

UNITARITY OF MINIMAL W -ALGEBRAS AND THEIR REPRESENTATIONS II: RAMOND SECTOR

VICTOR G. KAC
PIERLUIGI MÖSENER FRAJRIA
PAOLO PAPI

ABSTRACT. In this paper we study unitary Ramond twisted representations of minimal W -algebras. We classify all such irreducible highest weight representations with highest weight which is not Ramond extremal (unitarity in the Ramond extremal case, as well as in the untwisted extremal case, remains open). We compute the characters of these representations and deduce from them the denominator identities for all superconformal algebras in the Neveu-Schwarz and Ramond sector. Some of the results rely on conjectures about the properties of the quantum Hamiltonian reduction functor in the Ramond sector.

CONTENTS

1. Introduction	2
2. Twisted modules	5
3. Minimal W -algebra setup	9
4. Twisted highest weight modules over minimal W -algebras	12
5. The Zhu algebra in the Ramond sector	15
6. Ramond sector: necessary conditions for unitarity	21
7. Ramond sector: sufficient conditions for unitarity	34
8. Euler-Poincaré Characters	43
9. Unitarity between $A(k, \nu)$ and $B(k, \nu, \rho_R)$	48
10. Explicit conditions for unitarity for Ramond twisted modules $L^W(\nu, \ell)$	61
10.1. $\mathfrak{g} = \mathfrak{psl}(2 2)$	61
10.2. $\mathfrak{g} = \mathfrak{spo}(2 3)$	61
10.3. $\mathfrak{g} = \mathfrak{spo}(2 2r), r > 2$	62
10.4. $\mathfrak{g} = \mathfrak{spo}(2 2r+1), r > 1$	63
10.5. $\mathfrak{g} = D(2, 1; \frac{m}{n}), m, n \in \mathbb{N}, m, n$ coprime	63
10.6. $\mathfrak{g} = F(4)$	64
10.7. $\mathfrak{g} = G(3)$	65
11. Unitarity for Ramond extremal modules of the $N = 3$ and $N = 4$ superconformal algebras	65
11.1. $N = 3$	65
11.2. $N = 4$	66
12. The characters of massless Ramond twisted modules for minimal W -algebras	66
13. Denominator identities	69
13.1. $\mathfrak{g} = \mathfrak{spo}(2 N), N = 0, 1, 2, k_0 = -\frac{1}{2}$	71
13.2. $\mathfrak{g} = \mathfrak{psl}(2 2), k_0 = -1$	75
13.3. $\mathfrak{g} = \mathfrak{spo}(2 3), k_0 = -\frac{1}{2}$	75
13.4. $\mathfrak{g} = D(2, 1; 1) = \mathfrak{spo}(2 4), k_0 = -\frac{1}{2}$	76
13.5. $\mathfrak{g} = \mathfrak{spo}(2 2r), r > 2, k_0 = -\frac{1}{2}$	76

13.6.	$\mathfrak{g} = spo(2 2r + 1), r \geq 2, k_0 = -\frac{1}{2}$	76
13.7.	$\mathfrak{g} = F(4), k_0 = -\frac{2}{3}$	77
13.8.	$\mathfrak{g} = G(3), k_0 = -\frac{3}{4}$	77
14.	Appendix. Denominator identity for minimal W -algebras of Deligne series	78
	Acknowledgements	79
	References	79

1. INTRODUCTION

The problem of unitarity of highest weight representations of infinite-dimensional Lie algebras and superalgebras has been a hot topic in the 1980s and 1990s in Mathematics and Physics (see e.g. [1] for a list of references). With the advent of vertex algebra theory it was realized that all these Lie (super)algebras arise naturally in the context of a special class of vertex algebras, called the simple quantum affine W -algebras, obtained by quantum Hamiltonian reduction from affine vertex algebras of level k , starting from the datum $(\mathfrak{g}, \mathfrak{s})$ where $\mathfrak{g}$ is a *basic* Lie superalgebra and $\mathfrak{s}$ is an sl_2 subalgebra. The most important among these W -algebras are the minimal ones, denoted by $W_k^{\min}(\mathfrak{g})$, corresponding to the “minimal” choice of $\mathfrak{s}$. They include all superconformal algebras, which play an important role in quantum physics.

This paper is a continuation of our paper [15] on classification of unitary minimal (quantum affine) W -algebras $W_k^{\min}(\mathfrak{g})$ and their (non-twisted) representations. In that paper we singled out necessary conditions for unitarity and divided the irreducible highest weight representations satisfying these conditions into *extremal* and *non-extremal ones*. We proved unitarity of non-extremal representations and conjectured that extremal representations are unitary, completing thereby the classification of unitary highest weight representations. Support to our conjecture is given by the cases of $N = 3$ and $N = 4$ superconformal algebra.

In the present paper we study Ramond twisted unitary representations of minimal W -algebras. We introduce (suitably modified) notions of extremal and non extremal modules and we prove unitarity of non-extremal representations, under (for a certain subclass) the conjectural validity of exactness properties for the twisted quantum Hamiltonian reduction functor. Again we conjecture that extremal representations are unitary, and again we have unitarity for extremal representations of the $N = 3$ and $N = 4$ superconformal algebra. Another indirect support to the properties of this functor that we are assuming is given by classical combinatorial identities derived from character formulas by using this functor.

Let us discuss in more detail the setting and the results of the paper. Let $\mathfrak{g}$ be a simple finite-dimensional Lie superalgebra, over $\mathbb{C}$, with a reductive even part $\mathfrak{g}_0$ and invariant non-degenerate symmetric bilinear form $(\cdot|\cdot)$, with restriction to $\mathfrak{g}_0$ non-degenerate. Let $\mathfrak{s} = \text{Span}\{e, x, f\}$, where $[e, f] = x, [x, e] = e, [x, f] = -f$ be an sl_2 subalgebra of $\mathfrak{g}_0$. To the datum $(\mathfrak{g}, \mathfrak{s}, k \in \mathbb{C})$ one associates the universal quantum affine W -algebra $W^k(\mathfrak{g}, \mathfrak{s})$ of level k by the quantum Hamiltonian reduction [16], [19]. If k is different from the critical level k_{crit} , the vertex algebra $W^k(\mathfrak{g}, \mathfrak{s})$ has a unique maximal ideal, and the quotient by this ideal is a simple W -algebra, denoted by $W_k(\mathfrak{g}, \mathfrak{s})$.

The *minimal* W -algebras correspond to the choice of $\mathfrak{s}$, called minimal, for which the adx -eigenspace decomposition is of the form

$$(1.1) \quad \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1, \quad \text{where } \mathfrak{g}_{-1} = \mathbb{C}f, \mathfrak{g}_1 = \mathbb{C}e.$$

We normalize the bilinear form $(\cdot|\cdot)$ by the condition $(x|x) = \frac{1}{2}$. Then $k_{crit} = -h^\vee$, where $h^\vee$ is half of the eigenvalue of the Casimir operator on $\mathfrak{g}$. The decomposition (1.1) and the numbers $h^\vee$ are listed in [19, Tables 1-3].

In order to define unitarity of a W -algebra, one needs a conjugate linear involution ϕ of $\mathfrak{g}$, which fixes the subalgebra $\mathfrak{s}$ pointwise. Then, provided that $k \in \mathbb{R}$, ϕ induces a conjugate linear involution of the vertex algebra $W^k(\mathfrak{g}, \mathfrak{s})$, and it descends to $W_k(\mathfrak{g}, \mathfrak{s})$.

It is proved in [15, Proposition 7.2] that for minimal $\mathfrak{s} \subset \mathfrak{g}$ and a non-collapsing level $k \in \mathbb{R}$, any conjugate linear involution ϕ of the vertex algebra $W^k(\mathfrak{g}, \mathfrak{s})$ is necessarily induced by a conjugate linear involution ϕ of $\mathfrak{g}$ fixing $\mathfrak{s}$. (Recall that k is called a collapsing level if $W_k(\mathfrak{g}, \mathfrak{s})$ is isomorphic to its affine part.) Moreover, it is proved in [15, Proposition 8.9] that the vertex algebra $W_k(\mathfrak{g}, \mathfrak{s})$ is unitary only if the centralizer $\mathfrak{g}^\natural$ of $\mathfrak{s}$ in $\mathfrak{g}$ is a semisimple subalgebra of $\mathfrak{g}_0$, and the conjugate linear involution ϕ is *almost compact*, i.e. it restricts to a compact involution of $\mathfrak{g}^\natural$, and it leaves $\{e, x, f\}$ fixed. We write $\mathfrak{g}^\natural = \oplus_i \mathfrak{g}_i^\natural$, where $\mathfrak{g}_i^\natural$ are simple components of $\mathfrak{g}^\natural$.

We prove in [15] that an almost compact conjugate linear involution of $\mathfrak{g}$ exists if and only if $\mathfrak{g}$ is from the following lists:

$$(1.2) \quad psl(2|2), \quad spo(2|m) \text{ for } m \geq 0, D(2, 1; a) \text{ for } a \in \mathbb{R}, F(4), G(3);$$

$$(1.3) \quad sl(2|m) \text{ for } m \geq 3, osp(4|m) \text{ for } m > 2 \text{ even},$$

and it is essentially unique. Moreover, in these cases $\mathfrak{g}$ admits a unique, up to conjugation, minimal sl_2 -subalgebra $\mathfrak{s}$. We denote the corresponding minimal universal W -algebra of level k by $W_{\min}^k(\mathfrak{g})$, and its simple quotient by $W_k^{\min}(\mathfrak{g})$. Recall that, for $k \neq k_{crit}$, the vertex algebra $W_{\min}^k(\mathfrak{g})$ is conformal with Virasoro field $L = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$, and it is strongly and freely generated by the operators L_n , $n \in \mathbb{Z}$, and the Fourier coefficients of the primary fields $J^{\{a\}}(z) = \sum_{n \in \mathbb{Z}} J_n^{\{a\}} z^{-n-1}$, $a \in \mathfrak{g}^\natural$, of conformal weight 1, and $G^{\{u\}}(z) = \sum_{n \in \frac{1}{2} + \mathbb{Z}} G_n^{\{u\}} z^{-n-\frac{3}{2}}$, $u \in \mathfrak{g}_{-1/2}$, of conformal weight $\frac{3}{2}$ [19, Theorems 4.1 and 5.1].

Note that, as in [15], we exclude the case of $\mathfrak{g} = spo(2|m)$, $m = 0, 1$, and 2, since in these cases the W -algebra $W_{\min}^k(\mathfrak{g})$ is the universal Virasoro, Neveu-Schwarz, and $N = 2$ vertex algebra, respectively, for which unitarity of non-twisted and twisted modules is well understood.

A non-degenerate Hermitian form H on a module M with finite-dimensional L_0 eigenspaces over a conformal vertex algebra is called ϕ -invariant if it defines an isomorphism of M with its restricted dual. For $W_{\min}^k(\mathfrak{g})$ this is equivalent to the following conditions [7], [14]:

$$L_n^* = L_{-n}, \quad (J_n^{\{a\}})^* = J_{-n}^{\{\phi(a)\}}, \quad (G_n^{\{u\}})^* = G_{-n}^{\{\phi(u)\}}.$$

We proved in [15, Proposition 8.19] that, for $k \neq k_{crit}$, the minimal W -algebra $W_k^{\min}(\mathfrak{g})$ is not unitary for $\mathfrak{g}$ from the list (1.3), except when $\mathfrak{g} = sl(2|m)$, $m \geq 3$, and the level is the collapsing level $k = -1$. Furthermore, we proved in [15, Corollary 11.2] that, for $\mathfrak{g}$ from the list (1.2), the vertex algebra $W_k^{\min}(\mathfrak{g})$ is non-trivial unitary for $k \neq k_{crit}$ if and only if k lies in the *unitary range*, given in the following Table 1, along with $k = k_{crit}$ and $k = k_0$ for which $\dim W_k^{\min}(\mathfrak{g}) = 1$:

$\mathfrak{g}$	unitary range	k_{crit}	k_0
$psl(2 2)$	$-(\mathbb{N} + 1)$	0	-1
$spo(2 3)$	$-\frac{1}{4}(\mathbb{N} + 2)$	$-\frac{1}{2}$	$-\frac{1}{2}$
$spo(2 m), m \geq 4$	$-\frac{1}{2}(\mathbb{N} + 1)$	$\frac{m}{2} - 2$	$-\frac{1}{2}$
$D(2, 1; \frac{m}{n})$	$-\frac{mn}{m+n}\mathbb{N}, m, n \in \mathbb{N} \text{ coprime}, (m, n) \neq (1, 1)$	0	none
$F(4)$	$-\frac{2}{3}(\mathbb{N} + 1)$	2	$-\frac{2}{3}$
$G(3)$	$-\frac{3}{4}(\mathbb{N} + 1)$	$\frac{3}{2}$	$-\frac{3}{4}$

TABLE 1

In our paper [15], we also studied unitarity of irreducible highest weight $W_{\min}^k(\mathfrak{g})$ -modules $L^W(\nu, \ell_0)$, where $\mathfrak{g}$ is one of the Lie superalgebras from Table 1 (with the exception of $spo(2|m)$, $m \leq 2$), and k lies in the unitary range. These modules are parametrized by pairs $\nu \in (\mathfrak{h}_{\mathbb{R}}^{\natural})^*$ and $\ell_0 \in \mathbb{R}$. We proved that unitarity of $L^W(\nu, \ell_0)$ holds if and only if the following condition holds:

- (a) the affine levels $M_i(k)$ for $\mathfrak{g}_i^{\natural}$ are non-negative integers;
- (b) $\nu \in P_k^+ = \{\text{dominant integral weights for } \mathfrak{g}^{\natural} \text{ such that } \nu(\theta_i^{\vee}) \leq M_i(k)\}$ where θ_i are highest roots of $\mathfrak{g}_i^{\natural}$.
- (c) $\ell_0 \geq A(k, \nu)$, where $A(k, \nu)$ is defined in [15, formula (8.11)], and $\ell_0 = A(k, \nu)$ if ν is an extremal weight (i.e. $\nu(\theta_i^{\vee}) > M_i(k) + \chi_i$ for some i , χ_i being displayed in [15, Table 2]), except that the unitarity of $L^W(\nu, \ell_0)$ when ν is an extremal weight and $\ell_0 = A(k, \nu)$ is still an open question.

Actually in [15] we studied unitarity of the $W_{\min}^k(\mathfrak{g})$ -modules; however it has been proved in [1, Theorem 5.1] that any unitary $W_{\min}^k(\mathfrak{g})$ -module descends to the simple W -algebra $W_{\min}^k(\mathfrak{g})$.

The study of unitarity of the Ramond twisted irreducible highest weight modules over the vertex algebra $W_{\min}^k(\mathfrak{g})$, where k is in the unitary range, proceeds along similar lines. The main difference is that in the Ramond sector one has to consider separately two cases: when $\frac{1}{2}\theta$ is not a root of $\mathfrak{g}$, and when it is a root, where θ is the highest root of $\mathfrak{g}$. In both cases the necessary conditions of unitarity are similar to the above conditions (a), (b), (c), except that in condition (c) the constant $A(k, \nu)$ is replaced by the one given by (6.31), which we denote here by A^{tw} , and the notion of an extremal weight needs to be replaced by that of a Ramond extremal weight, defined by (9.3). See Section 6 for details.

As in [15, Section 10], we find sufficient conditions of unitarity of Ramond twisted irreducible highest weight modules over $W_{\min}^k(\mathfrak{g})$ by using its free field realization, introduced in [19, Theorem 5.2] and the Ramond twisted version of the Fairlie type modification. As a result, we prove unitarity for ℓ_0 larger than a certain constant B , defined by (7.25), in the cases when ν is not Ramond extremal (see Section 7).

It turns out that $B = A^{\text{tw}}$ in the cases when $\theta/2$ is a root of $\mathfrak{g}$ (see Lemma 9.4 (1)), which completes the proof of unitarity when ν is not Ramond extremal.

However, in the case when $\theta/2$ is not a root of $\mathfrak{g}$, $B = A^{\text{tw}}$ only for some very special weights ν (see Corollary 9.5). Generically one has that $B > A^{\text{tw}}$, and we need to use Proposition 8.5 on Euler-Poincaré characters, instead of determinants of ϕ -invariant Hermitian forms for twisted $W_{\min}^k(\mathfrak{g})$ -modules [20], as in [15, Section 11] for the non-twisted sector. At this point we need to use Conjecture 9.11, which claims that Arakawa's results [3] on properties of the

quantum Hamiltonian reduction functor can be extended to the Ramond twisted case. See Section 9 for details.

Note that, due to Corollary 9.10, analogous to that in [1], all unitary irreducible non-twisted or twisted highest weight modules over $W_{\min}^k(\mathfrak{g})$ descend to $W_k^{\min}(\mathfrak{g})$.

In Section 10 conditions for unitarity of Ramond twisted irreducible highest weight modules over $W_{\min}^k(\mathfrak{g})$ are exhibited in all cases, except for the well known cases of $\mathfrak{g} = sl_2$, $spo(2|1)$, and $spo(2|2)$, corresponding to Virasoro, Neveu-Schwarz, and $N = 2$ vertex algebras.

In Section 11 we prove unitarity of Ramond extremal modules over the $N = 3$ and $N = 4$ vertex algebras. The analysis of extremal modules for the big $N = 4$ superconformal algebra will appear in a forthcoming publication. For other unitary minimal W -algebras the problem of unitarity of Ramond extremal modules remains open.

In Section 12, we compute the characters of the Ramond twisted irreducible highest weight modules $W_k^{\min}(\mathfrak{g})$, when k is in the unitary range. As in the non-twisted case [15], there are two cases to consider. In the first case, called *massive*, the Ramond twisted $W_k^{\min}(\mathfrak{g})$ -module is obtained by twisted quantum Hamiltonian reduction of typical modules over the corresponding affine Lie algebra, and in the second case, called *massless*, from the maximally atypical ones. The corresponding character formulas are obtained by twisted quantum Hamiltonian reduction, using the properties conjectured in Conjecture 9.11, and they are given by Theorem 9.13 and Theorem 12.4 respectively.

In Section 13, using the character formulas for massless representations in the case of level k_0 when $\dim W_{k_0}^{\min}(\mathfrak{g}) = 1$, we find the denominator identities for $W_{\min}^k(\mathfrak{g})$. As a result, we recover the classical identities of Euler, Gauss and Ramanujan, and find some new identities.

In the Appendix we discuss a denominator identity for minimal W -algebras of Deligne series by exploiting a recent result [5] about certain $\hat{\mathfrak{g}}$ -modules of negative integer level.

Throughout the paper the base field is $\mathbb{C}$, and $\mathbb{Z}_+$ and $\mathbb{N}$ stand for the set of non-negative and positive integers, respectively.

2. TWISTED MODULES

We will denote by p the parity in a vector superspace, and let $p(a, b) = (-1)^{p(a)p(b)}$. Let R be a Lie conformal superalgebra over $\mathbb{C}$ with infinitesimal translation operator T and λ -bracket

$$(2.1) \quad [a_\lambda b] = \sum_{j \in \mathbb{Z}_+} \frac{\lambda^j}{j!} a_{(j)} b.$$

Let σ be a diagonalizable automorphism of R . We shall always assume that all eigenvalues of σ have modulus 1. We have:

$$(2.2) \quad R = \bigoplus_{\bar{\mu} \in \mathbb{R}/\mathbb{Z}} R^{\bar{\mu}}, \text{ where } R^{\bar{\mu}} = \{a \in R \mid \sigma(a) = e^{2\pi i \bar{\mu}} a\}.$$

Here and further $\bar{\mu}$ denotes the coset $\mu + \mathbb{Z}$ of $\mu \in \mathbb{R}$. Consider the subspace $\bigoplus_{\mu \in \mathbb{R}} (R^{\bar{\mu}} \otimes t^\mu)$ of $R[t^{\mathbb{R}}]$; it is $T \otimes 1 + 1 \otimes \partial_t$ -invariant. We associate to the pair (R, σ) the σ -twisted Lie superalgebra

$$(2.3) \quad \text{Lie}(R, \sigma) = \left(\bigoplus_{\mu \in \mathbb{R}} (R^{\bar{\mu}} \otimes t^\mu) \right) / \text{Image}(T \otimes 1 + 1 \otimes \partial_t),$$

endowed with the following (well-defined) bracket, where $a_{(\mu)}$ stands for the image of $a \otimes t^\mu$ in $\text{Lie}(R, \sigma)$:

$$(2.4) \quad [a_{(\mu)}, b_{(\nu)}] = \sum_{j \in \mathbb{Z}_+} \binom{\mu}{j} (a_{(j)} b)_{(\mu+\nu-j)}, \quad \mu, \nu \in \mathbb{R}.$$

A $\text{Lie}(R, \sigma)$ -module M is said to be *restricted* if, for each $m \in M$,

$$a_{(\mu)} m = 0 \text{ for } \mu \gg 0.$$

Let $V(R)$ be the universal enveloping vertex algebra of R . By the universality property of $V(R)$, σ extends to define an automorphism of $V(R)$. Since $V(R)$ is generated by R , it is clear that σ is diagonalizable on $V(R)$ with modulus one eigenvalues.

If V is a vertex algebra and σ is a diagonalizable automorphism of V with modulus one eigenvalues, then we write

$$V = \bigoplus_{\bar{\mu} \in \mathbb{R}/\mathbb{Z}} V^{\bar{\mu}}$$

to be its eigenspace decomposition. Recall that a σ -twisted module M over V is a linear map $a \rightarrow Y^M(a, z) = \sum_{\mu \in \bar{\mu}} a_{(\mu)}^M z^{-\mu-1}$ ($a \in V^{\bar{\mu}}$) where $a_{(\mu)}^M \in \text{End} M$ and for any $v \in M$, $a_{(\mu)}^M v = 0$ if $\mu \gg 0$, satisfying

$$(2.5) \quad |0\rangle_{(\mu)}^M = \delta_{\mu, -1} I_M,$$

$$(2.6) \quad \sum_{j \in \mathbb{Z}_+} \binom{\mu}{j} (a_{(n+j)} b)_{(\mu+\nu-j)}^M v \\ = \sum_{j \in \mathbb{Z}_+} (-1)^j \binom{n}{j} (a_{(\mu+n-j)}^M b_{(\nu+j)}^M - p(a, b) (-1)^n b_{(\nu+n-j)}^M a_{(\mu+j)}^M) v,$$

$$(2.7) \quad (Ta)_{(\mu)}^M = -\mu a_{(\mu-1)}^M.$$

where $a \in V^{\bar{\mu}}$, $b \in V^{\bar{\nu}}$, $n \in \mathbb{Z}$. Specializing (2.6) to $n = 0$ one obtains (2.4). It follows from (2.4) that a σ -twisted $V(R)$ -module that satisfies (2.7) is naturally a restricted $\text{Lie}(R, \sigma)$ -module. The construction given in [22, §3] shows that the converse also holds: if M is a $\text{Lie}(R, \sigma)$ -module, let, for $a \in R^{\bar{\mu}}$, $a_{(\mu)}^M$ to be the operator on M given by the action of $a_{(\mu)}$. For completeness we give a detailed proof of Li's result in a slightly more general setting (since we consider also infinite order automorphisms). Define the quantum fields

$$(2.8) \quad a^M(z) \equiv Y^M(a, z) = \sum_{\mu \in \bar{\mu}} a_{(\mu)}^M z^{-\mu-1}, \quad a \in R^{\bar{\mu}}.$$

Proposition 2.1. *The assignment $a \mapsto Y^M(a, z)$ given by (2.8), extends to define the structure of a σ -twisted $V(R)$ -module on M such that (2.7) holds for all $a \in V(R)$.*

Proof. Fix $\bar{\mu} \in \mathbb{R}/\mathbb{Z}$ and let

$$F(M, \bar{\mu}) = \left\{ \sum_{\mu \in \bar{\mu}} a_{(\mu)}^M z^{-\mu-1} \mid a_{(\mu)}^M \in \text{End}(M), \ a_{(\mu)}^M m = 0 \text{ for } m \in M, \ \mu \gg 0 \right\}.$$

If $a \in R^{\bar{\mu}}$, then $a^M(z)$ lies in $F(M, \bar{\mu})$. For $a(z) \in F(M, \bar{\mu})$ choose $\mu \in \bar{\mu}$ and define an n -product by setting

$$(2.9) \quad a(z)_{(n)} b(z) = \text{Res}_{z_1} \text{Res}_{z_0} i_{z_1, z_0} \left(\frac{z_1 - z_0}{z} \right)^\mu z_0^n X,$$

where

$$X = \sum_{n \in \mathbb{Z}} z_0^{-n-1} (i_{z_1, z}(z_1 - z)^n a(z_1) b(z) - (-1)^n p(a, b) i_{z, z_1}(z - z_1)^n b(z) a(z_1)).$$

As usual, i_{z_0, z_1} stands for the expansion in the domain $|z_0| > |z_1|$. Note that $a(z)_{(n)} b(z) \in F(M, \bar{\mu} + \bar{\nu})$. Similarly to [22, Remark 3.8], one shows that this definition does not depend on the choice of μ . Recall from [22, Definition 3.2] the definition of locality for twisted quantum fields and remark that Dong's Lemma holds in this setting (cf. [22, Proposition 3.9]). Hence the maximal local family A containing $a^M(z)$, $a \in R^{\bar{\mu}}$, $\bar{\mu} \in \mathbb{R}/\mathbb{Z}$, is a vertex algebra and the map on A defined by setting $\sigma(a(z)) = e^{2\pi\sqrt{-1}\bar{\mu}} a(z)$ for $a(z) \in F(M, \bar{\mu}) \cap A$ is an automorphism of A . It is then clear that M is a σ -twisted representation of A .

It is therefore enough to show that A is a quotient of $V(R)$: consider the special case $a(z_1) = Y^M(a, z_1)$, $b(z) = Y^M(b, z)$, $a \in R^{\bar{\mu}}$, $b \in R$. Since M is a twisted representation of A , (2.6) holds. We rewrite it as in [6, (2.43)] using the twisted delta functions

$$\delta_{\bar{\mu}}(z - w) := z^{-1} \sum_{\mu \in \bar{\mu}} \left(\frac{w}{z}\right)^{\mu}.$$

Using standard properties of formal calculus, we obtain

$$\begin{aligned}
X &= \sum_{n \in \mathbb{Z}} z_0^{-n-1} (i_{z_1, z}(z_1 - z)^n a(z_1) b(z) - p(a, b) i_{z, z_1}(z_1 - z)^n b(z) a(z_1)) \\
&= \sum_{n \in \mathbb{Z}} z_0^{-n-1} \left(\sum_{j \in \mathbb{Z}_+} Y^M(a_{(n+j)} b, z) \partial_z^j \delta_{\bar{\mu}}(z_1 - z) / j! \right) \\
&= \sum_{n \in \mathbb{Z}} z_0^{-n-1} \left(\sum_{j \in \mathbb{Z}_+} \sum_{m \in \bar{\gamma}} Y^M(a_{(n+j)} b, z) \partial_z^j z^m z_1^{-1-m} / j! \right) \\
&= \left(\sum_{j \in \mathbb{Z}_+} \sum_{m \in \bar{\gamma}} Y^M \left(\sum_{n \in \mathbb{Z}} z_0^{-n-1} a_{(n+j)} b, z \right) \partial_z^j z^m z_1^{-1-m} / j! \right) \\
&= \left(\sum_{j \in \mathbb{Z}_+} z_0^j \sum_{m \in \mathbb{Z}} Y^M(Y(a, z_0) b, z) \partial_z^j z^{m+\gamma_a} z_1^{-1-m-\gamma_a} / j! \right) \\
&= Y^M(Y(a, z_0) b, z) \sum_{m \in \mathbb{Z}} \sum_{j \in \mathbb{Z}_+} \frac{1}{j!} z_1^{-1-m-\gamma_a} \partial_{z_0}^j (z + z_0)_{|z_0=0}^{m+\gamma_a} z_0^j \\
&= Y^M(Y(a, z_0) b, z) \sum_{m \in \mathbb{Z}} z_1^{-1} i_{z, z_0} \left(\frac{z + z_0}{z_1} \right)^m i_{z, z_0} \left(\frac{z + z_0}{z_1} \right)^{\gamma_a} \\
(2.10) \quad &= Y^M(Y(a, z_0) b, z) z^{-1} \sum_{m \in \mathbb{Z}} i_{z_1, z_0} \left(\frac{z_1 - z_0}{z} \right)^m i_{z_1, z_0} \left(\frac{z_1 - z_0}{z} \right)^{-\gamma_a}.
\end{aligned}$$

Using (2.10) we have

$$\begin{aligned}
Y^M(a, z)_{(j)} Y^M(b, z) &= \text{Res}_{z_1} \text{Res}_{z_0} i_{z_1, z_0} \left(\frac{z_1 - z_0}{z} \right)^{\gamma_a} z_0^j X \\
&= \text{Res}_{z_1} \text{Res}_{z_0} z_0^j Y^M(Y(a, z_0)b, z) z^{-1} \sum_{m \in \mathbb{Z}} i_{z_1, z_0} \left(\frac{z_1 - z_0}{z} \right)^m \\
&= \text{Res}_{z_1} \text{Res}_{z_0} z_0^j Y^M(Y(a, z_0)b, z) z_1^{-1} \sum_{m \in \mathbb{Z}} i_{z, z_0} \left(\frac{z + z_0}{z_1} \right)^m \\
&= \text{Res}_{z_0} z_0^j Y^M(Y(a, z_0)b, z) = Y(a_{(j)}b, z).
\end{aligned}$$

We have proven that the map $a \mapsto Y^M(a, z)$ is a Lie conformal superalgebra homomorphism $R \rightarrow A$. By the universality property of $V(R)$ there is a vertex algebra homomorphism $V(R) \rightarrow A$ extending $a \mapsto Y^M(a, z)$, $a \in R$. $\square$

Example 2.2. Let A be a superspace with a non-degenerate skew-supersymmetric bilinear form $\langle \cdot, \cdot \rangle$. Let $R = (\mathbb{C}[T] \otimes A) \oplus \mathbb{C}K$ be the Lie conformal superalgebra with λ -bracket

$$[a_\lambda b] = \langle a, b \rangle K, \quad a, b \in A.$$

Let $V(R)$ be the universal vertex algebra of R and set $F(A) = V(R)/(K - |0\rangle)$. Let σ be a linear diagonalizable map on A with modulus one eigenvalues and such that $\langle \sigma(a), \sigma(b) \rangle = \langle a, b \rangle$. Write $A = \bigoplus_{\bar{\mu} \in \mathbb{R}/\mathbb{Z}} A^{\bar{\mu}}$ for the corresponding eigenspace decomposition. In this case

$$\begin{aligned}
\text{Lie}(R, \sigma) &= \left(\bigoplus_{\mu \in \mathbb{R}} (\mathbb{C}[T] \otimes A^{\bar{\mu}}) t^\mu \oplus \mathbb{C}K[t^{\pm 1}] \right) / \text{Image}(T \otimes 1 + 1 \otimes \partial_t) \\
&= \left(\bigoplus_{\mu \in \mathbb{R}} (\mathbb{C}[T] \otimes A^{\bar{\mu}}) t^\mu \right) / \text{Image}(T \otimes 1 + 1 \otimes \partial_t) \oplus \mathbb{C}K.
\end{aligned}$$

From now on we shall drop $\otimes$ sign. Consider the superspace

$$\tilde{A} = \sum_{\mu \in \mathbb{R}} A^{\bar{\mu}} t^\mu,$$

and Lie superalgebra

$$\hat{A}^{\text{tw}} = \tilde{A} \oplus \mathbb{C}K.$$

with bracket

$$[at^\mu + \alpha K, bt^\nu + \beta K] = \delta_{\mu+\nu, -1} \langle a, b \rangle K, \quad a \in A^{\bar{\mu}}, b \in A^{\bar{\nu}}, \alpha, \beta \in \mathbb{C}.$$

Then the map

$$(2.11) \quad \left(\frac{T^r}{r!} \otimes a \right) t^\mu \mapsto (-1)^r \binom{\mu}{r} a t^{\mu-r}, \quad K \mapsto K$$

extends to a Lie superalgebra isomorphism $\text{Lie}(R, \sigma) \cong \hat{A}^{\text{tw}}$.

