SAPIENZA
UNIVERSITÀ DI ROMA

DIPARTIMENTO DI MATEMATICA GUIDO CASTELNUOVO

PhD Thesis in Mathematics

On critical points of solutions of elliptic equations

Advisor: Prof. Massimo Grossi

Candidate: Fabio De Regibus

XXXV cycle

Abstract

In this thesis we deal with qualitative properties of solutions of the semilinear elliptic problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$ is a smooth domain and $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.

A classical problem concerns the study of the shape of u related to the one of the domain. In particular we investigate the number of critical points of u with respect to the convexity of Ω. Both the cases of positive and sign-changing solutions are treated.

Contents

Introduction

Let u be a solution of the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is smooth and $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, is a smooth and bounded domain. A classical problem in the qualitative study of solutions of the preceding argument concerns the shape of u, which is known to be strongly influenced by the geometry of the domain Ω and by the nonlinearity f. In particular we are interested in the number of critical points of u and in the geometry of its superlevel sets $\{u > c\}$, with $c \in \mathbb{R}$. Moreover, both positive and sign-changing solutions will be considered.

Since the literature is very wide it is impossible to give here a complete list of references, so we mainly focus on the results which are more strictly related to the rest of the thesis. We refer to Chapter 1 for a more detailed discussion in the case of positive solutions and to Section 5.1 in Chapter 5 for the sign-changing ones. Finally we mention the recent surveys [Mag16, Gro21] and the monograph [Kaw85a].

A first interesting result linking the geometry and the topology of the domain with the geometry of the solution u can be deduced from the Poincaré-Hopf Theorem, for instance see [Mil65]. In particular it follows that if u is a positive solution of (1) with isolated critical points $\boldsymbol{x_1}, \ldots, \boldsymbol{x_k}$, then one can prove

$$\sum_{i=1}^{k} \operatorname{ind}(\nabla u, \boldsymbol{x_i}) = (-1)^N \chi(\Omega),$$

where $\chi(\Omega)$ is the Euler characteristic of Ω (let us point out that the preceding formula is general and does not depend on the fact that u is a solution of an elliptic equation). Hence we can see that there are relationships between the topology of the domain and the number of critical points. Furthermore, it is natural to ask when the sum reduce to a minimal number of elements. Hence, if Ω is a contractible domain we have $\chi(\Omega) = 1$ and then we can investigate when $k = 1$.

Let us also mention that by a classical result in Lusternik-Schnirelmann theory, see for instance [Str08], we get a lower bound on the number of critical points. Indeed, given a smooth function $u : \Omega \to \mathbb{R}$ solution of problem(1), then

$$\sharp \{ \boldsymbol{x} \in \Omega \mid \nabla u(\boldsymbol{x}) = \boldsymbol{0} \} \geq \operatorname{cat}(\Omega),$$

where $\operatorname{cat}(\Omega)$ is the Lusternik-Schnirelmann category of Ω. Then if the topology of Ω is not trivial (for instance if the domain contains holes) we can have cat$(\Omega) \geq 2$. We are not going to treat the case when Ω is not simply connected, but let us just mention, for example, that a well studied case is the one of convex rings, we refer to Subsection 1.1.2 for a list of references on this topic, while the paper [GL20] investigate the case of domains with a small hole.

Let us start by examining the case when u is a positive solution of problem (1) on a simply connected domain Ω. A first important result has been proved by Makar-Limanov for the torsion problem, i.e. $f \equiv 1$. In [ML71], he proves that if Ω is a smooth, bounded and convex domain in $\mathbb{R}^2$, then the solution of the torsion problem has a unique nondegenerate critical point. Moreover, it is *quasiconcave*, i.e. the superlevel sets $\{u > c\}$ are convex for all $c \in \mathbb{R}$.

The same result is true for the first Dirichlet eigenfunction, $f(u) = \lambda u$, as it was proved by Brascamp and Lieb [BL76] (see also the paper by Acker, Payne and Philippin [APP81]).

A very general result on the uniqueness of the critical point of solutions of (1) can be deduced from the seminal paper [GNN79] by Gidas, Ni and Nirenberg. Indeed, if f is a Lipschitz continuous function and $\Omega \subseteq \mathbb{R}^N$ is a smooth and bounded domain which is symmetric with respect to the plane $x_i = 0$ for any $i = 1, \ldots, N$ and convex with respect to any direction $x_1, \ldots, x_N$, then u has exactly one critical point and moreover the superlevel sets are star-shaped with respect to the origin. We point out that it is still an open problem to prove when u is quasiconcave or not, see the work by Hamel, Nadirashvili and Sire [HNS16].

Some conjectures claim that the symmetry assumptions can be removed. An interesting contribution in this direction is the result in [CC98] by Cabré and Chanillo, see Theorem 1.1.9, where the uniqueness of the critical point is proved for semi-stable solutions in planar domains with strictly positive curvature of the boundary $\partial\Omega$.

We recall that a solution u of problem (1) is said to be *(semi-)stable* if the linearized operator at u is (non negative) positive definite, see Definition 1.1.8. In Chapter 2, which collects the results from [DRGM21], the theorem is extended allowing the curvature of the boundary to vanish somewhere, see Theorem 2.1.1. Furthermore, we also give an alternative proof of the result for strictly positive curvature.

If $\partial\Omega$ contains points with negative (mean) curvature the situation may change drastically, even if we consider solutions on domains which are not far from being convex. Indeed, not only the uniqueness of the critical point is lost, but it is not even possible to have any bound on the number of critical points. In [GG22], see also Theorem 1.2.1 for the precise steatement, Gladiali and Grossi prove that there exists a family of bounded domains Ω_ε in $\mathbb{R}^2$ and a solution u_ε to the torsion problem in Ω_ε such that u_ε has at least k maximum points with $k \geq 2$, while the domain Ω_ε is star-shaped, locally converges to the convex strip $\mathbb{R} \times (-1, 1)$, the curvature of the boundary is positive everywhere except to a portion and its minimum goes to 0 as $\varepsilon \to 0$. It is important to point out that this work also shows that if we consider a star-shaped domain, then the superlevel sets are not star-shaped too, in general.

Then the result has been extended to higher dimensions and to stable solutions of more general nonlinearities in [DRG22a], see Theorem 3.1.1 in Chapter 3. In particular the case $f = \lambda g$ where $\lambda > 0$, g is smooth and satisfies

$$g : \mathbb{R} \to \mathbb{R} \text{ is increasing and convex,}$$
$$g(0) > 0,$$

is considered.

Then the situation can be completely described for $N = 2$ as follows: if the curvature of the boundary of the domain is non negative, then stable solutions have exactly one critical point, while as soon as the curvature becomes negative somewhere, then we can find stable solutions with an arbitrary (finite) large number of critical points.

What happens for $N \geq 3$? It is still an open problem to determine if the (semi-)stable solutions admit exactly one critical point whenever the domain is convex. Our contribution in this context consists in showing that even if the (mean) curvature of the boundary $\partial\Omega$ is strictly positive everywhere, we can build a family of domains converging to a convex cylinder such that the solutions of the torsion problem on them admit an arbitrary large number of critical point. This is stated in Theorem 3.1.2 and it is still proved in the same paper as before. We refer to Chapter 3 for all the details.

It is natural to ask if it is still possible to recover uniqueness of the critical point even in non convex domains. In Chapter 4 we give some of results in this optic.

First of all we consider the Poisson problem

$$\begin{cases} -\Delta u = f(\boldsymbol{x}) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f : \Omega \to \mathbb{R}$ is a regular positive function and Ω is a smooth bounded domain, $\boldsymbol{x} \in \Omega$. We find a condition involving both the function f and the geometry of the domain Ω (the curvature of the boundary) to ensure uniqueness of the critical point. This is stated in Theorem 4.1.2 which may apply also for non convex domains.

Then we deal with small perturbations of bounded and convex domains. We show that if we fix a smooth, bounded and convex domain $\Omega \subseteq \mathbb{R}^2$, and we consider another domain $\widetilde{\Omega}$ sufficiently close to it, see Definition 4.1.4, then we have that semi-stable solutions of problem (1) on $\widetilde{\Omega}$ admits exactly one critical point, even if $\widetilde{\Omega}$ is not convex. We refer to Theorem 4.1.5 for the precise statement. Note that, as explained before, if we do not assume that the limit domain is bounded then the result is no longer true.

In the last part of the thesis we deal with the case of sign-changing solutions. To our knowledge there are no results in the literature. So our starting point is the classical problem of the second Dirichlet eigenfunction of the Laplacian in dimension $N = 2$, that is we consider the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda_2 u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where λ_2 is the second eigenvalue of the Laplace operator and u a corresponding eigenfunction. It is known that u change sign exactly once in virtue of the Courant

Nodal Domain Theorem, see [CH53]. The geometry and location of the nodal line $\Lambda = \overline{\{\, \boldsymbol{x} \in \Omega : u(\boldsymbol{x}) = 0 \,\}}$ has received a lot of interest. A very famous conjecture about the nodal line says that Λ musts touch $\partial\Omega$ at exactly two points for all (simply connected) domains: different versions of the conjecture had been stated by Payne in [Pay67] and by Yau in [Yau82]. The conjecture has been proved in convex domains by Melas in [Mel92], see also the paper [Ale94] by Alessandrini. We refer to Section 5.1 for an overview on this problem.

Of course the computation of the number of critical points of the second eigenfunction is strongly influenced by the geometry of the nodal line. If it is a closed curve contained in Ω we expect many critical points, otherwise two is the minimum number.

The first contribution in Chapter 5 is given by Theorem 5.2.2, where we prove that the second Dirichlet eigenfunction in a convex domain with large eccentricity has exactly two non degenerate critical points in Ω: a maximum and a minimum. Let us recall that the eccentricity of a planar domain is defined as

$$\mathrm{ecc}(\Omega) = \frac{\mathrm{diam}(\Omega)}{\mathrm{inradius}(\Omega)},$$

where inradius(Ω) is the radius of the largest ball contained in Ω. These domains were considered by Jerison in [Jer95a] and also in collaboration with Grieser in [GJ96] where the location of the nodal line Λ was characterized.

Finally we also deal with convex perturbation of rectangles, still studied by Grieser and Jerison in [GJ09]. In this case it is possible to see that the m-th eigenfunction has exactly m critical points in Ω, see Theorem 5.2.3.

The original contributions of Chapter 5 can be found in the paper [DRG22b].

We conclude this part of the introduction by pointing out that it is clearly interesting to consider different kind of differential operators in problem (1) and some works can be found in the literature. Even if not in the interest of this thesis let us mention that results in the spirit of the ones quoted before had been proved for more general elliptic operators, for the p-Laplacian operator $-\Delta_p$ and also for fully nonlinear ones. We refer to the end of Chapter 1 for some references. The case of non local operators seems to be, at the moment, less explored, see next section for a short discussion about this topic.

Some open problems

There are still lots of open problem about the geometric properties of solutions of elliptic problems. About future developments strictly connected with the results described above, it will be interesting to investigate the existence of geometric conditions on the domain Ω to limit the number of critical points to avoid phenomena as the ones described in [GG22, DRG22a]. Moreover, another future direction of research could be to replace stability with semi-stability in the construction showed in Theorem 3.1.1.

Another problem it would be very interesting to examine concerns the case of non local operators, and a good starting point could be the fractional Laplacian. As the author knowledge, very few is known about the number of critical points and the shape of the level sets of solutions of non local problem, even for the simplest cases. Let us focus on the general problem

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2}$$

for $s \in (0,1)$. We refer to [DNPV12, RO16, ROS14a, SV12, SV13] and the reference therein, for the theory about fractional spaces and Dirichlet problems with the fractional Laplacian operator (included existence, regularity etc. etc.).

The case of the torsion problem $f \equiv 1$ has been faced by Kulczycki in [Kul17] where he proves, through the Caffarelli-Silvestre extension [CS07], that if Ω is a planar, smooth, bounded and convex set, then the fractional torsion function is concave on Ω for $s = 1/2$.

Then it is natural to ask if it is possible to extend the preceding result and in particular if the result by Cabré and Chanillo in [CC98] for semi-stable solution can be extended to the fractional case. Note that in [ROS14b], Ros-Oton and Serra treat the case of semi-stable solutions of problem (2) extending the theory for the classical Laplacian (see Appendix A for a short resume of the classical case). It is important to point out that the non local nature of the operator could be very hard to overcome. Furthermore, the bad boundary regularity of the solutions is another fact that has to be taken into account and finally a fundamental step in the work of Cabré and Chanillo is based on the good behavior of the nodal lines of solutions of elliptic equation (see [CF85b], for instance), which can be worse in the fractional case, as shown in [STT20].

Organization of the work

The thesis is organized as follows: in Chapter 1 we give a short survey about known result on uniqueness of critical point and quasiconcavity of positive solutions of problem (1). The last section of the chapter is devoted to a collection of counterexamples to the uniqueness of critical point and to quasiconcavity.

In the sequent chapter we deal with Theorem 2.1.1, that is the extension of Cabré and Chanillo's theorem from the case of domain with strictly positive curvature of the boundary of the domain, to the case of non negative curvature.

In Chapter 3 we extend Gladiali and Grossi's Theorem 1.2.1 in Theorem 3.1.1 and we show that in dimension $N \geq 3$ it is not enough to ask for striclty positive curvature of the boundary to ensure uniqueness of the critical point. This is Theorem 3.1.2.

The proofs of Theorem 4.1.2 on the Poisson problem and of Theorem 4.1.5 on small perturbation of convex domain can be found in Chapter 4.

Finally, in Chapter 5 we treat the case of sign-changing solutions focusing on the Dirichlet eigenfunctions. After a short section on known results about the nodal line conjecture, we state and prove Theorem 5.2.2 and Theorem 5.2.3 about the number of critical points of eigenfunctions in convex domains with large eccentricity.

The appendix is divided into two parts: in the first one we resume some well known fact about stability of solutions of elliptic problems, while in the second one we prove some technical results needed in Chapter 3.

Notations

We adopt the following notations:

- unless different indication, $\boldsymbol{x} := (x_1, \dots, x_N)$ is a point in the euclidean space $\mathbb{R}^N$, with orthonormal base $\boldsymbol{e_1}, \dots, \boldsymbol{e_N}$, for $N \geq 1$;
- $\mathbf{0} := (0, \dots, 0) \in \mathbb{R}^N$ denotes the origin of the euclidean space $\mathbb{R}^N$;
- $B_r(\boldsymbol{x})$ denotes the open ball of radius $r > 0$ and center $\boldsymbol{x} \in \mathbb{R}^N$;
- for a given set $\Omega \subseteq \mathbb{R}^N$ we denote its Lebesgue measure as $|\Omega|$, while the counting measure is denoted with the symbol $\sharp$;
- if the closure of the set ω is contained in another set Ω we write $\omega \subset\subset \Omega$;
- we write $A \triangle B$ for the symmetric difference of two sets A and B, i.e. $(A \setminus B) \cup (B \setminus A)$;
- for a given function $u : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}$, the (possibly weak) partial derivative with respect to the direction $\boldsymbol{e_i}$ will be denoted by u_{x_i} or $\partial_{x_i} u$ for all $i = 1, \dots, N$ (anologous notation for higher order derivatives);
- if $\Omega \subseteq \mathbb{R}^N$ is a regular enough domain we denote by ν the other normal unit vector to its boundary $\partial\Omega$, in particular the other normal derivative will be denoted by u_ν or $\partial_\nu u$;
- given a domain $\Omega \subseteq \mathbb{R}^N$ and an elliptic differential operator L we denote by $\lambda_k(L, \Omega)$ the k-th eigenvalue of the operator in Ω with zero Dirichlet boundary conditions.

Finally, for classical results about partial differential equations - such as maximum principles, regularity theory and others - we refer, for instance, to the textbooks [GT01, Eva10, Jos13].

Chapter 1

On the number of critical points of positive solutions

For $N \geq 2$, let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. Assume u is a classical solution of the following problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given function.

As explained in the Introduction, we are interested in qualitative properties of the solutions of the preceding problem and in particular we want to examine the number of critical points of u and the shape of the superlevel sets $\{u > c\}$, where $c \in \mathbb{R}$. To this end, it is useful to recall the following notions of concavity.

Definition 1.0.1. A function $u : \Omega \to \mathbb{R}$ is said to be

(i) *concave* if $u(\mu \boldsymbol{x_1} + (1-\mu)\boldsymbol{x_2}) \leq \mu u(\boldsymbol{x_1}) - (1-\mu)u(\boldsymbol{x_2})$ for all $\boldsymbol{x_1}, \boldsymbol{x_2} \in \Omega$ and $\mu \in (0,1)$, if the preceding inequality is strict we say that u is *strictly concave*,

(ii) α*-concave* if u^α is concave for $\alpha \in (0,1)$ (see [Ken85] for a more general definition and related properties),

(iii) *log-concave* if $\log u$ is concave,

(iv) *quasiconcave* if the superlevel sets $\{u > c\}$ are convex for all $c \in \mathbb{R}$.

The following is a trivial, but important, remark.

Remark 1.0.2. 1) If $u \in \mathcal{C}^2(\Omega)$, then convexity of u is clearly equivalent to to the fact that the hessian matrix of u is negative semidefinite everywhere in Ω. The Hessian of u is negative definite everywhere if and only if u is strictly concave.

2) It holds

$$(i) \implies (ii) \implies (iii) \implies (iv),$$

but the reverse implications are not true in general.

3) If the superlevel sets $\{u > c\}$ are star-shaped for all $c \in \mathbb{R}$, then uniqueness of the critical point holds. In particular, quasiconcavity implies uniqueness of critical point.

4) More generally, given a monotone function h, we have that h-concavity implies quasiconcavity, where the definition of h-concavity is the trivial generalization of the previous ones.

It is natural to ask if the superlevel sets $\{u > c\}$ inherit the geometric properties of the domain. In particular, if Ω is convex, since it can be clearly seen as the superlevel set $\{u > 0\}$, one can investigate if this convexity property is preserved for all $c > 0$, see also [Lio81, Remark 3]. Anyway, proving the preceding property can be a very hard task and indeed it is not always true, see Section 1.2. Hence, taking into account that, as pointed out in the previous remark, quasiconcavity implies uniqueness of the critical point, clearly a weaker property that it is interesting to study is the uniqueness the critical point and its dependence on the geometry of the domain. That being said, in this chapter, we will mainly focus on the case where Ω is a convex domain.

The chapter is organized as follows: in the next section we take in consideration positive solutions of problem (1.1) and we give a short survey about results where uniqueness of the critical point is proved to be true. Then in Section 1.1.1 we describe the technique related to te Concavity Maximum Principle which allows to prove quasicancavity and in particular uniqueness of the critical point. In the last section we deal with a series of results where it is shown that uniqueness does not hold or the superlevel sets are not convex.

1.1 Uniqueness of the critical point

The number of critical points of (positive) solutions of problem (1.1) strongly depends on the geometry and the topology of the domain. The next two theorems are useful to understand this dependence.

Before stating the first one let us recall that given a smooth vector field V defined on a neighbourhood of $\boldsymbol{x} \in \mathbb{R}^N$, which is an isolated zero of V, the *index* of V in $\boldsymbol{x}$ is

$$\operatorname{ind}(V, \boldsymbol{x}) := \deg\left(V, B_r(\boldsymbol{x}), \mathbf{0}\right),$$

for suitably small $r > 0$ and where deg denotes the Browner degree (see, for instance, the classical texts [Llo78, Dei85] as references on topological degree. The Poincaré-Hopf Theorem can be stated as follows, see [Mil65].

Theorem 1.1.1 (Poincaré-Hopf Theorem)**.** *Given a smooth and bounded domain $\Omega \subseteq \mathbb{R}^N$, and a smooth vector field $V : \Omega \to \mathbb{R}^N$, assume that $\boldsymbol{x_1}, \ldots, \boldsymbol{x_k}$ are all the zeros of V, they are isolated and that $V(\boldsymbol{x}) \cdot \nu(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in \partial\Omega$. Then it holds*

$$\sum_{i=1}^{k} \operatorname{ind}(V, \boldsymbol{x_i}) = \chi(\Omega),$$

where $\chi(\Omega)$ is the Euler characteristic of Ω.

Hence if u solves problem (1.1) and we set $V = \nabla u$ we can easily derive the following corollary that gives a constraint on the number of critical points which depends only on the topology of the domain Ω.

Corollary 1.1.2. *Assume u is a positive solution of problem* (1.1) *and it has k isolated critical points $\boldsymbol{x_1}, \ldots, \boldsymbol{x_k}$. Then*

$$\sum_{i=1}^{k} \operatorname{ind}(\nabla u, \boldsymbol{x_i}) = (-1)^N \chi(\Omega).$$

Let us point out that if, for instance, f is analytic, then the critical points of u are isolated, see Theorem 1.1.11.

The second theorem gives a lower bound on the number of critical points in terms of the Lusternik-Schnirelmann category of the domain Ω. Let us recall some basic fact in this theory which are here adapted to our purposes. We refer for instance to the books [AM07, Str08] for a more general treatment. First of all the *Lusternik-Schnirelmann category* of Ω is given by

$$\operatorname{cat}(\Omega) := \sup \{ \operatorname{cat}_\Omega(K) \mid K \subseteq \Omega \text{ is compact} \},$$

where for all compact sets $K \subseteq \Omega$ one has

$$\operatorname{cat}_\Omega(K) := \min \left\{ \ell \in \mathbb{N} \,|\, K \subseteq \bigcup_{i=1}^{\ell} A_i, \text{ where } A_i \text{ are contractible in } \Omega \right\}.$$

Then we have the following result.

Theorem 1.1.3. *Let u be a positive solution of problem* (1.1)*, then*

$$\sharp \{ \boldsymbol{x} \in \Omega \mid \nabla u(\boldsymbol{x}) = \boldsymbol{0} \} \geq \operatorname{cat}(\Omega).$$

In particular, the preceding results tell us that if we want to find solutions with only one critical point we need to assume the domain Ω is contractible.

Now, we can start with the description of some results about quasiconcavity and uniqueness of the critical point. One of the first has been obtained for the torsion problem, i.e. $f \equiv 1$, by Makar-Limanov.

Theorem 1.1.4 ([ML71])**.** *Let $\Omega \subseteq \mathbb{R}^2$ be a convex domain and u be a solution of the torsion problem*

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then u has a unique critical point, a nondenerate maximum. Moreover, u is quasiconcave.

See also [Kim06] for the location of the maximum point in convex domains close to the ball. Quasiconcavity, and more generally $\frac{1}{2}$-concavity, of the torsion problem is true also in higher dimension, see Remark 1.1.21.

Before going on, for $N \geq 2$, let us recall that the *mean curvature* of the boundary of Ω is given by

$$\mathfrak{K} := \frac{1}{N} \operatorname{tr}(d\nu),$$

where ν is the unit other normal vector field, which is well defined and regular if the domain is assumed to be regular enough.

Another important case that has been studied is the first Dirichlet eigenfunction, which it is well known, is the only - up to a multiplicative factor - positive one. A first partial result on the convexity of level sets under some symmetry assumption can be found in Payne's work [Pay73]. Then the quasiconcavity has been proved removing the symmetry assumptions by Brascamp and Lieb in [BL76], but we refer to the work [APP81] by Acker, Payne and Philippin for a proof which is more familiar with PDEs methods.

Theorem 1.1.5 ([BL76, APP81]). *Let Ω be a convex domains and let u be the first Dirichlet eigenfunction of the Laplacian, i.e. u solves*

$$\begin{cases} -\Delta u = \lambda_1 u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda_1 := \lambda_1(-\Delta, \Omega) > 0$ is the first eigenvalue. Then u has a unique critical point, a nondenerate maximum. Moreover, u is quasiconcave.

The proof in [APP81] holds only in dimension 2 under the stronger assumption $\mathfrak{K} > 0$ on $\partial\Omega$, but it gives upper and lover bound for the curvature of all level sets. More precisely, one can prove that the first eigenfunction is log-concave, for instance see [CS82]. Moreover, in domains with large eccentricity, the localization of the critical point and a convergence result to a suitable function related to the geometry of the domain itself can be found in the work [GJ98] by Grieser and Jerison.

Other interesting estimates about the location of the maxima points both for the torsion function, both for the first eigenfunction had been proved by Magnanini and Poggesi in [MP21].

The preceding results give a very good description of the shape of the solution u provided Ω is a convex set, but they hold for very specific type of f. A first result for a more general class of nonlinearities f has been proved by Sperb in [Spe75], under some additional symmetry assumptions.

In this context, that is if we consider domains satisfying some symmetry assumptions, a very important result is the fundamental theorem by Gidas, Ni and Nirenberg in [GNN79] which allows to consider very general nonlinearities in any dimension.

Theorem 1.1.6 ([GNN79]). *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain which is symmetric with respect to the plane $x_i = 0$ for any $i = 1, \dots, N$ and convex with respect to any directions $x_1, \dots, x_N$. Suppose that u is a positive solution to* (1.1) *where f is a locally Lipschitz function. Then*

(i) u is symmetric with respect to $x_1, \dots, x_N$,

(ii) $\frac{\partial u}{\partial x_i} < 0$ for $x_i > 0$ and all $i = 1, \dots, N$.

As a consequence of the preceding theorem, it easily follows that u admits exactly one critical point and, moreover, all the superlevel sets are star-shaped. See also [Kaw83]. In general, as proved in [HNS16], it is not true that $\{u > c\}$ is convex for all $c \in \mathbb{R}$ if Ω si convex, see Theorem 1.2.3 and Remark 1.2.4.

To remove the symmetry assumptions in Theorem 1.1.6 is not an easy task. In this context, a partial result has been proved by Lions in [Lio81], where the technique of the the parabolic flow is used. Another result is due to Kennington, where in particular it is proved that for $\lambda > 0$ and $0 < p < 1$, if $f(u) = \lambda u^p$, the uniqueness of the critical point hold. See the following theorem.

Theorem 1.1.7 ([Ken85]). *Let $\Omega \subseteq \mathbb{R}^N$ be a smooth, bounded and convex domain and let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ be a solution of problem* (1.1)*, where $f : (0, +\infty) \to (0, +\infty)$ satisfies, for some $\alpha \in (0, 1)$, the two following conditions*

(i) $t^{\alpha-1} f(t)$ is strictly decreasing with respect to t,

(ii) $t^{(3\alpha-1)/\alpha} f(t^{1/\alpha})$, or equivalently $t^{(1-2\alpha)/\alpha} f(t^{-1/\alpha})$, is concave with respect to t, or $(1-2\alpha)(1-3\alpha)f(t)+(5\alpha-1)f'(t)+t^2 f''(t) \leq 0$ for twice differentiable f.

Then u is α-concave. In particular, if $f(t) := \lambda t^p$ for some constant $c > 0$ and $p \in (0, 1)$, then u is $\frac{1-p}{2}$-concave.

In the same paper more general results can be found. They involve the equation $-\Delta u = f(x, u)$ and also some results about large solutions. See also [Kea85] for related results only in dimension 2.

Finally, a very important result, that was proved by Cabré and Chanillo in [CC98], allows to avoid all the symmetry assumptions on Ω and to consider very general f, but it is assumed that u is a semi-stable solution.

Let us recall the following definition of (semi-)stability.

Definition 1.1.8. A function u is a *(semi-)stable* solution of the problem (1.1) if the linearized operator at u is positive (nonnegative) definite, i.e. if for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$ one has

$$\int_\Omega |\nabla \varphi|^2 - \int_\Omega f'(u)|\varphi|^2 > (\geq) 0,$$

or equivalently if the first eigenvalue of the linearized operator $-\Delta - f'(u)$ in Ω is positive (nonnegative).

In Appendix A a short resume about stability of solutions can be found.

Theorem 1.1.9 ([CC98]). *For $N = 2$, assume $\mathfrak{K} > 0$ on $\partial\Omega$, i.e. the boundary of Ω has strictly positive curvature everywhere. Suppose $f \geq 0$ and that u is a semi-stable solution to* (1.1)*. Then u has a unique nondegenerate critical point.*

Remark 1.1.10. 1) The preceding result was extended allowing $\partial\Omega$ to have points with zero curvature in [DRGM21], see Chapter 2.

2) The theorem does not hold in higher dimension under only the assumption that the mean curvature of the boundary is strictly positive, see Theorem 3.1.2 and the related discussion. It is an open problem to determine if or if not (semi-)stable solutions in convex domains admit a unique critical point for $N \geq 3$.

Let us also mention the following result due to Alessandrini and Magnanini where the authors show that critical points are isolated in dimension 2 and moreover they give a constraint on the kind of critical points.

Theorem 1.1.11 ([AM92]). *Let $\Omega \subseteq \mathbb{R}^2$, be a smooth, bounded and simply connected domain and let u be a solution of problem* (1.1)*, where $f > 0$ is analytic. Then the critical points of u are isolated and one has*

$$\sharp\{\textit{maxima of } u\} - \sharp\{\textit{saddle of } u\} = 1.$$

In the remaining part of this section we collect some other result where uniqueness of the critical point holds true for particular nonlinearities f.

Remark 1.1.12. Before going on let us briefly recall the situation for $f(u) = \lambda u^p$, $\lambda > 0$, $0 \le p < \frac{N+2}{N-2}$ if $N \ge 3$ or $p \ge 0$ if $N = 2$. If Ω is convex then

1) $p = 0$: i.e. the torsion problem. In this case we know the u is $\frac{1}{2}$-concave, see Remark 1.1.21 below.

2) $0 < p < 1$: in this case we know the u is $\frac{1-p}{2}$-concave.

3) $p = 1$: i.e. the eigenfunction problem. In this case $\lambda = \lambda_1(-\Delta, \Omega)$ and we know the u is log-concave.

4) $p > 1$: much less is known in this case. Some partial results can be found below. Let us point out that Theorem 1.1.9 can not be applied, since $p > 1$ implies that the solutions are not semi-stable.

For least energy solutions in the case $f(u) = \lambda u^p$ with $p > 1$, Lin proved the following result.

Theorem 1.1.13 ([Lin94])**.** *Let $\Omega \subseteq \mathbb{R}^2$ be a smooth, bounded and strictly convex domain and let u be the least energy solution of*

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $p > 1$. Then $u^{-\frac{p-1}{2}}$ is convex in Ω. In particular u is quasiconcave and uniqueness of the critical point holds true.

The case $f(u) = \lambda u^p$ with p close to the critical value $\frac{N+2}{N-2}$ has been studied in a series of paper by Gladiali, Grossi, Molle and Takahashi, where under suitable assumptions they prove uniqueness of the critical point and star-shapeness or convexity of the superlevel sets. The results are summarized in the following theorem.

Theorem 1.1.14 ([GM03, GG04b, GT10])**.** *Let $\Omega \subseteq \mathbb{R}^N$, with $N \ge 3$, be a smooth, bounded and convex domain and let u_ε be a solution of*

$$\begin{cases} -\Delta u = \lambda u^{\frac{N+2}{N-2} - \varepsilon} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Mreover, assume one of the following situation is verified

(i) $\lambda = N(N-2)$ *and*

$$\lim_{\varepsilon \to 0} \frac{\int_\Omega |\nabla u_\varepsilon|^2}{\left(\int_\Omega |u_\varepsilon|^{2^*}\right)^{2/2^*}} = S,$$

where S is the best constant in the Sobolev embedding and $2^* := 2N/(N-2)$;

(ii) $N \ge 4$ *and* $\lambda = 1$.

Then for ε small enough there exists a unique maximum point $\boldsymbol{x}_{\varepsilon}$ which is the only critical point of u_ε and the superlevels of u_ε are strictly star-shaped with respect to $\boldsymbol{x}_{\varepsilon}$.

Finally, if (i) is satisfied and if Ω has strictly positive Gauss curvature at any point of its boundary, then the level sets of u_ε have strictly positive Gauss curvature at any point that is not $\boldsymbol{x}_{\varepsilon}$. In particular, the superlevel sets are strictly convex.

Also the case of non stable solutions for Gelfand problem $f(u) = \lambda e^u$, when λ is close to 0 has been studied by Gladiali, Grossi and Takahashi.

Theorem 1.1.15 ([GG04a, GT10])**.** *Let $\Omega \subseteq \mathbb{R}^2$, be a smooth, bounded and convex domain and for $\lambda > 0$ let u_λ be a solution of*

$$\begin{cases} -\Delta u = \lambda e^u & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega, \end{cases}$$

such that $\|u_\lambda\|_{L^\infty(\Omega)} \to +\infty$ as $\lambda \to 0$. Then for λ small enough, the maximum point $\boldsymbol{x}_{\boldsymbol{\lambda}}$ is the only critical point of u_λ and the superlevels of u_λ are strictly star-shaped with respect to $\boldsymbol{x}_{\boldsymbol{\lambda}}$.

If $\partial\Omega$ has strictly positive curvature at any point, then the level sets of u_λ are strictly convex.

Remark 1.1.16. 1) The paper [GG04a] deals also with the more general problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where f is positive, increasing, convex and such that

$$f(0) > 0, \qquad \lim_{s \to +\infty} \frac{f(s)}{s} = +\infty,$$

for small values of $\lambda > 0$.

2) Finally, we refer also to the works [GG04b, EMG04] for other results involving $f(u) = \lambda u^p - \mu u$, with $\lambda, \mu > 0$ and $p > 1$.

1.1.1 Concavity Maximum Principle

An important tool to prove quasiconcavity and in particular uniqueness of the critical point is given by the Concavity Maximum Principle. Here we briefly describe his main features. Before it, we start this section recalling some classical strategies that can be adopted to prove the convexity of the level sets for solutions of problem (1.1), following Kawohl's book [Kaw85a]. Then after the statement of the Concavity Maximum Principle we show how it can be used to prove 1/2-half concavity of the solution of the torsion problem in convex domains.

Given a convex domain $\Omega \subseteq \mathbb{R}^N$, as explained in [Kaw85a], the most common strategies one can try to prove that the solutions of problem (1.1) are quasiconcave are the following ones:

1) show that u is concave, or that u is h-concave for a suitable monotone function h;

2) show that the principal curvatures of all superlevel sets are non negative, see for instance [APP81].

For the first case some sub-strategies can be adopted.

a) *Parabolic flow*: the linear parabolic operator $\frac{\partial}{\partial t} - \Delta$ under homogeneous Dirichlet boundary conditions preserves some concavity properties of initial datum u_0. Studying the asymptotic behavior as $t \to +\infty$ of the solution of the initial value problem, one can desume properties on the solutions of semilinear elliptic problems, see for instance [BL76, Lio81, Ken88, LV08].

b) *Concavity Maximum Principle*: a continuous function $v : \Omega \to \mathbb{R}$ is convex if and only if

$$v\left(\frac{\boldsymbol{x_1}+\boldsymbol{x_2}}{2}\right) - \frac{v(\boldsymbol{x_1})+v(\boldsymbol{x_2})}{2} \leq 0, \quad \text{for all } (\boldsymbol{x_1}, \boldsymbol{x_2}) \in \Omega \times \Omega.$$

One can verify the preceding inequality by means of a suitable maximum principle. This approach has been first introduced in [Kor83a], then it was independently exploited in [Kor83b, CS82] and generalized in [Ken85]. See also [Kaw85a, Kaw85b, Kaw86, GP93].

c) *Constant rank theorem*: under suitable assumptions if u solves and equation of the form $\Delta u = g(u, \nabla u, \Delta u)$ and if the Hessian of u si nonnegative definite, then it has constant rank. Then a *continuation method* can be applied, see for instance [CF85a, KL87, Lin94]

Let us refer to the book [Kaw85a] for the description of other possible techniques and for a more complete list of references.

