 2}, 3768
14	(193, 2){1, 2}, 24	(564, 1){3; 1, 2}, 126	(186, 1){2, 3}, 206	(38, 3){2}, 1000	(231, 2){2; 1, 4}, 3962
15	(268, 1){2}, 30	(719, 1){1}, 127	(39, 4){1, 2}, 220	(67, 6){1, 3; 2, 3}, 952	(231, 1){3}, 4044
16	(403, 1){1, 2}, 28	(647, 1){1}, 136	(258, 1){1, 3}, 222	(8, 7){2}, 974	(204, 1){2}, 3852
17	(88, 3){2; 3}, 28	(196, 1){}, 142	(56, 3){3}, 288	(171, 1){}, 1060	(46, 3){3}, 4186
18	(499, 1){1}, 31	(271, 3){1; 2}, 146	(112, 3){3; 1, 2}, 226	(234, 1){3}, 1082	(598, 1){1; 3}, 3858
19	(326, 1){2}, 33	(252, 1){1, 3}, 156	(28, 3){}, 240	(492, 1){3; 1, 2}, 1034	(693, 1){3; 1, 2}, 4170
20	(547, 1){1}, 33	(526, 1){2}, 164	(203, 2){1, 3}, 190	(87, 4){2; 3}, 1096	(362, 1){2}, 3972
21	(258, 1){3}, 34	(154, 1){}, 152	(264, 1){2, 3}, 272	(116, 3){1, 3; 2, 3}, 1030	(693, 2){3; 2, 4; 1, 2}, 4170
22	(422, 1){2}, 38	(189, 2){1, 3}, 174	(188, 1){}, 246	(201, 2){1, 3}, 1062	(1242, 1){1; 3; 2}, 4048
23	(364, 1){2, 3}, 36	**(887, 1){1}, 180**	(17, 8){2}, 244	(386, 1){2}, 1096	(306, 1){2}, 3928
24	(1122, 1){1, 3; 2, 4; 3, 4}, 31	**(206, 3){2; 3}, 188**	(1989, 1){1; 3; 2}, 204	(399, 2){3, 4; 1, 2, 3}, 1030	(77, 6){1, 2; 1, 3, 4}, 3970
25	(684, 1){2; 3}, 30	**(572, 1){2, 3}, 204**	(11, 9){2}, 210	(589, 2){1, 3; 2, 3}, 1065	(668, 1){2}, 4332
26	(739, 1){1}, 38	(1294, 1){1; 2}, 188	(744, 1){1, 3; 2, 3}, 222	(458, 1){2}, 1114	(681, 2){2; 1, 3}, 3936
27	(489, 1){2}, 36	**(574, 1){2, 3}, 204**	(774, 1){2; 1, 3}, 204	(546, 1){2, 4; 2, 3}, 1088	(414, 1){3}, 4186
28	(787, 1){1}, 35	**(23, 7){1, 2}, 202**	(287, 2){3}, 218	(464, 1){2}, 1218	(209, 6){3; 2, 4; 1, 2}, 4220
29	(1094, 1){1; 2}, 37	**(1342, 1){1; 2, 3}, 205**	(186, 1){}, 248	(696, 1){1; 3}, 1288	(261, 2){3}, 4202
30	(556, 1){2}, 39	**(1007, 1){1, 2}, 216**	(372, 1){2, 3}, 234	(323, 2){2}, 1032	(339, 2){1, 3}, 4106
31	(518, 1){2, 3}, 40	**(564, 1){3}, 228**	(297, 2){1, 3}, 232	(231, 2){4}, 1268	(231, 2){1, 4}, 4484
32	(1227, 1){1; 2}, 35	**(1246, 1){1; 2, 3}, 214**	(97, 4){2}, 208	(579, 1){2}, 1150	(39, 8){1; 2, 3}, 3976
33	(402, 1){3}, 42	**(924, 1){1, 2; 1, 3, 4}, 226**	(154, 3){1; 3}, 218	(603, 2){1, 3; 2, 3}, 1240	(184, 3){1; 3}, 4242
34	(614, 1){2}, 41	**(1223, 1){1}, 241**	(12, 7){1, 2}, 242	(16, 7){2}, 1260	(92, 3){3}, 4686
35	(1304, 1){1; 2}, 39	**(958, 1){2}, 242**	(133, 6){1, 4; 2, 4}, 244	(234, 1){}, 1204	(598, 1){3}, 4604
36	(214, 3){2; 3}, 45	**(1271, 1){1, 2}, 238**	(954, 1){2; 3}, 218	(592, 1){2}, 1202	(1557, 1){1; 2}, 4110
37	(662, 1){2}, 46	**(316, 3){2; 3}, 243**	(133, 6){1, 3; 1, 4}, 266	(468, 1){3}, 1296	(302, 1){}, 4284
38	(1334, 1){1; 2, 3}, 40	**(593, 2){1, 2}, 232**	(663, 2){1, 3; 1, 4}, 254	(687, 1){2}, 1268	(789, 2){1, 3; 2, 3}, 4232
39	(652, 1){2}, 48	**(1774, 1){1; 2}, 260**	(112, 3){3}, 336	(87, 4){3}, 1328	(522, 1){2, 3}, 4134
40	(541, 2){1, 2}, 40	**(814, 1){2, 3}, 272**	(609, 2){1, 3; 2, 3}, 252	(492, 1){3}, 1326	(609, 2){4; 2, 3}, 4122
41	(1003, 1){1, 2}, 46	**(861, 1){2, 3}, 260**	(56, 3){}, 288	(1547, 1){1; 2}, 1168	(418, 3){2; 1, 3; 3, 4}, 4261
42	(516, 1){3}, 48	**(1006, 1){2}, 296**	(663, 2){4; 3}, 274	(1158, 1){2; 3}, 1182	(418, 3){1; 3; 4}, 4128
43	(758, 1){2}, 50	**(1343, 1){1, 2}, 276**	(528, 1){2, 3}, 324	(232, 3){1, 3; 2, 3}, 1400	(778, 1){2}, 4650
44	(431, 3){1; 2}, 37	**(1487, 1){1}, 289**	(1232, 1){2; 3}, 254	(469, 2){1, 3}, 1298	(924, 1){2; 1, 4}, 4406
45	(197, 6){2; 1, 3}, 40	**(696, 1){2, 3}, 280**	(501, 2){3}, 266	(1158, 1){3; 1, 2}, 1288	(693, 2){1, 4; 2, 3, 4}, 4808
46	(1099, 1){1, 2}, 46	**(1052, 1){2}, 306**	(924, 1){2, 4; 2, 3}, 294	(9, 11){2}, 1438	(7, 13){1}, 4514
47	(1766, 1){1; 2}, 43	**(1149, 1){2}, 298**	(891, 1){2}, 282	(77, 6){2, 4}, 1196	(963, 1){2}, 4414
48	(844, 1){2}, 57	**(2014, 1){1; 2, 3}, 306**	(497, 2){1, 3}, 254	(772, 1){2}, 1466	(903, 2){1, 3; 2, 3, 4}, 4246
49	(748, 1){2, 3}, 48	**(1244, 1){2}, 315**	(612, 1){3}, 294	(589, 2){1, 3}, 1250	(1242, 1){1; 3}, 4718
50	(1688, 1){1; 2}, 46	**(1198, 1){2}, 306**	(2, 39){1, 3; 2, 3}, 232	(134, 3){2}, 1168	(786, 1){3}, 4324

Table B.6
Table for $\max\{|X(n_0, n_{\mathrm{ns}})/\langle W_{d_1}, \ldots, W_{d_t}\rangle(\mathbb{F}_q)|\}$ with $q = 7^k$, $k = 1, 2, 3, 4, 5$ and $n_0 n_{\mathrm{ns}}^2 \leq 10^4$.