Extend $\langle \cdot, \cdot \rangle$ to $\sum_{\mu \in \mathbb{R}} A^{\bar{\mu}} t^\mu$ by

$$\langle at^\nu, bt^\mu \rangle = \delta_{\nu+\mu, -1} \langle a, b \rangle,$$

and consider the corresponding Clifford algebra $Cl(\tilde{A}, \langle \cdot, \cdot \rangle)$. Choose a maximal isotropic subspace U of $A^{-1/2}$ and set

$$\tilde{A}^+ = \left(Ut^{-1/2} \right) \oplus \sum_{\mu > -\frac{1}{2}} A^{\bar{\mu}} t^{\mu}.$$

Let

$$F(A, \sigma) = Cl(\tilde{A}, \langle \cdot, \cdot \rangle) / (Cl(\tilde{A}, \langle \cdot, \cdot \rangle) \tilde{A}^+).$$

We can extend the natural action of $\tilde{A}$ on $F(A, \sigma)$ to $\hat{A}^{\text{tw}}$ by letting K act by $I_{F(A, \sigma)}$. Under the identification (2.11), Proposition 2.1 gives a σ -twisted representation of $V(R)$ on $F(A, \sigma)$ with K acting by $I_{F(A, \sigma)}$, hence a σ -twisted representation of $F(A)$.

Example 2.3. Let $\mathfrak{g}$ be either a simple Lie superalgebra or an (even) abelian Lie algebra. Assume furthermore that $\mathfrak{g}$ is equipped with an even supersymmetric non-degenerate invariant bilinear form $(\cdot | \cdot)$. Fix $k \in \mathbb{C}$ and let $R = \mathbb{C}[T] \otimes \mathfrak{g} \oplus \mathbb{C}K$ be the Lie conformal algebra with λ -bracket

$$[a_\lambda b] = [a, b] + \lambda k(a|b)K.$$

The vertex algebra $V(R)/(K - k|0\rangle)$ is called the universal affine vertex algebra of level k associated to $\mathfrak{g}$ and it is denoted by $V^k(\mathfrak{g})$. Let σ be an automorphism of $\mathfrak{g}$ with modulus one eigenvalues such that $(\sigma(a)|\sigma(b)) = (a|b)$, and let

$$\mathfrak{g} = \bigoplus_{\bar{\mu} \in \mathbb{R}/\mathbb{Z}} \mathfrak{g}^{\bar{\mu}}, \text{ where } \mathfrak{g}^{\bar{\mu}} = \{a \in \mathfrak{g} \mid \sigma(a) = e^{2\pi i \bar{\mu}} a\}$$

be its eigenspace decomposition. Let

$$\tilde{\mathfrak{g}}^{\text{tw}} = \sum_{\mu \in \mathbb{R}} \mathfrak{g}^{\bar{\mu}} t^{\mu}$$

be the corresponding twisted loop algebra, and consider the central extension

$$\hat{\mathfrak{g}}^{\text{tw}'} = \tilde{\mathfrak{g}}^{\text{tw}} \oplus \mathbb{C}K$$

of $\tilde{\mathfrak{g}}$ with bracket

$$(2.12) \quad [at^{\mu}, bt^{\nu}] = [a, b]t^{\mu+\nu} + \mu \delta_{\mu, -\nu} (a|b)K, \quad a \in \mathfrak{g}^{\bar{\mu}}, b \in \mathfrak{g}^{\bar{\nu}}.$$

As in Example 2.2, we can identify $\text{Lie}(R, \sigma)$ with $\hat{\mathfrak{g}}^{\text{tw}'}$ via (2.11).

Let (M, π_M) be a restricted $\hat{\mathfrak{g}}^{\text{tw}'}$ -module such that K acts by kI_M and define σ -twisted fields

$$Y^M(a, z) = \sum_{\mu \in \mathbb{R}} \pi_M(at^{\mu})z^{-\mu-1}, \text{ where } a \in \mathfrak{g}^{\bar{\mu}}.$$

Since (2.4) in this case is (2.12), we see, by Proposition 2.1, that M is a σ -twisted module over the vertex algebra $V^k(\mathfrak{g})$.

3. MINIMAL W -ALGEBRA SETUP

Let $\mathfrak{g}$ be a basic simple finite-dimensional Lie superalgebra such that

$$(3.1) \quad \mathfrak{g}_{\bar{0}} = \mathfrak{s} \oplus \mathfrak{g}^{\natural}.$$

where $\mathfrak{s} \cong \mathfrak{sl}_2$ and $\mathfrak{g}^{\natural}$ is the centralizer of $\mathfrak{s}$ in $\mathfrak{g}$. This corresponds to consider $\mathfrak{g}$ as in Table 2 of [19]. Let $\{e, x, f\}$ be an $\mathfrak{sl}_2$ -triple for $\mathfrak{s}$, i.e. $\mathfrak{s} = \text{span}(e, x, f)$, and $[x, e] = e, [x, f] = -f, [e, f] = x$. Let

$$(3.2) \quad \mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$$

be the adx -eigenspace decomposition of $\mathfrak{g}$. Thus

$$\mathfrak{g} = \mathbb{C}f + \mathfrak{g}_{-1/2} + \mathfrak{g}_0 + \mathfrak{g}_{1/2} + \mathbb{C}e,$$

with

$$\mathfrak{g}_0 = \mathbb{C}x \oplus \mathfrak{g}^{\natural}.$$

Note that our assumptions imply that $\mathfrak{g}_{\pm 1/2}$ are purely odd.

We will also assume, as in [15], that $\mathfrak{g}^{\natural}$ is not abelian; this condition rules out $\mathfrak{g} = \mathfrak{spo}(2|m)$, $m = 0, 1, 2$. Since we are interested in unitary W -algebras, according to [15], we may exclude $\mathfrak{g} = \mathfrak{sl}(2|m)$ and $\mathfrak{osp}(4|m)$ with $m > 2$ from consideration. Then

$$(3.3) \quad \mathfrak{g}^{\natural} = \bigoplus_{i=1}^s \mathfrak{g}_i^{\natural}$$

is the decomposition of $\mathfrak{g}^{\natural}$ into the direct sum of simple ideals (where $s \leq 2$).

Recall that $\mathfrak{g}$ carries an even invariant non-degenerate supersymmetric bilinear form $(\cdot | \cdot)$ that we normalize by requiring that $(x|x) = \frac{1}{2}$. An important role is played by the following bilinear forms $\langle \cdot, \cdot \rangle_{\text{ne}}$ on $\mathfrak{g}_{1/2}$ and $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}_{-1/2}$:

$$(3.4) \quad \langle a, b \rangle_{\text{ne}} = (f|[a, b]), \quad \langle a, b \rangle = (e|[a, b]),$$

which are symmetric and non-degenerate. Denote by A_{ne} the vector superspace $\mathfrak{g}_{-1/2}$ with the bilinear form $\langle \cdot, \cdot \rangle$ and by A_{ch} the vector superspace $\Pi(\mathfrak{g}_{<0} + \mathfrak{g}_{<0}^*)$ with the skewsupersymmetric bilinear form given by pairing (Π is the parity reversing functor). We use

Let $h^{\vee}$ be the dual Coxeter number of $\mathfrak{g}$. Let $\mathcal{C}(\mathfrak{g})$ be the vertex algebra $V^k(\mathfrak{g}) \otimes F(A_{\text{ch}}) \otimes F(A_{\text{ne}})$. Let $d \in \mathcal{C}(\mathfrak{g})$ be as in [19, Section 1]. Since $[d_{\lambda}d] = 0$, $d_{(0)}^2 = 0$ on $\mathcal{C}(\mathfrak{g})$. The homology $(H_{\bullet}(\mathcal{C}(\mathfrak{g}), d_{(0)}))$ is the vertex algebra $W_{\min}^k(\mathfrak{g})$, called the *universal minimal W -algebra of level k* , associated to the pair $(\mathfrak{g}, \mathfrak{s})$ (cf [19]). If $k \neq -h^{\vee}$, this vertex algebra has a unique simple quotient $W_k^{\min}(\mathfrak{g})$, since $W_{\min}^k(\mathfrak{g})$, when $k \neq -h^{\vee}$, is a conformal vertex algebra with conformal vector L , given in [19, (2.2)]. Furthermore, this vertex algebra is strongly and freely generated by L , primary elements $J^{\{a\}}$, $a \in \mathfrak{g}^{\natural}$, of conformal weight 1, and primary odd elements $G^{\{u\}}$, $u \in \mathfrak{g}_{-1/2}$, of conformal weight $3/2$ [19, Theorem 5.1].

Fix an automorphism σ of $\mathfrak{g}$ with the following three properties:

$$(3.5) \quad \sigma(x) = x, \quad \sigma(f) = f;$$

$$(3.6) \quad (\sigma(a)|\sigma(b)) = (a|b), \quad a, b \in \mathfrak{g};$$

$$(3.7) \quad \sigma \text{ is diagonalizable and all its eigenvalues have modulus 1.}$$

Then σ defines automorphisms of vertex algebras $V^k(\mathfrak{g})$, $F(A_{\text{ch}})$, $F(A_{\text{ne}})$, hence an automorphism of $\mathcal{C}(\mathfrak{g})$. Since $\sigma(d) = d$, σ induces an automorphism of $W_{\min}^k(\mathfrak{g})$ (and also of $W_k^{\min}(\mathfrak{g})$).

We want to apply the construction of Example 2.2 to A_{ch} and A_{ne} . For this we introduce the following $\frac{1}{2}\mathbb{Z}$ -graded subalgebra of $\mathfrak{g}$:

$$(3.8) \quad \mathfrak{g}(\sigma) = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j(\sigma), \quad \text{where } \mathfrak{g}_j(\sigma) = \{a \in \mathfrak{g}_j | \sigma(a) = (-1)^{2j}a\}.$$

The $\frac{1}{2}\mathbb{Z}$ -gradation (3.8) looks as follows:

$$\mathfrak{g}(\sigma) = \mathbb{C}f + \mathfrak{g}_{-1/2}^{-\sigma} + \mathfrak{g}_0^{\sigma} + \mathfrak{g}_{1/2}^{-\sigma} + \mathbb{C}e.$$

Fix a σ -stable Cartan subalgebra $\mathfrak{h}^{\natural}$ of $\mathfrak{g}^{\natural}$. Then $\mathfrak{h} = \mathfrak{h}^{\natural} \oplus \mathbb{C}x$ is a σ -stable Cartan subalgebra of $\mathfrak{g}$. Define $\theta \in \mathfrak{h}^*$ by $\theta(\mathfrak{h}^{\natural}) = 0$ and $\theta(x) = 1$. Observe that θ is the weight of e and $(\theta|\theta) = 2$.

Since $\mathfrak{g}_0^\sigma = (\mathfrak{g}^\natural)^\sigma + \mathbb{C}x$, it follows that there exists an element $h_0 \in (\mathfrak{h}^\natural)^\sigma$ such that

- the eigenvalues of $\text{ad } h_0$ are real,
- h_0 is a regular element of $(\mathfrak{g}^\natural)^\sigma$,
- the 0-th eigenspace of $\text{ad } h_0$ on $\mathfrak{g}_{1/2}^{-\sigma}$ (resp. $\mathfrak{g}_{-1/2}^{-\sigma}$) is either 0 or $\mathbb{C}e_{\theta/2}$ (resp. $\mathbb{C}e_{-\theta/2}$).

Here $e_{\theta/2}$ is a root vector of $\mathfrak{g}(\sigma)$ and $\theta/2$ stands for the restriction of $\theta/2$ to $\mathfrak{h}^\sigma$.

Let $\mathfrak{n}(\sigma)_+$ (resp. $\mathfrak{n}(\sigma)_-$) be the span of all eigenvectors of $\text{ad } h_0$ with positive (resp. negative) eigenvalues and the vectors $f = e_{-\theta}$ and $e_{-\theta/2}$ (resp. $e = e_\theta$ and $e_{\theta/2}$). Then

$$(3.9) \quad \mathfrak{g}(\sigma) = \mathfrak{n}(\sigma)_- \oplus \mathfrak{h}^\sigma \oplus \mathfrak{n}(\sigma)_+.$$

Set $\mathfrak{n}_j(\sigma)_\pm = \mathfrak{n}(\sigma)_\pm \cap \mathfrak{g}_j(\sigma)$. Then the following properties hold:

- (1) $\mathfrak{n}(\sigma)_\pm$ are isotropic with respect to $(\cdot | \cdot)$, and are nilpotent subalgebras normalized by $\mathfrak{h}^\sigma$,
- (2) $f \in \mathfrak{n}(\sigma)_+$,
- (3) $\mathfrak{n}_{1/2}(\sigma)_+$ is a maximal isotropic subspace of $\mathfrak{g}_{1/2}(\sigma)$ with respect to $\langle \cdot, \cdot \rangle_{\text{ne}}$ and $\mathfrak{n}_{1/2}(\sigma)_-$ is a maximal isotropic subspace of $\mathfrak{g}_{1/2}(\sigma)$ with respect to $\langle \cdot, \cdot \rangle$.
- (4) $\mathfrak{n}_{1/2}(\sigma)_-$ is a direct sum of a maximal isotropic subspace $\mathfrak{n}_{1/2}(\sigma)'_-$ of $\mathfrak{g}_{1/2}(\sigma)$ with respect to $\langle \cdot, \cdot \rangle_{\text{ne}}$ and of a subspace $\mathfrak{g}_{1/2}^0(\sigma)$ (at most 1-dimensional), normalized by $\mathfrak{h}^\sigma$.
- (5) $\mathfrak{n}_{-1/2}(\sigma)_+$ is a direct sum of a maximal isotropic subspace $\mathfrak{n}_{-1/2}(\sigma)'_+$ of $\mathfrak{g}_{-1/2}(\sigma)$ with respect to $\langle \cdot, \cdot \rangle$ and of a subspace $\mathfrak{g}_{-1/2}^0(\sigma)$ (at most 1-dimensional), normalized by $\mathfrak{h}^\sigma$.

We thus have the following decompositions:

$$(3.10) \quad \mathfrak{g}_{1/2}(\sigma) = \mathfrak{n}_{1/2}(\sigma)_+ + \mathfrak{g}_{1/2}^0(\sigma) + \mathfrak{n}_{1/2}(\sigma)'_- ,$$

and

$$(3.11) \quad \mathfrak{g}_{-1/2}(\sigma) = \mathfrak{n}_{-1/2}(\sigma)'_+ + \mathfrak{g}_{-1/2}^0(\sigma) + \mathfrak{n}_{-1/2}(\sigma)_- .$$

Set

$$(3.12) \quad \epsilon(\sigma) = \dim \mathfrak{g}_{1/2}^0(\sigma) = \dim \mathfrak{g}_{-1/2}^0(\sigma).$$

Then $\epsilon(\sigma) = 0$ or 1 and $\epsilon(\sigma) \neq 0$ iff $\dim \mathfrak{g}_{1/2}(\sigma)$ ($= \dim \mathfrak{g}_{-1/2}(\sigma)$) is odd.

Note also that in the decomposition (3.10), $\mathfrak{n}_{1/2}(\sigma)_+$ (resp. $\mathfrak{n}_{1/2}(\sigma)'_-$) is the span of all eigenvectors of $\text{ad } h_0$ with positive (resp. negative) eigenvalues and $\epsilon(\sigma) \neq 0$ iff $\theta/2$ is a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and $\sigma(e_{\theta/2}) = -e_{\theta/2}$.

Following [20], we let $F(A_{ne}, \sigma)$ be the σ -twisted $F(A_{ne})$ -module constructed as in Example 2.2 with $U = \mathfrak{n}_{1/2}(\sigma)_+$ as maximal isotropic subspace of $A_{ne}^{-1/2} = \mathfrak{g}_{1/2}^{-\sigma}$. Similarly, we let $F(A_{ch}, \sigma)$ be the σ -twisted $F(A_{ch})$ -module constructed using $U = \mathfrak{n}_{1/2}(\sigma)_+ \oplus (\mathfrak{n}_{1/2}(\sigma)_-)^*$ as maximal isotropic subspace of $A_{ch}^{-1/2}$.

Given a σ -twisted module M of $V^k(\mathfrak{g})$, then

$$\mathcal{C}(M) = M \otimes F(A_{ch}, \sigma) \otimes F(A_{ne}, \sigma)$$

is a σ -twisted module over the vertex algebra $\mathcal{C}(\mathfrak{g})$. It has the *charge decomposition*

$$\mathcal{C}(M) = \bigoplus_{j \in \mathbb{Z}} \mathcal{C}_j(M),$$

defined by

$$\text{charge } M = \text{charge } F(A_{ne}, \sigma) = 0$$

and

$$\text{charge}(\mathfrak{n}_{1/2}(\sigma)_+ \oplus \mathfrak{n}_{1/2}(\sigma)_-^*) = -\text{charge}(\mathfrak{n}_{1/2}(\sigma)_+^* \oplus \mathfrak{n}_{1/2}(\sigma)_-) = 1.$$

Let

$$C(\mathfrak{g}) = \bigoplus_{\bar{\mu} \in \mathbb{R}/\mathbb{Z}} C(\mathfrak{g})^{\bar{\mu}}$$

be the eigenspace decompositions for σ , and

$$W_{\min}^k(\mathfrak{g}) = \bigoplus_{\bar{\mu} \in \mathbb{R}/\mathbb{Z}} W_{\min}^k(\mathfrak{g})^{\bar{\mu}}$$

the corresponding decomposition of its homology.

Since $[d_\lambda d] = 0$, it follows from (2.4) that $(d_{(0)}^{\text{tw}})^2 = 0$. Let $H(M) = \sum_{j \in \mathbb{Z}} H_j(M)$ be the homology of the complex $(\mathcal{C}(M), d_{(0)}^{\text{tw}})$, with the $\mathbb{Z}$ -grading induced by the charge decomposition.

If $a \in \mathcal{C}(\mathfrak{g})$ then

$$[d_{(0)}^{\text{tw}}, a_{(\mu)}^{\text{tw}}] = (d_{(0)} a)_{(\mu)}^{\text{tw}}.$$

In particular, if $d_{(0)} a = 0$ and $d_{(0)}^{\text{tw}} m = 0$, then $d_{(0)}^{\text{tw}}(a_{(\mu)}^{\text{tw}} m) = 0$ and, if $m = d_{(0)}^{\text{tw}} m'$, then $a_{(\mu)}^{\text{tw}} m = (-1)^{p(a)} d_{(0)}^{\text{tw}}(a_{(\mu)}^{\text{tw}} m')$. Moreover, if $a = d_{(0)} a'$, then $a_{(\mu)}^{\text{tw}} m = d_{(0)}^{\text{tw}}((a')_{(\mu)}^{\text{tw}} m)$. Therefore the quantum fields

$$Y^{H(M)}(a, z) = \sum_{\bar{\mu} \in \mathbb{R}/\mathbb{Z}} a_{(\mu)}^{\text{tw}} z^{-\mu-1}, \quad a \in W_{\min}^k(\mathfrak{g})^{\bar{\mu}},$$

are well defined and define the structure of a σ -twisted $W_{\min}^k(\mathfrak{g})$ -module on $H(M)$. This module is called the quantum Hamiltonian reduction of the σ -twisted $V^k(\mathfrak{g})$ -module M .

If $a \in W_{\min}^k(\mathfrak{g})^{\bar{\mu}}$ with conformal weight Δ_a and M is a σ -twisted module, we write the field $Y^M(a, z)$ as

$$Y^M(a, z) = \sum_{n \in \bar{\mu} - \Delta_a} a_n^M z^{-n - \Delta_a}.$$

With this notation, (2.6) can be rewritten in its graded version:

$$\begin{aligned} (3.13) \quad & \sum_{j \in \mathbb{Z}_+} \binom{m + \Delta_a - 1}{j} (a_{(n+j)} b)_{m+k}^M \\ &= \sum_{j \in \mathbb{Z}_+} (-1)^j \binom{n}{j} (a_{m+n-j}^M b_{k+j-n}^M - p(a, b) (-1)^n b_{k-j}^M a_{m+j}^M), \end{aligned}$$

where $a \in W_{\min}^k(\mathfrak{g})^{\bar{\mu}}$, $b \in W_{\min}^k(\mathfrak{g})^{\bar{\nu}}$, $m \in \bar{\mu} - \Delta_a$, $n \in \mathbb{Z}$, $k \in \bar{\nu} - \Delta_b$. Note that, putting $n = 0$, (3.13) becomes the (twisted) commutator formula

$$(3.14) \quad [a_m^M, b_k^M] = \sum_{j \in \mathbb{Z}_+} \binom{m + \Delta_a - 1}{j} (a_{(j)} b)_{m+k}^M.$$

4. TWISTED HIGHEST WEIGHT MODULES OVER MINIMAL W -ALGEBRAS

Recall from Example 2.3 the Lie superalgebras $\tilde{\mathfrak{g}}$ and $\hat{\mathfrak{g}}^{\text{tw}'}$. Let $D = -L_0^{\mathfrak{g}, \text{tw}}$. Recall [20] that we have $(a \in \mathfrak{g}^{\bar{\mu}})$:

$$[D, at^\mu] = \mu(at^\mu), \quad [D, K] = 0.$$

As usual, we shall consider the extension

$$\hat{\mathfrak{g}}^{\text{tw}} = \hat{\mathfrak{g}}^{\text{tw}'} \rtimes \mathbb{C}D$$

of $\widehat{\mathfrak{g}}^{\text{tw}'}$. The decomposition (3.9) induces a triangular decomposition of the Lie superalgebra $\widehat{\mathfrak{g}}^{\text{tw}}$:

$$(4.1) \quad \widehat{\mathfrak{g}}^{\text{tw}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+,$$

where

$$(4.2) \quad \widehat{\mathfrak{h}} = \mathfrak{h}^\sigma \oplus \mathbb{C}K \oplus \mathbb{C}D,$$

$$(4.3) \quad \widehat{\mathfrak{n}}_+ = \sum_{j \in \frac{1}{2}\mathbb{Z}} (\mathfrak{n}_j(\sigma)_+ t^{-j} + \sum_{\substack{\mu \in \mathbb{R} \\ j+\mu > 0}} \mathfrak{g}_j^{\bar{\mu}} t^\mu),$$

$$(4.4) \quad \widehat{\mathfrak{n}}_- = \sum_{j \in \frac{1}{2}\mathbb{Z}} (\mathfrak{n}_j(\sigma)_- t^{-j} + \sum_{\substack{\mu \in \mathbb{R} \\ j+\mu < 0}} \mathfrak{g}_j^{\bar{\mu}} t^\mu).$$

Recall that, given a triangular decomposition (4.1), a *highest weight module* over the Lie superalgebra $\widehat{\mathfrak{g}}^{\text{tw}}$ with highest weight $\widehat{\Lambda} \in \widehat{\mathfrak{h}}^*$ is a $\widehat{\mathfrak{g}}^{\text{tw}}$ -module M which admits a non-zero vector $v_{\widehat{\Lambda}}$ with the properties:

- (1) $h v_{\widehat{\Lambda}} = \widehat{\Lambda}(h) v_{\widehat{\Lambda}}, h \in \widehat{\mathfrak{h}},$
- (2) $\widehat{\mathfrak{n}}_+ v_{\widehat{\Lambda}} = 0,$
- (3) $U(\widehat{\mathfrak{n}}_-) v_{\widehat{\Lambda}} = M.$

Note that a highest weight module M is graded by the eigenspace decomposition corresponding to the action of D . Since, by (4.4), the eigenvalues of the action of D have real parts bounded above, it is clear that a highest weight module M is restricted. Moreover M has level k if and only if

$$(4.5) \quad \widehat{\Lambda}(K) = k.$$

In particular the highest weight modules of highest weight $\widehat{\Lambda}$ such that (4.5) holds are σ -twisted $V^k(\mathfrak{g})$ -modules.

Let $\{u_i\}_{i \in S}$ be a basis of $\mathfrak{g}$, compatible with the decomposition (3.2), where S is the index set. For $j \neq 0$ let S_j denote the subset of indices of S which corresponds to a basis of $\mathfrak{g}_j$, and denote by $S' \subset S$ the subset of indices of the part of the basis $\{u_i\}_{i \in S}$ of $\mathfrak{g}$, which is a basis of $\mathfrak{g} \bmod \mathfrak{h}^\sigma$. Let

$$(4.6) \quad s_{u_i} = \min\{n \mid u_i t^n \text{ is non-zero and lies in } \widehat{\mathfrak{n}}_+\} \text{ for } i \in S', s_h = 1 \text{ for } h \in \mathfrak{h}^\sigma.$$

Since, by (3.5), each summand $\mathfrak{g}_j$ of the gradation (3.2) is σ -invariant, we have its σ -eigenspace decomposition:

$$\mathfrak{g}_j = \oplus_{\bar{\mu} \in \mathbb{R}/\mathbb{Z}} \mathfrak{g}_j^{\bar{\mu}}, \text{ where } \mathfrak{g}_j^{\bar{\mu}} = \{a \in \mathfrak{g}_j \mid \sigma(a) = e^{2\pi i \bar{\mu}} a\}.$$

Hence for a basis element $u_i \in \mathfrak{g}_{m_i}^{\bar{\mu}_i}$ we can rewrite formula (4.6) for $s_i = s_{u_i}$ ($i \in S'$) as follows:

$$(4.7) \quad s_i = \begin{cases} \min\{n \in \bar{\mu}_i \mid n > -m_i\} & \text{if } u_i \notin \mathfrak{n}(\sigma)_+, \\ -m_i & \text{if } u_i \in \mathfrak{n}(\sigma)_+. \end{cases}$$

It is easy to see that for a dual basis element $u^i \in \mathfrak{g}_{-m_i}^{-\bar{\mu}_i}$ we have for $s^i = s_{u^i}$:

$$(4.8) \quad s^i = 1 - s_i \text{ for all } i \in S'.$$

We extend this definition to $F(A_{ne}, \sigma)$ and $F(A_{ch}, \sigma)$ as follows. Let us relabel the basis $\{u_i\}_{i \in S_{-1/2}}$ of $A_{ne} = \mathfrak{g}_{-1/2}$ as $\{\Phi_i\}_{i \in S_{-1/2}}$ and the pair of dual bases $(\{u_i\}_{i \in S_{<0}}, \{u_i^*\}_{i \in S_{<0}})$,

of $A_{ch} = \mathfrak{g}_{<0} \oplus \mathfrak{g}_{<0}^*$ as $\{\varphi_i\}, \{\varphi_i^*\}$. Then

$$(4.9) \quad s_{\Phi_i} = s_i \ (i \in S_{1/2}), \ s_{\varphi_i} = s_i, \ s_{\varphi_i^*} = 1 - s_i \ (i \in S_+).$$

It is easy to see that we have

$$(4.10) \quad s_{\Phi_i} = \mp 1/2 \text{ if } \Phi_i \in \mathfrak{n}_{1/2}(\sigma)_{\pm}, \ |s_{\Phi_i}| < 1/2 \text{ otherwise.}$$

$$(4.11) \quad s_{\Phi_i} + s_{\Phi^i} = \delta_{i,i_0}, \text{ where } \langle \Phi_{i_0}, \Phi_{i_0} \rangle_{ne} \neq 0.$$

For a σ -twisted $W_{\min}^k(\mathfrak{g})$ -module M , a vector $m \in M$ is called cyclic if polynomials in the operators $J_n^{\{a\}, \text{tw}}$, with $a \in \mathfrak{g}^{\natural}$, $n \in \mathbb{Z}$, $G_n^{\{v\}, \text{tw}}$, with $v \in \mathfrak{g}_{-1/2}$, $n \in \frac{1}{2} + \mathbb{Z}$, and L_n^{tw} with $n \in \mathbb{Z}$ applied to m span M . A vector $m \in M$ such that there are $\ell_0 \in \mathbb{C}$ and $\lambda \in ((\mathfrak{h}^{\natural})^{\sigma})^*$ for which

$$(4.12) \quad L_0^{\text{tw}}(v_{\lambda, \ell_0}) = \ell_0 v_{\lambda, \ell_0},$$

$$(4.13) \quad J_0^{\{a\}, \text{tw}} v_{\lambda, \ell_0} = \lambda(a) v_{\lambda, \ell_0} \text{ if } a \in (\mathfrak{h}^{\natural})^{\sigma},$$

is called a weight vector and the pair (λ, ℓ_0) is called the weight of m .

