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The impact of ambiguity on dynamic portfolio selection in the epsilon-contaminated binomial market model

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ABSTRACT

We consider dynamic portfolio selection under ambiguity in the classical multi-period binomial market model. Ambiguity is incorporated in the real-world probability measure through an epsilon-contamination, that gives rise to a completely monotone capacity conveying a pessimistic investor's ambiguous beliefs. The dynamic portfolio selection problem is formulated as a Choquet expected utility maximization problem on the final wealth. Then, the optimal final wealth is proved to be a function of the final stock price: this allows a dimension reduction of the problem, switching from an exponential to a linear size with respect to the number of periods. Finally, an explicit characterization of the optimal final wealth is given in the case of a constant relative risk aversion utility function and the interaction between the ambiguity and the relative risk aversion parameters is investigated.

1. Introduction

Classical financial market models (see, e.g., Černý, 2009; Munk, 2013; Pliska, 1997) deal with uncertainty by referring to a single probability measure $\mathbf{P}$, that encodes the beliefs of market agents, assumed to share the same probabilistic opinions. The probability measure $\mathbf{P}$ has a fundamental role since it is used for computing the expected utility functional, that represents agents' preferences on random payoffs. Nevertheless, the presence of unobserved variables, partial information, or misspecified variables often makes it no longer possible to handle uncertainty through a single probability measure $\mathbf{P}$. A typical example is the celebrated experiment in Ellsberg (1961), where an urn with partially specified composition is considered, that induces a class of compatible probability measures.

Decision problems where uncertainty is summarized by a class $\mathcal{P}$ of probability measures are customarily referred to as decisions under *ambiguity* (see, e.g., Etner et al., 2012 and Gilboa & Marinacci, 2013). More generally, ambiguity can be modeled either through a class of probability measures $\mathcal{P}$ or through a non-additive uncertainty measure ν (see, e.g., Gilboa, 2009). The two approaches are generally not equivalent, but in case ν is (at least) 2-monotone (Grabisch, 2016; Walley, 1991) the two approaches reduce to Choquet expected utility (Gilboa & Schmeidler, 1989; Schmeidler, 1986).

Switching to a set of probabilities $\mathcal{P}$ in place of the single probability measure $\mathbf{P}$ allows to address situations where a single market agent is not completely convinced of the hypotheses behind $\mathbf{P}$, like independence or association among some variables, thus he/she aims to test their robustness. A distinguished model is the *epsilon-contamination* of a reference probability measure $\mathbf{P}$, which is the class $\mathcal{P}_{\mathbf{P},\epsilon}$ of all probability measures of the type $\mathbf{P}' = (1-\epsilon)\mathbf{P} + \epsilon\mathbf{P}''$, where $\mathbf{P}''$ is any probability measure on the same space, and $\epsilon \in [0,1)$ is a fixed ambiguity parameter. The epsilon-contamination class, also referred to as *linear-vacuous mixture model*, gives rise to the lower envelope $\nu_{\mathbf{P},\epsilon} = \min \mathcal{P}_{\mathbf{P},\epsilon}$, which is a completely monotone capacity (see Sections 2.9.2 and 3.3.5 in Walley (1991) or Section 5.1 in Montes et al., 2020). We also notice that $\nu_{\mathbf{P},\epsilon}$ is a particular nearly-linear model, according to Pelessoni et al. (2021). Furthermore, since the Choquet integral with respect to $\nu_{\mathbf{P},\epsilon}$ turns out to be the lower expectation with respect to probability measures in $\mathcal{P}_{\mathbf{P},\epsilon}$, this class permits to model investors' preferences consistent with the Choquet expected utility theory. In particular, the classical expected utility theory is recovered when $\epsilon = 0$.

In recent years a growing interest has been addressed towards ambiguity in behavioral finance and portfolio selection (see, e.g., Anantanasuwong et al., 2019), with the aim of proposing more robust and more realistic models. In this paper, we refer to the classical multi-period binomial model (Černý, 2009; Pliska, 1997), which considers a market formed by a non-dividend-paying stock and a risk-free bond, whose prices evolve over a discrete set of times, with a finite horizon T. We formalize a dynamic portfolio selection problem under ambiguity, referring to an epsilon-contamination of the probability measure $\mathbf{P}$

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related to a multi-period binomial market model, and to a strictly increasing and strictly concave utility function. Since the probability $\mathbf{P}$ in the multi-period binomial market model is completely determined by a parameter p, the *epsilon-contaminated binomial model* is parameterized by p and the ambiguity parameter ϵ, and is denoted by $\mathcal{P}_{p,\epsilon}$.

The completeness of the market allows to formulate the dynamic portfolio selection in terms of the final wealth V_T, reachable with a fixed initial amount V_0. Due to the no-arbitrage condition, discounted wealth processes can be represented as martingales with respect to an "artificial" *risk-neutral probability measure* $\mathbf{Q}$ (see, e.g., Černý, 2009; Pliska, 1997). On the other hand, the agent's subjective probability measure $\mathbf{P}$ is usually dubbed as *real-world probability measure* in mathematical finance literature (see, e.g., Chapter 4 in Munk, 2013), to distinguish it from $\mathbf{Q}$.

A systematically pessimistic attitude towards ambiguity is taken for the market agent which is assumed to be risk averse and ambiguity averse. The problem amounts to finding a final wealth maximizing the corresponding Choquet expected utility functional. As shown in Gilboa (2009) (see also Grabisch, 2016), an agent which is a Choquet expected utility maximizer is actually an expected utility maximinimizer, with respect to the epsilon-contamination class.

In the one-period case, a portfolio selection problem under ambiguity via maximinimization has been faced in Pflug and Pohl (2018), Pflug and Wozabal (2007). In the quoted papers, the authors take a ball of probability measures around a reference probability measure $\mathbf{P}$ (on $\mathbb{R}^m$) with respect to the Wasserstein distance of order 1 (see, e.g., Villani, 2009), and constrain the feasible portfolios by avoiding short sales and respecting some risk measurement constraints. Thus, besides the modeling of ambiguity, their problem is rather different from ours as we consider a discrete multi-period setting and allow for short sales. Nevertheless, a similarity of our proposal with the quoted one can be singled out since the class of probability measures $\mathcal{P}_{p,\epsilon}$ we consider can be viewed as a ball around $\mathbf{P}$ generated by a suitable pseudo-distance (Montes et al., 2020).

A deeper similarity of our portfolio selection problem can be found with the problem formulated in Appendix C of Jin and Yu Zhou (2008), that requires to maximize a Choquet expected utility over the set of final wealth. The quoted problem differs from ours for working in a continuous-time setting and with a non-atomic probability measure $\mathbf{P}$, which is distorted by applying a differentiable automorphism of the unit interval, generally resulting only in a capacity. Such approach for dealing with ambiguity is not comparable with our setting, in general, and the true goal of the quoted reference is to face portfolio selection when agent's preferences agree with the cumulative prospect theory of Tversky and Kahneman (1992). This is the same setting of papers by Bi et al. (2018), Harris and Mazibas (2022).

It turns out that introducing ambiguity via the epsilon-contamination class allows to achieve manageable algorithms to solve the problem. Indeed, as highlighted in Jin and Yu Zhou (2008), maximizing the Choquet expected utility over the final wealth is a computationally hard problem, that cannot be reduced to dynamic programming, in general.

Assuming uniformly distributed stock returns, some preliminary results concerning the problem faced in this paper appear in Antonini et al. (2020), where a characterization relying on the resolution of an optimization problem of dimension 2^T is provided. Therefore, results appearing in Antonini et al. (2020) cannot be used in practice for a large number of periods T.

In this paper, we prove that our Choquet expected utility maximization problem admits a unique optimal final wealth V_T^*, that can be expressed as a suitable function of the stock value at the maturity S_T, i.e., $V_T^* = \varphi(S_T)$. The problem reduces to a family of linearly constrained concave problems with $T+1$ unknowns, which is equivalent to a single non-linearly constrained problem with a linear objective, over $T+2$ unknowns.

To the best of our knowledge, this is the first time the epsilon-contamination model is used for dynamic portfolio selection. Moreover, the dimension reduction we obtain essentially relies on the properties of $\mathcal{P}_{p,\epsilon}$, due to the simple structure of its extreme points and their connection with the Choquet integral (see Sections 3 and 4).

Focusing on a constant relative risk aversion (CRRA) utility function with relative risk aversion parameter $\gamma > 0$, we further provide a characterization of the optimal solution V_T^* and introduce an algorithm that finds the exact optimal solution by solving $T+1$ combinatorial optimization problems. Finally, in the CRRA case, we study the impact of the contamination parameter ϵ on the optimal portfolio, showing the presence of a threshold ϵ^* above which the optimal self-financing strategy reduces to 0 for all times: this highlights that with such values of ϵ the ambiguity in the model is so high to make the risk-free portfolio the most suitable choice. In particular, we investigate the effect of the risk aversion parameter γ on ϵ^* showing the existence of two thresholds γ_1^*, γ_2^* with $\gamma_1^* \leq \gamma_2^*$, delimiting different interactions between ambiguity and risk aversion, that result in different optimal portfolio choice behaviors.

The paper is structured as follows. Section 2 contains preliminaries on the dynamic portfolio selection problem in the classical binomial market model. Section 3 introduces the epsilon-contaminated binomial market model and formulates dynamic portfolio selection as a Choquet expected utility maximization problem. The same section shows that the optimal final wealth is a function of the stock price at time T. Section 4 uses the latter property to equivalently reformulate the initial problem either in terms of a family of $T+1$ linearly constrained concave problems on $T+1$ unknowns or as a unique non-linearly constrained problem on $T+2$ unknowns with a linear objective. Then, an explicit characterization of the optimal final wealth is given in the case of a CRRA utility function with parameter $\gamma > 0$. Finally, Section 5 analyzes the interaction between the ambiguity parameter ϵ and the risk aversion parameter γ, while Section 6 draws our conclusions and future perspectives. Proofs are collected in Appendix.

2. Dynamic portfolio selection in the binomial market model

The paper refers to the classical multi-period binomial market model (see, e.g., Černý, 2009; Pliska, 1997). Such model considers a perfect (competitive and frictionless) market under no-arbitrage, where two basic securities are traded: a non-dividend-paying stock and a risk-free bond.

For a finite horizon $T \in \mathbb{N}$, we denote by S_t and B_t the prices of the stock and the bond, respectively, at time $t \in \{0, \dots, T\}$. The stochastic process $\{S_0, \dots, S_T\}$ and the deterministic process $\{B_0, \dots, B_T\}$ are such that $S_0 = s > 0$, $B_0 = 1$, and for $t = 1, \dots, T$, the returns are

$$\frac{S_t}{S_{t-1}} = \begin{cases} u, & \text{with probability } p \\ d, & \text{with probability } 1-p \end{cases} \qquad \text{and} \qquad \frac{B_t}{B_{t-1}} = (1+r),$$

where $u > d > 0$ are the "up" and "down" stock price coefficients, r is the risk-free interest rate over each period, satisfying $u > (1+r) > d$, and $p \in (0,1)$ is the probability of an "up" movement for the stock price. Thus, for $t = 1, \dots, T$, we have that

$$S_t = S_0 \prod_{n=1}^{t} \frac{S_n}{S_{n-1}} \qquad \text{and} \qquad B_t = (1+r)^t,$$

assuring that both price processes are strictly positive, in compliance with the limited liability assumption for securities (see, e.g., Munk, 2013). Notice that the trajectories of $\{S_0, \dots, S_T\}$ can be represented graphically on a recombining binomial tree.