g	$\mathbb{F}_7$	$\mathbb{F}_{7^2}$	$\mathbb{F}_{7^3}$	$\mathbb{F}_{7^4}$	$\mathbb{F}_{7^5}$
1	(1, 15){1; 2}, 13	(1, 8){}, 64	(31, 2){2}, 380	(1, 6){2}, 2496	(1, 10){2}, 17050
2	(1, 15){1, 2}, 16	(1, 16){1}, 78	(1, 9){}, 412	(1, 12){2}, 2590	(8, 5){1; 2}, 17292
3	(13, 6){1, 2, 3}, 18	(9, 4){1}, 92	(33, 2){3}, 446	(29, 2){2}, 2684	(4, 5){2}, 17534
4	(137, 1){1}, 21	(54, 1){}, 102	(165, 1){2; 3}, 440	(16, 3){2}, 2778	(22, 3){1; 3}, 17770
5	(2, 15){2; 3}, 24	(52, 1){}, 112	(88, 1){2}, 440	(29, 2){}, 2808	(11, 3){}, 18012
6	(73, 3){1; 2}, 26	(156, 1){2, 3}, 122	(33, 4){2; 3}, 480	(102, 1){3}, 2774	(190, 1){2, 3}, 17830
7	(257, 1){1}, 26	(1, 16){}, 148	(20, 3){2}, 474	(87, 2){1, 2, 3}, 2954	(8, 5){2}, 18496
8	(136, 1){2}, 24	(431, 1){1}, 138	(2, 9){}, 508	(304, 1){1; 2}, 2831	(110, 1){2}, 18496
9	(97, 3){1; 2}, 29	(151, 3){1; 2}, 167	(114, 1){3}, 528	(152, 1){2}, 2904	**(99, 1){}, 18968**
10	(2, 15){2, 3}, 28	(27, 4){1}, 153	(176, 1){2}, 536	(19, 6){2}, 2976	(209, 1){1}, 18143
11	(353, 1){1}, 33	**(382, 1){2}, 176**	(165, 2){4; 3}, 556	(117, 1){}, 3052	(34, 3){2}, 18724
12	(226, 1){2}, 36	**(718, 1){1; 2}, 171**	(342, 1){3; 1, 2}, 548	(250, 1){2}, 2893	(12, 5){2, 3}, 18242
13	(4, 15){2; 3}, 36	**(599, 1){1}, 184**	(31, 6){3}, 504	(232, 1){1, 2}, 3112	(198, 1){1, 2, 3}, 18496
14	(73, 3){1, 2}, 34	**(110, 3){2; 3, 4}, 196**	(330, 1){4; 3}, 510	(83, 2){}, 3002	(59, 4){2}, 18520
15	(274, 1){2}, 40	**(478, 1){2}, 214**	(40, 3){2}, 584	(268, 1){2}, 3104	(110, 1){}, 19216
16	(204, 1){3}, 36	(380, 1){2, 3}, 222	(398, 1){2}, 526	(204, 1){3}, 3090	(44, 3){3}, 18732
17	(706, 1){1; 2}, 35	(46, 5){2; 3}, 216	(1, 60){1; 3}, 532	(335, 1){2}, 3228	(185, 1){}, 18804
18	(785, 1){1; 2}, 33	**(179, 4){1; 2}, 224**	(228, 1){3}, 584	(304, 1){2}, 3118	(110, 3){4; 2, 3}, 18974
20	(97, 3){1, 2}, 40	(46, 5){1, 3; 2, 3}, 246	(528, 1){3; 1, 2}, 530	(362, 1){2}, 3201	(935, 1){1; 2, 3}, 18902
21	(193, 3){1; 2}, 41	(426, 1){3}, 252	(264, 1){3}, 632	(282, 1){1, 2, 3}, 3096	(207, 1){}, 19568
22	(317, 2){1, 2}, 38	(139, 6){2; 1, 3}, 255	(188, 1){}, 594	(356, 1){2}, 3174	(754, 1){3; 1, 2}, 19126
23	(617, 1){1}, 45	(911, 1){1}, 264	(990, 1){1; 3; 4}, 566	(282, 1){1, 3}, 3348	(429, 2){2; 3, 4}, 18884
24	(146, 3){2; 3}, 44	(206, 3){2; 3}, 272	(71, 4){2}, 540	(670, 1){3; 1, 2}, 3290	(808, 1){1; 2}, 19211
25	(4, 15){2, 3}, 42	(717, 1){2}, 284	(27, 5){2}, 660	(117, 2){}, 3408	(404, 1){2}, 19362
26	(466, 1){2}, 48	(718, 1){2}, 312	(374, 1){2}, 566	(885, 2){3; 2, 4; 1, 2}, 3237	(370, 1){1}, 19586
27	(8, 15){2; 3}, 48	(69, 5){2; 3}, 288	(342, 1){3}, 824	(232, 1){}, 3340	(8, 15){2; 3}, 19488
28	(452, 1){2}, 54	(764, 1){2}, 330	(513, 1){2}, 558	(464, 1){2}, 3498	(479, 2){2}, 19743
29	(737, 1){1, 2}, 48	(16, 5){}, 304	(40, 3){}, 592	(885, 2){4; 1, 3; 2, 3}, 3461	(198, 1){}, 19936
30	(822, 1){3; 1, 2}, 43	(319, 3){1; 2, 3}, 320	(1320, 1){3; 4; 1, 2}, 610	(804, 1){3; 1, 2}, 3386	(465, 2){1, 3; 1, 4}, 18782
31	(946, 1){1, 3; 2, 3}, 48	(1146, 1){3; 1, 2}, 330	(660, 1){4; 3}, 696	(80, 3){2}, 3396	(297, 1){}, 20352
32	(386, 3){1; 3; 2}, 47	(1079, 1){1, 2}, 332	(495, 2){3; 1, 4}, 606	(31, 12){2; 3}, 3496	(374, 3){1; 2; 3, 4}, 19718
33	(194, 3){2; 3}, 56	(862, 1){2}, 370	(43, 5){2}, 662	(804, 1){2; 3}, 3578	(440, 1){2}, 20168
34	(548, 1){2}, 60	(220, 3){2; 3, 4}, 366	(176, 3){1, 3; 2, 3}, 612	(335, 2){3}, 3622	(494, 1){3}, 19650
35	(771, 1){2}, 52	(645, 2){4; 1, 3}, 366	(248, 3){1, 3; 2, 3}, 568	(234, 1){}, 3412	(377, 2){3}, 19382
36	(977, 1){1}, 56	(302, 3){2; 3}, 392	(327, 2){1, 3}, 716	(2145, 1){2; 1; 4}, 3352	(302, 3){2; 3}, 19580
37	(1097, 1){1}, 54	(316, 3){2; 3}, 408	(9, 20){1; 3}, 784	(333, 2){1, 3}, 3408	(528, 1){1, 3}, 19864
38	(534, 1){3}, 48	(440, 3){2; 1, 3; 3, 4}, 370	(978, 1){2; 3}, 578	(555, 2){3; 1, 2}, 3318	(663, 2){2; 3, 4}, 19568
39	(2033, 1){1; 2}, 51	(46, 5){2, 3}, 404	(1056, 1){3; 1, 2}, 728	(155, 6){2; 4}, 3590	(663, 2){3, 4; 1, 2, 3}, 20412
40	(706, 1){2}, 64	(92, 5){2; 3}, 402	(845, 2){1, 3; 2, 3}, 588	(351, 2){1, 3}, 3792	(2064, 1){1; 3; 2}, 19478
41	(1193, 1){1}, 60	(1077, 1){2}, 412	(19, 9){2}, 600	(31, 8){1}, 3474	(663, 2){2, 4; 2, 3}, 20048
42	(1426, 1){2; 3}, 54	(701, 2){1, 2}, 400	(610, 1){3}, 544	(2136, 1){1; 3; 2}, 3446	(22, 15){1; 3; 4}, 19234
43	(1217, 1){1}, 57	**(387, 2){3}, 444**	(528, 1){3}, 872	(1068, 1){2; 3}, 3612	(580, 1){3}, 20766
44	(1426, 1){1, 3; 2, 3}, 56	(761, 2){1, 2}, 408	(1443, 1){1, 3; 2, 3}, 526	(1160, 1){2; 3}, 3598	(1654, 1){1; 2}, 19418
45	(802, 1){2}, 56	(451, 3){1; 2, 3}, 422	(1056, 1){2; 3}, 760	(724, 1){2}, 3992	(76, 5){2; 3}, 20462
46	(1474, 1){1; 2, 3}, 56	(1052, 1){2}, 450	(752, 1){1}, 594	(831, 1){2}, 3964	(1140, 1){2; 3, 4}, 19872
47	(754, 1){2, 3}, 68	(600, 1){1, 2}, 436	(1980, 1){3; 4; 1, 2}, 592	(1128, 1){1, 3; 2, 3}, 3588	(927, 2){1, 3; 2, 3}, 19816
48	(317, 4){1; 2}, 62	(1102, 1){2, 3}, 466	(990, 1){2, 4; 2, 3}, 746	(577, 2){2}, 3474	(754, 1){3}, 20284
49	(2, 39){2; 3}, 48	(852, 1){3}, 480	(545, 2){1, 3}, 662	(1139, 1){2}, 3470	(36, 5){3}, 20880
50	(146, 3){2, 3}, 60	**(1198, 1){2}, 506**	(1485, 1){2; 3}, 622	(670, 1){3}, 4084	(808, 1){2}, 21084

Table B.7
Table for $\max\{|X(n_0, n_{\rm ns})/\langle W_{d_1}, \dots, W_{d_t}\rangle(\mathbb{F}_q)|\}$ with $q = 11^k$, $k = 1, 2, 3, 4, 5$ and $n_0 n_{\rm ns}^2 \le 10^4$.