We define a highest weight module for $W_{\min}^k(\mathfrak{g})$ as follows:

Definition 4.1. A σ -twisted $W_{\min}^k(\mathfrak{g})$ -module M is called a *highest weight module* of highest weight (λ, ℓ_0) if there exists a cyclic weight vector $v_{\lambda, \ell_0} \in M$ of weight (λ, ℓ_0) such that

$$(4.14) \quad J_m^{\{a\}, \text{tw}} v_{\lambda, \ell_0} = G_m^{\{v\}, \text{tw}} v_{\lambda, \ell_0} = L_m^{\text{tw}} v_{\lambda, \ell_0} = 0 \text{ if } m > 0,$$

$$(4.15) \quad J_0^{\{a\}, \text{tw}} v_{\lambda, \ell_0} = 0 \text{ if } a \in \mathfrak{n}_0(\sigma)_+,$$

$$(4.16) \quad G_0^{\{u\}, \text{tw}} v_{\lambda, \ell_0} = 0 \text{ if } u \in \mathfrak{n}_{-1/2}(\sigma)'_+.$$

The vector v_{λ, ℓ_0} is called a highest weight vector.

Let V be a conformal vertex algebra strongly generated by elements $\{J^{\{i\}}\}_{i \in I}$, where $J^{\{i\}}$ has conformal weight $\Delta(i) \in \mathbb{R}$. Let M be a σ -twisted positive energy module (i.e. the eigenvalues of L_0^{tw} are bounded below). Write, for shortness, $J_m^{\{i\}}$ instead of $(J^{\{i\}})_m^M$. Then, by the commutator formula (3.14), we have:

$$(4.17) \quad [J_m^{\{i\}}, J_n^{\{j\}}] = \sum_{\vec{s}, \vec{t}} c_{m,n}^{ij}(\vec{s}, \vec{t}) (J_{t_1}^{\{s_1\}} J_{t_2}^{\{s_2\}} \dots),$$

where $c_{m,n}^{ij} \in \mathbb{C}$ and for each term of this sum we have:

$$(4.18) \quad t_r \in \mathbb{Z} - \Delta(s_r), \ \sum_r t_r = m + n \text{ and } t_1 \leq t_2 \leq \dots$$

Denote by $\mathcal{A}$ the unital associative superalgebra generated by $J_m^{\{i\}}$ ($i \in I, m \in \mathbb{Z} - \Delta(i)$) inside $\text{End}(M)$. Note that, though the sum in the R.H.S. of (4.17) may not be finite, by (4.18) and the fact that M is a positive energy V -module, the R.H.S. of (4.17) makes sense as an element of $\text{End}(M)$.

Let $\tilde{\mathcal{A}}_-$, $\tilde{\mathcal{A}}_+$ and $\tilde{\mathcal{A}}_0$ be the subalgebras of $\mathcal{A}$ generated by the $J_m^{\{i\}}$ with $m < 0$, $m > 0$ and $m = 0$, respectively. It follows from (4.17) and (4.18) that

$$(4.19) \quad \mathcal{A} = \tilde{\mathcal{A}}_- \tilde{\mathcal{A}}_0 \tilde{\mathcal{A}}_+.$$

The previous discussion proves the following result.

Lemma 4.2. *Let M be a σ -twisted highest weight $W_{\min}^k(\mathfrak{g})$ -module and let v be a highest weight vector for M . Then M is spanned by*

$$(4.20) \quad \{P_{j_1}P_{j_2}\cdots P_{j_t}v \mid j_1 < j_2 < \cdots < j_t \leq 0\} \text{ with } P_j \in \mathcal{M}_j, \text{ where}$$

$$\mathcal{M}_j = \{(J_j^{\{a_{i_1}\}})^{m_1} \cdots (J_j^{\{a_{i_t}\}})^{m_t} G_j^{\{u_{i_1}\}} \cdots G_j^{\{u_{i_t}\}} L_j^k\}, \text{ if } j < 0,$$

$$\mathcal{M}_0 = \{(J_0^{\{a_{i_1}\}})^{m_1} \cdots (J_0^{\{a_{i_t}\}})^{m_t} G_0^{\{u_{i_1}\}} \cdots G_0^{\{u_{i_t}\}} : a_{i_s} \in \mathfrak{n}_0(\sigma)_-, u_{i_r} \in (\mathfrak{n}_{-1/2}(\sigma)_- \oplus \mathfrak{g}_{-1/2}^0(\sigma))\}.$$

We say that a highest weight $W_{\min}^k(\mathfrak{g})$ -module M of highest weight (λ, ℓ_0) is a *Verma module* (denoted by $M^W(\nu, \ell_0)$) if the elements in (4.20) form a basis of M . The following proposition summarizes various results proven in Proposition 3.1, Theorem 3.1, and Proposition 4.1 of [20].

Proposition 4.3. *Let M be a σ -twisted highest weight module over $V^k(\mathfrak{g})$ with highest weight $\hat{\Lambda}$ and highest weight vector $v_{\hat{\Lambda}}$. Set $\Lambda = \hat{\Lambda}|_{\mathfrak{h}^\sigma}$ and*

$$(4.21) \quad \gamma' = \frac{1}{2} \sum_{\alpha \in S'} (-1)^{p(\alpha)} s_\alpha \alpha, \quad \gamma_{1/2} = \frac{1}{2} \sum_{\alpha \in S_{1/2}} (-1)^{p(\alpha)} s_\alpha \alpha.$$

Then

- (1) $d_0^{\text{tw}}(v_{\hat{\Lambda}} \otimes 1 \otimes 1) = 0$,
- (2) *if the homology class $[v_{\hat{\Lambda}} \otimes 1 \otimes 1]$ is non-zero, then it is a highest weight vector of the highest weight module $W_{\min}^k(\mathfrak{g})[v_{\hat{\Lambda}} \otimes 1 \otimes 1]$ whose weight is (λ, ℓ) , where*

$$(4.22) \quad \ell = \frac{1}{2(k + h^\vee)} ((\Lambda|\Lambda) - 2(\Lambda|\gamma')) - \Lambda(x) + s_{fg} + s_{gh},$$

with

$$s_{fg} = -\frac{k}{4(k + h^\vee)} \sum_{\alpha \in S'} (-1)^{p(\alpha)} s_\alpha (s_\alpha - 1), \quad s_{gh} = \frac{1}{4} \sum_{\alpha \in S_{1/2}} (-1)^{p(\alpha)} s_\alpha^2,$$

and

$$(4.23) \quad \lambda = (\Lambda - \gamma_{1/2})|_{\mathfrak{h}^\natural}.$$

- (3) *If $M(\hat{\Lambda})$ is a σ -twisted Verma module over $V^k(\mathfrak{g})$, then $H_j(M(\hat{\Lambda})) = 0$ for $j \neq 0$ and $H_0(M(\hat{\Lambda}))$ is the σ -twisted Verma module over $W_{\min}^k(\mathfrak{g})$ of highest weight (λ, ℓ) given by (4.22), (4.23).*

5. THE ZHU ALGEBRA IN THE RAMOND SECTOR

Let σ_R be the automorphism of $\mathfrak{g}$ given by $\sigma_R(a) = (-1)^{p(a)}a$, which clearly satisfies the properties (3.5), (3.6), (3.7). In this case

$$\mathfrak{g}(\sigma_R) = \mathfrak{g}, \quad \mathfrak{g}^{\sigma_R} = \mathfrak{g}_0 = \mathfrak{g}^\natural \oplus \mathbb{C}x, \quad \mathfrak{g}_{\pm 1/2}^{-\sigma_R} = \mathfrak{g}_{\pm 1/2}, \quad \mathfrak{h}^{\sigma_R} = \mathfrak{h} = \mathfrak{h}^\natural \oplus \mathbb{C}x.$$

The main purpose of this section is to compute the Zhu algebra $Zhu_{\sigma_R}(W_{\min}^k(\mathfrak{g}))$, which will be denoted Zhu_R for short, define its Verma and irreducible highest weight modules, and check the existence of an invariant even Hermitian form on Verma modules.

Note that the grading induced by σ_R is the same as the $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ -grading induced by L_0 as described in Example 2.12 of [6]. It follows that the Zhu algebra Zhu_R is the algebra $Zhu_{L_0}(W_{\min}^k(\mathfrak{g}))$ described in [14, Section 7] which we now recall. Let $\pi_Z : W_{\min}^k(\mathfrak{g}) \rightarrow Zhu_R$

be the canonical projection; then the map $\mathfrak{g}^{\natural} \oplus \mathfrak{g}_{-1/2} \oplus \mathbb{C}L \rightarrow Zhu_R$ defined by $a \mapsto \pi_Z(J^{\{a\}})$, $a \in \mathfrak{g}^{\natural}$, $v \mapsto \pi_Z(G^{\{v\}})$, $v \in \mathfrak{g}_{-1/2}$, $L \mapsto \pi_Z(L)$ is a linear isomorphism onto a set of generators.

Moreover the commutation relations among the generators are as follows (here $[\cdot, \cdot]_{\mathfrak{g}}$ denotes the bracket in $\mathfrak{g}$, while $[\cdot, \cdot]$ is the bracket in Zhu_R).

- (1) L is a central element,
- (2) $[a, b] = [a, b]_{\mathfrak{g}}$ if $a, b \in \mathfrak{g}^{\natural}$,
- (3) $[a, v] = [a, v]_{\mathfrak{g}}$ if $a \in \mathfrak{g}^{\natural}$ and $v \in \mathfrak{g}_{-1/2}$,
- (4) If $u, v \in \mathfrak{g}_{-1/2}$ then

$$(5.1) \quad [u, v] = \langle u, v \rangle \left(\sum_{\alpha=1}^{\dim \mathfrak{g}^{\natural}} a^{\alpha} * a_{\alpha} - 2(k + h^{\vee})L - \frac{1}{2}p(k) \right) \\ + \sum_{\alpha, \beta=1}^{\dim \mathfrak{g}^{\natural}} \langle [a_{\alpha}, u]_{\mathfrak{g}}, [v, a^{\beta}]_{\mathfrak{g}} \rangle (a^{\alpha} * a_{\beta} + a_{\beta} * a^{\alpha}),$$

where $p(k)$ is a monic quadratic polynomial defined in [2] and $\{a_{\alpha}\}, \{a^{\alpha}\}$ are dual bases of $\mathfrak{g}^{\natural}$. By (2) and (3) above, we can drop the subscript $\mathfrak{g}$ from the bracket $[\cdot, \cdot]_{\mathfrak{g}}$. Let $L' = 2(k + h^{\vee})L + \frac{1}{2}p(k)$. Then Zhu_R is independent of k if $k \neq -h^{\vee}$.

Let

$$m_+ = \dim \mathfrak{n}_{-1/2}(\sigma_R)'_+, \quad m_- = \dim \mathfrak{n}_{-1/2}(\sigma_R)_- + \epsilon(\sigma_R), \\ n = \dim \mathfrak{n}_0(\sigma_R)_- = \dim \mathfrak{n}_0(\sigma_R)_+, \quad r = \dim \mathfrak{h}^{\natural}.$$

Choose $\{v_i^+ \mid i = 1, \dots, m_+\}$ (resp. $\{v_i^- \mid i = 1, \dots, m_-\}$) as a basis of $\mathfrak{n}_{-1/2}(\sigma_R)'_+$ (resp. $\mathfrak{n}_{-1/2}(\sigma_R)_- \oplus \mathfrak{g}_{-1/2}^0(\sigma_R)$). Also assume that $v_{m_-}^- \in \mathfrak{g}_{-1/2}^0(\sigma_R)$ when $\epsilon(\sigma_R) = 1$ (see (3.10)). Similarly choose $\{a_i^{\pm} \mid i = 1, \dots, n\}$ bases of $\mathfrak{n}_0(\sigma_R)_{\pm}$ and $\{h_i \mid i = 1, \dots, r\}$ a basis of $\mathfrak{h}^{\natural}$. If

$$\mathbf{p} = (p_1, \dots, p_{m_{\pm}}) \in \{0, 1\}^{m_{\pm}}, \quad \mathbf{q} = (q_1, \dots, q_n) \in \mathbb{Z}_+^n, \quad \mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}_+^r,$$

given $\lambda \in (\mathfrak{h}^{\natural})^*$, set

$$a_{\pm}^{\mathbf{q}} = (a_1^{\pm})^{q_1} \dots (a_n^{\pm})^{q_n}, \quad v_{\pm}^{\mathbf{p}} = (v_1^{\pm})^{p_1} \dots (v_{m_{\pm}}^{\pm})^{p_{m_{\pm}}}, \\ h(\lambda)^{\mathbf{k}} = (h_1 - \lambda(h_1))^{k_1} \dots (h_r - \lambda(h_r))^{k_r}.$$

Theorem 3.25 of [6] implies that a basis of Zhu_R is given by

$$\{a_-^{\mathbf{q}^-} * v_-^{\mathbf{p}^-} * h(0)^{\mathbf{k}} * v_+^{\mathbf{p}^+} * a_+^{\mathbf{q}^+} * L^k\},$$

where $*$ denotes the product in Zhu_R [6]. It follows that, for all $\lambda \in (\mathfrak{h}^{\natural})^*$ and $\ell_0 \in \mathbb{C}$, the set

$$(5.2) \quad \mathcal{B}_{\lambda, \ell_0} = \{a_-^{\mathbf{q}^-} * v_-^{\mathbf{p}^-} * h(\lambda)^{\mathbf{k}} * v_+^{\mathbf{p}^+} * a_+^{\mathbf{q}^+} * (L - \ell_0)^k\}.$$

is also a basis of Zhu_R .

By a Zhu_R -module we mean a representation of Zhu_R as an associative superalgebra. A Zhu_R -module M is called a highest weight module of highest weight (λ, ℓ_0) if M is generated by a vector v_{λ, ℓ_0} such that

$$(5.3) \quad Lv_{\lambda, \ell_0} = \ell_0 v_{\lambda, \ell_0},$$

$$(5.4) \quad av_{\lambda, \ell_0} = \lambda(a)v_{\lambda, \ell_0} \text{ if } a \in \mathfrak{h}^{\natural},$$

$$(5.5) \quad av_{\lambda, \ell_0} = 0 \text{ if } a \in \mathfrak{n}_0(\sigma_R)_+,$$

$$(5.6) \quad uv_{\lambda, \ell_0} = 0 \text{ if } u \in \mathfrak{n}_{-1/2}(\sigma_R)'_+.$$

Remark 5.1. Since the set $\mathcal{B}_{\lambda, \ell_0}$ is a basis of Zhu_R it is clear that, if M is a highest weight vector of highest weight (λ, ℓ_0) and v_{λ, ℓ_0} is a highest weight vector, then M is spanned by

$$(5.7) \quad \{a_-^{\mathbf{q}-} * v_-^{\mathbf{p}-} \cdot v_{\lambda, \ell_0}\}.$$

Set $S_{-1/2}^0 = \emptyset$ if $\epsilon(\sigma_R) = 0$ and $S_{-1/2}^0 = \{m_-\}$ if $\epsilon(\sigma_R) = 1$. Since the set in (5.7) is a set of linear generators for M , the weight space of M of weight (λ, ℓ_0) is

$$M_{\lambda, \ell_0} = \text{span}(v_{\lambda, \ell_0}, v_{\gamma}^- v_{\lambda, \ell_0} \mid \gamma \in S_{-1/2}^0).$$

Since $\mathfrak{h}^{\natural} \oplus \mathbb{C}L$ is even we have

$$M_{\lambda, \ell_0} = (M_{\lambda, \ell_0})_{\bar{0}} \oplus (M_{\lambda, \ell_0})_{\bar{1}}.$$

Decompose v_{λ, ℓ_0} accordingly:

$$v_{\lambda, \ell_0} = (v_{\lambda, \ell_0})_{\bar{0}} + (v_{\lambda, \ell_0})_{\bar{1}}.$$

If $v_{\gamma}^- v_{\lambda, \ell_0} = 0$ then v_{λ, ℓ_0} is either even or odd. If $v_{\gamma}^- v_{\lambda, \ell_0} \neq 0$ then there is $\bar{i}$ such that $v_{\gamma}^- (v_{\lambda, \ell_0})_{\bar{i}} \neq 0$. Then

$$(5.8) \quad M_{\lambda, \ell_0} = \text{span}((v_{\lambda, \ell_0})_{\bar{i}}, v_{\gamma}^- (v_{\lambda, \ell_0})_{\bar{i}}).$$

In particular $(v_{\lambda, \ell_0})_{\bar{i}}$ generates M . The outcome is that we can always assume that the highest weight vector is either even or odd. From now on this will be always assumed.

The following two lemmas record easy consequences of Remark 5.1.

Lemma 5.2. *If M is a highest weight module over Zhu_R , then it admits a unique maximal proper submodule.*

Proof. If N is a proper submodule, since v_{γ}^- is odd and N is $\mathbb{Z}_2$ -graded, by (5.8) we have $N \cap M_{\lambda, \ell_0} \subset \mathbb{C}v_{\gamma}^- v_{\lambda, \ell_0}$, $\gamma \in S_{-1/2}^0$. This implies that the sum of all proper submodules is still proper. The statement follows. $\square$

Lemma 5.3. *Let M be an irreducible finite-dimensional module over Zhu_R . Then M is a highest weight module.*

Proof. Since M is irreducible and L is central, L acts by $\ell_0 I_M$ for some $\ell_0 \in \mathbb{C}$. Since M is finite dimensional and $\mathfrak{g}^{\natural}$ is a semisimple Lie algebra, $\mathfrak{h}^{\natural}$ acts semisimply on it. Let M_0 be the eigenspace corresponding to the maximal eigenvalue for the action of h_0 . M_0 is clearly $\mathfrak{h}^{\natural}$ -stable, hence we can choose a weight λ such that $(M_0)_{\lambda, \ell_0} \neq \{0\}$. As in Remark 5.1, we have $((M_0)_{\lambda, \ell_0})_{\bar{i}} \neq \{0\}$ for some $\bar{i}$, so we can choose a nonzero vector v_{λ, ℓ_0} in it. Clearly (5.5) and (5.6) hold. By construction (5.3) and (5.4) hold. Since M is irreducible, v_{λ, ℓ_0} generates M . $\square$

A Verma module of highest weight (λ, ℓ_0) for Zhu_R is a highest weight module $M^Z(\lambda, \ell_0)$ of highest weight (λ, ℓ_0) such that the set of linear generators given in (5.7) is a basis of M .

Lemma 5.4. *For all pairs (λ, ℓ_0) , a Verma module over Zhu_R of highest weight (λ, ℓ_0) exists.*

Proof. Given the pair (λ, ℓ_0) , let $\mathbb{C}_{\lambda, \ell_0}$ be the one dimensional representation of $\mathfrak{h}^{\natural} \oplus \mathbb{C}L \oplus \mathfrak{n}_0(\sigma_r)_+$ given by

$$(h + cL + n).1 = \lambda(h) + c\ell_0, \quad h \in \mathfrak{h}^{\natural}, \quad n \in \mathfrak{n}_0(\sigma_r)_+.$$

Set $\mathfrak{b}^{\natural} = \mathfrak{h}^{\natural} \oplus \mathfrak{n}_0(\sigma_r)_+$ and

$$\text{Ind}(\mathbb{C}_{\lambda, \ell_0}) = Zhu_R \otimes_{U(\mathfrak{b}^{\natural}) \otimes \mathbb{C}[L]} \mathbb{C}_{\lambda, \ell_0}.$$

For simplicity, consider $\mathbb{C}_{\lambda, \ell_0}$ as a purely even space. The $\mathbb{Z}_2$ -gradings on $\mathbb{C}_{\lambda, \ell_0}$ and Zhu_R define a $\mathbb{Z}_2$ -grading on $Zhu_R \otimes \mathbb{C}_{\lambda, \ell_0}$ by $p(a \otimes m) = p(a) + p(m)$ and, since $U(\mathfrak{b}^\natural) \otimes \mathbb{C}[L]$ is even, hence graded, the $\mathbb{Z}_2$ -grading pushes down to $Ind(\mathbb{C}_{\lambda, \ell_0})$ making the latter a Zhu_R -module.

Since Zhu_R is free as a right $U(\mathfrak{b}^\natural) \otimes \mathbb{C}[L]$ -module, a basis of $Ind(\mathbb{C}_{\lambda, \ell_0})$ is given by

$$(5.9) \quad \{a_-^{\mathbf{q}^-} * v_-^{\mathbf{p}^-} * v_+^{\mathbf{p}^+} \otimes 1\}.$$

Set $v_{\lambda+2\rho_R, \ell_0} = v_1^+ * \cdots * v_{m_+}^+ \otimes 1$. Then

$$M^Z(\lambda + 2\rho_R, \ell_0) = Zhu_R \cdot v_{\lambda+2\rho_R, \ell_0}$$

is a Verma module. Indeed, since the set (5.9) generates $Ind(\mathbb{C}_{\lambda, \ell_0})$, the eigenvalues of h_0 on $Ind(\mathbb{C}_{\lambda, \ell_0})$ are less or equal to $(\lambda + 2\rho_R)(h_0)$. It follows that $v_{\lambda+2\rho_R, \ell_0}$ is a highest weight vector of weight $(\lambda + 2\rho_R, \ell_0)$. This implies that the set

$$\{a_-^{\mathbf{q}^-} * v_-^{\mathbf{p}^-} \cdot v_{\lambda+2\rho_R, \ell_0}\}$$

generates $M^Z(\lambda + 2\rho_R, \ell_0)$ and, since (5.9) is a basis of $Ind(\mathbb{C}_{\lambda, \ell_0})$, it is linearly independent. $\square$

We now show that the modules $M^Z(\lambda, \ell_0)$ satisfy the usual universal property for highest weight modules. If $\mathbf{m} = (m_1, \dots, m_t) \in \mathbb{Z}_+^t$, we set $|\mathbf{m}| = \sum_{i=1}^t m_i$.

Lemma 5.5. *The annihilator of $v_{\lambda, \ell_0} \in M^Z(\lambda, \ell_0)$ in Zhu_R is*

$$J_{\lambda, \ell_0} = \text{span}(a_-^{\mathbf{q}^-} * v_-^{\mathbf{p}^-} * h(\lambda)^{\mathbf{k}} * v_+^{\mathbf{p}^+} * a_+^{\mathbf{q}^+} * (L - \ell_0)^k \mid |\mathbf{k}| + |\mathbf{q}_+| + |\mathbf{p}_+| + k > 0).$$

In particular J_{λ, ℓ_0} is a left ideal and

$$M^Z(\lambda, \ell_0) \simeq Zhu_R / J_{\lambda, \ell_0}.$$

Proof. The set $\{a_-^{\mathbf{q}^-} * v_-^{\mathbf{p}^-} \mid \mathbf{q}_- \in \mathbb{Z}_+^n, \mathbf{p}_- \in \{0, 1\}^{m_-}\}$ completes the set

$$\{a_-^{\mathbf{q}^-} * v_-^{\mathbf{p}^-} * h(\lambda)^{\mathbf{k}} * v_+^{\mathbf{p}^+} * a_+^{\mathbf{q}^+} * (L - \ell_0)^k \mid |\mathbf{k}| + |\mathbf{q}_+| + |\mathbf{p}_+| + k > 0\}$$

to the basis $\mathcal{B}_{\lambda, \ell_0}$ of Zhu_R . $\square$

Corollary 5.6. *The Verma module $M^Z(\lambda, \ell_0)$ is the universal highest weight module over Zhu_R of highest weight (λ, ℓ_0) . In particular its unique simple quotient $L^Z(\lambda, \ell_0)$ is the unique irreducible highest weight module of highest weight (λ, ℓ_0) .*

Proof. If M is a highest weight module of highest weight (λ, ℓ_0) and highest weight vector v_{λ, ℓ_0} then $J_{\lambda, \ell_0} \cdot v_{\lambda, \ell_0} = \{0\}$. $\square$

Let ϕ be an almost compact conjugate linear involution of $\mathfrak{g}$. Recall also that there is a conjugate linear anti-involution ω defined on generators by

$$\omega(L) = L, \quad \omega(a) = -\phi(a), \quad a \in \mathfrak{g}^\natural, \quad \omega(v) = \phi(v), \quad v \in \mathfrak{g}_{-1/2}.$$

Fix a pair (λ, ℓ_0) with λ purely imaginary (i. e. $\lambda(\phi(h)) = -\overline{\lambda(h)}$) and ℓ_0 real. We now define an invariant Hermitian form on $M^Z(\lambda, \ell_0)$.