To describe the main features of the Concavity Maximum Principle, let us introduce the following definition.

Definition 1.1.17. Let u be defined in $\overline{\Omega}$ where Ω is a bounded and convex domain in $\mathbb{R}^N$. Then for $\boldsymbol{x_1}, \boldsymbol{x_2} \in \overline{\Omega}$ and $\mu \in [0,1]$ the *concavity function* is given by

$$\mathfrak{C}_u(\boldsymbol{x_1}, \boldsymbol{x_2}, \mu) := u(\mu\boldsymbol{x_2} + (1-\mu)\boldsymbol{x_1}) - \mu u(\boldsymbol{x_2}) - (1-\mu)u(\boldsymbol{x_1}).$$

Remark 1.1.18. As explained in [Kor83a], the concavity function $\mathfrak{C}_u(\boldsymbol{x_1}, \boldsymbol{x_2}, \mu)$ is the difference height, with sign, between the graph of u and the line segment joining $(\boldsymbol{x_1}, u(\boldsymbol{x_1}))$ to $(\boldsymbol{x_2}, u(\boldsymbol{x_2}))$, above the point $\mu\boldsymbol{x_2} + (1-\mu)\boldsymbol{x_1}$. In particular, the function u is convex if and only if $\mathfrak{C}_u \leq 0$ for all $\boldsymbol{x_1}, \boldsymbol{x_2} \in \Omega$ and $\mu \in (0,1)$. Clearly, it is enough to consider $\mu = 1/2$.

The following theorem is a maximum principle for the concavity function. Let us point out that more general statements can be found in the original papers and in the book [Kaw85a]. We refer also to the paper [Ken85], by Kennington.

Theorem 1.1.19 (Concavity Maximum Principle)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded and convex domain and let $u \in \mathcal{C}^2(\overline{\Omega})$ be a solution of*

$$\sum_{ij=1}^{N} a_{ij}(\nabla u(x))u_{ij}(x)u = b(\boldsymbol{x}, u, \nabla u),$$

where for all $p \in \mathbb{R}^N$ the matrix $(a_{ij}(p))_{ij}$ is symmetric and positive semidefinite. Moreover assume that *(i)* $b(\boldsymbol{x},\cdot,p)$ *is strictly decreasing, for all* $\boldsymbol{x}\in\Omega$, *for all* $p\in\mathbb{R}^N$,

(ii) $b(\cdot,\cdot,p)$ *is harmonic concave for all* $p\in\mathbb{R}^N$, *that is* $(b(\cdot,\cdot,p))^{-1}$ *is convex, for all* $p\in\mathbb{R}^N$, *jointly in the first two arguments.*

Then if the maximum of $\mathfrak{C}_u$ *is positive, it is not attained in* $\Omega\times\Omega\times[0,1]$.

Remark 1.1.20. A variant of the Concavity Maximum Principle can be proved also for parabolic equations. See [Kor83b], for instance.

The Concavity Maximum Principle allows to prove log and power concavity of solutions of some elliptic problems. To show how, a good example can be the proof of the fact that the torsion function is $\frac{1}{2}$-concave if $\Omega\subseteq\mathbb{R}^N$ is a strictly convex domain. Here, we sketch the one that can be found in the book [Kaw85a]. Given such a Ω let u solves

$$\begin{cases}-\Delta u=1 & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega.\end{cases}$$

Then, setting $v:=-\sqrt{u}$, it is easy to see that v satisfies the following problem

$$\begin{cases}\Delta v=-\frac{1}{v}\left(|\nabla v|^2+\frac{1}{2}\right) & \text{in } \Omega\\ v<0 & \text{in } \Omega\\ v=0 & \text{on } \partial\Omega.\end{cases}$$

Thus, it is immediate to verify that

$$\begin{aligned} b:\Omega\times\mathbb{R}\times\mathbb{R}^N &\longrightarrow \mathbb{R}\\ (\boldsymbol{x},t,p) &\longmapsto -\frac{1}{t}\left(|p|^2+\frac{1}{2}\right),\end{aligned}$$

satisfies assumptions (i) and (ii) of Theorem 1.1.19. As a consequence, if the maximum of $\mathfrak{C}_v$ is positive, then it is not attained in $\Omega\times\Omega\times[0,1]$. Finally, it is not difficult to show that the maximum of $\mathfrak{C}_v$ on $\partial(\Omega\times\Omega)\times[0,1]$ can not be positive, see [Kaw85a, Lemma 3.12], and then $\mathfrak{C}_v\le 0$ in $\Omega\times\Omega\times(0,1)$. Hence, taking into account Remark 1.1.18, we can conclude that v is convex, that is $\sqrt{u}$ is concave in Ω, as claimed.

Remark 1.1.21. We just showed that, as a consequence of the Concavity Maximum Principle in Theorem 1.1.19, one can prove $\frac{1}{2}$-concavity of the solution of the torsion function in convex domains, generalizing Theorem 1.1.4. Note that it holds in any dimension $N\ge 2$. See also Theorem 4.1.1.

In the literature we can find a lot of works where - possibly slightly different versions of - the Concavity Maximum Principle are used to prove quasiconcavity of functions. Among them let us quote the following papers by Korevaar [Kor83b], Caffarelli and Spruck [CS82], Kennington [Ken85] and Kawohl [Kaw85b].

The results apply, for instance, to the Schrödinger eigenvalue problem

$$-\Delta u+V(\boldsymbol{x})u=\lambda u,$$

with V convex, $\lambda>0$ and to

$$-\Delta u=ug(u),$$

where $g'(u)\le 0$ and $g'(u)+ug''(u)\le 0$. A typical example is $g(u)=\lambda-\mu u^p$ with $p,\mu>0$.

1.1.2 Related problems

A related problem that has been extensively studied concerns the case of convex rings. We say that Ω is a *convex ring* if

$$\Omega := \Omega_1 \setminus \overline{\Omega}_2,$$

where $\Omega_1, \Omega_2 \subseteq \mathbb{R}^N$ are two non empty, bounded and convex domains such that $\Omega_2 \subset\subset \Omega_1$. Hence, given $M > 0$, it is natural to investigate when u solution of the semilinear elliptic problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega_1 \\ u = M & \text{on } \partial\Omega_2, \end{cases}$$

inherits the geometry of the domain, and in particular when u is quasiconcave. Clearly here the notion of quasiconcavity has to be slighty modified in $\{u > c\} \cup \overline{\Omega}_2$ is convex for all $c \in \mathbb{R}$. For this problem we refer to [Gab57, CS82, CF85a, Kaw85a, Kor90, DK93, CS03, Sal05, CS06, BLS09, Gre09, HNS16] and the references therein.

We point out that some result can be found also for different boundary conditions and different differential operators. We list here some references, but we emphasize that the list does not pretend at all to be exhaustive.

About other boundary conditions we mention [Sak90] for Neumann and Robin nonlinear problems and still in the Robin case the works [ACH20, CF21] for the first eigenfunction and the torsion problem. Some result for non homogeneus Dirichlet boundary conditions can be found in [Ale87, DLT18].

An important work in the case of the p-Laplacian is [Sak87] by Sakaguchi. Convexity of viscosity solutions for some fully nonlinear elliptic equations has been established by Alvarez, Lasry and Lions in [ALL97].

1.2 Some counterexamples

In this section we collect some known result where uniqueness of the critical point does not hold or in general where the geometry of the solution does not inherit the one of the domain.

Let us start by recalling that in non convex domains we can not expect uniqueness of the critical point for solutions of problem (1.1) as illustrated by the well known case of a dumbbell domain. Indeed if we consider Ω to be a domain given by two disjoint circular disks of radius one linked by a thin tube of width ε, then if ε is small enough, we have that the solution of the torsion problem has at least two maxima. A proof of this fact can be found in [Spe75], for instance.

A more interesting and delicate situation concerns the case of domains which are close to be convex: it can be shown that the number of critical points can be large as we want. We refer to the following theorem proved by Gladiali and Grossi in [GG22] for the torsion problem and to Remark 1.2.2. Let us recall that if Ω is convex then the solution of the torsion problem is quasiconcave and in particular it has a unique critical point.

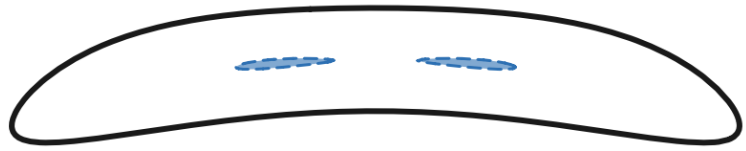

Figure 1.1: The domain Ω_ε and in blue the superlevel set $\{u_\varepsilon > c\}$ in Theorem 1.2.1 for $k = 2$.

Theorem 1.2.1 ([GG22]). *For any $k \in \mathbb{N}$, there exists a family of bounded domains Ω_ε in $\mathbb{R}^2$ and a solution u_ε to*

$$\begin{cases} -\Delta u_\varepsilon = 1 & \text{in } \Omega_\varepsilon \\ u_\varepsilon > 0 & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \end{cases}$$

such that for ε small enough

(i) Ω_ε is star-shaped with respect to an interior point;

(ii) Ω_ε locally converges to the strip $\mathcal{S} := \{ (x_1, x_2) \in \mathbb{R}^2 \mid -1 < x_2 < 1 \}$ for $\varepsilon \to 0$, i.e. for all compact set $K \subseteq \mathbb{R}^2$ it holds $|K \cap (\mathcal{S} \triangle \Omega_\varepsilon)| \to 0$ as $\varepsilon \to 0$;

(iii) The curvature of $\partial\Omega_\varepsilon$ change sign once and

$$\min_{\partial\Omega_\varepsilon} \mathfrak{K} \to 0 \quad \text{as} \quad \varepsilon \to 0;$$

(iv) for suitable $c > 0$, the superlevel set $\{u_\varepsilon > c\}$ is composed by at least k different connected components, in particular u_ε has at least k maximum points, see Figure 1.1.

Remark 1.2.2. 1) As a consequence of this result, we will get that the condition $\mathfrak{K} \geq 0$ is sharp to get uniqueness of the critical point for convex $\Omega \subseteq \mathbb{R}^2$, see Remark 2.1.2 for more details.

Moreover, the preceding theorem shows that as soon as the curvature of the boundary of Ω is negative, then not only we loose the uniqueness of the critical point, but also we can find solutions with an arbitrary large (finite) number of critical points.

2) It is interesting to note that if we consider small perturbations of a given bounded convex set, then the critical point is unique even in domains with negative curvature, see Theorem 4.1.5.

3) A similar result has been proved in [DRG22a] for $N \geq 2$ and for more general nonlinearities, see Chapter 3.

As discussed in the preceding section, one may ask if every positive solutions of (1.1) in a convex domain is quasiconcave: indeed, among the others, this is the case for $f \equiv 1$ or $f(u) = \lambda_1(-\Delta, \Omega)u$. However, the answer is negative and it was proved by Hamel, Nadirshavili and Sire in the following theorem.

Theorem 1.2.3 ([HNS16]). *In dimension $N = 2$, there are some smooth bounded convex domains Ω and some $f \in \mathcal{C}^\infty([0, +\infty))$ such that $f \geq 1$ and such that the problem*

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits both a quasiconcave solution v and a solution u which is not quasiconcave.

Remark 1.2.4. 1) The domains of the preceding theorem satisfy the symmetry assumptions of the Gidas, Ni, Nirenberg's Theorem 1.1.6 and then both the solutions admits exactly one critical point.

2) It is an open question if the semi-stability of the solution in a convex domain is a sufficient condition to have quasiconcavity. Indeed it is not know if the solution u of the preceding theorem is stable or not.

3) It is then also natural to investigate what happens if the domain is star-shaped. Are all the superlevel sets of the solutions star-shaped? Also in this case the answer is negative in general: indeed it is not true for the torsion problem, as can be easily deduced by Theorem 1.2.1 by Gladiali and Grossi.

We conclude this part by mentioning that in [Kaw94] Kawhol asks if given a domain Ω which is convex in the x_1 direction, then also the superlevel sets of the first eigenfunctions are convex in the x_1 direction. The answer is negative, and the counterexample has been illustrated by Weth in [Wet11].

Theorem 1.2.5 ([Wet11]). *There exist bounded domains $\Omega \subseteq \mathbb{R}^2$ such that they are convex in the x_1 direction, but the first Dirichlet eigenfunction has superlevel sets which are not.*

Remark 1.2.6. The same conclusion holds true also for the torsion problem, see [Wet11].

Chapter 2

Convex domains with vanishing curvature

The chapter is devoted to the extension of Cabré-Chanillo's Theorem 1.1.9 from the case of domains whose boundaries have strictly positive curvature to the one where the curvature of $\partial\Omega$ is allowed to vanish somewhere.

The results can be found in [DRGM21].

2.1 Main result

In this chapter $N = 2$. For convenience we write coordinates as (x, y) instead of (x_1, x_2). We recall that we are considering classical solutions of

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u > 0) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $f : (0, +\infty) \to \mathbb{R}$ is a smooth nonlinearity. The main result of this chapter is the following.

Theorem 2.1.1. *Assume $f(0) \geq 0$ and let $\Omega \subseteq \mathbb{R}^2$ be a smooth bounded convex domain whose boundary has nonnegative curvature and such that the subset of zero-curvature consists of isolated points or segments.*
If u is a semi-stable solution of (2.1) then u has a unique critical point $\boldsymbol{x_0}$. Moreover $\boldsymbol{x_0}$ is a nondegenerate maximum point of u.

We point out that the extension of Theorem 1.1.9 when the curvature of $\partial\Omega$ vanishes somewhere is nontrivial. Indeed the proof in Theorem 1.1.9 does not work in this case because the vector field Z considered at page 7 in [CC98] is not well defined as the curvature of $\partial\Omega$ vanishes.
Our main idea is to consider the vector field $T : \overline{\Omega} \to \mathbb{R}^2$ - already used in [CC98] - given by

$$T(q) = (u_{yy}(q)u_x(q) - u_{xy}(q)u_y(q), u_{xx}(q)u_y(q) - u_{xy}(q)u_x(q)), \tag{2.2}$$

for $q \in \overline{\Omega}$. One of the main tool of our proof is to compute the *topological degree* $\deg(T, \Omega, \mathbf{0})$ of T. In particular if the curvature of $\partial\Omega$ is positive then we have that (see Lemma 2.2.2)

$$\deg(T, \Omega, \mathbf{0}) = 1.$$

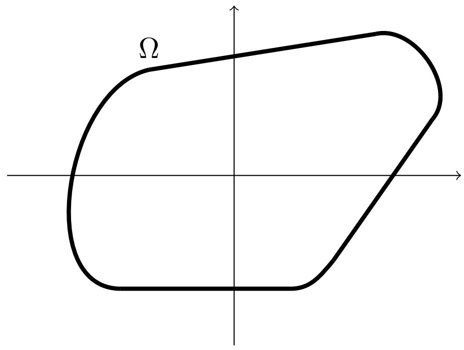

Figure 2.1: Example of domain for which Theorem 2.1.1 holds.

A deeper analysis of the degree of T concerns the *index* of the zeros of T. It is not difficult to see (Lemma 2.2.3) that if $T(q) = 0$ then either q is a critical point of u or q is a critical point of the directional derivative

$$\frac{\partial u}{\partial \sigma} := u_y(q)u_x - u_x(q)u_y.$$

In Lemma 2.2.5 we will compute the index in both cases. This is one of the crucial steps of the proof. We remark that this approach provides a quicker proof of Theorem 1.1.9 because it simplifies the topological approach at pages $6-8$ in [CC98].
Eventually a careful analysis of the critical points of $\frac{\partial u}{\partial \sigma}$ on $\partial\Omega$ and of the nodal lines of $\frac{\partial u}{\partial \sigma}$ in Ω ends the proof,

Remark 2.1.2. We stress that our result is sharp in the sense that, as proved in [GG22], it is possible to construct a bounded domain $\Omega \subseteq \mathbb{R}^2$ and a function u verifying

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

such that u admits an arbitrarily large number of critical points. Here $\partial\Omega$ admits points with negative curvature but close to 0 as we want and Ω locally converges to a convex domain, see Theorem 1.2.1 for the precise statement.

The rest of the chapter is organized as follows: in Section 2.2 we recall some preliminary results, basically proved in [CC98], and prove the main properties of the vector field T. In Section 2.3 we prove Theorem 2.1.1.

2.2 Preliminary results

Let $\Omega \subseteq \mathbb{R}^2$ be a smooth bounded domain and let u be a solution to (2.1) where $f \in \mathcal{C}^{1,\alpha}(\mathbb{R}^+, \mathbb{R}^+)$ with $\alpha \in (0, 1]$. Recall that $u \in \mathcal{C}^3(\overline{\Omega})$ by the standard regularity theory.

As in [CC98], we introduce the following notation: for every $\theta \in [0, 2\pi)$ we write $\boldsymbol{e_\theta} = (\cos\theta, \sin\theta)$ and we set

$$u_\theta := \cos\theta\, u_x + \sin\theta\, u_y = \frac{\partial u}{\partial \boldsymbol{e_\theta}},$$
$$N_\theta := \{\, p \in \overline{\Omega} \mid u_\theta(p) = 0 \,\},$$
$$M_\theta := \{\, p \in N_\theta \mid \nabla u_\theta(p) = \mathbf{0} \,\}.$$

Moreover, if the set $\{\, u > c \,\}$ is smooth then the curvature of tis boundary is given by

$$\mathfrak{K} := -\frac{u_{yy}u_x^2 - 2u_{xy}u_xu_y + u_{xx}u_y^2}{|\nabla u|^3}.$$

The following result tells us that the nodal sets of a semi-stable solution of (2.1) are smooth curves without self intersection and every critical point of u is nondegenerate.

Proposition 2.2.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a smooth bounded convex domain whose boundary has nonnegative curvature and such that the curvature vanishes only at isolated points. Assume that $f(0) \geq 0$. If u is a semi-stable solution of* (2.1) *then for every $\theta \in [0, 2\pi)$, the nodal set N_θ is a smooth curve in $\overline{\Omega}$ without self-intersection which hits $\partial\Omega$ at the two end points of N_θ. Moreover in any critical point of u the Hessian has rank* 2.

Proof. The proof is given in [CC98] in the case of positive curvature. For reader's convenience first we report the key steps of the proof in [CC98] and next we add the case of zero curvature at isolated points of the boundary.

For any $\theta \in [0, 2\pi)$ we have that

i) around any $p \in (N_\theta \cap \Omega) \setminus M_\theta$ the nodal set N_θ is a smooth curve;

ii) if $p \in M_\theta \cap \Omega$, then N_θ consists of at least two smooth curves intersecting transversally at p (here the proof uses a result in [CF85b]);

iii) there is no nonempty domain $H \subseteq \Omega$ such that $\partial H \subseteq N_\theta$ (where the boundary of H is considered as a subset of $\mathbb{R}^2$);

iv) if $p_i \in N_\theta \cap \partial\Omega$ by the implicit function theorem one has that around p_i, N_θ is a smooth curve intersecting $\partial\Omega$ transversally in p_i;

v) if $N_\theta \cap \partial\Omega = \{p_1, p_2\}$ then $M_\theta = \emptyset$ and any critical point of u verifies that $\mathrm{Hess}_u(p)$ has rank 2.

We stress that all the above properties hold for semi-stable solutions u in *any* domain Ω.

Now we consider the case where the curvature of the boundary vanishes at isolated points. By the compactness of $\partial\Omega$ and the smoothness of u we have that the curvature $\mathfrak{K}$ vanishes only at finitely many points of $\partial\Omega$.

Assume $\mathfrak{K}(p_1) = 0$ and $\mathfrak{K}(p_2) > 0$. If there exists $\rho > 0$ such that $N_\theta \cap B_\rho(p_1) \cap \Omega = \emptyset$ then the nodal curve N_θ starting from p_2 has to enclose a nonempty domain $H \subseteq \Omega$ with $\partial H \subseteq N_\theta$, but this yields to a contradiction. Otherwise N_θ consists of at least one curve starting from p_1 and disjoint from $\partial\Omega$ (since $N_\theta \cap \partial\Omega = \{p_1, p_2\}$). If there are more then one curve this implies again that there exists a subdomain H as before and this is a contradiction. Hence as in [CC98], around p_1 we have that N_θ is a smooth curve intersecting $\partial\Omega$ in p_1.

If $\mathfrak{K}(p_1) = \mathfrak{K}(p_2) = 0$ we can argue similarly to get that around each p_i, N_θ is a smooth curve intersecting $\partial\Omega$ in p_i, for $i = 1, 2$. Note that if there exists $\rho > 0$ such that $N_\theta \cap B_\rho(p_1) \cap \Omega = N_\theta \cap B_\rho(p_2) \cap \Omega = \emptyset$ then, since $N_\theta \cap \Omega$ is nonempty by the fact that there exists at least a critical point of u in Ω, the nodal set enclose again a domain as before.

The remaining claims of the proposition follow arguing as in [CC98]. □

For u solution of (2.1), consider the map $T : \overline{\Omega} \to \mathbb{R}^2$ given by

$$T(q) = (u_{yy}(q)u_x(q) - u_{xy}(q)u_y(q), u_{xx}(q)u_y(q) - u_{xy}(q)u_x(q)).$$

Since $u \in \mathcal{C}^3(\overline{\Omega})$, T is of class $\mathcal{C}^1$. In next lemmata we state some important properties of the vector field T.

Lemma 2.2.2. *Let $D \subseteq \Omega$ be a smooth convex domain such that ∂D has positive curvature. Then* $\deg(D, T, \mathbf{0}) = 1$.

Proof. Let $p = (x_p, y_p) \in \Omega$ and consider the homotopy

$$\begin{aligned} H : [0,1] \times \overline{\Omega} &\to \mathbb{R}^2 \\ (t, q) &\mapsto tT(q) + (1-t)(q-p). \end{aligned}$$

H is an admissible homotopy, i.e. $H\big(t, (x, y)\big) \neq \mathbf{0}$ for any $t \in [0, 1]$ and $(x, y) \in \partial\Omega$. Otherwise, there would exist $\tau \in [0, 1]$ and $\overline{q} = (\overline{x}, \overline{y}) \in \partial\Omega$ such that $H(\tau, \overline{q}) = 0$, i.e.

$$\begin{cases} \tau(u_{yy}(\overline{q})u_x(\overline{q}) - u_{xy}(\overline{q})u_y(\overline{q})) = (\tau - 1)(\overline{x} - x_p) \\ \tau(u_{xx}(\overline{q})u_y(\overline{q}) - u_{xy}(\overline{q})u_x(\overline{q})) = (\tau - 1)(\overline{y} - y_p). \end{cases} \tag{2.3}$$

Then, multiplying the first equation by $u_x(\overline{q})$, the second by $u_y(\overline{q})$ and summing we get

$$\begin{aligned} \tau(u_{yy}(\overline{q})u_x^2(\overline{q}) - 2u_{xy}(\overline{q})u_x(\overline{q})u_y(\overline{q}) + u_{xx}(\overline{q})u_y^2(\overline{q})) \\ = (\tau - 1)[(\overline{x} - x_p)u_x(\overline{q}) + (\overline{y} - y_p)u_y(\overline{q})], \end{aligned}$$

and writing $\nu = (\nu_x, \nu_y)$ for the unit normal exterior vector at $\overline{q}$, it follows

$$-\tau\mathfrak{K}(\overline{q})|\nabla u(\overline{q})|^3 = (\tau - 1)u_\nu(\overline{q})[(\overline{x} - x_p)\nu_x + (\overline{y} - y_p)\nu_y]. \tag{2.4}$$

Since Ω is strictly star-shaped with respect to the point p we have $(\overline{x} - x_p)\nu_x + (\overline{y} - y_p)\nu_y > 0$. Since $\mathfrak{K}(\overline{q}) > 0$ and $u_\nu(\overline{q}) < 0$ by (2.4) we get $\tau = 0$ and thanks to (2.3) this yields to $\overline{q} = p$ which is clearly a contradiction.

Then H is an admissible homotopy and so we conclude

$$\deg(\Omega, T, \mathbf{0}) = \deg(\Omega, Id - p, \mathbf{0}) = 1.$$ □

Lemma 2.2.3. *If $q \in \Omega$ is such that $T(q) = \mathbf{0}$ then either*

q is a critical point for u,

or

there exists $\theta \in [0, 2\pi)$ such that $q \in M_\theta$.

Proof. Of course if q is a critical point for u then $T(q) = \mathbf{0}$. So suppose that q is not a critical point for u and consider $\theta \in [0, 2\pi)$ such that $(\cos\theta, \sin\theta) = \left(\frac{u_y(q)}{\sqrt{u_x^2(q)+u_y^2(q)}}, -\frac{u_x(q)}{\sqrt{u_x^2(q)+u_y^2(q)}}\right)$. Then it is straightforward to verify that

$$u_\theta = \cos\theta u_x + \sin\theta u_y$$

satisfies $u_\theta(q) = 0$ and $\nabla u_\theta(q) = \mathbf{0}$. Hence $q \in M_\theta$. □

Remark 2.2.4. We point out that if $q \in M_\theta$ then up to a rotation we can assume that

$$u_x(q) = u_{xx}(q) = u_{xy}(q) = 0. \tag{2.5}$$

From now if q is an isolated zero of the vector field T, let us recall that $\mathrm{ind}(T, q) = \deg\left(T, B_r(q), \mathbf{0}\right)$, for some $r > 0$ small enough.

Lemma 2.2.5. *Let $q \in \Omega$ be such that $T(q) = \mathbf{0}$. Then we have that*

(i) if q is a nondegenerate critical point for u, then $\mathrm{ind}(T, q) = 1$*;*

(ii) if $q \in M_\theta$ for some $\theta \in [0, 2\pi)$ and it is a nondegenerate critical point for u_θ then $\mathrm{ind}(T, q) = -1$.

Proof. One has

$$\mathrm{Jac}_T = \begin{pmatrix} u_{xx}u_{yy} - u_{xy}^2 + u_x u_{xyy} - u_y u_{xxy} & u_x u_{yyy} - u_y u_{xyy} \\ u_y u_{xxx} - u_x u_{xxy} & u_{xx}u_{yy} - u_{xy}^2 + u_y u_{xxy} - u_x u_{xyy} \end{pmatrix}.$$

If q is a critical point for u we have

$$\det \mathrm{Jac}_T(q) = (\det \mathrm{Hess}_u(q))^2,$$

and since it is nondegenerate we get that $\mathrm{ind}(T, q) = 1$.
On the other hand if $q \in M_\theta$ by Remark 2.2.4 we have that (2.5) holds and then

$$\begin{aligned} \det \mathrm{Jac}_T(q) &= -u_y^2(q)(u_{xxy}^2(q) - u_{xxx}(q)u_{xyy}(q)) \\ &= -u_y^2(q)(u_{xxy}^2(q) + u_{xxx}^2(q)), \end{aligned}$$

where the last equality follows differentiating (2.1). Finally the nondegeneracy of q for ∇u_θ gives that $u_{xxy}^2(q) + u_{xxx}^2(q) \neq 0$ and the claim follows. □

As remarked in the introduction, the previous lemma gives a simplified proof of Cabré-Chanillo's result.

Proof of Theorem 1.1.9. By Proposition 2.2.1 we have that $M_\theta = \emptyset$ for any $\theta \in [0, 2\pi)$. Hence if $T(q) = 0$ then q is a critical point of u. Moreover it is nondegenerate, otherwise $u \in M_\theta$ for some $\theta \in [0, 2\pi)$.

Finally by Lemma 2.2.2 and Lemma 2.2.5 we have

$$1 = \deg(T, \Omega, \mathbf{0}) = \sum_{q \text{ such that } \nabla u = 0} \mathrm{ind}(T, q) = \sharp\{\text{critical points of } u\},$$

which gives the claim. □

Corollary 2.2.6. *Let $D \subseteq \overline{\Omega}$ be such that $M_\theta \cap D = \emptyset$ for all $\theta \in [0, 2\pi)$ and $\mathbf{0} \notin T(\partial D)$. If $\deg(D, T, \mathbf{0}) = 1$, then u has exactly one critical point in D which is a maximum with negative definite Hessian.*

Proof. Considering that $M_\theta \cap D = \emptyset$ for all $\theta \in [0, 2\pi)$ implies that the Hessian of u has maximal rank in every critical point in D, the proof easily follows from Remark 2.2.4 and Lemma 2.2.5. ☐

We end this section by proving a technical result which will be useful later. The proof is essentially the one in [Maj].

Lemma 2.2.7. *Let $\Omega \subseteq \{ (x, y) \in \mathbb{R}^2 \mid y < 0 \}$ be a bounded smooth domain such that $\partial\Omega$ is tangent to the x-axis at $\mathbf{0}$. Let $F \in \mathcal{C}^2(\overline{\Omega}, \mathbb{R})$ with*

$$F(\mathbf{0}) = F_x(\mathbf{0}) = F_y(\mathbf{0}) = F_{xy}(\mathbf{0}) = 0, \quad F_{xx}(\mathbf{0}) < 0, \quad \textit{and} \quad F_{yy}(\mathbf{0}) > 0.$$

Then there exist $\delta, \eta > 0$ and two functions $Y_1 \in \mathcal{C}^1([-\delta, 0], [-\eta, 0])$ and $Y_2 \in \mathcal{C}^1([0, \delta], [-\eta, 0])$ such that

(i) $Y_1(0) = Y_2(0) = 0$,

(ii) $Y_1'(0) = \sqrt{-\frac{F_{xx}(\mathbf{0})}{F_{yy}(\mathbf{0})}}$, $Y_2'(0) = -\sqrt{-\frac{F_{xx}(\mathbf{0})}{F_{yy}(\mathbf{0})}}$,

(iii) $F(x, y) = 0$ *if and only if* $y = Y_1(x)$ *for* $x \in [-\delta, 0]$ *and* $y = Y_2(x)$ *for* $x \in [0, \delta]$.

Proof. Since $F_{xx}(\mathbf{0}) < 0$ and $F_{yy}(\mathbf{0}) > 0$, there exists $\eta > 0$ such that

$$F_{xx}(x, y) < 0 \quad \text{and} \quad F_{yy}(x, y) > 0, \quad \text{for } (x, y) \in \overline{\Omega} \cap [-\eta, \eta] \times [-\eta, 0].$$

From now on we work with $(x, y) \in \overline{\Omega} \cap [-\eta, \eta] \times [-\eta, 0]$. Since $F_x(\mathbf{0}) = 0$ and $F_{xx}(\mathbf{0}) < 0$ it follows that

$$F(x, y) < 0, \quad \text{for } (x, y) \in \partial\Omega \text{ with } 0 < |x| \leq \eta',$$

where $0 < \eta' \leq \eta$. Moreover, by the strictly convexity of the function

$$y \mapsto F(x, y), \quad \text{for } x \text{ fixed},$$

and the fact that $F_y(\mathbf{0}) = 0$, one has

$$F(0, y) > 0, \quad \text{for } y \neq 0.$$

In particular $F(0, -\eta') > 0$ and by continuity $F(x, -\eta') > 0$ for $-\delta < x < \delta$, where $0 < \delta \leq \eta'$. Now, for $-\delta < x \leq 0$ the function

$$y \mapsto F(x, y),$$

is such that

$$F(x, \overline{y}) < 0, \text{ for } (x, \overline{y}) \in \partial\Omega \quad F(x, -\eta') > 0, \quad F_{yy}(x, \cdot) > 0,$$

and since $\overline{y} > -\eta'$ then there exists a unique $y_x \in (\overline{y}, -\eta')$ such that $F(x, y_x) = 0$; furthermore, since $F_{yy}(x, y) > 0$, one has $F_y(x, y_x) < 0$. Then the zero set of F is given by a continuous function $Y_1 : [-\delta, 0] \to [-\eta, 0]$ (where the continuity in 0 holds since we can choose η arbitrarily small). From the implicit function theorem we have $Y_1 \in \mathcal{C}^0([-\delta,0]) \cap \mathcal{C}^1([-\delta,0))$ with

$$F_x(x, Y_1(x)) + F_y(x, Y_1(x))Y_1'(x) = 0, \quad \text{for } x \neq 0. \tag{2.6}$$

Moreover, from

$$F(x,y) = \frac{1}{2}(F_{xx}(\mathbf{0}) + o(1))x^2 + \frac{1}{2}(F_{yy}(\mathbf{0}) + o(1))y^2, \tag{2.7}$$

we deduce that $(x, Y_1(x))$ belongs to a cone around the line $y = -\frac{F_{xx}(\mathbf{0})}{F_{yy}(\mathbf{0})}x$ and it is differentiable at 0 with $Y_1'(0) = \sqrt{-\frac{F_{xx}(\mathbf{0})}{F_{yy}(\mathbf{0})}}$. Indeed, for $y = Y_1(x)$ in (2.7) we obtain

$$Y_1(x)^2 = -\frac{F_{xx}(\mathbf{0})}{F_{yy}(\mathbf{0}) + o(1)}x^2 + \frac{x^2}{F_{yy}(\mathbf{0}) + o(1)}o(1),$$

then

$$Y_1(x) = \left(\sqrt{-\frac{F_{xx}(\mathbf{0})}{F_{yy}(\mathbf{0})}} + o(1)\right)x, \tag{2.8}$$

and for $x \to 0$ we get the claim. Moreover, we have

$$\begin{aligned} F_x(x,y) &= F_{xx}(\mathbf{0})x + o(x+y), \\ F_y(x,y) &= F_{yy}(\mathbf{0})y + o(x+y), \end{aligned}$$

and then from (2.6) and (2.8) it follows

$$\begin{aligned} Y_1'(x) &= -\frac{F_x(x, Y_1(x))}{F_y(x, Y_1(x))} \\ &= -\frac{F_{xx}(\mathbf{0})x + o(x) + o(Y_1(x))}{F_{yy}(\mathbf{0})Y_1(x) + o(x) + o(Y_1(x))} \\ &= -\frac{F_{xx}(\mathbf{0}) + o(1)}{F_{yy}(\mathbf{0})\sqrt{-\frac{F_{xx}(\mathbf{0})}{F_{yy}(\mathbf{0})}} + o(1)} = \sqrt{-\frac{F_{xx}(\mathbf{0})}{F_{yy}(\mathbf{0})}} + o(1) \quad \text{for } x \to 0, \end{aligned}$$

that is $Y_1 \in \mathcal{C}^1([-\delta,0])$. For $0 \leq x < \delta$ we can argue analogously. □

Remark 2.2.8. A similar result can be proved for interior points of the domain, see [Maj].