g	$\mathbb{F}_{11}$	$\mathbb{F}_{11^2}$	$\mathbb{F}_{11^3}$	$\mathbb{F}_{11^4}$	$\mathbb{F}_{11^5}$
1	(10, 3){2; 3}, 18	(1, 6){2}, 144	(19, 1){}, 1404	(19, 2){2}, 14875	(5, 12){1; 3; 2}, 161854
2	(91, 2){2; 1, 3}, 22	(1, 12){2}, 166	(48, 1){2}, 1468	(161, 1){1; 2}, 15118	(2, 7){2}, 162656
3	(109, 1){1}, 25	(5, 6){2}, 188	(105, 2){4; 1, 3}, 1516	(27, 4){1; 2}, 15287	(52, 1){2}, 163458
4	(74, 1){2}, 27	(1, 15){1}, 210	(86, 1){2}, 1600	(111, 2){2; 1, 3}, 15520	(39, 2){3}, 164260
5	(181, 1){1}, 32	(1, 30){2; 3}, 232	(13, 6){1, 3}, 1516	(378, 1){1; 3; 2}, 15699	(3, 7){2}, 165062
6	(122, 1){2}, 32	(103, 3){1; 2}, 239	(1, 21){2}, 1622	(84, 1){3}, 15590	(78, 1){3}, 165864
7	(229, 1){1}, 37	(1, 24){2}, 276	(65, 2){1, 3}, 1590	(37, 6){2; 1, 3}, 15690	**(6, 7){3; 1, 2}, 166666**
8	(362, 1){1; 2}, 35	(240, 1){2; 3}, 266	(76, 1){}, 1740	(266, 1){2; 3}, 15673	(234, 1){2; 3}, 165864
9	(58, 3){2; 3}, 36	(1, 30){2}, 320	**(172, 1){2}, 1812**	(126, 1){3}, 16496	**(312, 1){2; 3}, 167544**
10	(458, 1){1; 2}, 38	(108, 1){}, 306	(546, 1){3; 2, 4; 1, 2}, 1618	(27, 4){1}, 16308	**(312, 1){1; 3}, 168744**
11	(349, 1){1}, 43	(112, 1){}, 316	(141, 2){1, 3}, 1740	(192, 1){1}, 16012	(78, 1){}, 166888
12	(218, 1){2}, 44	**(156, 1){3}, 338**	(159, 2){1, 3}, 1828	(61, 6){1, 3; 2, 3}, 16156	**(156, 1){3}, 170676**
13	(37, 5){1; 2}, 40	(25, 6){2}, 408	(546, 1){2; 4; 3}, 1642	(378, 1){2; 3}, 17241	(6, 7){1, 2, 3}, 167620
14	(26, 5){2; 3}, 44	(910, 1){3; 2, 4; 1, 2}, 347	(87, 4){1, 3; 2, 3}, 1652	(562, 1){1; 2}, 16181	(95, 4){3; 1, 2}, 167234
15	(244, 1){2}, 48	(153, 1){}, 380	(273, 1){1}, 1724	(378, 1){1; 3}, 16800	(24, 7){1; 3; 2}, 166581
16	(41, 12){1; 3; 2}, 47	**(9, 7){2}, 414**	(70, 3){2, 4; 2, 3}, 1838	(333, 2){3; 1, 2}, 16126	**(6, 7){3}, 172432**
17	(698, 1){1; 2}, 47	**(480, 1){3; 1, 2}, 416**	(152, 1){}, 1880	(246, 1){2, 3}, 16820	(936, 1){1; 3; 2}, 168592
18	(58, 3){2, 3}, 48	(234, 1){3}, 422	(228, 1){2, 3}, 1740	(532, 1){2; 3}, 16751	(294, 1){1, 2, 3}, 167696
19	(794, 1){1; 2}, 47	(64, 3){2}, 540	(602, 1){3; 1, 2}, 1758	(252, 1){3}, 17244	(468, 1){2; 3}, 170676
20	(362, 1){2}, 61	(622, 1){2}, 438	(115, 4){1, 3; 2, 3}, 1806	(104, 3){1, 3; 2, 3}, 16426	(294, 1){1, 2}, 167848
21	(59, 6){1, 2, 3}, 44	**(256, 1){}, 464**	(344, 1){2}, 1896	(45, 4){1}, 17048	(294, 1){3}, 172508
22	(116, 3){2; 3}, 54	(400, 1){2}, 462	(195, 4){3; 2, 4; 1, 2}, 1850	(840, 1){2; 4; 1, 3}, 16818	(267, 4){1; 3; 2}, 167708
23	(661, 1){1}, 62	(21, 5){1, 3}, 472	(1365, 2){2; 4; 3, 5; 1, 5}, 1824	(741, 2){4; 1, 3; 2, 3}, 16682	(156, 1){}, 172572
24	(1189, 1){1; 2}, 54	**(306, 1){1, 3}, 530**	(2, 21){3}, 1800	(507, 2){2; 1, 3}, 16858	(18, 7){2; 3}, 172432
25	(565, 1){1, 2}, 54	(416, 1){2}, 544	(380, 1){3}, 1824	(673, 1){1}, 16916	**(312, 1){3}, 180048**
26	(458, 1){2}, 69	(624, 1){1; 3}, 502	(969, 1){2; 3}, 1808	(82, 3){1, 2, 3}, 16506	(624, 1){1; 3}, 170676
27	(436, 1){2}, 66	**(342, 1){3}, 584**	(190, 1){}, 1800	(2490, 1){1; 2; 3; 4}, 16742	(1143, 1){1; 2}, 168884
28	(269, 3){1; 2}, 64	(764, 1){2}, 546	(172, 3){2; 3}, 1812	(756, 1){3; 1, 2}, 18060	(762, 1){1; 3}, 168264
29	(26, 5){2, 3}, 68	(1295, 2){3; 4; 1, 2}, 535	(630, 1){2, 4; 2, 3}, 1838	(630, 1){4; 2, 3}, 17694	(203, 3){2; 3}, 171650
30	(366, 1){3}, 64	(115, 6){1, 3; 2, 4}, 558	(1260, 1){1, 3; 2, 4; 3, 4}, 1742	(1512, 1){1; 3; 2}, 17203	(762, 1){3; 1, 2}, 169876
31	(202, 3){1; 2, 3}, 60	(125, 6){2; 1, 3}, 576	(35, 8){1; 3}, 1864	(378, 1){3}, 19164	(6, 7){}, 174416
32	(78, 5){3; 4; 1, 2}, 56	(670, 1){2, 3}, 582	(140, 3){3, 4; 1, 2, 3}, 1892	(135, 4){1; 3}, 16766	(24, 7){1; 3}, 172546
33	(52, 5){2; 3}, 66	(525, 1){1, 3}, 624	(70, 3){3, 4}, 1916	(280, 3){1; 3; 4}, 17420	(882, 1){2; 3}, 172508
34	(901, 1){1, 2}, 72	(12, 7){3}, 624	(140, 3){2, 4; 2, 3}, 2196	(135, 4){1; 2}, 17466	**(12, 7){3}, 180450**
35	(31, 8){1, 2}, 62	(115, 6){1, 3; 1, 4}, 620	(228, 1){}, 2028	(421, 2){2}, 17478	(598, 1){3}, 170088
36	(444, 1){3}, 60	(2080, 1){1; 3; 2}, 658	(273, 4){2; 1, 3; 3, 4}, 1770	(208, 3){1, 3; 2, 3}, 17354	(584, 1){2}, 172324
37	(1021, 1){1}, 77	(468, 1){3}, 720	(456, 1){2}, 2072	(492, 1){2, 3}, 17910	(24, 7){2; 3}, 174120
38	(687, 1){2}, 70	(1560, 1){3; 4; 1, 2}, 650	(646, 1){2, 3}, 1998	(259, 3){1; 3}, 17752	(582, 1){1, 3}, 171872
39	(1130, 1){1, 3; 2, 3}, 69	(520, 1){3}, 708	(183, 2){}, 1972	(1064, 1){2; 3}, 17788	(936, 1){2; 3}, 180048
40	(698, 1){2}, 84	(1040, 1){2; 3}, 714	(1292, 1){1, 3; 2, 3}, 1986	(665, 2){2; 3, 4}, 17640	(1316, 1){2; 3}, 172542
41	(602, 1){2, 3}, 80	**(128, 3){2}, 880**	(516, 1){2}, 1938	(504, 1){3}, 18544	(294, 1){}, 174568
42	(1086, 1){3; 1, 2}, 69	(1545, 2){2; 3; 1, 4}, 692	(423, 2){1, 3}, 2172	(1, 90){1, 3; 2, 3}, 17578	(582, 1){2, 3}, 172000
43	(116, 3){2, 3}, 72	(400, 1){}, 732	(1032, 1){2; 3}, 2068	(27, 8){1}, 18330	(14, 15){4; 2, 3}, 170984
44	(74, 5){2; 3}, 78	(1966, 1){1; 2}, 697	(712, 1){2}, 1852	(181, 3){1}, 17758	(582, 1){1, 2, 3}, 171080
45	(724, 1){2}, 90	(18, 5){}, 800	(1615, 1){1, 3; 2, 3}, 1966	(840, 1){4; 2, 3}, 17984	(588, 1){3}, 179250
46	(921, 2){2; 1, 3}, 64	(1278, 1){1; 3}, 742	(1365, 2){2; 3, 4, 5; 1, 5}, 2104	(1134, 1){3; 1, 2}, 18708	(475, 2){2, 3}, 171508
47	(1189, 1){1, 2}, 80	(1248, 1){1; 3}, 804	(477, 2){1, 3}, 2164	(1890, 1){2; 4; 3}, 18179	(1545, 2){1, 3; 1, 2; 1, 4}, 174090
48	(794, 1){2}, 75	(1102, 1){2, 3}, 830	(1510, 1){1, 3; 2, 3}, 2048	(1134, 1){1; 3}, 18816	(267, 4){1; 3}, 173222
49	(250, 3){2; 3}, 73	(1244, 1){2}, 840	(1365, 2){2, 4, 5; 1, 3, 5; 2, 3}, 2092	(1890, 1){2; 4; 1, 3}, 18261	(312, 1){}, 180336
50	(842, 1){2}, 85	(36, 7){3; 1, 2}, 836	(1365, 2){3; 2, 4, 5; 1, 2, 5}, 1898	(842, 1){2}, 17711	(531, 2){1, 3}, 174084

Table B.8
Table for $\max\{|X(n_0, n_{\rm ns})/\langle W_{d_1}, \dots, W_{d_t}\rangle(\mathbb{F}_q)|\}$ with $q = 13^k$, $k = 1, 2, 3, 4, 5$ and $n_0 n_{\rm ns}^2 \le 10^4$.

g	$\mathbb{F}_{13}$	$\mathbb{F}_{13^2}$	$\mathbb{F}_{13^3}$	$\mathbb{F}_{13^4}$	$\mathbb{F}_{13^5}$
1	(1, 21){1; 2}, 21	(1, 7){}, 196	(2, 5){2}, 2290	(1, 15){1; 2}, 28899	(1, 10){2}, 372496
2	(1, 30){2; 1, 3}, 26	(2, 7){2}, 222	(22, 1){}, 2382	(17, 6){3; 1, 2}, 29236	(11, 6){3; 1, 2}, 373698
3	(43, 3){1; 2}, 29	(2, 9){1, 2}, 245	(5, 6){2}, 2408	(43, 3){1; 2}, 29573	(9, 4){1}, 374900
4	(214, 1){1; 2}, 31	(133, 1){2}, 270	(5, 12){1, 3; 2, 3}, 2500	(86, 1){2}, 29840	(126, 1){3; 1, 2}, 376094
5	(11, 5){1, 2}, 34	(266, 1){3; 1, 2}, 295	(56, 3){2; 3}, 2548	(69, 2){1, 3}, 30078	(96, 1){2}, 377296
6	(67, 3){1; 2}, 41	(97, 2){1, 2}, 310	(220, 1){2; 1, 3}, 2508	(203, 1){1, 2}, 29944	(11, 6){1, 2}, 376094
7	(43, 3){1, 2}, 37	(4, 9){1, 2}, 339	(5, 8){2}, 2600	(50, 3){2; 3}, 30426	(15, 4){1}, 377304
8	(137, 2){1, 2}, 36	**(2, 9){}, 364**	(110, 1){1}, 2670	(132, 1){1, 2, 3}, 30082	(5, 14){1, 3; 2, 3}, 376752
9	(214, 1){2}, 46	(45, 2){}, 368	(209, 2){1, 3; 2, 3}, 2610	(172, 1){2}, 30719	**(96, 1){}, 382096**
10	(347, 1){1}, 45	(37, 6){2; 3}, 390	(399, 2){3; 2, 4; 1, 2}, 2576	(210, 1){2; 3}, 31012	**(189, 1){2}, 382887**
11	(323, 1){1, 2}, 50	(55, 6){1, 3; 2, 3, 4}, 396	(5, 12){2, 3}, 2600	(231, 1){2, 3}, 30376	(99, 2){1, 2}, 379692
12	(86, 3){2; 3}, 50	(23, 6){2}, 422	(319, 1){2}, 2646	(71, 2){}, 31162	(395, 1){1, 2}, 378790
13	(67, 3){1, 2}, 52	**(9, 8){1}, 456**	(220, 1){1, 3}, 2862	(410, 1){1; 3}, 30826	**(378, 1){3; 1, 2}, 385275**
14	(443, 1){1}, 56	**(133, 3){1; 3}, 442**	(220, 1){1, 2, 3}, 2700	(1410, 1){1; 2; 3; 4}, 30560	(7, 30){4; 1, 3; 2, 3}, 379942
15	(358, 1){2}, 50	(244, 1){2}, 482	(110, 1){}, 2984	(16, 5){1}, 30980	**(378, 1){1; 3}, 384496**
16	(332, 1){2}, 54	(291, 1){2}, 488	(112, 3){2; 3}, 2910	(69, 4){2; 3}, 31380	(297, 1){1}, 380118
17	(310, 1){2, 3}, 56	**(21, 8){1; 2}, 500**	(474, 1){3; 1, 2}, 2892	(387, 1){1, 2}, 31376	(552, 1){2; 3}, 381044
18	(646, 1){2; 3}, 57	(271, 3){1; 2}, 524	(474, 1){1; 3}, 3030	(71, 3){1}, 31154	(118, 3){2; 1, 3}, 382460
19	(134, 3){1; 2, 3}, 58	**(4, 9){}, 582**	(320, 1){2}, 2984	(132, 1){}, 31548	(252, 1){3}, 384492
20	(642, 1){3; 1, 2}, 57	(1140, 1){1, 3; 2, 4; 3, 4}, 546	(58, 3){2}, 2614	(213, 2){1, 2, 3}, 31262	(79, 3){1}, 383320
21	(134, 3){2; 3}, 71	(22, 7){1, 3; 2, 3}, 558	(56, 3){2}, 2936	(344, 1){2}, 31580	(45, 4){1}, 386432
22	(428, 1){2}, 69	**(300, 1){3}, 594**	(112, 3){1; 2}, 2922	(420, 1){2; 3}, 32754	(1155, 1){2; 4; 3}, 384556
23	(406, 1){2, 3}, 70	**(714, 1){4; 3}, 594**	(158, 3){3; 1, 2}, 2898	(31, 10){3; 1, 2}, 31608	(282, 1){1, 3}, 382804
24	(683, 1){1}, 70	(732, 1){2; 3}, 626	(146, 3){2; 3}, 2762	(71, 4){2}, 31694	(714, 1){2, 4; 2, 3}, 382120
25	(44, 5){2; 3}, 60	**(180, 1){}, 672**	(158, 3){1; 3}, 2810	(203, 3){2; 1, 3}, 31694	(1295, 1){2; 1, 3}, 384408
26	(86, 3){2, 3}, 68	(755, 2){3; 1, 2}, 634	(505, 2){2; 3}, 2824	(1254, 1){1; 3; 2, 4}, 32249	(424, 1){2}, 382246
27	(803, 1){1, 2}, 64	(115, 6){2; 1, 3, 4}, 644	(113, 3){1}, 2942	(142, 3){1; 2}, 32030	(187, 3){1; 2}, 382800
28	(172, 3){2; 3}, 75	(1151, 1){1}, 670	(4, 21){2; 3}, 2898	(172, 3){2; 3}, 32979	(756, 1){3; 1, 2}, 387315
29	(33, 5){2, 3}, 60	(578, 1){1}, 682	(15, 8){3}, 3368	(16, 5){}, 32776	(323, 2){3}, 384860
30	(344, 3){1; 3; 2}, 64	**(488, 1){2}, 730**	(804, 1){3; 1, 2}, 2934	(344, 3){1; 3; 2}, 32088	(158, 3){1; 2}, 385836
31	(609, 1){2, 3}, 68	**(567, 1){2}, 744**	(220, 1){}, 3372	(69, 4){3}, 32168	**(378, 1){3}, 398892**
32	(498, 1){3}, 72	(291, 2){1, 3}, 750	(1320, 1){1; 3; 2, 4}, 2892	(1290, 1){4; 1, 3; 2, 3}, 32277	(470, 1){2, 3}, 385320
33	(253, 6){2; 3; 1, 4}, 66	(1319, 1){1}, 744	(804, 1){2; 3}, 3308	(1410, 1){3; 4; 1, 2}, 32599	(288, 1){}, 391712
34	(1030, 1){2; 3}, 67	**(450, 1){3}, 798**	(660, 1){3; 2, 4}, 3342	(284, 1){}, 33606	(297, 2){1, 2}, 385938
35	(947, 1){1}, 82	(532, 1){3}, 796	(133, 6){1, 4; 2, 4}, 3122	(568, 1){1}, 32916	(518, 1){2}, 388916
36	(694, 1){2}, 86	(302, 3){2; 3}, 758	(948, 1){3; 1, 2}, 3086	(71, 6){1, 2}, 32562	(418, 3){2; 4; 1, 3}, 386492
37	(1366, 1){1; 2}, 76	(220, 3){1, 3; 2, 3, 4}, 800	(196, 3){2; 3}, 3204	(1, 54){2}, 32331	(1155, 1){2; 1, 3, 4}, 386510
38	(646, 1){2, 3}, 94	(912, 1){2; 3}, 854	(133, 6){1, 2; 1, 3, 4}, 2918	(609, 2){4; 3}, 32838	(316, 1){}, 388882
39	(930, 1){2, 4; 2, 3}, 66	(510, 1){2, 3, 4}, 820	(474, 1){3}, 3736	(609, 1){3}, 32324	(158, 3){2, 3}, 386888
40	(620, 1){2, 3}, 84	(1020, 1){3; 2, 4}, 852	(627, 2){1, 3; 1, 4}, 3226	(1060, 1){2; 3}, 32952	(15, 14){1, 3; 1, 4}, 389258
41	(1187, 1){1}, 83	(1630, 1){1, 3; 2, 3}, 858	(640, 1){2}, 3368	(616, 1){2}, 33344	(480, 1){2}, 386912
42	(1163, 1){1}, 81	(1455, 1){3; 1, 2}, 868	(1245, 2){3; 2, 4; 1, 2}, 2974	(516, 1){3}, 32882	(1155, 2){1, 2; 4, 5; 3, 5}, 385670
43	(1283, 1){1}, 84	(27, 8){1}, 933	(112, 3){2}, 3660	(420, 1){3}, 33252	(289, 2){}, 391428
44	(134, 3){2, 3}, 94	(94, 5){1, 3; 2, 3}, 884	(1160, 1){2; 3}, 2970	(207, 4){3; 1, 2}, 32632	(118, 3){2}, 388810
45	(1367, 1){1}, 74	(315, 2){1, 4}, 992	(5, 24){2, 3}, 3220	(840, 1){2; 3}, 33280	(282, 1){}, 392568