Let $p^\natural : Zhu_R \rightarrow U(\mathfrak{g}^\natural) \otimes \mathbb{C}[L]$ be the projection with respect to the decomposition

$$Zhu_R = \text{span}(a_-^{\mathbf{q}^-} * v_-^{\mathbf{p}^-} * h(0)^{\mathbf{k}} * v_+^{\mathbf{p}^+} * a_+^{\mathbf{q}^+} * L^k \mid |\mathbf{p}_-| + |\mathbf{p}_+| > 0) \oplus U(\mathfrak{g}^\natural) \otimes \mathbb{C}[L].$$

Let $p^{\mathfrak{h}^\natural}$ be the usual Harish-Chandra projection from $U(\mathfrak{g}^\natural)$ to $U(\mathfrak{h}^\natural)$. Set

$$p^Z = (p^{\mathfrak{h}^\natural} \otimes I_{\mathbb{C}[L]}) \circ p^\natural.$$

Consider the pair (λ, ℓ_0) as an element of $(\mathfrak{h}^\natural \oplus \mathbb{C}L)^*$ via $(\lambda, \ell_0)(h + L) = \lambda(h) + \ell_0$. View $U(\mathfrak{h}^\natural) \otimes \mathbb{C}[L]$ as the polynomial algebra on $(\mathfrak{h}^\natural \oplus \mathbb{C}L)^*$ and define a sesquilinear form

$$H : Zhu_R \times Zhu_R \rightarrow \mathbb{C}$$

by setting

$$H(a, b) = \overline{p^Z(\omega(b)a)(\lambda, \ell_0)}.$$

This form is obviously ω -invariant:

$$H(a, bc) = \overline{p^Z(\omega(bc)a)(\lambda, \ell_0)} = \overline{p^Z(\omega(c)\omega(b)a)(\lambda, \ell_0)} = H(\omega(b)a, c).$$

We now prove that the form $H(\cdot, \cdot)$ can be chosen to be Hermitian: since the eigenvalues of $ad(h_0)$ on $\mathfrak{g}$ are real we can assume that $\phi(h_0) = -h_0$. With this assumption we have

$$\phi(\mathfrak{n}_0(\sigma_R)_+) = \mathfrak{n}_0(\sigma_R)_-, \quad \phi(\mathfrak{n}_{-1/2}(\sigma_R)_+) = \mathfrak{n}_{-1/2}(\sigma_R)_-, \quad \phi(\mathfrak{g}_{-1/2}^0(\sigma_R)) = \mathfrak{g}_{-1/2}^0(\sigma_R),$$

hence we can choose the bases $\{v_i^\pm\}$ so that $\omega(v_i^+) = v_{m_- - \epsilon(\sigma_R) + 1 - i}^-$ and, if $\epsilon(\sigma_R) = 1$, $\omega(v_{m_-}^-) = v_{m_-}^-$. With this choice of bases it is clear that, if $\gamma \in S_{-1/2}^0$,

$$\omega(a_-^{\mathfrak{q}^-} * v_-^{\mathbf{p}^-} * h(0)^{\mathbf{k}} * v_+^{\mathbf{p}^+} * a_+^{\mathfrak{q}^+} * L^k) = \omega(a_+^{\mathfrak{q}^+} * v_-^{\mathbf{p}^+} * \omega(h(0)^{\mathbf{k}}) * v_\gamma^{\mathbf{p}^-} * v_+^{\mathbf{p}^+} * \omega(a_+^{\mathfrak{q}^+}) * L^k$$

where if $\mathbf{p} \in \{0, 1\}^{m_\pm}$ we let $\mathbf{p}' = (p_{m_+}, \dots, p_1)$. Since $[h, v_\gamma] = 0$ we obtain

$$\omega(a_-^{\mathfrak{q}^-} * v_-^{\mathbf{p}^-} * h(0)^{\mathbf{k}} * v_+^{\mathbf{p}^+} * a_+^{\mathfrak{q}^+} * L^k) = \omega(a_+^{\mathfrak{q}^+} * v_-^{\mathbf{p}^+} * v_\gamma^{\mathbf{p}^-} * \omega(h(0)^{\mathbf{k}}) * v_+^{\mathbf{p}^+} * \omega(a_-^{\mathfrak{q}^-}) * L^k$$

It follows that $p^\natural \circ \omega = \omega \circ p^\natural$. Similarly one has $\omega \circ p^{\mathfrak{h}^\natural} = p^{\mathfrak{h}^\natural} \circ \omega$ so

$$p^Z \circ \omega = \omega \circ p^Z.$$

We are finally ready to compute

$$H(b, a) = \overline{p^Z(\omega(a)b)(\lambda, \ell_0)} = \overline{p^Z(\omega(\omega(b)a))(\lambda, \ell_0)} = \overline{\omega(p^Z(\omega(b)a))(\lambda, \ell_0)},$$

hence we need only to check that

$$\overline{\omega(p^Z(\omega(b)a))(\lambda, \ell_0)} = p^Z(\omega(b)a)(\lambda, \ell_0) = \overline{H(a, b)}.$$

To check this last equality it is enough to prove that

$$\overline{\omega(h(0)^{\mathbf{k}}L^k)(\lambda, L_0)} = (h(0)^{\mathbf{k}}L^k)(\lambda, L_0).$$

Indeed, since λ is purely imaginary and ℓ_0 is real,

$$\begin{aligned} \overline{\omega(h(0)^{\mathbf{k}}L^k)(\lambda, L_0)} &= \overline{\lambda(\omega(h_1))^{k_1} \dots \lambda(\omega(h_r))^{k_r} \ell_0^k} = \overline{\lambda(-\phi(h_1))^{k_1} \dots \lambda(-\phi(h_r))^{k_r} \ell_0^k} \\ &= \overline{\lambda(h_1)^{k_1} \dots \lambda(h_r)^{k_r} \ell_0^k} = \lambda(h_1)^{k_1} \dots \lambda(h_r)^{k_r} \ell_0^k = (h(0)^{\mathbf{k}}L^k)(\lambda, L_0). \end{aligned}$$

We now show that $H(\cdot, \cdot)$ can be pushed down to define a Hermitian form on $M^Z(\lambda, \ell_0)$. We need to check that

$$H(J_{\lambda, \ell_0}, Zhu_R) = H(Zhu_R, J_{\lambda, \ell_0}) = 0.$$

Since the form is Hermitian it is enough to show that

$$H(J_{\lambda, \ell_0}, Zhu_R) = 0.$$

Since J_{λ, ℓ_0} is a left ideal and $H(a, b) = \overline{p^Z(\omega(a)b)(\lambda, \ell_0)}$, it is enough to show that, if $a \in J_{\lambda, \ell_0}$, then $p^Z(a)(\lambda, \ell_0) = 0$. This is easily checked: if $|\mathbf{q}_-| + |\mathbf{p}_-| + |\mathbf{q}_+| + |\mathbf{p}_+| > 0$ then

$$p^Z(a_-^{\mathfrak{q}^-} * v_-^{\mathbf{p}^-} * h(\lambda)^{\mathbf{k}} * v_+^{\mathbf{p}^+} * a_+^{\mathfrak{q}^+} * (L - \ell_0)^k) = 0.$$

If $|\mathbf{q}_-| = |\mathbf{p}_-| = |\mathbf{q}_+| = |\mathbf{p}_+| = 0$, then

$$p^Z(a_-^{\mathbf{q}_-} * v_-^{\mathbf{p}_-} * h(\lambda)^{\mathbf{k}} * v_+^{\mathbf{p}_+} * a_+^{\mathbf{q}_+} * (L - \ell_0)^k)(\lambda, \ell_0) = h(\lambda)^{\mathbf{k}}(L - \ell_0)^k(\lambda, \ell_0) = 0.$$

We summarize the above discussion in the following result. Recall that a Hermitian form $H(\cdot, \cdot)$ on a superspace V is called even if $H(V_0, V_1) = 0$.

Proposition 5.7. *If λ is purely imaginary and $\ell_0 \in \mathbb{R}$, then there is a unique even Hermitian form $H_{\lambda, \ell_0}(\cdot, \cdot)$ on $M^Z(\lambda, \ell_0)$ such that $H_{\lambda, \ell_0}(v_{\lambda, \ell_0}, v_{\lambda, \ell_0}) = 1$.*

Furthermore, the radical of $H_{\lambda, \ell_0}(\cdot, \cdot)$ is the unique maximal $\mathbb{Z}_2$ -graded proper submodule of $M^Z(\lambda, \ell_0)$, hence $H_{\lambda, \ell_0}(\cdot, \cdot)$ induces the unique nondegenerate invariant even Hermitian form on $L^Z(\lambda, \ell_0)$ such that $H_{\lambda, \ell_0}(v_{\lambda, \ell_0}, v_{\lambda, \ell_0}) = 1$.

Proof. The invariant Hermitian form $H(\cdot, \cdot)$ defined above has clearly the property that $H(1, 1) = 1$. Moreover, since $p^Z(a) = 0$ if a is odd, $H(\cdot, \cdot)$ is even. Suppose that $H'(\cdot, \cdot)$ is another even invariant Hermitian form on $M^Z(\lambda, \ell_0)$ such that $H'(v_{\lambda, \ell_0}, v_{\lambda, \ell_0}) = 1$. Set $(H - H')(\cdot, \cdot) = H(\cdot, \cdot) - H'(\cdot, \cdot)$. Clearly $(H - H')(\cdot, \cdot)$ is an invariant even Hermitian form. Moreover $(H - H')(v_{\lambda, \ell_0}, v_{\lambda, \ell_0}) = 0$ and, since $(H - H')(\cdot, \cdot)$ is even,

$$(H - H')(v_{\lambda, \ell_0}, v_{\gamma}^- v_{\lambda, \ell_0}) = (H - H')(v_{\gamma}^- v_{\lambda, \ell_0}, v_{\lambda, \ell_0}) = 0.$$

Observe that $\omega(v_{\gamma}^- v_{\lambda, \ell_0}) \in M^Z(\lambda, \ell_0)_{\lambda, \ell_0}$ and it has the same parity of v_{λ, ℓ_0} . Since $M^Z(\lambda, \ell_0)_{\lambda, \ell_0} = \text{span}(v_{\gamma}^- v_{\lambda, \ell_0}, v_{\lambda, \ell_0})$, it follows that $\omega(v_{\gamma}^- v_{\lambda, \ell_0}) = \text{const.} v_{\lambda, \ell_0}$. Thus

$$(H - H')(v_{\gamma}^- v_{\lambda, \ell_0}, v_{\gamma}^- v_{\lambda, \ell_0}) = (H - H')(\omega(v_{\gamma}^- v_{\lambda, \ell_0}) v_{\gamma}^- v_{\lambda, \ell_0}, v_{\lambda, \ell_0}) = \overline{\text{const.}} (H - H')(v_{\lambda, \ell_0}, v_{\lambda, \ell_0}) = 0.$$

The upshot is that $(H - H')(\cdot, \cdot)|_{M^Z(\lambda, \ell_0)_{\lambda, \ell_0} \times M^Z(\lambda, \ell_0)_{\lambda, \ell_0}} = 0$. By invariance, if $(\lambda, \ell_0) \neq (\nu, \ell)$,

$$(H - H')(M^Z(\lambda, \ell_0)_{\lambda, \ell_0}, M^Z(\lambda, \ell_0)_{\nu, \ell}) = 0$$

so we conclude that $(H - H')(M^Z(\lambda, \ell_0)_{\lambda, \ell_0}, M^Z(\lambda, \ell_0)) = 0$. Since $M^Z(\lambda, \ell_0)_{\lambda, \ell_0}$ generates $M^Z(\lambda, \ell_0)$ we deduce $(H - H')(\cdot, \cdot) = 0$ so $H'(\cdot, \cdot) = H(\cdot, \cdot)$.

It remains to prove that the radical of the form is the unique maximal $\mathbb{Z}_2$ -graded proper submodule of $M^Z(\lambda, \ell_0)$. By invariance of the form, the radical is a submodule of $M^Z(\lambda, \ell_0)$ and, since $H(1, 1) = 1$, it is proper. If N is a proper $\mathbb{Z}_2$ -graded submodule of $M^Z(\lambda, \ell_0)$, then $N \cap M^Z(\lambda, \ell_0)_{\lambda, \ell_0} \subset \mathbb{C} v_{\gamma}^- v_{\lambda, \ell_0}$. Since H is even, $H(N, v_{\lambda, \ell_0}) = 0$ and, since v_{λ, ℓ_0} generates $M^Z(\lambda, \ell_0)_{\lambda, \ell_0}$, N is in the radical of the form. $\square$

Remark 5.8. Universal minimal W -algebras $W_{\min}^k(\mathfrak{g})$ are defined for arbitrary simple basic finite-dimensional Lie superalgebras [16], [19], but in general the grading (3.2) is not compatible with parity. The automorphism σ_R in general is defined by

$$\sigma_R(a) = (-1)^{2j} a, \text{ if } a \in \mathfrak{g}_j \text{ in (3.2).}$$

Then the Zhu algebra Zhu_R is still defined by the above commutation relations [14]. On the other hand, it is isomorphic to the finite W -algebra, associated to a minimal sl_2 , studied in [24] for simple Lie algebras. Actually, this result holds for all quantum affine W -algebras [6, Example 2.12]. Since Zhu_R is independent of k , we obtain from (5.1) that in cases when $\mathfrak{g}$ is a simple Lie algebra, Premet's constant in [24] is $c_0 = -\frac{1}{2}p(-h^\vee)$.

6. RAMOND SECTOR: NECESSARY CONDITIONS FOR UNITARITY

Let Δ and $\Delta^\natural$ be the sets of roots of $\mathfrak{g}$ and $\mathfrak{g}^\natural$ respectively. As in [15, Table 1], we choose a specific set Π of simple roots for $\mathfrak{g}$. This set Π has the property that, letting $\Pi_{\text{odd}} = \{\alpha \in \Pi \mid \alpha \text{ is odd}\}$, then $\Pi^\natural := \Pi \setminus \Pi_{\text{odd}}$ is a set of simple roots for $\mathfrak{g}^\natural$. We let Δ^+ and $(\Delta^\natural)^+$ denote the sets of positive roots in Δ and $\Delta^\natural$ corresponding to Π and $\Pi^\natural$ respectively. We also let ξ be the highest weight (with respect to $\Pi^\natural$) of $\mathfrak{g}_{1/2}$ as a $\mathfrak{g}^\natural$ -module. Since $\mathfrak{g}_{-1/2}$ is isomorphic to $\mathfrak{g}_{1/2}$ as a $\mathfrak{g}^\natural$ -module, we necessarily have that $\xi = -\alpha|_{\mathfrak{h}^\natural}$ for all $\alpha \in \Pi_{\text{odd}}$.

Let $\widehat{\Delta}^{\text{tw}}$ denote the set of roots of $\widehat{\mathfrak{g}}^{\text{tw}}$; explicitly

$$\widehat{\Delta}^{\text{tw}} = \{s\delta + \alpha \mid \alpha \in \Delta^\natural \cup \{\theta\}, s \in \mathbb{Z}\} \cup \{s\delta + \alpha \mid \alpha \in \Delta_{1/2}, s \in \frac{1}{2} + \mathbb{Z}\} \cup \{s\delta \mid s \in \pm\mathbb{N}\}.$$

We also set $\widehat{\Delta}^\natural = \{s\delta + \alpha \mid \alpha \in \Delta^\natural \mid s \in \mathbb{Z}\} \cup \{s\delta \mid s \in \pm\mathbb{N}\}$.

We specialize to $\sigma = \sigma_R$ the triangular decomposition given in (4.1), (4.3), and (4.4). Let $\widehat{\Delta}_+^{\text{tw}}, \widehat{\Pi}$ be the corresponding sets of positive and simple roots and set $\widehat{\Delta}_+^\natural = \widehat{\Delta}^\natural \cap \widehat{\Delta}_+^{\text{tw}}$.

Let $\Delta_i = \{\alpha \in \Delta \mid \alpha(x) = i\}$ and $\overline{\Delta}_i = (\Delta_i)|_{\mathfrak{h}^\natural}$. Since $\text{ad } f$ is an isomorphism of $\mathfrak{g}^\natural$ -modules between $\mathfrak{g}_{1/2}$ and $\mathfrak{g}_{-1/2}$ it follows that $\Delta_{\pm 1/2} = \{\pm \frac{1}{2}\theta + \eta \mid \eta \in \overline{\Delta}_{1/2}\}$. Therefore $\overline{\Delta}_{1/2} = \overline{\Delta}_{-1/2} = -\overline{\Delta}_{1/2}$.

The set $\overline{\Delta}_{1/2}$ is naturally partially ordered setting $\beta \prec \gamma$ if and only if $\gamma - \beta$ is a sum of roots in $\Pi^\natural$.

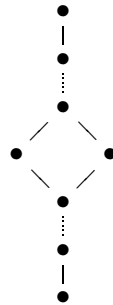
Recall from Section 3 that the triangular decomposition (4.1) depends on the choice of a regular element $h_0 \in \mathfrak{h}^\natural$. We choose h_0 such that $\alpha(h_0) > 0$ for all $\alpha \in \Pi^\natural$, so that $\widehat{\Delta}_+^\natural$ is the set of positive roots in $\widehat{\Delta}^\natural$ whose corresponding set of simple roots is $\widehat{\Pi}^\natural := \Pi^\natural \cup \{\delta - \theta_i \mid 1 \leq i \leq s\}$.

Set

$$(6.1) \quad \overline{\Delta}_{1/2}^+ = \{\eta \in \overline{\Delta}_{1/2} \mid \eta(h_0) > 0\}.$$

Below we give an explicit description of the set $\overline{\Delta}_{1/2}$ in all cases. A direct check using this data shows that three cases occur:

- (1) Case 1: $\mathfrak{g} = \mathfrak{psl}(2|2)$. In this case $\overline{\Delta}_{1/2} = \{\xi, \xi - \theta_1 = -\xi\}$. It follows that $\overline{\Delta}_{1/2}^+ = \{\xi\}$. Set $\eta_{\min} = \xi$. It is obvious that $\eta_{\min}$ is the unique minimal element of $\overline{\Delta}_{1/2}^+$.
- (2) Case 2: $0 \in \overline{\Delta}_{1/2}$. In this case $\prec$ is a total order. It follows that $\overline{\Delta}_{1/2}^+ = \{\eta \in \overline{\Delta}_{1/2} \mid \eta \succ 0\}$. Set $\eta_{\min}$ to be its minimum.
- (3) Case 3: $\mathfrak{g} \neq \mathfrak{psl}(2|2)$ and $0 \notin \overline{\Delta}_{1/2}$. In this case the Hasse diagram of the poset $\overline{\Delta}_{1/2}$ is as follows:



If $\eta_{\min}$ is a minimal element of $\overline{\Delta}_{1/2}^+$, then $\overline{\Delta}_{1/2}^+ \supset \{\eta \in \overline{\Delta}_{1/2} \mid \eta \succeq \eta_{\min}\}$. Let η_1, η_2 be the two non comparable elements in $\overline{\Delta}_{1/2}$. The map $\eta \mapsto -\eta$ is the unique order reversing involution without fixed points of the diagram. It follows that $\eta_1 = -\eta_2$,

hence $\{\eta_1, \eta_2\} \cap \overline{\Delta}_{1/2}^+$ has exactly one element. It is then clear that $\eta_{\min}$ is the unique element of $\{\eta_1, \eta_2\} \cap \overline{\Delta}_{1/2}^+$ (hence $\overline{\Delta}_{1/2}^+$ has a unique minimal element) and that $\overline{\Delta}_{1/2}^+ = \{\eta \in \overline{\Delta}_{1/2}^+ \mid \eta \succeq \eta_{\min}\}$.

The outcome of this discussion is that, if $\mathfrak{g}$ is of type $\mathfrak{spo}(2|2r), F(4), D(2, 1; a)$, then there are two choices of $\widehat{\Delta}_+^{\text{tw}}$; one for each choice of $\eta_{\min}$. In all the other cases there is only one choice for $\widehat{\Delta}_+^{\text{tw}}$. Remarkably, in all cases $\overline{\Delta}_{1/2}^+$ has a unique minimal element $\eta_{\min}$.

We now describe explicitly case by case, for each choice of $\widehat{\Delta}_+^{\text{tw}}$, the set $\overline{\Delta}_{1/2}$, the set $\overline{\Delta}_{1/2}^+$ with its minimal element $\eta_{\min}$, and the set of simple roots $\widehat{\Pi}$.

$\mathfrak{g} = \mathfrak{psl}(2|2)$. In this case $\overline{\Delta}_{1/2} = \{\xi, -\xi\}$ with $\xi = \frac{\delta_1 - \delta_2}{2}$, $\overline{\Delta}_{1/2}^+ = \{\xi\}$, $\eta_{\min} = \xi$ and

$$\widehat{\Pi} = \{\delta - (\epsilon_1 - \epsilon_2), -\frac{1}{2}\delta + \delta_1 - \epsilon_2, \delta - (\delta_1 - \delta_2), -\frac{1}{2}\delta + \epsilon_1 - \delta_2\}.$$

The set $\widehat{\Pi}$ is actually a set of simple roots for $\widehat{\mathfrak{g}}^{\text{tw}}$ and the two simple isotropic roots are linearly independent (their restrictions to $\widehat{\mathfrak{h}}$ are equal).

$\mathfrak{g} = \mathfrak{spo}(2|3)$. In this case $\overline{\Delta}_{1/2} = \{\xi, 0, -\xi\}$ with $\xi = \epsilon_1$, $\overline{\Delta}_{1/2}^+ = \{\xi\}$, $\eta_{\min} = \xi$ and

$$\widehat{\Pi} = \{-\frac{\delta}{2} + \delta_1 + \epsilon_1, \frac{\delta}{2} - \delta_1, \frac{\delta}{2} + \delta_1 - \epsilon_1\}.$$

$\mathfrak{g} = \mathfrak{spo}(2|2r), r > 2$. In this case $\overline{\Delta}_{1/2} = \{\pm\epsilon_i \mid i = 1, \dots, r\}$ with $\xi = \epsilon_1$. There are two choices for $\overline{\Delta}_{1/2}^+$: either $\overline{\Delta}_{1/2}^+ = \{\epsilon_i \mid i = 1, \dots, r\}$ or $\overline{\Delta}_{1/2}^+ = \{\epsilon_i \mid i = 1, \dots, r-1\} \cup \{-\epsilon_r\}$.

For the first choice $\eta_{\min} = \epsilon_r$ and

$$\widehat{\Pi} = \{\delta - (\epsilon_1 + \epsilon_2), \epsilon_1 - \epsilon_2, \dots, \epsilon_{r-2} - \epsilon_{r-1}, \epsilon_{r-1} - \epsilon_r, -\frac{1}{2}\delta + \delta_1 + \epsilon_r, \delta - 2\delta_1\}.$$

For the second choice $\eta_{\min} = -\epsilon_r$ and

$$\widehat{\Pi} = \{\delta - (\epsilon_1 + \epsilon_2), \epsilon_1 - \epsilon_2, \dots, \epsilon_{r-2} - \epsilon_{r-1}, \epsilon_{r-1} + \epsilon_r, -\frac{1}{2}\delta + \delta_1 - \epsilon_r, \delta - 2\delta_1\}.$$

$\mathfrak{g} = \mathfrak{spo}(2|2r+1), r > 1$. In this case $\overline{\Delta}_{1/2} = \{\pm\epsilon_i \mid i = 1, \dots, r\} \cup \{0\}$, $\xi = \epsilon_1$, $\overline{\Delta}_{1/2}^+ = \{\epsilon_i \mid i = 1, \dots, r\}$, $\eta_{\min} = \epsilon_r$ and

$$\widehat{\Pi} = \{\delta - (\epsilon_1 + \epsilon_2), \epsilon_1 - \epsilon_2, \dots, \epsilon_{r-2} - \epsilon_{r-1}, \epsilon_{r-1} - \epsilon_r, -\frac{1}{2}\delta + \delta_1 + \epsilon_r, \frac{1}{2}\delta - \delta_1\}.$$

$\mathfrak{g} = \mathbf{D}(2, 1; \mathbf{a})$. In this case $\overline{\Delta}_{1/2} = \{\pm\epsilon_2 \pm \epsilon_3\}$ with $\xi = \epsilon_2 + \epsilon_3$. There are two choices for $\overline{\Delta}_{1/2}^+$: either $\overline{\Delta}_{1/2}^+ = \{\epsilon_2 + \epsilon_3, \epsilon_2 - \epsilon_3\}$ or $\overline{\Delta}_{1/2}^+ = \{\epsilon_2 + \epsilon_3, -\epsilon_2 + \epsilon_3\}$.

For the first choice $\eta_{\min} = \epsilon_2 - \epsilon_3$ and

$$\widehat{\Pi} = \{-\frac{1}{2}\delta + \epsilon_1 + \epsilon_2 - \epsilon_3, 2\epsilon_3, \delta - 2\epsilon_2, \delta - 2\epsilon_1\}.$$

For the second choice $\eta_{\min} = -\epsilon_2 + \epsilon_3$ and

$$\widehat{\Pi} = \{-\frac{1}{2}\delta + \epsilon_1 - \epsilon_2 + \epsilon_3, 2\epsilon_2, \delta - 2\epsilon_3, \delta - 2\epsilon_1\}.$$

$\mathfrak{g} = \mathbf{F}(4)$. In this case $\overline{\Delta}_{1/2} = \{\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3)\}$ with $\xi = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3)$. There are two choices for $\overline{\Delta}_{1/2}^+$: either

$$\overline{\Delta}_{1/2}^+ = \{\frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3), \frac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3), \frac{1}{2}(\epsilon_1 - \epsilon_2 + \epsilon_3), \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3)\}$$

or

$$\overline{\Delta}_{1/2}^+ = \{\frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3), \frac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3), \frac{1}{2}(\epsilon_1 - \epsilon_2 + \epsilon_3), \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3)\}.$$

For the first choice $\eta_{\min} = \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3)$ and

$$\hat{\Pi} = \{\delta - (\epsilon_1 + \epsilon_2), \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, -\frac{1}{2}(\delta - \delta_1 + \epsilon_1 - \epsilon_2 - \epsilon_3), \delta - \delta_1\}.$$

For the second choice $\eta_{\min} = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3)$ and

$$\hat{\Pi} = \{\delta - (\epsilon_1 + \epsilon_2), \epsilon_2 - \epsilon_3, \epsilon_3, -\frac{1}{2}(\delta - \delta_1 - \epsilon_1 + \epsilon_2 + \epsilon_3), \delta - \delta_1\}.$$

$\mathfrak{g} = \mathbf{G}(\mathbf{3})$. In this case $\overline{\Delta}_{1/2} = \{\pm\epsilon_1, \pm\epsilon_2, \pm(\epsilon_2 + \epsilon_1)\} \cup \{0\}$, $\xi = \epsilon_2 + \epsilon_1$, $\overline{\Delta}_{1/2}^+ = \{\epsilon_1, \epsilon_2, \epsilon_2 + \epsilon_1\}$, $\eta_{\min} = \epsilon_1$ and

$$\hat{\Pi} = \{-\frac{\delta}{2} + \delta_1 + \epsilon_1, \frac{\delta}{2} - \delta_1, \epsilon_2 - \epsilon_1, \delta - \epsilon_1 - 2\epsilon_2\}.$$

Let M be a σ_R -twisted $W_{\min}^k(\mathfrak{g})$ -module. If $v \in \mathfrak{g}_{-1/2}$, then

$$Y^M(G^{\{v\}}, z) = \sum_{m \in \frac{1}{2} + \mathbb{Z}} G_{(m)}^{\{v\}, \text{tw}} z^{-m-1} = \sum_{m \in \mathbb{Z}} G_m^{\{v\}, \text{tw}} z^{-m-3/2}.$$

In this section and the following ones we will write $G_m^{\{v\}, \text{tw}} = G_m^{\{v\}}$, $J_m^{\{a\}, \text{tw}} = J_m^{\{a\}}$ for short.

Next, we would like to provide a more precise description of the action of $G_0^{\{v\}}$ on the highest weight vector of a highest weight module M . For computing it, we use a *good choice* $\{X_\alpha \mid \alpha \in \Delta\}$ of root vectors as in [15], and choose as a basis of $\mathfrak{g}_{-1/2}$ the vectors

$$(6.2) \quad u_\alpha = X_\alpha + \sqrt{-1}N_{-\theta, \alpha}X_{\alpha-\theta}, \quad \alpha \in \Delta^+, \alpha \text{ odd}.$$

Here $N_{\alpha, \beta}$ are structure constants w.r.t. $\{X_\alpha\}$. We observe that we can choose $u_\gamma = X_\gamma$ for $\gamma \in \Delta^\natural$, so that $u^\gamma = X_{-\gamma}$. We now check that the $N_{\alpha, \beta}$ are the structure constants w.r.t $\{u_\alpha\}$. Compute, for $\gamma \in \Delta^\natural$ and α odd positive,

$$\begin{aligned} [X_\gamma, u_\alpha] &= [X_\gamma, X_\alpha + \sqrt{-1}N_{-\theta, \alpha}X_{\alpha-\theta}] \\ &= N_{\gamma, \alpha}X_{\gamma+\alpha} + \sqrt{-1}N_{-\theta, \alpha}N_{\gamma, \alpha-\theta}X_{\gamma+\alpha-\theta}. \end{aligned}$$

Since this vector lies in $\mathfrak{g}_{-1/2}$, it is a linear combination of the u_α 's, and this forces

$$(6.3) \quad [X_\gamma, u_\alpha] = N_{\gamma, \alpha}u_{\gamma+\alpha}.$$

Substituting $u_{\gamma+\alpha}$ by its expression (6.2), we deduce that

$$(6.4) \quad N_{-\theta, \alpha}N_{\gamma, \alpha-\theta} = N_{-\theta, \gamma+\alpha}N_{\gamma, \alpha}.$$

Let ϕ be an almost compact conjugate linear involution of $\mathfrak{g}$ [15, Definition 1.1].