All the processes are defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0,\dots,T}, \mathbf{P})$, where $\Omega = \{1, \dots, 2^T\}$, $\mathcal{F} = 2^\Omega$ with 2^Ω the power set of Ω, and $\mathcal{F}_t$ is the algebra generated by random variables $\{S_0, \dots, S_t\}$, for $t = 0, \dots, T$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. As usual $\mathbf{E}^{\mathbf{P}}$ denotes the expected value with respect to $\mathbf{P}$.

Assuming that the returns $\frac{S_1}{S_0}, \dots, \frac{S_T}{S_{T-1}}$ are i.i.d. random variables, the probability $\mathbf{P}$ is completely singled out by the parameter p, and the process $\{S_0, \dots, S_T\}$ is a multiplicative binomial process since

$$\mathbf{P}(S_t = u^k d^{t-k} s) = \frac{t!}{k!(t-k)!} p^k (1-p)^{t-k},$$

where S_t ranges in $\mathcal{S}_t = \{u^k d^{t-k} s \ : \ k = 0, \ldots, t\}$.

Let $V_0 \in \mathbb{R}$ be an initial wealth. A *self-financing strategy* $\{\theta_0, \ldots, \theta_{T-1}\}$ is an adapted process such that θ_t is the (random) number of shares of stock to buy (if positive) or short-sell (if negative) at time t up to time $t+1$ (Černý, 2009), that determines an adapted wealth process $\{V_0, \ldots, V_T\}$, where, for $t = 0, \ldots, T-1$,

$$V_{t+1} = (1+r)V_t + \theta_t S_t \left(\frac{S_{t+1}}{S_t} - (1+r) \right). \tag{1}$$

In turn, $V_t - \theta_t S_t$ is the amount of money invested in the bond from time t up to time $t+1$.

This market model is said to be *complete*, i.e., there is a unique risk-neutral probability measure $\mathbf{Q}$ on $\mathcal{F}$, equivalent to $\mathbf{P}$, such that the discounted wealth process of any self-financing strategy is a martingale under $\mathbf{Q}$:

$$\frac{V_t}{(1+r)^t} = \mathbf{E}_t^{\mathbf{Q}} \left[\frac{V_T}{(1+r)^T} \right], \tag{2}$$

for $t = 0, \ldots, T$, where $\mathbf{E}_t^{\mathbf{Q}}[\cdot] = \mathbf{E}^{\mathbf{Q}}[\cdot|\mathcal{F}_t]$ and $\mathbf{E}_0^{\mathbf{Q}} \equiv \mathbf{E}^{\mathbf{Q}}$. Completeness implies that every payoff $V_T \in \mathbb{R}^{\Omega}$ depending only on the stock price history can be replicated by a dynamic self-financing strategy $\{\theta_0, \ldots, \theta_{T-1}\}$ and its unique no-arbitrage price at time $t = 0$ is determined by Eq. (2), since

$$V_0 = \frac{\mathbf{E}^{\mathbf{Q}}[V_T]}{(1+r)^T}. \tag{3}$$

Notice that the process $\{S_0, \ldots, S_T\}$ is still a multiplicative binomial process under $\mathbf{Q}$, completely characterized by the parameter

$$q = \frac{(1+r) - d}{u - d} \in (0, 1). \tag{4}$$

Both $\mathbf{P}$ and $\mathbf{Q}$ can be explicitly defined by identifying every state $i \in \Omega$ with the path of the stock price evolution corresponding to the T-digit binary expansion of number $i-1$, in which ones are interpreted as "up" movements and zeros as "down" movements. Denoting by $\kappa(i)$ the number of "up" movements and by $T - \kappa(i)$ the number of "down" movements, it holds that (we avoid braces to simplify writing)

$$\mathbf{P}(i) = p^{\kappa(i)}(1-p)^{T-\kappa(i)} \quad \text{and} \quad \mathbf{Q}(i) = q^{\kappa(i)}(1-q)^{T-\kappa(i)}, \tag{5}$$

showing that both $\mathbf{P}$ and $\mathbf{Q}$ are strictly positive on $\mathcal{F} \setminus \{\emptyset\}$.

We point out that $\mathbf{Q}$ is an "artificial" probability measure implied by the no-arbitrage condition, whose unique purpose is to represent a wealth process as in Eq. (2).

In the classical binomial market model, the probability measure $\mathbf{P}$ is assumed to encode the beliefs of an investor and is commonly estimated from historical data. In this paper we suppose that our investor has ambiguous beliefs, meaning that, rather than having a single probability $\mathbf{P}$, he/she actually considers a closed (in the product topology) class of probabilities $\mathcal{P}$, as highlighted in the introduction.

3. Modeling ambiguity through the epsilon-contamination model

We deal with the binomial market model by introducing ambiguity to get an *epsilon-contaminated binomial market model*. Given the real-world probability $\mathbf{P}$ defined on $\mathcal{F}$ as in Section 2 (which is completely singled out by p) and $\epsilon \in (0,1)$, the corresponding *epsilon-contamination model* (see, e.g., Huber, 1981) is the class of probability measures on $\mathcal{F}$ defined as

$$\mathcal{P}_{p,\epsilon} = \{\mathbf{P}' = (1-\epsilon)\mathbf{P} + \epsilon \mathbf{P}'' \ : \ \mathbf{P}'' \text{ is a probability measure on } \mathcal{F}\},$$

whose lower envelope $\nu_{p,\epsilon} = \min \mathcal{P}_{p,\epsilon}$ is defined on $\mathcal{F}$ as

$$\nu_{p,\epsilon}(A) = \begin{cases} (1-\epsilon)\mathbf{P}(A), & \text{if } A \neq \Omega, \\ 1, & \text{if } A = \Omega. \end{cases} \tag{6}$$

It turns out that $\nu_{p,\epsilon}$ is a completely monotone capacity (see Section 5.1 in Montes et al., 2020), i.e., it satisfies:

(i) $\nu_{p,\epsilon}(\emptyset) = 0$ and $\nu_{p,\epsilon}(\Omega) = 1$;

(ii) $\nu_{p,\epsilon}\left(\bigcup_{i=1}^n E_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \nu_{p,\epsilon}\left(\bigcap_{i \in I} E_i\right)$, for every $n \geq 2$ and for every $E_1, \ldots, E_n \in \mathcal{F}$.

Notice that, the case $\epsilon = 0$ corresponds to absence of ambiguity, since $\mathcal{P}_{p,\epsilon} = \{\mathbf{P}\}$ and $\nu_{p,\epsilon} = \mathbf{P}$. Therefore, to cover also this possibility we take $\epsilon \in [0,1)$.

For every permutation σ of Ω, we define a probability measure $\mathbf{P}^\sigma$ on $\mathcal{F}$, whose value on the singletons (we avoid braces to simplify writing) is

$$\mathbf{P}^\sigma(\sigma(i)) = \nu_{p,\epsilon}(E_i^\sigma) - \nu_{p,\epsilon}(E_{i+1}^\sigma), \tag{7}$$

where $E_i^\sigma = \{\sigma(i), \ldots, \sigma(2^T)\}$, for all $i \in \Omega$, and $E_{2^T+1}^\sigma = \emptyset$. In what follows Σ denotes the set of all permutations of Ω.

It turns out (see, e.g., Chapter 3 in Grabisch, 2016) that probabilities $\mathbf{P}^\sigma$'s are the extreme points of $\mathcal{P}_{p,\epsilon}$, that is $\mathrm{ext}(\mathcal{P}_{p,\epsilon}) = \{\mathbf{P}^\sigma \ : \ \sigma \in \Sigma\}$. This holds since $\mathcal{P}_{p,\epsilon}$ coincides with the *core* induced by $\nu_{p,\epsilon}$, i.e., with the set of probability measures on $\mathcal{F}$ dominating $\nu_{p,\epsilon}$.

As follows by Section 3.6.3 in Walley (1991) (see also Section 5.1 in Montes et al., 2020), only the first element of a permutation matters, in the sense that if two permutations $\sigma, \sigma' \in \Sigma$ are such that $\sigma(1) = \sigma'(1)$, then $\mathbf{P}^\sigma = \mathbf{P}^{\sigma'}$. Therefore, we can focus on a minimal subset $\Sigma' = \{\sigma_1, \ldots, \sigma_{2^T}\}$, where $\sigma_h(1) = h$ for all $h \in \Omega$, to recover $\mathrm{ext}(\mathcal{P}_{p,\epsilon})$: a subset Σ' can be obtained choosing, for every $h \in \Omega$, a permutation σ_h having h in its first position. Indeed, for every permutation σ of Ω it holds that

$$\mathbf{P}^\sigma(\sigma(i)) = \begin{cases} (1-\epsilon)\mathbf{P}(\sigma(i)) + \epsilon, & \text{if } i = 1, \\ (1-\epsilon)\mathbf{P}(\sigma(i)), & \text{otherwise.} \end{cases} \tag{8}$$

In what follows, we denote $\mathbb{R}_{++} = (0, +\infty)$. We consider a utility function $U : \mathbb{R}_{++} \to \mathbb{R}$ satisfying the following requirements:

(A) U is continuously differentiable, strictly increasing, strictly concave, $\lim_{x \to 0^+} U'(x) = +\infty$ and $U'(\mathbb{R}_{++}) = \mathbb{R}_{++}$.

By **(A)** the derivative U' is continuous and strictly decreasing, thus it is invertible with inverse function $(U')^{-1}$.

A typical example is a *constant relative risk aversion (CRRA)* utility function defined, for $\gamma > 0$, as

$$U_\gamma(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, & \gamma \neq 1, \\ \ln x, & \gamma = 1, \end{cases} \quad \text{for } x > 0. \tag{9}$$

The parameter γ expresses the *relative risk aversion* of the utility function U_γ and the higher γ the more risk-averse is the investor (see, e.g., Section 5.6.1 in Munk, 2013).

For every random variable $V_T \in \mathbb{R}_{++}^{\Omega}$, we can define the functional

$$\mathbf{CEU}_{U,p,\epsilon}[V_T] = \oint U(V_T)\,\mathrm{d}\nu_{p,\epsilon}, \tag{10}$$

where the integral on the right side is a Choquet integral. In particular, since U is strictly increasing, for all $V_T \in \mathbb{R}_{++}^{\Omega}$, if σ is a permutation of Ω such that $V_T(\sigma(1)) \leq \ldots \leq V_T(\sigma(2^T))$ then the functional $\mathbf{CEU}_{U,p,\epsilon}$ can be expressed (see Chapter 4 in Grabisch, 2016) as follows

$$\mathbf{CEU}_{U,p,\epsilon}[V_T] = \sum_{k=1}^{2^T} \mathbf{P}^\sigma(k) U(V_T(k)). \tag{11}$$

Notice that the functional $\mathbf{CEU}_{U,p,\epsilon}$ represents the agent's preferences on the set of final wealth $\mathbb{R}_{++}^{\Omega}$. Therefore, maximizing $\mathbf{CEU}_{U,p,\epsilon}$ we are choosing the final wealth that is more preferred under ambiguity.

Proposition 1. *The functional* $\mathbf{CEU}_{U,p,\epsilon}$ *is concave, that is, for all* $V_T, V_T' \in \mathbb{R}_{++}^{\Omega}$ *and all* $\alpha \in [0,1]$, *it holds*

$$\mathbf{CEU}_{U,p,\epsilon}[\alpha V_T + (1-\alpha)V_T'] \geq \alpha \mathbf{CEU}_{U,p,\epsilon}[V_T] + (1-\alpha)\mathbf{CEU}_{U,p,\epsilon}[V_T'].$$

By the main theorem in Schmeidler (1986), we also have that

$$\mathbf{CEU}_{U,p,\epsilon}[V_T] = \min_{\mathbf{P}' \in \mathcal{P}_{p,\epsilon}} \mathbf{E}^{\mathbf{P}'}[U(V_T)], \tag{12}$$

therefore, $\mathbf{CEU}_{U,p,\epsilon}$ is a *lower expected utility* (Walley, 1991) and maximizing it we are actually applying a *maximin criterion* of choice (Troffaes, 2007).