(continued on next page)

Table B.8 (*continued*)

g	$\mathbb{F}_{13}$	$\mathbb{F}_{13^2}$	$\mathbb{F}_{13^3}$	$\mathbb{F}_{13^4}$	$\mathbb{F}_{13^5}$
46	(268, 3){2; 3}, 102	(2590, 1){1; 4; 2, 3}, 902	(1206, 1){3; 1, 2}, 3063	(924, 1){4; 3}, 32718	(2323, 1){1; 2}, 388185
47	(536, 3){1; 3; 2}, 79	(987, 2){4; 1, 3}, 940	(1081, 2){1, 3; 2, 3}, 3132	(1155, 2){1, 4; 2, 3, 4; 5}, 33686	(1155, 1){2; 3, 4}, 392646
48	(886, 1){2}, 110	**(873, 1){2}, 1058**	(705, 2){3; 2, 4}, 2970	(1155, 2){3; 2, 4; 1, 2, 5}, 33204	(776, 1){2}, 385894
49	(44, 5){2, 3}, 90	(225, 2){}, 1040	(158, 3){3}, 3472	(1242, 1){1, 3; 2, 3}, 33030	(594, 1){1, 3}, 389172
50	(66, 5){2, 4; 2, 3}, 76	(1290, 1){3, 4; 1, 2, 3}, 984	(292, 3){3; 1, 2}, 2830	(284, 3){2; 1, 3}, 32706	(181, 4){1, 2}, 388226

Table B.9
Table for $\max\{|X(n_0, n_{\mathrm{ns}})/\langle W_{d_1}, \ldots, W_{d_t}\rangle(\mathbb{F}_q)|\}$ with $q = 17^k$, $k = 1, 2, 3, 4, 5$ and $n_0 n_{\mathrm{ns}}^2 \leq 10^4$.