Lemma 6.1. *For $\gamma \in \Delta^\natural$, α odd positive, set $c_{\gamma, \alpha} = N_{\gamma, -\alpha}N_{\alpha, -\gamma}$. Then*

$$(6.5) \quad \langle [u_\gamma, \phi(u_\alpha)], [u_\alpha, u^\gamma] \rangle = \langle \phi(u_\alpha), u_\alpha \rangle c_{\gamma, \alpha}.$$

Proof. To compute (6.5) we use the following formulas

$$\phi(u_\alpha) = -N_{-\theta, \alpha}u_{\theta-\alpha}, \quad \langle u_\alpha, u_\beta \rangle = -(N_{-\theta, \alpha} + N_{-\theta, \beta})\delta_{\theta-\alpha, \beta}.$$

The first one is given in Lemma 5.3 of [15] while the second is (5.12) of [15]. In particular we have

$$(6.6) \quad \langle \phi(u_\alpha), u_\alpha \rangle = -N_{-\theta, \alpha}\langle u_{\theta-\alpha}, u_\alpha \rangle = N_{-\theta, \alpha}(N_{-\theta, \theta-\alpha} + N_{-\theta, \alpha}).$$

It follows that

$$\begin{aligned}
\langle [u_\gamma, \phi(u_\alpha)], [u_\alpha, u^\gamma] \rangle &= \langle [X_\gamma, \phi(u_\alpha)], [u_\alpha, X_{-\gamma}] \rangle \\
&= \langle [X_\gamma, -N_{-\theta, \alpha} u_{\theta-\alpha}], [u_\alpha, X_{-\gamma}] \rangle \\
&= N_{-\theta, \alpha} N_{\gamma, \theta-\alpha} N_{-\gamma, \alpha} \langle u_{\gamma+\theta-\alpha}, u_{-\gamma+\alpha} \rangle \\
(6.7) \quad &= -(N_{-\theta, \gamma+\theta-\alpha} + N_{-\theta, -\gamma+\alpha}) N_{-\theta, \alpha} N_{\gamma, \theta-\alpha} N_{-\gamma, \alpha}.
\end{aligned}$$

Using (6.4) for γ and $-\gamma$, formula (6.7) becomes

$$(6.8) \quad \langle [u_\gamma, \phi(u_\alpha)], [u_\alpha, u^\gamma] \rangle = -N_{-\theta, \alpha}^2 N_{-\gamma, \alpha-\theta} N_{\gamma, \theta-\alpha} - N_{-\theta, \alpha} N_{-\theta, \theta-\alpha} N_{\gamma, -\alpha} N_{-\gamma, \alpha}$$

$$(6.9) \quad = N_{-\theta, \alpha}^2 c_{-\gamma, \theta-\alpha} + N_{-\theta, \alpha} N_{-\theta, \theta-\alpha} c_{\gamma, \alpha}.$$

We want to prove that $N_{-\gamma, \alpha-\theta} N_{\gamma, \theta-\alpha} = N_{\gamma, -\alpha} N_{-\gamma, \alpha}$. Observe that

$$[X_{-\theta}, [X_\theta, X_{-\alpha}]] = X_{-\alpha}$$

hence

$$[X_\gamma, [X_{-\theta}, [X_\theta, X_{-\alpha}]]] = N_{\gamma, -\alpha} X_{-\alpha}.$$

On the other hand

$$[X_\gamma, [X_\theta, X_{-\alpha}]] = N_{\gamma, \theta-\alpha} [X_\theta, X_{-\alpha}]$$

so

$$[X_\gamma, [X_{-\theta}, [X_\theta, X_{-\alpha}]]] = N_{\gamma, \theta-\alpha} [X_{-\theta}, [X_\theta, X_{-\alpha}]] = N_{\gamma, \theta-\alpha} X_{-\alpha}.$$

thus

$$N_{\gamma, -\alpha} = N_{\gamma, \theta-\alpha}.$$

Similarly one has

$$N_{-\gamma, \alpha} = N_{-\gamma, \alpha-\theta}.$$

This implies $c_{-\gamma, \theta-\alpha} = c_{\gamma, \alpha}$; (6.9) now reads

$$(6.10) \quad \langle [u_\gamma, \phi(u_\alpha)], [u_\alpha, u^\gamma] \rangle = c_{\gamma, \alpha} (N_{-\theta, \alpha}^2 + N_{-\theta, \alpha} N_{-\theta, \theta-\alpha}).$$

Substituting (6.6) into (6.10) yields (6.5). $\square$

Let v be a weight vector of the $\mathfrak{g}^\natural$ -module $\mathfrak{g}_{-1/2}$ of weight η . Observe that $\eta = \alpha|_{\mathfrak{h}^\natural}$, with $\alpha = \frac{1}{2}\theta + \eta$, a positive odd root. If $\gamma \in \Delta^\natural$, set $c_{\gamma, \eta} = c_{\gamma, \alpha}$.

Lemma 6.2. *If $\eta \neq 0$ then*

$$c_{\gamma, \eta} = \begin{cases} (\gamma|\eta) & \text{if } (\gamma|\eta) \leq 0 \\ 0 & \text{otherwise} \end{cases}.$$

If $\eta = 0$ then

$$c_{\gamma, \eta} = \begin{cases} -\frac{1}{2} & \text{if } \gamma \text{ is a root from } \Delta^\natural \text{ of minimal length,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $\eta \neq 0$ then $\alpha = \frac{1}{2}\theta + \eta$ is an isotropic root so we can use formula [11, (4.6)]: if $\alpha - \gamma$ is a root then

$$c_{\gamma, \eta} = (\alpha|\gamma) = (\eta|\gamma).$$

If $(\alpha|\gamma) < 0$ then $\alpha - \gamma$ is a root so $c_{\gamma, \eta} = (\eta|\gamma)$ in this case. If $(\alpha|\gamma) = 0$ and $\alpha - \gamma$ is a root, then also $\alpha + \gamma$ is a root and they are both nonisotropic. This implies $\alpha - \gamma = \theta/2 = \alpha + \gamma$ which is absurd. So, if $(\alpha|\gamma) = 0$ then $c_{\gamma, \eta} = (\eta|\gamma)$ as well. If $(\alpha|\gamma) > 0$ then $\alpha - \gamma$ cannot be isotropic, for, in such a case,

$$2(\alpha|\gamma) = (\gamma|\gamma) < 0.$$

It follows that $\alpha - \gamma$ is not isotropic. Since $\alpha + \gamma$ is a root, it must be isotropic by the argument above. It follows that

$$2(\alpha|\gamma) = -(\gamma|\gamma)$$

and

$$(\alpha - \gamma|\alpha - \gamma) = -2(\alpha|\gamma) + (\gamma|\gamma) = 2(\gamma|\gamma) < 0.$$

On the other hand $\alpha - \gamma = \theta/2$ so

$$(\alpha - \gamma|\alpha - \gamma) = \frac{1}{2}.$$

The only possibility is that, when $(\alpha|\gamma) > 0$, then $\alpha - \gamma$ is not a root hence $c_{\gamma,\eta} = 0$.

If $\eta = 0$ then $\theta/2$ is a root, hence $\mathfrak{g}^\natural$ acts on $\mathfrak{g}_{-1/2}$ as the little adjoint representation. If v has weight 0 and γ is a root of $\mathfrak{g}^\natural$ then $[X_\gamma, v]$ has weight γ , so, if γ is long, $N_{\gamma, -\theta/2} = 0$ so $c_{\gamma,0} = 0$. If γ is short, then

$$c_{\gamma,0} = N_{\theta/2, -\gamma} N_{\gamma, -\theta/2}$$

and use formula (4.6) from [11] to compute it. Since γ is short, the $\theta/2$ -strings through $\pm\gamma$ are given by $\pm\gamma - \theta/2, \pm\gamma, \pm\gamma + \theta/2$, hence $N_{\theta/2, -\gamma} N_{\gamma, -\theta/2} = N_{\theta/2, \gamma} N_{-\gamma, -\theta/2} = -(\theta/2|\theta/2) = -1/2$. $\square$

Lemma 6.3. *Let M be a σ_R -twisted highest weight $W_{\min}^k(\mathfrak{g})$ -module with highest weight (ν, ℓ_0) and highest weight vector v_{ν, ℓ_0} . Let $v \in \mathfrak{g}_{-1/2}$ be a weight vector for $\mathfrak{h}^\natural$ of weight η . Then*

$$(6.11) \quad \sum_{\alpha, \beta} \langle [u_\alpha, \phi(v)], [v, u^\beta] \rangle J_0^{\{u^\alpha\}} J_0^{\{u^\beta\}} v_{\nu, \ell_0} = \langle \phi(v), v \rangle \left((\eta|\nu)^2 - \sum_{\gamma < 0} c_{\gamma, \eta} (\nu|\gamma) \right) v_{\nu, \ell_0},$$

$$(6.12) \quad \sum_{\alpha, \beta} \langle [u_\alpha, \phi(v)], [v, u^\beta] \rangle J_0^{\{u^\beta\}} J_0^{\{u^\alpha\}} v_{\nu, \ell_0} = \langle \phi(v), v \rangle \left((\eta|\nu)^2 + \sum_{\gamma > 0} c_{\gamma, \eta} (\nu|\gamma) \right) v_{\nu, \ell_0}.$$

Proof. Note that $v = cu_\eta$ for some non-zero constant c . Using Lemma 6.1, we have

$$\begin{aligned}
& \sum_{\gamma, \beta} \langle [u_\gamma, \phi(v)], [v, u^\beta] \rangle J_0^{\{u^\gamma\}} J_0^{\{u^\beta\}} v_{\nu, \ell_0} \\
&= \sum_{i, j} \langle [u_i, \phi(v)], [v, u^j] \rangle J_0^{\{u^i\}} J_0^{\{u^j\}} v_{\nu, \ell_0} + \sum_{\gamma < 0} \langle [u_\gamma, \phi(v)], [v, u^\gamma] \rangle J_0^{\{u^\gamma\}} J_0^{\{u^\gamma\}} v_{\nu, \ell_0} \\
&= \langle \phi(v), v \rangle \sum_{i, j} \eta(u^j) \eta(u_i) \nu(u_j) \nu(u^i) v_{\nu, \ell_0} + \sum_{\gamma < 0} \langle [u_\gamma, \phi(v)], [v, u^\gamma] \rangle J_0^{\{u^\gamma\}} J_0^{\{u^\gamma\}} v_{\nu, \ell_0} \\
&= \langle \phi(v), v \rangle (\eta|\nu)^2 v_{\nu, \ell_0} + \sum_{\gamma < 0} \langle [u_\gamma, \phi(v)], [v, u^\gamma] \rangle J_0^{\{u^\gamma\}} J_0^{\{u^\gamma\}} v_{\nu, \ell_0} \\
&= \langle \phi(v), v \rangle (\eta|\nu)^2 v_{\nu, \ell_0} + \sum_{\gamma < 0} \langle [u_\gamma, \phi(v)], [v, u^\gamma] \rangle [J_0^{\{u^\gamma\}}, J_0^{\{u^\gamma\}}] v_{\nu, \ell_0} \\
&= \left(\langle \phi(v), v \rangle (\eta|\nu)^2 - \sum_{\gamma < 0} \langle [u_\gamma, \phi(v)], [v, u^\gamma] \rangle (\nu|\gamma) \right) v_{\nu, \ell_0} \\
&= \left(\langle \phi(v), v \rangle (\eta|\nu)^2 - \sum_{\gamma < 0} \langle [u_\gamma, \phi(cu_\eta)], [cu^\eta, u^\gamma] \rangle (\nu|\gamma) \right) v_{\nu, \ell_0} \\
&\stackrel{(6.5)}{=} \left(\langle \phi(v), v \rangle (\eta|\nu)^2 - |c|^2 \sum_{\gamma < 0} \langle \phi(u_\eta), u_\eta \rangle c_{\gamma, \eta} (\nu|\gamma) \right) v_{\nu, \ell_0} \\
&= \left(\langle \phi(v), v \rangle (\eta|\nu)^2 - \sum_{\gamma < 0} \langle \phi(v), v \rangle c_{\gamma, \eta} (\nu|\gamma) \right) v_{\nu, \ell_0}.
\end{aligned}$$

This proves (6.11). For (6.12) we have

$$\begin{aligned}
& \sum_{\gamma, \beta} \langle [u_\gamma, \phi(v)], [v, u^\beta] \rangle J_0^{\{u^\beta\}} J_0^{\{u^\gamma\}} v_{\nu, \ell_0} = \\
& \sum_{i, j} \langle [u_i, \phi(v)], [v, u^j] \rangle J_0^{\{u^j\}} J_0^{\{u^i\}} v_{\nu, \ell_0} + \sum_{\gamma > 0} \langle [u_\gamma, \phi(v)], [v, u^\gamma] \rangle J_0^{\{u^\gamma\}} J_0^{\{u^\gamma\}} v_{\nu, \ell_0} \\
&= \left((\eta|\nu)^2 \langle \phi(v), v \rangle + \sum_{\gamma > 0} \langle [u_\gamma, \phi(v)], [v, u^\gamma] \rangle (\nu|\gamma) \right) v_{\nu, \ell_0}.
\end{aligned}$$

We conclude as above. $\square$

Lemma 6.4. *With the hypothesis of Lemma 6.3, we have*

$$\begin{aligned}
& [G_0^{\{\phi(v)\}}, G_0^{\{v\}}] v_{\nu, \ell_0} = \\
& \langle \phi(v), v \rangle \left((-2(k + h^\vee)\ell_0 + (\nu|\nu + 2\rho^\natural) - \tfrac{1}{2}p(k) + 2(\eta|\nu)^2 + \sum_{\gamma > 0} c_{\gamma, \eta} (\nu|\gamma) - \sum_{\gamma < 0} c_{\gamma, \eta} (\nu|\gamma)) \right) v_{\nu, \ell_0}.
\end{aligned}$$

Proof. Using Borchers' commutator formula

$$\begin{aligned}
[G_0^{\{\phi(v)\}}, G_0^{\{v\}}] &= \sum_j \binom{1/2}{j} (G^{\{\phi(v)\}}_{(j)} G^{\{v\}})_0 \\
&= (G^{\{\phi(v)\}}_{(0)} G^{\{v\}})_0 + 1/2 (G^{\{\phi(v)\}}_{(1)} G^{\{v\}}) - 1/8 (G^{\{\phi(v)\}}_{(2)} G^{\{v\}}).
\end{aligned}$$

and recalling that [2]

$$\begin{aligned}
 (6.13) \quad [G^{\{u\}}_{\lambda} G^{\{v\}}] &= -2(k + h^{\vee})\langle u, v \rangle L + \langle u, v \rangle \sum_{\alpha=1}^{\dim \mathfrak{g}^{\natural}} : J^{\{u^{\alpha}\}} J^{\{u_{\alpha}\}} : \\
 &\quad + 2 \sum_{\alpha, \beta} \langle [u_{\alpha}, u], [v, u^{\beta}] \rangle : J^{\{u^{\alpha}\}} J^{\{u_{\beta}\}} : + 2(k+1)(\partial + 2\lambda) J^{\{[e, u], v\}^{\natural}} \\
 &\quad + 2\lambda \sum_{\alpha, \beta} \langle [u_{\alpha}, u], [v, u^{\beta}] \rangle J^{\{u^{\alpha}, u_{\beta}\}} + 2\lambda^2 \langle u, v \rangle p(k) |0\rangle,
 \end{aligned}$$

we get

$$\begin{aligned}
 [G_0^{\{u\}}, G_0^{\{v\}}] v_{\nu, \ell_0} &= \langle u, v \rangle (-2(k + h^{\vee})\ell_0 + (\nu|\nu + 2\rho^{\natural})) v_{\nu, \ell_0} \\
 &\quad + 2 \sum_{\alpha, \beta} \langle [u_{\alpha}, u], [v, u^{\beta}] \rangle : J^{\{u^{\alpha}\}} J^{\{u_{\beta}\}} :_0 v_{\nu, \ell_0} \\
 &\quad + \sum_{\alpha, \beta} \langle [u_{\alpha}, u], [v, u^{\beta}] \rangle J_0^{\{u^{\alpha}, u_{\beta}\}} v_{\nu, \ell_0} - \frac{1}{2} \langle u, v \rangle p(k) v_{\nu, \ell_0} = \\
 &\quad \langle u, v \rangle (-2(k + h^{\vee})\ell_0 + (\nu|\nu + 2\rho^{\natural})) v_{\nu, \ell_0} + 2 \sum_{\alpha, \beta} \langle [u_{\alpha}, u], [v, u^{\beta}] \rangle J_0^{\{u_{\beta}\}} J_0^{\{u^{\alpha}\}} v_{\nu, \ell_0} \\
 &\quad + \sum_{\alpha, \beta} \langle [u_{\alpha}, u], [v, u^{\beta}] \rangle (J_0^{\{u^{\alpha}\}} J_0^{\{u_{\beta}\}} - J_0^{\{u_{\beta}\}} J_0^{\{u^{\alpha}\}}) v_{\nu, \ell_0} - \frac{1}{2} \langle u, v \rangle p(k) v_{\nu, \ell_0} = \\
 &\quad \langle u, v \rangle (-2(k + h^{\vee})\ell_0 + (\nu|\nu + 2\rho^{\natural}) - \frac{1}{2}p(k)) v_{\nu, \ell_0} \\
 &\quad + \sum_{\alpha, \beta} \langle [u_{\alpha}, u], [v, u^{\beta}] \rangle J_0^{\{u_{\beta}\}} J_0^{\{u^{\alpha}\}} v_{\nu, \ell_0} + \sum_{\alpha, \beta} \langle [u_{\alpha}, u], [v, u^{\beta}] \rangle J_0^{\{u^{\alpha}\}} J_0^{\{u_{\beta}\}} v_{\nu, \ell_0}.
 \end{aligned}$$

Now apply Lemma 6.3. □

Let $\overline{\Delta}_{1/2}^+$ be the set defined in (6.1) and observe that, by construction, $\overline{\Delta}_{1/2}^+ = \{\eta \in (\mathfrak{h}^{\natural})^* \mid \eta \text{ is a } \mathfrak{h}^{\natural}\text{-weight of } \mathfrak{n}_{-1/2}(\sigma_R)_+\}'$. Set

$$(6.14) \quad \rho_R = \rho(\mathfrak{n}_{1/2}(\sigma_R)_+)|_{\mathfrak{h}^{\natural}} = \frac{1}{2} \sum_{\mathfrak{g}_{\alpha} \subset \mathfrak{n}_{1/2}(\sigma_R)_+} \alpha|_{\mathfrak{h}^{\natural}} = \frac{1}{2} \sum_{\eta \in \overline{\Delta}_{1/2}^+} \eta.$$

Applying the formulas of Lemma 6.2 we obtain the following refinement of Lemma 6.4.

Lemma 6.5. *Under the hypothesis of Lemma 6.3, we have*

(1) *Assume that $\theta/2 \in \Delta$ and let $v \in \mathfrak{g}_{-1/2}$ be a vector of weight 0. Then*

$$G_0^{\{\phi(v)\}} G_0^{\{v\}} v_{\nu, \ell_0} = 2\langle \phi(v), v \rangle \left((-2(k + h^{\vee})\ell_0 + (\nu|\nu + 2\rho^{\natural} - \rho_R)) - \frac{1}{2}p(k) \right) v_{\nu, \ell_0}.$$

(2) *Assume that $v \in \mathfrak{g}_{-1/2}$ is a vector of weight $\eta \neq 0$. Then*

$$\begin{aligned}
 &[G_0^{\{\phi(v)\}}, G_0^{\{v\}}] v_{\nu, \ell_0} \\
 &= \langle \phi(v), v \rangle \left((-2(k + h^{\vee})\ell_0 + (\nu|\nu + 2\rho^{\natural}) - \frac{1}{2}p(k) + 2(\eta|\nu)^2) \right) v_{\nu, \ell_0} \\
 &\quad + \left(\sum_{\gamma > 0, (\gamma|\eta) \leq 0} (\gamma|\eta)(\nu|\gamma) - \sum_{\gamma > 0, (\gamma|\eta) \geq 0} (\gamma|\eta)(\nu|\gamma) \right) v_{\nu, \ell_0}.
 \end{aligned}$$

Proof. To prove (1), recall that in this case, as already observed in the proof of Lemma 6.2, $\mathfrak{g}^{\natural}$ acts on $\mathfrak{g}_{-1/2}$ as the little adjoint representation, so its nonzero weights are precisely the roots in $\Delta^{\natural}$ of minimal length. Since we chose h_0 so that $\alpha(h_0) > 0$ for $\alpha \in \Delta_+^{\natural}$, it follows immediately from (6.1) that

$$(6.15) \quad \rho_R = \sum_{\gamma \in \Delta_+^{\natural}, \gamma \text{ short}} \gamma.$$

By Lemma 6.4 and Lemma 6.2,

$$\begin{aligned} & [G_0^{\{\phi(v)\}}, G_0^{\{v\}}]v_{\nu, \ell_0} \\ &= \langle \phi(v), v \rangle \left((-2(k + h^{\vee})\ell_0 + (\nu|\nu + 2\rho^{\natural}) - \tfrac{1}{2}p(k) - \sum_{\gamma \in \Delta_+^{\natural}, \gamma \text{ short}} (\nu|\gamma)) v_{\nu, \ell_0} \right) \\ &= \langle \phi(v), v \rangle \left((-2(k + h^{\vee})\ell_0 + (\nu|\nu + 2\rho^{\natural}) - \tfrac{1}{2}p(k) - 2(\nu|\rho_R)) v_{\nu, \ell_0} \right). \end{aligned}$$

Now observe that $\phi(v) = hv$, so that $[G_0^{\{\phi(v)\}}, G_0^{\{v\}}] = \tfrac{1}{2}h(G_0^{\{v\}})^2 = \tfrac{1}{2}G_0^{\{\phi(v)\}}G_0^{\{v\}}$.

If v has weight $\eta \neq 0$ then, by Lemma 6.2,

$$\begin{aligned} \sum_{\gamma > 0} c_{\gamma, \eta}(\nu|\gamma) - \sum_{\gamma < 0} c_{\gamma, \eta}(\nu|\gamma) &= \sum_{\gamma > 0, (\gamma|\eta) \leq 0} (\gamma|\eta)(\nu|\gamma) - \sum_{\gamma < 0, (\gamma|\eta) \leq 0} (\gamma|\eta)(\nu|\gamma) \\ &= \sum_{\gamma > 0, (\gamma|\eta) \leq 0} (\gamma|\eta)(\nu|\gamma) - \sum_{\gamma > 0, (\gamma|\eta) \geq 0} (\gamma|\eta)(\nu|\gamma). \end{aligned}$$

Thus (2) follows from Lemma 6.4. $\square$

Let $L^W(\lambda, \ell_0)$ be the irreducible σ_R -twisted positive energy $W_{\min}^k(\mathfrak{g})$ -module such that $L^W(\lambda, \ell_0)_0 = L^Z(\lambda, \ell_0)$ (see Theorem 2.30 of [6]). It is clear that $L^W(\lambda, \ell_0)$ is a highest weight module of highest weight (λ, ℓ_0) . Conversely, if M is an irreducible highest weight module of highest weight (λ, ℓ_0) , then, by Lemma 4.20, the grading given by the action of L_0 defines the structure of a positive energy module. By Theorem 2.30 of [6], M_0 is irreducible. Since clearly M_0 is a highest weight module of highest weight (λ, ℓ_0) , by Corollary 5.6, $M_0 = L^Z(\lambda, \ell_0)$ hence, by [6, Theorem 2.30] again, $M = L^W(\lambda, \ell_0)$. This shows that $L^W(\lambda, \ell_0)$ is the unique irreducible σ_R -twisted highest weight $W_{\min}^k(\mathfrak{g})$ -module of highest weight (λ, ℓ_0) .

Consider (ν, ℓ_0) with ν purely imaginary and ℓ_0 real. Combining Proposition 5.7 and [14, Proposition 6.7], we get the existence of a unique even invariant Hermitian form $H(\cdot, \cdot)$ on $L^W(\nu, \ell_0)$ such that $H(v_{\nu, \ell_0}, v_{\nu, \ell_0}) = 1$ (v_{ν, ℓ_0} is a highest weight vector).

Proposition 6.6. *Let $L^W(\nu, \ell_0)$ be the irreducible σ_R -twisted unitary highest weight module with highest weight vector v_{ν, ℓ_0} and the Hermitian form $H(\cdot, \cdot)$, and set $\|a\|^2 = H(a, a)$.*

(1) *Let $v \in \mathfrak{n}_{-1/2}(\sigma)_+'$ be a vector of $\mathfrak{h}^{\natural}$ -weight η . Then*

$$\begin{aligned} & \|G_0^{\{\phi(v)\}}v_{\nu, \ell_0}\|^2 = \\ &= \langle \phi(v), v \rangle \left((-2(k + h^{\vee})\ell_0 + (\nu|\nu + 2\rho^{\natural}) - \tfrac{1}{2}p(k) + 2(\eta|\nu)^2) \right) \\ &+ \langle \phi(v), v \rangle \left(\sum_{\gamma > 0, (\gamma|\eta) \leq 0} (\gamma|\eta)(\nu|\gamma) - \sum_{\gamma > 0, (\gamma|\eta) \geq 0} (\gamma|\eta)(\nu|\gamma) \right). \end{aligned}$$

(2) Assume that $\theta/2 \in \Delta$. Then

$$\|G_0^{\{u_{\theta/2}\}} v_{\nu, \ell_0}\|^2 = 2\langle \phi(u_{\theta/2}), u_{\theta/2} \rangle \left((-2(k + h^\vee)\ell_0 + (\nu|\nu + 2(\rho^\natural - \rho_R)) - \frac{1}{2}p(k)) \right).$$

Proof. We have

(6.16)

$$H(G_0^{\{\phi(v)\}} v_{\nu, \ell_0}, G_0^{\{\phi(v)\}} v_{\nu, \ell_0}) = H(G_0^{\{v\}} G_0^{\{\phi(v)\}} v_{\nu, \ell_0}, v_{\nu, \ell_0}) = H([G_0^{\{\phi(v)\}}, G_0^{\{v\}}] v_{\nu, \ell_0}, v_{\nu, \ell_0}).$$

Now apply Lemma 6.4 to get (1). As for (2), we have

$$(6.17) \quad H(G_0^{\{u_{\theta/2}\}} v_{\nu, \ell_0}, G_0^{\{u_{\theta/2}\}} v_{\nu, \ell_0}) = H((G_0^{\{\phi(u_{\theta/2})\}} (G_0^{\{u_{\theta/2}\}}) v_{\nu, \ell_0}, v_{\nu, \ell_0}),$$

and we can apply (1) in Lemma 6.5. $\square$

If $L^W(\nu, \ell_0)$ is as in Proposition 6.6, then, since $\langle \phi(v), v \rangle > 0$, by (1) of this proposition we have

$$(6.18) \quad \ell_0 \geq \frac{1}{2(k + h^\vee)} ((\nu|\nu + 2\rho^\natural) - \frac{1}{2}p(k) + F_\nu(\eta)),$$

where

$$(6.19) \quad F_\nu(\eta) = 2(\eta|\nu)^2 + \sum_{\gamma > 0, (\gamma|\eta) \leq 0} (\gamma|\eta)(\nu|\gamma) - \sum_{\gamma > 0, (\gamma|\eta) \geq 0} (\gamma|\eta)(\nu|\gamma).$$

Similarly, if $\eta = 0$ is a weight of $\mathfrak{g}_{-1/2}$, then, by (2) of Proposition 6.6, we have

$$(6.20) \quad \ell_0 \geq \frac{1}{2(k + h^\vee)} ((\nu|\nu + 2(\rho^\natural - \rho_R)) - \frac{1}{2}p(k)).$$

Since $k + h^\vee < 0$, the maximal value of the right hand side of (6.18) is achieved for η such that $F_\nu(\eta)$ is minimal.

Let $P^+ \subset (\mathfrak{h}^\natural)^*$ be the set of dominant integral weights for $\mathfrak{g}^\natural$. We compute below the minimal value of $F_\nu(\eta)$ for $\nu \in P^+$ and $\eta \in \overline{\Delta}_{1/2}^+$ via a case by case inspection. In the cases where $\theta/2$ is an odd root (so that $\eta = 0$ is a weight for $\mathfrak{g}_{-1/2}$) we also show that $-2(\nu|\rho_R) \leq F_\nu(\eta)$, hence (6.20) gives the best bound. We gave above an explicit description of the set $\overline{\Delta}_{1/2}^+$, hence we can compute explicit expressions for ρ_R that we list in Tables 2 and 3, along with the values of $\eta_{\min}$, $\rho^\natural$, and ξ . In Tables 2 and 3 we denote by ω_j^i the fundamental weights of the simple ideal $\mathfrak{g}_i^\natural$ of $\mathfrak{g}^\natural$. We drop the superscript i if $\mathfrak{g}^\natural$ is simple.

$\mathfrak{g}$	$psl(2 2)$	$spo(2 2r+1), r \geq 1$	$G(3)$
$\eta_{\min}$	$\frac{\delta_1 - \delta_2}{2}$	ϵ_r	ϵ_1
ρ_R	ω_1	ω_r	ω_1
$\rho^\natural$	$\frac{\delta_1 - \delta_2}{2}$	$\sum_{i=1}^r (r - i + \frac{1}{2}) \epsilon_i$	$2\epsilon_1 + 3\epsilon_2$
ξ	$\frac{\delta_1 - \delta_2}{2}$	ϵ_1	$\epsilon_1 + \epsilon_2$

TABLE 2. Cases with only one choice for $\overline{\Delta}_{1/2}^+$.

$psl(2|2)$. In this case $\Delta_+^\natural = \{\delta_1 - \delta_2\}$ and $\nu = r/2(\delta_1 - \delta_2)$. Then

$$(6.21) \quad \min_{\eta} F_\nu(\eta) = F_\nu(\xi) = \frac{1}{2}r^2 + r.$$

$\mathfrak{g}$	$\mathfrak{spo}(2 2r)$	$D(2, 1; \frac{m}{n})$	$F(4)$
$\eta_{\min}$	$\epsilon_r, -\epsilon_r$	$\epsilon_2 - \epsilon_3, -\epsilon_2 + \epsilon_3$	$\frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3), \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3)$
ρ_R	ω_r, ω_{r-1}	ω_1^1, ω_2^2	ω_1, ω_3
$\rho^{\natural}$	$\sum_{i=1}^r (r-i)\epsilon_i$	$\epsilon_2 + \epsilon_3$	$\frac{5}{2}\epsilon_1 + \frac{3}{2}\epsilon_2 + \frac{1}{2}\epsilon_3$
ξ	ϵ_1	$\epsilon_2 + \epsilon_3$	$\frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3)$

TABLE 3. Cases with two choices for $\overline{\Delta}_{1/2}^+$.

$\mathfrak{spo}(2|3)$. In this case $\Delta_+^{\natural} = \{\epsilon_1\}$ and $\nu = \frac{r}{2}\epsilon_1$. Then

$$(6.22) \quad \min_{\eta} F_{\nu}(\eta) = F_{\nu}(\xi) = \frac{r^2 + r}{8} = \frac{r}{4} + \frac{1}{8}r(r-1).$$

Since in this case $\eta = 0$ is a weight of $\mathfrak{g}_{-1/2}$, we need also to compute

$$(6.23) \quad -2(\nu|\rho_R) = \frac{r}{4},$$

which gives the minimal value of $F_{\nu}(\eta)$.