2.3 Proof of Theorem 2.1.1

In this section we prove the main results of the chapter. First of all we fix the assumptions on the domain Ω.

(H_Ω) Ω is a convex domain such that the curvature is zero at a single point of its boundary and positive elsewhere. Up to a rotation and a translation we assume $\Omega \subseteq \{ (x,y) \in \mathbb{R}^2 \mid y < 0 \}$ such that $\partial\Omega$ is tangent to the x-axis in $\mathbf{0}$, which is the only point where the curvature is zero.

Theorem 2.3.1. *Suppose that Ω satisfies (H_Ω). If u is a semi-stable solution of* (2.1) *then u has a unique critical point $\boldsymbol{x_0}$. Moreover $\boldsymbol{x_0}$ is the maximum of u and it has negative definite Hessian.*

To prove the theorem, we need the following auxiliary lemma.

Lemma 2.3.2. *Suppose that Ω satisfies (H_Ω), u is a semi-stable solution of* (2.1) *and $u_{xy}(\mathbf{0}) = 0$. Then $\mathfrak{K}_y(\mathbf{0}) < 0$.*

Proof. From assumption $\mathfrak{K}(\mathbf{0}) = 0$ and from $u_x(\mathbf{0}) = 0$ we easily get that $u_{xx}(\mathbf{0}) = 0$. It follows that
$$\mathfrak{K}_y(\mathbf{0}) = -\frac{u_{xxy}(\mathbf{0})u_y^2(\mathbf{0})}{|\nabla u|^3}.$$
We claim that
$$u_{xxy}(\mathbf{0}) > 0, \tag{2.9}$$
which ends the proof since $u_y(\mathbf{0}) = u_\nu(\mathbf{0}) < 0$ by the Hopf boundary lemma. In order to prove (2.9) we divide the proof in four steps.

Step 1: $u_{xxy}(\mathbf{0}) \neq 0$.
Since $u_y(\mathbf{0}) \neq 0$, by the implicit function theorem we get that near the origin one has $u(x, y) = 0$ if and only if $y = \varphi(x)$, for some function φ. In particular, by the assumptions on the boundary of Ω, we have $\varphi(0) = \varphi'(0) = \varphi''(0) = \varphi'''(0) = 0$ and differentiating $u(x, \varphi(x)) = 0$ we deduce $u_{xxx}(\mathbf{0}) = 0$. Moreover, differentiating (2.1), we get that also $u_{xyy}(\mathbf{0}) = 0$. If $u_{xxy}(\mathbf{0}) = 0$, then the Taylor expansion of u_x in a neighborhood of $\mathbf{0}$ becomes
$$u_x(x, y) = \text{homogeneous harmonic polynomial of order three} + O\big((x^2 + y^2)^2\big),$$
This means that locally $N_0 = \{\, u_x = 0 \,\}$ consists of at least three branches of curves and at least two must be entering in Ω, a contradiction with Proposition 2.2.1.

Step 2: parametrization of N_0 near the origin.
Let $F(x, y) = u_x(x, y)$ with $(x, y) \in \Omega$, then up to a rotation and eventually changing sign one has
$$F(\mathbf{0}) = F_x(\mathbf{0}) = F_y(\mathbf{0}) = F_{xy}(\mathbf{0}) = 0, \quad -F_{xx}(\mathbf{0}) = F_{yy}(\mathbf{0}) > 0.$$
Then we can apply Lemma 2.2.7 and taking into account (2.8) and the fact that $u_x = 0$ consist in exactly one branch entering in Ω from $\mathbf{0}$, the nodal curve N_0 can be parametrized as
$$\mathcal{C} = \begin{cases} x = g(t) \\ y = t \end{cases} \qquad t \in [-\delta, 0],$$
for some $\delta > 0$ and $g(t) = o(t)$.

Step 3: $u_{xx}(g(t), t) \leq 0$ for t close to 0.
Let $(\overline{x}, \overline{y}) \in \partial\Omega$ close to $\mathbf{0}$ with $\overline{x} < 0$ and $(g(\overline{y}), \overline{y}) \in \mathcal{C}$. Since $\overline{x} < 0$, $u_x(x, \overline{y}) > 0$ for $\overline{x} \leq x < g(\overline{y})$ and $u_x(g(\overline{y}), \overline{y}) = 0$ we derive that $u_{xx}(g(\overline{y}), \overline{y}) \leq 0$.

Step 4: end of the proof.
Set $H(t) := u_{xx}(g(t), t)$ for $t \in [-\delta, 0]$. By the previous step we have that $H(t) \leq 0$ and by the assumptions on Ω one has $H(0) = 0$. Hence
$$H'(0) \geq 0.$$

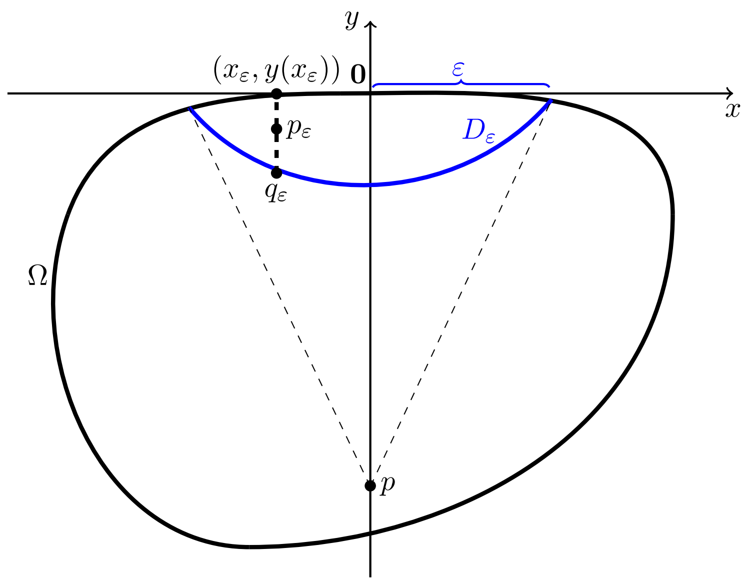

Figure 2.2: The construction of *Case 2* in the proof of Theorem 2.3.1

Finally, $H'(t) = u_{xxx}(g(t),t)g'(t) + u_{xxy}(g(t),t)$ and so $0 \le H'(0) = u_{xxy}(\mathbf{0})$ which gives the claim thanks to Step 1. □

We can now conclude the proof of Theorem 2.3.1.

Proof of Theorem 2.3.1. As remarked we have $u_x(\mathbf{0}) = u_{xx}(\mathbf{0}) = 0$ and $u_y(\mathbf{0}) < 0$. We now distinguish the two cases. according to whether $u_{xy}(\mathbf{0})$ vanishes or not.

Case 1: $u_{xy}(\mathbf{0}) \neq 0$.
Similarly as in the proof of Lemma 2.2.2 we consider the map $T : \Omega \to \mathbb{R}^2$ defined in (2.2) and the homotopy

$$H : [0,1] \times \overline{\Omega} \to \mathbb{R}^2$$
$$(t,q) \mapsto tT(q) + (1-t)(q-p),$$

for some $p = (x_p, y_p) \in \Omega$. Let us show that H is an admissible homotopy. Otherwise there would exist $\tau \in [0,1]$ and $\overline{q} = (\overline{x}, \overline{y}) \in \partial\Omega$ such that (2.3) and (2.4) hold. Then we deduce $\mathfrak{K}(\overline{q}) = 0$ and $\tau = 1$, which thanks to the first equation of (2.3) yields to $-u_y(\mathbf{0})u_{xy}(\mathbf{0}) = 0$: a contradiction. Then H is an admissible homotopy and so

$$\deg(\Omega, T, \mathbf{0}) = \deg(\Omega, Id - p, \mathbf{0}) = 1.$$

The claim follows from Corollary 2.2.6.

Case 2: $u_{xy}(\mathbf{0}) = 0$.
This case is more delicate because $T(\mathbf{0}) = \mathbf{0}$ and since $\mathbf{0} \in \partial\Omega$ the degree of T is not well defined. For this reason we introduce $\Omega_\varepsilon := \Omega \setminus \overline{D}_\varepsilon$ where D_ε is contained in a ball of radius $\varepsilon > 0$ centered in the origin and it is chosen in such a way that

$$|\nabla u| \neq 0, \quad \text{in } \overline{D}_\varepsilon \cap \overline{\Omega}, \tag{2.10}$$

and

$$\Omega_\varepsilon \text{ is star-shaped with respect to some } p = (x_p, y_p), \tag{2.11}$$

(such an ε exists by the Hopf boundary lemma). Now, consider again the map $T \in \mathcal{C}^1(\overline{\Omega}_\varepsilon, \mathbb{R}^2)$. In this way the degree of T is well defined and if the homotopy

$$H_\varepsilon : [0,1] \times \overline{\Omega}_\varepsilon \to \mathbb{R}^2 \tag{2.12}$$

$$(t, q) \mapsto tT(q) + (1-t)(q-p), \tag{2.13}$$

is admissible then we deduce

$$\deg(\Omega_\varepsilon, T, \mathbf{0}) = 1.$$

Assume, by contradiction, that the homotopy H_ε is not admissible. Hence, there exist $\tau_\varepsilon \in [0,1]$ and $q_\varepsilon = (x_\varepsilon, y_\varepsilon) \in \partial\Omega_\varepsilon$ such that $H_\varepsilon(\tau_\varepsilon, q_\varepsilon) = 0$, i.e.

$$\begin{cases} \tau_\varepsilon(u_{yy}(q_\varepsilon)u_x(q_\varepsilon) - u_{xy}(q_\varepsilon)u_y(q_\varepsilon)) = (\tau_\varepsilon - 1)(x_\varepsilon - x_p) \\ \tau_\varepsilon(u_{xx}(q_\varepsilon)u_y(q_\varepsilon) - u_{xy}(q_\varepsilon)u_x(q_\varepsilon)) = (\tau_\varepsilon - 1)(y_\varepsilon - y_p). \end{cases} \tag{2.14}$$

Proceeding as in the previous step we get

$$-\tau_\varepsilon \mathfrak{K}(q_\varepsilon)|\nabla u(q_\varepsilon|^3 = (\tau_\varepsilon - 1)[(x_\varepsilon - x_p)u_x(q_\varepsilon) + (y_\varepsilon - y_p)u_y(q_\varepsilon)]. \tag{2.15}$$

Since for all $q = (x, y) \in \partial\Omega$, writing again $\nu = (\nu_x, \nu_y)$ for the unit normal exterior vector in q, we have

$$(x - x_p)u_x(q) + (y - y_p)u_y(q) = u_\nu(q)[(x - x_p)\nu_x + (y - y_p)\nu_y] < 0,$$

by continuity it follows that $(x_\varepsilon - x_p)u_x(q_\varepsilon) + (y_\varepsilon - y_p)u_y(q_\varepsilon) < 0$ for all $q_\varepsilon \in \partial\Omega_\varepsilon$. Since $\mathfrak{K} > 0$ on $\partial\Omega \setminus \{\mathbf{0}\}$, from (2.15) it follows that $q_\varepsilon \in \partial\overline{D}_\varepsilon \cap \Omega$ and $\mathfrak{K}(q_\varepsilon) \le 0$. Then the vertical line $x = x_\varepsilon$ hits $\partial\Omega$ in a unique point $(x_\varepsilon, y(x_\varepsilon))$, with $y(x_\varepsilon) > y_\varepsilon$. Since $\mathfrak{K}(x_\varepsilon, y(x_\varepsilon)) \ge 0$, there exists $p_\varepsilon \in D_\varepsilon \cap \Omega$ such that $\mathfrak{K}_y(p_\varepsilon) \ge 0$ and for $\varepsilon \to 0$ we have $\mathfrak{K}_y(\mathbf{0}) \ge 0$, but this is in contradiction with Lemma 2.3.2.

Since $\deg(\Omega_\varepsilon, T, \mathbf{0}) = 1$ it is possible to apply Corollary 2.2.6 to get that there exists exactly one critical point in Ω_ε (a maximum with negative definite Hessian). Moreover by the assumptions on ε that there are no critical points in $\overline{B}_\varepsilon \cap \Omega$ and the claim follows. □

We now treat domains where the curvature vanishes at a segment of its boundary.

Theorem 2.3.3. *Let $\Omega \subseteq \{ (x, y) \in \mathbb{R}^2 \mid y < 0 \}$ be a smooth open bounded, convex domain, whose boundary has nonnegative curvature and such that the zero curvature set is an interval Γ on the x-axis. If u is a semi-stable solution of* (2.1) *then u has a unique critical point $\boldsymbol{x_0}$. Moreover $\boldsymbol{x_0}$ is the maximum of u and it has negative definite Hessian.*

Proof. We want to argue as in Proposition 2.2.1 to show that for every $\theta \in (0, \pi) \cap (\pi, 2\pi)$, the nodal set N_θ is a smooth curve in $\overline{\Omega}$ homeomorphic to the closed interval $[0,1]$ (without self-intersection) which intersects $\partial\Omega$ at the two end points of N_θ and $M_\theta = \emptyset$.
First we recall that there exists a unique point $p \in \partial\Omega \setminus \Gamma$ such that $u_x(p) = 0$ and in a neighborhood of p, N_0 is a smooth curve that intesects $\partial\Omega$ transversally at p.

Next we claim that there exists at least a point $\xi \in \Gamma$ such that around that point N_0 consists exactly of 2 branches of curves: the first is Γ while the second intersects $\partial\Omega$ at ξ. Indeed if there exists $\varepsilon > 0$ such that $\Omega \cap N_0 \cap (\mathbb{R} \times (-\varepsilon, 0]) = \emptyset$ then the nodal curve N_0 starting from p has to enclose a nonempty domain $H \subseteq \Omega$ with $\partial H \subseteq N_0$, but this yields a contradiction by Proposition 2.2.1. Analogously we get that we cannot have more than one branch of N_0 exiting at ξ, otherwise we create again such a domain $H \subseteq \Omega$.

Finally we have the uniqueness of $\xi \in \Gamma$ such that N_0 consists of one curve starting from ξ and disjoint from $\partial\Omega$. Indeed if there exists another point $\eta \in \Gamma$ with the same property we can argue as before to get the existence of $H \subseteq \Omega$ which yields a contradiction.

So we have proved that N_0 is the union of two smooth curves in $\overline{\Omega}$: one is the subset of the boundary Γ and the other is homeomorphic to the closed interval $[0,1]$ (without self-intersection) and intersects $\partial\Omega$ at the two end points. They intersect each other only in the point ξ. Moreover we have $M_\theta = \emptyset$ and in any critical point $q \in \Omega$ of u the Hessian has rank 2 (same argument of Proposition 2.2.1).

Up to a translation we can assume $\xi = \mathbf{0}$. We point out that for all $q \in \Gamma$, by the Hopf boundary lemma and $\mathfrak{K} = 0$, there holds

$$u_x = 0, \quad u_{xx} = 0, \quad u_y < 0.$$

Moreover, if $q \neq \mathbf{0}$ one has $u_{xy} \neq 0$ otherwise, locally, u_x is an harmonic polynomial of degree at least 2 and this implies that there exists a branch of N_0 entering in Ω, a contradiction with the uniqueness of the point $\xi = \mathbf{0}$ with this property.

As in the proof of Case 2 of Theorem 2.3.1, let $\varepsilon > 0$ such that (2.10) and (2.11) are verified. Furthermore, if the homotopy defined in (2.12) - (2.13) is not admissible, then (2.14) and (2.15) still hold for some $\tau_\varepsilon \in [0,1]$ and $q_\varepsilon = (x_\varepsilon, y_\varepsilon) \in \partial\Omega_\varepsilon$.

Let us prove that $u_{xxy}(\mathbf{0}) > 0$ (this implies that $\mathfrak{K}_y(\mathbf{0}) = -\frac{u_y^2(\mathbf{0})u_{xxy}(\mathbf{0})}{|\nabla u|^3} < 0$). Indeed, since $u_{xy} \neq 0$ on $\Gamma \setminus \{\mathbf{0}\}$ we have $u_{xy} > 0$ on $\{(x,y) \in \Gamma \mid x < 0\}$ and $u_{xy} < 0$ on $\{(x,y) \in \Gamma \mid x > 0\}$. It follows $u_{xxy}(\mathbf{0}) \geq 0$, but if equality holds we can argue as in the proof of Lemma 2.3.2 to get a contradiction.

Next we can repeat the same argument as in Case 2 of the proof of Theorem 2.3.1 getting that the homotopy is admissible and $\deg(\Omega_\varepsilon, T, \mathbf{0}) = 1$. Finally we apply Corollary 2.2.6 to conclude as in the proof of Theorem 2.3.1. □

Remark 2.3.4. We observe that if $\partial\Omega$ contains more then one component homeomorphic to an interval, then they are in a finite number: $I_1, \ldots, I_m$. Moreover, since the domain is convex, they are parallel at most at pairs. Then also in this case it is possible to prove that for every $\theta \in [0, 2\pi)$, the nodal set N_θ is a smooth curve in $\overline{\Omega}$ homeomorphic to the closed interval $[0,1]$ (without self-intersection) which intersects $\partial\Omega$ at the two end points of N_θ and in any critical point $q \in \Omega$ of u the Hessian has rank 2.

Finally the proof of Theorem 2.1.1 easy follows.

Proof of Theorem 2.1.1. Let $\mathcal{K} := \{ p \in \partial\Omega \mid \mathfrak{K}(p) = 0,\ u_{\nu t}(p) = 0 \}$ where ν is the normal exterior unit vector and t the tangent one. Then define

$$\Omega_\varepsilon := \Omega \setminus \bigcup_{q \in \mathcal{K}} \overline{D_\varepsilon(q)},$$

where $D_\varepsilon(q)$ is contained in a ball of radius $\varepsilon > 0$ centered at q and they are chosen in such a way that

$$\begin{gathered} D_\varepsilon(q_1) \cap D_\varepsilon(q_2) = \emptyset, \quad \text{for all } q_1, q_2 \in \mathcal{K}, \\ \Omega_\varepsilon \text{ is star-shaped with respect to some } p, \\ |\nabla u| \neq 0, \quad \text{in } \bigcup_{q \in \mathcal{K}} \overline{D_\varepsilon(q)} \cap \overline{\Omega}. \end{gathered}$$

The proof follows arguing as in the preceding theorems. $\square$

Chapter 3

Stable solutions in bounded strip-like domains

The aim of this chapter is twofold: first we want to extend Gladiali and Grossi's Theorem 1.2.1 to more general nonlinearities and to any dimension. On the other hand we want to investigate the role of the curvature of $\partial\Omega$ in higher dimensions. In particular we prove that there exists a family of domains close to be convex and whose boundary has positive mean curvature, such that the solution of the torsion problem admits at least k critical points, with $k \in \mathbb{N}$ arbitrarily large.

The results of this chapter can be found in [DRG22a].

3.1 Main results

We consider the following problem

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$ and g is a smooth nonlinearity.

Concerning the generalization of Gladiali and Grossi's Theorem 1.2.1 to more general nonlinearities, let us assume that the nonlinearity has the form $g := \lambda f$ where $\lambda > 0$, f is smooth and satisfies

$$f : \mathbb{R} \to \mathbb{R} \ \text{ is increasing and convex,} \tag{3.2}$$

$$f(0) > 0. \tag{3.3}$$

In this setting it is well known that there exists $\lambda^*(\Omega) > 0$ such that for all $\lambda \in (0, \lambda^*(\Omega))$ the problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.4}$$

admits a positive stable solution. We refer to Appendix A for a very short overview on stability of solutions.

Finally, let us denote by $\mathcal{S}$ the strip $\mathcal{S} := \{ (x, y) \in \mathbb{R}^N \times \mathbb{R} \mid -1 < y < 1 \}$, for $N \geq 1$. Our first result claims that, if f satisfies (3.2) and (3.3), then there exists a family of bounded smooth domains Ω_ε "close" to the strip $\mathcal{S}$ and a solution u_ε to problem (3.4) with k maximum points, $k \geq 2$. The precise statement follows.

Theorem 3.1.1. *Assume that f satisfies* (3.2) *and* (3.3). *Then for any $\lambda \in (0, \lambda^*(-1, 1))$ and for all $k \in \mathbb{N}$ there exists a family of smooth and bounded domains $\Omega_\varepsilon \subseteq \mathbb{R}^{N+1}$ such that*

(i) *Ω_ε is star-shaped with respect to the origin and symmetric with respect to the hyperplanes $x_j = 0$ for $j = 1, \dots, N$ and $y = 0$;*

(ii) *Ω_ε locally converges to the strip $\mathcal{S}$ for $\varepsilon \to 0$, i.e. for all compact sets $K \subseteq \mathbb{R}^{N+1}$ it holds $|K \cap (\mathcal{S} \triangle \Omega_\varepsilon)| \to 0$ as $\varepsilon \to 0$;*

(iii) *$\lambda^*(\Omega_\varepsilon) \geq \lambda^*(-1, 1)$ for ε small enough;*

(iv) *if u_ε is the stable solution of problem* (3.4) *in Ω_ε for some $0 < \lambda < \lambda^*(\Omega_\varepsilon)$, then u_ε has at least k maximum points.*

Let us give an idea of Theorem 3.1.1. The assumptions on f imply that there exists a *stable* solution u_0 of the following ODE

$$\begin{cases} -u'' = \lambda f(u) & \text{in } (-1, 1) \\ u > 0 & \text{in } (-1, 1) \\ u(\pm 1) = 0. \end{cases}$$

Next, for a small $\sigma > 0$ let u_σ be an appropriate extension of u_0 to a slightly larger interval $(-1 - \sigma, 1 + \sigma)$ and denote by $\varphi : \mathbb{R}^{N+1} \to \mathbb{R}$ a suitable solution of the following PDE

$$-\Delta v = \lambda f'(u_\sigma(y)) v, \quad \text{in } \mathbb{R}^N \times (-1 - \sigma, 1 + \sigma). \tag{3.5}$$

Of course (3.5) can be solved using the classical separation of variables method.

Our domain Ω_ε will be the connected component of $\{ u_\sigma + \varepsilon\varphi > 0 \}$ containing the origin and the solution u_ε the *stable* solution to (3.4) with $\Omega = \Omega_\varepsilon$. Finally we show that u_ε is close to $u_0 + \varepsilon\varphi$ on the compact sets of Ω_ε and, since it will be proved that this last function admits k nondegenerate critical points then (*iv*) follows.

We point out that it is possible to prove a slight more general result for problem (3.1) without assuming (3.3), see Remark 3.2.10.

It is important to remark that our construction only works for *stable* solutions to (3.1). Indeed, even for the case of the first eigenfunction of the Laplacian (where the first eigenvalue of the linearized problem is zero), we are not able to construct a domain Ω_ε as in Theorem 3.1.1. This will be discussed in Remark 3.2.11. We do not know if in this case there exists a pair $(\Omega_\varepsilon, u_\varepsilon)$ as in Theorem 3.1.1.

Next let us discuss the role of the curvature of $\partial\Omega$ for solutions to (3.1) in higher dimensions. We will focus on the particular case of the torsion problem, i.e. $g \equiv 1$ in (3.1). By Makar-Limanov's Theorem 1.1.4, if $N = 2$ and the curvature of $\partial\Omega$ is positive then the solution u admits exactly one critical point (see Chapter 2 if the curvature vanishes somewhere). So a couple of questions naturally arise:

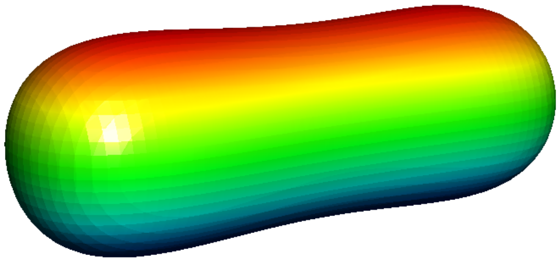

Figure 3.1: The domain Ω_ε in Theorem 3.1.2 for $N = 2$ and $k = 2$.

if $N \geq 3$ what is the positive curvature of the boundary a sufficient condition on $\partial\Omega$ to ensures the uniqueness of the critical point for the torsion problem? What about (semi-)stable solutions for more general nonlinearities?

In the second part of this chapter we give a contribution to these questions showing that the answer to the first one is negative and that the *positive mean curvature* of $\partial\Omega$ is not the correct extension to higher dimensions.

Indeed, for any $k \geq 2$, we will construct a domain $\Omega_\varepsilon \subseteq \mathbb{R}^N$ with $N \geq 3$ and positive mean curvature of the boundary, close to a convex one and a solution u_ε of the torsion problem in Ω_ε such that u_ε has at least k critical points. Actually we suspect that the correct condition which implies the uniqueness of the critical point for (semi-)stable solutions is that all *principal curvatures* have to be positive. However we have no result to support this idea.

The construction of the pair $(\Omega_\varepsilon, u_\varepsilon)$ is similar to the one in Theorem 3.1.1, but Ω_ε turns out to be a suitable perturbation of the infinite cylinder $\mathcal{C} := \{\, (x, y) \in \mathbb{R} \times \mathbb{R}^N \mid |y|^2 < 1 \,\}$, for $N \geq 2$. The result is the following.

Theorem 3.1.2. *Let $N \geq 2$. For any $k \in \mathbb{N}$ there exists a family of smooth and bounded domains $\Omega_\varepsilon \subseteq \mathbb{R}^{N+1}$ and smooth positive functions $u_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ such that*

(i) *Ω_ε is star-shaped with respect to an interior point;*

(ii) *Ω_ε locally converges to the cylinder $\mathcal{C}$ for $\varepsilon \to 0$, i.e. for all compact sets $K \subseteq \mathbb{R}^{N+1}$ it holds $|K \cap (\mathcal{C} \triangle \Omega_\varepsilon)| \to 0$ as $\varepsilon \to 0$;*

(iii) *the mean curvature of $\partial\Omega_\varepsilon$ is positive;*

(iv) *u_ε solves the torsion problem*

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega_\varepsilon \\ u = 0 & \text{on } \partial\Omega_\varepsilon; \end{cases}$$

(v) *u_ε has at least k nondegenerate maximum points.*

As in Theorem 3.1.1 we have that $u_\varepsilon = u_0 + \varepsilon\varphi$ where $u_0 = \frac{1}{2N}\left(1 - |y|^2\right)$ is a solution of the torsion problem in the unit ball in $\mathbb{R}^N$ and φ turns to be a harmonic function in the whole $\mathbb{R}^{N+1}$.

Then we take Ω_ε as in Theorem 3.1.1, while our solution will directly be $u_\varepsilon = u_0 + \varepsilon\varphi$. Since the set Ω_ε turns out to be a small perturbation of the cylinder $\mathcal{C}$, which boundary has positive mean curvature, then (*iii*) of Theorem 3.1.2 follows. Note that, unlike as in Theorem 3.1.1, here the pair $(\Omega_\varepsilon, u_\varepsilon)$ is explicitly computed.

The chapter is organized as follows: the next section is devoted to the proof of Theorem 3.1.1, while Theorem 3.1.2 will be proved in Section 3.3. The detailed proof of some claims in Section 3.2 and Section 3.3 can be found in the Appendix B.

3.2 Proof of Theorem 3.1.1

In this section we take $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $y \in \mathbb{R}$ and we assume the hypothesis of Theorem 3.1.1. The proof works as follows: we first construct a suitable domain Ω_ε which verifies the claim of Theorem 3.1.1 and next we prove that the stable solution of problem (3.4) satisfies the claim (*iv*) in Theorem 3.1.1.

The first step in the construction of the domain Ω_ε is to introduce a solution u_0 of the 1-dimensional problem

$$\begin{cases} -u'' = \lambda f(u) & \text{in } (-1,1) \\ u > 0 & \text{in } (-1,1) \\ u(\pm 1) = 0. \end{cases} \tag{3.6}$$

By the assumption on f such a solution exists and by elementary argument it can be extended to verify

$$\begin{cases} -u'' = \lambda f(u) & \text{in } (-1-\sigma, 1+\sigma) \\ u > 0 & \text{in } (-1,1) \\ u(\pm 1) = 0 & \\ u < 0 & \text{in } [-1-\sigma, -1) \cup (1, 1+\sigma] \end{cases}$$

for $\sigma > 0$ and small. We denote by u_σ this extension, and let us point out that $u_\sigma = u_0$ in $(-1,1)$.

Since u_0 is a stable solution we have that the first eigenvalue of the linearized operator

$$-\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \lambda f'(u_0(y)), \tag{3.7}$$

in $(-1,1)$ with Dirichlet boundary conditions is strictly positive. Then, up to choose a smaller σ, also the first eigenvalue of (3.7) in $(-1-\sigma, 1+\sigma)$ is strictly positive. We denote it by μ_0.

Next ingredient in the construction of Ω_ε involves a solution of a suitable linearized problem in the strip $\mathbb{R}^N \times (-1-\sigma, 1+\sigma)$. To do this we need to study the following ODE.

Lemma 3.2.1. *For $\mu \in (0, \mu_0)$ there exists a solution ω_μ of the ordinary equation*

$$\begin{cases} -\omega'' - \lambda f'(u_\sigma(y))\omega = \mu\omega & \text{in } (-1-\sigma, 1+\sigma) \\ \omega_\mu(0) = 1 \end{cases}$$

such that

(i) $\omega_\mu > 0$ *in* $[-1-\sigma, 1+\sigma]$,

(ii) ω_μ *is symmetric with respect to* 0,

(iii) $y\omega'_\mu(y) < 0$ *for all* $y \neq 0$.

Proof. Fix $\mu \in (0, \mu_\sigma)$ and let ω be the solution of

$$\begin{cases} -\omega'' - \lambda f'(u_\sigma(y))\omega = \mu\omega & \text{in } (-1-\sigma, 1+\sigma) \\ \omega(\pm(1+\sigma)) = 1. \end{cases}$$

Since $\mu < \mu_\sigma$, by the maximum principle we know that $\omega > 0$ in $(-1-\sigma, 1+\sigma)$. Taking into account the symmetry of u_σ and the maximum principle we get that $\omega(y) = \omega(-y)$ and then (ii) follows.

Moreover, from $f' \geq 0$ we deduce $\omega'' < 0$ in $[-1-\sigma, 1+\sigma]$ and then 0 turns out to be a maximum point. The strictly concavity of ω tells also that $\omega'(y) < 0$ for $y > 0$ and, together to the symmetry of the function, this yields (iii). To conclude the proof set $\omega_\mu = \omega/\omega(0)$. □

3.2.1 Construction of the domain Ω_ε

Now, for some $n = n(k) \in \mathbb{N}$, let $\mu_1, \ldots, \mu_n \in \mathbb{R}$ be such that

$$\frac{\mu_0}{4} > \mu_1 > \cdots > \mu_n > 0, \tag{3.8}$$

and for $i = 1, \ldots, n$

$$\omega_i(y) := \omega_{\mu_i}(y), \quad y \in (-1-\sigma, 1+\sigma),$$

the function given by Lemma 3.2.1. From now on, we consider σ fixed.

Given $(t, y) \in \mathbb{R} \times (-1-\sigma, 1+\sigma)$, we define

$$\tilde{\varphi}(t, y) := \sum_{i=1}^{n} \alpha_i \cosh(\sqrt{\mu_i}t)\omega_i(y),$$

for some $\alpha_i \in \mathbb{R}$ which will be fixed later. A straightforward computation shows that $\tilde{\varphi}$ is a solution of the linearized problem

$$-\Delta v = \lambda f'(u_\sigma(y))v, \quad \text{in } \mathbb{R} \times (-1-\sigma, 1+\sigma)\,.$$

We set $\alpha_1 = -1$ while we choose $\alpha_2, \ldots, \alpha_n$ in such a way that the function $\tilde{\varphi}(t, 0) = \sum_{i=1}^n \alpha_i \cosh(\sqrt{\mu_i}t)$ has k nondegenerate maximum points $t_1, \ldots, t_k$. We point out that it is always possible to do this, see Lemma B.1 in the Appendix for the details. Finally, for $(x_1, \ldots, x_N, y) \in \mathbb{R}^N \times (-1-\sigma, 1+\sigma)$ we define

$$\varphi(x_1, \ldots, x_N, y) := \sum_{j=1}^{N} \tilde{\varphi}(x_j, y) = \sum_{j=1}^{N}\sum_{i=1}^{n} \alpha_i \cosh(\sqrt{\mu_i}x_j)\omega_i(y) \tag{3.9}$$

which solves

$$-\Delta v = \lambda f'(u_\sigma(y))v, \quad \text{in } \mathbb{R}^N \times (-1-\sigma, 1+\sigma)\,.$$

We point out that, for ε small enough,

$$u_0(0) + \varepsilon\varphi(0, \ldots, 0, 0) > 0,$$

and we denote by

$$\boxed{\Omega_\varepsilon \text{ the connected component of } \{\, u_\sigma + \varepsilon\varphi > 0 \,\} \text{ containing the origin.}}$$

The following lemma proves some properties of the set Ω_ε. The proof follows [GG22].

Lemma 3.2.2. *The set Ω_ε satisfies the following properties.*

(i) $\Omega_\varepsilon \subseteq R_\varepsilon$ *for ε small enough, with* $R_\varepsilon := [-M_\varepsilon, M_\varepsilon]^N \times [-1-\eta, 1+\eta]$ *where*

$$M_\varepsilon := \frac{1}{\sqrt{\mu_1}} \log\left(\frac{3\,\|u_0\|_{L^\infty(-1-\eta,1+\eta)}}{\varepsilon\omega_1(1+\eta)}\right),$$

and $\eta \in (0,\sigma)$ as small as we want.

(ii) $\Omega_\varepsilon \supseteq [t_1, t_k]^N \times \{0\}$.

(iii) Let $(x^\varepsilon, y^\varepsilon) \in \partial\Omega_\varepsilon$ for ε small enough. Then, if $|x^\varepsilon| \le C$ we have

$$y^\varepsilon = \pm 1 + o(1), \tag{3.10}$$

and if $|x^\varepsilon| \to +\infty$ we have

$$\sum_{j=1}^{N} \cosh(\sqrt{\mu_1}x_j^\varepsilon) = \frac{u_0(y^\varepsilon)}{\varepsilon\omega_1(y^\varepsilon)}(1+o(1)). \tag{3.11}$$

In particular Ω_ε locally converges to the strip $\mathcal{S} = \mathbb{R}^N \times (-1,1)$ for $\varepsilon \to 0$.

(iv) Ω_ε is symmetric with respect to the hyperplanes $x_j = 0$ for $j = 1,\dots,N$ and $y = 0$. Moreover, it is a smooth and star-shaped domain with respect to the origin for ε small enough.