g	$\mathbb{F}_{17}$	$\mathbb{F}_{17^2}$	$\mathbb{F}_{17^3}$	$\mathbb{F}_{17^4}$	$\mathbb{F}_{17^5}$
1	(190, 1){1; 3; 2}, 25	(1, 6){2}, 324	(7, 6){3; 1, 2}, 5054	(1, 10){2}, 84096	(1, 14){1, 2}, 1422141
2	(5, 4){2}, 30	(1, 12){2}, 358	(87, 1){2}, 5166	(3, 10){2; 3}, 84670	(13, 4){1, 2}, 1424424
3	(127, 1){1}, 36	(71, 2){2}, 392	(4, 5){2}, 5256	(20, 3){1, 3; 2, 3}, 85244	(33, 1){}, 1426584
4	(80, 1){2}, 42	(16, 3){2}, 426	(70, 1){2}, 5350	(60, 1){3}, 85818	(159, 2){1, 3; 2, 3}, 1429140
5	(23, 6){3; 1, 2}, 38	(7, 12){2; 1, 3}, 448	(155, 1){2}, 5390	(20, 3){1, 2}, 85496	(52, 1){}, 1430904
6	(46, 3){2; 3}, 46	(22, 3){1, 2, 3}, 470	(47, 4){1; 2}, 5372	(24, 5){1; 3; 2}, 86345	(104, 1){2}, 1432818
7	(135, 2){3; 1, 2}, 43	(16, 3){}, 516	(55, 3){1; 3}, 5572	(365, 1){1; 2}, 86577	(215, 2){1, 3; 2, 3}, 1432388
8	(206, 1){2}, 50	(225, 1){1, 2}, 534	(99, 2){1, 3}, 5526	(206, 1){2}, 86662	(286, 1){3; 1, 2}, 1435502
9	(160, 1){2}, 48	(99, 1){}, 560	(93, 2){2}, 5602	(120, 1){3}, 86896	**(82, 1){}, 1438108**
10	(31, 5){1; 2}, 53	(108, 1){}, 630	(93, 2){2, 3}, 5498	(264, 1){1; 3}, 88046	**(208, 1){1, 2}, 1436834**
11	(254, 1){2}, 62	(49, 3){1}, 592	(190, 1){3}, 5620	(112, 1){}, 87148	**(104, 1){}, 1438748**
12	(184, 3){1; 3; 2}, 56	(1, 28){2}, 630	(169, 2){2}, 5616	(231, 2){3; 1, 4}, 86954	**(208, 1){2}, 1445778**
13	(92, 3){2; 3}, 66	**(144, 1){}, 696**	(495, 1){1; 3}, 5668	(74, 3){1, 3; 2, 3}, 87310	(13, 8){1, 2}, 1439158
14	(193, 2){1, 2}, 56	(13, 12){2; 3}, 676	(297, 1){2}, 5664	(365, 2){2; 1, 3}, 87288	(143, 2){3}, 1437974
15	(105, 2){4}, 52	(16, 7){1; 2}, 703	(155, 1){}, 5632	(88, 3){1, 3; 2, 3}, 87972	**(80, 3){1; 3}, 1445778**
16	(463, 1){1}, 64	(208, 3){1; 3; 2}, 762	(70, 3){2; 3}, 5782	(333, 1){1, 2}, 87602	(645, 2){1, 3; 2, 4; 3, 4}, 1439512
17	(487, 1){1}, 72	(104, 3){1; 2, 3}, 754	(55, 3){3}, 6100	(222, 1){2}, 88710	(286, 1){1, 2, 3}, 1444260
18	(412, 1){2}, 75	**(271, 3){1; 2}, 746**	(314, 1){2}, 5861	(412, 1){2}, 88715	(328, 1){1, 2}, 1449322
19	(607, 1){1}, 66	(225, 1){}, 836	(229, 2){2}, 5708	(240, 1){3}, 89052	(164, 1){}, 1449636
20	(142, 3){2; 3}, 60	**(104, 3){2; 3}, 862**	(61, 4){2}, 5794	(475, 2){2; 1, 3}, 89350	(858, 1){4; 1, 3; 2, 3}, 1442726
21	(446, 1){2}, 72	**(45, 4){1}, 824**	(110, 3){4; 1, 2, 3}, 6104	(759, 2){2; 3; 1, 4}, 88774	(270, 1){3}, 1446644
22	(23, 9){1; 2}, 70	(860, 1){2; 3}, 864	(110, 3){4; 3}, 5846	(453, 2){3; 1, 2}, 89354	(188, 1){}, 1442166
23	(583, 1){1, 2}, 80	(1102, 1){1, 3; 2, 3}, 842	(465, 1){2, 3}, 6016	(208, 1){}, 88324	(208, 1){}, 1454436
24	(508, 1){2}, 93	**(109, 3){2}, 880**	(898, 1){1; 2}, 5851	(429, 2){4; 1, 2, 3}, 88786	(485, 1){2}, 1441184
25	(92, 3){2, 3}, 84	**(49, 6){2}, 990**	(205, 6){4; 1, 3; 2, 3}, 5948	(730, 1){2; 3}, 90199	(416, 1){2}, 1458896
26	(399, 2){2, 3; 1, 2, 4}, 70	(343, 2){2}, 884	(507, 2){2; 3}, 6080	(515, 1){2}, 89642	(1203, 1){1; 2}, 1444846
27	(263, 3){1; 2}, 72	(370, 1){3}, 970	(141, 4){1; 3}, 6246	(448, 1){1}, 89420	(429, 1){3}, 1454252
28	(494, 1){2, 3}, 94	(130, 3){3; 2, 4}, 956	(681, 1){2}, 5880	(429, 2){4; 2, 3}, 88602	(188, 3){2; 3}, 1446690
29	(184, 3){2; 3}, 88	**(198, 1){}, 1000**	(495, 1){3}, 6292	(259, 3){2; 1, 3}, 88558	(720, 1){1; 3}, 1449606
30	(823, 1){1}, 95	(962, 1){1; 3}, 967	(372, 1){2, 3}, 6126	(549, 1){2}, 92918	(1015, 1){2; 3}, 1448331
31	(368, 3){1; 3; 2}, 70	**(1820, 1){2; 4; 3}, 1038**	(164, 3){2; 3}, 6018	(19, 10){1, 3}, 90728	(80, 3){3}, 1461144
32	(1166, 1){1; 2, 3}, 83	(32, 7){1; 2}, 1054	(205, 6){2; 4; 3}, 6220	(444, 1){1, 2, 3}, 90182	(656, 1){1, 2}, 1455706
33	(62, 5){2; 3}, 82	(25, 8){1}, 1038	(88, 3){3}, 6240	(1554, 1){1; 2; 3, 4}, 89329	(42, 5){4; 1, 3}, 1452724
34	(254, 3){1; 2, 3}, 76	**(16, 7){2}, 1260**	(140, 3){2; 3}, 6102	(1030, 1){2; 3}, 92738	(206, 3){1; 3}, 1446024
35	(1166, 1){2; 3}, 87	**(208, 3){1; 2, 3}, 1152**	(55, 6){1, 4}, 6488	(247, 3){1; 3}, 92680	(208, 3){1; 2, 3}, 1456994
36	(590, 1){2, 3}, 92	**(208, 3){1, 3; 2, 3}, 1182**	(403, 2){2}, 6010	(444, 1){1, 3}, 88950	(208, 3){1, 3; 2, 3}, 1457620
37	(734, 1){2}, 94	(324, 1){}, 1224	(320, 1){}, 6552	(1030, 1){3; 1, 2}, 91540	(858, 1){4; 1, 2, 3}, 1459186
38	(889, 2){2; 1, 3}, 72	**(416, 3){1; 3; 2}, 1224**	(77, 10){2; 4; 3}, 6394	(532, 1){2}, 91222	(572, 1){1, 3}, 1453414
39	(526, 3){1; 3; 2}, 79	**(104, 3){2, 3}, 1284**	(628, 1){2}, 6738	(711, 1){2}, 93212	(328, 1){}, 1469476
40	(7, 13){1, 2}, 78	(208, 3){2; 3}, 1434	(15, 14){1, 4; 2, 4}, 5998	(22, 15){1; 2, 3; 3, 4}, 90084	(572, 1){3}, 1463658
41	(1087, 1){1}, 104	(1848, 1){2; 4; 3}, 1200	(143, 3){2}, 6332	(504, 1){3}, 92080	(858, 1){4; 1, 3}, 1461312
42	(1646, 1){1; 2}, 101	(516, 1){3}, 1250	(41, 12){2; 3}, 6350	(1860, 1){2; 4; 1, 3}, 90206	(346, 1){}, 1458362
43	(23, 9){1, 2}, 90	(648, 1){2}, 1308	(990, 1){4; 1, 2, 3}, 6384	(1068, 1){2; 3}, 92988	(2030, 1){1; 3; 4}, 1454311
44	(568, 3){1; 3; 2}, 80	(61, 12){2; 1, 3}, 1306	(1860, 1){2; 1, 3; 3, 4}, 5924	(534, 1){2}, 89410	(1705, 2){1, 3; 2, 4; 3, 4}, 1460002
45	(892, 1){2}, 105	(18, 5){}, 1392	(990, 1){4; 1, 3}, 6376	(475, 2){1, 3}, 95564	(5, 24){1, 3}, 1465300
46	(14, 11){2; 3}, 82	(417, 2){3}, 1266	(594, 1){3}, 6378	(411, 2){3}, 92478	(540, 1){3}, 1457166
47	(133, 4){1, 2, 3}, 90	(4991, 1){1; 3; 2}, 1312	(1485, 1){3; 1, 2}, 6416	(759, 2){3; 1, 4}, 92752	(832, 1){1, 2}, 1473644
48	(1327, 1){1}, 110	(795, 2){3, 4; 1, 2, 3}, 1344	(235, 4){1; 3}, 6272	(763, 3){1; 3; 2}, 90002	(1705, 1){1; 3}, 1453916
49	(1303, 1){1}, 119	(5, 42){2; 1, 4}, 1398	(220, 3){4; 1, 2}, 6482	(1550, 1){2; 3}, 91704	(416, 1){}, 1476512
50	(23, 15){1; 2, 3}, 92	(1290, 1){4; 3}, 1396	(531, 2){1, 3}, 6702	(1460, 1){3; 1, 2}, 92540	(659, 2){2}, 1447175

Table B.10
Table for $\max\{|X(n_0, n_{\mathrm{ns}})/\langle W_{d_1}, \ldots, W_{d_t}\rangle(\mathbb{F}_q)|\}$ with $q = 19^k$, $k = 1, 2, 3, 4, 5$ and $n_0 n_{\mathrm{ns}}^2 \leq 10^4$.

g	$\mathbb{F}_{19}$	$\mathbb{F}_{19^2}$	$\mathbb{F}_{19^3}$	$\mathbb{F}_{19^4}$	$\mathbb{F}_{19^5}$
1	(1, 24){1; 2}, 28	(1, 7){}, 400	(1, 12){1}, 7024	(11, 3){1; 2}, 131040	(23, 2){1, 2}, 2478982
2	(130, 1){1, 3; 2, 3}, 34	(22, 1){}, 438	(30, 1){2}, 7188	(5, 7){1; 2}, 131758	(191, 1){1}, 2482286
3	(149, 1){1}, 38	(33, 1){}, 476	(9, 4){1}, 7352	(23, 4){1; 2}, 132476	(46, 1){1}, 2484746
4	(305, 1){1; 2}, 39	(66, 1){1, 2}, 514	(143, 2){1, 3; 2, 3}, 7333	(130, 1){2; 3}, 133194	(262, 1){1; 2}, 2487889
5	(33, 4){1, 3; 2, 3}, 44	(55, 1){}, 536	(9, 4){}, 7352	(55, 6){3; 4; 1, 2}, 133048	(46, 1){}, 2490510
6	(221, 1){1, 2}, 48	(73, 3){1; 2}, 560	(58, 1){}, 7420	(104, 1){2}, 133478	(103, 3){1; 2}, 2490728
7	(298, 1){1; 2}, 47	(215, 2){1, 3; 2, 3}, 600	(135, 1){1}, 7464	(65, 2){1, 2}, 134460	(11, 10){3; 1, 2}, 2492010
8	(179, 2){1, 2}, 47	(2, 9){}, 616	(315, 1){2; 3}, 7534	(260, 1){2; 1, 3}, 134026	(183, 2){2; 1, 3}, 2492026
9	(85, 3){1; 2, 3}, 50	(128, 1){}, 688	(21, 4){1}, 7676	(130, 1){2}, 134816	(11, 10){2; 3}, 2494874
10	(195, 2){4; 1, 3}, 52	**(319, 2){1, 3; 2, 3}, 703**	(189, 1){2}, 7848	(260, 1){2; 3}, 135582	**(92, 1){}, 2499156**
11	(389, 1){1}, 64	(117, 1){}, 724	(13, 12){1; 2}, 7612	(13, 10){2; 3}, 134932	**(34, 3){2}, 2501908**
12	(442, 1){1, 3; 2, 3}, 64	(3, 14){3}, 750	(28, 5){2; 3}, 7646	(130, 3){1; 4; 2, 3}, 136218	(51, 4){1; 2}, 2500509
13	(212, 1){2}, 60	(335, 2){2; 3}, 743	(9, 8){1}, 7900	(24, 5){1; 3}, 135354	(275, 2){2; 3}, 2502798
14	(509, 1){1}, 65	(438, 1){1; 3}, 790	(354, 1){1; 2}, 7987	(130, 3){2; 4; 3}, 138038	(336, 1){2; 3}, 2496276
15	(298, 1){2}, 73	(147, 2){3}, 816	(378, 1){1; 3}, 7676	(279, 1){2}, 136520	(153, 1){}, 2503472
16	(53, 5){1; 2}, 62	(73, 3){2}, 870	(525, 1){1; 2}, 7682	(780, 1){2; 4; 3}, 137010	(103, 3){1, 2}, 2504656
17	(71, 4){1, 2}, 62	(1, 60){1; 3; 2, 3}, 856	(130, 1){}, 7836	(106, 3){1, 3; 2, 3}, 135986	(246, 1){2, 3}, 2504038
18	(447, 1){2}, 70	(118, 3){2; 1, 3}, 869	(9, 14){1, 3; 2, 3}, 7740	(390, 1){4; 3}, 137262	(112, 3){3; 1, 2}, 2495544
19	(357, 2){2, 3; 1, 2, 4}, 66	(1, 60){1; 2}, 900	(210, 1){1, 2}, 8004	(260, 1){2}, 138036	(286, 1){3}, 2504088
20	(606, 1){3; 1, 2}, 72	(622, 1){2}, 898	(21, 8){1; 3}, 8004	(520, 1){2; 3}, 137706	(136, 3){2; 1, 3}, 2507788
21	(229, 3){1; 2}, 74	(294, 1){1}, 990	(315, 1){3}, 8408	(134, 3){2; 3}, 136514	(184, 1){}, 2510840
22	(394, 1){2}, 71	(146, 3){3; 1, 2}, 954	(700, 1){1, 3; 2, 3}, 8086	(300, 1){3}, 138078	(275, 2){1, 2, 3}, 2512012
23	(10, 7){2, 3}, 72	**(3, 14){}, 988**	(117, 4){1; 2, 3}, 8252	(18, 5){3}, 137176	(34, 3){}, 2516432
24	(401, 2){1, 2}, 74	(219, 2){1, 3}, 1050	(444, 1){2, 3}, 8240	(23, 12){1; 2}, 136514	(206, 3){1; 2, 3}, 2517472
25	(404, 1){2}, 72	(1518, 1){1; 2; 3, 4}, 1054	(13, 12){1}, 8312	(416, 1){2}, 140432	(68, 3){2}, 2521824
26	(442, 1){2, 3}, 92	(146, 3){1; 3}, 1042	(28, 5){2, 3}, 8548	(546, 1){2; 4}, 137452	(136, 3){2; 3}, 2511764
27	(821, 1){1}, 88	(976, 1){1; 2}, 1116	(117, 4){1; 3}, 8156	(935, 1){2; 3}, 138692	(275, 2){3}, 2515812
28	(348, 1){3}, 84	(1518, 1){2; 4; 3}, 1095	(236, 1){}, 8190	(130, 3){4; 2, 3}, 142106	(275, 2){2}, 2514876
29	(106, 5){1; 3; 2}, 82	**(198, 1){}, 1216**	(63, 4){1}, 8656	(130, 3){2; 4}, 142264	(1537, 1){1; 2}, 2518632
30	(894, 1){3; 1, 2}, 84	**(488, 1){2}, 1202**	(745, 2){1, 3; 2, 3}, 7784	(260, 3){4; 1, 3; 2, 3}, 142262	(549, 2){1, 3; 2, 3}, 2511548
31	(884, 1){2; 3}, 78	**(9, 7){}, 1260**	(945, 1){2; 3}, 8862	(260, 3){2; 4; 3}, 141924	(153, 4){1; 2, 3}, 2512864
32	(869, 1){1, 2}, 98	**(770, 1){2; 3}, 1236**	(28, 5){3}, 8140	(685, 2){2; 1, 3}, 135856	(492, 1){1, 3}, 2510936
33	(663, 1){2, 3}, 94	(73, 6){1, 3}, 1256	(390, 1){2, 3, 4}, 8200	(24, 5){3}, 139664	(309, 1){}, 2516440
34	(807, 1){2}, 82	**(506, 1){2, 3}, 1300**	(28, 5){1, 3}, 8072	(450, 1){3}, 139794	(1196, 1){2; 1, 3}, 2511098