Given an initial wealth $V_0 > 0$, our aim is to select a self-financing strategy $\{\theta_0, \ldots, \theta_{T-1}\}$ resulting in a final wealth $V_T \in \mathbb{R}^{\Omega}_{++}$, solving

$$\max_{\theta_0,\ldots,\theta_{T-1}} \mathbf{CEU}_{U,p,\epsilon}[V_T]. \tag{13}$$

Taking into account (3), which is due to the completeness of the market, the above problem can be rewritten maximizing over the final wealth random variables V_T's that can be reached with the fixed initial wealth V_0

$$\text{maximize } \mathbf{CEU}_{U,p,\epsilon}[V_T] \text{ subject to:} \\ \begin{cases} \mathbf{E}^{\mathbf{Q}}[V_T] - (1+r)^T V_0 = 0, \\ V_T \in \mathbb{R}^{\Omega}_{++}. \end{cases} \tag{14}$$

Notice that problem (13) seeks a stochastic process $\{\theta_0, \ldots, \theta_{T-1}\}$, which is a self-financing strategy, while problem (14) looks for a random variable V_T, which is a final wealth.

By Proposition 1, the objective function in (14) is a concave function on $\mathbb{R}^{\Omega}_{++}$, subject to a linear constraint, therefore every local maximum is a global maximum and, further, the set of global maxima is convex (Boyd & Vandenberghe, 2004).

Theorem 1. *The following statements hold:*

(i) there exists an optimal solution $V_T^* \in \mathbb{R}^{\Omega}_{++}$ of problem (14) and such optimal solution is unique;
(ii) if V_T^* is the optimal solution of problem (14), then there exists a function $\varphi : \mathcal{S}_T \to \mathbb{R}_{++}$, such that $V_T^* = \varphi(S_T)$.

Notice that the optimal solution V_T^* of problem (14) corresponds to a unique self-financing strategy $\{\theta_0^*, \ldots, \theta_{T-1}^*\}$ by (1), which is the optimal solution of problem (13), and vice versa.

Remark 1. Theorem 1 has a deep computational impact, since it implies that V_T^* can be simply regarded as a function defined on $\mathcal{S}_T$, being it constant on $\{S_T = s_T\}$, for all $s_T \in \mathcal{S}_T$. As a byproduct, the optimal wealth process $\{V_0^*, \ldots, V_T^*\}$ turns out to be a Markov process under $\mathbf{Q}$ and can be represented on a recombining binomial tree. Therefore, despite ambiguity, the final optimal wealth V_T^* can be regarded as the payoff of a European derivative on the stock.

4. Dimension reduction of the problem

Let V_T^* be the optimal final wealth (solution of problem (14)), which can be seen as a European derivative on the stock by Theorem 1, i.e., $V_T^* = \varphi(S_T)$. Taking into account (8), problem (14) can be reformulated by referring to $T+1$ unknowns and $T+1$ permutations. To do so, let $\Theta = \{1, \ldots, T+1\}$ and index the range $\mathcal{S}_T = \{s_T^i : i \in \Theta\}$ of S_T in such a way that $s_T^1 < s_T^2 < \cdots < s_T^{T+1}$. Then $\mathbf{P}$, $\mathbf{Q}$ and $\nu_{p,\epsilon}$ can be restricted to the algebra generated by the random variable S_T which is isomorphic to 2^{Θ}.

From now on, we use the symbols $\hat{\mathbf{P}}$, $\hat{\mathbf{Q}}$ and $\hat{\nu}_{p,\epsilon}$ to denote the restrictions on the algebra generated by S_T, the latter identified with 2^{Θ}. Random variables $X \in \mathbb{R}^{\Omega}$ which are constant on the atoms of the algebra generated by S_T are seen as elements of $\mathbb{R}^{\Theta}$ writing $\hat{X}$ to stress this fact, while we denote by Π the set of permutations of Θ. For every $i \in \Theta$ it holds that

$$\hat{\mathbf{P}}(i) = \mathbf{P}(S_T = s_T^i) \quad \text{and} \quad \hat{\mathbf{Q}}(i) = \mathbf{Q}(S_T = s_T^i), \tag{15}$$

while other values on 2^{Θ} are found by additivity.

Let $\hat{\mathcal{P}}_{p,\epsilon}$ be the epsilon-contamination class of $\hat{\mathbf{P}}$ and $\hat{\nu}_{p,\epsilon} = \min \hat{\mathcal{P}}_{p,\epsilon}$, where the minimum is pointwise over 2^{Θ}. Since $\hat{\nu}_{p,\epsilon}$ is a completely monotone capacity, we still have that the set of extreme points of $\hat{\mathcal{P}}_{p,\epsilon}$ is

$$\text{ext}(\hat{\mathcal{P}}_{p,\epsilon}) = \{\hat{\mathbf{P}}^{\pi} : \pi \in \Pi\}, \tag{16}$$

and, as an immediate consequence of Eq. (8), we derive that

$$\hat{\mathbf{P}}^{\pi}(\pi(i)) = \begin{cases} (1-\epsilon)\hat{\mathbf{P}}(\pi(i)) + \epsilon, & \text{if } i = 1, \\ (1-\epsilon)\hat{\mathbf{P}}(\pi(i)), & \text{otherwise.} \end{cases} \tag{17}$$

In turn, this allows us to conclude that also in this reduced case $\text{ext}(\hat{\mathcal{P}}_{p,\epsilon})$ can be recovered by a minimal subset of permutations $\Pi' = \{\pi_1, \ldots, \pi_{T+1}\}$, where $\pi_h(1) = h$ for all $h \in \Theta$.

Problem (14) is shown to be equivalent both to a family of linearly constrained concave problems on $T+1$ unknowns and to a single problem with $T+1$ non-linear constraints and an extra scalar variable.

Theorem 2. *For $V_T^* \in \mathbb{R}^{\Omega}_{++}$, the following statements are equivalent:*

(i) V_T^* solves problem (14);
(ii) $\hat{V}_T^*$ attains $\max_{\pi_h \in \Pi'} \mathbf{E}^{\hat{\mathbf{P}}^{\pi_h}}[U(\hat{V}_T^{\pi_h})]$, where $\hat{V}_T^{\pi_h}$ is an optimal solution of the problem

$$\text{maximize } \left[\sum_{k=1}^{T+1} \hat{\mathbf{P}}^{\pi_h}(k) U(\hat{V}_T(k))\right] \text{ subject to:} \\ \begin{cases} \sum_{k=1}^{T+1} \hat{\mathbf{Q}}(k)\hat{V}_T(k) - (1+r)^T V_0 = 0, \\ \hat{V}_T(\pi_h(1)) - \hat{V}_T(\pi_h(i)) \le 0, \quad \text{for all } i \in \Theta \setminus \{1\}, \\ \hat{V}_T \in \mathbb{R}^{\Theta}_{++}. \end{cases} \tag{18}$$

(iii) $\hat{V}_T^*$ solves the problem

$$\text{maximize } c \text{ subject to:} \\ \begin{cases} \sum_{k=1}^{T+1} \hat{\mathbf{P}}^{\pi_h}(k) U(\hat{V}_T(k)) \ge c, & \text{for } h = 1, \ldots, T+1, \\ \sum_{k=1}^{T+1} \hat{\mathbf{Q}}(k)\hat{V}_T(k) - (1+r)^T V_0 = 0, \\ \hat{V}_T \in \mathbb{R}^{\Theta}_{++}. \end{cases} \tag{19}$$

Problem (18) maximizes the expected utility with respect to $\hat{\mathbf{P}}^{\pi_h}$, by fixing a permutation π_h of Θ, such that $\pi_h(1) = h$. In particular, the first and the third constraints assure that all the considered $\hat{V}_T$'s are a feasible final wealth, given the initial wealth V_0, while the second constraint restricts to those $\hat{V}_T$'s whose minimum value is reached at $\pi_h(1) = h$. In turn, this guarantees that the Choquet expected utility with respect to $\hat{\nu}_{p,\epsilon}$ of such $\hat{V}_T$'s coincides with the expected utility computed with respect to $\hat{\mathbf{P}}^{\pi_h}$. Thus, varying $\pi_h \in \Pi'$ and taking the maximum, we are actually maximizing the Choquet expected utility.

On the other hand, problem (19) relies on the fact that $\text{ext}(\hat{\mathcal{P}}_{p,\epsilon})$ reduces to $\{\hat{\mathbf{P}}^{\pi_h} : h = 1, \ldots, T+1\}$ and the Choquet expected utility of any $\hat{V}_T$ with respect to $\hat{\nu}_{p,\epsilon}$ coincides with the minimum of expected utilities with respect to the $\hat{\mathbf{P}}^{\pi_h}$'s. Thus, the second and third constraints assure that all the considered $\hat{V}_T$'s are a feasible final wealth, given the initial wealth V_0, while maximizing the lower bound in the first constraint we are actually maximizing the Choquet expected utility.

4.1. The particular case of a CRRA utility function

The aim of this section is to show that when the utility function is of CRRA type we can provide an explicit expression of the optimal solution of problem (14), based on the dimension reduction proved in Theorem 2. In view of the dimension reduction of the problem, below we provide a characterization of the optimal solution viewing it as an element of $\mathbb{R}^{\Theta}_{++}$.

Let π be a permutation of Θ and $I \subseteq \Theta \setminus \{1\}$. If $I \neq \emptyset$, for all $i \in I$, consider the constants

$$A_i^{\pi,I} = \frac{\widehat{\mathbf{P}}^{\pi}(\pi(i))}{\sum_{k\in I\cup\{1\}}\widehat{\mathbf{P}}^{\pi}(\pi(k))}\left[\left(\frac{\widehat{\mathbf{Q}}(\pi(i))}{\widehat{\mathbf{P}}^{\pi}(\pi(i))} - \frac{\widehat{\mathbf{Q}}(\pi(1))}{\widehat{\mathbf{P}}^{\pi}(\pi(1))}\right)\left(\sum_{k\in(I\cup\{1\})\setminus\{i\}}\widehat{\mathbf{P}}^{\pi}(\pi(k))\right) + \sum_{k\in I\setminus\{i\}}\left(\widehat{\mathbf{P}}^{\pi}(\pi(k))\left(\frac{\widehat{\mathbf{Q}}(\pi(1))}{\widehat{\mathbf{P}}^{\pi}(\pi(1))} - \frac{\widehat{\mathbf{Q}}(\pi(k))}{\widehat{\mathbf{P}}^{\pi}(\pi(k))}\right)\right)\right], \tag{20}$$

where summations over an empty set are intended to be 0.