$\mathfrak{spo}(2|2\mathbf{r}), \mathbf{r} > 2$. In this case $\Delta_+^{\natural} = \{\epsilon_i \pm \epsilon_j, 1 \leq i < j \leq r\}$. Since ν is dominant integral, $\nu = \sum_i m_i \epsilon_i$, $m_i \in \frac{1}{2} + \mathbb{Z}$ or $m_i \in \mathbb{Z}, m_1 \geq \dots \geq m_{r-1} \geq |m_r|$. If $\eta = \epsilon_i$, then

$$\begin{aligned} F_{\nu}(\eta) &= m_i^2/2 - 1/2 \sum_{\gamma \in \{\epsilon_s + \epsilon_i, i \neq s\} \cup \{\epsilon_i - \epsilon_s, s > i\}} (\nu|\gamma) - \frac{1}{2} \sum_{\gamma \in \{\epsilon_s - \epsilon_i, s < i\}} (\nu|\gamma) \\ &= m_i^2/2 - 1/2 \left(\sum_{\gamma \in \{\epsilon_s \pm \epsilon_i, s < i\}} (\nu|\gamma) + \sum_{\gamma \in \{\epsilon_i \pm \epsilon_s, s > i\}} (\nu|\gamma) \right) \\ &= m_i^2/2 + 1/4 \left(\sum_{s < i} (m_s \pm m_i) + \sum_{s > i} (m_i \pm m_s) \right) = m_i^2/2 + 1/2 \sum_{s < i} m_s + 1/2(r-i)m_i. \end{aligned}$$

The minimum value for $F_{\nu}(\eta)$ is achieved if $\eta = \pm \epsilon_r$ and it is

$$(6.24) \quad F_{\nu}(\eta) = \frac{1}{2}(m_r^2 + \sum_{s < r} m_s).$$

$\mathfrak{spo}(2|2\mathbf{r} + 1), \mathbf{r} > 1$. $\Delta_+^{\natural} = \{\epsilon_i \pm \epsilon_j, 1 \leq i < j \leq r\} \cup \{\epsilon_i\}$, $\nu = \sum m_i \epsilon_i$, $m_i \in \frac{1}{2} + \mathbb{Z}$ or $m_i \in \mathbb{Z}, m_1 \geq \dots \geq m_r \geq 0$. If $\eta = \epsilon_i$, then

$$\begin{aligned} F_{\nu}(\eta) &= m_i^2/2 - 1/2 \sum_{\gamma \in \{\epsilon_s + \epsilon_i, i \neq s\} \cup \{\epsilon_i - \epsilon_s, s > i\} \cup \{\epsilon_i\}} (\nu|\gamma) - \frac{1}{2} \sum_{\gamma \in \{\epsilon_s - \epsilon_i, s < i\}} (\nu|\gamma) \\ &= m_i^2/2 + 1/4 \left(\sum_{s < i} (m_s \pm m_i) + m_i + \sum_{s > i} (m_i \pm m_s) \right) \\ &= m_i^2/2 + 1/2 \sum_{s < i} m_s + 1/2(r-i+\frac{1}{2})m_i. \end{aligned}$$

The minimum is achieved when $i = r$ and it is

$$(6.25) \quad \min_{\eta} F_{\nu}(\eta) = \frac{1}{2}(m_r^2 + \sum_{s < r} m_s + \frac{1}{2}m_r) = \frac{1}{2} \sum_{s \leq r} m_s + \frac{1}{2}m_r(m_r - \frac{1}{2}).$$

Since in this case $\eta = 0$ is a weight of $\mathfrak{g}_{-1/2}$, we need also to compute

$$(6.26) \quad -2(\nu|\rho_R) = -\sum_{i=1}^r (\nu|\epsilon_i) = \frac{1}{2} \sum_{s \leq r} m_s,$$

which gives the minimal value of $F_\nu(\eta)$.

D(2, 1; a). $\Delta_+^\natural = \{2\epsilon_2, 2\epsilon_3\}$, $\nu = m_1\epsilon_1 + m_2\epsilon_2$, $m_i \in \mathbb{Z}_+$. Then a computer computation shows that the minimum of $F_\nu(\eta)$ is attained at $\epsilon_2 - \epsilon_3$ and at $-\epsilon_2 + \epsilon_3$ and its value is

$$(6.27) \quad \min_{\eta} F_\nu(\eta) = \frac{(m_1 - am_2)^2 + 2(m_1 + a^2m_2)}{2(1+a)^2}.$$

F(4). $\Delta_+^\natural = \{\epsilon_i \pm \epsilon_j, 1 \leq i < j \leq 3\} \cup \{\epsilon_i\}$. Since ν is dominant integral, $\nu = \sum m_i \epsilon_i$, $m_i \in \frac{1}{2} + \mathbb{Z}$ or $m_i \in \mathbb{Z}$, $m_1 \geq m_2 \geq r_3 \geq 0$. A computer computation shows that the minimum of $F_\nu(\eta)$ is attained when $\eta = \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3)$ and when $\eta = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3)$. Its value is

$$(6.28) \quad \min_{\eta} F_\nu(\eta) = \frac{2}{9}((-m_1 + m_2 + m_3)^2 + 5m_1 + m_2 + m_3).$$

G(3). We have $\Delta_+^\natural = \{\epsilon_1, \epsilon_2 - \epsilon_1, \epsilon_2, \epsilon_2 + \epsilon_1, \epsilon_2 + 2\epsilon_1, 2\epsilon_2 + \epsilon_1\}$, $\nu = m_1\epsilon_1 + m_2\epsilon_2$, $2m_1 \geq m_2 \geq m_1$, $m_i \in \mathbb{Z}_+$. A computer computation shows that the minimum of $F_\nu(\eta)$ is attained when $\eta = \epsilon_1$ and

$$\min_{\eta} F_\nu(\eta) = \frac{1}{8}(2m_1 - m_2)(2m_1 - m_2 - 1) + \frac{1}{2}(m_1 + m_2).$$

Since in this case $\eta = 0$ is a weight of $\mathfrak{g}_{-1/2}$, we need also to compute

$$(6.29) \quad -2(\nu|\rho_R) = \frac{1}{2}(m_1 + m_2) \leq \min_{\eta} F_\nu(\eta).$$

Our direct inspection shows the following fact

Proposition 6.7. *We have $\min_{\eta} F_\nu(\eta) = F_\nu(\eta_{\min})$, and this value is independent from the choice of the set $\bar{\Delta}_{1/2}^+$.*

Recall [19, Theorem 2.1], [15, Section 7] that there is an embedding in $W_{\min}^k(\mathfrak{g})$ of the universal affine vertex algebra $\bigotimes_i V^{M_i(k)}(\mathfrak{g}_i^\natural)$, where the $M_i(k)$, $i = 1, \dots, s$, are given in Table 4, along with other quantities, explained in [15, § 7.2]. If $L^W(\nu, \ell_0)$ is unitary, then $\bigotimes_i V^{M_i(k)}(\mathfrak{g}_i^\natural) \cdot v_{\nu, \ell_0}$ is a unitary representation of the latter affine vertex algebra, hence ν lies in P_k^+ , where

$$(6.30) \quad P_k^+ = \left\{ \nu \in P^+ \mid \nu(\theta_i^\vee) \leq M_i(k) \text{ for all } i = 1, \dots, s \right\},$$

where P^+ is the set of dominant integral weights of $\mathfrak{g}^\natural$.

We summarize our findings in the following statement:

Proposition 6.8. *Let $\nu \in P_k^+$ and $\ell_0 \in \mathbb{R}$. Set*

$$(6.31) \quad A(k, \nu) = \begin{cases} \frac{1}{2(k+h^\vee)}((\nu|\nu + 2(\rho^\natural - \rho_R)) - \frac{1}{2}p(k)) & \text{if } \theta/2 \in \Delta, \\ \frac{1}{2(k+h^\vee)}\left((\nu|\nu + 2\rho^\natural) - \frac{1}{2}p(k) + F_\nu(\eta_{\min})\right) & \text{otherwise.} \end{cases}$$

If $L^W(\nu, \ell_0)$ is unitary, then

$$\ell_0 \geq A(k, \nu).$$

Lemma 6.9. *If $L^W(\nu, \ell_0)$ is unitary, then $\ell_0 = A(k, \nu)$ in the following cases:*

$\mathfrak{g}$	$\mathfrak{g}^\natural$	$h^\vee$	$\bar{h}_i^\vee$	$M_i(k)$	χ_i	$u_i = (\theta_i \theta_i)$
$psl(2 2)$	sl_2	-2	0	$-k-1$	-1	-2
$spo(2 3)$	sl_2	$1/2$	$-1/2$	$-4k-2$	-2	$-1/2$
$spo(2 m), m \geq 4$	so_m	$2 - m/2$	$1 - m/2$	$-2k-1$	-1	-1
$D(2, 1; a)$	$sl_2 \oplus sl_2$	0	$-\frac{2}{1+a}, -\frac{2a}{1+a}$	$-(1+a)k-1, -\frac{1+a}{a}k-1$	$-1, -1$	$-\frac{2}{1+a}, -\frac{2a}{1+a}$
$F(4)$	so_7	-2	$-10/3$	$-\frac{3}{2}k-1$	-1	$-4/3$
$G(3)$	G_2	$-3/2$	-3	$-\frac{4}{3}k-1$	-1	$-3/2$

TABLE 4. Numerical information

- (1) $\mathfrak{g} \neq spo(2|3), D(2, 1; a)$: $\nu \in P_k^+$ such that $\nu - \rho_R \notin P^+$.
- (2) $\mathfrak{g} = spo(2|3)$: $\nu = 0$ and $\nu = \frac{M_1(k)}{2}\epsilon_1$.
- (3) $\mathfrak{g} = D(2, 1; a)$: $\nu = r\epsilon_3$ and $\nu = r\epsilon_2 + M_2(k)\epsilon_3$, where $r = 0, \dots, M_1(k)$ (resp. $\nu = r\epsilon_2$ and $\nu = M_1(k)\epsilon_2 + r\epsilon_3$, where $r = 0, \dots, M_2(k)$) if $\rho_R = \omega_1^1 = \epsilon_2$ (resp. $\rho_R = \omega_1^2 = \epsilon_3$).

Proof. Assume $\nu - \rho_R \notin P^+$. We first discuss the cases when $\theta/2$ is a root, namely $\mathfrak{g} = spo(2|2r+1)$ and $\mathfrak{g} = G(3)$. Since $\rho_R = \omega_{j_0}^{i_0}$ for some i_0, j_0 , then $\nu = \sum_{i,j} m_j^i \omega_j^i$ with $m_{j_0}^{i_0} = 0$. Let $\bar{\alpha}$ be the simple root corresponding to $\omega_{j_0}^{i_0}$ (i. e. $\omega_{j_0}^{i_0}(\bar{\alpha}^\vee) = \delta_{\alpha, \bar{\alpha}}$ for all simple roots α of $\Delta^\natural$). Explicitly $\bar{\alpha} = \epsilon_r$ for $\mathfrak{g} = spo(2|2r+1)$ while $\bar{\alpha} = \epsilon_1$ for $\mathfrak{g} = G(3)$. Then $J_0^{\{X-\bar{\alpha}\}}$ acts trivially on v_{ν, ℓ_0} . We note that $\bar{\alpha} = \eta_{\min}$. If $v \in \mathfrak{g}_{-1/2}$ has weight $\eta_{\min}$, then $G_0^{\{v\}} v_{\nu, \ell_0} = 0$. Note that v is a root vector for the root $-\theta/2 + \eta_{\min}$, hence $[X_{-\bar{\alpha}}, v]$ is a root vector for the root $-\theta/2$, thus $w = [X_{-\bar{\alpha}}, v]$ is a nonzero multiple of $u_{\theta/2}$. Since

$$0 = J_0^{\{X-\bar{\alpha}\}} G_0^{\{v\}} v_{\nu, \ell_0} = G_0^{\{[X-\bar{\alpha}, v]\}} v_{\nu, \ell_0} = G_0^{\{w\}} v_{\nu, \ell_0} = \text{const.} G_0^{\{u_{\theta/2}\}} v_{\nu, \ell_0},$$

we deduce that $\|G_0^{\{u_{\theta/2}\}} v_{\nu, \ell_0}\|^2 = 0$ and Proposition 6.6 implies $\ell_0 = A(k, \nu)$.

If $\mathfrak{g} = D(2, 1; a)$ and $\eta_{\min} = \epsilon_2 - \epsilon_3$, then $\rho_R = \omega_1^1 = \epsilon_2$. Let $\bar{\alpha} = 2\epsilon_2$. Since $\nu - \rho_R \notin P^+$, $\nu = r_3\epsilon_3$, so $X_{-\bar{\alpha}} v_{\nu, \ell_0} = 0$. If $v \in \mathfrak{g}_{-1/2}$ has weight $\eta_{\min}$, then $G_0^{\{v\}} v_{\nu, \ell_0} = 0$. Note that v is a root vector for the root $-\epsilon_1 + \epsilon_2 - \epsilon_3$, hence $[X_{-\bar{\alpha}}, v]$ is a root vector for the root $-\epsilon_1 - \epsilon_2 - \epsilon_3$, thus $w = [X_{-\bar{\alpha}}, v] \neq 0$. Moreover $w \in \mathfrak{g}_{-1/2}$ is a vector of weight $-\eta_{\min}$ thus $w = \text{const.} \phi(v)$. Therefore

$$0 = J_0^{\{X-\bar{\alpha}\}} G_0^{\{v\}} v_{\nu, \ell_0} = G_0^{\{[X-\bar{\alpha}, v]\}} v_{\nu, \ell_0} = G_0^{\{w\}} v_{\nu, \ell_0} = \text{const.} G_0^{\{\phi(v)\}} v_{\nu, \ell_0}.$$

We deduce that $\|G_0^{\{\phi(v)\}} v_{\nu, \ell_0}\|^2 = 0$ and Proposition 6.6 implies $\ell_0 = A(k, \nu)$ also in this case. The argument for $\mathfrak{g} = D(2, 1; a)$, $\eta_{\min} = -\epsilon_2 + \epsilon_3$, and $\nu = r_2\epsilon_2$ is completely analogous using $\bar{\alpha} = 2\epsilon_3$.

We now turn to the remaining cases where the argument is similar but somewhat more complicated: we claim that for $v \in \mathfrak{n}_{-1/2}(\sigma_R)_+$ of weight $\eta_{\min}$, we have

$$(6.32) \quad G_0^{\{\phi(v)\}} v_{\nu, \ell_0} = 0.$$

Indeed, if $\mathfrak{g} = psl(2|2)$, then $\nu - \rho_R \notin P^+$ means that $\nu = 0$, thus $J_0^{\{a\}} v_{\nu, \ell_0} = 0$ for all $a \in \mathfrak{g}^\natural$. We know that $G_0^{\{u\}} v_{\nu, \ell_0} = 0$ for all $u \in \mathfrak{n}_{-1/2}(\sigma_R)_+$. Since $[J_0^{\{a\}}, G_0^{\{vu\}}] = G_0^{\{[a, u]\}}$, then $G_0^{\{[a, u]\}} v_{\nu, \ell_0} = J_0^{\{a\}} G_0^{\{u\}} v_{\nu, \ell_0}$. Since $ad(U(\mathfrak{g}^\natural))\mathfrak{n}_{-1/2}(\sigma_R) = \mathfrak{g}_{-1/2}$, we see that $G_0^{\{u\}} v_{\nu, \ell_0} = 0$ for all $u \in \mathfrak{g}_{-1/2}$, thus, in particular, (6.32) holds in this case.

We now prove (6.32) in the cases $\mathfrak{g} = \mathfrak{spo}(2|2r)$, $\mathfrak{g} = F(4)$. Since $\rho_R = \omega_{j_0}^{i_0}$ let $\bar{\alpha}$, as above, be the simple root corresponding to $\omega_{j_0}^{i_0}$. Since $J_0^{\{X-\bar{\alpha}\}}$ acts trivially on v_{ν, ℓ_0} and $G_0^{\{v\}} v_{\nu, \ell_0} = 0$ if $v \in \mathfrak{g}_{-1/2}$ has weight $\eta_{\min}$,

$$0 = J_0^{\{X-\bar{\alpha}\}} G_0^{\{v\}} v_{\nu, \ell_0} = G_0^{\{[X-\bar{\alpha}, v]\}} v_{\nu, \ell_0}.$$

Let $\beta = -\frac{1}{2}\theta + \eta_{\min}$. This is the root of $\mathfrak{g}_{-1/2}$ such that $\beta|_{\mathfrak{h}^\natural} = \eta_{\min}$. We check in each case that

- (i) $\beta - \bar{\alpha}$ is a root,
- (ii) there exists a positive root $\gamma \in \Delta^\natural$ such that $\beta - \bar{\alpha} + \gamma = -\theta/2 - \eta_{\min}$.

If (i), (ii) hold, then $[X-\bar{\alpha}, v] \neq 0$ and both $[X_\gamma, [X-\bar{\alpha}, v]]$ and $\phi(v)$ are in $\mathfrak{g}_{\beta'}$ with $\beta' = -\frac{1}{2}\theta - \eta_{\min}$. Since all the roots have multiplicity one, this implies that $[X_\gamma, [X-\bar{\alpha}, v]] = \text{const.} \phi(v)$. Since γ is positive $X_\gamma v_{\nu, \ell_0} = 0$, hence

$$0 = J_0^{\{X_\gamma\}} J_0^{\{[X-\bar{\alpha}, v]\}} G_0^{\{v\}} v_{\nu, \ell_0} = G_0^{\{[X_\gamma, [X-\bar{\alpha}, v]]\}} v_{\nu, \ell_0} = \text{const.} G_0^{\{\phi(v)\}} v_{\nu, \ell_0},$$

so that (6.32) holds.

The following list describes for each case the roots $\bar{\alpha}$, β , $\beta - \bar{\alpha}$, and γ .

- Case 1: $\mathfrak{so}(2|2r)$, $r > 2$, $\eta_{\min} = \epsilon_r$, $\rho_R = \omega_r$, $\bar{\alpha} = \epsilon_{r-1} + \epsilon_r$, $\beta = -\delta_1 + \epsilon_r$, $\beta - \bar{\alpha} = -\delta_1 - \epsilon_{r-1}$, $\gamma = \epsilon_{r-1} - \epsilon_r$.
- Case 2: $\mathfrak{so}(2|2r)$, $r > 2$, $\eta_{\min} = -\epsilon_r$, $\rho_R = \omega_{r-1}$, $\bar{\alpha} = \epsilon_{r-1} - \epsilon_r$, $\beta = -\delta_1 - \epsilon_r$, $\beta - \bar{\alpha} = -\delta_1 - \epsilon_{r-1}$, $\gamma = \epsilon_{r-1} + \epsilon_r$.
- Case 3: $F(4)$, $\eta_{\min} = \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3)$, $\rho_R = \omega_3$, $\bar{\alpha} = \epsilon_3$, $\beta = \frac{1}{2}(-\delta_1 - \epsilon_1 + \epsilon_2 + \epsilon_3)$, $\beta - \bar{\alpha} = \frac{1}{2}(-\delta_1 - \epsilon_1 + \epsilon_2 - \epsilon_3)$, $\gamma = \epsilon_1 - \epsilon_2$.
- Case 4: $F(4)$, $\eta_{\min} = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3)$, $\rho_R = \omega_1$, $\bar{\alpha} = \epsilon_1 - \epsilon_2$, $\beta = \frac{1}{2}(-\delta_1 + \epsilon_1 - \epsilon_2 - \epsilon_3)$, $\beta - \bar{\alpha} = \frac{1}{2}(-\delta_1 - \epsilon_1 + \epsilon_2 - \epsilon_3)$, $\gamma = \epsilon_3$.

Having established (6.32), by Proposition 6.6, $\|G_0^{\{\phi(v)\}} v_{0, \ell_0}\|^2 = 0$ implies $\ell_0 = A(k, 0)$.

It remains only to check the statement for $\mathfrak{g} = \mathfrak{spo}(2|3)$ or $\mathfrak{g} = D(2, 1; a)$ and $\nu = \sum_i \frac{r_i}{2} \theta_i$ with $r_i = M_i(k)$ for some i . Since

$$\begin{aligned} J_1^{\{X-\theta_i\}} J_{-1}^{\{X_{\theta_i}\}} v_{\nu, \ell_0} &= [J_1^{\{X-\theta_i\}}, J_{-1}^{\{X_{\theta_i}\}}] v_{\nu, \ell_0} = -(\nu|\theta_i) + M_i(k) \frac{(\theta_1|\theta_i)}{2} \\ &= \frac{(\theta_1|\theta_i)}{2} (-(\nu|\theta_i^\vee) + M_i(k)) = 0, \end{aligned}$$

we see that

$$\begin{aligned} \|J_{-1}^{X_{\theta_i}} v_{\nu, \ell_0}\|^2 &= H(J_{-1}^{\{X_{\epsilon_1}\}} v_{\nu, \ell_0}, J_{-1}^{\{X_{\epsilon_1}\}} v_{\nu, \ell_0}) = -H(J_1^{\{\phi(X_{\epsilon_1})\}} J_{-1}^{\{X_{\epsilon_1}\}} v_{\nu, \ell_0}, v_{\nu, \ell_0}) \\ &= -\text{const.} H(J_1^{\{X_{-\epsilon_1}\}} J_{-1}^{\{X_{\epsilon_1}\}} v_{\nu, \ell_0}, v_{\nu, \ell_0}) = 0. \end{aligned}$$

By unitarity we deduce

$$(6.33) \quad J_{-1}^{\{X_{\theta_i}\}} v_{\nu, \ell_0} = 0.$$

If $\mathfrak{g} = \mathfrak{spo}(2|3)$ then (6.33) implies

$$0 = G_1^{\{\phi(u_{\delta_1+\epsilon_1})\}} J_{-1}^{\{X_{\epsilon_1}\}} v_{\nu, \ell_0} = [G_1^{\{\phi(u_{\delta_1+\epsilon_1})\}}, J_{-1}^{\{X_{\epsilon_1}\}}] v_{\nu, \ell_0} = -G_0^{\{[X_{\epsilon_1}, \phi(u_{\delta_1+\epsilon_1})]\}} v_{\nu, \ell_0}$$

so, since $[X_{\epsilon_1}, \phi(u_{\delta_1+\epsilon_1})] = \text{const.} u_{\delta_1}$, we obtain

$$G_0^{\{u_{\delta_1}\}} v_{\nu, \ell_0} = 0.$$

Again Proposition 6.6 implies that $\ell_0 = A(k, \nu)$.

If $\mathfrak{g} = D(2, 1; a)$, then observe that $\eta_{\min} = \xi - \theta_i$. Let $u_\xi \in \mathfrak{g}_{-1/2}$ be a vector of weight $-\xi$. Note that u_ξ is a root vector for the root $-\theta/2 - \xi$. Since $\theta_i + (-\theta/2 - \xi) = -\theta/2 - \eta_{\min}$ is a root, we see that $[X_{\theta_i}, u_\xi]$ is a nonzero vector in $\mathfrak{g}_{-1/2}$ of weight $-\eta_{\min}$. If $v \in \mathfrak{g}_{-1/2}$ has weight $\eta_{\min}$, it follows that $[X_{\theta_i}, u_\xi] = \text{const.} \phi(v)$. By (6.33),

$$0 = G_1^{\{u_\xi\}} J_{-1}^{\{X_{\theta_i}\}} v_{\nu, \ell_0} = [G_1^{\{u_\xi\}}, J_{-1}^{\{X_{\theta_i}\}}] v_{\nu, \ell_0} = -G_0^{\{[X_{\theta_i}, u_\xi]\}} v_{\nu, \ell_0} = \text{const.} G_0^{\{\phi(v)\}} v_{\nu, \ell_0}.$$

We can therefore conclude using Proposition 6.6. $\square$

We can finally state the necessary conditions for unitarity.

Theorem 6.10. *Let $L^W(\nu, \ell_0)$ be a unitary σ_R -twisted $W_{\min}^k(\mathfrak{g})$ -module, where k is in the unitary range. Then*

- (1) $\nu \in P_k^+$,
- (2) $\ell_0 \geq A(k, \nu)$,
- (3) If ν is as in Lemma 6.9 (1), (2), (3), then $\ell_0 = A(k, \nu)$.

7. RAMOND SECTOR: SUFFICIENT CONDITIONS FOR UNITARITY

Recall that in Section 3 we constructed the twisted $F(\mathfrak{g}_{1/2})$ -module $F(\mathfrak{g}_{1/2}, \sigma_R)$. Our first task is to show that this module is unitary.

Fix bases $\{w_\gamma\}, \{w^\gamma\}$ of $\mathfrak{g}_{1/2}$ dual w.r.t. $\langle \cdot, \cdot \rangle_{\text{ne}}$, with γ ranging over $S_{1/2}$. Recall that there is an embedding of superspaces $u \mapsto \Phi_u$ of $\mathfrak{g}_{1/2}$ into $F(\mathfrak{g}_{1/2})$. Sometimes we write Φ_α (resp. Φ^α) as a shortcut for Φ_{w_α} (resp. Φ_{w^α}).

We can assume that $S_{1/2} = S_{1/2}^+ \cup S_{1/2}^-$ with $\{w_\gamma \mid \gamma \in S_{1/2}^\pm\}$ basis of $\mathfrak{n}_{1/2}(\sigma)_\pm$. We will also need a more refined notation $S_{1/2}^- = S_{1/2}^{-,0} \cup S_{1/2}^{-, '}$ to label bases of $\mathfrak{g}_{1/2}^0(\sigma_R), \mathfrak{n}_{1/2}(\sigma_R)'$ (see (3.10)). We

Lemma 7.1. *The map $t^{-1/2}\Phi_u \mapsto [\Phi_u]$ extends to an isomorphism between $Cl(\mathfrak{g}_{1/2})$ and $Zhu_{\sigma_R}(F(\mathfrak{g}_{1/2}))$.*

Proof. Recall that $*$ denotes the product in $Zhu_{\sigma_R}(F(\mathfrak{g}_{1/2}))$. By [6, Theorem 2.13],

$$[[\Phi_u], [\Phi_v]] = \sum_j \binom{-1/2}{j} (\Phi_u)_{(j)} (\Phi_v) = \langle u, v \rangle = [t^{-1/2}\Phi_u, t^{-1/2}\Phi_v],$$

so the map $t^{-1/2}\Phi_u \mapsto [\Phi_u]$ extends to a homomorphism from $Cl(\mathfrak{g}_{1/2})$ to $Zhu_{\sigma_R}(F(\mathfrak{g}_{1/2}))$.

We prove by induction on Δ_a that $[a]$ is in the subalgebra of $Zhu_{\sigma_R}(F(\mathfrak{g}_{1/2}))$ spanned by $[\Phi_u]$, $u \in \mathfrak{g}_{1/2}$, for all $a \in F(\mathfrak{g}_{1/2})$. If $a = |0\rangle$ the claim holds. Now assume $a =: T^k \Phi_u b$ with $\Delta_b < \Delta_a$. Note that by [6, Theorem 2.13 (b)]

$$: T^k \Phi_u b : + : T^{k-1} \Phi_u T b : = T \left(: T^{k-1} \Phi_u b : \right) \equiv \text{const} : T^{k-1} \Phi_u b :,$$

hence we can assume $k = 0$. Since

$$(\Phi_u)_{(-1, \sigma_R)} b = \sum_{j \in \mathbb{Z}_+} \binom{\frac{1}{2}}{j} (\Phi_u)_{(-1+j)} b =: \Phi_u a : + \sum_{j \in \mathbb{N}} \binom{\frac{1}{2}}{j} (\Phi_u)_{(-1+j)} b,$$

we obtain

$$[a] = [: \Phi_u b :] = [\Phi_u] * [b] - \sum_{j \in \mathbb{N}} \binom{\frac{1}{2}}{j} [(\Phi_u)_{(-1+j)} b],$$

and the claim follows by induction. It follows that the homomorphism from $Cl(\mathfrak{g}_{1/2})$ to $Zhu_{\sigma_R}(F(\mathfrak{g}_{1/2}))$ is surjective.