Table B.10 (*continued*)

g	$\mathbb{F}_{19}$	$\mathbb{F}_{19^2}$	$\mathbb{F}_{19^3}$	$\mathbb{F}_{19^4}$	$\mathbb{F}_{19^5}$
35	$(170,3)\{2;3,4\},94$	$(98,3)\{1,2\},1256$	$(645,2)\{1,3;1,4\},8422$	$(22,9)\{2;1,3\},139489$	$(63,10)\{3;2,4;1,2\},2513220$
36	$(218,3)\{2;3\},92$	$(657,1)\{2\},1490$	$(681,2)\{2;3\},8025$	$(906,1)\{2;3\},141771$	$(553,3)\{1;3;2\},2508416$
37	$(596,1)\{2\},108$	$\mathbf{(9,14)\{3\},1452}$	$(626,1)\{2\},8692$	$(1560,1)\{2;4;3\},141368$	$(1100,1)\{2;3\},2524158$
38	$(1061,1)\{1\},112$	$(391,3)\{3;1,2\},1356$	$(525,2)\{1,3;1,4\},8040$	$(1560,1)\{3;4;1,2\},140806$	$(644,1)\{1,2,3\},2511048$
39	$(301,3)\{2;1,3\},78$	$(930,1)\{2,4;2,3\},1332$	$(21,8)\{1\},8444$	$(520,1)\{2\},141252$	$(246,1)\{\},2525140$
40	$(37,15)\{1;3;2\},90$	$\mathbf{(1020,1)\{4;2,3\},1440}$	$(15,14)\{2;1,4\},8360$	$(780,1)\{4;3\},143874$	$(572,1)\{3\},2522796$
41	$(611,2)\{1,2,3\},86$	$(441,1)\{\},1480$	$(315,1)\{\},8752$	$(45,8)\{1;2,3\},139504$	$(927,1)\{1,2\},2530204$
42	$(1229,1)\{1\},103$	$(381,2)\{1,3\},1410$	$(451,2)\{3\},8152$	$(25,7)\{2\},139658$	$(2406,1)\{1;3;2\},2514753$
43	$(778,1)\{2\},125$	$(420,1)\{3\},1428$	$(700,1)\{2,3\},9036$	$(778,1)\{2\},141021$	$(580,1)\{2,3\},2521938$
44	$(2501,1)\{1;2\},97$	$(94,5)\{1,3;2,3\},1528$	$(1274,1)\{1;3\},8486$	$(1599,2)\{3;4;1,2\},141049$	$(966,1)\{1,2;1,3,4\},2515774$
45	$(421,3)\{1;2\},116$	$(719,2)\{2\},1458$	$(1400,1)\{1;2,3\},8168$	$(558,1)\{3\},141752$	$(322,1)\{\},2531308$
46	$(1642,1)\{1;2\},101$	$(8,11)\{2\},1542$	$(14,11)\{2;3\},8832$	$(924,1)\{4;3\},140142$	$(1680,1)\{2;4;3\},2521700$
47	$(229,3)\{1,2\},96$	$(14,9)\{3\},1528$	$(21,20)\{1;2,3;3,4\},8770$	$(930,1)\{4;1\},140160$	$(306,1)\{\},2545480$
48	$(1490,1)\{3;1,2\},96$	$(398,3)\{2;3\},1640$	$(1890,1)\{2;1;4\},8006$	$(2538,1)\{1;3;2\},138845$	$(578,3)\{1;3;2\},2517253$
49	$(788,1)\{2\},102$	$(441,2)\{3\},1668$	$(350,1)\{\},8636$	$(36,5)\{3\},145608$	$(68,3)\{\},2543304$
50	$(922,1)\{2\},113$	$\mathbf{(1314,1)\{3;1,2\},1750}$	$(21,20)\{2;1;4\},8572$	$(2992,1)\{1;3;2\},140498$	$(606,1)\{1,3\},2522604$

Appendix C. Tables for the maximal curves

In the following tables (C.11, C.12, C.13), we list the quotient curves $X(n_0, n_{\rm ns})/\langle W_{d_1}, \ldots, W_{d_k}\rangle$ with genus $g \geq 2$ achieving the maximal number of points for a certain pair (genus, field), as discussed in Section 5. The curves are grouped by their real Weil polynomial $h_W(t)$ that is displayed as well. The notation for the curves is the same as in the tables of Appendix B. The two new maximal curves of Section 4 are in bold.

Table C.11
Maximal curves over $\mathbb{F}_q$ of genus $g = 2$.

g	q	$h_W(x)$	Curves
2	2	x^2+3x+1	$(67,1)\{1\},\ (73,1)\{1\},\ (93,1)\{1\},\ (103,1)\{1\},\ (115,1)\{1\},\ (133,1)\{1\}$
2	2^2	x^2+5x+9	$(23,1)\{\},\ (29,3)\{1;2\},\ (31,1)\{\},\ (31,3)\{1,2\},\ (69,1)\{1,2\},\ (87,1)\{2\},\ (107,1)\{1\},\ (125,1)\{1\},\ (161,1)\{1;2\},$ $(167,1)\{1\},\ (177,1)\{1;2\},\ (191,1)\{1\},\ (205,1)\{1;2\},\ (213,1)\{1;2\},\ (221,1)\{1;2\},\ (287,1)\{1;2\},\ (299,1)\{1;2\}$
2	2^7	$x^2+43x+713$	$(23,1)\{\},\ (29,3)\{1;2\},\ (31,3)\{1,2\},\ (69,1)\{1,2\},\ (107,1)\{1\},\ (125,1)\{1\},\ (161,1)\{1;2\},\ (167,1)\{1\},$ $(177,1)\{1;2\},\ (191,1)\{1\},\ (205,1)\{1;2\},\ (213,1)\{1;2\},\ (221,1)\{1;2\},\ (287,1)\{1;2\}$
2	3	x^2+4x+2 x^2+4x+3 x^2+4x+4	$(85,1)\{1;2\},\ (85,2)\{1;2;3\},\ (170,1)\{1;2;3\}$ $(37,2)\{1,2\},\ (53,2)\{1,2\},\ (58,1)\{2\},\ (77,1)\{1,2\},\ (77,2)\{1;2,3\},\ (77,2)\{2;1,3\},\ (106,1)\{1;2\},\ (116,1)\{1;2\}$ $(28,1)\{\},\ (40,1)\{2\},\ (43,2)\{1,2\},\ (65,2)\{2;1,3\},\ (122,1)\{1;2\}$
2	3^2	$x^2+10x+37$	$(2,5)\{\},\ (4,5)\{1,2\},\ (4,5)\{1\},\ (7,5)\{1,2\},\ (7,10)\{1;2,3\},\ (11,2)\{\},\ (11,4)\{2\},\ (13,4)\{1,2\},\ (14,5)\{1;2;3\},$ $(22,1)\{\},\ (29,4)\{1;2\},\ (35,2)\{1,2\}\ (44,1)\{1\},\ (50,1)\{\},\ (89,2)\{1,2\},\ (100,1)\{1\},\ (103,1)\{1\},\ (103,2)\{1;2\},$ $(110,1)\{1,3;2,3\},\ (115,1)\{1;2\},\ (115,2)\{1;2;3\},\ (121,1)\{1\},\ (121,2)\{1;2\},\ (142,1)\{2\},\ (143,1)\{1,2\},$ $(143,2)\{1;2,3\},\ (158,1)\{1;2\},\ (161,1)\{1;2\},\ (161,2)\{1;2;3\},\ (166,1)\{1;2\},\ (184,1)\{1;2\},\ (190,1)\{2;3\},$ $(191,1)\{1\},\ (191,2)\{1;2\},\ (205,1)\{1;2\},\ (205,2)\{1;2;3\},\ (206,1)\{1;2\},\ (230,1)\{1;2;3\},\ (284,1)\{1;2\},$ $(286,1)\{1;2;3\},\ (380,1)\{1;2;3\}$
2	3^5	$x^2+62x+1441$	$(7,5)\{1,2\},\ (7,10)\{1;2,3\},\ (13,4)\{1,2\},\ (14,5)\{1;2;3\},\ (22,1)\{\},\ (44,1)\{1\},\ (89,2)\{1,2\},\ (103,1)\{1\},$ $(103,2)\{1;2\},\ (110,1)\{1,3;2,3\},\ (115,1)\{1;2\},\ (115,2)\{1;2;3\},\ (121,1)\{1\},\ (121,2)\{1;2\},\ (143,1)\{1,2\},$ $(143,2)\{1;2,3\},\ (158,1)\{1;2\},\ (161,1)\{1;2\},\ (161,2)\{1;2;3\},\ (166,1)\{1;2\},\ (190,1)\{2;3\},\ (191,1)\{1\},$ $(191,2)\{1;2\},\ (205,1)\{1;2\},\ (205,2)\{1;2;3\},\ (206,1)\{1;2\},\ (230,1)\{1;2;3\},\ (286,1)\{1;2;3\},\ (380,1)\{1;2;3\}$
2	5	x^2+6x+8 x^2+6x+9	$(11,6)\{3;1,2\},\ (21,4)\{2;1,3\},\ (22,3)\{1;2,3\},\ (37,2)\{1,2\},\ (102,1)\{1;2,3\},\ (129,1)\{1;2\},\ (129,2)\{1;2;3\}$ $(57,2)\{2;1,3\},\ (61,2)\{1,2\},\ (67,1)\{1\},\ (67,2)\{1;2\},\ (88,1)\{1;2\},\ (91,1)\{1,2\},\ (91,2)\{1;2,3\},\ (134,1)\{1;2\}$
2	5^2	$x^2+20x+140$	$(1,12)\{2\},\ (2,7)\{1\},\ (7,6)\{2,3\},\ (8,3)\{2\},\ (13,3)\{1\},\ (13,6)\{1;2\},\ (27,2)\{2\},\ (28,1)\{\},\ (28,3)\{1,3;2,3\},$ $(47,2)\{2\},\ (51,4)\{1;2;3\},\ (98,1)\{1,2\}$
2	5^3	$x^2+44x+724$	$(14,3)\{1;3\}$
2	7	$x^2+8x+15$ $x^2+8x+16$	$(1,15)\{1,2\},\ (1,30)\{1;2,3\},\ (57,2)\{3;1,2\}$ $(8,3)\{2\}$
2	7^2	$x^2+28x+280$	$(1,16)\{1\},\ (30,1)\{2\},\ (48,1)\{2\},\ (48,1)\{1\},\ (60,1)\{1;2\}$
2	11^2	$x^2+44x+704$	$(1,12)\{2\},\ (1,30)\{2;1,3\},\ (5,4)\{2\},\ (5,6)\{2;3\},\ (7,4)\{2\},\ (8,3)\{2\},\ (15,2)\{3\},\ (21,4)\{3;1,2\},\ (25,2)\{2\},$ $(27,2)\{2\},\ (28,1)\{\},\ (40,1)\{2\},\ (60,1)\{3;1,2\},\ (102,1)\{1;2,3\},\ (119,2)\{1,3;2,3\},\ (167,1)\{1\},\ (167,2)\{1;2\}$
2	11^5	$x^2+1604x+965284$	$(2,7)\{2\},\ (3,7)\{1;2\},\ (3,14)\{1;2;3\},\ (4,7)\{1;2\},\ (6,7)\{1;2;3\},\ (31,3)\{1,2\},\ (31,6)\{1;2,3\},\ (39,1)\{2\},\ (39,2)\{1;3\},$ $(78,1)\{1;3\},\ (101,2)\{1,2\},\ (104,1)\{1;2\},\ (117,1)\{1;2\},\ (117,2)\{1;2;3\},\ (147,1)\{1;2\},\ (147,2)\{1;2;3\}$
2	13	$x^2+12x+35$ $x^2+12x+36$	$(1,30)\{2;1,3\}$ $(107,1)\{1\},\ (107,2)\{1;2\}$
2	13^2	$x^2+52x+988$	$(2,7)\{2\},\ (4,7)\{1;2\},\ (23,3)\{1;2\},\ (23,6)\{1;2;3\},\ (46,3)\{1;2;3\}$
2	17^2	$x^2+12x+36$	$(1,12)\{2\},\ (1,30)\{2;1,3\},\ (7,6)\{2,3\},\ (8,3)\{2\},\ (27,2)\{2\},\ (54,1)\{1\},\ (161,1)\{1;2\},\ (161,2)\{1;2;3\},$ $(191,1)\{1\},\ (191,2)\{1;2\}$
2	19^2	$x^2+76x+2128$	$(22,1)\{\},\ (29,3)\{1;2\},\ (29,4)\{1;2\},\ (29,6)\{1;2;3\},\ (33,1)\{1\},\ (33,2)\{1;2\},\ (44,1)\{1\},\ (66,1)\{1;2\},$ $(66,1)\{1,3;2,3\},\ (66,1)\{2;1,3\},\ (121,1)\{1\},\ (121,2)\{1;2\}$