The above constants are used to define the following weights:

$$\lambda_1^{\pi,I} = U'_\gamma\left(\frac{(1+r)^T V_0}{\sum_{k=1}^{T+1}\widehat{\mathbf{Q}}(\pi(k))(U'_\gamma)^{-1}\left(\frac{1}{\widehat{\mathbf{P}}^{\pi}(\pi(k))}\left(\widehat{\mathbf{Q}}(\pi(k)) + \mathbf{1}_{\{1\}}(k)\left(\sum_{i\in I}A_i^{\pi,I}\right) - \sum_{i\in I}\mathbf{1}_{\{i\}}(k)A_i^{\pi,I}\right)\right)}\right), \tag{21}$$

where $\mathbf{1}_{\{i\}}$ is the indicator function of the singleton $\{i\}$, while, for all $i \in I$, set

$$\lambda_i^{\pi,I} = A_i^{\pi,I}\lambda_1^{\pi,I}, \tag{22}$$

and $\lambda_i^{\pi,I} = 0$, for all $i \in \Theta \setminus (I \cup \{1\})$. Notice that, if $I = \emptyset$, then the two inner summations involving I in the definition of $\lambda_1^{\pi,I}$ are set to 0: in this case, only $\lambda_1^{\pi,I}$ will be non-null.

Theorem 3. *Let U_γ be a CRRA utility function with $\gamma > 0$. For $p \in (0,1)$ and $T \geq 1$, a random variable $\widehat{V}_T \in \mathbb{R}^{\Theta}_{++}$ can be mapped to the optimal solution $V_T \in \mathbb{R}^{\Omega}_{++}$ of problem (14) if and only if there is a permutation π of Θ and a subset $I \subseteq \Theta \setminus \{1\}$ inducing the weights $\lambda_i^{\pi,I}$'s such that the following conditions hold:*

(i) $\lambda_i^{\pi,I} \geq 0$, *for all* $i \in I$;
(ii) $\widehat{V}_T(\pi(1)) = (U'_\gamma)^{-1}\left(\frac{1}{\widehat{\mathbf{P}}^{\pi}(\pi(1))}\left(\widehat{\mathbf{Q}}(\pi(1))\lambda_1^{\pi,I} + \sum_{k=2}^{T+1}\lambda_k^{\pi,I}\right)\right)$ *and* $\widehat{V}_T(\pi(i)) = (U'_\gamma)^{-1}\left(\frac{1}{\widehat{\mathbf{P}}^{\pi}(\pi(i))}\left(\widehat{\mathbf{Q}}(\pi(i))\lambda_1^{\pi,I} - \lambda_i^{\pi,I}\right)\right)$, *for all* $i \in \Theta\setminus\{1\}$;
(iii) $\widehat{V}_T(\pi(1)) \leq \widehat{V}_T(\pi(i))$, *for all* $i \in \Theta \setminus (I \cup \{1\})$;

and there is no other permutation π' of Θ and no other subset $I' \subseteq \Theta \setminus \{1\}$ determining the weights $\lambda^{\pi',I'}$'s and the random variable $\widehat{V}'_T \in \mathbb{R}^{\Theta}_{++}$ satisfying (i)–(iii) such that $\mathbf{CEU}_{U_\gamma,p,\epsilon}[V'_T] > \mathbf{CEU}_{U_\gamma,p,\epsilon}[V_T]$.

In case of a CRRA utility function U_γ, Theorem 3 allows to find the analytic expression of the optimal solution of problem (14), reducing it to $T+1$ combinatorial optimization problems with $T+1$ unknowns, each corresponding to a permutation π_h of Θ such that $\pi_h(1) = h$, for all $h \in \Theta$. The procedure derived by Theorem 3 is reported in Algorithm 1: for each of the permutations $\pi_1, \dots, \pi_{T+1}$ of Θ, one needs to find a subset $I \subseteq \Theta \setminus \{1\}$ satisfying conditions *(i)–(iii)*. Therefore, in the worst case we need to perform $(T+1)2^T$ controls of conditions *(i)–(iii)*. This procedure, though resulting in an exact solution, is computationally hard and is applicable for small values of T only. Solving the equivalent problem (19) through a non-linear global solver, though only returning an approximate solution in general, turns out to be more efficient and allows to deal with large values of T.

The following example shows an application of Algorithm 1, where we calibrate the model on the non-dividend-paying stock META and a US T-bill maturing in one month.

Example 1. We identify $t = 0$ with the date 2023-06-30 and consider the daily closing price evolution of META stock in the first 6 months of year 2023, whose graph is displayed in Fig. 1. Data are taken from Yahoo! Finance and accessed in Python through the yfinance library.

Algorithm 1 Combinatorial solution of (14) for a CRRA utility function

▷ *input:* $p, u, d, r, V_0, S_0, T, \epsilon, \gamma$, parameters
▷ *output:* V_T^*, optimal solution of (14)

Compute probability distributions $\widehat{\mathbf{P}}$ and $\widehat{\mathbf{Q}}$ on Θ as in (15)
Fix permutations $\pi_1, \dots, \pi_{T+1}$ of Θ such that $\pi_h(1) = h$
$CEU^{best} := -\infty$
$\widehat{V}_T^{best} := 0$
for $h = 1, \dots, T+1$ **do**
  Compute probability distribution $\widehat{\mathbf{P}}^{\pi_h}$ on Θ as in (17)
  for $I \subseteq \Theta \setminus \{1\}$ **do**
    Compute $A_i^{\pi_h,I}$ as in (20), for all $i \in I$
    Compute $\lambda_1^{\pi_h,I}$ as in (21)
    Compute $\lambda_i^{\pi_h,I}$ as in (22), for all $i \in I$
    $\lambda_i^{\pi_h,I} = 0$, for all $i \in \Theta \setminus (\{1\} \cup I)$
    Compute $\widehat{V}_T$ as in *(ii)* of Theorem 3
    if $\widehat{V}_T \in \mathbb{R}^{\Theta}_{++}$ and *(i)* and *(iii)* of Theorem 3 are satisfied **then**
      $CEU^{new} := \mathbf{E}^{\widehat{\mathbf{P}}^{\pi_h}}[U_\gamma(\widehat{V}_T)]$
      if $CEU^{new} > CEU^{best}$ **then**
        $CEU^{best} := CEU^{new}$
        $\widehat{V}_T^{best} := \widehat{V}_T$
      end if
    end if
  end for
end for
Map $\widehat{V}_T^{best}$ in $V_T^* \in \mathbb{R}^{\Omega}_{++}$ constant on $\{\{S_T = s_T^i\},\ i \in \Theta\}$
return V_T^*

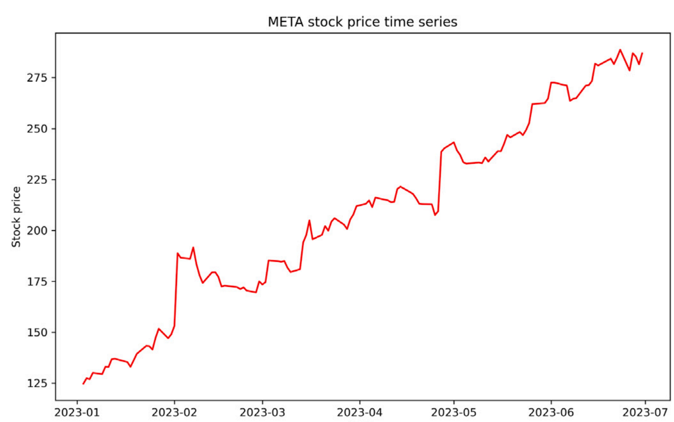


Fig. 1. META stock price time series (daily closing prices) from 2023-01-03 to 2023-06-30.

We refer to a trading year composed by 250 days, so, daily periods have length $\Delta t = \frac{1}{250}$ years. At the date 2023-06-30, the META stock price is $S_0 = 286.98$ dollars and we further consider a US T-bill maturing in 1 month, such that $r = (1.0508)^{\Delta t} - 1$ per period.

Parameters u, d, and p can be estimated from the META stock price time series, under the assumption that daily log-returns are i.i.d. and normal (see, e.g., Chapters 13 and 15 in Hull, 2018). Given the available 124 price observations $\{m_0, \dots, m_{123}\}$ of META stock, we compute the corresponding log-returns $\{\ell_1, \dots, \ell_{123}\}$, where $\ell_n = \ln\frac{m_n}{m_{n-1}}$, for $n = 1, \dots, 123$. Next, set $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$, where $\sigma = \sqrt{250}\sigma_{day}$ and σ_{day}^2 is the sample variance of $\{\ell_1, \dots, \ell_{123}\}$. Finally, p is estimated as the relative frequency of positive values in the set of log-returns $\{\ell_1, \dots, \ell_{123}\}$. The estimation procedure results in $u = 1.0292$, $d = 0.9717$, and $p = 0.5528$, so, by Eq. (4) we get $q = 0.4963$.

Next, at date 2023-06-30 corresponding to time $t = 0$, we suppose to invest an initial wealth $V_0 = 1000$ dollars in the META stock and in

Table 1
Optimal final wealth V_T^* as a function $\varphi(S_T)$.

S_T	$d^5 286.98$	$ud^4 286.98$	$u^2d^3 286.98$	$u^3d^2 286.98$	$u^4d 286.98$	$u^5 286.98$
V_T^*	877.09	877.09	928.76	1040.45	1165.57	1305.73

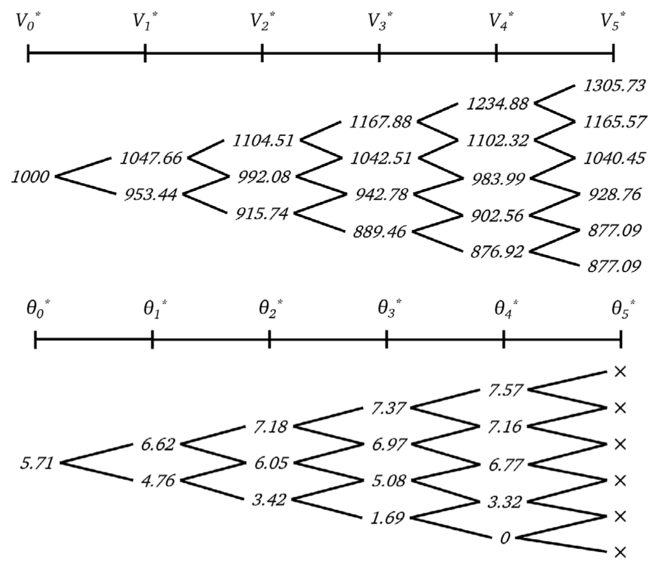


Fig. 2. Recombining binomial tree representations of the processes $\{V_0^*, \ldots, V_5^*\}$ and $\{\theta_0^*, \ldots, \theta_4^*\}$.

Table 2
$\text{MAE}(\tilde{V}_T^*, V_T^*)$ as a function of T, for $T \in \{5, 10, 15, 20\}$.

T	5	10	15	20
$\text{MAE}(\tilde{V}_T^*, V_T^*)$	0.0108	0.1049	0.3040	137.5346

the US T-bill. We take a CRRA utility function with parameter $\gamma = 2$, a contamination parameter $\epsilon = 0.02$, and consider a trading week by taking $T = 5$ periods. The choice of both parameters γ and ϵ is a crucial part and a preliminary tuning procedure should be executed. In this example, $\gamma = 2$ singles out a quite risk averse agent, while $\epsilon = 0.02$ indicates that the agent is rather convinced of the reference $\mathbf{P}$, to which a $1 - \epsilon = 0.98$ weight is assigned. Section 5 carries out a sensitivity analysis and investigates the interaction between agent's risk and ambiguity attitudes when performing the optimal portfolio selection.

Applying Algorithm 1 we obtain that the optimal final wealth is V_T^*, whose values are reported in Table 1, seen as a function $\varphi(S_T)$, according to Theorem 3.