We now prove that the homomorphism is injective. Set

$$L^{\geq}(A, \sigma_R) = \bigoplus_{\mu \in -\frac{1}{2} + \mathbb{Z}_+} t^{\mu} \mathfrak{g}_{1/2} \subset L(\mathfrak{g}_{1/2}, \sigma_R).$$

Consider $Cl(\mathfrak{g}_{1/2})$ as a $L^{\geq}(A, \sigma_R) \oplus \mathbb{C}K$ -module by letting $t^{-1/2} \mathfrak{g}_{1/2}$ act by left multiplication, $t^{\mu} \mathfrak{g}_{1/2}$ act trivially if $\mu > 0$, and K act by $I_{Cl(\mathfrak{g}_{1/2})}$. Let

$$(7.1) \quad M = Ind_{L^{\geq}(\mathfrak{g}_{1/2}, \sigma_R)}^{\widehat{L}(\mathfrak{g}_{1/2}, \sigma_R)}(Cl(\mathfrak{g}_{1/2})).$$

By Proposition 2.1, M is a σ_R -twisted $F(\mathfrak{g}_{1/2})$ -module. Recall that

$$L^{ne} = \frac{1}{2} \sum_{\alpha \in S_{1/2}} : (T\Phi^{\alpha}) \Phi_{\alpha} :,$$

and that M is spanned by monomials

$$(t^{\mu_1} \Phi_1)(t^{\mu_2} \Phi_2) \cdots (t^{\mu_k} \Phi_k), \quad \mu_i \in -(\frac{1}{2} + \mathbb{Z}_+).$$

Since

$$[L_0^{ne, tw}, t^{\mu} \Phi_u] = -(\mu + \frac{1}{2}) t^{\mu} \Phi_u,$$

the action of $L_0^{ne, tw}$ is semisimple and the eigenspace decomposition gives a grading

$$M = \bigoplus_{n \geq 0} M_n,$$

turning M into a positive energy representation. Since

$$M_0 = span((t^{-\frac{1}{2}} \Phi_1)(t^{-\frac{1}{2}} \Phi_2) \cdots (t^{-\frac{1}{2}} \Phi_k)) = Cl(\mathfrak{g}_{1/2})$$

and $(\Phi_u)_0 = (\Phi_u)_{(-1/2)}$, the action of $(\Phi_u)_0$ is given by left multiplication by $t^{-1/2} \Phi_u$. Since the action of $Cl(\mathfrak{g}_{1/2})$ on itself by left multiplication is faithful and this action factors to $Zhu_{\sigma_R}(F(\mathfrak{g}_{1/2}))$ we deduce that the homomorphism from $Cl(\mathfrak{g}_{1/2})$ to $Zhu_{\sigma_R}(F(\mathfrak{g}_{1/2}))$ is injective. $\square$

Lemma 7.2. *Consider $L_0^{ne, tw} - \frac{1}{16} \dim \mathfrak{g}_{1/2}$ as an operator on $F(\mathfrak{g}_{1/2}, \sigma_R)$. Then the action is semisimple with eigenvalues in $\frac{1}{2} \mathbb{Z}_+$. The corresponding eigenspace decomposition*

$$(7.2) \quad F(\mathfrak{g}_{1/2}, \sigma_R) = \bigoplus_{n \in \frac{1}{2} \mathbb{Z}_+} F(\mathfrak{g}_{1/2}, \sigma_R)_n$$

gives to $F(\mathfrak{g}_{1/2}, \sigma_R)$ the structure of a σ_R -twisted positive energy representation of $F(\mathfrak{g}_{1/2})$.

The space $F(\mathfrak{g}_{1/2}, \sigma_R)_0$ of minimal energy is naturally isomorphic, as a $Zhu_R(F(\mathfrak{g}_{1/2}))$ -module, to the Clifford module

$$(7.3) \quad CM = Cl(t^{-1/2} \mathfrak{g}_{1/2}) / (Cl(t^{-1/2} \mathfrak{g}_{1/2}) \mathfrak{n}_{1/2}(\sigma_R)_+).$$

Proof. Recall that

$$L^{ne} = \frac{1}{2} \sum_{\alpha \in S_{1/2}} : (T\Phi^{\alpha}) \Phi_{\alpha} :.$$

We claim that

$$(7.4) \quad L_0^{ne, tw} \cdot 1 = \frac{1}{16} \dim \mathfrak{g}_{1/2}.$$

To check (7.4), start with (2.6) putting $n = -1, \mu = \nu = 1/2$:

$$\begin{aligned} & ((T\Phi^\alpha)_{(-1)}\Phi_\alpha)_{(1)} \cdot 1 + \binom{1/2}{2} ((T\Phi^\alpha)_{(1)}\Phi_\alpha)_{(-1)} \cdot 1 \\ &= \sum_{j \in \mathbb{Z}_+} (1/2 + j) (\Phi_\alpha)_{(-1/2-j)} (\Phi^\alpha)_{(-1/2+j)} \cdot 1, \end{aligned}$$

which gives

$$((T\Phi^\alpha)_{(-1)}\Phi_\alpha)_{(1)} \cdot 1 + 1/8 = \frac{1}{2} (\Phi_\alpha)_{(-1/2)} (\Phi^\alpha)_{(-1/2)} \cdot 1.$$

Assume $\mathfrak{g}_\alpha \subset \mathfrak{n}_{1/2}(\sigma_R)_+$. Then

$$\frac{1}{2} (\Phi_\alpha)_{(-1/2)} (\Phi^\alpha)_{(-1/2)} \cdot 1 = \frac{1}{2} [(\Phi_\alpha)_{(-1/2)}, (\Phi^\alpha)_{(-1/2)}] = 1/2.$$

Assume $\mathfrak{g}_\alpha \subset \mathfrak{g}_{1/2}^0(\sigma_R)$. Then

$$\frac{1}{2} (\Phi_\alpha)_{(-1/2)} (\Phi^\alpha)_{(-1/2)} \cdot 1 = \frac{1}{4} [(\Phi_\alpha)_{(-1/2)}, (\Phi^\alpha)_{(-1/2)}] = 1/4.$$

Summing up

$$L_0^{ne,tw} \cdot 1 = \frac{1}{2} (-\frac{1}{8} \dim \mathfrak{g}_{1/2} + \frac{1}{4} \cdot 2 \cdot \dim \mathfrak{n}_{1/2}(\sigma_R)_+ + \frac{1}{4} \epsilon(\sigma_R)) = \frac{1}{16} \dim \mathfrak{g}_{1/2}.$$

Since $F(\mathfrak{g}_{1/2}, \sigma_R)$ is generated by monomials

$$(7.5) \quad (t^{\mu_1} \Phi_1)(t^{\mu_2} \Phi_2) \cdots (t^{\mu_k} \Phi_k), \quad \mu_i \in -(\frac{1}{2} + \mathbb{Z}_+)$$

and

$$[L_0^{ne,tw}, t^\mu \Phi_u] = -(\mu + \frac{1}{2}) t^\mu \Phi_u$$

(7.2) follows readily.

The space of minimal energy is

$$F(\mathfrak{g}_{1/2}, \sigma_R)_0 = \text{span}((t^{-\frac{1}{2}} \Phi_1)(t^{-\frac{1}{2}} \Phi_2) \cdots (t^{-\frac{1}{2}} \Phi_k)).$$

On $F(\mathfrak{g}_{1/2}, \sigma_R)_0$, $[\Phi_u] \in \text{Zhu}_R(F(\mathfrak{g}_{1/2}))$ acts by the action of $(\Phi_u)_0 = (\Phi_u)_{(-1/2)}$, which is left multiplication by $t^{-\frac{1}{2}} \Phi_u$. $\square$

Let ϕ be an almost compact conjugate linear involution of $\mathfrak{g}$. Set

$$\omega = e^{L_1} g, \quad g(a) = e^{-\pi \sqrt{-1}(\Delta_a + \frac{1}{2} p(a))} \phi(a),$$

so that

$$\omega([\Phi_u]) = -[\Phi_{\phi(u)}].$$

Lemma 7.3. $F(\mathfrak{g}_{1/2}, \sigma_R)$ is a unitary σ_R -twisted $F(\mathfrak{g}_{1/2})$ -module.

Proof. To prove the existence of a ω -invariant Hermitian form on $F(\mathfrak{g}_{1/2}, \sigma_R)$, we use [14, Proposition 6.7]. According to this result, it suffices to show that there exists an invariant Hermitian form on $F(\mathfrak{g}_{1/2}, \sigma_R)_0$, which in Lemma 7.2 is identified with the Clifford module CM (7.3). We identify as vector spaces CM and $\bigwedge \mathfrak{n}_{1/2}(\sigma_R)_-$ via $u_1 \wedge \cdots \wedge u_k \mapsto (t^{-1/2} \Phi_{u_1}) \cdots (t^{-1/2} \Phi_{u_k})$, $u_i \in \mathfrak{n}_{1/2}(\sigma_R)_-$. Define an Hermitian form on $\bigwedge \mathfrak{n}_{1/2}(\sigma_R)_-$ by determinants:

$$(7.6) \quad (w_{i_1} \wedge w_{i_2} \cdots \wedge w_{i_h}, w_{j_1} \wedge w_{j_2} \cdots \wedge w_{j_k}) = \delta_{hk} (-1)^k \det(\langle \phi(w_{i_r}), w_{j_s} \rangle_{ne}).$$

We now check that it is ω -invariant. Since ϕ is an almost compact involution, we have that $\phi(h_0) = -h_0$, hence $\phi(\mathfrak{n}_{1/2}(\sigma_R)_+) = \mathfrak{n}'_{1/2}(\sigma_R)_-$. Consider $u_1 \in \mathfrak{n}_{1/2}(\sigma_R)_-$ (resp. $u_1 \in$

$\mathfrak{n}_{1/2}(\sigma_R)_+$); note that $t^{-1/2}\Phi_{u_1} \cdot t^{-1/2}\Phi_{u_2} \dots t^{-1/2}\Phi_{u_k}$ maps, under our identifications, to $u_1 \wedge \dots \wedge u_k$ (resp. $\sum_r (-1)^{r+1} \langle u_1, u_r \rangle_{ne} u_1 \wedge \dots \widehat{u_r} \wedge \dots u_k$). If $u_1 \in \mathfrak{n}_{1/2}(\sigma_R)_+$

$$\begin{aligned} (\phi(u_1) \wedge u_2 \wedge u_3 \dots \wedge u_k, w_1 \wedge w_2 \dots \wedge w_k) &= (-1)^k \det(\langle \phi(u_i), w_j \rangle_{ne}) \\ &= (-1)^k \sum_r (-1)^{r+1} \langle u_1, w_r \rangle_{ne} \det((\langle \phi(u_i), w_r \rangle_{ne}))_{1r} \\ &= -(u_2 \wedge u_3 \dots \wedge u_k, u_1 \cdot w_1 \wedge w_2 \dots \wedge w_k). \end{aligned}$$

If $u_1 \in \mathfrak{n}_{1/2}(\sigma_R)_-$ one argues similarly.

Recall from [15] that $\langle \phi(\cdot), \cdot \rangle_{ne}$ is negative definite on $\mathfrak{g}_{1/2}$. Let $\{u_1, \dots, u_k\}$ be a basis of $\mathfrak{n}_{1/2}(\sigma_R)_-$ such that $\langle \phi(u_i), u_j \rangle_{ne} = -\delta_{ij}$; then $\{u_{i_1} \wedge \dots \wedge u_{i_k} \mid i_1 < \dots < i_k\}$ is an orthonormal basis of $\wedge \mathfrak{n}_{1/2}(\sigma_R)_-$ for the Hermitian form (7.6). We now check that the invariant form $H(\cdot, \cdot)$ on $F(\mathfrak{g}_{1/2}, \sigma_R)$ which restricts to (7.6) on $F(\mathfrak{g}_{1/2}, \sigma_R)_0$ is positive definite. Let $\{v_1, \dots, v_s\}$ be a basis of $\mathfrak{g}_{1/2}$ such that $\langle \phi(v_i), v_j \rangle_{ne} = -\delta_{ij}$; let $\{w_1, \dots, w_t\}$ be an orthonormal basis of $F(\mathfrak{g}_{1/2}, \sigma_R)_0$. Then the vectors

$$(7.7) \quad (t^{\mu_1} \Phi_{v_{i_1}})(t^{\mu_2} \Phi_{v_{i_2}}) \dots (t^{\mu_k} \Phi_{v_{i_k}}) w_j, \quad \mu_i \in -(\tfrac{1}{2} + \mathbb{N}), 1 \leq j \leq t,$$

where the pairs (μ_j, i_j) are ordered lexicographically, form a basis of $F(\mathfrak{g}_{1/2}, \sigma_R)$. We claim that this basis is orthonormal. Indeed, the Φ_u are primary, so $H(m, t^\mu \Phi_u m') = -H(t^{-\mu} \Phi_{\phi(u)} m, m')$. It follows that

$$\begin{aligned} &H((t^{\mu_1} \Phi_{v_{i_1}})(t^{\mu_2} \Phi_{v_{i_2}}) \dots (t^{\mu_k} \Phi_{v_{i_k}}) w_j, (t^{\nu_1} \Phi_{v_{j_1}})(t^{\nu_2} \Phi_{v_{j_2}}) \dots (t^{\nu_k} \Phi_{v_{j_k}}) w_s) \\ &= H((t^{-\nu_k} \Phi_{-\phi(v_{j_k})}) \dots (t^{-\nu_2} \Phi_{-\phi(v_{j_2})})(t^{-\nu_1} \Phi_{-\phi(v_{j_1})})(t^{\mu_1} \Phi_{v_{i_1}})(t^{\mu_2} \Phi_{v_{i_2}}) \dots (t^{\mu_k} \Phi_{v_{i_k}}) w_j, w_s) \\ &= \prod_i \delta_{\mu_i \nu_i} \prod_r \delta_{i_r j_r} \delta_j s. \end{aligned}$$

□

Our second task is to construct unitary σ_R -twisted representations of $W_{\min}^k(\mathfrak{g})$ using the free field realization introduced in [19, Theorem 5.2]; it is the embedding

$$\Psi : W_{\min}^k(\mathfrak{g}) \rightarrow \mathcal{V}^k := V^{k+h^\vee}(\mathbb{C}x) \otimes V^{\alpha_k}(\mathfrak{g}^\natural) \otimes F(\mathfrak{g}_{1/2})$$

explicitly given on the generators of $W_{\min}^k(\mathfrak{g})$ by

$$(7.8) \quad J^{\{b\}} \mapsto b + \frac{1}{2} \sum_{\alpha \in S_{1/2}} : \Phi^\alpha \Phi_{[w_\alpha, b]} : (b \in \mathfrak{g}^\natural),$$

$$(7.9) \quad \begin{aligned} G^{\{v\}} &\mapsto \sum_{\alpha \in S_{1/2}} : [v, w_\alpha] \Phi^\alpha : - (k+1) \sum_{\alpha \in S_{1/2}} (v|w_\alpha) T \Phi^\alpha \\ &\quad + \frac{1}{3} \sum_{\alpha, \beta \in S_{1/2}} : \Phi^\alpha \Phi^\beta \Phi_{[w_\beta, [w_\alpha, v]]} : (v \in \mathfrak{g}_{-1/2}), \end{aligned}$$

$$(7.10) \quad L \mapsto \frac{1}{2(k+h^\vee)} \sum_{\alpha \in S_0} : a_\alpha a^\alpha : + \frac{k+1}{k+h^\vee} T x + \frac{1}{2} \sum_{\alpha \in S_{1/2}} : (T \Phi^\alpha) \Phi_\alpha : .$$

Using the normalization

$$(7.11) \quad a = \sqrt{-1} \frac{\sqrt{2}}{\sqrt{|k+h^\vee|}} x,$$

we have $V^{k+h^\vee}(\mathbb{C}x) = M(1)$. Here $M(1)$ is the Heisenberg vertex algebra (free boson) generated by the element a with λ -bracket

$$[a_\lambda a] = \lambda.$$

Recall that we denoted by ξ the highest weight of the $\mathfrak{g}^\natural$ -module $\mathfrak{g}_{-1/2}$. Recall from [15] that an element $\nu \in P_k^+$ is called an *extremal weight* if $\nu + \xi$ doesn't lie in P_k^+ . Let $\nu \in P_k^+$ be a non extremal weight. Let $L(\nu)$ be the irreducible $V^{\alpha_k}(\mathfrak{g}^\natural)$ -module of highest weight ν . Fix $\mu \in \mathbb{C}$ and let $M(1, \mu)$ be the Verma module for $M(1)$ such that a_0 acts as the multiplication by μ . Clearly $Id_{M(1)} \otimes Id_{V^{\alpha_k}(\mathfrak{g}^\natural)} \otimes \sigma_R$ is an automorphism of $\mathcal{V}^k$ and

$$(7.12) \quad M(1, \mu) \otimes L(\nu) \otimes F(\mathfrak{g}_{1/2}, \sigma_R)$$

is a $Id_{M(1)} \otimes Id_{V^{\alpha_k}(\mathfrak{g}^\natural)} \otimes \sigma_R$ -twisted $\mathcal{V}^k$ -module.

Since $\Psi \circ \sigma_R = (Id_{M(1)} \otimes Id_{V^{\alpha_k}(\mathfrak{g}^\natural)} \otimes \sigma_R) \circ \Psi$, the restriction to $\Psi(W_{\min}^k(\mathfrak{g}))$ defines the structure of a σ_R -twisted $W_{\min}^k(\mathfrak{g})$ -module on the $\mathcal{V}^k$ -module in (7.12).

Set

$$(7.13) \quad s_k = \sqrt{-1} \frac{(k+1)}{\sqrt{2|k+h^\vee|}}.$$

Proposition 7.4. *Fix $\mu \in \mathbb{C}$ and $\nu \in (\mathfrak{h}^\natural)^*$. Set*

$$v(\mu, \nu) = v_\mu \otimes v_\nu \otimes 1,$$

and

$$(7.14) \quad N(\mu, \nu) = \Psi(W_{\min}^k(\mathfrak{g})) \cdot v(\mu, \nu) \subset M(1, \mu) \otimes L(\nu) \otimes F(\mathfrak{g}_{1/2}, \sigma_R).$$

If

$$(7.15) \quad \ell_0(\mu, \nu) = \frac{\mu^2}{2} - s_k \mu + \frac{(\nu|\nu + 2\rho^\natural)}{2(k+h^\vee)} + \frac{1}{16} \dim \mathfrak{g}_{1/2},$$

then $N(\mu, \nu)$ is a highest weight $W_{\min}^k(\mathfrak{g})$ -module with highest weight vector $v(\mu, \nu)$, of highest weight $(\nu + \rho_R, \ell_0(\mu, \nu))$.

Proof. By (2.6) we have

$$(7.16) \quad : \Phi^\alpha \Phi_{[w_\alpha, b]} :_r^{\text{tw}} = \frac{1}{2} \langle w^\alpha, [w_\alpha, b] \rangle_{ne} \delta_{r,0} + \sum_{j \in \mathbb{Z}_+} ((\Phi^\alpha)_{-1-j}^{\text{tw}} (\Phi_{[w_\alpha, b]}^{\text{tw}})_{r+j+1} - (\Phi_{[w_\alpha, b]}^{\text{tw}})_{r-j} (\Phi^\alpha)_j^{\text{tw}}),$$

$$(7.17) \quad : T\Phi^\alpha \Phi_\alpha :_r^{\text{tw}} = \sum_{j \in \mathbb{Z}_+} (\frac{1}{2} + j) (\Phi^\alpha)_{-1-j}^{\text{tw}} (\Phi_\alpha)_{r+j+1}^{\text{tw}} + (\Phi_\alpha)_{r-j}^{\text{tw}} (\Phi^\alpha)_j^{\text{tw}} - 1/8 \delta_{r,0},$$

$$(7.18) \quad : [v, w_\alpha] \Phi^\alpha :_r^{\text{tw}} = \sum_{j \in \mathbb{Z}} [v, w_\alpha]_{-j} (\Phi^\alpha)_{r+j}^{\text{tw}},$$

$$(7.19) \quad : \Phi^\alpha \Phi^\beta \Phi_{[w_\beta, [w_\alpha, v]]} :_r^{\text{tw}} = \frac{1}{2} \langle w^\alpha, w^\beta \rangle_{ne} (\Phi_{[w_\beta, [w_\alpha, v]]}^{\text{tw}})_r - \frac{1}{2} \langle w^\alpha, [w_\beta, [w_\alpha, v]] \rangle_{ne} (\Phi^\beta)_r^{\text{tw}} \\ + \sum_{j \in \mathbb{Z}_+} ((\Phi^\alpha)_{-1-j}^{\text{tw}} : \Phi^\beta \Phi_{[w_\beta, [w_\alpha, v]]} :_{r+j+1}^{\text{tw}} + : \Phi^\beta \Phi_{[w_\beta, [w_\alpha, v]]} :_{r-j}^{\text{tw}} (\Phi^\alpha)_j^{\text{tw}}).$$

We start by checking (4.14). If $r > 0$,

$$\begin{aligned}
\Psi(J^{\{b\}})_r v(\mu, \nu) &= v_\mu \otimes b_r v_\nu \otimes 1 + \frac{1}{2} \sum_{\alpha \in S_{1/2}} v_\mu \otimes v_\nu \otimes : \Phi^\alpha \Phi_{[w_\alpha, b]} :_r^{\text{tw}} 1 \\
&= \frac{1}{2} \sum_{\alpha \in S_{1/2}, j \in \mathbb{Z}_+} (v_\mu \otimes v_\nu \otimes (\Phi^\alpha)_{-j-1}^{\text{tw}} (\Phi_{[w_\alpha, b]})_{r+j+1}^{\text{tw}} - v_\mu \otimes v_\nu \otimes (\Phi_{[w_\alpha, b]})_{r-j}^{\text{tw}} (\Phi^\alpha)_j^{\text{tw}}) 1 \\
&= -\frac{1}{2} \sum_{\alpha \in S_{1/2}} v_\mu \otimes v_\nu \otimes (\Phi_{[w_\alpha, b]})_r^{\text{tw}} (\Phi^\alpha)_0^{\text{tw}} 1 = \frac{1}{2} \sum_{\alpha \in S_{1/2}} v_\mu \otimes v_\nu \otimes (\Phi^\alpha)_0^{\text{tw}} (\Phi_{[w_\alpha, b]})_r^{\text{tw}} 1 = 0.
\end{aligned}$$

Likewise

$$\begin{aligned}
\Psi(G^{\{v\}})_r v(\mu, \nu) &= \\
&\sum_{j \in \mathbb{Z}} [v, w_\alpha]_{-j}^{\text{tw}} (v_\mu \otimes v_\nu) \otimes (\Phi^\alpha)_{r+j}^{\text{tw}} 1 - (k+1) \sum_{\alpha \in S_{1/2}} (v|w_\alpha) v_\mu \otimes v_\nu \otimes (T\Phi^\alpha)_r^{\text{tw}} 1 \\
&+ \frac{1}{6} \sum_{\alpha, \beta \in S_{1/2}} v_\mu \otimes v_\nu \otimes (\langle w^\alpha, w^\beta \rangle_{ne} (\Phi_{[w_\beta, [w_\alpha, v]]})_r^{\text{tw}} - \frac{1}{2} \langle w^\alpha, [w_\beta, [w_\alpha, v]] \rangle_{ne} (\Phi^\beta)_r^{\text{tw}}) 1 \\
&+ \frac{1}{3} \sum_{\alpha, \beta \in S_{1/2}, j \in \mathbb{Z}_+} v_\mu \otimes v_\nu \otimes ((\Phi^\alpha)_{-1-j}^{\text{tw}} : \Phi^\beta \Phi_{[w_\beta, [w_\alpha, v]]} :_{r+j+1}^{\text{tw}} + : \Phi^\beta \Phi_{[w_\beta, [w_\alpha, v]]} :_{r-j}^{\text{tw}} (\Phi^\alpha)_j^{\text{tw}}) 1 \\
&= (k+1) \sum_{\alpha \in S_{1/2}} (v|w_\alpha) (r + \frac{1}{2}) v_\mu \otimes v_\nu \otimes (\Phi^\alpha)_r^{\text{tw}} 1 + \frac{1}{3} \sum_{\alpha, \beta \in S_{1/2}} v_\mu \otimes v_\nu \otimes : \Phi^\beta \Phi_{[w_\beta, [w_\alpha, v]]} :_r^{\text{tw}} (\Phi^\alpha)_0^{\text{tw}} 1 \\
&= \frac{1}{3} \sum_{\alpha, \beta \in S_{1/2}} v_\mu \otimes v_\nu \otimes : \Phi^\beta \Phi_{[w_\beta, [w_\alpha, v]]} :_r^{\text{tw}} (\Phi^\alpha)_0^{\text{tw}} 1 \\
&= \frac{1}{3} \sum_{\alpha, \beta \in S_{1/2}, j \in \mathbb{Z}_+} v_\mu \otimes v_\nu \otimes ((\Phi^\beta)_{-j-1}^{\text{tw}} (\Phi_{[w_\beta, [w_\alpha, v]]})_{r+j+1}^{\text{tw}} - (\Phi_{[w_\beta, [w_\alpha, v]]})_{r-j}^{\text{tw}} (\Phi^\beta)_j^{\text{tw}}) (\Phi^\alpha)_0^{\text{tw}} 1 \\
&= -\frac{1}{3} \sum_{\alpha, \beta \in S_{1/2}} v_\mu \otimes v_\nu \otimes (\Phi_{[w_\beta, [w_\alpha, v]]})_r^{\text{tw}} (\Phi^\beta)_0^{\text{tw}} (\Phi^\alpha)_0^{\text{tw}} 1 = 0.
\end{aligned}$$

Finally

$$\begin{aligned}
\Psi(L)_r v(\mu, \nu) &= \frac{1}{2(k+h^\vee)} \sum_{\alpha \in S_0} : a_\alpha a^\alpha :_r (v_\mu \otimes v_\nu) \otimes 1 + \frac{k+1}{k+h^\vee} (Tx)_r v_\mu \otimes v_\nu \otimes 1 \\
&+ \frac{1}{2} \sum_{\alpha \in S_{1/2}} v_\mu \otimes v_\nu \otimes : (T\Phi^\alpha) \Phi_\alpha :_r^{\text{tw}} 1 \\
&= -(r+1) \frac{k+1}{k+h^\vee} x_r v_\mu \otimes v_\nu \otimes 1 + \frac{1}{2} \sum_{\alpha \in S_{1/2}} v_\mu \otimes v_\nu \otimes : (T\Phi^\alpha) \Phi_\alpha :_r^{\text{tw}} 1 \\
&= \frac{1}{2} \sum_{\alpha \in S_{1/2}, j \in \mathbb{Z}_+} (\frac{1}{2} + j) v_\mu \otimes v_\nu \otimes ((\Phi^\alpha)_{-1-j}^{\text{tw}} (\Phi_\alpha)_{r+j+1}^{\text{tw}} + (\Phi_\alpha)_{r-j}^{\text{tw}} (\Phi^\alpha)_j^{\text{tw}}) 1 = 0.
\end{aligned}$$

This concludes the check that (4.14) holds. Let us now verify (4.15): if $a \in \mathfrak{g}^\natural$, then

$$(7.20) \quad \Psi(J^{\{a\}})_0 v(\mu, \nu) = v_\mu \otimes a_0 v_\nu \otimes 1 + \frac{1}{2} \sum_{\alpha \in S_{1/2}} v_\mu \otimes v_\nu \otimes : \Phi^\alpha \Phi_{[w_\alpha, a]} :_0^{\text{tw}} 1.$$

If, in particular, $a \in \mathfrak{n}_0(\sigma_R)_+$ then, since $\langle w^\alpha, [w_\alpha, a] \rangle_{ne} = 0$,

$$\begin{aligned} \Psi(J^{\{a\}})_0 v(\mu, \nu) &= \frac{1}{2} \sum_{\alpha \in S_{1/2}, j \in \mathbb{Z}_+} (v_\mu \otimes v_\nu \otimes ((\Phi^\alpha)^{\text{tw}}_{-j-1}(\Phi_{[w_\alpha, a]})^{\text{tw}}_{j+1} - (\Phi_{[w_\alpha, a]}^{\text{tw}}_{-j}(\Phi^\alpha)^{\text{tw}}_j)1 \\ &= -\frac{1}{2} \sum_{\alpha \in S_{1/2}} (v_\mu \otimes v_\nu \otimes (\Phi_{[w_\alpha, a]}^{\text{tw}}_0(\Phi^\alpha)^{\text{tw}}_0)1. \end{aligned}$$

If $w_\alpha \in \mathfrak{n}_{1/2}(\sigma_R)'$, then $w^\alpha \in \mathfrak{n}_{1/2}(\sigma_R)_+$ and $(\Phi^\alpha)_0^{\text{tw}}1 = 0$. Since $\langle w^\alpha, [w_\alpha, a] \rangle_{ne} = 0$,

$$(\Phi_{[w_\alpha, a]}^{\text{tw}}_0(\Phi^\alpha)^{\text{tw}}_0)1 = -(\Phi^\alpha)_0^{\text{tw}}(\Phi_{[w_\alpha, a]}^{\text{tw}}_0)1.$$

If $w_\alpha \notin \mathfrak{n}_{1/2}(\sigma_R)'$, then $[w_\alpha, a] \in \mathfrak{n}_{1/2}(\sigma_R)_+$ so $(\Phi_{[w_\alpha, a]}^{\text{tw}}_0)1 = 0$.