Table C.12
Maximal curves over $\mathbb{F}_q$ of genus $3 \leq g \leq 4$.

g	q	$h_W(x)$	Curves
3	2	x^3+4x^2+3x-1	$(97,1)\{1\}$
3	2^2d	$x^3+9x^2+33x+63$	$(17,5)\{1;2\},\ (45,1)\{\},\ (55,3)\{2;1,3\},\ (61,3)\{1;2\},\ (63,1)\{2\},\ (79,3)\{1;2\},\ (105,1)\{2,3\},$ $(195,1)\{1;2,3\},\ (249,1)\{1;2\}$
3	3	x^3+6x^2+9x	$(52,1)\{2\}$
3	3^2	$x^3+18x^2+126x+432$	$(64,1)\{\},\ (91,2)\{3\},\ (124,1)\{2\},\ (238,1)\{1,3;2,3\}$
3	5^2	$x^3+30x^2+360x+2200$	$(19,12)\{1;2;3\}$
3	5^3	$x^3+66x^2+1812x+26488$	$(14,3)\{3\},\ (28,3)\{1;3\}$
3	7^2	$x^3+42x^2+714x+6272$	$(9,4)\{1\},\ (45,1)\{\},\ (45,2)\{1\},\ (48,1)\{\},\ (62,3)\{2;3\},\ (64,1)\{\},\ (96,1)\{1,2\},\ (124,3)\{1;2;3\}$
3	11^2	$x^3+66x^2+1782x+25168$	$(5,6)\{2\},\ (5,6)\{2,3\},\ (15,4)\{3;1,2\},\ (25,2)\{\},\ (45,2)\{2;3\},\ (56,1)\{1\},\ (64,1)\{\},$ $(69,2)\{2;3\},\ (105,4)\{1;2;3;4\},\ (120,1)\{3;1,2\}$

(*continued on next page*)

Table C.12 (*continued*)

g	q	$h_W(x)$	Curves
3	11^5	$x^3 + 2406x^2 + 2412732x + 1290774088$	$(52,1)\{2\},\ (78,1)\{3;1,2\},\ (156,1)\{2;3\},\ (312,1)\{1;2;3\}$
3	17^2	$x^3 + 102x^2 + 4284x + 94792$	$(71,2)\{2\}$
3	19^2	$x^3 + 114x^2 + 5358x + 132848$	$(33,1)\{\},\ (33,2)\{1\},\ (55,1)\{1\},\ (55,2)\{1;2\},\ (64,1)\{\},\ (66,1)\{1,2,3\},\ (110,1)\{2;1,3\}$
4	2^2	$x^4 + 10x^3 + 41x^2 + 96x + 172$ $x^4 + 10x^3 + 41x^2 + 100x + 188$ $x^4 + 10x^3 + 43x^2 + 110x + 205$	$(1,11)\{\},\ (1,21)\{1,2\},\ (25,3)\{2\},\ (135,1)\{1,2\},\ (225,1)\{1;2\},\ (231,1)\{2;1,3\},\ (285,1)\{1;2,3\}$ $(7,15)\{1;2;3\},\ (81,1)\{\},\ (177,1)\{2\},\ (261,1)\{1;2\}$ $(29,3)\{1,2\},\ (161,1)\{1,2\},\ (251,1)\{1\},\ (321,1)\{1;2\},\ (341,1)\{1;2\},\ (483,1)\{1;2;3\}$
4	2^7	$x^4 + 86x^3 + 3275x^2 + 72154x + 1007077$	$(29,3)\{1,2\},\ (161,1)\{1,2\},\ (251,1)\{1\},\ (321,1)\{1;2\},\ (341,1)\{1;2\},\ (483,1)\{1;2;3\}$
4	3^2	$x^4 + 20x^3 + 174x^2 + 860x + 2713$	$(14,5)\{1;2,3\},\ (44,1)\{\},\ (110,1)\{2,3\},\ (115,2)\{3;1,2\},\ (161,2)\{3;1,2\},$ $(220,1)\{1;2,3\},\ (286,1)\{1;2,3\}$
4	5^2	$x^4 + 40x^3 + 680x^2 + 6400x + 36800$	$(16,3)\{2\},\ (27,2)\{\},\ (28,3)\{2,3\},\ (98,1)\{1\},\ (188,1)\{2\}$
4	7^2	$x^4 + 52x^3 + 1176x^2 + 15176x + 123872$ $x^4 + 52x^3 + 1182x^2 + 15340x + 125497$	$(60,1)\{2\},\ (120,1)\{2;1,3\}$ $(54,1)\{\},\ (108,1)\{1\}$
4	11^2	$x^4 + 88x^3 + 3344x^2 + 71632x + 950576$	$(1,15)\{1\},\ (1,30)\{1;2\},\ (15,4)\{2;3\},\ (16,3)\{2\},\ (25,3)\{1\},\ (25,6)\{1;2\},\ (25,6)\{2;1,3\},$ $(27,2)\{\},\ (32,3)\{1;2\},\ (60,1)\{3\},\ (80,1)\{2\}$
4	11^5	$x^4 + 3208x^3 + 4503384x^2+$ $+3613247392x + 1812300626896$	$(39,2)\{3\}$
4	17^2	$x^4 + 136x^3 + 8024x^2 + 268192x + 5571920$	$(16,3)\{2\},\ (27,2)\{\},\ (31,6)\{2;3\},\ (54,1)\{\},\ (108,1)\{2\},\ (108,1)\{1,2\},\ (108,1)\{1\}$
4	19^2	$x^4 + 152x^3 + 10032x^2 + 375440x + 8739088$	$(66,1)\{1,2\}$

Table C.13
Maximal curves over $\mathbb{F}_q$ of genus $g \geq 5$.