Proof. In order to prove (i) we show that $u_0 + \varepsilon\varphi < 0$ on ∂R_ε. First let us consider the case where $x = (x_1,\dots,x_N) \in [-M_\varepsilon, M_\varepsilon]^N$ is such that $x_j = \pm M_\varepsilon$ for some $j = 1,\dots,N$ and $y \in [-1-\eta, 1+\eta]$. Hence, recalling (3.9), one has

$$\begin{aligned} u_0(y) + \varepsilon\varphi(x,y) &\le \|u_0\|_{L^\infty(-1-\eta,1+\eta)} - \varepsilon\frac{3\,\|u_0\|_{L^\infty(-1-\eta,1+\eta)}}{\varepsilon}(1+o(1)) \\ &\le -\|u_0\|_{L^\infty(-1-\eta,1+\eta)} < 0, \end{aligned}$$

as $\varepsilon \to 0$.
Next let $(x,y) \in \{\,(x,y) \in \mathbb{R}^{N+1} \mid x \in [-M_\varepsilon, M_\varepsilon]^N,\ y = \pm(1+\eta)\,\}$ and observe that since $\omega_i > 0$ for $y \in [-1-\eta, 1+\eta]$ for all $i = 1,\dots,n$ and $\alpha_1 = -1$ we get

$$\sup_{\mathbb{R}^N \times [-1-\eta,1+\eta]} \varphi = C \in \mathbb{R}.$$

Finally, we have

$$u(x,y) \le u_0(\pm(1+\eta)) + C\varepsilon < \frac{u_0(\pm(1+\eta))}{2} < 0,$$

for ε small enough. Then (i) follows.

Concerning (ii), if ε satisfies

$$\varepsilon N \max_{t\in[t_1,t_k]}\left[\sum_{i=1}^{n}\alpha_i\cosh(\sqrt{\mu_i}t)\right]^{-} < \frac{u_0(0)}{2},$$

where $[\,\cdot\,]^{-}$ denotes the negative part, then we get

$$u_0+\varepsilon\varphi \geq u_0-\varepsilon\varphi^{-} > \frac{u_0(0)}{2},$$

and so $[t_1,t_k]^N\times\{0\}\subseteq\Omega_\varepsilon$.

To prove (iii) note that from $u(x^\varepsilon,y^\varepsilon)=0$ on $\partial\Omega_\varepsilon$ we have

$$u_0(y^\varepsilon) = -\varepsilon\varphi(x^\varepsilon,y^\varepsilon).$$

If $|x^\varepsilon|\leq C$ then φ is uniformly bounded with respect to $\varepsilon\to 0$ and then $u_0(y^\varepsilon)\to 0$ which yields to $y^\varepsilon\to\pm 1$.
On the other hand, if $|x^\varepsilon|\to+\infty$ we have (recall that $\alpha_1=-1$)

$$-\,u_0(y^\varepsilon) = -\varepsilon\sum_{j=1}^{N}\cosh(\sqrt{\mu_1}x_j^\varepsilon)\omega_1(y^\varepsilon)(1+o(1)), \tag{3.12}$$

which gives (3.11). Moreover, since the right hand side of equation (3.12) is strictly negative we get $u_0(y^\varepsilon)>0$ that implies $|y^\varepsilon|\leq 1$.

The symmetry properties of the domain immediately follow from the ones of φ and u_σ. To show the star-shapeness with respect to the origin, it is enough to prove that there exists $\alpha>0$ such that

$$y\partial_y u_\sigma(y)+\varepsilon\sum_{j=1}^{N}x_j\partial_{x_j}\varphi(x,y)+\varepsilon y\partial_y\varphi(x,y)\leq -\alpha<0, \quad \text{for all } (x,y)\in\partial\Omega_\varepsilon.$$

Since u_σ solves (3.6) we have that

$$y\partial_y u_\sigma(y)<0, \quad \text{in } R_\varepsilon\setminus\{\,y=0\,\}\,.$$

If $(x^\varepsilon,y^\varepsilon)\in\partial\Omega_\varepsilon$ is such that $|x^\varepsilon|\leq C$ as $\varepsilon\to 0$, then from (3.10) follows

$$y^\varepsilon\partial_y u_\sigma(y^\varepsilon) = \pm\partial_y u_0(\pm 1)(1+o(1)) = \partial_y u_0(1)(1+o(1))<0.$$

In this case, since the derivatives of φ are uniformly bounded with respect to ε, it easily follows

$$\begin{aligned} y\partial_y u_0(y^\varepsilon)+\varepsilon\sum_{j=1}^{N}x_j^\varepsilon\partial_{x_j}\varphi(x^\varepsilon,y^\varepsilon)+\varepsilon y\partial_y\varphi(x^\varepsilon,y^\varepsilon) &= \partial_y u_0(1)(1+o(1))+O(\varepsilon)\\ &\leq \frac{1}{2}\partial_y u_0(1)<0,\end{aligned}$$

for ε small enough.
On the other hand, if $|x^\varepsilon|\to+\infty$, let $\{j_1,\dots,j_m\}\subseteq\{1,\dots,N\}$ be such that $|x_j|\to+\infty$ if and only if $j=j_h$ for some $h=1,\dots,m$. Then one gets

$$\begin{aligned} & y^\varepsilon\partial_y u_0(y^\varepsilon)+\varepsilon\sum_{j=1}^{N}x_j^\varepsilon\partial_{x_j}\varphi(x^\varepsilon,y^\varepsilon)+\varepsilon y^\varepsilon\partial_y\varphi(x^\varepsilon,y^\varepsilon)\\ &= y^\varepsilon\partial_y u_0(y^\varepsilon)+\varepsilon\sum_{j=1}^{N}\sum_{i=1}^{n}\alpha_i\Big(\sqrt{\mu_i}x_j^\varepsilon\sinh(\sqrt{\mu_i}x_j^\varepsilon)\omega_i(y^\varepsilon)+\cosh(\sqrt{\mu_i}x_j^\varepsilon)y^\varepsilon\partial_y\omega_i(y^\varepsilon)\Big)\\ &\leq y^\varepsilon\partial_y u_0(y^\varepsilon)-\frac{\varepsilon}{2}\sum_{h=1}^{m}\sqrt{\mu_1}x_{j_h}^\varepsilon\sinh(\sqrt{\mu_1}x_{j_h}^\varepsilon)\omega_1(y^\varepsilon)\big((1+o(1)\big).\end{aligned} \tag{3.13}$$

For $h = 1, \dots, m$ we have that $-x_{j_h} \sinh(\sqrt{\mu_1} x_{j_h}) \le -\cosh(\sqrt{\mu_1} x_{j_h})$ and then

$$\begin{aligned} -\sum_{h=1}^{m} x^\varepsilon_{j_h} \sinh(\sqrt{\mu_1} x^\varepsilon_{j_h})\big((1+o(1)\big) &\le -\sum_{h=1}^{m} \cosh(\sqrt{\mu_1} x^\varepsilon_{j_h})\big((1+o(1)\big) \\ &= -\sum_{j=1}^{N} \cosh(\sqrt{\mu_1} x^\varepsilon_{j})\big((1+o(1)\big). \end{aligned}$$

So we have that (3.13) becomes

$$\begin{aligned} & y^\varepsilon \partial_y u_0(y^\varepsilon) + \varepsilon \sum_{j=1}^{N} x^\varepsilon_j \partial_{x_j} \varphi(x^\varepsilon, y^\varepsilon) + \varepsilon y^\varepsilon \partial_y \varphi(x^\varepsilon, y^\varepsilon) \\ & \qquad \le \; y^\varepsilon \partial_y u_0(y^\varepsilon) - \frac{\varepsilon}{2} \sqrt{\mu_1} \omega_1(y^\varepsilon) \sum_{j=1}^{N} \cosh(\sqrt{\mu_1} x^\varepsilon_j)(1+o(1)) \\ & \qquad \overset{(3.11)}{\le} \; y^\varepsilon \partial_y u_0(y^\varepsilon) - \frac{\sqrt{\mu_1}}{2} u_0(y^\varepsilon)(1+o(1)) \\ & \qquad \le \; y^\varepsilon \partial_y u_0(y^\varepsilon) - \frac{\sqrt{\mu_1}}{4} u_0(y^\varepsilon), \end{aligned}$$

and if $y^\varepsilon \partial_y u_0(y^\varepsilon) - \frac{\sqrt{\mu_1}}{4} u_0(y^\varepsilon) \to 0$, since both terms are nonpositive, then they both go to 0. This implies $y^\varepsilon \to 0$ in the first term, and $y^\varepsilon \to 1$ in the second one, a contradiction.

Hence $y^\varepsilon \partial_y u_0(y^\varepsilon) - \frac{\sqrt{\mu_1}}{4} u_0(y^\varepsilon) \le -\tilde{\alpha}$. Finally, for

$$\alpha = \min\left\{ -\frac{1}{2} \partial_y u_0(1), \tilde{\alpha} \right\},$$

we have the claim.

Of course $y \partial_y u_0(y) + \varepsilon \sum_{j=1}^{N} x_j \partial_{x_j} \varphi(x,y) + \varepsilon y \partial_y \varphi(x,y) \neq 0$ on $\partial\Omega_\varepsilon$ implies that $\partial\Omega_\varepsilon$ is a smooth set. □

Next lemma tells us that the function $u_0 + \varepsilon\varphi$ has many critical points.

Lemma 3.2.3. *The function $u_0 + \varepsilon\varphi$ has at least k different nondegenerate local maxima in Ω_ε for ε small enough.*

Proof. Set $U = u_0 + \varepsilon\varphi$ and let $t_1 < \cdots < t_k$ be local, nondegenerate maxima for $\tilde{\varphi}(t,0) = \sum_{i=1}^{n} \alpha_i \cosh(\sqrt{\mu_i} t)$. Then a straightforward computation gives

$$\nabla U(t_m, \dots, t_m, 0) = 0.$$

Next, observing that $\partial_{yy} u_0(0) = -\lambda f(u_0(0)) < 0$ we have

$$\begin{aligned} \partial_{yy} U(t_m, \dots, t_m, 0) &= \partial_{yy} u_0(0) + \varepsilon \sum_{j=1}^{N} \sum_{i=1}^{k} \alpha_i \cosh(\sqrt{\mu_i} t_m) \partial_{yy} \omega_i(0) \\ &< -\frac{\lambda}{2} f(u_0(0)) < 0, \end{aligned} \tag{3.14}$$

for ε small enough and for all $m = 1, \dots, k$. Finally in $(t_m, \dots, t_m, 0)$ one has

$$\begin{aligned}
&\partial_{x_j x_j} U = \varepsilon \sum_{i=1}^{k} \alpha_i \mu_i \cosh(\sqrt{\mu_i} t_m) < 0, \\
&\partial_{x_\ell x_j} U = 0, \qquad \text{for all } \ell \neq j, \\
&\partial_{x_j y} U = \varepsilon \sum_{i=1}^{k} \alpha_i \sqrt{\mu_i} \sinh(\sqrt{\mu_i} t_m) \partial_y \omega_i(0) = 0,
\end{aligned}$$

which, together to (3.14) show us that the Hessian matrix of U is negative definite in $(t_m, \dots, t_m, 0)$ for all $m = 1, \dots, k$ and the proof is complete. □

Now we prove that problem (3.4) admits a stable solution in the domain Ω_ε for the values of λ we are considering.

Lemma 3.2.4. *For ε small enough, it holds*

$$\lambda^*(\Omega_\varepsilon) \geq \lambda^*(-1, 1).$$

Proof. Let us write $\lambda^* = \lambda^*(-1, 1)$ for simplicity. For $\eta > 0$ small enough we have

$$\lambda^*_\eta = \lambda^*(-1 - \eta, 1 + \eta) = \frac{\lambda^*}{(1 + \eta)^2} > \lambda,$$

and by u^*_η the solution of

$$\begin{cases} -u'' = \lambda^*_\eta f(u) & \text{in } (-1 - \eta, 1 + \eta) \\ u > 0 & \text{in } (-1 - \eta, 1 + \eta) \\ u(\pm(1 + \eta)) = 0. \end{cases}$$

Now, let ε so small that $\Omega_\varepsilon \subseteq \mathbb{R}^N \times (-1 - \eta, 1 - \eta)$, then u^*_η is a *supersolution* of problem

$$\begin{cases} -u'' = \lambda^*_\eta f(u) & \text{in } \Omega_\varepsilon \\ u > 0 & \text{in } \Omega_\varepsilon \\ u = 0 & \text{on } \partial\Omega_\varepsilon \end{cases}$$

that is $-\Delta u^*_\eta \geq \lambda^*_\eta f(u^*_\eta)$ in Ω_ε and $u^*_\eta \geq 0$ on $\partial\Omega_\varepsilon$ (here we follows the notations in [Ban80]). Then [Ban80, Theorem 4.7] ensures that $\lambda^*(\Omega_\varepsilon) \geq \lambda^*_\eta > \lambda$. □

Finally, for $\varepsilon > 0$, we define

$$\boxed{u_\varepsilon \text{ as a stable solution of problem (3.4) in } \Omega_\varepsilon.} \tag{3.15}$$

3.2.2 Properties of the function u_ε

Before stating the main properties of the solution u_ε we compute the eigenvalues of a related operator. The proof uses the classical separation of variables.

Lemma 3.2.5. *Denote by $\mu_{1,\sigma}(R)$ the first eigenvalue of the operator $-\Delta - \lambda f'(u_\sigma(y))$ in the rectangle*

$$R := \prod_{j}^{N} (a_j, b_j) \times (-1 - \sigma, 1 + \sigma),$$

with $u_{|\partial R} = 0$*, where* $a_j < b_j$ *for all* $j = 1, \dots, N$*. Then*

$$\mu_{1,\sigma}(R) = \mu_\sigma + \sum_{j=1}^{N} \left(\frac{\pi}{b_j - a_j} \right)^2 > \mu_\sigma.$$

Proof. Fix $\mu \in \mathbb{R}$ and let A_j and B be positive solutions of

$$\begin{cases} A_j''(t) = c_j A_j(t) & \text{in } (a_j, b_j) \\ A_j(a_j) = A_j(b_j) = 0 \end{cases} \tag{3.16}$$

and

$$\begin{cases} -B''(y) - (\lambda f'(u_\sigma(y)) + \mu) B(y) = \sum_{j=1}^{N} c_j B(y) & \text{in } (-1-\sigma, 1+\sigma) \\ B(\pm(1+\sigma)) = 0 \end{cases} \tag{3.17}$$

for some $c_j \in \mathbb{R}$. We have that the solution of (3.16) is given by

$$A_j(t) = \alpha \sin\left(\sqrt{-c_j}(t - a_j)\right),$$

with $\alpha \in \mathbb{R}$ and

$$c_j = -\left(\frac{\pi}{b_j - a_j} \right)^2 < 0,$$

and from (3.17) it follows

$$\sum_{j=1}^{N} c_j + \mu = \mu_\sigma.$$

Finally, since

$$v(x, y) = B(y) \prod_{j}^{N} A_j(x_j),$$

solves

$$\begin{cases} -\Delta v - \lambda f'(u_\sigma(y)) v = \mu v & \text{in } R \\ v = 0 & \text{on } \partial R \end{cases}$$

and $v > 0$ we conclude that

$$\mu_{1,\sigma}(R) = \mu = \mu_\sigma - \sum_{j=1}^{N} c_j = \mu_\sigma + \sum_{j=1}^{N} \left(\frac{\pi}{b_j - a_j} \right)^2 > \mu_\sigma. \qquad \square$$

Remark 3.2.6. From (i) of Lemma 3.2.2 and the previous lemma, one has that the first eigenvalue of the operator $-\Delta - \lambda f'(u_\sigma(y))$ with Dirichlet boundary conditions in Ω_ε is strictly positive.

The rest of the section is devoted to show that the solution u_ε defined in (3.15) is close to $u_0 + \varepsilon\varphi$ as $\varepsilon \to 0$. By Lemma 3.2.3 then (iv) of Theorem 3.1.1 follows.

Let us start with the following bound for u_ε.

Lemma 3.2.7. *There exists a function* $h : (0, +\infty) \to (0, +\infty)$ *such that* $h(\varepsilon) \to 0$ *for* $\varepsilon \to 0$ *and* $u_\varepsilon - u_0 \le h(\varepsilon)$ *in* Ω_ε *uniformly with respect to* $(x, y) \in \Omega_\varepsilon$*.*

Proof. For $\eta > 0$, let u_η be the stable solution of

$$\begin{cases} -u'' = \lambda f(u) & \text{in } (-1-\eta, 1+\eta) \\ u > 0 & \text{in } (-1-\eta, 1+\eta) \\ u(\pm(1+\eta)) = 0. \end{cases}$$

For ε small enough such that $\Omega_\varepsilon \subseteq \mathbb{R}^N \times (-1-\eta, 1+\eta)$, from the convexity of f we have

$$\begin{cases} -\Delta(u_\varepsilon - u_\eta) = \lambda\left(f(u_\varepsilon) - f(u_\eta)\right) \le \lambda f'(u_\varepsilon)(u_\varepsilon - u_\eta) & \text{in } \Omega_\varepsilon \\ u_\varepsilon - u_\eta < 0 & \text{on } \partial\Omega_\varepsilon \end{cases}$$

and then from the stability of u_ε we can apply the maximum principle to deduce $u_\varepsilon \le u_\eta$ in Ω_ε. For $(x,y) \in \Omega_\varepsilon$, by the maximum principle applied to $u_\eta - u_0$ we get

$$u_\varepsilon(x,y) - u_0(y) \le u_\eta(y) - u_0(y) \le \max(u_\eta - u_0)_{|y=\pm(1+\eta)} = -u_0(1+\eta).$$

Next let us define the function $h(\varepsilon)$ as follows: for any $\varepsilon > 0$ let $\eta(\varepsilon)$ be the smallest positive number such that $\Omega_\varepsilon \subseteq \mathbb{R}^N \times (-1-\eta(\varepsilon), 1+\eta(\varepsilon))$. By the properties of Ω_ε we have that $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Finally, as $\varepsilon \to 0$

$$h(\varepsilon) = -u_0(1+\eta(\varepsilon)) \to 0,$$

which gives the claim. □

Next lemma gives a first approximation of the closeness of u_ε to $u_0 + \varepsilon\varphi$. It will be improved later.

Lemma 3.2.8. *Given* $\psi_\varepsilon := \frac{u_\varepsilon - u_\sigma - \varepsilon\varphi}{\varepsilon}$ *one has* $0 \le \psi_\varepsilon < \bar\psi$ *in* Ω_ε *for* ε *small enough, where*

$$\bar\psi(x,y) := \sum_{j=1}^{N}\sum_{i=1}^{n} |\alpha_i| \left(\omega_i(y) - C_i\right) \cosh(\sqrt{\mu_i} x_j),$$

with $0 < C_i < \inf\limits_{(-1-\eta,1+\eta)} \omega_i$ *for all* $i = 1, \dots, k$ *and* $0 < \eta < \sigma$ *small, fixed.*

Proof. Using the convexity of f we have

$$-\Delta\psi_\varepsilon - \lambda f'(u_\sigma)\psi_\varepsilon \ge 0.$$

Moreover, $\psi_\varepsilon = 0$ on $\partial\Omega_\varepsilon$ and taking into account Remark 3.2.6 we can apply the maximum principle to get $\psi_\varepsilon > 0$ in Ω_ε.

Again from the convexity of f we have

$$\begin{aligned} -\Delta\psi_\varepsilon - \lambda f'(u_\varepsilon)\psi_\varepsilon &\le \lambda\left(f'(u_\varepsilon) - f'(u_\sigma)\right)\varphi \\ &= \lambda \sum_{j=1}^{N}\sum_{i=1}^{n} \alpha_i \left(f'(u_\varepsilon) - f'(u_\sigma)\right) \cosh(\sqrt{\mu_i} x_j)\omega_i(y). \qquad (3.18) \end{aligned}$$

From the definition of C_i it holds $\bar\psi > 0$ on $\overline{\Omega}_\varepsilon$. Furthermore, in Ω_ε we have that $\bar\psi$ verifies

$$-\Delta\bar\psi = \sum_{j=1}^{N}\sum_{i=1}^{n} |\alpha_i| \left(\lambda f'(u_\sigma)\omega_i(y) + \mu_i C_i\right) \cosh(\sqrt{\mu_i} x_j),$$

and then

$$-\Delta\bar{\psi} - \lambda f'(u_\varepsilon)\bar{\psi}$$
$$= \sum_{j=1}^{N}\sum_{i=1}^{n}|\alpha_i| \left[\lambda\left(f'(u_\sigma) - f'(u_\varepsilon)\right)\omega_i(y) + (\lambda f'(u_\varepsilon) + \mu_i)C_i\right]\cosh(\sqrt{\mu_i}x_j). \tag{3.19}$$

Moreover

$$f'(u_\varepsilon) - f'(u_\sigma) = f''\left(t_\varepsilon u_\varepsilon + (1-t_\varepsilon)u_\sigma\right)(u_\varepsilon - u_\sigma),$$

with $t_\varepsilon = t_\varepsilon(x,y) \in (0,1)$.
From Lemma 3.2.7 we have $u_\varepsilon - u_0 \leq h(\varepsilon)$ with $h > 0$ and $h \to 0$ as $\varepsilon \to 0$. Since f'' is positive and $t_\varepsilon u_\varepsilon + (1-t_\varepsilon)u_\sigma$ is bounded uniformly with respect to ε we get

$$\lambda\left(f'(u_\varepsilon) - f'(u_0)\right) \leq Ch(\varepsilon),$$

for some $C > 0$. Finally from (3.18) and (3.19) we deduce that

$$\begin{aligned}
&-\Delta(\psi_\varepsilon - \bar{\psi}) - \lambda f'(u_\varepsilon)(\psi_\varepsilon - \bar{\psi})\\
&\leq \sum_{j=1}^{N}\sum_{i=1}^{n}\left[(|\alpha_i| + \alpha_i)\lambda(f'(u_\varepsilon) - f'(u_0))\omega_i(y) - |\alpha_i|(\lambda f'(u_\varepsilon) + \mu_i)C_i\right]\cosh(\sqrt{\mu_i}x_j)\\
&\leq \sum_{j=1}^{N}\sum_{i=1}^{n}\Big[(|\alpha_i| + \alpha_i)Ch(\varepsilon) - \underbrace{|\alpha_i|(\lambda f'(u_\varepsilon) + \mu_i)C_i}_{\leq -|\alpha_i|\mu_i C_i}\Big]\cosh(\sqrt{\mu_i}x_j) \leq 0,
\end{aligned}$$

for ε small enough, which gives

$$\begin{cases}-\Delta(\psi_\varepsilon - \bar{\psi}) - \lambda f'(u_\varepsilon)(\psi_\varepsilon - \bar{\psi}) \leq 0 & \text{in } \Omega_\varepsilon\\ \psi_\varepsilon - \bar{\psi} < 0 & \text{on } \partial\Omega_\varepsilon\end{cases}$$

and the maximum principle provides $\psi_\varepsilon - \bar{\psi} < 0$ in Ω_ε. □

Next lemma gives us the final estimate. Here it will be crucial to choose the coefficients μ_i as in (3.8).

Lemma 3.2.9. *Let*

$$\Psi_\varepsilon := \frac{u_\varepsilon - u_0 - \varepsilon\varphi}{\varepsilon^2}.$$

Then in every $K \subset\subset \Omega_\varepsilon$ one has $|\Psi_\varepsilon| \leq C$, for some $C = C(K) > 0$ and ε small enough.

Proof. Let us denote by C any positive constant which does not depend on ε. Consider the function $F(\varepsilon) = f(u_0 + \varepsilon\varphi + \varepsilon^2\Psi_\varepsilon)$. Then for ε small there exists $t_\varepsilon = t_\varepsilon(x,y) \in (0,1)$ such that

$$\begin{aligned}
f(u_\varepsilon) = F(\varepsilon) = f(u_0) + \varepsilon f'(u_0)\varphi + \frac{\varepsilon^2}{2}f''(u_0)\varphi^2 + \varepsilon^2 f'(u_0)\Psi_\varepsilon +&\\
+ \frac{\varepsilon^3}{6}f'''(u_0 + t_\varepsilon\varepsilon\varphi + t_\varepsilon^2\varepsilon^2\Psi_\varepsilon)(\varphi + 2t_\varepsilon\varepsilon\Psi_\varepsilon)^2 +&\\
+ \varepsilon^3 f''(u_0 + t_\varepsilon\varepsilon\varphi + t_\varepsilon^2\varepsilon^2\Psi_\varepsilon)(\varphi + 2t_\varepsilon\varepsilon\Psi_\varepsilon)\Psi_\varepsilon.&
\end{aligned} \tag{3.20}$$

From the previous lemma we have that $0 \le \varepsilon\Psi_\varepsilon \le \bar{\psi} \le C\sum_{j=1}^{N}\cosh(\sqrt{\mu_1}x_j)$. From Lemma 3.2.2, $|x_j| \le C\log(1/\varepsilon)$ for all $j = 1,\dots,N$ and then

$$\left|u_0 + t_\varepsilon\varepsilon\varphi + t_\varepsilon^2\varepsilon^2\Psi_\varepsilon\right| \le C, \quad \text{in } \Omega_\varepsilon.$$

In Ω_ε, taking into account (3.20), we have the following inequality

$$\begin{aligned} f(u_\varepsilon) - f(u_0) - \varepsilon f'(u_0)\varphi &\le C\varepsilon^2\left(\varphi^2 + \varepsilon(\varphi + 2\bar{\psi})^2 + (\varphi + 2\bar{\psi})\bar{\psi}\right) + \varepsilon^2 f'(u_0)\Psi_\varepsilon \\ &\le \frac{C_\infty}{\lambda}\varepsilon^2\sum_{j=1}^{N}\cosh(2\sqrt{\mu_1}x_j) + \varepsilon^2 f'(u_0)\Psi_\varepsilon, \end{aligned}$$

for some $C_\infty > 0$, that implies

$$-\Delta\Psi_\varepsilon - \lambda f'(u_0)\Psi_\varepsilon \le C_\infty\sum_{j=1}^{N}\cosh(2\sqrt{\mu_1}x_j). \tag{3.21}$$

Fix $\mu_\infty = 4\mu_1$. Note that $\mu_\infty < \mu_0$ thanks to (3.8). Then taking into account Lemma 3.2.1 set $\omega_\infty = \omega_{\mu_\infty}$ and for $(x,y) \in \mathbb{R}^N \times (1-\sigma, 1+\sigma)$ consider

$$\psi_\infty(x,y) = \frac{C_\infty}{c_\infty\mu_\infty}\sum_{j=1}^{N}(\omega_\infty(y) - c_\infty)\cosh(\sqrt{\mu_\infty}x_j),$$

where $0 < c_\infty < \inf\limits_{(-1-\sigma,1+\sigma)}\omega_\infty$.

Clearly $\psi_\infty > 0$ in $\overline{\Omega}_\varepsilon$ and ψ_∞ satisfies the following inequality

$$\begin{aligned} -\Delta\psi_\infty - \lambda f'(u_0)\psi_\infty &= \frac{C_\infty}{c_\infty\mu_\infty}\sum_{j=1}^{N}c_\infty\left(\mu_\infty + \lambda f'(u_0)\right)\cosh(\sqrt{\mu_\infty}x_j) \\ &\ge C_\infty\sum_{j=1}^{N}\cosh(2\sqrt{\mu_1}x_j), \end{aligned}$$

which together to (3.21) gives

$$\begin{cases} -\Delta(\Psi_\varepsilon - \psi_\infty) - \lambda f'(u_0)(\Psi_\varepsilon - \psi_\infty) \le 0 & \text{in } \Omega_\varepsilon \\ \Psi_\varepsilon - \psi_\infty < 0 & \text{on } \partial\Omega_\varepsilon \end{cases}$$

and again the maximum principle provides $\Psi_\varepsilon - \psi_\infty < 0$ in Ω_ε. For $C(K) = \max_K \psi_\infty$ the proof is complete. □

3.2.3 Proof of Theorem 3.1.1

Proof. We have that *(i)* and *(ii)* follow by *(iv)* and *(iii)* of Lemma 3.2.2 respectively. The proof of *(iii)* is given in Lemma 3.2.4.

Let us prove *(iv)*. By Lemma 3.2.3 we have that $u_0 + \varepsilon\varphi$ admits k strict maxima points. Fix a compact set $K \subset\subset \Omega_\varepsilon$ containing such points. On the other hand Lemma 3.2.9 implies $u_\varepsilon = u_0 + \varepsilon\varphi + O(\varepsilon^2)$ in K and so the claim follows. □

Remark 3.2.10. We can prove a little more general version of Theorem 3.1.1: indeed assumption (3.3) can be dropped and we can simply ask that there exists u_0 stable solution of

$$\begin{cases} -u'' = g(u) & \text{in } (-1,1) \\ u > 0 & \text{in } (-1,1) \\ u(\pm 1) = 0. \end{cases}$$

Finally we build Ω_ε as before and then ask for the existence of a stable solution u_ε of problem (3.1) in Ω_ε.

Remark 3.2.11. Let us show that the assumption that u_ε is a *stable* solution is crucial in our construction. To do this let us assume $N = 1$ for simplicity and consider $f(t) = \lambda_1 t$, where λ_1 is the first eigenvalue of the Dirichlet problem. In this case the first eigenvalue of the linearized problem at the first eigenfunction is 0. Let us see that it is not possible to construct a domain Ω_ε as in the previous section. Indeed if we argue as before we have that $u_0(y) = \cos\left(\frac{\pi}{2}y\right)$ is the solution of

$$\begin{cases} -u'' = \frac{\pi^2}{4}u & \text{in } (-1,1) \\ u > 0 & \text{in } (-1,1) \\ u(\pm 1) = 0. \end{cases}$$

Now, for $n \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$ (again with $\alpha_1 = -1$) and $\mu_i > 0$ for $i = 1, \dots, n$, we have that

$$\varphi(x,y) = \sum_{i=1}^{n} \alpha_i \cosh(\sqrt{\mu_i}x) \cos\left(\sqrt{\pi^2/4 + \mu_i y}\right),$$

solves the linearized problem, i.e.

$$-\Delta\varphi = \frac{\pi^2}{4}\varphi \quad \text{in } \mathbb{R}^2,$$

As for the general case we observe that $u_0(0) + \varepsilon\varphi(0,0) > 0$ for ε small enough and then we set $\Omega_\varepsilon = \{\, u_0 + \varepsilon\varphi > 0 \,\}$. Now for any $\mu_1 > 0$ set

$$\bar{y} = \frac{\frac{\pi}{2}}{\sqrt{\pi^2/4 + \mu_1}} \in (0,1),$$

and then we can find $\delta > 0$ sufficiently small such that if ε is small enough it holds

$$\mathbb{R} \times \{\, y = \bar{y} + \delta \,\} \subseteq \Omega_\varepsilon,$$

showing that the domain Ω_ε is not bounded. This shows that our construction fails.

3.3 The torsion problem: proof of Theorem 3.1.2

In this section we take $x \in \mathbb{R}$ and $y = (y_1, \dots, y_N) \in \mathbb{R}^N$ and we assume the hypothesis of Theorem 3.1.2. We build a solution u_ε of the torsion problem ($g(u) \equiv$ constant) with k maximum points in a domain Ω_ε whose boundary has *positive mean curvature.* Here the domain Ω_ε and the function u_ε are similar to the ones defined in Section 3.2.

Let us start by introducing the following function $u_\varepsilon : \mathbb{R}^{N+1} \to \mathbb{R}$, given by

$$u_\varepsilon(x,y) := u_0(y) + \varepsilon\varphi(x,y) \quad x \in \mathbb{R},\ y \in \mathbb{R}^N,$$

where

$$u_0(y) := \frac{1}{2}\sum_{j=1}^{N}\left(1 - y_j^2\right) = \frac{1}{2}\left(N - |y|^2\right),$$

which solves

$$\begin{cases} -\Delta u = N & \text{in } \mathcal{C} \\ u = 0 & \text{on } \partial\mathcal{C} \end{cases}$$

in the cylinder $\mathcal{C} = \{(x,y) \in \mathbb{R}^{N+1} | |y|^2 < N\}$. Finally φ is a harmonic function in the whole $\mathbb{R}^{N+1}$ defined by

$$\varphi(x,y) := \sum_{j=1}^{N} v(x, y_j),$$

where $v(t,s) := \Re(F_k(t+is))$, for $t, s \in \mathbb{R}$ with

$$\begin{aligned} F_k(t+is) &:= -\prod_{\ell=1}^{k}\left[(t - t_\ell + is)(t + t_\ell + is)\right] \\ &= -\prod_{\ell=1}^{k}\left(t^2 - s^2 - t_\ell^2 + 2its\right), \qquad \text{for } 0 < t_1 < \cdots < t_k, \end{aligned}$$

and $\Re(\,\cdot\,)$ stands for the real part of a complex function. Note that v is symmetric with respect to both $\{t = 0\}$ and $\{s = 0\}$ and it can be written as

$$v(t,s) = -\sum_{h=0}^{2k} a_h P_h(t,s), \tag{3.22}$$

where P_h is a harmonic polynomial of degree h, $a_{2k} = 1$ and

$$P_{2k}(t,s) = \sum_{\ell=0}^{k} b_\ell t^{2k-2\ell} s^{2\ell}, \quad b_0 = b_k = 1. \tag{3.23}$$

Resuming, we have that for $x \in \mathbb{R}$ and $y \in \mathbb{R}^N$

$$\begin{aligned} u_\varepsilon(x,y) &= u_0(y) + \varepsilon\varphi(x,y) \\ &= \frac{1}{2}\left(N - |y|^2\right) + \varepsilon\sum_{j=1}^{N} v(x, y_j) \\ &= \frac{1}{2}\sum_{j=1}^{N}\left(1 - y_j^2\right) - \varepsilon\sum_{j=1}^{N}\sum_{h=0}^{2k} a_h P_h(x, y_j). \end{aligned}$$

Since $F_k : \mathbb{C} \to \mathbb{C}$ is holomorphic, it easily follows that φ is harmonic and then u_ε satisfies $-\Delta u_\varepsilon = N$. Finally, we point out that $\partial_{y_i y_j} u_\varepsilon = 0$ for all $i \neq j$.

3.3.1 Preliminary results

In this section we show some properties of the function u_ε and of the domain Ω_ε that we are going to define.

As in Section 3.2 we point out that

$$u_\varepsilon(0,0,\dots,0) = \frac{N}{2} + \varepsilon\sum_{j=1}^{N} v(0,0) \geq \frac{N}{4} > 0,$$

for ε small enough and we denote by Ω_ε the connected component of the superlevel set $\{\, u_0 + \varepsilon\varphi > 0 \,\}$ containing the origin.

The following lemma proves some properties of the set Ω_ε.

Lemma 3.3.1. *The set Ω_ε satisfies the following properties.*

(i) $\Omega_\varepsilon \subseteq C_\varepsilon$ *for ε small enough, where*

$$C_\varepsilon := \left\{ (x,y) \in \mathbb{R}^{N+1} \,\middle|\, x \in (-M_\varepsilon, M_\varepsilon),\ |y|^2 < N(1+\eta)^2 \right\},$$

for some $0 < \eta < 1$, and $M_\varepsilon := \varepsilon^{-\frac{1}{2k}}$.

(ii) $\Omega_\varepsilon \supseteq [-t_k, t_k] \times \{0\}^N$.