Using the martingale property with respect to $\mathbf{Q}$ of the wealth process $\{V_0^*, \ldots, V_5^*\}$, that is $V_t^* = \frac{\mathbf{E}_t^{\mathbf{Q}}[V_5^*]}{(1+r)^{5-t}}$, for $t = 0, \ldots, 4$, we can recover the optimal self-financing strategy $\{\theta_0^*, \ldots, \theta_4^*\}$ through Eq. (1). In agreement with Theorem 1, the optimal wealth process $\{V_0^*, \ldots, V_5^*\}$ can be represented on a recombining binomial tree and the same holds for the optimal self-financing strategy $\{\theta_0^*, \ldots, \theta_4^*\}$, as shown in Fig. 2.

We point out that, though Algorithm 1 provides an exact solution, its execution time grows exponentially with T. On the other hand, by statement *(iii)* of Theorem 2, a much faster approach is to solve the non-linear problem (19) by relying on a global non-linear solver, like `bonmin` (Bonmin, 2023). The resulting optimal final wealth $\tilde{V}_T^*$ obtained with the `bonmin` solver will be an approximation of the true optimal V_T^*. Table 2 shows the mean absolute error $\text{MAE}(\tilde{V}_T^*, V_T^*)$, for $T \in \{5, 10, 15, 20\}$, highlighting an increasing trend with respect to T.

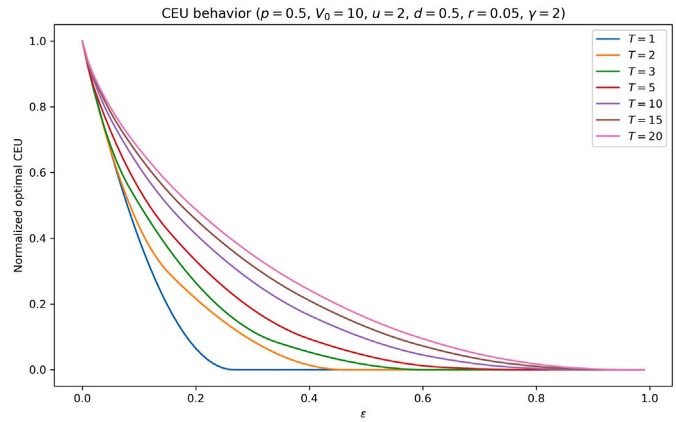


Fig. 3. Normalized optimal value of $\mathbf{CEU}_{U_\gamma,p,\epsilon}$, seen as a function of $\epsilon \in [0, 1)$.

5. The impact of the ambiguity parameter on the optimal portfolio

The aim of this section is to investigate the impact of ϵ on the composition of the optimal portfolio. Here, we restrict to a CRRA utility function U_γ: the aim is to analyze the interaction between parameters ϵ and γ to highlight the mutual influence between ambiguity and risk aversion in the context of dynamic portfolio selection.

We first prove that, whenever $p = q = \frac{(1+r)-d}{u-d}$, then the optimization problem (14) becomes trivial and the result is independent of ϵ and γ.

Proposition 2. *If $p = q = \frac{(1+r)-d}{u-d}$, then the optimal solution of (14) is $V_T^* = (1+r)^T V_0$ and the optimal value is $\mathbf{CEU}_{U_\gamma,p,\epsilon}[V_T^*] = U_\gamma((1+r)^T V_0)$.*

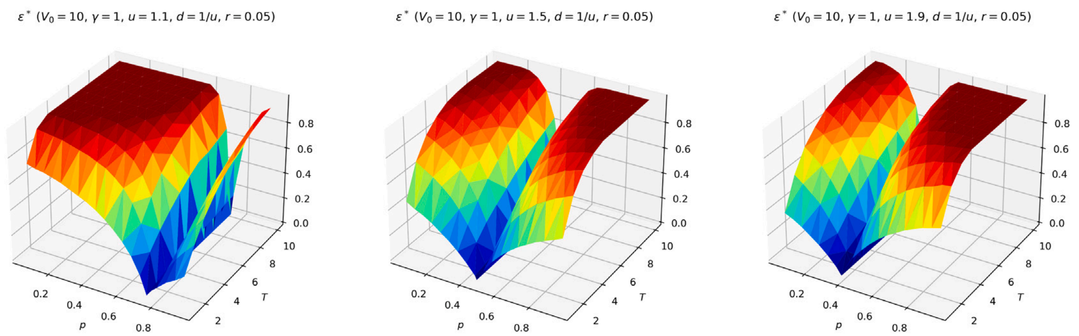
Remark 2. In the limit case $p = q = \frac{(1+r)-d}{u-d}$, the optimal portfolio does not depend on ϵ nor on γ and always reduces to a strategy in which $\theta_t^* = 0$, for $t = 0, \ldots, T-1$, i.e., we have only risk-free bond investments in all the periods. From a financial point of view, this means that, if the real-world probability p coincides with the risk-neutral probability q, the agent always chooses a completely risk-free investment, independently of his/her ambiguity and risk attitudes.

Now, we consider the case in which $p \neq q = \frac{(1+r)-d}{u-d}$. For fixed V_0, u, d, r, and p, the optimal value of $\mathbf{CEU}_{U_\gamma,p,\epsilon}$ seen as a function of ϵ is non-increasing with maximum value at $\epsilon = 0$ and minimum value $U_\gamma((1+r)^T V_0)$, which is the optimal value we get when the optimal solution is $V_T^* = (1+r)^T V_0$. Since the range of $\mathbf{CEU}_{U_\gamma,p,\epsilon}$ depends on T, we normalize it so as to range in $[0, 1]$. For every T, the normalized value 1 corresponds to the optimal value we get for $\epsilon = 0$, that is without ambiguity, while the normalized value 0 corresponds to $U_\gamma((1+r)^T V_0)$.

Fig. 3 shows the normalized optimal value of $\mathbf{CEU}_{U_\gamma,p,\epsilon}$, seen as a function of $\epsilon \in [0, 1)$, for fixed $p = \frac{1}{2}$, $V_0 = 10$, $u = 2$, $d = \frac{1}{u}$, $r = 0.05$ and a CRRA utility function with $\gamma = 2$.

Fig. 3 highlights the existence of a value $\epsilon^* = \epsilon^*(\gamma, T, p, u, d, r, V_0)$ above which the optimal self-financing strategy reduces to 0 for all times. In a sense, we may think at ϵ^* as a threshold above which the ambiguity incorporated in the real-world probability $\mathbf{P}$ through the epsilon-contamination is so high to make the risk-free portfolio the most suitable choice. A numerical approximation of ϵ^* can be achieved by applying a suitable bisection algorithm. The following toy example has the purpose of showing that ϵ^* is a true threshold after which the $\mathbf{CEU}_{U_\gamma,p,\epsilon}$ functional is stuck on its minimum value, and is not an asymptotic value.

Example 2. Take $T = 1$, $p = \frac{1}{2}$, $V_0 = 10$, $u = 2$, $d = \frac{1}{u}$, $r = 0.05$ and a CRRA utility function with $\gamma = 2$. This implies that $q = \frac{11}{30}$. The



(a) $u = 1.1$ (b) $u = 1.5$ (c) $u = 1.9$

Fig. 4. Threshold ϵ^* seen as a function of p and T, for fixed $V_0 = 10$, $\gamma = 1$, and $r = 0.05$.

normalized optimal value of $\mathbf{CEU}_{U_\gamma,p,\epsilon}$ seen as function of $\epsilon \in [0,1)$ is the blue line in Fig. 3. By applying a bisection algorithm we derive that $\epsilon^* \in [0.27, 0.28]$, so, for every $0.28 \leq \epsilon < 1$ the optimal solution turns out to be the constant $V_1^\epsilon = (1+r)V_0 = 10.5$.

To see this, let $\epsilon = 0.28 = \frac{7}{25}$ and consider the reduced index set $\Theta = \{1,2\}$ related to Algorithm 1, for which there are only two permutations: $\pi_1 = \langle 1,2\rangle$ and $\pi_2 = \langle 2,1\rangle$ with $\widehat{\mathbf{P}}^{\pi_1} \equiv \left(\frac{9}{25}, \frac{16}{25}\right)$ and $\widehat{\mathbf{P}}^{\pi_2} \equiv \left(\frac{16}{25}, \frac{9}{25}\right)$ over Θ.

For π_1 the only subset of $\Theta \setminus \{1\}$ satisfying *(i)–(iii)* is $I = \{2\}$ with: $\widehat{V}_1^{\pi_1}(1) = 10.5$, $\widehat{V}_1^{\pi_1}(2) = 10.5$, $\mathbf{CEU}_{U_\gamma,p,\epsilon}[V_1^{\pi_1}] = -\frac{1}{10.5}$.

For π_2 the only subset of $\Theta \setminus \{1\}$ satisfying *(i)–(iii)* is $I = \{2\}$ with: $\widehat{V}_1^{\pi_2}(1) = 10.5$, $\widehat{V}_1^{\pi_2}(2) = 10.5$, $\mathbf{CEU}_{U_\gamma,p,\epsilon}[V_1^{\pi_2}] = -\frac{1}{10.5}$.

Therefore, taking $\epsilon = 0.28$ the optimal reduced solution is the constant $\widehat{V}_1^{0.28} = 10.5$, and the corresponding optimal solution in $\mathbb{R}^{\Omega}_{++}$ is still the constant $V_1^{0.28} = 10.5$. Now, referring to the proof of Proposition 2, it is immediate to conclude that the optimal solution is $V_1^\epsilon = 10.5$, for all $0.28 \leq \epsilon < 1$.

Fig. 4 shows the interpolated ϵ^* surface seen as a function of p and T, where p ranges in $[0.05, 0.95]$ with a 0.05 step and T ranges in $\{1,\ldots,10\}$. The surface is computed for fixed $\gamma = 1$, $V_0 = 10$, $r = 0.05$, and varying u in $\{1.1, 1.5, 1.9\}$, where $d = \frac{1}{u}$. In all the graphs, the steep valley in blue is obtained for choices of p close to $q = \frac{(1+r)-d}{u-d}$. Indeed, in the particular case $p = q = \frac{(1+r)-d}{u-d}$ we get $\epsilon^* = 0$ constantly, in agreement with Proposition 2.

The graphs in Fig. 4 show that, for a fixed p, ϵ^* is non-decreasing with respect to T, in particular, for values of p not close to $q = \frac{(1+r)-d}{u-d}$, then ϵ^* tends to get close to 1 fast, yet for a time horizon $T \geq 5$.

From a financial point of view, this behavior can be interpreted by saying that a risk averse and ambiguity averse investor has a threshold ϵ^* after which he/she switches to a completely risk-free portfolio. Such a threshold becomes higher and higher with the time horizon T. In a sense, if we have more time to adjust our portfolio, then ambiguity aversion becomes less compelling in the portfolio composition.

On the other hand, if we have a probability p close to q, then the initial value of ϵ^* is lower and its growth is much slower. In this case, having a real-world probability p close to the risk-neutral probability q makes the agent very sensitive to ambiguity: low deviations from $\mathbf{P}$ (i.e., a value of ϵ greater than ϵ^*) makes the agent favor the completely risk-free alternative. At the same time, agent's sensitivity to ambiguity in this case requires a much larger time horizon to mitigate the effect of his/her ambiguity aversion.