g	q	$h_W(x)$	Curves
5	2^2	$x^5 + 12x^4 + 62x^3 + 192x^2 + 436x + 864$	$(209,1)\{1,2\},\ (645,1)\{1;2;3\}$
5	11^2	$x^5 + 110x^4 + 5390x^3 + 154880x^2 + 2895530x + 37097632$	$(1,30)\{2;3\},\ (120,1)\{1;3\}$
5	11^5	$x^5 + 4010x^4 + 7237240x^3 + 7741577680x^2 + 5435351766080x + 2617242553904032$	$(3,7)\{2\},\ (3,14)\{1;3\},\ (156,1)\{3;1,2\}$
6	2^2	$x^6 + 15x^5 + 102x^4 + 425x^3 + 1254x^2 + 2925x + 6021$	$(299,1)\{1,2\}$
6	11^5	$x^6 + 4812x^5 + 10614300x^4 + 14191614560x^3+$ $+12809583966480x^2 + 8223105443361792x + 3849697906629612544$	$(78,1)\{3\},\ (104,1)\{2\},\ (156,1)\{1;3\}$
7	2	$x^7 + 7x^6 + 14x^5 - 2x^4 - 30x^3 - 16x^2 + 12x + 8$	$(187,1)\{1,2\}$
7	2^2	$x^7 + 16x^6 + 116x^5 + 520x^4 + 1685x^3 + 4424x^2 + 10310x + 22880$	$(189,1)\{1,2\},\ (455,1)\{1;2,3\}$
7	7^2	$x^7 + 98x^6 + 4410x^5 + 120736x^4 + 2248022x^3 + 30194976x^2 + 303918580x + 2367501248$	$(1,16)\{\}$
7	11^2	$x^7 + 154x^6 + 10934x^5 + 474320x^4 + 14051730x^3 + 301274512x^2 + 4843013868x + 59903475808$	$(1,24)\{2\},\ (25,6)\{2;3\},\ (64,3)\{1;2\}$
7	11^5	$x^7 + 5614x^6 + 14634564x^5 + 23479207640x^4 + 25900587487040x^3+$ $+20781817917390432x^2 + 12507502676802272608x + 5736056081814890238208$	$\mathbf{(6,7)\{3;1,2\}}$
9	11^2	$x^9 + 198x^8 + 18414x^7 + 1068672x^6 + 43379226x^5 + 1308681792x^4+$ $+30454499196x^3 + 560623333248x^2 + 8321568916914x + 101330949451072$	$(1,30)\{2\},\ (32,3)\{2\}$
10	2^2	$x^{10} + 22x^9 + 227x^8 + 1488x^7 + 7099x^6 + 26790x^5 + 84949x^4+$ $+237900x^3 + 614564x^2 + 1518856x + 3689816$	$(13,15)\{1;2;3\}$
		$x^{10} + 22x^9 + 227x^8 + 1476x^7 + 6907x^6 + 25398x^5 + 78709x^4+$ $+217680x^3 + 561476x^2 + 1395072x + 3415000$	$(225,1)\{2\}$
10	3^2	$x^{10} + 44x^9 + 924x^8 + 12336x^7 + 117822x^6 + 860472x^5 + 5028348x^4+$ $+24382968x^3 + 101458377x^2 + 374775308x + 1272392128$	$(220,1)\{2,3\}$
10	17^2	$x^{10} + 340x^9 + 54740x^8 + 5548800x^7 + 397201600x^6 + 21349618368x^5+$ $+894614065600x^4 + 29960920387200x^3 + 816165461399840x^2$ $+ + 18327354677661760x + 343126733171222720$	$(108,1)\{\}$
12	11^5	$x^{12} + 9624x^{11} + 44383944x^{10} + 130535252320x^9 + 274862630948400x^8+$ $+440993755828543104x^7 + 560170020610594432896x^6 + 577840514161469228879616x^5+$ $+492617240971238592986103360x^4 + 351365791059712247073125157750x^3+$ $+211472505871262576194557806678120x^2 + 108000279404968696858606142571834266x+$ $+46952382003014183461596141477941970132$	$\mathbf{(156,1)\{3\}}$
13	11^2	$x^{13} + 286x^{12} + 39182x^{11} + 3422848x^{10} + 214132490x^9 + 10215137504x^8+$ $+386406235644x^7 + 11901007829568x^6 + 304192864120002x^5 + 6546901285290080x^4+$ $+120036129906428164x^3 + 1893927675352782784x^2 + 25962335404137817828x+$ $+312310047332950820032$	$(25,6)\{2\}$
19	11^2	$x^{19} + 418x^{18} + 84854x^{17} + 11145552x^{16} + 1064837026x^{15} + 78868905456x^{14}+$ $+4713619036588x^{13} + 233624987892384x^{12} + 9795244689058202x^{11}+$ $+352657670048703536x^{10} + 11031136579467494292x^9 + 302625311123193214816x^8+$ $+7338131279130078069588x^7 + 158317948152609796199264x^6+$ $+3056722777699641740353752x^5 + 53095079109227126428981568x^4+$ $+833873235186826014803382474x^3 + 11900141855044287421899255472x^2+$ $+155111515626413902614636636040x + 1856794761749706495782311526412$	$(64,3)\{2\}$

Data availability

The research described in this article relies on publicly available data from the websites LMFDB [35] and manypoints.org [27]. We thank the respective teams for maintaining these valuable resources.

References

[1] S. Anni, E. Assaf, E. Lorenzo García, On smooth plane models for modular curves of Shimura type, Res. Number Theory 9 (2) (2023) 21.
[2] A.O.L. Atkin, J. Lehner, Hecke operators on $\Gamma_0(m)$, Math. Ann. 185 (1970) 134–160.
[3] M. Akbas, D. Singerman, The normalizer of $\Gamma_0(N)$ in $\mathrm{PSL}(2, \mathbf{R})$, Glasg. Math. J. 32 (3) (1990) 317–327.

[4] J. Balakrishnan, et al., Explicit Chabauty-Kim for the split Cartan modular curve of level 13, Ann. Math. (2) 189 (3) (2019) 885–944.
[5] J.S. Balakrishnan, et al., Rational points on the non-split Cartan modular curve of level 27 and quadratic Chabauty over number fields, arXiv preprint, arXiv:2501.07833, 2025.
[6] F. Bars, The group structure of the normalizer of $\Gamma_0(N)$ after Atkin-Lehner, Commun. Algebra 36 (6) (2008) 2160–2170.
[7] B. Baran, An exceptional isomorphism between modular curves of level 13, J. Number Theory 145 (2014) 273–300.
[8] A.J. Best, et al., Computing classical modular forms, in: J.S. Balakrishnan, et al. (Eds.), Arithmetic Geometry, Number Theory, and Computation, Springer International Publishing, Cham, 2021, pp. 131–213.
[9] M.H. Baker, E. González-Jiménez, J. González, B. Poonen, Finiteness results for modular curves of genus at least 2, Am. J. Math. 127 (6) (2005) 1325–1387.
[10] D. Bartoli, M. Giulietti, M. Kawakita, M. Montanucci, New examples of maximal curves with low genus, Finite Fields Appl. 68 (2020) 101744.
[11] M. Baker, Y. Hasegawa, Automorphisms of $X_0^*(p)$, J. Number Theory 100 (1) (2003) 72–87.
[12] W. Bosma, J. Cannon, C. Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput. 24 (3-4) (1997) 235–265, Computational algebra and number theory (London, 1993).
[13] I. Chen, Jacobians of modular curves associated to normalizers of Cartan subgroups of level p^n, C. R. Math. Acad. Sci. Paris 339 (3) (2004) 187–192.
[14] I. Chen, The Jacobians of non-split Cartan modular curves, Proc. Lond. Math. Soc. (3) 77 (1) (1998) 1–38.
[15] V. Cam, F. Özbudak, Curves with many rational points via Atkin–Lehner involution, Bol. Soc. Mat. Mex. 31 (2025).
[16] V. Dose, J. Fernández, J. González, R. Schoof, The automorphism group of the non-split Cartan modular curve of level 11, J. Algebra 417 (2014) 95–102.
[17] V. Dose, G. Lido, P. Mercuri, Automorphisms of Cartan modular curves of prime and composite level, Algebra Number Theory 16 (6) (2022) 1423–1461.
[18] V. Dose, G. Lido, P. Mercuri, C. Stirpe, Modular curves with many points over finite fields, J. Algebra 635 (2023) 790–821.
[19] V. Dose, P. Mercuri, C. Stirpe, Double covers of Cartan modular curves, J. Number Theory 195 (2019) 96–114.
[20] V. Dose, On the automorphisms of the nonsplit Cartan modular curves of prime level, Nagoya Math. J. 224 (1) (2016) 74–92.
[21] P. Deligne, M. Rapoport, Les schémas de modules de courbes elliptiques, in: Modular Functions of One Variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), in: Lecture Notes in Math., vol. 349, Springer, 1973, pp. 143–316.
[22] F. Diamond, J. Shurman, A First Course in Modular Forms, Graduate Texts in Mathematics, vol. 228, Springer-Verlag, New York, 2005, pp. xvi+436.
[23] B. Edixhoven, P. Parent, Semistable reduction of modular curves associated with maximal subgroups in prime level, Doc. Math. 26 (2021) 231–269.
[24] L. Furio, D. Lombardo, Serre's uniformity question and proper subgroups of $C_{ns}^+(p)$, arXiv:2305.17780 [math.NT], 2023.
[25] S. Frengley, Congruences of elliptic curves arising from nonsurjective mod N Galois representations, Math. Comp. 92 (339) (2023) 409–450.
[26] S.D. Galbraith, Rational points on $X_0^+(p)$, Exp. Math. 8 (4) (1999) 311–318.
[27] G. van der Geer, E. Howe, K. Lauter, C. Ritzenthaler, Tables of curves with many points, http://manypoints.org.
[28] G. van der Geer, E. Howe, K. Lauter, C. Ritzenthaler, Tables of curves with many points: the case $\mathbb{F}_{2^7}$ and $g = 4$, https://web.archive.org/web/20240913142835/https://manypoints.org//Details.aspx?g=4&q=128.
[29] R. Gupta, E.A. Mendoza, L. Quoos, Reciprocal polynomials and curves with many points over a finite field, Res. Number Theory 9 (3) (2023) 60.
[30] J. González, Constraints on the automorphism group of a curve, J. Théor. Nr. Bordx. 29 (2) (2017) 535–548.
[31] E.W. Howe, Curves of medium genus with many points, Finite Fields Appl. 47 (2017) 145–160.
[32] E. Kani, Endomorphisms of Jacobians of modular curves, Arch. Math. 91 (3) (2008) 226–237.
[33] D. Kohen, A. Pacetti, Heegner points on Cartan non-split curves, Can. J. Math. 68 (2) (2016) 422–444.

[34] Q. Liu, D. Lorenzini, Models of curves and finite covers, Compos. Math. 118 (1) (1999) 61–102.
[35] The LMFDB Collaboration, The l-functions and modular forms database, http://www.lmfdb.org, 2024. (Accessed 12 April 2024). Online.
[36] P. Mercuri, Equations and rational points of the modular curves $X_0^+(p)$, Ramanujan J. 47 (2) (2018) 291–308.
[37] Magma shared code for computations of maximal curves, https://github.com/mercuri-pietro/Modular-Curves/tree/main/Many_points_modular_curves.
[38] P. Michaud-Rodgers, Quadratic points on non-split Cartan modular curves, Int. J. Number Theory 18 (2) (2022) 245–267.
[39] T. Miyake, Modular Forms, Springer Monographs in Mathematics, Springer, Berlin, Heidelberg, 2005, pp. x+338.
[40] P. Mercuri, R. Schoof, Modular forms invariant under non-split Cartan subgroups, Math. Comput. 89 (324) (2020) 1969–1991.
[41] K.V. Nguyen, M.-H. Saito, d-gonality of modular curves and bounding torsions, arXiv:alg-geom/9603024 [alg-geom], 1996.
[42] A. Rigato, Uniqueness of low genus optimal curves over $\mathbb{F}_2$, in: D. Kohel, R. Rolland (Eds.), Arithmetic, Geometry, Cryptography and Coding Theory 2009, in: Contemp. Math., vol. 521, Amer. Math. Soc., Providence, 2010, pp. 87–105.
[43] B. de Smit, B. Edixhoven, Sur un résultat d'Imin Chen, Math. Res. Lett. 7 (2-3) (2000) 147–153.
[44] M. Shimura, Defining equations of modular curves $X_0(N)$, Tokyo J. Math. 18 (2) (1995) 443–456.
[45] H. Stichtenoth, Algebraic Function Fields and Codes, second, Graduate Texts in Mathematics, vol. 254, Springer-Verlag, Berlin, 2009, pp. xiv+355.
[46] T. Youssefi, Inegalite relative des genres (and an appendix with Michel Matignon, Manuscr. Math. 78 (1993) 111–128.
[47] D. Zywina, Computing actions on cusp forms, arXiv:2001.07270 [math.NT], 2021.
[48] Magma shared code to compute cusp forms, https://github.com/davidzywina/OpenImage/tree/master/main.