(iii) Let $(x^\varepsilon, y^\varepsilon) \in \partial\Omega_\varepsilon$. If $|y^\varepsilon| \to 0$ then we have

$$|x^\varepsilon| = (2\varepsilon)^{-\frac{1}{2k}}(1 + o(1)) \to +\infty. \tag{3.24}$$

On the other hand, if $|x^\varepsilon| \le C$, then

$$|y^\varepsilon|^2 \to N.$$

(iv) Ω_ε is symmetric with respect to the hyperplanes $x = 0$ and $y_j = 0$ for $j = 1, \dots, N$. Moreover, it is a smooth and star-shaped domain with respect to the origin for ε small enough.

Proof. To prove (i) we firstly show that

$$u_\varepsilon \le -1/2, \quad \text{on } \left\{ (x,y) \in \mathbb{R}^{N+1} \,\middle|\, x = \pm M_\varepsilon,\ |y|^2 < N(1+\eta)^2 \right\}, \tag{3.25}$$

for ε small enough. Indeed by (3.23) we get

$$\varepsilon P_{2k}(\pm M_\varepsilon, s) = \varepsilon \sum_{\ell=0}^{k} b_\ell \left(\varepsilon^{-\frac{1}{2k}}\right)^{2k-2\ell} s^{2\ell} = 1 + o(1), \quad \text{as } \varepsilon \to 0,$$

uniformly with respect to $|s| < \sqrt{N}(1+\eta)$. Similarly we have

$$\varepsilon P_h(\pm M_\varepsilon, s) = o(1), \quad \text{for all } 0 \le h \le 2k-1.$$

Finally, for $x = \pm M_\varepsilon$ and $|y|^2 \le N(1+\eta)^2$ we have

$$u_\varepsilon(x,y) \le \frac{N}{2} + \varepsilon \sum_{j=1}^{N} v(\pm M_\varepsilon, y_j)(1 + o(1)) = \frac{N}{2} - N + o(1) \le -\frac{1}{2}.$$

On the other hand by (3.22) and since $a_{2k} = 1$ we get

$$\sup_{t\in\mathbb{R}} \max_{s \in [-\sqrt{N}(1+\eta), \sqrt{N}(1+\eta)]} v(t,s) = C \in \mathbb{R}.$$

Then for all $(x,y) \in \overline{C}_\varepsilon$ with $|y|^2 = N(1+\eta)^2$ we obtain

$$u_\varepsilon(x,y) = -\frac{N}{2}\eta^2 - N\eta + \varepsilon \sum_{j=1}^{N} v(x, y_j) < -\frac{N}{2}\eta^2 < 0,$$

for ε small enough which together to (3.25) proves (i).

Concerning (ii), we know that the origin belongs to Ω_ε and since u_ε is continuous, then Ω_ε is an open and connected set. Finally if ε satisfies

$$\varepsilon < \frac{u_0(0,\dots,0)}{\max_{x\in[-t_k,t_k]}(-\varphi(x,0,\dots,0))},$$

then $[-t_k,t_k]\times\{0\}^N \subseteq \Omega_\varepsilon$.

In order to prove (iii), let $(x^\varepsilon,y^\varepsilon)\in\partial\Omega_\varepsilon$. Then one has

$$\frac{1}{2}\left(N-|y^\varepsilon|^2\right) = -\varepsilon\sum_{j=1}^N v(x^\varepsilon,y_j^\varepsilon). \tag{3.26}$$

If $|x^\varepsilon|\le C$, $v(x^\varepsilon,y_j^\varepsilon)$ is bounded and then we easily get $|y^\varepsilon|^2\to N$.

Then we can assume $|x^\varepsilon|\to+\infty$. In particular, for all $j=1,\dots,N$, it holds $v(x^\varepsilon,y_j^\varepsilon) = -(x^\varepsilon)^{2k}(1+o(1))$ and from (3.26) we get

$$(x^\varepsilon)^{2k} = \frac{1}{2}\left(1-\frac{|y^\varepsilon|^2}{N}\right)\varepsilon^{-1}(1+o(1)) = \frac{1}{2}\varepsilon^{-1}(1+o(1)),$$

and in particular (3.24) holds.

The symmetry properties of the domain immediately follow from the ones of u_ε. Then to finish the proof it is enough to prove that there exists $\alpha>0$ such that

$$x\partial_x u_\varepsilon + \sum_{j=1}^N y_j\partial_{y_j}u_\varepsilon \le -\alpha < 0, \quad \text{for all } (x,y)\in\partial\Omega_\varepsilon.$$

We have

$$x\partial_x u_\varepsilon + \sum_{j=1}^N y_j\partial_{y_j}u_\varepsilon = -\sum_{j=1}^N y_j^2 + \varepsilon\sum_{j=1}^N\left(xv_t(x,y_j)+y_jv_s(x,y_j)\right).$$

On the other hand since $u_\varepsilon(x,y)=0$ on $\partial\Omega_\varepsilon$ we have

$$\sum_{j=1}^N y_j^2 = N + 2\varepsilon\sum_{j=1}^N v(x,y_j),$$

and then

$$x\partial_x u_\varepsilon + \sum_{j=1}^N y_j\partial_{y_j}u_\varepsilon = -N + \varepsilon\sum_{j=1}^N\left(xv_t(x,y_j)+y_jv_s(x,y_j)-2v(x,y_j)\right).$$

Since we have that

$$\begin{aligned} tv_t(t,s)+sv_s(t,s)-2v(t,s) &= -\sum_{h=0}^{2k} a_h\left(t\partial_t P_h(t,s)+s\partial_s P_h(t,s)-2P_h(t,s)\right)\\ &= -\sum_{h=0}^{2k}(h-2)a_hP_h(t,s)\to-\infty, \end{aligned}$$

for $|t|\to+\infty$ uniformly with respect to $|s|<\sqrt{N}(1+\eta)$, it follows

$$\sup_{(t,s)\in\mathbb{R}\times[-\sqrt{N}(1+\eta),\sqrt{N}(1+\eta)]} tv_t(t,s)+sv_s(t,s)-2v(t,s) = d < +\infty,$$

and then

$$\sum_{j=1}^{N}\left(xv_t(x,y_j)+y_jv_s(x,y_j)-2v(x,y_j)\right)\le Nd<+\infty.$$

Finally

$$\sup_{\partial\Omega_\varepsilon}\left(x\partial_x u_\varepsilon+\sum_{j=1}^{N}y_j\partial_{y_j}u_\varepsilon\right)\le -N+o(1)\le -\frac{N}{2},$$

for ε small enough. Of course $x\partial_x u_\varepsilon+\sum_{j=1}^{N}y_j\partial_{y_j}u_\varepsilon\neq 0$ on $\partial\Omega_\varepsilon$ implies that $\partial\Omega_\varepsilon$ is a smooth hypersurface. □

Remark 3.3.2. In particular from (iii) of Lemma 3.3.1 we deduce that Ω_ε locally converges to the cylinder $\mathcal{C}=\{\,(x,y)\in\mathbb{R}^{N+1}\mid |y|^2<N\,\}$.

Equation (3.24) will be useful in the computation of the curvature of $\partial\Omega_\varepsilon$ in next subsection.

Lemma 3.3.3. *The function u_ε has at least k different nondegenerate local maxima in Ω_ε for ε small enough.*

Proof. The proof is similar to the one of Lemma 3.2.3.

For

$$q(t)=\Re\left(F_k(t+i0)\right)=-\prod_{\ell=1}^{k}(t-t_\ell)(t+t_\ell)=v(t,0),$$

we have $q(t)=0$ if and only if $t=\pm t_\ell$ for some $\ell=1,\dots,k$ and $q(t)\to-\infty$ as $|t|\to+\infty$. Now assume k even, the case k odd follows by minor changes. Then there exist $\bar t_\ell\in(t_{2\ell+1},t_{2\ell+2})$ with $\ell=0,\dots,k/2$ such that

$$q'(\bar t_\ell)=0,\quad\text{and}\quad q''(\bar t_\ell)<0\quad\text{for all }\ell=0,\dots,k/2,$$

see also Lemma B.2.

Moreover, from the definition of v, since every time a power of s appears then it is an even power, we get that $\partial_s v(t,0)=\partial_{ts}v(t,0)=0$ for all $t\in\mathbb{R}$. Then a straightforward computation gives

$$\nabla u_\varepsilon(\bar t_\ell,0,\dots,0)=0.$$

Next, for all $j=1,\dots,N$ and for all $\ell=0,\dots,k/2$, we have

$$\partial_{y_jy_j}u_\varepsilon(\bar t_\ell,0,\dots,0)=-1+\varepsilon\partial_{ss}v(\bar t_\ell,0)<0,\tag{3.27}$$

for ε small enough. Finally in $(\bar t_\ell,0,\dots,0)$ one has

$$\begin{aligned}\partial_{xx}u_\varepsilon&=\varepsilon Nq''(\bar t_\ell,0)<0,\\ \partial_{y_iy_j}u_\varepsilon&=0,\qquad\text{for all } i\neq j,\\ \partial_{xy_j}u_\varepsilon&=\varepsilon\partial_{ts}v(\bar t_\ell,0)=0,\end{aligned}$$

which, together to (3.27), shows us that the Hessian matrix of u_ε is negative definite in $(\bar t_\ell,0,\dots,0)$ for all $\ell=0,\dots,k/2$ and the proof is complete since u_ε is even in the x variable. □

Remark 3.3.4. We point out that Ω_ε is not convex. Indeed, we know from Lemma 3.3.1 that the domain is symmetric with respect to $\{x=0\}$ and $\{y_j=0\}$ for all $j=1,\dots,N$ and by the well known result by [GNN79], see Theorem 1.1.6, the domain cannot be convex otherwise every solution of problem (3.1) has exactly one critical point in contradiction with Lemma 3.3.3.

3.3.2 Curvature of the domain

In this section we prove that the domain Ω_ε previously defined has *positive mean curvature*.

Let us start by a technical lemma that gives us an explicit formula to compute the mean curvature for manifolds which are preimage of a regular value of real functions. The proof is postponed to the Appendix.

Lemma 3.3.5. *Let $\Sigma = F^{-1}(0)$, for some $F \in \mathcal{C}^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. Assume 0 is a regular value for F and $F_{y_i y_j} = 0$ for all $i \neq j$. Then the* mean curvature *of Σ is given by*

$$\mathfrak{K} = -\frac{1}{N|\nabla F|^3}\left[\sum_{j=1}^{N}\left(F_x^2 F_{y_j y_j} - 2F_x F_{y_j} F_{xy_j} + F_{y_j}^2 F_{xx}\right) + \sum_{j=1}^{N} F_{y_j}^2 \sum_{\substack{\ell=1\\ \ell\neq j}}^{N} F_{y_\ell y_\ell}\right].$$

Finally, we are able to compute the mean curvature of the boundary of the domain.

Lemma 3.3.6. *The mean curvature of the boundary of Ω_ε is strictly positive everywhere.*

Proof. We will apply the previous lemma to $F(x, y) = u_\varepsilon(x, y)$. Note that $\nabla u_\varepsilon \neq 0$ on $\partial\Omega_\varepsilon$ from (iv) of Lemma 3.3.1. Let $(x^\varepsilon, y^\varepsilon) \in \partial\Omega_\varepsilon$ and from the asymptotic behavior of the derivatives of $v(t, y)$ for $t \to \infty$ we have

$$\begin{aligned}
v_t &= -2kt^{2k-1}(1 + o(1)), & v_s &= c_k t^{2k-2} s(1 + o(1)),\\
v_{tt} &= -2k(2k-1)t^{2k-2}(1 + o(1)), & v_{ts} &= c_k' t^{2k-3} s(1 + o(1)),\\
v_{ss} &= c_k t^{2k-2}(1 + o(1)),
\end{aligned}$$

and from the estimate $|x^\varepsilon| \leq \varepsilon^{-\frac{1}{2k}}$ we get that for all $j = 1, \dots, N$ the following quantities

$$\varepsilon v_t(x^\varepsilon, y_j^\varepsilon), \quad \varepsilon v_s(x^\varepsilon, y_j^\varepsilon), \quad \varepsilon v_{tt}(x^\varepsilon, y_j^\varepsilon), \quad \varepsilon v_{ts}(x^\varepsilon, y_j^\varepsilon), \quad \varepsilon v_{ss}(x^\varepsilon, y_j^\varepsilon),$$

go to 0 as $\varepsilon \to 0$.

Then we proceed by considering the cases $|y^\varepsilon| \not\to 0$ and $|y^\varepsilon| \to 0$.

Case $|y^\varepsilon| \not\to 0$.
We point out that for ε small enough there exists $j \in \{1, \dots, N\}$ such that $\partial_{y_j} u_\varepsilon \neq 0$, otherwise $|y^\varepsilon| \to 0$. Then from Lemma 3.3.5 we have

$$\mathfrak{K} = -\frac{-(N-1)|y^\varepsilon|^2(1 + o(1))}{N\left(|y^\varepsilon|^2(1 + o(1))\right)^{\frac{3}{2}}} = \frac{N-1}{N|y^\varepsilon|}(1 + o(1)) > 0.$$

Note that the assumption $N \geq 2$ is crucial. Indeed if $N = 1$ the curvature changes sign, see [GG22].

Case $|y^\varepsilon| \to 0$.
In this case, by (3.24) we have that $x^\varepsilon \to +\infty$ and for all $j = 1, \dots, N$ fixed $\partial_{y_j} u_\varepsilon = o(1)$. Recalling (3.24) again, the following estimates hold true

$$\begin{aligned}
(\partial_{y_j} u_\varepsilon)^2 \partial_{xx} u_\varepsilon &= o(\varepsilon^{1-\frac{2k-2}{2k}}) = o(\varepsilon^{\frac{1}{k}}),\\
\partial_x u_\varepsilon \partial_{y_j} u_\varepsilon \partial_{xy_j} u_\varepsilon &= o\left(\varepsilon^{1-\frac{2k-1}{2k}} \varepsilon^{1-\frac{2k-3}{2k}}\right) = o(\varepsilon^{\frac{2}{k}}) = o(\varepsilon^{\frac{1}{k}}),\\
(\partial_x u_\varepsilon)^2 \partial_{y_j y_j} u_\varepsilon &= -\left(-2Nk\varepsilon(x^\varepsilon)^{2k-1}\right)^2 (1 + o(1))\\
&= -2^{\frac{1}{k}} N^2 k^2 \varepsilon^{\frac{1}{k}} (1 + o(1)).
\end{aligned}$$

This yields

$$(\partial_{y_j} u_\varepsilon)^2 \partial_{xx} u_\varepsilon - 2\partial_x u_\varepsilon \partial_{y_j} u_\varepsilon \partial_{xy_j} u_\varepsilon + (\partial_x u_\varepsilon)^2 \partial_{y_j y_j} u_\varepsilon = -2^{\frac{1}{k}} N^2 k^2 \varepsilon^{\frac{1}{k}} (1 + o(1)). \tag{3.28}$$

Moreover by similar computations

$$\sum_{j=1}^{N} (\partial_{y_j} u_\varepsilon)^2 \sum_{\substack{\iota=1 \\ \iota \neq j}}^{N} \partial_{y_\iota y_\iota} u_\varepsilon = -(N-1)(1 + o(1)) \sum_{j=1}^{N} (\partial_{y_j} u_\varepsilon)^2 \leq 0. \tag{3.29}$$

Finally, we can apply Lemma 3.3.5, and putting together (3.28) and (3.29) we have

$$\begin{aligned} -N|\nabla u_\varepsilon|^3 \mathfrak{K} &\leq \sum_{j=1}^{N} \Big((\partial_{y_j} u_\varepsilon)^2 \partial_{xx} u_\varepsilon - 2\partial_x u_\varepsilon \partial_{y_j} u_\varepsilon \partial_{xy_j} u_\varepsilon + (\partial_x u_\varepsilon)^2 \partial_{y_j y_j} u_\varepsilon \Big) \\ &= -2^{\frac{1}{k}} N^3 k^2 \varepsilon^{\frac{1}{k}} (1 + o(1)) < 0, \end{aligned}$$

that is $\mathfrak{K} > 0$. □

3.3.3 Proof of Theorem 3.1.2

Proof. Up to a dilatation of the domain, the claims follow from Lemma 3.3.1, Lemma 3.3.3 and Lemma 3.3.6 considering u_ε / N. □

Remark 3.3.7. It is also possible to treat the case $x = (x_1, \ldots, x_M) \in \mathbb{R}^M$, with $M > 1$, in such a way that the domain Ω_ε grows in M directions. The proof works replacing the function u_ε by the following one

$$\tilde{u}_\varepsilon(x, y) := \frac{1}{2} \sum_{j=1}^{N} \left(1 - y_j^2\right) + \varepsilon \sum_{i=1}^{M} \sum_{j=1}^{N} v(x_i, y_j).$$

The computations are very similar to the case $M = 1$. It is not difficult to generalize Lemma 3.3.5 taking into account that $\partial_{x_i x_h} u_\varepsilon = 0$ for all $i \neq h$.

Chapter 4

Uniqueness of critical point in non convex domains

In this short chapter we want to show that sometimes it is possible to recover uniqueness of critical point for solutions of elliptic equations even in non convex domains and without symmetry assumptions. To this end we first work on the Poisson problem and then also on the nonlinear problem $-\Delta u = f(u)$.

The results had been obtained in collaboration with Luca Battaglia and Massimo Grossi.

4.1 Main results

Let $\Omega \subseteq \mathbb{R}^2$ be a smooth and bounded domain and consider the following problem

$$\begin{cases} -\Delta u = f\left((x,y),u\right) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a smooth function. Here and in the rest of the chapter, the generic point of $\mathbb{R}^2$ is denoted using coordinates (x, y) instead of (x_1, x_2).

All the uniqueness of the critical point results quoted in the preceding chapters hold in convex domain and it is known that, in general, we can not expect uniqueness of the critical point in non convex domains, see Section 1.2. Then it is natural to ask whether it is possible to recover the uniqueness in (possibly) non convex domain, under suitable assumptions.

In the first part of this chapter we examine the Poisson problem

$$\begin{cases} -\Delta u = f(x,y) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $f : \overline{\Omega} \to \mathbb{R}$ is a smooth positive function and $\Omega \subseteq \mathbb{R}^2$ is a smooth, bounded and simply connected domain. Let us recall that for the Poisson problem it is possible to prove uniqueness of the critical point in convex domain, under suitable assumption on the function f as showed by Kennington, see next theorem.

Theorem 4.1.1 ([Ken85]). *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded and convex domain and let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ satisfy*

$$\begin{cases} -\Delta u = f(\boldsymbol{x}) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f : \Omega \to (0, +\infty)$ is β-concave Then u is $\frac{\beta}{1+2\beta}$-concave. Moreover, if f is constant, then u is $\frac{1}{2}$-concave.

We have the following result.

Theorem 4.1.2. *Assume $f > 0$ in $\overline{\Omega}$ and*

$$\Delta(\log f) = 0, \quad \text{in } \Omega, \tag{4.2}$$

and

$$\frac{1}{2}\frac{\partial f}{\partial \nu} + \mathfrak{K} f \geq 0, \quad \text{on } \partial\Omega, \tag{4.3}$$

where ν is the outnormal unit vector to $\partial\Omega$ and $\mathfrak{K} = \mathrm{tr}(\mathrm{d}\nu)/N$ is its curvature. If u is the solution of problem (4.1), then it has a unique nondegenerate critical point $(x_0, y_0) \in \Omega$.

Remark 4.1.3. 1) Theorem 4.1.2 holds even if Ω is not convex. For instance, if we consider $f(x, y) := e^{2x}$, then equation (4.2) is trivially satisfied while equation (4.3) is satisfied if Ω is such that

$$\nu_x \geq -\mathfrak{K}, \quad \text{on } \partial\Omega, \tag{4.4}$$

where we write $\nu := (\nu_x, \nu_y)$. This condition can be verified by non convex domains Ω, see Figure 4.1.

2) Let us also point out that we can not drop assumption (4.2) or (4.3), otherwise we can loose the uniqueness of the critical point: see Remark 4.2.4 and Remark 4.2.5 for the details.

3) The preceding theorem can be seen as a generalization of Makar-Limanov's Theorem 1.1.4 for the torsion problem. Indeed, for $f \equiv 1$ and Ω convex it is easy to see that assumptions (4.2) and (4.3) are trivially satisfied.

The proof of the theorem works as follows: we firstly show that under assumptions (4.2) and (4.3) we can construct a conformal map T from Ω to a bounded and convex subsets of $\mathbb{C}$ and such that $|T'|^2$ is exactly f, see Proposition 4.2.1. From this we can find a one to one correspondence between the critical points of u and the ones of the solution of the torsion problem on the image $T(\Omega)$. Hence, the claim follows thanks to Makar-Limanov's Theorem 1.1.4.

In the second part of the chapter, we came back to the nonlinear problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

where Ω is still a smooth bounded and convex domain in $\mathbb{R}^2$ and $f : \mathbb{R} \to \mathbb{R}$ is a smooth nonlinearity.

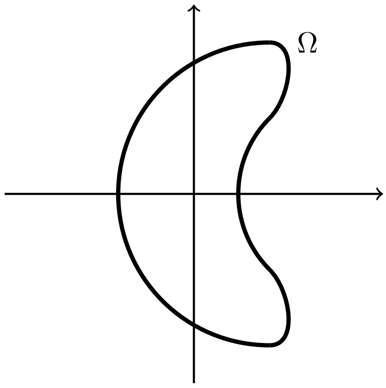

Figure 4.1: Example of domain which satisfies (4.4).

We are interested in investigating the number of critical points of solutions of the preceding problem when Ω is close to be a convex domain.

We recall that Theorem 1.1.9 and Theorem 2.1.1 imply that if Ω is convex then a semi-stable solution of (4.5) admits exactly one critical point which turns out to be nondegenerate. Then we can ask what happens if we consider domains which are (possibly) non convex, but close to a convex one.

First of all let us recall that for a convex domain $\Omega \subseteq \mathbb{R}^N$, with $\mathbf{0} \in \Omega$ and with no empty interior, up to a dilatation, we can find a Lipschitz function $\chi_\Omega : \mathbb{S}^{N-1} \to (0 + \infty)$ such that

$$\Omega = \left\{ tP((1 + \chi_\Omega(P)) \;\middle|\; P \in \mathbb{S}^{N-1}, \quad t \in [0, 1) \right\},$$

furthermore, if we assume Ω to be of class $\mathcal{C}^k$ then $\chi_\Omega \in \mathcal{C}^k(\mathbb{S}^{N-1})$.

Hence let us give the following definition of convergence of a family of smooth sets to a smooth and convex one.

Definition 4.1.4. Given a bounded and convex set $\Omega \subseteq \mathbb{R}^N$ of class $\mathcal{C}^k$ and with no empty interior, we say that the family $(\Omega_\varepsilon)_\varepsilon \subseteq \mathbb{R}^N$ of bounded sets of class $\mathcal{C}^k$ *converges to the convex set* Ω for $\varepsilon \to 0$ - and we write $\Omega_\varepsilon \to \Omega$ for $\varepsilon \to 0$ - if there exists a family of functions $(\chi_{\Omega_\varepsilon})_\varepsilon \subseteq \mathcal{C}^k(\mathbb{S}^{N-1})$ such that

$$\Omega_\varepsilon = \left\{ tP((1 + \chi_{\Omega_\varepsilon}(P)) \;\middle|\; P \in \mathbb{S}^{N-1}, \quad t \in [0, 1) \right\},$$

and

$$\|\chi_{\Omega_\varepsilon} - \chi_\Omega\|_{\mathcal{C}^k(\mathbb{S}^{N-1})} \to 0, \quad \text{as } \varepsilon \to 0.$$

Now, let us fix a smooth and convex domain $\Omega \subseteq \mathbb{R}^2$, with $\mathbf{0} \in \Omega$ and we consider a family of domains Ω_ε that are smooth and such that $\Omega_\varepsilon \to \Omega$ for $\varepsilon \to 0$, at least in $\mathcal{C}^4$ sense, according to the preceding definition.

The following result holds true.

Theorem 4.1.5. *Let u_ε be a semi-stable solution the following problem*

$$\begin{cases} -\Delta u_\varepsilon = f(u_\varepsilon) & \text{in } \Omega_\varepsilon \\ u_\varepsilon > 0 & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{4.6}$$

with $f(0) \geq 0$ and assume that $\|u_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq C$ for some $C > 0$. Then u_ε has a unique nondegenerate critical point $p_\varepsilon \in \Omega_\varepsilon$. Moreover $p_\varepsilon \to p_c$ where $p_c \in \Omega$ is the unique critical point of a semi-stable solution u of problem (4.5) *in* Ω.

Remark 4.1.6. 1) Let us point out that if the limit domain is convex, but unbounded, then the preceding result does not hold. Indeed, it is possible to build a family of domains which converges to the strip $\mathcal{S} = \mathbb{R} \times (-1, 1)$ and such that the corresponding solutions have an arbitrary large (finite) number of critical points. See Theorem 1.2.1 and Theorem 3.1.1.

2) Let us point out that the hypothesis that $\|u_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}$ is uniformly bounded is always satisfied if, for instance, we assume that the nonlinearity has the form $f(u) = \lambda g(u)$, g is smooth and satisfies (3.2) and (3.3), that are

$$g : \mathbb{R} \to \mathbb{R} \text{ is increasing and convex,}$$
$$g(0) > 0,$$

and $\lambda \in (0, \lambda^*(\Omega))$, as in Chapter 3. See Remark 4.3.1 for the details.

To prove Theorem 4.1.5 we show that u_ε converges to u, the solution of probem (4.5) in Ω, and then the claim can be deduced by Theorem 2.1.1.

The chapter is organized as follows: in the next section we prove Theorem 4.1.2, and we conclude it by showing that if at least one between (4.2) and (4.3) does not hold, then Theorem 4.1.2 may fail. In Section 4.3 we prove Theorem 4.1.5.

4.2 The Poisson problem

In this section we prove Theorem 4.1.2. Up to the end of the section we identify $\mathbb{R}^2$ with $\mathbb{C}$ and we write $z := x + iy$.

Assume $f : \overline{\Omega} \to \mathbb{R}$ be positive and such that assumptions (4.2) and (4.3) are satisfied. Then the following proposition holds true.

Proposition 4.2.1. *There exist a holomorphic function $T : \Omega \to \mathbb{C}$ such that*

(i) $|T'|^2 = f$ *in* Ω,

(ii) $T(\Omega)$ *is bounded and convex,*

(iii) there exist a holomorphic function $\tau : T(\Omega) \to \mathbb{C}$ *such that* $\tau = T^{-1}$.

Proof. Without loss of generality, we can assume $\mathbf{0} \in \Omega$. Since $\log f$ is harmonic in Ω by (4.2) and Ω is simply connected, we can find a holomorphic function $w : \Omega \to \mathbb{R}$ such that

$$\Re(w) = \frac{1}{2} \log f.$$

Moreover, since $f > 0$ in $\overline{\Omega}$, w is holomorphic up to the boundary of Ω. Hence, also the function $\mathfrak{t} := e^w$ is holomorphic in $\overline{\Omega}$ and if we decompose it by modulus and principal argument we have

$$\mathfrak{t} = |\mathfrak{t}| e^{i\Theta}.$$

Finally

$$\frac{1}{2} \log f = \Re(w) = \Re(\log \mathfrak{t}) = \Re\left(\log(|\mathfrak{t}| e^{i\Theta})\right) = \log|\mathfrak{t}|,$$

which yields to

$$|\mathfrak{t}|^2 = f. \tag{4.7}$$

Then, since Ω is simply connected and $\mathfrak{t}$ is holomorphic, we can define T as the primitive of $\mathfrak{t}$ such that $T(\mathbf{0}) = \mathbf{0}$. Clearly T is holomorphic and (4.7) implies $|T'|^2 = f$, proving (i).

To prove (ii) note that since T is continuous up to the boundary of Ω we have that $|T|$ is bounded and then $T(\Omega)$ is. To show the second claim we recall that the curvature $\widetilde{\mathfrak{K}}$ of the boundary of $T(\Omega)$, in $\zeta \in \partial T(\Omega)$ is given by (see [Nee97, equation (23) pag. 234])

$$\widetilde{\mathfrak{K}}(\zeta) = \frac{1}{|T'(z)|}\left(\Im\left(\frac{t(z)T''(z)}{T'(z)}\right) + \mathfrak{K}(z)\right), \tag{4.8}$$

where $z \in \partial\Omega$ satisifes $T(z) = \zeta$ and $t(z)$ is the unit tangent vector to $\partial\Omega$ in z. Here, since Ω is simply connected we have that its boundary $\partial\Omega$ is a Jordan curve and we assume it is orientated is such a way that the winding number satisfies

$$W_{\partial\Omega}(z) := \frac{1}{2\pi i}\int_{\partial\Omega}\frac{d\mathbf{z}}{\mathbf{z}-z} = \begin{cases} 1 & \text{if } z \in \Omega \\ 0 & \text{if } z \in \mathbb{C}\setminus\overline{\Omega}. \end{cases}$$

Thanks to this convention the unit tangent vector t is uniquely determined by the orientation of $\partial\Omega$. If we write $T := h + ig$ we have

$$T' = h_x - ih_y, \quad \text{and} \quad T'' = h_{xx} - ih_{xy},$$

and then, writing $t = x_t + iy_t$, one has

$$\begin{aligned}
\Im\left(\frac{tT''}{T'}\right) &= \Im\left(\frac{(x_t + iy_t)(h_{xx} - ih_{xy})}{h_x - ih_y}\right) \\
&= \Im\left(\frac{x_t h_{xx} + y_t h_{xy} + i(-x_t h_{xy} + y_t h_{xx})}{h_x - ih_y}\right) \\
&= \frac{x_t(h_y h_{xx} - h_x h_{xy}) + y_t(h_y h_{xy} + h_x h_{xx})}{h_x^2 + h_y^2} \\
&= \frac{y_t(h_x h_{xx} + h_y h_{xy}) - x_t(h_x h_{xy} + h_y h_{yy})}{h_x^2 + h_y^2}.
\end{aligned} \tag{4.9}$$

Taking into account

$$\begin{aligned}
\partial_x f &= \partial_x(h_x^2 + h_y^2) = 2h_x h_{xx} + 2h_y h_{xy}, \\
\partial_y f &= \partial_y(h_x^2 + h_y^2) = 2h_x h_{xy} + 2h_y h_{yy},
\end{aligned}$$

equation (4.9) becomes

$$\Im\left(\frac{tT''}{T'}\right) = \frac{y_t f_x - x_t f_y}{2f} = \frac{\nu\cdot\nabla f}{2f},$$

where $\nu = (y_t, -x_t)$. Finally, the previous equation and (4.8) imply

$$|T'|\widetilde{\mathfrak{K}} = \Im\left(\frac{tT''}{T'}\right) + \mathfrak{K} = \frac{f_\nu + 2\mathfrak{K}f}{2f} \geq 0,$$

where the last inequality holds true by (4.3). Then $\widetilde{\mathfrak{K}} \geq 0$ and $T(\Omega)$ is convex.

Since T is proper, $T' \neq 0$ and $T(\Omega)$ is simply connected, [Gor72, Theorem B] tells us that T is invertible. Finally, the inverse is holomorphic by the Open Mapping Theorem and (iii) follows. $\square$

Remark 4.2.2. In particular T is a conformal map, indeed $|T'|^2 = f > 0$ in $\overline{\Omega}$ and then Ω and $T(\Omega)$ are conformally equivalent.

Remark 4.2.3. The function T can be written in a more explicit way by setting

$$T'(z) = \mathfrak{t}(x+iy) := \frac{1}{\mathfrak{t}(\mathbf{0})} f\left(\frac{x+iy}{2}, \frac{y-ix}{2}\right).$$

See [Sha04, equation (4.3)].

We can now prove Theorem 4.1.2.

Proof of Theorem 4.1.2. Let us denote $\Lambda := T(\Omega)$ with coordinates $\zeta := \xi + i\eta$ and set

$$v(\xi,\eta) := u(\tau(\xi,\eta)),$$

where we recall that $\tau = T^{-1}$. If we write $\tau := \varphi + i\psi$ we have

$$\begin{aligned}
\partial_\xi v &= u_x\varphi_\xi + u_y\psi_\xi,\\
\partial_\eta v &= u_x\varphi_\eta + u_y\psi_\eta,
\end{aligned}$$

and

$$\begin{aligned}
\partial_{\xi\xi} v &= u_{xx}\varphi_\xi^2 + 2u_{xy}\varphi_\xi\psi_\xi + u_{yy}\psi_\xi^2 + u_x\varphi_{\xi\xi} + u_y\psi_{\xi\xi},\\
\partial_{\eta\eta} v &= u_{xx}\varphi_\eta^2 + 2u_{xy}\varphi_\eta\psi_\eta + u_{yy}\psi_\eta^2 + u_x\varphi_{\eta\eta} + u_y\psi_{\eta\eta}.
\end{aligned}$$

Hence by the Cauchy-Riemann equations one has

$$\Delta v = u_{xx}|\nabla\varphi|^2 + 2u_{xy}\nabla\varphi\nabla\psi + u_{yy}|\nabla\psi|^2 + u_x\Delta\varphi + u_y\Delta\psi = \Delta u|\tau'|^2 = -f|\tau'|^2,$$

and then by (i) of Proposition 4.2.1 we get

$$-\Delta v = f|T'|^{-2} = 1,$$

that is v is the solution of the torsion problem in Λ, i.e.

$$\begin{cases} -\Delta v = 1 & \text{in } \Lambda\\ v = 0 & \text{on } \partial\Lambda. \end{cases}$$

Thank to Theorem 1.1.4, v has a unique nondegenerate critical point $(\xi_0,\eta_0) \in \Lambda$ and then $(x_0,y_0) := \tau(\xi_0,\eta_0) \in \Omega$ is the unique critical point of u. To show the nondegeneracy of (x_0,y_0), since $u_x(x_0,y_0) = u_y(x_0,y_0) = 0$ one has

$$\begin{aligned}
\partial_{\xi\xi} v(\tau(\xi_0,\eta_0)) &= u_{xx}\varphi_\xi^2 + 2u_{xy}\varphi_\xi\psi_\xi + u_{yy}\psi_\xi^2,\\
\partial_{\xi\eta} v(\tau(\xi_0,\eta_0)) &= u_{xx}\varphi_\xi\varphi_\eta + u_{xy}(\varphi_\xi\psi_\eta + \varphi_\eta\psi_\xi) + u_{yy}\psi_\xi\psi_\eta,\\
\partial_{\eta\eta} v(\tau(\xi_0,\eta_0)) &= u_{xx}\varphi_\eta^2 + 2u_{xy}\varphi_\eta\psi_\eta + u_{yy}\psi_\eta^2.
\end{aligned}$$

Then

$$\begin{aligned}
\partial_{\xi\xi} v\partial_{\eta\eta} v = u_{xx}^2\varphi_\xi^2\varphi_\eta^2 &+ 2u_{xx}u_{xy}\varphi_\xi^2\varphi_\eta\psi_\eta + u_{xx}u_{yy}\varphi_\xi^2\psi_\eta^2+\\
&+ 2u_{xx}u_{xy}\varphi_\xi\varphi_\eta^2\psi_\xi + 4u_{xy}^2\varphi_\xi\varphi_\eta\psi_\xi\psi_\eta + 2u_{xy}u_{yy}\varphi_\xi\psi_\xi\psi_\eta^2+\\
&+ u_{xx}u_{yy}\varphi_\eta^2\psi_\xi^2 + 2u_{xy}u_{yy}\varphi_\eta\psi_\xi^2\psi_\eta + u_{yy}^2\psi_\xi^2\psi_\eta^2,
\end{aligned}$$

and

$$(\partial_{\xi\eta}v)^2 = u_{xx}^2\varphi_\xi^2\varphi_\eta^2 + u_{xy}^2(\varphi_\xi\psi_\eta + \varphi_\eta\psi_\xi)^2 + u_{yy}^2\psi_\xi^2\psi_\eta^2 + 2u_{xx}u_{yy}\varphi_\xi\varphi_\eta\psi_\xi\psi_\eta + \\ +2u_{xx}u_{xy}\varphi_\xi^2\varphi_\eta\psi_\eta + 2u_{xx}u_{xy}\varphi_\xi\varphi_\eta^2\psi_\xi + 2u_{xy}u_{yy}\varphi_\xi\psi_\xi\psi_\eta^2 + 2u_{xy}u_{yy}\varphi_\eta\psi_\xi^2\psi_\eta.$$

Finally we have that

$$v_{\xi\xi}v_{\eta\eta} - v_{\xi\eta}^2 = (u_{xx}u_{yy} - u_{xy}^2)(\varphi_\xi\psi_\eta - \varphi_\eta\psi_\xi)^2,$$

and since (ξ_0, η_0) is nondegenerate, the same holds true for (x_0, y_0). □

4.2.1 Final remarks

We conclude this section by showing that if at least one between (4.2) and (4.3) does not hold, then Theorem 4.1.2 may fail.