We consider the evolution of ϵ^* seen as a function of the time horizon T in the case p is close to q. Taking $V_0 = 10$, $r = 0.05$, $u = 2$ and $d = \frac{1}{2}$, we have that $q = \frac{11}{30}$. Fig. 5 depicts the evolution of ϵ^* for $p = q \pm \delta$ with $\delta \in \{0.01, 0.02, 0.03\}$ and T ranging in $\{5,\ldots,50\}$ with a 5 step. Results are obtained by solving (19) through the `bonmin` solver and then applying a bisection algorithm to estimate ϵ^*. The graph shows that, for $\gamma = 2$, for increasing δ we have an increasing initial value

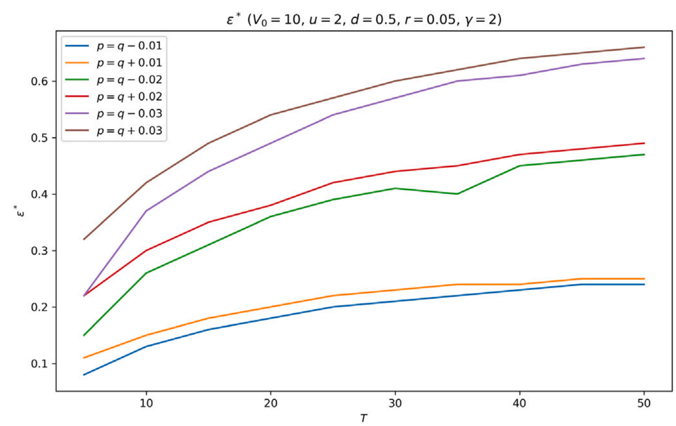


Fig. 5. Threshold ϵ^* seen as a function of T, for fixed $V_0 = 10$, $\gamma = 2$, $r = 0.05$ and $p = q \pm \delta$ with $\delta \in \{0.01, 0.02, 0.03\}$.

at $T = 5$ and an higher growth with a resulting higher final value at $T = 50$.

Notice that, for fixed values of u and d, the interest rate r determines the value of q as in Eq. (4): in turn, r has a direct effect on ϵ^*, since the closeness of p to q affects ϵ^*. For $V_0 = 10$, $\gamma = 1$, $u = 2$, $d = \frac{1}{u}$, and $T = 5$, Fig. 6 shows the interpolated surface of ϵ^* seen as a function of p and r together with its contour lines, where p ranges in $[0.05, 0.95]$ with a 0.05 step and r ranges in $[0, 0.2]$ with a 0.02 step. The graph highlights that ϵ^* is 0 on the line $p = q = \frac{(1+r)-d}{u-d}$, in agreement with Proposition 2.

We finally investigate the impact of the risk aversion parameter γ on the ambiguity threshold ϵ^*. For that we plot ϵ^* as a function of γ ranging in $[0.25, 4]$ with a 0.25 step, for fixed $V_0 = 100$, $r = 0.05$, $u = 2$, $d = 0.5$, and $p = 0.8$. Fig. 7 shows that, for values of γ lower than a threshold $\gamma_1^* = \gamma_1^*(\epsilon, T, p, u, d, r, V_0)$, the ambiguity threshold ϵ^* is positive and constant, while, for higher γ, there is a rapid transition of ϵ^* towards 0. The same figure shows that the higher the time horizon T, the higher is the threshold ϵ^* for the same value of γ. It is also possible to notice that for γ higher than a threshold $\gamma_2^* = \gamma_2^*(\epsilon, T, p, u, d, r, V_0)$ the ambiguity threshold ϵ^* collapses to 0. This facts suggest the following interpretation:

- when γ is lower than γ_1^*, the ambiguity threshold ϵ^* to switch to a completely risk-free investment is not influenced by risk aversion: only ambiguity aversion plays a role;
- when γ ranges between γ_1^* and γ_2^*, there is an interaction between risk aversion and ambiguity aversion that materializes with an ϵ^* decreasing with respect to γ: increasing γ we switch to a completely risk-free investment for lower values of ϵ;

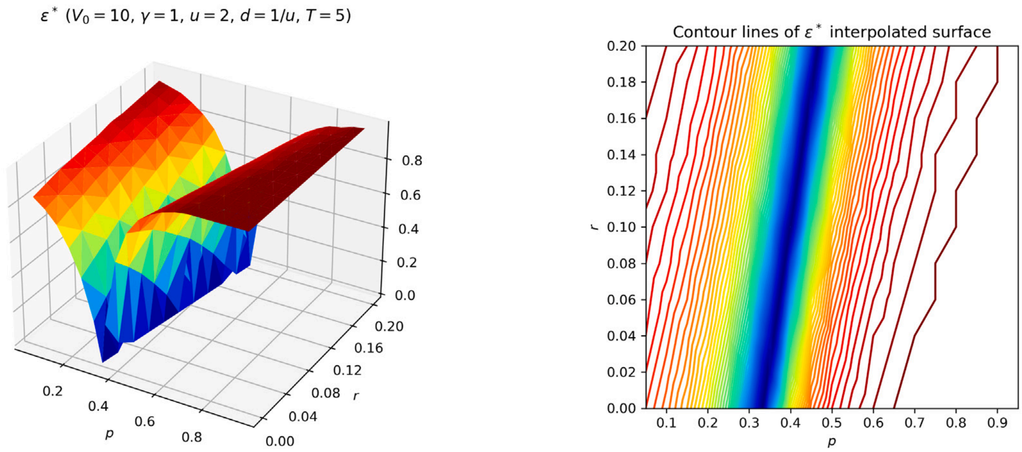


Fig. 6. Threshold ϵ^* seen as a function of p and r, for fixed $V_0 = 10$, $\gamma = 1$, $u = 2$, $d = \frac{1}{u}$, and $T = 5$.

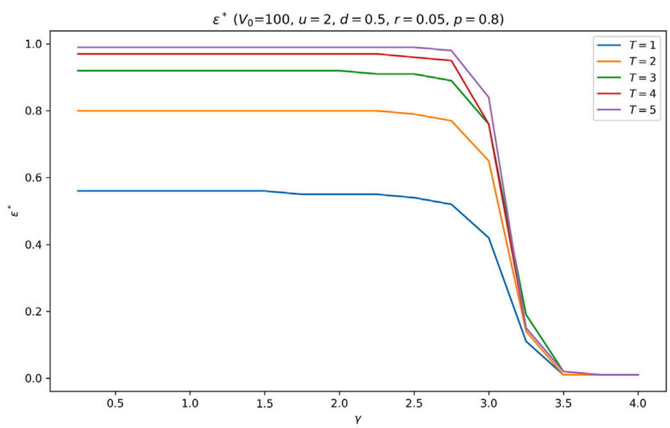


Fig. 7. Threshold ϵ^* seen as a function of γ, for fixed $V_0 = 100$, $r = 0.05$, $u = 2$, $d = 0.5$, and $p = 0.8$.

- when γ is higher than γ_2^*, the ambiguity threshold ϵ^* to switch to a completely risk-free investment turns to 0: in this case the risk-aversion of the investor is so high to make any level of ambiguity not tolerable.

6. Conclusions and future works

This paper introduces the epsilon-contaminated binomial model and investigates dynamic portfolio selection in this market. Under usual regularity conditions on the utility function and referring to the lower envelope of the epsilon-contamination class, the problem is formulated as a Choquet expected utility maximization on the final wealth. With this choice the investor is assumed to be both risk averse and ambiguity averse.

The particular structure of the epsilon-contamination class allows us to prove that the optimal final wealth is a function of the stock final price. In turn, this last fact permits a reduction of the problem dimension as the optimal value process can be represented on a recombining binomial tree. Under a CRRA utility function, a complete characterization of the optimal solution is provided and this is used to analyze the interaction between ambiguity aversion and risk aversion.

The sensitivity analysis highlights an effect of ambiguity on optimal dynamic portfolio selection. The initial investment conditions of an agent in our market model are expressed by the level of relative risk aversion γ, the time horizon T, the stock evolution parameters p, u, d, the risk-neutral interest rate r, and the initial wealth V_0. Such investment conditions single out an ambiguity threshold ϵ^*, which expresses the maximum tolerance to ambiguity for the agent. Such a threshold is shown to be sensitive to γ and we actually find out a region comprised between two values γ_1^*, γ_2^* with $\gamma_1^* \leq \gamma_2^*$, where there is a great interaction between ambiguity aversion and risk aversion, since we switch from a fixed ambiguity sensitivity to complete ambiguity intolerance. Such analysis sheds new light on the interplay between the two distinguished attitudes of ambiguity aversion and risk aversion. In particular, as long as $p \neq q$, $\gamma \leq 2$, and $T \geq 5$, the ambiguity parameter ϵ can assume even values close to 1, without switching to a completely risk-free optimal portfolio. So, the agent is more tolerant concerning ambiguity when he/she is not very risk-averse and has enough time to carry out his/her investment.

As is well-known (see, e.g., Chapter 6 in Černý, 2009), under a suitable choice of parameters, the multi-period binomial market model converges in distribution to the Black–Scholes market model, that is based on the geometric Brownian motion. Therefore, our epsilon-contaminated binomial market model provides a discrete-time approximation of an epsilon-contaminated Black–Scholes market model. Hence, the analysis of the impact of ϵ in the dynamic portfolio selection can be envisaged also in the continuous-time setting, and it will be the aim of future research.

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Appendix. Proofs

Proof of Proposition 1. The proof goes along the line of the proof of Proposition 1 in Antonini et al. (2020) by replacing a CRRA utility function U_γ, where $\gamma > 0$ and $\gamma \neq 1$, with a general utility U satisfying **(A)**. □

Proof of Theorem 1. Statement *(i)*. By (11) the computation of $\mathbf{CEU}_{U,p,\epsilon}[V_T]$ depends on a permutation σ such that the values of V_T are increasingly ordered, therefore problem (14) can be decomposed in a family of optimization problems, each indexed by a permutation

$\sigma \in \Sigma$

$$\begin{aligned}&\text{maximize } \left[\sum_{k=1}^{2^T} \mathbf{P}^\sigma(k) U(V_T(k))\right] \text{ subject to:}\\ &\begin{cases}\displaystyle\sum_{k=1}^{2^T} \mathbf{Q}(k) V_T(k) - (1+r)^T V_0 = 0,\\ V_T(\sigma(i-1)) - V_T(\sigma(i)) \leq 0, \qquad \text{for all } i \in \Omega \setminus \{1\},\\ V_T \in \mathbb{R}^\Omega_{++}.\end{cases}\end{aligned} \tag{A.1}$$

We first show that every problem (A.1) has an optimal solution and such optimal solution is unique. The subset $\mathcal{V}^\sigma$ of $\mathbb{R}^\Omega$ satisfying the equality and inequality constraints in (A.1) is closed, and $\mathcal{V}^\sigma \cap \mathbb{R}^\Omega_{++} \neq \emptyset$, since $V_T = (1+r)^T V_0 \in \mathcal{V}^\sigma \cap \mathbb{R}^\Omega_{++}$. Proceeding as in the proof of Theorem 2.12 in Pascucci and Runggaldier (2009), we have that problem (A.1) has an optimal solution V_T^σ, and such optimal solution is unique since the objective function in (A.1) is strictly concave (Boyd & Vandenberghe, 2004), as U is strictly concave. In turn, problem (14) has an optimal solution V_T^* obtained by selecting V_T^σ for a permutation σ where the objective function is maximum. Suppose there are two permutations σ, σ' attaining the maximum and such that $V_T^\sigma \neq V_T^{\sigma'}$. Since the set of optimal solutions of (14) is convex, this implies that, for all $\alpha \in [0,1]$, $V_T^\alpha = \alpha V_T^\sigma + (1-\alpha) V_T^{\sigma'}$ is an optimal solution of (14). So, since the number of permutations of Ω is finite, we can find an α^* such that $V_T^{\alpha^*}$ solves (A.1) for a permutation σ'' but $V_T^{\alpha^*} \neq V_T^{\sigma''}$, reaching a contradiction. This implies that problem (14) has a unique optimal solution.