Remark 4.2.4. If $f > 0$ in $\overline{\Omega}$, satisfies (4.2), but does not satisfy (4.3) then the solution of the Poisson problem (4.1) can have more than one critical point.

Indeed in Theorem 1.2.1 it is shown that for any $\delta > 0$, there exists a star-shaped domain $\Omega := \Omega(\delta)$ such that the solution of the torsion problem, i.e. $f \equiv 1$, admits at least two critical points. Moreover one has $\mathfrak{K}_{|\partial\Omega} \geq -\delta$ and it is negative somewhere. Then $f > 0$ in $\overline{\Omega}$, (4.2) is satisfied but (4.3) is not.

Remark 4.2.5. If $f > 0$ in $\overline{\Omega}$, satisfies (4.3), but does not satisfy (4.2) then the solution of the Poisson problem (4.1) can have more than one critical point.

Indeed, as a consequence of [EPW06, Theorem 1.1], one has that if $p > 1$ is large enough there exists a solution u of the following Hénon problem

$$\begin{cases} -\Delta u = (x^2 + y^2)^\alpha |u|^p & \text{in } B \\ u > 0 & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases}$$

with $\alpha > 0$, $B := B_1(\mathbf{0})$, and there exist $q_1, q_2 \in B$ such that

$$\max_{B \setminus \bigcup_{i=1}^2 B_{2\delta}(q_i)} u \leq \frac{\sqrt{e}}{4}, \quad \text{and} \quad \sup_{B_\delta(q_i)} u \geq \frac{\sqrt{e}}{2}, \quad i = 1, 2,$$

for some $0 < \delta < \frac{\operatorname{dist}(q_1, q_2)}{4}$. Then let v be the solution of the torsion problem in B with Dirichlet boundary conditions and set

$$u_\varepsilon := u + \varepsilon v, \quad 0 < \varepsilon < \frac{\sqrt{e}}{4\,\|v\|_\infty},$$

which solves

$$\begin{cases} -\Delta u_\varepsilon = f_\varepsilon(x, y) & \text{in } B \\ u_\varepsilon > 0 & \text{in } B \\ u_\varepsilon = 0 & \text{on } \partial B, \end{cases}$$

with $f_\varepsilon(x, y) := (x^2 + y^2)^\alpha |u|^p + \varepsilon > 0$ in $\overline{\Omega}$. Then $u = 0$ on $\partial\Omega$ implies

$$\frac{1}{2}\frac{\partial f_\varepsilon}{\partial \nu} + \mathfrak{K} f_\varepsilon = 0 + \varepsilon > 0, \quad \text{on } \partial\Omega,$$

that is (4.3) is satisfied. Moreover, it is easy to see that u_ε admits at least one critical point in $B_\delta(q_i)$ for $i = 1, 2$.

Now let $q \in B_\delta(q_1) \setminus \{\mathbf{0}\}$ be a critical point for u such that $u(q) \geq \frac{\sqrt{e}}{2}$. Then one has

$$\begin{aligned} & f_\varepsilon(q)^2 \Delta \log f_\varepsilon(q) \\ & \quad = f_\varepsilon(q)\Delta f_\varepsilon(q) - |\nabla f_\varepsilon(q)|^2 \\ & \quad = -p|q|^{6\alpha}|u(q)|^{3p-1} + 4\varepsilon\alpha^2|q|^{2(\alpha-1)}|u(q)|^p - \varepsilon p|q|^{4\alpha}|u(q)|^{2p-1} < 0, \end{aligned}$$

for ε small enough and then (4.2) is not satisfied.

4.3 The nonlinear problem

In this section we prove Theorem 4.1.5. Hence, let Ω be a fixed bounded and convex domain in $\mathbb{R}^2$, Ω_ε the family of smooth domains converging to Ω as $\varepsilon \to 0$ and u_ε the solution of problem (4.6), as in Theorem 4.1.5.

Proof of Theorem 4.1.5. Since Ω is convex we know that if u is a semi-stable solution of problem (4.5) in Ω, then it admits a unique nondegenerate critical point, we denote it by p_c. Hence, it is enough to show that for all multiindices α, with $|\alpha| \leq 2$ it holds

$$\sup_{\Omega_\varepsilon \cap \Omega} \left| D^\alpha (u_\varepsilon - u) \right| \to 0, \quad \text{for } \varepsilon \to 0. \tag{4.10}$$

Indeed, if $p_\varepsilon \in \Omega_\varepsilon$ is a critical point for u_ε, then $p_\varepsilon \to p_c$ as $\varepsilon \to 0$, and it is a nondenerate maximum thanks to (4.10). Then uniqueness follows from to the convergence to P_c and the nondegeneracy.

We prove (4.10) through several steps.

Step 1: there exists $C > 0$ such that $\|u_\varepsilon\|_{H^2(\Omega_\varepsilon)} \leq C$.
From classical regularity theory one has

$$\|u_\varepsilon\|_{H^{m+2}(\Omega_\varepsilon)} \leq C(\Omega_\varepsilon) \left(\|f(u_\varepsilon)\|_{H^m(\Omega_\varepsilon)} + \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \right),$$

but from the convergence of Ω_ε to Ω one can see that $C(\Omega_\varepsilon)$ does not really depend on ε. Then for $m = 0$ using the assumption $\|u_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq C$ we get the desired claim.

Step 2: $u_\varepsilon \rightharpoonup u$ in $H^1(\Omega^\rho)$, where u is a semi-stable solution of problem (4.5) *in Ω and $\Omega^\rho := \{ (x, y) \in \mathbb{R}^2 \mid (\frac{x}{\rho}, \frac{y}{\rho}) \in \Omega \}$.*
First of all from the convergence of Ω_ε to Ω we can find $\rho > 1$ such that $\Omega_\varepsilon \subseteq \Omega^\rho$. Since Ω_ε are smooth we can consider u_ε defined in Ω^ρ by means of zero extension outside Ω_ε and with a little abuse of notation we still denote such an extension by u_ε. Then from the previous step we have $u_\varepsilon \rightharpoonup u$ in $H^1(\Omega^\rho)$. Then it is easy to see that by means of the dominated convergence theorem it holds

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega f(u)\varphi, \quad \text{for all } \varphi \in C_0^\infty(\Omega),$$

and $u = 0$ on $\partial\Omega$ in trace sense. Finally for any $\xi \in \mathcal{C}_0^\infty(\Omega)$ again the dominated convergence theorem gives

$$\int_\Omega |\nabla\xi|^2 - \int_\Omega f'(u)\xi^2 = \lim_{\varepsilon\to 0} \int_\Omega |\nabla\xi|^2 - \int_\Omega f'(u_\varepsilon)\xi^2 \geq 0.$$

Hence we proved that u is a stable solution of problem (4.5) in Ω.

Step 3: end of the proof
From the convergence of Ω_ε to Ω we can find $r > 0$ and $q_1, \dots, q_k \in \partial\Omega$ such that

$$\partial\Omega \subseteq \bigcup_{i=1}^k B_r(q_i), \quad \text{and} \quad \partial\Omega_\varepsilon \subseteq \bigcup_{i=1}^k B_r(q_i),$$
$$\Omega \cap B_{2r}(q_i) = \{\, (x,y) \in B_{2r}(q_i) \mid y > \Gamma^i(x) \,\},$$
$$\Omega_\varepsilon \cap B_{2r}(q_i) = \{\, (x,y) \in B_{2r}(q_i) \mid y > \Gamma^i(x) + \gamma_\varepsilon^i(x) \,\},$$

where the last two relations hold up to a rotation and where $\Gamma^1, \dots, \Gamma^k, \gamma_\varepsilon^1, \dots, \gamma_\varepsilon^k$ are smooth functions such that

$$\gamma_\varepsilon^i \to 0 \quad \text{in } \mathcal{C}^4, \quad \text{for all } i = 1, \dots, k.$$

Now, let us fix $i = 1$: up to a translation we can assume $q_1 = \mathbf{0}$ and consider

$$\bar{u}_\varepsilon(x,y) := u_\varepsilon(x, y + \gamma_\varepsilon(x)), \quad \text{for all } (x,y) \in \Omega \cap B_{2r},$$

where we omitted the apex i and $B_{2r} := B_{2r}(\mathbf{0})$. Then one has $\bar{u}_\varepsilon = 0$ on $\partial\Omega \cap B_{2r}$ and since $\gamma_\varepsilon \to 0$ in $\mathcal{C}^2$ it holds

$$\sup_{\Omega_\varepsilon\cap\Omega\cap B_{2r}} |D^\alpha(u_\varepsilon - \bar{u}_\varepsilon)| \to 0, \quad \text{for } \varepsilon \to 0,$$

for all multiindices α, with $|\alpha| \leq 2$. Moreover, we have

$$\begin{cases} -\Delta(u - \bar{u}_\varepsilon) = h_\varepsilon & \text{in } \Omega \cap B_{2r} \\ u - \bar{u}_\varepsilon = 0 & \text{on } \partial\Omega \cap B_{2r}, \end{cases}$$

where

$$h_\varepsilon := f(u) - f(\bar{u}_\varepsilon) - 2\partial_{xy} u_{\varepsilon|(x,y+\gamma_\varepsilon(x))}\dot{\gamma}_\varepsilon - \partial_{yy} u_{\varepsilon|(x,y+\gamma_\varepsilon(x))}\dot{\gamma}_\varepsilon^2 - \partial_y u_{\varepsilon|(x,y+\gamma_\varepsilon(x))}\ddot{\gamma}_\varepsilon.$$

For $m \in \mathbb{N}$, by means of the mean value theorem and taking into account that u, u_ε and in turn $\bar{u}_\varepsilon$ are uniformly bounded, we have $|f(u) - f(\bar{u}_\varepsilon)| \leq C|u - \bar{u}_\varepsilon|$, then

$$\|h_\varepsilon\|_{H^m(\Omega\cap B_{2r})} \leq C\left(\|u - \bar{u}_\varepsilon\|_{H^m(\Omega\cap B_{2r})} + \|u_\varepsilon\|_{H^{m+2}(\Omega_\varepsilon)} \|\gamma_\varepsilon\|_{\mathcal{C}^{m+2}}\right).$$

Iterating the argument in *Step 1* and we can find $C > 0$ such that $\|u_\varepsilon\|_{H^{m+2}(\Omega_\varepsilon)} \leq C$ and

$$\|u - \bar{u}_\varepsilon\|_{L^2(\Omega\cap B_{2r})} \leq \|u - u_\varepsilon\|_{L^2(\Omega\cap B_{2r})} + C\|\bar{u}_\varepsilon - u_\varepsilon\|_{L^\infty(\Omega\cap B_{2r})} \to 0, \quad \text{for } \varepsilon \to 0,$$

thanks to the compact embedding of $H^1(\Omega^\rho)$ in $L^2(\Omega^\rho)$. Then classical boundary regularity theory gives $\|u - \bar{u}_\varepsilon\|_{\mathcal{C}^2(\Omega\cap B_r)} \to 0$ for $\varepsilon \to 0$ and in turn

$$\sup_{\Omega_\varepsilon\cap\Omega\cap B_r} |D^\alpha(u_\varepsilon - u)| \to 0, \quad \text{for } \varepsilon \to 0,$$

for all multiindices α, with $|\alpha| \leq 2$. To complete the proof of (4.10) it is enough to repeat the argument for all $i = 1, \dots, k$ and use interior regularity estimates taking into account that $|-\Delta(u - u_\varepsilon)| \leq C|u - u_\varepsilon|$, for some $C > 0$. □

Remark 4.3.1. Here we show that if we assume that the nonlinearity has the form $f(u) = \lambda g(u)$, g is smooth and satisfies (3.2) and (3.3), that are

$$g : \mathbb{R} \to \mathbb{R} \text{ is increasing and convex,}$$
$$g(0) > 0,$$

and $\lambda \in (0, \lambda^*(\Omega))$, then there exists $C > 0$ such that $\|u_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \le C$.

First of all note that, since $\Omega_\varepsilon \to \Omega$ then $\lambda^*(\Omega_\varepsilon) \to \lambda^*(\Omega)$ and then $\lambda < \lambda^*(\Omega_\varepsilon)$ for ε small enough.

Remember that from the convergence of Ω_ε to Ω we can find $\rho > 1$ such that $\Omega_\varepsilon \subseteq \Omega^\rho = \{\, (x, y) \in \mathbb{R}^2 \mid (x/\rho, y/\rho) \in \Omega \,\}$. Moreover under this set of assumptions one has $\lambda^*(\Omega^\rho) > \lambda$ for ε small enough. Hence if we consider the stable solution u^ρ of (4.5) in Ω^ρ - using the convexity of f - we have

$$\begin{cases} -\Delta(u_\varepsilon - u^\rho) = \lambda\,(f(u_\varepsilon) - f(u^\rho)) \le \lambda f'(u_\varepsilon)(u_\varepsilon - u^\rho) & \text{in } \Omega_\varepsilon \\ u_\varepsilon - u^\rho \le 0 & \text{on } \partial\Omega_\varepsilon \end{cases}$$

and then from the stability of u_ε we can apply the maximum principle to deduce $u_\varepsilon \le u^\rho \le \|u^\rho\|_{L^\infty(\Omega^\rho)}$ in Ω_ε.

Chapter 5

Sign-changing solutions: the Dirichlet eigenfunctions

In this chapter we are interested in the study of the number of critical points in the case of sign-changing solutions. Since to our knowledge there are no results in the literature, let us focus on the k-th Dirichlet eigenfunction. In particular, for $m \in \mathbb{N}$, let u_k be the solution of

$$\begin{cases} -\Delta u_k = \lambda_k u_k & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\lambda_k := \lambda_k(-\Delta, \Omega) > 0$ be the corresponding eigenvalue. A fundamental result is the Courant Nodal Domain Theorem, see [CH53], which tells us that u_k has at most k nodal domains. In particular, since the only eigenfunctions that does not change sign is the first one, it follows that u_2 has exactly two nodal domains. In this chapter we are going to prove that the second eigenfunction has exactly two critical points in planar convex domains with large eccentricity. Moreover, if we restrict our attention to a smaller class of domains we are able to establish the exact number of critical point for all the eigenfunctions.

We start the chapter recalling some known results about the nodal line conjecture for the second eigenfunction, then we state and prove our main theorems which can be found in [DRG22b].

5.1 The nodal line conjecture of the second eigenfunction

From now up to the end of this chapter we fix $N = 2$, unless different indication. Here we focus on the second eigenfunction, i.e.

$$\begin{cases} -\Delta u_2 = \lambda_2 u_2 & \text{in } \Omega \\ u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\Omega \subseteq \mathbb{R}^2$ is a given domain.

A first interesting topic concerns the nodal line of the second eigenfunction u_2. It was conjectured that the nodal line $\Lambda := \overline{\{\, \boldsymbol{x} \in \Omega \mid u_2(\boldsymbol{x}) = 0 \,\}}$ of the second eigenfunction in planar domains touches the boundary $\partial\Omega$ at exactly two points. In [Pay67] it was conjectured that it happens in any bounded domain, while in [Yau82] we find the conjecture only for convex domains. Finally in [HOHON97] - after disproving the claim in non simply connected domains, see Theorem 5.1.7 - the authors conjecture that it happens in all bounded planar simply connected domains.

Remark 5.1.1. If you are able to prove the conjecture, we strongly recommend you to read [Ale20, Section 6].

Remark 5.1.2. As pointed out in [Ple56], in the case of Neumann boundary conditions one has $\Lambda \cap \partial\Omega \neq \emptyset$ for any bounded and smooth domain $\Omega \subseteq \mathbb{R}^N$ for all $N \geq 2$: indeed it is an easy consequence of the fact that the second Neumann eigenvalue is smaller then the first Dirichlet one.

We start by recalling a results by Payne where under suitable symmetry assumptions on the domain Ω the nodal line conjecture is proved.

Theorem 5.1.3 ([Pay73])**.** *Let Ω be bounded, convex in x_1 and symmetric about the x_2 axis. Then u_2 cannot have an interior closed nodal curve.*

After this, the conjecture has been showed to be true also by Lin, see [Lin87], and Pütter, see [Pü90]. In both cases different symmetry assumptions are still needed.

In the paper [Lin87], the author also prove that in convex planar domains the multiplicity of the second eigenvalue is at most two.

Later on, another interesting result has been proved by Jerison in [Jer91], where the author considers convex domain without symmetry assumptions, but with large eccentricity, see Definition 5.2.1. We refer to Section 5.2, for other results for this kind of domains.

The nodal line conjecture was finally proved for convex domains by Melas under regularity assumptions on the boundary $\partial\Omega$.

Theorem 5.1.4 ([Mel92])**.** *Let Ω be a bounded convex domain with $\mathcal{C}^\infty$ boundary. Then the nodal line Λ of any second eigenfunctions u_2 must intersect the boundary $\partial\Omega$ at exactly two points.*

Remark 5.1.5. The smoothness assumption of the boundary can be dropped. The improved theorem was proved by Alessandrini in [Ale94]

A generalization of Theorem 5.1.3 to any dimension and to higher eigenfunction can be found in Damascelli [Dam00].

Theorem 5.1.6 ([Dam00])**.** *Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a bounded domain. If Ω is convex and symmetric with respect to m orthogonal directions, $1 \leq m \leq N$, and*

$$\Lambda_{u_k} := \overline{\{\, \boldsymbol{x} \in \Omega \mid u_k(\boldsymbol{x}) = 0 \,\}},$$

is the nodal set of the eigenfunction u_k, then $\Lambda_{u_k} \cap \partial\Omega \neq \emptyset$, for $2 \leq k \leq m+1$.

If $\Omega \subseteq \mathbb{R}^N$, with $N \geq 2$, then the nodal set of the second eigenfunction touches the boundary provided that the domain is convex and the eccentricity is large enough. This has been proved by Jerison in [Jer95c] and no symmetry assumptions are required.

It is also possible to prove the nodal line conjecture in some particular cases of non convex domains. For instance, it has been proved by Freitas and Krejčiřík in [FK08] for, possibly non convex, thin tubes. If the section of the tube is small enough then the conjecture holds. We refer to the original paper for the precise statements.

Moreover, under small perturbations of rectangles one can recover the validity of the nodal line conjecture as showed in [BCM21].

Finally, different symmetry assumptions could guarantee the validity of the nodal line conjecture also for concave or not simply connected domains, for instance see [YG13, Kiw18].

5.1.1 Counterexamples to the nodal line conjecture

We conclude this section by reporting some cases in which the nodal line conjecture does not hold.

First of all, the nodal line conjecture does not hold for general Schrödinger operators. Indeed, in the paper [LN88], Lin and Ni proved that it is possible to find a potential $V : B_1 \to \mathbb{R}$ such that for all $N \geq 2$ all the second eigenfunctions of $-\Delta + V(x)$ in B_1 with zero Dirichlet boundary data are radially symmetric. In particular the nodal line conjecture fails.

The nodal line Λ can be closed in non convex, not simply connected domains. The following result by Hoffmann-Ostenhof, Hoffmann-Ostenhof and Nadirashvili disprove the nodal line conjecture as stated by Payne in [Pay67].

Theorem 5.1.7 ([HOHON97])**.** *Let $0 < R_1 < R_2$ be such that*

$$\lambda_1(-\Delta, B_{R_1}) < \lambda_1(-\Delta, B_{R_2} \setminus \overline{B}_{R_1}) < \lambda_2(-\Delta, B_{R_1}).$$

For $n \in \mathbb{N}$ and $0 < \varepsilon < \pi/n$ let

$$\Omega_{n,\varepsilon} := B_{R_1} \cup B_{R_2} \setminus \overline{B}_{R_1} \cup \bigcup_{j=1}^{n-1} \left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \rho = R_1, \quad \theta \in \left(\frac{2\pi j}{n} - \varepsilon, \frac{2\pi j}{n} + \varepsilon \right) \right\},$$

where $x_1 = \rho \cos\theta$ and $x_2 = \rho \sin\theta$, see Figure 5.1. Then there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ and sufficiently small $\varepsilon = \varepsilon(n)$ the second eigenfunctions u_2 in $\Omega_{n,\varepsilon}$ has closed nodal line. That is $\Lambda \cap \partial\Omega_{n,\varepsilon} = \emptyset$.

The preceding results was generalized to $N \geq 2$ by Fournais [Fou01], see also [Ken13]. Clearly, it would be interesting to understand what is the smallest number of boundary components such that the nodal line is a closed one contained in the domain. In [DGSH21] it is shown that this number is at most 7, but the authors make no claim it is optimal.

Counterexamples to the nodal line conjecture can be found for Robin boundary conditions.

Theorem 5.1.8 ([Ken11])**.** *Fix $M, \beta > 0$. There exists a bounded, connected domain Ω with Lipschitz boundary such that $|\Omega| = M$ and the second Robin eigenfunction, i.e. v_2 solution of the problem*

$$\begin{cases} -\Delta v = \lambda_2 v & \text{in } \Omega \\ \frac{\partial v_2}{\partial \nu} + \beta v = 0 & \text{on } \partial\Omega, \end{cases}$$

is simple and satisfies

$$\{\, x \in \Omega \mid v_2(\boldsymbol{x}) < 0 \,\} \subset\subset \Omega.$$

In particular the nodal line does not touch the boundary of the domain.

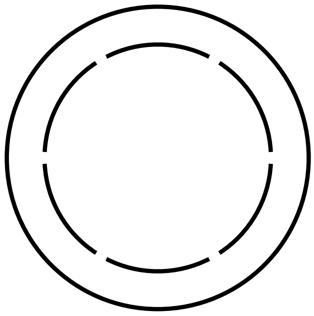

Figure 5.1: The domain $\Omega_{n,\varepsilon}$ in Theorem 5.1.7 with $n = 6$.

If the domain Ω is unbounded the nodal line Λ does not touch the boundary $\partial\Omega$ in general, even if the symmetry assumptions of Theorem 5.1.3 are satisfied, see the results by Freitas and Krejčiřík in [FK07].

Finally, on manifolds the nodal line conjecture fails in general, even for Robin boundary condition as showed by Freitas in [Fre02].

5.2 Large eccentricity domains: main results

In case of planar convex domains with large eccentricity it is possible to have more information about the nodal line. Here we follow the works of Jerison and Grieser and Jerison. From now up to the end of the chapter, the generic point of $\mathbb{R}^2$ will be denoted in coordinate with (x, y).

Let us recall the definition of eccentricity.

Definition 5.2.1. The *eccentricity* of a bounded domain Ω is given by

$$\operatorname{ecc}(\Omega) := \frac{\operatorname{diam}(\Omega)}{\operatorname{inradius}(\Omega)},$$

where

$$\operatorname{inradius}(\Omega) := \max\{r > 0 | B_r(\boldsymbol{x}) \subseteq \Omega, \quad \text{for some } \boldsymbol{x} \in \Omega\},$$

is the *inradius* of the domain.

Convex domains with large eccentricity were considered by Jerison in [Jer95a] and Grieser-Jerison in [GJ96] where the location of the nodal line Λ was characterized. In order to state their result we need to normalize the domain Ω in an appropriate way. First let us rotate Ω so that its projection on the y-axis has the shortest possible length, and then dilate so that this projection has length 1. Denote by M the length of the projection of Ω on the x-axis. Then $M \geq 1$, and M is essentially the diameter of Ω, see Figure 5.4. Moreover, since the inradius is close to 1, we have that also the eccentricity of the domain has order M. From now we denote by Ω_M a domain satisfying the previous properties and accordingly by u_M a solution to (5.1) in $\Omega = \Omega_M$ with Λ_M its nodal line.

Note that in this setting the domain Ω_M, as M grows, is close to the strip $\{ (x, y) \in \mathbb{R}^2 : 0 < y < 1 \}$ (in a suitable way).

In [Jer95a], Jerison studies such a class of domains and gives an estimate of the first two eigenvalues showing that they can be compared to the ones of the associated ordinary operator

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\pi^2}{h(x)^2},$$

with Dirichlet boundary conditions on the projection of Ω on the x-axis and where h denotes the cross section of Ω. Furthermore, he proves that if $\bar{x} \in \mathbb{R}$ is the zero of the second eigenfunction of the Schrödinger operator $-\mathrm{d}^2/\mathrm{d}x^2 + \pi^2/h(x)^2$, then the nodal line Λ lies in $\overline{\Omega} \cap (\bar{x} - K, \bar{x} + K) \times \mathbb{R}$ for some fixed $K > 0$.

This last result has ben improved in the subsequent work [GJ96] by Grieser and Jerison itself.

Theorem 5.2.1 ([GJ96])**.** *There is an absolute constat C such that the width of the nodal line Λ_M is at most C/M. In other words, up to translate Ω_M, one has*

$$(x, y) \in \Lambda_M \implies |x| < \frac{C}{M}.$$

This result is our starting point to compute the number of critical points of u_M in Ω_M. Indeed, since as the eccentricity of the domain grows the nodal line becomes close to a straight line, this implies that the nodal domains are not so far from being convex and then we can expect to have exactly one critical point for any of them. We have the following theorem.

Theorem 5.2.2. *For M large enough, the second eigenfunction u_M has exactly two critical points $P_M, Q_M \in \Omega_M$. Moreover P_M (say) is a nondegenerate maximum point while Q_M is a nondegenerate minimum. Finally $|P_M|, |Q_M| \to +\infty$ as $M \to +\infty$.*

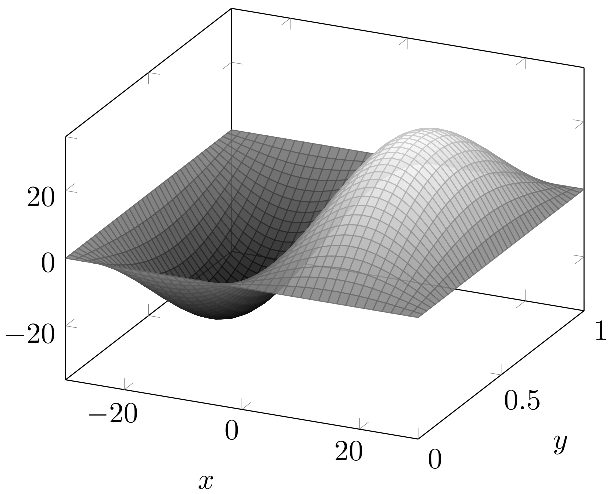

Figure 5.2: A graph of u_M for M large.

The theorem says that if the eccentricity of the domain is big enough (let us recall it has the same order of M), then we are able to compute the exact number of the critical points of the second eigenfunction. Moreover, the distance between the critical points and the nodal line becomes larger and larger as the eccentricity goes to infinity.

The proof of the previous theorem is splitted in two parts. In the first one we deduce, up to a suitable normalization, the convergence on compact sets of the eigenfuction u_M to the "limit" function $u_\infty(x, y) = A_0 x \sin(\pi y)$ where A_0 is a nonzero constant. This will be done combining some results in [GJ96] and [GJ09]. We stress that the choice of the normalization of the eigenfunction u_M is not a trivial issue, as already discussed in [Jer95a] and [GJ96].

The second part of the proof involves a topological argument: similarly to Chapter 2,we introduce the vector field $T : \Omega_M \cap \left\{x > \frac{1}{2}\right\} \to \mathbb{R}^2$

$$T(q) = (u_{yy}(q)u_x(q) - u_{xy}(q)u_y(q), u_{xx}(q)u_y(q) - u_{xy}(q)u_x(q)),$$

$q \in \Omega_M \cap \left\{x > \frac{1}{2}\right\}$, which allows to "count" the critical points of u_M. It will be proved that the vector field T is homotopic to the map $I - (x_0, y_0)$ with $(x_0, y_0) \in \Omega_M \cap \{ x > \frac{1}{2} \}$ (the same will be done in $\Omega_M \cap \{ x < -\frac{1}{2} \}$). This result will give the uniqueness and nondegeneracy of the critical point of u_M in the set where $u_M > 0$ and $u_M < 0$ respectively.
All these computations strongly use the convexity of the domain Ω_M and the convergence of u_M to u_∞. We stress that, although this convergence is only on compact sets, it will be enough to handle the computations in *all* the set Ω_M.

In the second part of the chapter we deal with a particular class of convex domains not included in the previous section, which are perturbation of rectangles converging to the strip. This family of domains has been studied in [GJ09] where Grieser and Jerison give a full asymptotic expansion for the k-th Dirichlet eigenvalue and for the associated eigenfunction (see Theorem 5.4.1 below).

Let $\varphi : [0, 1] \to [0, \infty)$ be a Lipschitz and concave function and for $M \in [0, \infty)$ set

$$\mathcal{R}_M := \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; 0 < y < 1, \; -\varphi(y) < x < M \right\}, \tag{5.2}$$

see Figure 5.3.

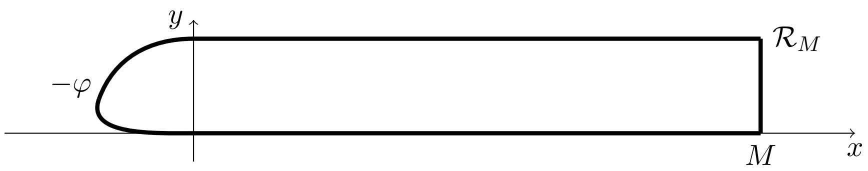

Figure 5.3: The domain $\mathcal{R}_M$.

Let $u_{k,M} \in \mathcal{C}^\infty(\mathcal{R}_M)$ be the k-th Dirichlet eigenfunction in $\mathcal{R}_M$ which solves

$$\begin{cases} -\Delta u_{k,M} = \lambda_{k,M} u_{k,M} & \text{in } \mathcal{R}_M \\ u_{k,M} = 0 & \text{on } \partial\mathcal{R}_M. \end{cases}$$

where $\lambda_{k,M} := \lambda_k(-\Delta, \mathcal{R}_M)$ is the k-th eigenvalue. In next theorem we prove the existence of exactly k critical points for $u_{k,M}$ in $\mathcal{R}_M$.

Theorem 5.2.3. *For M large enough, $u_{k,M}$ has exactly k nondegenerate critical points in the set $\mathcal{R}_M$. Moreover all of them are maxima and minima.*

Unlike Theorem 5.2.2, the proof of Theorem 5.2.3 is much easier and it strongly follows by the estimates proved in [GJ09].

The rest of the chapter is organized as follows: in the next section we prove Theorem 5.2.2, while in Section 5.4 we investigate the eigenfunctions on convex perturbations of long rectangles, proving Theorem 5.2.3.

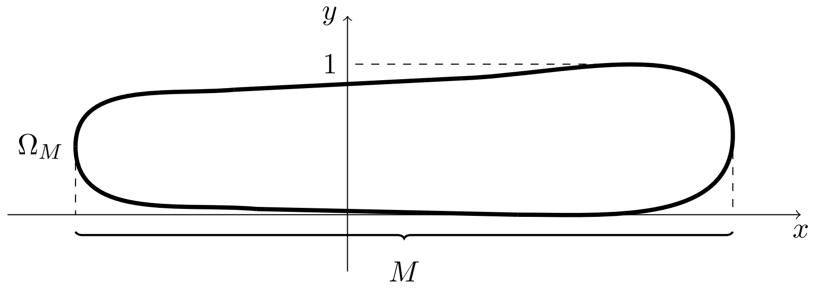

Figure 5.4: The set Ω_M.

5.3 Proof of Theorem 5.2.2

In this section we prove Theorem 5.2.2. To this end, in the next subsection we recall some notations and some results for the second eigenfunction on convex domain with large eccentricity from the papers of Jerison and Grieser, Jerison and in the next one we extrapolate the local convergence of u_M to u_∞ (see Proposition 5.3.4). Subsection 5.3.3 is devoted to the topological argument where we perform the computations involving the vector field T and we finally prove Theorem 5.2.2.

5.3.1 Preliminary results

Here we collect some results proved in [Jer95a, GJ96] (see also [Jer95b] for an overview of the problem). As we pointed out in Section 5.2, let us rotate Ω_M so that its projection on the y-axis has the shortest possible length, then dilate so that this projection has length 1. Denote by M the length of the projection of the domain on the x-axis, then $M \geq 1$, see Figure 5.4. Hence, we write

$$\Omega_M = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; f_{1,M}(x) < y < f_{2,M}(x), \quad x \in (a_M, b_M) \right\},$$

where $b_M - a_M = M$, $0 \leq f_{1,M} \leq f_{2,M} \leq 1$, and the height function of Ω_M is $h_M := f_{2,M} - f_{1,M}$. We require that

$$f_{1,M} \to 0 \text{ and } f_{2,M} \to 1 \text{ in } C^\infty_{loc}(\mathbb{R}) \text{ as } M \to +\infty.$$

By the convexity of Ω_M we have that $f''_{1,M} \leq 0$ and $f''_{2,M} \geq 0$. Our assumptions imply that the set Ω_M "converges" to the strip

$$\Omega_\infty := \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; 0 < y < 1 \right\},$$

(here we prefer to denote the strip by Ω_∞ and not by $\mathcal{S}$ to underline the convergence of Ω_M to it). More precisely we have that for all compact sets $K \subseteq \mathbb{R}^2$ one has $|(\Omega_M \triangle \Omega_\infty) \cap K| \to 0$. As expalined in Section 5.2 we know that the nodal line

$$\Lambda_M := \overline{\{ (x,y) \in \Omega_M \mid u_M(x,y) = 0 \}},$$

is close to the straight line $\{x = 0\}$, up to a translation (see Theorem 5.2.1 in the section above). Finally let $u_M \in \mathcal{C}^\infty(\Omega_M)$ be the solution of

$$\begin{cases} -\Delta u = \lambda_{2,M} u & \text{in } \Omega_M \\ u = 0 & \text{on } \partial\Omega_M, \end{cases} \tag{5.3}$$

wehre $\lambda_{2,M} := \lambda_2(-\Delta, \Omega_M)$. Moreover, for all $(x_0, y_0) \in \Lambda_M \cap \Omega_M$ we can assume that $u_M(x_0+1, y_0) > 0$ and $u_M(x_0-1, y_0) < 0$, that is $u_M > 0$ on the right of the nodal line and u_M is negative on the left.

Finally, let L_M be the length of the longest interval $\mathcal{I}_{L_M} \subseteq (a_M, b_M)$ such that

$$h_M(x) = f_{2,M} - f_{1,M} \geq 1 - \frac{1}{L_M^2}, \quad \text{in } \mathcal{I}_{L_M}.$$

The number L_M is related to the length of the rectangle contained in Ω_M with lowest first eigenvalue and it satisfies the following bounds (see [GJ96, Jer95b])

$$M^{1/3} \leq L_M \leq M. \tag{5.4}$$

For future convenience, we introduce for $n \in \mathbb{R}$ the sets

$$\Omega_M^n := \{\, (x,y) \in \Omega_M \mid -n < x < n \,\},$$

and

$$\Omega_\infty^n := \left\{\, (x,y) \in \mathbb{R}^2 \;\middle|\; -n < x < n,\ 0 < y < 1 \,\right\},$$

where we remember that $\Omega_\infty = \mathbb{R} \times (0,1)$ is the infinite strip of height 1. Since $0 \leq f_{1,M} \leq f_{2,M} \leq 1$, we have that the continuous embedding $H_0^1(\Omega_M) \hookrightarrow H_0^1(\Omega_\infty)$ holds true by means of zero extension outside Ω_M.

An important step to deduce good estimates for the eigenfunction u_M is to choose a correct normalization. So let us define $\widehat{u}_M$ as

$$\widehat{u}_M := L_M \frac{u_M}{||u_M||_\infty}.$$

With a little abuse of notation, in the following we will set

$$\widehat{u}_M = u_M.$$

From the results in [GJ96] we will deduce the following lemma.

Lemma 5.3.1. *There exists a positive constant C independent of M, such that*

$$|u_M(x,y)| \leq C(1+|x|), \quad \textit{for all } (x,y) \in \Omega_M, \tag{5.5}$$

and

$$|u_M(\pm 1, 1/2)| \geq \frac{1}{C}. \tag{5.6}$$

Proof. The first estimate (5.5) is proved in [GJ96, Theorem 4].

To prove (5.6), still recalling [GJ96], define the following function

$$\tilde{u}_M(x,y) := \psi_M(x) \sqrt{\frac{2}{h_M(x)}} \sin\left(\pi \frac{y - f_{1,M}(x)}{h_M(x)}\right),$$

where

$$\psi_M(x) := \sqrt{\frac{2}{h_M(x)}} \int_{f_{1,M}(x)}^{f_{2,M}(x)} \sin\left(\pi \frac{y - f_{1,M}(x)}{h_M(x)}\right) u_M(x,y)\, dy.$$

Note that $\tilde{u}_M(x,y) \sim \sqrt{2} \sin(\pi y) \psi_M(x)$ and $\psi_M(x) \sim \sqrt{2} \int_0^1 \sin(\pi y) u_M(x,y)\, dy$ if x is bounded. Finally, let $v_M := u_M - \tilde{u}_M$.

Now, let $C > 0$ be any positive constant independent from M which may vary in the rest of the proof and recall the following estimates. [GJ96, Equation (26)] tells us

$$|\psi_M(x)| \geq C|x|, \quad -2 < x < 2,$$

and [GJ96, Lemma 5] gives for all $(x, y) \in \Omega_M^2$

$$\begin{aligned}
&|v_M(x,y)| \\
&\leq \sqrt{\frac{2}{h_M(x)}} \sin\left(\pi \frac{y - f_{1,M}(x)}{h_M(x)}\right)\left(1 + |x| \left|\log\left(\sqrt{\frac{2}{h_M(x)}} \sin\left(\pi \frac{y - f_{1,M}(x)}{h_M(x)}\right)\right)\right|\right) L_M^{-3} \\
&\leq \frac{C}{L_M^3}.
\end{aligned}$$

Hence for $(x, y) \in \Omega_M^2$ one has

$$\begin{aligned}
|u_M(x,y)| &= |\tilde{u}_M(x,y) + v_M(x,y)| \\
&\geq \left|\psi_M(x)\sqrt{\frac{2}{h_M(x)}} \sin\left(\pi \frac{y - f_{1,M}(x)}{h_M(x)}\right)\right| - |v_M(x,y)| \\
&\geq |\psi_M(x)| \sin\left(\pi \frac{y - f_{1,M}(x)}{h_M(x)}\right) - \frac{C}{L_M^3} \\
&\geq C|x| \sin\left(\pi \frac{y - f_{1,M}(x)}{h_M(x)}\right) - \frac{C}{L_M^3}.
\end{aligned}$$

Finally, since for $M \to +\infty$ from (5.4) also $L_M \to +\infty$, one has $(\pm 1, (f_{1,M}(1) + f_{2,M}(1))/2) \to (\pm 1, 1/2)$, and then we have

$$\begin{aligned}
|u_M(\pm 1, 1/2)| &= |u_M(\pm 1, (f_{1,M}(1) + f_{2,M}(1))/2)| + o(1) \\
&\geq C|\pm 1|(1 + o(1)) \geq \frac{C}{2}.
\end{aligned}$$

□

Remark 5.3.2. From (5.5) one has

$$\|u_M\|_{L^\infty(\Omega_\infty^n)} \leq C(1 + k), \quad \text{for all } n \in \mathbb{N}. \tag{5.7}$$

The following lemma follows by the standard elliptic regularity theory.

Lemma 5.3.3. *For $m \in \mathbb{N}$, $f \in H^m(\Omega_M^{n+1})$, let $u \in H^1(\Omega_M^{n+1})$ be a weak solution of*

$$\begin{cases} -\Delta u = f & \text{in } \Omega_M^{n+1} \\ u = 0 & \text{on } \partial\Omega_M^{n+1} \setminus \{\, x = \pm(n+1) \,\}. \end{cases}$$

Then for $\delta \in (0,1)$ it holds

$$u \in H^{m+2}(\Omega_M^{n+\delta}),$$

with the estimate

$$\|u\|_{H^{m+2}(\Omega_M^{n+\delta})} \leq C\left(\|f\|_{H^m(\Omega_M^{n+1})} + \|u\|_{L^2(\Omega_M^{n+1})}\right),$$

for some $C > 0$ independent from M.

We point out that the independence from M follows from the convergence of Ω_M to Ω_∞.

5.3.2 The asymptotic behavior of u_M

In this section we study the limiting behavior of the solution u_M on compact sets. In particular, u_M converges to a function which is a solution in the whole strip Ω_∞.

Proposition 5.3.4. *Up to renormalize u_M, we have that for all multiindices α, with $|\alpha| \leq 2$ and fixed $n \in \mathbb{N}$, it holds*

$$\sup_{\overline{\Omega}_M \cap \{-n \leq x \leq n\}} \left| D^\alpha (u_M - A_0 x \sin(\pi y)) \right| = o(1), \quad \text{for } M \to +\infty, \tag{5.8}$$

for some suitable constant $A_0 \neq 0$.

The proof of the previous proposition is a consequence of the next two lemmata.

Lemma 5.3.5. *We have that there exists $u_\infty : \Omega_\infty \to \mathbb{R}$ such that for all multiindices α, with $|\alpha| \leq 2$ and fixed $n \in \mathbb{N}$, up to subsequences, one has*

$$\sup_{\overline{\Omega}_M \cap \{-n \leq x \leq n\}} \left| D^\alpha (u_M - u_\infty) \right| = o(1), \quad \text{for } M \to +\infty,$$

and u_∞ solves

$$\begin{cases} -\Delta u_\infty = \pi^2 u_\infty & \text{in } \Omega_\infty \\ u_\infty = 0 & \text{for } y = 0, 1. \end{cases}$$

Proof. In the proof of the lemma, convergence will be understood up to subsequences.

Fix $n \in \mathbb{N}$. From (5.7) and Lemma 5.3.3 we have

$$\|u_M\|_{H^2\left(\Omega_\infty^{n+\frac{1}{2}}\right)} \leq C(n),$$

for some $C(n) > 0$ and so there exists $u_\infty^n \in H^1\left(\Omega_\infty^{n+\frac{1}{2}}\right)$ such that

$$u_M \rightharpoonup u_\infty^n \quad \text{weakly in } H^1\left(\Omega_\infty^{n+\frac{1}{2}}\right).$$

Let us show that in $\Omega_\infty^{n+\frac{1}{2}}$ we have that $-\Delta u_\infty^n = \pi^2 u_\infty^n$ in weak sense. Indeed, for all $\varphi \in \mathcal{C}_0^\infty\left(\Omega_\infty^{n+\frac{1}{2}}\right)$ one has

$$\begin{aligned} \int_{\Omega_\infty^{n+\frac{1}{2}}} \nabla u_\infty^n \nabla \varphi &= \int_{\Omega_\infty^{n+\frac{1}{2}}} (\nabla u_\infty^n \nabla \varphi + u_\infty^n \varphi) - \int_{\Omega_\infty^{n+\frac{1}{2}}} u_\infty^n \varphi \\ &= \lim_M \int_{\Omega_\infty^{n+\frac{1}{2}}} (\nabla u_M \nabla \varphi + u_M \varphi) - \lim_M \int_{\Omega_\infty^{n+\frac{1}{2}}} u_M \varphi \\ &= \lim_M \int_{\Omega_\infty^{n+\frac{1}{2}}} \nabla u_M \nabla \varphi \\ &= \lim_M \lambda_{2,M} \int_{\Omega_\infty^{n+\frac{1}{2}}} u_M \varphi \\ &= \pi^2 \int_{\Omega_\infty^{n+\frac{1}{2}}} u_\infty^n \varphi. \end{aligned}$$

Moreover, it is not difficult to see that

$$u_\infty^n = 0, \quad \text{on } \partial\Omega_\infty^{n+\frac{1}{2}} \setminus \{ x = \pm(n+\frac{1}{2}) \},$$

and by Lemma 5.3.3 we obtain that $u_\infty^n \in \mathcal{C}^\infty\left(\Omega_\infty^{n+\frac{1}{3}}\right)$.

By (5.6) we deduce that $u_\infty^n \not\equiv 0$ in Ω_∞^n, and from the assumptions on the nodal lines of u_M one has $u_\infty^n(0,y) = 0$ for all $y \in (0,1)$.

Next we show the $\mathcal{C}^2$ convergence up to the boundary of Ω_M^n. Let us start by fixing a point $(x,0)$ with $-n < x < n$. From the assumption on Ω_M we can define the set

$$B(M) := \Omega_M \cap B_r(x,0) = \{ (x,y) \in B_r(x,0) \mid y > f_{1,M}(x) \},$$

for some $r > 0$ suitably small. Then, from the standard regularity theory we deduce that

$$\|u_M - u_\infty^n\|_{C^2(B_{1/2}(M))} \to 0, \quad \text{for } M \to +\infty.$$

where $B_{1/2}(M) := \Omega_M \cap B_{r/2}(x,0)$. To show $\mathcal{C}^2$ convergence in the whole Ω_∞^n it is enough to cover the segments $(-n,n)\times\{0\}$ and $(-n,n)\times\{1\}$ with finitely many balls.
Thus we have proved that for all $n \in \mathbb{N}$ we can find a function $u_\infty^n \in \mathcal{C}^\infty(\Omega_\infty^n)$ such that $u_M \to u_\infty^n$ in $\mathcal{C}^2(\Omega_\infty^n)$ and u_∞^n solves

$$\begin{cases} -\Delta u_\infty^n = \pi^2 u_\infty^n & \text{in } \Omega_\infty^n \\ u_\infty^n = 0 & \text{for } y = 0, 1. \end{cases}$$

By uniqueness of the limit we have $u_\infty^{n+1} = u_\infty^n$ in Ω_∞^n, and this allows us to define a $\mathcal{C}^2$ function in the whole strip Ω_∞ given by

$$u_\infty(x,y) := u_\infty^n(x,y), \quad \text{for } (x,y) \in \Omega_\infty^n,$$

which is a solution of

$$\begin{cases} -\Delta u_\infty = \pi^2 u_\infty & \text{in } \Omega_\infty \\ u_\infty = 0 & \text{for } y = 0, 1. \end{cases}$$

Moreover, from the corresponding properties of u_∞^n, note that $u_\infty(0,y) = 0$ for all $y \in (0,1)$ and $|u_\infty(\pm1, 1/2)| > 0$. □

To conclude the proof of Proposition 5.3.4 we must prove that $u_\infty(x,y) = A_0 x \sin(\pi y)$ for some $A_0 > 0$. This is a consequence of the next lemma.

Lemma 5.3.6. *The functions $u(x,y) = Ax\sin(\pi y)$ are the unique solutions of the problem*

$$\begin{cases} -\Delta u = \pi^2 u & \text{in } \Omega_\infty \\ u(0,y) = 0 & \text{for any } y \in [0,1] \\ u(x,0) = u(x,1) = 0 & \text{for any } x \in \mathbb{R} \\ |u(x,y)| \le C(1+|x|) & \text{for some constant } C > 0, \end{cases} \tag{5.9}$$

for any $A \in \mathbb{R}$.

Proof. Here we follow [GJ09, Lemma 6]. Let $u(x,y)$ be a solution to (5.9). Then for each fixed x its Fourier series is given by

$$u(x,y) = \sum_{j=1}^{\infty} A_j(x) \sin(j\pi y),$$

where

$$A_j(x) := 2\int_0^1 u(x,t) \sin(j\pi t)\, dt, \tag{5.10}$$

that is $A_1(x) = c_1 x + d_1$ and

$$A_j(x) = c_j e^{-\sqrt{j^2-1}\pi x} + d_j e^{\sqrt{j^2-1}\pi x}, \quad \text{for } j \geq 2,$$

with $c_j, d_j \in \mathbb{R}$ for all $j \geq 1$, see [GJ09, Lemma 6] for more details.

Then we evaluate (5.10) for $x = 0$ and taking into account that $u(0,y) = 0$ for all $y \in [0,1]$ we have

$$d_1 = A_1(0) = 2\int_0^1 u(0,y) \sin(\pi y)\, dy = 0,$$

and

$$c_j + d_j = A_j(0) = 2\int_0^1 u(0,y) \sin(j\pi y)\, dy = 0, \tag{5.11}$$

for $j \geq 2$.

By the definition of $A_j(x)$ and since u has growth at most linear we have that $d_j = 0$ for all $j \geq 2$. Hence (5.11) implies $c_j = 0$ for all $j \geq 2$ and then

$$\begin{aligned} u(x,y) &= \sum_{j=1}^{\infty} A_j(x) \sin(j\pi y) = A_1(x) \sin(\pi y) \\ &= (c_1 x + d_1) \sin(\pi y) = c_1 x \sin(\pi y), \end{aligned}$$

and the claim follows. □

Now we are in the position to give the proof of Proposition 5.3.4.

Proof of Proposition 5.3.4. By Lemma 5.3.5 u_M converges up to a subsequence to u_∞, let us show that $u_\infty(x,y) = A_0 x \sin(\pi y)$. First we observe that from inequality (5.5) in Lemma 5.3.1 we know that u_∞ has growth at most linear for $x \to \pm\infty$. Hence Lemma 5.3.6 applies and so $u_\infty(x,y) = Ax\sin(\pi y)$. Finally $A = A_0 = u_\infty(1, 1/2) > 0$. To conclude the proof we need to show that, up to renormalize some u_M the convergence holds for the whole sequence. By contradiction, assume that we can find a subsequence $(u_{M_\ell})_\ell \subseteq (u_M)_M$ not converging to u_∞ and $C > 0$ such that

$$\|u_{M_\ell} - A_0 x \sin(\pi y)\|_{L^\infty(\Omega_{M_\ell} \cap \{-n<x<n\})} \geq C.$$

Now, we can apply Lemma 5.3.5, and in turn Lemma 5.3.6, to the sequence $(u_{M_\ell})_\ell$ to find that, up to subsequences

$$\|u_{M_\ell} - A_1 x \sin(\pi y)\|_{L^\infty(\Omega_{M_\ell} \cap \{-n<x<n\})} \to 0, \quad \text{for } \ell \to +\infty,$$

for some $A_1 > 0$. Hence, up to multiply u_{M_ℓ} by A_0/A_1 we get $u_{M_\ell} \to u_\infty$, a contradiction. □

Remark 5.3.7. A consequence of (5.8) is that $\nabla u \neq 0$ in $\Omega_M \cap \{-1 < x < 1\}$. Note also that by the previous lemmata it is possible to deduce that in $\Lambda_M \cap \partial\Omega_M$ there are two nondegenerate saddle points. Indeed, from Theorem 5.2.1 the nodal line is contained in $\Omega_M \cap \{-1 < x < 1\}$ and [Lin87, Lemma 1.2] tells us that the two points in $\Lambda_M \cap \partial\Omega_M$ are critical points. Moreover, setting $\Lambda_M \cap \partial\Omega_M = \{ q_1, q_2 \}$ we have $q_1 = (o(1), 1 + o(1))$ and $q_2 = (o(1), o(1))$ and then from Proposition 5.3.4, writing $q_i := (x_{q_i}, y_{q_i})$, we get for $i = 1, 2$

$$\partial_{xx} u_M(q_i) = \partial_{xx}\left(A_0 x \sin(\pi y_{q_i})\right) + o(1) = 0 + o(1) = o(1),$$

and similarly one has

$$\begin{aligned}
\partial_{xy} u_M(q_i) &= \partial_{xy}\left(A_0 x \sin(\pi y_{q_i})\right) + o(1) \\
&= A_0 \pi \cos(\pi y_{q_i}) + o(1) = (-1)^i A_0 \pi + o(1), \\
\partial_{yy} u_M(q_i) &= \partial_{yy}\left(A_0 x_{q_i} \sin(\pi y_{q_i})\right) + o(1) = -A_0 \pi^2 x_{q_i} \sin(\pi y_{q_i}) + o(1) = o(1).
\end{aligned}$$

This yields to

$$\det \mathrm{Hess}_{u_M}(q_i) = o(1) - ((-1)^i A_0 \pi)^2 < 0,$$

and the claim follows.

5.3.3 The topological argument

Up to the end of this section let us write u instead of u_M for brevity. Let us use the notations from Section 2.2.

Let us point out that u_θ clearly solves $-\Delta u_\theta = \lambda_{2,M} u_\theta$ in Ω_M. Moreover, if the set $\{ u > c \}$ is smooth then we recall that the curvature of its bounday is given by

$$\mathfrak{K} = -\frac{u_{yy} u_x^2 - 2 u_{xy} u_x u_y + u_{xx} u_y^2}{|\nabla u|^3}.$$

Consider

$$\Omega_M' := \{ (x, y) \in \Omega_M \mid x > 1/2 \}.$$

In the next proposition we recall some properties of the sets M_θ and N_θ in Ω_M'.

Proposition 5.3.8. *We have that for every $\theta \in [0, \pi)$,*

(i) around any $p \in (N_\theta \cap \Omega_M') \setminus M_\theta$ the nodal set N_θ is a smooth curve;

(ii) if $p \in M_\theta \cap \Omega_M'$, then N_θ consists of at least two smooth curves intersecting transversally at p;

(iii) the monotonicity property of Dirichlet eigenvalues with respect to the domain implies that there is no nonempty domain $H \subseteq \Omega_M'$ such that $\partial H \subseteq N_\theta$ (where the boundary of H is considered as a subset of $\mathbb{R}^2$);

(iv) if $p \in (N_\theta \cap \partial(\Omega_M' \cap \Omega_M)) \setminus M_\theta$ by the implicit function theorem one has that around p, N_θ is a smooth curve intersecting $\partial\Omega_M'$ transversally in p.

Proof. See [CC98]. □

The following result tells us that for each $\theta \in [0, \pi)$ the nodal sets of u_θ is a smooth curve without self intersection and every critical point of u is nondegenerate.

Proposition 5.3.9. *For M large enough and for every $\theta \in [0, \pi)$, the nodal set N_θ of the partial derivative u_θ is a smooth curve in $\overline{\Omega'_M}$ without self-intersection which hits $\partial\Omega'_M$ exactly at two points. Moreover at any critical point of u in Ω'_M the Hessian matrix has rank 2.*

Proof. The proof uses Proposition 5.3.8 jointly with Proposition 5.3.4.

From the previous points, if we prove that
$a)$ $M_\theta = \emptyset$ on $N_\theta \cap \partial\Omega'_M$,
and
$b)$ $N_\theta \cap \partial\Omega'_M = \{p_1, p_2\}$,
we have the claim. Indeed if $a)$ and $b)$ hold then we cannot have self-intersections of N_θ otherwise (iii) of Proposition 5.3.8 fails. So $M_\theta = \emptyset$ and this fact jointly with (i) of Proposition 5.3.8 gives the smoothness of N_θ in Ω'_M. In order to prove $a)$ and $b)$ we will show that the following scenario holds:

- If θ is far away from 0 and π then N_θ intersect $\partial\Omega'_M$ exactly at *two* points, one of them belonging to $\partial\Omega_M$ and the other on the straight line $x = \frac{1}{2}$.
- If θ is close to 0 and π then N_θ intersect $\partial\Omega'_M$ exactly at *two* points, both belonging to the straight line $x = \frac{1}{2}$.
- In both cases N_θ intersect $\partial\Omega'_M$ transversely.

Now let us consider the two different situations.

Case 1: $a)$ and $b)$ hold for θ far away from 0 and π.
From the assumptions on Ω_M and taking into account that the curvature $\mathfrak{K}$ is positive, there exist $\delta_i := \delta_i(M) > 0$, with $\delta_i \to 0$ as $M \to +\infty$, for $i = 1, 2$, such that for $\theta \in (\delta_1(M), \pi - \delta_2(M))$ there exists a unique p_1 on $\partial\Omega_M$ with $x > 1/2$ such that the tangent vector of $\partial\Omega'_M$ at p_1 is parallel to e_θ.

It follows that $p_1 \in N_\theta$ and from $\mathfrak{K} > 0$ we get $p_1 \notin M_\theta$. Indeed

$$u_{\theta\theta}(p_1) = u_{tt}(p_1) = \mathfrak{K}(p_1) u_\nu(p_1) \neq 0,$$

where t denotes the unit tangent normal vector, ν the unit exterior vector and $u_\nu(p_1) \neq 0$ by the Hopf boundary lemma. Hence $p \in \left(N_\theta \cap \partial(\Omega'_M \cap \Omega_M)\right) \setminus M_\theta$ and (iv) of Proposition 5.3.8 implies that N_θ is a smooth curve intersecting $\partial(\Omega'_M \cap \Omega_M)$ transversely in p_1.

Next let us show that for $\theta \in (\delta_1(M), \pi - \delta_2(M))$ and $p = (1/2, y)$ we have that N_θ is a singleton. Taking into account (5.8), one has

$$\begin{aligned} 0 = u_\theta &= \cos\theta \partial_x u + \sin\theta \partial_y u \\ &= \cos\theta \partial_x \left(A_0 x \sin(\pi y)\right) + \sin\theta \partial_y \left(A_0 x \sin(\pi y)\right) + o(1) \\ &= A_0 \cos\theta \sin(\pi y) + A_0 \frac{\pi}{2} \sin\theta \cos(\pi y) + o(1), \end{aligned}$$

if and only if

$$\cot\theta = -\frac{\pi}{2} \cot(\pi y)(1 + o(1)),$$

which tells us that, for M sufficiently large, there exists exactly one point $p_2 = (1/2, y_\theta)$ such that $u_\theta(p_2) = 0$. Uniqueness of p_2 follows from $\mathcal{C}^1$ convergence of u_θ

given by Proposition 5.3.4. Moreover similar computations show that $p_2 \notin M_\theta$, indeed

$$\begin{aligned}\partial_x u_\theta &= \cos\theta \partial_{xx} u + \sin\theta \partial_{xy} u \\ &= \cos\theta \partial_{xx} \left(A_0 x \sin(\pi y)\right) + \sin\theta \partial_{xy} \left(A_0 x \sin(\pi y)\right) + o(1) \\ &= A_0 \pi \sin\theta \cos(\pi y) + o(1) \neq 0,\end{aligned}$$

for $y \neq 1/2 + o(1)$. If $y = 1/2 + o(1)$ one has

$$\begin{aligned}\partial_y u_\theta &= A_0 \pi \cos\theta \cos(\pi y) - A_0 \frac{\pi^2}{2} \sin\theta \sin(\pi y) + o(1) \\ &= -A_0 \frac{\pi^2}{2} \sin\theta + o(1) \neq 0.\end{aligned}$$

So $N_\theta \cap \partial\Omega'_M = \{p_1, p_2\}$ and $p_i \notin M_\theta$ for $i = 1, 2$; hence $a)$ and $b)$ hold for $\theta \in (\delta_1(M), \pi - \delta_2(M))$.

Case 2: a) and b) hold for θ close to 0 and π.
According to the notations of the previous case let us consider $\theta \in [0, \delta_1(M)) \cup (\pi - \delta_2(M), \pi)$. So in this case either $\theta \to 0$ or $\theta \to \pi$ as $M \to +\infty$.

Note that here we have that $N_\theta \cap \partial\Omega_M \cap \partial\Omega'_M = \emptyset$ and then we only have to study what happens on the straight line $x = \frac{1}{2}$. Moreover, Remark 5.3.7 implies the existence of at least a critical point in Ω'_M and then $N_\theta \cap \Omega'_M \neq \emptyset$. Since there are no intersections of N_θ with $\Omega_M \cap \partial\Omega'_M$ then necessarily N_θ intersects the straight line $x = \frac{1}{2}$, otherwise ∂N_θ is a closed curve contained in Ω'_M, a contradiction with $iii)$ in Proposition 5.3.8.

Next let us study the intersection of N_θ with $x = \frac{1}{2}$. Recalling that $u(x, y) \sim A_0 x \sin(\pi y)$ we get that $u_\theta(1/2, y) = 0$ if and only if

$$0 = u_\theta(1/2, y) = A_0 \underbrace{\cos\theta}_{\to \pm 1} \sin(\pi y) + \frac{A_0}{2} \underbrace{\sin\theta}_{= o(1)} \cos(\pi y) + o(1),$$

that implies

$$\sin(\pi y) + o(1) = 0,$$

and hence we have two solutions $y_1 = o(1)$ and $y_2 = 1 + o(1)$. Observe that the last equation admits *exactly* two solution by the $\mathcal{C}^1$ convergence of u_θ to $\partial_\theta \left(A_0 x \sin(\pi y)\right)$.
Finally let us show that both points $p_1 = \left(\frac{1}{2}, y_1\right)$ and $p_2 = \left(\frac{1}{2}, y_2\right)$ do not belong to M_θ. Indeed, for M large enough

$$\partial_y u_\theta(p_1) = \frac{A_0}{2} \pi + o(1) \neq 0 \quad \text{and} \quad \partial_y u_\theta(p_2) = -\frac{A_0}{2} \pi + o(1) \neq 0,$$

which shows that $p_1, p_2 \notin M_\theta$ and as before the implicit function theorem tells us that if $x = 1/2$ the nodal set N_θ is a smooth curve intersecting transversely the line $\{x = 1/2\}$ at p_1 and p_2. This ends the *Case* 2.

Hence we proved $a)$ and $b)$ for all $\theta \in [0, \pi)$.

Finally at any critical point of u we have that the Hessian matrix is nondegenerate otherwise we deduce that there exists θ such that $M_\theta \neq \emptyset$ contradicting $a)$. □

As in Chapter 2, for u solution of (5.3), consider the vector field $T : \overline{\Omega'_M} \to \mathbb{R}^2$ given by

$$T(q) := (u_{yy}(q)u_x(q) - u_{xy}(q)u_y(q), u_{xx}(q)u_y(q) - u_{xy}(q)u_x(q)), \quad q \in \Omega'_M.$$

By the smoothness of u we have that T is of class $\mathcal{C}^1$. T satisfies the same properties proved in Section 2.2 and in particular Corollary 2.2.6 holds true. Next we prove the uniqueness of critical point in Ω'_M.

Proposition 5.3.10. *For M large enough u_M has exactly one critical point in the set Ω'_M. In particular it is a nondegenerate maximum point.*

Proof. We want to apply Corollary 2.2.6. First of all note that $T \neq \mathbf{0}$ on $\partial\Omega'_M$. Indeed, in $\partial\Omega'_M \cap \partial\Omega_M$, $T = \mathbf{0}$ implies

$$\begin{aligned} -|\nabla u|^3 \mathfrak{K} &= u_{yy}u_x^2 - 2u_{xy}u_xu_y + u_{xx}u_y^2 \\ &= u_x\left(u_{yy}u_x - u_{xy}u_y\right) + u_y\left(u_{xx}u_{xy} - u_{xy}u_x\right) = 0, \end{aligned}$$

a contradiction with the Hopf boundary lemma and the assumption $\mathfrak{K} > 0$ on $\partial\Omega_M$.

On the other hand, for $p = (1/2, y)$, using (5.8), we have

$$\begin{aligned} u_xu_{yy} - u_yu_{xy} &= \partial_x\left(A_0x\sin(\pi y)\right)\partial_{yy}\left(A_0x\sin(\pi y)\right) + \\ &\quad - \partial_y\left(A_0x\sin(\pi y)\right)\partial_{xy}\left(A_0x\sin(\pi y)\right) + o(1) \\ &= -\frac{A_0^2\pi^2}{2}(1 + o(1)), \end{aligned} \tag{5.12}$$

and then $T \neq \mathbf{0}$.

So the degree of T is well defined and if for $p_0 := \left(1, \frac{1}{2}\right)$ the homotopy

$$\begin{aligned} H : [0,1] \times \overline{\Omega'}_M &\to \mathbb{R}^2 \\ (t, q) &\mapsto tT(q) + (1-t)(q - p_0), \end{aligned}$$

is admissible then we deduce

$$\deg(\Omega'_M, T, \mathbf{0}) = \deg(\Omega'_M, I - p_0, \mathbf{0}) = 1,$$

Assume, by contradiction, that the homotopy H is not admissible. Hence, there exist $\tau \in [0,1]$ and $q := (x_q, y_q) \in \partial\Omega'_M$ such that $H(\tau, q) = \mathbf{0}$, i.e.

$$\begin{cases} \tau(u_{yy}(q)u_x(q) - u_{xy}(q)u_y(q)) = (\tau - 1)(x_q - 1) \\ \tau(u_{xx}(q)u_y(q) - u_{xy}(q)u_x(q)) = (\tau - 1)(y_q - 1/2). \end{cases} \tag{5.13}$$

Then, multiplying the first equation by $u_x(q)$, the second by $u_y(q)$ and summing we get

$$-\tau\mathfrak{K}(q)|\nabla u(q)|^3 = (\tau - 1)[(x_q - 1)u_x(q) + (y_q - 1/2)u_y(q)]. \tag{5.14}$$

We want to show that (5.14) leads to a contradiction. First assume that $q \in \partial\Omega'_M \cap \partial\Omega_M$.

For $(x, y) \in \partial\Omega'_M \cap \partial\Omega_M$ denote by $\nu = (\nu_x, \nu_y)$ the unit normal exterior vector at q (consider ν as the exterior normal to $\partial\Omega_M$ if $x_q = 1/2$). Using that Ω'_M is star-shaped with respect to p_0 and the Hopf boundary lemma we have

$$(x_q - 1)u_x(q) + (y_q - 1/2)u_y(q) = u_\nu(q)[(x_q - 1)\nu_x + (y_q - 1/2)\nu_y] < 0.$$

Since $\mathfrak{K} > 0$ on $\partial\Omega'_M \cap \partial\Omega_M$, from (5.14) we get a contradiction. It follows that $q \notin \partial\Omega'_M \cap \partial\Omega_M$ and then $q = (1/2, y_q)$. From (5.12) and the first line of (5.13) we get

$$-\frac{A_0^2\pi^2}{2}\tau(1+o(1)) = (\tau-1)(1/2-1) = \frac{1-\tau}{2},$$

again a contradiction.

So $\deg(\Omega'_M, T, \mathbf{0}) = 1$ and by Corollary 2.2.6 we get that there exists exactly one critical point in Ω'_M: a maximum with negative definite Hessian. □

Similarly we can prove the following proposition.

Proposition 5.3.11. *For M big enough, u_M has exactly one critical point in the set $\{(x,y) \in \Omega_M | x < -1/2\}$. In particular, it is a nondegenerate minimum point.*

Finally the proof of Theorem 5.2.2 easily follows.

Proof of Theorem 5.2.2. The proof follows from Remark 5.3.7, Proposition 5.3.10 and Proposition 5.3.11. Observe that by the local convergence of u_M to the function $u_\infty(x,y) = A_0 x \sin(\pi y)$ we get that $|P_M|, |Q_M| \to +\infty$. □

5.4 Convex perturbations of rectangles: proof of Theorem 5.2.3

In this section we prove Theorem 5.2.3. We start recalling the asymptotic expansion of $u_{k,M}$ given in [GJ09].

Theorem 5.4.1 ([GJ09, Theorem 1])**.** *There is a number $a := a(\varphi) \in [0, \max\varphi]$ such that for each $k \in \mathbb{N}$ the k-th Dirichlet eigenvalue of $\mathcal{R}_M$, see* (5.2)*, satisfies*

$$\lambda_{k,M} = \pi^2 + \frac{k^2\pi^2}{(M+a(\varphi))^2} + O(M^{-5}), \quad M \to +\infty.$$

In particular, the eigenvalues $\lambda_{1,M}, \dots, \lambda_{k,M}$ of $\mathcal{R}_M$ are simple for M sufficiently large. The suitably rescaled eigenfunction $u_{k,M}$ satisfies, for all multiindices α,

$$\sup_{\substack{x>3\log M\\ 0<y<1}} |D^\alpha(u_{k,M}(x,y) - v_k(x,y))| = O(M^{-3}), \tag{5.15}$$

where

$$v_k(x,y) := \sin\left(k\pi\frac{x+a(\varphi)}{M+a(\varphi)}\right)\sin(\pi y),$$

and

$$\sup_{\substack{x\le 3\log M\\ 0<y<1}} |u_{k,M}(x,y)| = O(M^{-1}\log M).$$

We prove Theorem 5.2.3 for $k = 2$, the general case is a simple generalization as will be clear from the proof, see also Remark 5.4.3. We write $u_M = u_{2,M}$ and $v = v_2$ for brevity.