Statement *(ii)*. Let V_T^* be the optimal solution of problem (14), which exists and is unique by statement *(i)*. To show the existence of φ it is sufficient to show that V_T^* is constant on the elements of the partition $\{\{S_T = s_T\} \,:\, s_T \in \mathcal{S}_T\}$.

Suppose without loss of generality that there exists $s_T = u^h d^{T-h} s \in \mathcal{S}_T$ such that for distinct $i, j \in \{S_T = s_T\}$ we have that $V_T^*(i) < V_T^*(j)$. Notice that it cannot be $h = 0$ nor $h = T$, since in those cases $\{S_T = s_T\}$ reduces to a singleton. Let σ be a permutation of Ω such that $V_T^*(\sigma(1)) \leq \ldots \leq V_T^*(\sigma(2^T))$, therefore, we have that $\mathbf{CEU}_{U,p,\epsilon}[V_T^*] = \sum_{k=1}^{2^T} \mathbf{P}^\sigma(k) U(V_T^*(k))$.

Define a new random variable $V_T^{**} \in \mathbb{R}^\Omega_{++}$ by setting

$$V_T^{**}(k) = \begin{cases} V_T^*(k), & \text{for } k \notin \{i,j\},\\ \frac{V_T^*(i)}{2} + \frac{V_T^*(j)}{2} & \text{for } k \in \{i,j\}.\end{cases}$$

By Eq. (5), since $\kappa(i) = \kappa(j) = h$, we have that $\mathbf{P}(i) = \mathbf{P}(j) = p^h(1-p)^{T-h}$ and $\mathbf{Q}(i) = \mathbf{Q}(j) = q^h(1-q)^{T-h}$, thus it holds that

$$\begin{aligned} V_0(1+r)^T &= \sum_{k \notin \{i,j\}} \mathbf{Q}(k) V_T^*(k) + q^h(1-q)^{T-h}\left[V_T^*(i) + V_T^*(j)\right]\\ &= \sum_{k \notin \{i,j\}} \mathbf{Q}(k) V_T^*(k) + q^h(1-q)^{T-h} 2\left[\frac{V_T^*(i)}{2} + \frac{V_T^*(j)}{2}\right]\\ &= \sum_{k=1}^{2^T} \mathbf{Q}(k) V_T^{**}(k),\end{aligned}$$

that is V_T^{**} is a feasible solution of (14).

If $\sigma(1) \neq i$, then $\mathbf{P}^\sigma(i) = \mathbf{P}^\sigma(j) = (1-\epsilon)p^h(1-p)^{T-h}$, so

$$\begin{aligned} &\mathbf{CEU}_{U,p,\epsilon}[V_T^*] =\\ &= \sum_{k \notin \{i,j\}} \mathbf{P}^\sigma(k) U(V_T^*(k)) + (1-\epsilon)p^h(1-p)^{T-h}\left[U(V_T^*(i)) + U(V_T^*(j))\right]\\ &= \sum_{k \notin \{i,j\}} \mathbf{P}^\sigma(k) U(V_T^*(k)) + (1-\epsilon)p^h(1-p)^{T-h} 2\left[\frac{U(V_T^*(i))}{2} + \frac{U(V_T^*(j))}{2}\right]\\ &\leq \sum_{k \notin \{i,j\}} \mathbf{P}^\sigma(k) U(V_T^*(k)) + (1-\epsilon)p^h(1-p)^{T-h} 2\left[U\left(\frac{V_T^*(i)}{2} + \frac{V_T^*(j)}{2}\right)\right]\\ &= \mathbf{CEU}_{U,p,\epsilon}[V_T^{**}],\end{aligned}$$

where the last equality follows by Eq. (8) and the fact that $\sigma(1) \notin \{i,j\}$. This implies that V_T^{**} is a feasible solution different from V_T^* with a value of $\mathbf{CEU}_{U,p,\epsilon}$ which is greater than or equal to that of V_T^*, but since the optimal solution is unique we get a contradiction.

If $\sigma(1) = i$, we have that $\mathbf{P}^\sigma(i) = (1-\epsilon)p^h(1-p)^{T-h} + \epsilon$ and $\mathbf{P}^\sigma(j) = (1-\epsilon)p^h(1-p)^{T-h}$. Let l be the element of Ω such that $V_T^{**}(l)$ takes its minimum value. Notice that $V_T^*(i) \leq V_T^{**}(l) \leq \frac{V_T^*(i)}{2} + \frac{V_T^*(j)}{2}$. Hence, we have that $U(V_T^*(1)) \leq U(V_T^{**}(l))$ from which we get that

$$\begin{aligned} \mathbf{CEU}_{U,p,\epsilon}[V_T^*] &= \sum_{k \notin \{i,j\}} \mathbf{P}^\sigma(k) U(V_T^*(k))\\ &\quad + [(1-\epsilon)p^h(1-p)^{T-h} + \epsilon] U(V_T^*(i))\\ &\quad + (1-\epsilon)p^h(1-p)^{T-h} U(V_T^*(j))\\ &= \sum_{k \notin \{i,j\}} \mathbf{P}^\sigma(k) U(V_T^*(k)) + \epsilon U(V_T^*(i))\\ &\quad + (1-\epsilon)p^h(1-p)^{T-h}\left[U(V_T^*(i)) + U(V_T^*(j))\right]\\ &= \sum_{k \notin \{i,j\}} \mathbf{P}^\sigma(k) U(V_T^*(k)) + \epsilon U(V_T^*(i))\\ &\quad + (1-\epsilon)p^h(1-p)^{T-h} 2\left[\frac{U(V_T^*(i))}{2} + \frac{U(V_T^*(j))}{2}\right]\\ &\leq \sum_{k \notin \{i,j\}} \mathbf{P}^\sigma(k) U(V_T^*(k)) + \epsilon U(V_T^*(i))\\ &\quad + (1-\epsilon)p^h(1-p)^{T-h} 2\left[U\left(\frac{V_T^*(i)}{2} + \frac{V_T^*(j)}{2}\right)\right]\\ &\leq \sum_{k \notin \{i,j\}} \mathbf{P}^\sigma(k) U(V_T^*(k)) + \epsilon U(V_T^{**}(l))\\ &\quad + (1-\epsilon)p^h(1-p)^{T-h} 2\left[U\left(\frac{V_T^*(i)}{2} + \frac{V_T^*(j)}{2}\right)\right]\\ &= \mathbf{CEU}_{U,p,\epsilon}[V_T^{**}].\end{aligned}$$

Thus, also in this case we get that V_T^{**} is a feasible solution different from V_T^* with a value of $\mathbf{CEU}_{U,p,\epsilon}$ which is greater than or equal to that of V_T^*, but since the optimal solution is unique we get a contradiction. □

Proof of Theorem 2. *(i)* $\iff$ *(ii)*. By Theorem 1 and the fact that $\mathrm{ext}(\hat{\mathcal{P}}_{p,\epsilon})$ is determined by $\Pi' = \{\pi_1, \ldots, \pi_{T+1}\}$, for all $V_T \in \mathbb{R}^\Omega_{++}$ constant on the atoms of the algebra generated by S_T, it holds that

$$\mathbf{CEU}_{U,p,\epsilon}[V_T] = \min_{h=1,\ldots,T+1} \mathbf{E}^{\hat{\mathbf{P}}^{\pi_h}}[U(\hat{V}_T)],$$

where the expectations in the minimum are computed over Θ. Analogously to the proof of Theorem 1, for every permutation π_h, problem (18) has a unique optimal solution $\hat{V}_T^{\pi_h}$. Then, the optimal solution V_T^* for (14) can be found by selecting a $\hat{V}_T^{\pi_h}$ for a permutation π_h (possibly not unique) where the objective function is maximum and then mapping it to a function V_T^* in $\mathbb{R}^\Omega_{++}$ which is constant on the atoms of the algebra generated by S_T. This shows that problem (14) can be solved by solving $T+1$ problems (18). Thus, we easily get that *(i)* is equivalent to *(ii)*.

(i) $\iff$ *(iii)*. In the light of previous equivalence, varying $\hat{V}_T \in \mathbb{R}^\Theta_{++}$ and maximizing c such that $\mathbf{E}^{\hat{\mathbf{P}}^{\pi_h}}[U(V_T)] \geq c$, for $h = 1, \ldots, T+1$, we get a solution that maximizes $\mathbf{CEU}_{U,p,\epsilon}$ as well, and this allows to get equivalence between *(i)* and *(iii)*. □

Proof of Theorem 3. The proof is an adaptation of the proof of Theorem 1 in Antonini et al. (2020) based on the dimension reduction proved in Theorem 2. Let π be a permutation of Θ. We first show that $\hat{V}_T \in \mathbb{R}^\Theta_{++}$ is the optimal solution of the corresponding problem (18) if and only if there is a subset $I \subseteq \Theta \setminus \{1\}$ inducing weights $\lambda_i^{\pi,I}$'s satisfying *(i)–(iii)*.

Since reordering the elements of Θ according to a permutation π does not change the objective and the equality constraint in (18), below we assume that elements of $\hat{V}_T$ are ordered by π. Therefore, denote by

$$f(\hat{V}_T) = \sum_{k=1}^{T+1} \hat{\mathbf{P}}^\pi(\pi(k)) U_\gamma(\hat{V}_T(\pi(k))),$$

$$g(\widehat{V}_T) = \sum_{k=1}^{T+1} \widehat{\mathbf{Q}}(\pi(k))\widehat{V}_T(\pi(k)) - (1+r)^T V_0 = 0,$$

the objective function and the equality constraint in (18), which are, respectively, (strictly) concave and linear. We also have that all inequality constraints in (18) are linear, therefore, the Karush–Kuhn–Tucker (KKT) conditions are necessary and sufficient in this case (Boyd & Vandenberghe, 2004). Define the Lagrangian function

$$L(\widehat{V}_T, \lambda_1, \dots, \lambda_{T+1}) = f(\widehat{V}_T) - \lambda_1 g(\widehat{V}_T) - \sum_{k=2}^{T+1} \lambda_k(\widehat{V}_T(\pi(1)) - \widehat{V}_T(\pi(k))),$$

for which it holds

$$\frac{\partial L}{\partial \widehat{V}_T(\pi(1))} = \widehat{\mathbf{P}}^\pi(\pi(1))U'_\gamma(\widehat{V}_T(\pi(1))) - \lambda_1\widehat{\mathbf{Q}}(\pi(1)) - \sum_{k=2}^{T+1}\lambda_k,$$
$$\frac{\partial L}{\partial \widehat{V}_T(\pi(i))} = \widehat{\mathbf{P}}^\pi(\pi(i))U'_\gamma(\widehat{V}_T(\pi(i))) - \lambda_1\widehat{\mathbf{Q}}(\pi(i)) + \lambda_i, \quad \text{for all } i \in \Theta\setminus\{1\}.$$

Imposing the KKT conditions, we look for $\widehat{V}_T \in \mathbb{R}^\Theta_{++}$, $\lambda_1 \in \mathbb{R}$, and $\lambda_2, \dots, \lambda_{T+1} \geq 0$ such that $\frac{\partial L}{\partial \widehat{V}_T(\pi(k))} = 0$, for all $k \in \Theta$, $g(\widehat{V}_T) = 0$, $\widehat{V}_T(\pi(1)) \leq \widehat{V}_T(\pi(i))$ and $\lambda_i(\widehat{V}_T(\pi(1)) - \widehat{V}_T(\pi(i))) = 0$, for all $i \in \Theta\setminus\{1\}$.