For future convenience let us set

$$\begin{aligned} x_M &:= \frac{1}{2}(M+a) - a, \\ x_M^+ &:= \frac{1}{4}(M+a) - a, \\ x_M^- &:= \frac{3}{4}(M+a) - a, \\ x_M' &:= \frac{1}{12}(M+a) - a. \end{aligned}$$

Proposition 5.4.2. *For M big enough, the eigenfunction u_M has exactly one nondegenerate maximum point and one nondegenerate minimum point in the set $\mathcal{R}_M \cap \{x > 3\log M\}$.*

Proof. From (5.15) easily follows that u_M has a maximum point close to $(x_M^+, 1/2)$ and a minimum point close to $(x_M^-, 1/2)$. To show that they are the only ones and are nondegenerate, let $p := (x_p, y_p) \in \mathcal{R}_M \cap \{x > 3\log M\}$ be a critical point for u_M.

Then (5.15) implies that there exist a continuous and decreasing function $h : (0,+\infty) \to (0,+\infty)$ such that $\lim_{M\to+\infty} h(M) = 0$ and one of the following occurs

$$p \in B_{h(M)}(x_M^+, 1/2), \tag{5.16}$$
$$p \in B_{h(M)}(x_M^-, 1/2), \tag{5.17}$$
$$p \in B_{h(M)}(x_M, 0) \cap \Omega_M, \tag{5.18}$$
$$p \in B_{h(M)}(x_M, 1) \cap \Omega_M, \tag{5.19}$$
$$p \in B_{h(M)}(N, 0) \cap \Omega_M, \tag{5.20}$$
$$p \in B_{h(M)}(N, 1) \cap \Omega_M. \tag{5.21}$$

Assume (5.16), then from (5.15) one has

$$\begin{aligned} \partial_{xx} u_M(p) &= \partial_{xx} v(p) + O(M^{-3}) \\ &= -\frac{4\pi^2}{(M+a)^2} \sin(\pi/2)\sin(\pi/2)(1+o(1)) = -\frac{4\pi^2}{(M+a)^2}(1+o(1)), \end{aligned}$$

and similarly

$$\partial_{xy} u_M(p) = o(M^{-1}) \quad \text{and} \quad \partial_{yy} u_M(p) = -\pi^2(1+o(1)).$$

Hence p is a nondegenerate maximum point. Moreover, we can find $r > 0$ independent from M such that the following homotopy

$$\begin{aligned} H : [0,1] \times \overline{B_r(x_M^+, 1/2)} &\to \mathbb{R}^2 \\ (t,q) &\mapsto t\nabla u_M(q) + (1-t)\nabla v(q), \end{aligned}$$

is admissible for M big enough. Then

$$\deg(B_r(x_M^+, 1/2), \nabla u_M, \mathbf{0}) = \deg(B_r(x_M^+, 1/2), \nabla v, \mathbf{0}) = 1,$$

shows that there is exactly one critical point satisfying (5.16). If we assume (5.17), by similar computations, we obtain the existence of exactly one nondegenerate minimum point in $B_{h(M)}(x_M^-, 1/2)$.

Now assume (5.18) i.e. $p \in B_{h(M)}(x_M, 0) \cap \mathcal{R}_M$. Then the same computation as before tell us that p is a nondegenerate saddle point, indeed one has

$$\partial_{xx} u_M(p) = o(M^{-2}), \quad \partial_{xy} u_M(p) = -\frac{2\pi^2}{M+a}(1 + o(1)), \quad \partial_{yy} u_M(p) = o(1). \tag{5.22}$$

Now, if $\Lambda_M := \overline{\{(x,y) \in \mathcal{R}_M | u_M(x,y) = 0\}}$ is the nodal line of u_M, let $p_M := (\tilde{x}_M, 0) \in \partial\mathcal{R}_M \cap \Lambda_M$. Since $\mathcal{R}_M$ is convex we know from [Ale94, Theorem 1] that Λ_M intersects $\partial\mathcal{R}_M$ transversally at p_M. In particular $\partial_y u_M(p_M) = 0$ and then p_M is a critical point for u and (5.22) shows that it is a nondegenerate saddle point. Since both p and p_M are nondegenerate we can find $g(M) \in (0, h(M))$ such that $p \in B_{h(M)}(x_M, 0) \setminus \overline{B_{g(M)}(x_M, 0)}$, and for $r > 0$ suitably small and M big enough, since in every critical point in $\omega_M := B_r(x_M, 0) \setminus \overline{B_{g(M)}(x_M, 0)} \cap \Omega_M$ one has

$$\det \operatorname{Hess} u_M = -\left(\frac{2\pi^2}{M+a}\right)^2 (1 + o(1)) < 0,$$

thanks to (5.22), and since at least p belongs to ω_M it follows $\deg(\omega_M, \nabla u_M, \mathbf{0}) \leq -1$ and then

$$-1 \geq \deg(\omega_M, \nabla u_M, \mathbf{0}) = \deg(\omega_M, \nabla v, \mathbf{0}) = 0,$$

a contradiction.

The same argument shows that (5.19), (5.20) and (5.21) cannot occur and the proof is complete. □

Remark 5.4.3. In case $k > 2$, [Ale94, Theorem 1] still ensures that the nodal lines intersect the boundary $\partial\mathcal{R}_M$ transversally at $2k$ different points

Proposition 5.4.4. *For M big enough, u_M has no critical points in the set*

$$\mathcal{R}'_M := \{\, (x,y) \in \mathcal{R}_M \mid x < x'_M \,\}.$$

Proof. Let us point out that, from the estimate (5.15) and since $x'_M < x_M$, it follows $u_M > 0$ in $\mathcal{R}'_M$. By the domain monotonicity for Dirichlet eigenvalues one has $\lambda_1(\mathcal{R}'_M) > \lambda_{2,M}$ and then the operator $-\Delta - \lambda_{2,M}$ satisfies the maximum principle in $\mathcal{R}'_M$. From (5.15) one has for all $y \in (0,1)$

$$\partial_x u_M(x'_M, y) = \frac{2\pi}{M+a} \cos(\pi/6) \sin(\pi y)\,(1 + o(1)) \geq 0.$$

Therefore, $\partial_x u_M \geq 0$ on $\partial\mathcal{R}'_M$ and then the maximum principle gives $\partial_x u_M > 0$ on $\mathcal{R}'_M$. □

Proof of Theorem 5.2.3. The proof is an obvious consequence of Proposition 5.4.2 and Proposition 5.4.4. □

Appendix A

Stable solutions

In this Appendix we recall some well known result about stability of solutions for semilinear elliptic problems. For further details we refer - for instance - to the papers [CR75, MP80], to the books [Ban80, Dup11], and to the references therein.

We consider the following problem

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$ and g is a smooth nonlinearity.

Let us recall recall Definition 1.1.8: a function u is a *(semi-)stable* solution of the preceding problem if the linearized operator at u is positive (nonnegative) definite, i.e. if for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$ one has

$$\int_\Omega |\nabla\varphi|^2 - \int_\Omega g'(u)|\varphi|^2 > (\geq) 0,$$

or equivalently if the first eigenvalue of the linearized operator $-\Delta - g'(u)$ in Ω is positive (non negative).

If we write $G(u) := \int_0^u g(t)\,dt$, then the energy functional associated to the preceding problem is

$$E(u) := \int_\Omega |\nabla u|^2\,dx - \int_\Omega G(u)\,dx,$$

for $u \in H_0^1(\Omega)$, and it is known that the solutions are its critical points, that is they solve

$$0 = E'(u) = \frac{\mathrm{d}}{\mathrm{d}s} E(u + s\varphi)_{|s=0} = \int_\Omega |\nabla u|^2 - \int_\Omega g(u)\varphi, \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(\Omega).$$

Then looking at the stability of the solutions corresponds to study the sign of the second variation of the energy functional

$$E''(u) = \frac{\mathrm{d}^2}{\mathrm{d}s^2} E(u + s\varphi)_{|s=0} = \int_\Omega |\nabla\varphi|^2 - \int_\Omega g'(u)|\varphi|^2, \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(\Omega),$$

and hence u is a (semi-)stable solution if and only if the second variation is positive (non negative).

We now focus on the particular case $g := \lambda f$ where f is smooth and satisfies

$$\begin{gathered} f : \mathbb{R} \to \mathbb{R} \text{ is increasing and convex,} \\ f(0) > 0, \end{gathered}$$

that is we are interested in the following problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{A.1}$$

and $\lambda \in \mathbb{R}$ is positive. Typical examples of functions f satisfying the preceding assumptions are $f(u) = e^u$ and $f(u) = (1+u)^p$, for $p > 1$.

Theorem A.1. There exists $\lambda^* := \lambda^*(\Omega) > 0$ such that for all $\lambda \in (0, \lambda^*)$, problem (A.1) admits a minimal solution u_λ, furthermore it is a classical solution and it is stable. On the other hand, for all $\lambda > \lambda^*$ problem (A.1) admits no solution, neither in weak sense.

Proposition A.1. The following properties holds true:

(i) for $\lambda \in (0, \lambda^*)$, the stable solution u_λ is unique;

(ii) the function
$$\lambda \mapsto u_\lambda$$
is increasing and smooth;

(iii) for $\lambda = \lambda^*$, the function
$$u^*(x) = \lim_{\lambda \to \lambda^*} u_\lambda(x),$$
solves problem (A.1), at least in a weak sense.

The preceding results can be stated in a more general settings, but we just focused on the particular case treated in the thesis.

Remark A.1. Existence and uniqueness of non stable solution of problem (A.1) for $\lambda \in (0, \lambda^*)$ and boundedness of the extremal solution are problems which are not in the interest of this appendix. We just recall that very different situations may occur, in particular depending on the dimension N.

Appendix B

Technical lemmata

Here we prove some technical results needed in Chapter 3. Let us start by showing that there exist coefficients $\alpha_i \in \mathbb{R}$ such that the function introduced in subsection 3.2.1

$$F(t) = \sum_{i=1}^{n} \alpha_i \cosh(\sqrt{\mu_i} t),$$

admits k nondegenerate maxima points.

Lemma B.1. For $k \in \mathbb{N}$ fixed, there exists $n = n(k) \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that the function

$$F(t) = \sum_{i=1}^{n} \alpha_i \cosh(\sqrt{\mu_i} t),$$

admits k nondegenerate maxima points for $\alpha_1 = -1$.

Proof. Let $1 < \tau_1 < \cdots < \tau_k$. For some $n = n(k) \in \mathbb{N}$ consider a polynomial $P(t) = \sum_{j=1}^{n} a_j t^j$ such that

$$\begin{aligned} a_n &= -1 \\ P'(\tau_i) &= 0, \quad \text{for all } i = 1, \ldots, k, \\ P''(\tau_i) &< 0, \quad \text{for all } i = 1, \ldots, k. \end{aligned}$$

Let $0 < t_1 < \cdots < t_k$ be such that $\cosh(t_i) = \tau_i$ for all $i = 1, \ldots, k$ and define $h(t) = P(\cosh(t))$. Then we have

$$h'(t_i) = 0, \qquad h''(t_i) < 0,$$

that is $t_1, \ldots, t_k$ are nondegenerate maximum points for h. Up to a constant, from the binomial formula it is easy to see that for all $m \in \mathbb{N}$

$$(\cosh(t))^m = \sum_{\ell=1}^{m} c(m, \ell) \cosh(\ell t),$$

for suitable $c(m, \ell) > 0$, with $c(m, m) = 1$. Finally, for $\delta = \frac{\mu_0}{8n}$ the function

$$F(t) = \sum_{j=1}^{n} a_j \sum_{\ell=1}^{j} c(j, \ell) \cosh(\delta \ell t),$$

is the function we were looking for. We point out that from the choice of δ, (3.8) is satisfied. $\square$

Now we prove that the critical points of the function

$$q(t) = -\prod_{\ell=1}^{k}(t^2 - t_k^2), \quad \text{with } k \in \mathbb{N},\ \ k \geq 2 \text{ and } 0 < t_1 < \cdots < t_k,$$

are nondegenerate.

Lemma B.2. Let $q(t) = -\prod_{\ell=1}^{k}(t^2 - t_k^2)$ with $k \in \mathbb{N}$, $k \geq 2$ and $0 < t_1 < \cdots < t_k$. Then the critical points of q are nondegenerate.

Proof. Let $k > 2$ (the case $k = 2$ is left to the reader). A straightforward computation shows that $q'(0) = 0$ and $q''(0) \neq 0$. Now let $\tau \neq 0$ be such that $q'(\tau) = 0$. Of course $q(\tau) \neq 0$ and

$$0 = q'(\tau) = -2\tau \sum_{\ell=1}^{k} \prod_{\substack{h=1 \\ h\neq \ell}}^{k} (\tau^2 - t_h^2),$$

Finally, one has

$$\begin{aligned}
q''(\tau) &= -4\tau^2 \sum_{\ell=1}^{k} \sum_{\substack{h=1 \\ h\neq\ell}}^{k} \prod_{\substack{m=1 \\ m\neq\ell \\ m\neq h}}^{k} (\tau^2 - t_m^2) \\
&= -4\tau^2 \sum_{\ell=1}^{k} \frac{1}{(\tau^2 - t_\ell^2)} \sum_{\substack{h=1 \\ h\neq\ell}}^{k} \prod_{\substack{m=1 \\ m\neq h}}^{k} (\tau^2 - t_m^2) \\
&= -4\tau^2 \sum_{\ell=1}^{k} \frac{1}{(\tau^2 - t_\ell^2)} \left[\underbrace{\sum_{h=1}^{k} \prod_{\substack{m=1 \\ m\neq h}}^{k} (\tau^2 - t_m^2)}_{=0 \text{ since } q'(\tau)=0} - \prod_{\substack{m=1 \\ m\neq\ell}}^{k} (\tau^2 - t_m^2) \right] \\
&= 4\tau^2 \sum_{\ell=1}^{k} \frac{1}{(\tau^2 - t_\ell^2)} \prod_{\substack{m=1 \\ m\neq\ell}}^{k} (\tau^2 - t_m^2) \\
&= -4\tau^2 q(\tau) \sum_{\ell=1}^{k} \frac{1}{(\tau^2 - t_\ell^2)^2} \neq 0.
\end{aligned}$$

□

We conclude with the proof of Lemma 3.3.5 from Section 3.3, that is the formula for the mean curvature of $\Sigma = F^{-1}(0)$, where $F \in \mathcal{C}^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, 0 is a regular value for F and $F_{y_i y_j} = 0$ for all $i \neq j$.

Proof of Lemma 3.3.5 . Let $\Phi = \frac{1}{|\nabla F|}$ and consider the normal field

$$\mathbf{N} = -\Phi \cdot (F_x, F_{y_1}, \ldots, F_{y_N}).$$

Then the mean curvature of Σ is given by

$$\mathfrak{K}(p) = \frac{1}{N}\operatorname{tr}(d\mathbf{N}_p).$$

Taking into account that

$$\begin{aligned}
\Phi_x &= -\Phi^3 \left(F_x F_{xx} + \sum_{j=1}^{N} F_{y_j} F_{xy_j} \right), \\
\Phi_{y_j} &= -\Phi^3 \left(F_x F_{xy_j} + F_{y_j} F_{y_j y_j} \right),
\end{aligned}$$

one has

$$\begin{aligned}
-\mathrm{tr}(d\mathbf{N}_p) &= \Phi \Delta F + \Phi_x F_x + \sum_{j=1}^{N} \Phi_{y_j} F_{y_j} \\
&= \Phi^3 \left[|\nabla F|^2 \left(F_{xx} + \sum_{j=1}^{N} F_{y_j y_j} \right) - \left(F_x F_{xx} + \sum_{j=1}^{N} F_{y_j} F_{xy_j} \right) F_x \right. \\
&\quad \left. - \sum_{j=1}^{N} \left(F_x F_{xy_j} + F_{y_j} F_{y_j y_j} \right) F_{y_j} \right] \\
&= \Phi^3 \left[\sum_{j=1}^{N} \left(F_x^2 F_{y_j y_j} - 2 F_x F_{y_j} F_{xy_j} + F_{y_j}^2 F_{xx} \right) + \sum_{j=1}^{N} F_{y_j}^2 \sum_{\substack{\ell=1 \\ \ell \neq j}}^{N} F_{y_\ell y_\ell} \right]
\end{aligned}$$

which yields the claim. □

Bibliography

[ACH20] B. Andrews, J. Clutterbuck, and D. Hauer. Non-concavity of the Robin ground state. *Camb. J. Math.*, 8(2):243–310, 2020.

[Ale87] G. Alessandrini. Critical points of solutions of elliptic equations in two variables. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 14(2):229–256 (1988), 1987.

[Ale94] G. Alessandrini. Nodal lines of eigenfunctions of the fixed membrane problem in general convex domains. *Comment. Math. Helv.*, 69(1):142–154, 1994.

[Ale20] G. Alessandrini. A small collection of open problems. *Rend. Istit. Mat. Univ. Trieste*, 52:591–600, 2020.

[ALL97] O. Alvarez, J.-M. Lasry, and P.-L. Lions. Convex viscosity solutions and state constraints. *J. Math. Pures Appl. (9)*, 76(3):265–288, 1997.

[AM92] G. Alessandrini and R. Magnanini. The index of isolated critical points and solutions of elliptic equations in the plane. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 19(4):567–589, 1992.

[AM07] A. Ambrosetti and A. Malchiodi. *Nonlinear analysis and semilinear elliptic problems*, volume 104 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2007.

[APP81] A. Acker, L. E. Payne, and G. Philippin. On the convexity of level lines of the fundamental mode in the clamped membrane problem, and the existence of convex solutions in a related free boundary problem. *Z. Angew. Math. Phys.*, 32(6):683–694, 1981.

[Ban80] C. Bandle. *Isoperimetric inequalities and applications.* Pitman Boston, 1980.

[BCM21] T. Beck, Y. Canzani, and J. Marzuola. Nodal line estimates for the second Dirichlet eigenfunction. *J. Spectr. Theory*, 11(1):323–353, 2021.

[BL76] H. J. Brascamp and E. H. Lieb. On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Functional Analysis*, 22(4):366–389, 1976.

[BLS09] C. Bianchini, M. Longinetti, and P. Salani. Quasiconcave solutions to elliptic problems in convex rings. *Indiana Univ. Math. J.*, 58(4):1565–1589, 2009.

[CC98] X. Cabré and S. Chanillo. Stable solutions of semilinear elliptic problems in convex domains. *Selecta Math. (N.S.)*, 4(1):1–10, 1998.

[CF85a] L. A. Caffarelli and A. Friedman. Convexity of solutions of semilinear elliptic equations. *Duke Math. J.*, 52(2):431–456, 1985.

[CF85b] L. A. Caffarelli and A. Friedman. Partial regularity of the zero-set of solutions of linear and superlinear elliptic equations. *J. Differential Equations*, 60(3):420–433, 1985.

[CF21] G. Crasta and I. Fragalà. Concavity properties of solutions to Robin problems. *Camb. J. Math.*, 9(1):177–212, 2021.

[CH53] R. Courant and D. Hilbert. *Methods of mathematical physics. Vol. I.* Interscience Publishers, Inc., New York, N.Y., 1953.

[CR75] M. G. Crandall and P. H. Rabinowitz. Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems. *Arch. Rational Mech. Anal.*, 58(3):207–218, 1975.

[CS82] L. A. Caffarelli and J. Spruck. Convexity properties of solutions to some classical variational problems. *Comm. Partial Differential Equations*, 7(11):1337–1379, 1982.

[CS03] A. Colesanti and P. Salani. Quasi-concave envelope of a function and convexity of level sets of solutions to elliptic equations. *Math. Nachr.*, 258:3–15, 2003.

[CS06] P. Cuoghi and P. Salani. Convexity of level sets for solutions to nonlinear elliptic problems in convex rings. *Electron. J. Differential Equations*, pages No. 124, 12, 2006.

[CS07] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.

[Dam00] L. Damascelli. On the nodal set of the second eigenfunction of the Laplacian in symmetric domains in $\mathbb{R}^n$. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 11(3):175–181 (2001), 2000.

[Dei85] K. Deimling. *Nonlinear functional analysis*. Springer-Verlag, Berlin, 1985.

[DGSH21] J. Dahne, J. Gómez-Serrano, and K. Hou. A counterexample to Payne's nodal line conjecture with few holes. *Commun. Nonlinear Sci. Numer. Simul.*, 103:Paper No. 105957, 13, 2021.

[DK93] J. I. Diaz and B. Kawohl. On convexity and starshapedness of level sets for some nonlinear elliptic and parabolic problems on convex rings. *J. Math. Anal. Appl.*, 177(1):263–286, 1993.

[DLT18] H. Deng, H. Liu, and L. Tian. Critical points of solutions to a quasilinear elliptic equation with nonhomogeneous Dirichlet boundary conditions. *J. Differential Equations*, 265(9):4133–4157, 2018.

[DNPV12] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[DRG22a] F. De Regibus and M. Grossi. On the number of critical points of stable solutions in bounded strip-like domains. *J. Differential Equations*, 306:1–27, 2022.

[DRG22b] F. De Regibus and M. Grossi. On the number of critical points of the second eigenfunction of the Laplacian in convex planar domains. *J. Funct. Anal.*, 283(1):Paper No. 109496, 2022.

[DRGM21] F. De Regibus, M. Grossi, and D. Mukherjee. Uniqueness of the critical point for semi-stable solutions in $\mathbb{R}^2$. *Calc. Var. Partial Differential Equations*, 60(1):25, 2021.

[Dup11] L. Dupaigne. *Stable solutions of elliptic partial differential equations*, volume 143 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2011.

[EMG04] K. El Mehdi and M. Grossi. Asymptotic estimates and qualitative properties of an elliptic problem in dimension two. *Adv. Nonlinear Stud.*, 4(1):15–36, 2004.

[EPW06] P. Esposito, A. Pistoia, and J. Wei. Concentrating solutions for the Hénon equation in $\mathbb{R}^2$. *J. Anal. Math.*, 100:249–280, 2006.

[Eva10] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[FK07] P. Freitas and D. Krejčiřík. Unbounded planar domains whose second nodal line does not touch the boundary. *Math. Res. Lett.*, 14(1):107–111, 2007.

[FK08] P. Freitas and D. Krejčiřík. Location of the nodal set for thin curved tubes. *Indiana Univ. Math. J.*, 57(1):343–375, 2008.

[Fou01] S. Fournais. The nodal surface of the second eigenfunction of the Laplacian in $\mathbb{R}^D$ can be closed. *J. Differential Equations*, 173(1):145–159, 2001.

[Fre02] P. Freitas. Closed nodal lines and interior hot spots of the second eigenfunction of the Laplacian on surfaces. *Indiana Univ. Math. J.*, 51(2):305–316, 2002.

[Gab57] R. M. Gabriel. A result concerning convex level surfaces of 3-dimensional harmonic functions. *J. London Math. Soc.*, 32:286–294, 1957.

[GG04a] F. Gladiali and M. Grossi. Some results for the Gelfand's problem. *Comm. Partial Differential Equations*, 29(9-10):1335–1364, 2004.

[GG04b] F. Gladiali and M. Grossi. Strict convexity of level sets of solutions of some nonlinear elliptic equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 134(2):363–373, 2004.

[GG22] F. Gladiali and M. Grossi. On the number of critical points of solutions of semilinear equations in $\mathbb{R}^2$. *Amer. J. Math.*, 144(5):1221–1240, 2022.

[GJ96] D. Grieser and D. Jerison. Asymptotics of the first nodal line of a convex domain. *Invent. Math.*, 125(2):197–219, 1996.

[GJ98] D. Grieser and D. Jerison. The size of the first eigenfunction of a convex planar domain. *J. Amer. Math. Soc.*, 11(1):41–72, 1998.

[GJ09] D. Grieser and D. Jerison. Asymptotics of eigenfunctions on plane domains. *Pacific J. Math.*, 240(1):109–133, 2009.

[GL20] M. Grossi and P. Luo. On the number and location of critical points of solutions of nonlinear elliptic equations in domains with a small hole. *preprint ArXiv*, 2003.03643, 2020.

[GM03] M. Grossi and R. Molle. On the shape of the solutions of some semilinear elliptic problems. *Commun. Contemp. Math.*, 5(1):85–99, 2003.

[GNN79] B. Gidas, W. M. Ni, and L. Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.*, 68(3):209–243, 1979.

[Gor72] W. B. Gordon. On the diffeomorphisms of Euclidean space. *Amer. Math. Monthly*, 79:755–759, 1972.

[GP93] A. Greco and G. Porru. Convexity of solutions to some elliptic partial differential equations. *SIAM J. Math. Anal.*, 24(4):833–839, 1993.

[Gre09] A. Greco. Extremality conditions for the quasi-concavity function and applications. *Arch. Math. (Basel)*, 93(4):389–398, 2009.

[Gro21] M. Grossi. On the number of critical points of solutions of semilinear elliptic equations. *Electron. Res. Arch.*, 29(6):4215–4228, 2021.

[GT01] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[GT10] M. Grossi and F. Takahashi. Nonexistence of multi-bubble solutions to some elliptic equations on convex domains. *J. Funct. Anal.*, 259(4):904–917, 2010.

[HNS16] F. Hamel, N. Nadirashvili, and Y. Sire. Convexity of level sets for elliptic problems in convex domains or convex rings: two counterexamples. *Amer. J. Math.*, 138(2):499–527, 2016.

[HOHON97] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and N. Nadirashvili. The nodal line of the second eigenfunction of the Laplacian in $\mathbb{R}^2$ can be closed. *Duke Math. J.*, 90(3):631–640, 1997.

[Jer91] D. Jerison. The first nodal line of a convex planar domain. *Internat. Math. Res. Notices*, (1):1–5, 1991.

[Jer95a] D. Jerison. The diameter of the first nodal line of a convex domain. *Ann. of Math. (2)*, 141(1):1–33, 1995.

[Jer95b] D. Jerison. Eigenfunctions and harmonic functions in convex and concave domains. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 1108–1117. Birkhäuser, Basel, 1995.

[Jer95c] D. Jerison. The first nodal set of a convex domain. In *Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991)*, volume 42 of *Princeton Math. Ser.*, pages 225–249. Princeton Univ. Press, Princeton, NJ, 1995.

[Jos13] J. Jost. *Partial differential equations*, volume 214 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2013.

[Kaw83] B. Kawohl. A geometric property of level sets of solutions to semilinear elliptic Dirichlet problems. *Applicable Anal.*, 16(3):229–233, 1983.

[Kaw85a] B. Kawohl. *Rearrangements and convexity of level sets in PDE*, volume 1150 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1985.

[Kaw85b] B. Kawohl. When are solutions to nonlinear elliptic boundary value problems convex? *Comm. Partial Differential Equations*, 10(10):1213–1225, 1985.

[Kaw86] B. Kawohl. A remark on N. Korevaar's concavity maximum principle and on the asymptotic uniqueness of solutions to the plasma problem. *Math. Methods Appl. Sci.*, 8(1):93–101, 1986.

[Kaw94] B. Kawohl. Remarks on some old and current eigenvalue problems. In *Partial differential equations of elliptic type (Cortona, 1992)*, Sympos. Math., XXXV, pages 165–183. Cambridge Univ. Press, Cambridge, 1994.

[Kea85] G. Keady. The power concavity of solutions of some semilinear elliptic boundary value problems. *Bull. Austral. Math. Soc.*, 31(2):181–184, 1985.

[Ken85] A. U. Kennington. Power concavity and boundary value problems. *Indiana Univ. Math. J.*, 34(3):687–704, 1985.

[Ken88] A. U. Kennington. Convexity of level curves for an initial value problem. *J. Math. Anal. Appl.*, 133(2):324–330, 1988.

[Ken11] J. B. Kennedy. The nodal line of the second eigenfunction of the Robin Laplacian in $\mathbb{R}^2$ can be closed. *J. Differential Equations*, 251(12):3606–3624, 2011.

[Ken13] J. B. Kennedy. Closed nodal surfaces for simply connected domains in higher dimensions. *Indiana Univ. Math. J.*, 62(3):785–798, 2013.

[Kim06] S.-C. Kim. On the location of critical point for the Poisson equation in plane. *J. Math. Anal. Appl.*, 321(1):213–222, 2006.

[Kiw18] R. Kiwan. On the nodal set of a second Dirichlet eigenfunction in a doubly connected domain. *Ann. Fac. Sci. Toulouse Math. (6)*, 27(4):863–873, 2018.

[KL87] N. J. Korevaar and J. L. Lewis. Convex solutions of certain elliptic equations have constant rank Hessians. *Arch. Rational Mech. Anal.*, 97(1):19–32, 1987.

[Kor83a] N. J. Korevaar. Capillary surface continuity above irregular domains. *Comm. Partial Differential Equations*, 8(3):213–245, 1983.

[Kor83b] N. J. Korevaar. Convex solutions to nonlinear elliptic and parabolic boundary value problems. *Indiana Univ. Math. J.*, 32(4):603–614, 1983.

[Kor90] N. J. Korevaar. Convexity of level sets for solutions to elliptic ring problems. *Comm. Partial Differential Equations*, 15(4):541–556, 1990.

[Kul17] T. Kulczycki. On concavity of solutions of the Dirichlet problem for the equation $(-\Delta)^{1/2}\varphi = 1$ in convex planar regions. *J. Eur. Math. Soc. (JEMS)*, 19(5):1361–1420, 2017.

[Lin87] C. S. Lin. On the second eigenfunctions of the Laplacian in $\mathbb{R}^2$. *Comm. Math. Phys.*, 111(2):161–166, 1987.

[Lin94] C. S. Lin. Uniqueness of least energy solutions to a semilinear elliptic equation in $\mathbb{R}^2$. *Manuscripta Math.*, 84(1):13–19, 1994.

[Lio81] P.-L. Lions. Two geometrical properties of solutions of semilinear problems. *Applicable Anal.*, 12(4):267–272, 1981.

[Llo78] N. G. Lloyd. *Degree theory.* Cambridge Tracts in Mathematics, No. 73. Cambridge University Press, Cambridge-New York-Melbourne, 1978.

[LN88] C. S. Lin and W. M. Ni. A counterexample to the nodal domain conjecture and a related semilinear equation. *Proc. Amer. Math. Soc.*, 102(2):271–277, 1988.

[LV08] K.-a. Lee and J. L. Vázquez. Parabolic approach to nonlinear elliptic eigenvalue problems. *Adv. Math.*, 219(6):2006–2028, 2008.

[Mag16] R. Magnanini. An introduction to the study of critical points of solutions of elliptic and parabolic equations. *Rend. Istit. Mat. Univ. Trieste*, 48:121–166, 2016.

[Maj] P. Majer. Implicit function theorem at a singular point? MathOverflow. URL:https://mathoverflow.net/q/85529 (version: 2019-08-04).

[Mel92] A. D. Melas. On the nodal line of the second eigenfunction of the Laplacian in $\mathbb{R}^2$. *J. Differential Geom.*, 35(1):255–263, 1992.

[Mil65] J. W. Milnor. *Topology from the differentiable viewpoint.* University Press of Virginia, Charlottesville, Va., 1965.

[ML71] L. G. Makar-Limanov. The solution of the Dirichlet problem for the equation $\Delta u = -1$ in a convex region. *Mat. Zametki*, 9:89–92, 1971.

[MP80] F. Mignot and J. P. Puel. Sur une classe de problèmes non linéaires avec non linéairité positive, croissante, convexe. *Comm. Partial Differential Equations*, 5(8):791–836, 1980.

[MP21] R. Magnanini and G. Poggesi. The location of hot spots and other extremal points. *Math. Ann.*, 2021.

[Nee97] T. Needham. *Visual complex analysis.* The Clarendon Press, Oxford University Press, New York, 1997.

[Pay67] L. E. Payne. Isoperimetric inequalities and their applications. *SIAM Rev.*, 9:453–488, 1967.

[Pay73] L. E. Payne. On two conjectures in the fixed membrane eigenvalue problem. *Z. Angew. Math. Phys.*, 24:721–729, 1973.

[Ple56] Å. Pleijel. Remarks on Courant's nodal line theorem. *Comm. Pure Appl. Math.*, 9:543–550, 1956.

[Pü90] R. Pütter. On the nodal lines of second eigenfunctions of the fixed membrane problem. *Comment. Math. Helv.*, 65(1):96–103, 1990.

[RO16] X. Ros-Oton. Nonlocal elliptic equations in bounded domains: a survey. *Publ. Mat.*, 60(1):3–26, 2016.

[ROS14a] X. Ros-Oton and J. Serra. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl. (9)*, 101(3):275–302, 2014.

[ROS14b] X. Ros-Oton and J. Serra. The extremal solution for the fractional Laplacian. *Calc. Var. Partial Differential Equations*, 50(3-4):723–750, 2014.

[Sak87] S. Sakaguchi. Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 14(3):403–421 (1988), 1987.

[Sak90] S. Sakaguchi. Uniqueness of the critical point of the solutions to some semilinear elliptic boundary value problems in $\mathbb{R}^2$. *Trans. Amer. Math. Soc.*, 319(1):179–190, 1990.

[Sal05] P. Salani. Starshapedness of level sets of solutions to elliptic PDEs. *Appl. Anal.*, 84(12):1185–1197, 2005.

[Sha04] W. T. Shaw. Recovering holomorphic functions from their real or imaginary parts without the Cauchy-Riemann equations. *SIAM Rev.*, 46(4):717–728, 2004.

[Spe75] R. P. Sperb. Extension of two theorems of Payne to some nonlinear Dirichlet problems. *Z. Angew. Math. Phys.*, 26(6):721–726, 1975.

[Str08] M. Struwe. *Variational methods*, volume 34. Springer-Verlag, Berlin, fourth edition, 2008.

[STT20] Y. Sire, S. Terracini, and G. Tortone. On the nodal set of solutions to degenerate or singular elliptic equations with an application to s-harmonic functions. *J. Math. Pures Appl. (9)*, 143:376–441, 2020.

[SV12] R. Servadei and E. Valdinoci. Mountain pass solutions for non-local elliptic operators. *J. Math. Anal. Appl.*, 389(2):887–898, 2012.

[SV13] R. Servadei and E. Valdinoci. Variational methods for non-local operators of elliptic type. *Discrete Contin. Dyn. Syst.*, 33(5):2105–2137, 2013.

[Wet11] T. Weth. On the lack of directional quasiconcavity of the fundamental mode in the clamped membrane problem. *Arch. Math. (Basel)*, 97(4):365–372, 2011.

[Yau82] S. T. Yau. Problem section. In *Seminar on Differential Geometry*, volume 102 of *Ann. of Math. Stud.*, pages 669–706. Princeton Univ. Press, Princeton, N.J., 1982.

[YG13] D.-H. Yang and B.-Z. Guo. On nodal line of the second eigenfunction of the Laplacian over concave domains in $\mathbb{R}^2$. *J. Syst. Sci. Complex.*, 26(3):483–488, 2013.