By $\frac{\partial L}{\partial \widehat{V}_T(\pi(k))} = 0$, for all $k \in \Theta$, we derive

$$\widehat{V}_T(\pi(1)) = (U'_\gamma)^{-1}\left(\frac{1}{\widehat{\mathbf{P}}^\pi(\pi(1))}\left(\lambda_1\widehat{\mathbf{Q}}(\pi(1)) + \sum_{k=2}^{T+1}\lambda_k\right)\right),$$
$$\widehat{V}_T(\pi(i)) = (U'_\gamma)^{-1}\left(\frac{1}{\widehat{\mathbf{P}}^\pi(\pi(i))}\left(\lambda_1\widehat{\mathbf{Q}}(\pi(i)) - \lambda_i\right)\right), \quad \text{for all } i \in \Theta\setminus\{1\}.$$

Moreover, by the complementary slackness conditions $\lambda_i(\widehat{V}_T(\pi(1)) - \widehat{V}_T(\pi(i))) = 0$, for all $i \in \Theta\setminus\{1\}$, there must exist $I \subseteq \Theta\setminus\{1\}$ such that $\lambda_i = 0$, for all $i \in \Theta\setminus(I\cup\{1\})$, while, for all $i \in I$, $\widehat{V}_T(\pi(1)) - \widehat{V}_T(\pi(i)) = 0$.

The case $I = \emptyset$ is trivial, thus suppose $I \neq \emptyset$. For every $i \in I$, equation $\widehat{V}_T(\pi(1)) - \widehat{V}_T(\pi(i)) = 0$ holds if and only if

$$\left(\frac{\widehat{\mathbf{Q}}(\pi(1))}{\widehat{\mathbf{P}}^\pi(\pi(1))} - \frac{\widehat{\mathbf{Q}}(\pi(i))}{\widehat{\mathbf{P}}^\pi(\pi(i))}\right)\lambda_1 + \left(\frac{1}{\widehat{\mathbf{P}}^\pi(\pi(1))} + \frac{1}{\widehat{\mathbf{P}}^\pi(\pi(i))}\right)\lambda_i + \frac{1}{\widehat{\mathbf{P}}^\pi(\pi(1))}\sum_{k\in I\setminus\{i\}}\lambda_k = 0.$$

Choose an enumeration of $I\cup\{1\} = \{i_1, i_2, \dots, i_n\}$ with $i_1 = 1$. Then the above equations give rise to the homogeneous linear system $\mathbf{A}\mathbf{x} = \mathbf{0}$, whose unknown is the column vector $\mathbf{x} = [\lambda_1\ \lambda_{i_2}\ \cdots\ \lambda_{i_n}]^T \in \mathbb{R}^n$ and whose coefficient matrix is $\mathbf{A} = \left[\mathbf{q}|\mathbf{B}\right] \in \mathbb{R}^{(n-1)\times n}$ with

$$\mathbf{q} = \left[\left(\frac{\widehat{\mathbf{Q}}(\pi(1))}{\widehat{\mathbf{P}}^\pi(\pi(1))} - \frac{\widehat{\mathbf{Q}}(\pi(i_2))}{\widehat{\mathbf{P}}^\pi(\pi(i_2))}\right)\ \left(\frac{\widehat{\mathbf{Q}}(\pi(1))}{\widehat{\mathbf{P}}^\pi(\pi(1))} - \frac{\widehat{\mathbf{Q}}(\pi(i_3))}{\widehat{\mathbf{P}}^\pi(\pi(i_3))}\right)\ \cdots\ \left(\frac{\widehat{\mathbf{Q}}(\pi(1))}{\widehat{\mathbf{P}}^\pi(\pi(1))} - \frac{\widehat{\mathbf{Q}}(\pi(i_n))}{\widehat{\mathbf{P}}^\pi(\pi(i_n))}\right)\right]^T,$$
$$\mathbf{B} = \mathbf{C} + \mathbf{D},$$

where $\mathbf{C}, \mathbf{D} \in \mathbb{R}^{(n-1)\times(n-1)}$, $\mathbf{C}$ is a constant matrix with all entries equal to $\frac{1}{\widehat{\mathbf{P}}^\pi(\pi(1))}$ and $\mathbf{D}$ is the diagonal matrix whose diagonal contains the elements $\frac{1}{\widehat{\mathbf{P}}^\pi(\pi(i_2))}, \dots, \frac{1}{\widehat{\mathbf{P}}^\pi(\pi(i_n))}$. Subtracting the first row of $\mathbf{B}$ from all other rows and applying the Laplace expansion to the resulting matrix along the first row we get that

$$\det\mathbf{B} = \det(\mathbf{C}+\mathbf{D}) = \sum_{j=1}^n \widehat{\mathbf{P}}^\pi(\pi(i_j)) \Bigg/ \prod_{j=1}^n \widehat{\mathbf{P}}^\pi(\pi(i_j)),$$

and since $\det\mathbf{B} \neq 0$, we have that $\operatorname{rank}\mathbf{B} = n-1$ and the system admits non-trivial solutions, depending on one real parameter that we identify with λ_1. Now, apply Cramer's rule to the reduced system $\mathbf{B}\mathbf{y} = -\lambda_1\mathbf{q}$ with unknown the column vector $\mathbf{y} = \left[\lambda_{i_2}\ \cdots\ \lambda_{i_n}\right]^T \in \mathbb{R}^{n-1}$.

For $j = 2, \dots, n$, denote by $\mathbf{B}_{j-1}$ the matrix obtained by substituting the $(j-1)$-th column of $\mathbf{B}$ with the vector $-\lambda_1\mathbf{q}$. Applying the Laplace expansion along the $(j-1)$-th column of $\mathbf{B}_{j-1}$ and noticing that all minors can be transformed (by swapping rows and keeping track of sign changes) in the sum of a constant matrix and a diagonal matrix (possibly with a zero on the diagonal), we have

$$\det\mathbf{B}_{j-1} = \left[\left(\frac{\widehat{\mathbf{Q}}(\pi(i_j))}{\widehat{\mathbf{P}}^\pi(\pi(i_j))} - \frac{\widehat{\mathbf{Q}}(\pi(1))}{\widehat{\mathbf{P}}^\pi(\pi(1))}\right)\frac{\sum_{k\in(I\cup\{1\})\setminus\{i_j\}}\widehat{\mathbf{P}}^\pi(\pi(k))}{\prod_{k\in(I\cup\{1\})\setminus\{i_j\}}\widehat{\mathbf{P}}^\pi(\pi(k))} + \sum_{k\in I\setminus\{i_j\}}\left(\left(\frac{\widehat{\mathbf{Q}}(\pi(1))}{\widehat{\mathbf{P}}^\pi(\pi(1))} - \frac{\widehat{\mathbf{Q}}(\pi(k))}{\widehat{\mathbf{P}}^\pi(\pi(k))}\right)\frac{1}{\prod_{s\in(I\cup\{1\})\setminus\{i_j,k\}}\widehat{\mathbf{P}}^\pi(\pi(s))}\right)\right]\lambda_1$$

therefore, $\lambda_{i_j} = \frac{\det\mathbf{B}_{j-1}}{\det\mathbf{B}} = A^{\pi,I}_{i_j}\lambda_1$.

Substituting in $g(\widehat{V}_T) = 0$ the expressions of $\widehat{V}_T(\pi(k))$, for all $k \in \Theta$, and λ_i, for all $i \in \Theta\setminus\{1\}$, we get for λ_1 the expression of $\lambda^{\pi,I}_1$, thus λ_i coincides with $\lambda^{\pi,I}_i$. Hence, $\widehat{V}_T \in \mathbb{R}^\Theta_{++}$ is the optimal solution for the problem (18) if and only if there exists $I \subseteq \Theta\setminus\{1\}$ inducing weights $\lambda^{\pi,I}_i$'s satisfying *(i)–(iii)*.

For every $h \in \Theta$, let π_h be a permutation of Θ such that $\pi_h(1) = h$ and denote by $\widehat{V}^{\pi_h}_T$ the optimal solution of problem (18) for π_h. By the definition of the Choquet integral (Grabisch, 2016), problem (18) is equivalent to maximizing $\mathbf{CEU}_{U_\gamma,p,\epsilon}$ over optimal solutions of the family of problems (18), indexed by all permutations of Θ. In turn, since, for every permutations π, π' of Θ such that $\pi(1) = \pi'(1)$, it holds $\widehat{\mathbf{P}}^\pi = \widehat{\mathbf{P}}^{\pi'}$, such maximization can be reduced to maximizing $\mathbf{CEU}_{U,p,\epsilon}$ over optimal solutions of the family of problems (18), indexed by permutations $\pi_1, \dots, \pi_{T+1}$ of Θ. This finally proves the theorem. □

Proof of Proposition 2. For fixed U_γ, p, u, d, r and T, we show that the optimal value of $\mathbf{CEU}_{U_\gamma,p,\epsilon}$ is non-increasing with respect to ϵ, seen as a function of $\epsilon \in [0,1)$. Therefore, let $0 \leq \epsilon < \epsilon' < 1$ and let $V^\epsilon_T, V^{\epsilon'}_T$ be the optimal solutions of (14) obtained for ϵ, ϵ', respectively. Since $\mathcal{P}_{\epsilon,p} \subset \mathcal{P}_{\epsilon',p}$, by (12) we have that $\mathbf{CEU}_{U_\gamma,p,\epsilon}[V^\epsilon_T] \geq \mathbf{CEU}_{U_\gamma,p,\epsilon'}[V^\epsilon_T]$ and $\mathbf{CEU}_{U_\gamma,p,\epsilon}[V^{\epsilon'}_T] \geq \mathbf{CEU}_{U_\gamma,p,\epsilon'}[V^{\epsilon'}_T]$. Moreover, since V^ϵ_T attains the maximum of $\mathbf{CEU}_{U_\gamma,p,\epsilon}$, then we have that $\mathbf{CEU}_{U_\gamma,p,\epsilon}[V^\epsilon_T] \geq \mathbf{CEU}_{U_\gamma,p,\epsilon}[V^{\epsilon'}_T]$, so we obtain that $\mathbf{CEU}_{U_\gamma,p,\epsilon}[V^\epsilon_T] \geq \mathbf{CEU}_{U_\gamma,p,\epsilon'}[V^{\epsilon'}_T]$. In turn, this implies that the maximum optimal value of $\mathbf{CEU}_{U_\gamma,p,\epsilon}$ is reached at $\epsilon = 0$, i.e., in absence of ambiguity.

In the case $\epsilon = 0$, if $p = q$, by Theorem 3 we obtain that the optimal solution is $V^0_T = (1+r)^T V_0$, for which we have $\mathbf{CEU}_{U_\gamma,p,0}[V^0_T] = U_\gamma((1+r)^T V_0)$. Now, since the optimal value of $\mathbf{CEU}_{U_\gamma,p,\epsilon}$ is non-increasing with respect to ϵ, for every $\epsilon \in (0,1)$ we get that $\mathbf{CEU}_{U_\gamma,p,\epsilon}[V^\epsilon_T] \leq U_\gamma((1+r)^T V_0)$. Moreover, since $V_T = (1+r)^T V_0 \in \mathbb{R}^\Omega_{++}$ and $\mathbf{CEU}_{U_\gamma,p,\epsilon}[V_T] = U_\gamma((1+r)^T V_0)$ we get that the optimal solution must be $V^\epsilon_T = (1+r)^T V_0$, for all $\epsilon \in (0,1)$. Finally, this implies that the optimal value of $\mathbf{CEU}_{U_\gamma,p,\epsilon}$ is constantly equal to $U_\gamma((1+r)^T V_0)$ for all $\epsilon \in [0,1)$. □

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