We now turn to checking (4.12): if $h \in \mathfrak{h}^\natural$ then

$$\begin{aligned} \Psi(J^{\{h\}})_0 v(\mu, \nu) &= v_\mu \otimes h_0 v_\nu \otimes 1 + \frac{1}{2} \sum_{\alpha \in S_{1/2}} v_\mu \otimes v_\nu \otimes : \Phi^\alpha \Phi_{[w_\alpha, h]}^{\text{tw}} :_0^{\text{tw}} 1 \\ &= \nu(h) v_\mu \otimes v_\nu \otimes 1 + \frac{1}{2} \sum_{\alpha \in S_{1/2}} \alpha(h) (v_\mu \otimes v_\nu \otimes (\Phi^\alpha)_0^{\text{tw}}(\Phi^\alpha)_0^{\text{tw}}1 - \frac{1}{4} \sum_{\alpha \in S_{1/2}} \alpha(h) (v_\mu \otimes v_\nu \otimes 1) \\ &= \nu(h) v_\mu \otimes v_\nu \otimes 1 + \frac{1}{2} \sum_{\alpha \in S_{1/2}^+} \alpha(h) (v_\mu \otimes v_\nu \otimes (\Phi^\alpha)_0^{\text{tw}}(\Phi^\alpha)_0^{\text{tw}}1 - \frac{1}{4} \sum_{\alpha \in S_{1/2}} \alpha(h) (v_\mu \otimes v_\nu \otimes 1) \\ &= \nu(h) v_\mu \otimes v_\nu \otimes 1 + (\frac{1}{2} \sum_{\alpha \in S_{1/2}^+} \alpha(h) - \frac{1}{4} \sum_{\alpha \in S_{1/2}} \alpha(h)) (v_\mu \otimes v_\nu \otimes 1), \end{aligned}$$

hence

$$(7.21) \quad \Psi(J^{\{h\}})_0 v(\mu, \nu) = (\nu + \rho_R)(h) (v_\mu \otimes v_\nu \otimes 1).$$

We now compute ℓ_0 checking (4.13): recall from [15, (9.8)] that

$$\Psi(L) = \frac{1}{2} : aa : + s_k T a + L^{\mathfrak{g}^\natural} + L^{ne},$$

and it is easy to check that $\frac{1}{2} : aa :_0 + s_k (Ta)_0 v_\mu = \frac{1}{2} \mu^2 - s_k \mu$. Using (7.4), one readily obtains

$$L_0(v_\mu \otimes v_\nu \otimes 1) = \left(\frac{\mu^2}{2} - s_k \mu + \frac{(\nu|\nu + 2\rho^\natural)}{2(k + h^\vee)} + \frac{1}{16} \dim \mathfrak{g}_{1/2} \right) (v_\mu \otimes v_\nu \otimes 1).$$

It remains only to check (4.16): if $v \in \mathfrak{n}_{-1/2}(\sigma_R)_+$ then

$$\begin{aligned} &\Psi(G^{\{v\}})_0 (v_\mu \otimes v_\nu \otimes 1) \\ &= \sum_{\alpha \in S_{1/2}} : [v, w_\alpha] \Phi^\alpha :_0^{\text{tw}} (v_\mu \otimes v_\nu \otimes 1) - (k+1) \sum_{\alpha \in S_{1/2}} (v|w_\alpha) v_\mu \otimes v_\nu \otimes (T\Phi^\alpha)_0^{\text{tw}} 1 \\ &\quad + \frac{1}{3} \sum_{\alpha, \beta \in S_{1/2}} v_\mu \otimes v_\nu \otimes : \Phi^\alpha \Phi^\beta \Phi_{[w_\beta, [w_\alpha, v]]} :_0^{\text{tw}} 1 \\ &= \sum_{\alpha \in S_{1/2}, j \in \mathbb{Z}} [v, w_\alpha]_{-j} (v_\mu \otimes v_\nu) \otimes (\Phi^\alpha)_j^{\text{tw}} 1 + \frac{k+1}{2} \sum_{\alpha \in S_{1/2}} (v|w_\alpha) v_\mu \otimes v_\nu \otimes (\Phi^\alpha)_0^{\text{tw}} 1 \\ &\quad + \frac{1}{3} \sum_{\alpha, \beta \in S_{1/2}} v_\mu \otimes v_\nu \otimes : \Phi^\alpha \Phi^\beta \Phi_{[w_\beta, [w_\alpha, v]]} :_0^{\text{tw}} 1, \end{aligned}$$

so that

$$\begin{aligned} \Psi(G^{\{v\}})_0(v_\mu \otimes v_\nu \otimes 1) &= \sum_{\alpha \in S_{1/2}^+ \cup S_{1/2}^{-,0}} [v, w_\alpha]_0(v_\mu \otimes v_\nu) \otimes (\Phi^\alpha)_0^{\text{tw}} 1 \\ &+ \frac{k+1}{2} \sum_{\alpha \in S_{1/2}^+ \cup S_{1/2}^{-,0}} (v|w_\alpha) v_\mu \otimes v_\nu \otimes (\Phi^\alpha)_0^{\text{tw}} 1 + \frac{1}{3} \sum_{\alpha, \beta \in S_{1/2}} v_\mu \otimes v_\nu \otimes : \Phi^\alpha \Phi^\beta \Phi_{[w_\beta, [w_\alpha, v]]} :_0^{\text{tw}} 1. \end{aligned}$$

Note that h_0 has positive eigenvalues on v (which we can assume to be a root vector) and non-negative eigenvalues on w_α , $\alpha \in S_{1/2}^+ \cup S_{1/2}^{-,0}$. Hence $[v, w_\alpha] \in \mathfrak{n}_0(\sigma_R)_+$, and the first summand vanishes; moreover, for the same reason, $(v|w_\alpha) = 0$, and the second summand vanishes. We are left with evaluating the third summand.

$$\begin{aligned} (7.22) \quad & \sum_{\alpha, \beta \in S_{1/2}} : \Phi^\alpha \Phi^\beta \Phi_{[w_\beta, [w_\alpha, v]]} :_0^{\text{tw}} 1 = \\ & \frac{1}{2} \sum_{\alpha, \beta \in S_{1/2}} (\langle w^\alpha, w^\beta \rangle_{ne} (\Phi_{[w_\beta, [w_\alpha, v]]})_0^{\text{tw}} - \langle w^\alpha, [w_\beta, [w_\alpha, v]] \rangle_{ne} (\Phi^\beta)_0^{\text{tw}}) 1 \\ (7.23) \quad & + \sum_{\alpha, \beta \in S_{1/2}} \sum_{j \in \mathbb{Z}_+} ((\Phi^\alpha)_{-1-j}^{\text{tw}} : \Phi^\beta \Phi_{[w_\beta, [w_\alpha, v]]} :_{j+1}^{\text{tw}} + : \Phi^\beta \Phi_{[w_\beta, [w_\alpha, v]]} :_{-j}^{\text{tw}} (\Phi^\alpha)_j^{\text{tw}}) 1. \end{aligned}$$

We evaluate the summands in (7.23). The first is

$$\sum_{j \in \mathbb{Z}_+} ((\Phi^\alpha)_{-1-j}^{\text{tw}} : \Phi^\beta \Phi_{[w_\beta, [w_\alpha, v]]} :_{j+1}^{\text{tw}}) 1 = 0.$$

The second is

$$\begin{aligned} & \sum_{j \in \mathbb{Z}_+} : \Phi^\beta \Phi_{[w_\beta, [w_\alpha, v]]} :_{-j}^{\text{tw}} (\Phi^\alpha)_j^{\text{tw}} 1 = : \Phi^\beta \Phi_{[w_\beta, [w_\alpha, v]]} :_0^{\text{tw}} (\Phi^\alpha)_0^{\text{tw}} 1 = \\ & \left(\frac{1}{2} \langle w^\beta, [w_\beta, [w_\alpha, v]] \rangle_{ne} + \sum_{j \in \mathbb{Z}_+} ((\Phi^\beta)_{-1-j}^{\text{tw}} (\Phi_{[w_\beta, [w_\alpha, v]]})_{j+1}^{\text{tw}} - (\Phi_{[w_\beta, [w_\alpha, v]]})_{-j}^{\text{tw}} (\Phi^\beta)_j^{\text{tw}}) \right) (\Phi^\alpha)_0^{\text{tw}} 1 \\ & = \left(\frac{1}{2} \langle w^\beta, [w_\beta, [w_\alpha, v]] \rangle_{ne} - (\Phi_{[w_\beta, [w_\alpha, v]]})_0^{\text{tw}} (\Phi^\beta)_0^{\text{tw}} \right) (\Phi^\alpha)_0^{\text{tw}} 1. \end{aligned}$$

Hence (7.22) becomes

$$\begin{aligned} & \sum_{\alpha, \beta \in S_{1/2}} : \Phi^\alpha \Phi^\beta \Phi_{[w_\beta, [w_\alpha, v]]} :_0^{\text{tw}} 1 = \\ & \frac{1}{2} \sum_{\alpha, \beta \in S_{1/2}} (\langle w^\alpha, w^\beta \rangle_{ne} (\Phi_{[w_\beta, [w_\alpha, v]]})_0^{\text{tw}} - \langle w^\alpha, [w_\beta, [w_\alpha, v]] \rangle_{ne} (\Phi^\beta)_0^{\text{tw}}) 1 \\ & + \sum_{\alpha, \beta \in S_{1/2}} \left(\frac{1}{2} \langle w^\beta, [w_\beta, [w_\alpha, v]] \rangle_{ne} - (\Phi_{[w_\beta, [w_\alpha, v]]})_0^{\text{tw}} (\Phi^\beta)_0^{\text{tw}} \right) (\Phi^\alpha)_0^{\text{tw}} 1. \end{aligned}$$

The first term equals $\frac{1}{2} \sum_{\alpha \in S_{1/2}} ((\Phi_{[w^\alpha, [w_\alpha, v]]})_0^{\text{tw}} 1$, which vanishes since

$$[w^\alpha, [w_\alpha, v]] \in \mathfrak{n}_{1/2}(\sigma_R)_+.$$

For the second term, note that we may assume $\beta \in S_{1/2}^+ \cup S_{1/2}^{0,-}$; but then $\langle w^\alpha, [w_\beta, [w_\alpha, v]] \rangle_{ne} = 0$, since $[w_\beta, [w_\alpha, v]]$ cannot have weight α . The third term is handled in the same way. The last term equals

$$(7.24) \quad \sum_{\alpha, \beta \in S_{1/2}^+ \cup S_{1/2}^{0,-}, \alpha \neq -\beta} (\Phi_{[w_\beta, [w_\alpha, v]]})_0^{\text{tw}} (\Phi^\beta)_0^{\text{tw}} (\Phi^\alpha)_0^{\text{tw}} 1 + \sum_{\alpha \in S_{1/2}^+ \cup S_{1/2}^{0,-}} (\Phi_{[w_{-\alpha}, [w_\alpha, v]]})_0^{\text{tw}} (\Phi^{-\alpha})_0^{\text{tw}} (\Phi^\alpha)_0^{\text{tw}} 1.$$

The first term in (7.24) vanishes since

$$(\Phi_{[w_\beta, [w_\alpha, v]]})_0^{\text{tw}} (\Phi^\beta)_0^{\text{tw}} (\Phi^\alpha)_0^{\text{tw}} 1 = (\Phi^\beta)_0^{\text{tw}} (\Phi^\alpha)_0^{\text{tw}} (\Phi_{[w_\beta, [w_\alpha, v]]})_0^{\text{tw}} 1 = 0.$$

For the second term

$$\begin{aligned} & \sum_{\alpha \in S_{1/2}^+ \cup S_{1/2}^{0,-}} (\Phi_{[w^\alpha, [w_\alpha, v]]})_0^{\text{tw}} (\Phi_\alpha)_0^{\text{tw}} (\Phi^\alpha)_0^{\text{tw}} 1 \\ &= \sum_{\alpha \in S_{1/2}^{0,-}} (\Phi_{[w^\alpha, [w_\alpha, v]]})_0^{\text{tw}} ((\Phi^\alpha)_0^{\text{tw}})^2 1 + \sum_{\alpha \in S_{1/2}^+} (\Phi_{[w^\alpha, [w_\alpha, v]]})_0^{\text{tw}} 1 \\ &= \frac{1}{2} \sum_{\alpha \in S_{1/2}^{0,-}} (\Phi_{[w^\alpha, [w_\alpha, v]]})_0^{\text{tw}} 1 + \sum_{\alpha \in S_{1/2}^+} (\Phi_{[w^\alpha, [w_\alpha, v]]})_0^{\text{tw}} 1 = 0. \end{aligned}$$

In the last equality we used that $((\Phi^\alpha)_0^{\text{tw}})^2 = \frac{1}{2}[(\Phi^\alpha)_0^{\text{tw}}, (\Phi^\alpha)_0^{\text{tw}}] = \frac{1}{2}$. $\square$

Recall [15] that $\mu \in P_k^+$ is called *extremal* if $\mu(\theta_i^\vee) \leq M_i(k) + \chi_i$ for all i (see Table 4).

Theorem 7.5. *Let*

$$(7.25) \quad B(k, \nu, \rho_R) = -\frac{(k+1)^2}{4(k+h^\vee)} + \frac{(\nu - \rho_R | \nu - \rho_R + 2\rho^\natural)}{2(k+h^\vee)} + \frac{1}{16} \dim \mathfrak{g}_{1/2}.$$

If $\nu - \rho_R \in P_k^+$ is non extremal then, for all $\ell \geq B(k, \nu, \rho_R)$, the σ_R -twisted $W_{\min}^k(\mathfrak{g})$ -module $L^W(\nu, \ell_0)$ is unitary. In particular, $\nu \in P_k^+$.

Proof. Fix $\mu \in \mathbb{R}$. By Proposition 7.4, $N(\mu + s_k, \nu - \rho_R)$ is a highest weight module with highest weight $(\nu, \ell_0(\mu + s_k, \nu))$ (cf. (7.15)) so that, using (7.13),

$$\ell_0(\mu + s_k, \nu) = \frac{\mu^2}{2} - \frac{(k+1)^2}{4(k+h^\vee)} + \frac{(\nu - \rho_R | \nu - \rho_R + 2\rho^\natural)}{2(k+h^\vee)} + \frac{1}{16} \dim \mathfrak{g}_{1/2}.$$

Hence, for all $\ell \geq B(k, \nu, \rho_R)$, choosing $\mu = 2\sqrt{\ell_0 - B(k, \nu, \rho_R)}$, $N(\mu + s_k, \nu - \rho_R)$ is a highest weight module with highest weight (ν, ℓ) .

Since $M_i(k) + \chi_i \in \mathbb{Z}_+$ and $(\nu - \rho_R)(\theta_i^\vee) \leq M_i(k) + \chi_i$ for all i , $L(\nu - \rho_R)$ is integrable for $V^{\alpha_k}(\mathfrak{g}^\natural)$, hence it is an irreducible unitary highest weight $V^{\alpha_k}(\mathfrak{g}^\natural)$ -module. Let

$$(\cdot, \cdot)_{\mu+s_k} = H_{\mu+s_k} \otimes H^{\mathfrak{g}^\natural} \otimes H^F,$$

where $H_{\mu+s_k}$ is the invariant hermitian form on $M(1, \mu + s_k)$, $H^{\mathfrak{g}^\natural}$ is the invariant hermitian form on $L(\nu - \rho_R)$ and H^F is the invariant Hermitian form on $F(\mathfrak{g}_{1/2}, \sigma_R)$ constructed in Lemma 7.3. By [15, Proposition 9.2], $(\cdot, \cdot)_{\mu+s_k}$ is a ϕ -invariant form on $N(\mu + s_k, \nu - \rho_R)$.

Since $H_{\mu+s_k}$, $H^{\mathfrak{g}^\natural}$, and H^F are all positive definite, $(\cdot, \cdot)_{\mu+s_k}$ is positive definite as well hence, by restriction, $N(\mu + s_k, \nu - \rho_R)$ is unitary. In particular, $V^{\beta_k}(\mathfrak{g}^\natural)(v_\mu \otimes v_{\nu-\rho_R} \otimes 1)$ is unitary, hence it has to be integrable, so that $\nu \in P_k^+$. $\square$

Corollary 7.6. (1) *If $\mathfrak{g} \neq \mathfrak{spo}(2|3)$, $D(2, 1; a)$ and $\nu \in P_k^+$, $\nu - \rho_R \in P^+$, $\ell \geq B(k, \nu, \rho_R)$, then the σ_R -twisted $W_{\min}^k(\mathfrak{g})$ -module $L^W(\nu, \ell)$ is unitary.*

- (2) If $\mathfrak{g} = \mathfrak{spo}(2|3)$, then for all $\nu \in P_k^+$, $\nu \neq 0$, $\frac{M_1(k)}{2}\epsilon_1$ and $\ell \geq B(k, \nu, \rho_R)$, there the σ_R -twisted $W_{\min}^k(\mathfrak{g})$ -module $L^W(\nu, \ell)$ is unitary.
- (3) If $\mathfrak{g} = D(2, 1; a)$, then for all $\nu \in P_k^+$ which do not lie in $\{r_3\epsilon_3, r_2\epsilon_2 + M_2(k)\epsilon_3\}$ if $\rho_R = \epsilon_2$ or in $\{r_2\epsilon_2, M_1(k)\epsilon_2 + r_3\epsilon_3\}$ if $\rho_R = \epsilon_3$ then, for $\ell \geq B(k, \nu, \rho_R)$, the σ_R -twisted $W_{\min}^k(\mathfrak{g})$ -module $L^W(\nu, \ell)$ is unitary.

Proof. (1). Using Table 4, observe that $\rho_R(\theta_1^\vee) = -\chi_1$, Then, if $\nu \in P_k^+$, we have

$$(\nu - \rho_R)(\theta_i^\vee) = \nu(\theta_i^\vee) - \rho_R(\theta_i^\vee) \leq M_1(k) + \chi_1,$$

so $\nu - \rho_R$ is not extremal; being dominant by assumption, it lies in P_k^+ , hence we can apply Theorem 7.5.

(2). We have $\nu = \frac{r}{2}\epsilon_1$, $1 \leq r \leq M_1(k) - 1$, so $(\nu - \rho_R)(\theta_i^\vee) = \frac{r-1}{2}\epsilon_1$ belongs to P_k^+ and it is not extremal.

(3). We have $\nu = r_2\epsilon_2 + r_3\epsilon_3$, $r_2, r_3 \in \mathbb{Z}_+$, $r_2 \leq M_1(k)$, $r_3 \leq M_2(k)$. So if $\rho_R = \epsilon_2$ (resp. $\rho_R = \epsilon_3$) $\nu - \rho_R$ is not dominant precisely when $\nu = r_3\epsilon_3$ (resp. $\nu = r_2\epsilon_2$) and it is extremal if $\nu = r_2\epsilon_2 + M_2(k)\epsilon_3$ (resp. $\nu = M_1(k)\epsilon_2 + r_3\epsilon_3$). $\square$

8. EULER-POINCARÉ CHARACTERS

First of all we specialize to the Ramond sector the results of Proposition 4.3. Recall from (3.12) the definition of ϵ .

Lemma 8.1. *If $\sigma = \sigma_R$, then*

$$\gamma' = 2\rho_R - \rho - \frac{1}{2}\epsilon(\sigma_R)\theta/2, \quad \gamma_{1/2} = \rho_R - \frac{1}{4}\epsilon(\sigma_R)\theta/2$$

and

$$s_{fg} = \frac{k}{8(k + \hbar^\vee)}(\dim \mathfrak{g}_{1/2} - 2\epsilon(\sigma_R)), \quad s_{gh} = -\frac{1}{16}\dim \mathfrak{g}_{1/2}.$$

Proof. Specializing (4.7) to σ_R we find

$$s_e = 0, \quad s_{u_i} = -\frac{1}{2}, \text{ if } u_i \in \mathfrak{n}_{1/2}(\sigma_R)_+, \quad s_{u_i} = \frac{1}{2} \text{ if } u_i \in \mathfrak{n}_{1/2}(\sigma_R)_-, \quad s_{a_i} = 0 \text{ if } a_i \in \mathfrak{n}_0(\sigma_R)_+,$$

and

$$s_{a_i} = 1 \text{ if } a_i \in \mathfrak{n}_0(\sigma_R)_-, \quad s_{u_i} = \frac{1}{2} \text{ if } u_i \in \mathfrak{n}_{-1/2}(\sigma_R)_+, \quad s_{u_i} = \frac{3}{2} \text{ if } u_i \in \mathfrak{n}_{-1/2}(\sigma_R)_-, \quad s_f = 1.$$

so

$$\gamma' = \frac{1}{4} \sum_{\mathfrak{g}_\alpha \subset \mathfrak{n}_{1/2}(\sigma_R)_+} \alpha - \frac{1}{4} \sum_{\mathfrak{g}_\alpha \subset \mathfrak{n}_{1/2}(\sigma_R)_-} \alpha + \frac{1}{2} \sum_{\mathfrak{g}_\alpha \subset \mathfrak{n}_0(\sigma_R)_-} \alpha - \frac{1}{4} \sum_{\mathfrak{g}_\alpha \subset \mathfrak{n}_{-1/2}(\sigma_R)_+} \alpha - \frac{3}{4} \sum_{\mathfrak{g}_\alpha \subset \mathfrak{n}_{-1/2}(\sigma_R)_-} \alpha - \frac{1}{2}\theta.$$

Since:

- $\mathfrak{g}_\alpha \subset \mathfrak{n}_{1/2}(\sigma_R)_+$ if and only if $\alpha = \theta/2 + \eta$ with $\eta \in \overline{\Delta}_{1/2}^+$,
- $\mathfrak{g}_\alpha \subset \mathfrak{n}_{-1/2}(\sigma_R)_+$ if and only if $\mathfrak{g}_\alpha \subset \mathfrak{g}_{-1/2}^0(\sigma_R)$ or $\alpha = -\theta/2 + \eta$ with $\eta \in \overline{\Delta}_{1/2}^+$,
- $\mathfrak{g}_\alpha \subset \mathfrak{n}_{1/2}(\sigma_R)_-$ if and only if $\mathfrak{g}_\alpha \subset \mathfrak{g}_{1/2}^0(\sigma_R)$ or $\alpha = \theta/2 - \eta$ with $\eta \in \overline{\Delta}_{1/2}^+$,
- $\mathfrak{g}_\alpha \subset \mathfrak{n}_{-1/2}(\sigma_R)_-$ if and only if $\alpha = -\theta/2 - \eta$ with $\eta \in \overline{\Delta}_{1/2}^+$,

we can write

$$\begin{aligned} \gamma' &= \frac{1}{4} \sum_{\eta \in \overline{\Delta}_{1/2}^+} (\theta/2 + \eta) - \frac{1}{4} \sum_{\eta \in \overline{\Delta}_{1/2}^+} (\theta/2 - \eta) - \frac{1}{4} \epsilon(\sigma_R) \theta/2 \\ &\quad - \frac{1}{2} \sum_{\alpha \in \mathfrak{n}_0(\sigma_R)_+} \alpha - \frac{1}{4} \sum_{\eta \in \overline{\Delta}_{1/2}^+} (-\theta/2 + \eta) + \frac{1}{4} \epsilon(\sigma_R) \theta/2 - \frac{3}{4} \sum_{\eta \in \overline{\Delta}_{1/2}^+} (-\theta/2 - \eta) - \theta/2. \end{aligned}$$

By an obvious calculation we find

$$\begin{aligned} \gamma' &= \sum_{\eta \in \overline{\Delta}_{1/2}^+} \eta - \frac{1}{2} \sum_{\alpha \in \mathfrak{n}_0(\sigma_R)_+} \alpha + (\dim \mathfrak{n}_{1/2}(\sigma_R)_+ - 1) \theta/2 \\ &= 2\rho_R - \rho^\natural + \frac{1}{2} (\dim \mathfrak{g}_{1/2} - \epsilon(\sigma_R) - 2) \theta/2 \\ &= 2\rho_R - \rho^\natural + (-1 + \frac{1}{2} \dim \mathfrak{g}_{1/2}) \theta/2 - \frac{1}{2} \epsilon(\sigma_R) \theta/2 \\ &= 2\rho_R - \rho - \frac{1}{2} \epsilon(\sigma_R) \theta/2. \end{aligned}$$

Likewise $\gamma_{1/2}$ specializes to

$$\begin{aligned} \gamma_{1/2} &= \frac{1}{4} \sum_{\mathfrak{g}_\alpha \subset \mathfrak{n}_{1/2}(\sigma_R)_+} \alpha - \frac{1}{4} \sum_{\mathfrak{g}_\alpha \subset \mathfrak{n}_{1/2}(\sigma_R)_-} \alpha \\ &= \frac{1}{4} \sum_{\eta \in \overline{\Delta}_{1/2}^+} (\theta/2 + \eta) - \frac{1}{4} \sum_{\eta \in \overline{\Delta}_{1/2}^+} (\theta/2 - \eta) - \frac{1}{4} \epsilon(\sigma_R) \theta/2 = \rho_R - \frac{1}{4} \epsilon(\sigma_R) \theta/2. \end{aligned}$$

Finally

$$\begin{aligned} s_{fg} &= \frac{k}{4(k + h^\vee)} \left(\sum_{\mathfrak{g}_\alpha \subset \mathfrak{n}_{1/2}(\sigma_R)_+} 3/4 - \sum_{\mathfrak{g}_\alpha \subset \mathfrak{n}_{1/2}(\sigma_R)_-} 1/4 - \sum_{\mathfrak{g}_\alpha \subset \mathfrak{n}_{-1/2}(\sigma_R)_+} 1/4 + \sum_{\mathfrak{g}_\alpha \subset \mathfrak{n}_{-1/2}(\sigma_R)_-} 3/4 \right) \\ &= \frac{k}{8(k + h^\vee)} (3 \dim \mathfrak{n}_{1/2}(\sigma_R)_+ - \dim \mathfrak{n}_{1/2}(\sigma_R)_-) = \frac{k}{8(k + h^\vee)} (\dim \mathfrak{g}_{1/2} - 2\epsilon(\sigma_R)), \end{aligned}$$

and

$$s_{gh} = -\frac{1}{4} \sum_{\alpha \in S_{1/2}} s_\alpha^2 = -\frac{1}{16} \dim \mathfrak{g}_{1/2}.$$

□

Corollary 8.2. *If $\widehat{\Lambda} \in \widehat{\mathfrak{h}}^*$, then $H_j(M(\widehat{\Lambda})) = 0$ if $j \neq 0$ and $H_0(M(\widehat{\Lambda})) = M^W(\nu, \ell(\widehat{\Lambda}))$ with*

$$\begin{aligned} \ell(\widehat{\Lambda}) &= \frac{1}{2(k + h^\vee)} \left((\widehat{\Lambda}|_{\mathfrak{h}^\natural} | \widehat{\Lambda}|_{\mathfrak{h}^\natural}) - 2(\widehat{\Lambda}| 2\rho_R - \rho - \frac{1}{2} \epsilon(\sigma_R) \theta/2) \right) - \widehat{\Lambda}(x + D) \\ (8.1) \quad &+ \frac{k}{8(k + h^\vee)} (\dim \mathfrak{g}_{1/2} - 2\epsilon(\sigma_R)) - \frac{1}{16} \dim \mathfrak{g}_{1/2}. \end{aligned}$$

and $\nu = \widehat{\Lambda}|_{\mathfrak{h}^\natural} - \rho_R$.

Proof. Using the calculations in Lemma 8.1, specialize to $\sigma = \sigma_R$ the results of Proposition 4.3, which summarizes Proposition 3.1, Theorem 3.1, and Proposition 4.1 of [20]. □

By [20, Corollary 3.2] and [20, (4.12)], the Weyl vector corresponding to $\widehat{\Pi}$ is

$$(8.2) \quad \widehat{\rho}^{\text{tw}} = -\gamma' + h^\vee \Lambda_0 = -2\rho_R + \rho + \frac{1}{2}\epsilon(\sigma_R)\theta/2 + h^\vee \Lambda_0.$$

It follows that (8.1) can be rewritten as

$$(8.3) \quad \ell(\widehat{\Lambda}) = \frac{\|\widehat{\Lambda} + \widehat{\rho}^{\text{tw}}\|^2 - \|\widehat{\rho}^{\text{tw}}\|^2}{2(k + h^\vee)} - \widehat{\Lambda}(x + D) + a(k) = \frac{(\widehat{\Lambda}|\widehat{\Lambda} + 2\widehat{\rho}^{\text{tw}})}{2(k + h^\vee)} - \widehat{\Lambda}(x + D) + a(k),$$

where

$$(8.4) \quad a(k) = \frac{k}{8(k + h^\vee)}(\dim \mathfrak{g}_{1/2} - 2\epsilon(\sigma_R)) - \frac{1}{16} \dim \mathfrak{g}_{1/2}.$$

Next, we introduce the framework for computation of characters of highest weight modules, following [10]. Let $\mathcal{B}$ be a basis of $\mathbb{C} \oplus (\mathfrak{h}^\natural)^*$. In the set of formal series $\sum_{(z,\mu) \in \mathbb{C} \oplus (\mathfrak{h}^\natural)^*} b_{(z,\mu)} q^z e^\mu$ with $b_{(z,\mu)} \in \mathbb{Q}$, we consider the algebra $\mathcal{R}(\mathcal{B})$ of finite linear combinations of series of the form $\sum_{(n,\mu) \in \mathbb{Z}_+ \mathcal{B}} b_{(n,\mu)} q^{z-n} e^{\lambda-\mu}$. Given a series $Y = \sum_{(z,\mu) \in \mathbb{C} \oplus (\mathfrak{h}^\natural)^*} b_{(z,\mu)} q^z e^\mu$, we define the support of Y as the set $\text{Supp } Y = \{(z, \mu) \mid b_{(z,\mu)} \neq 0\}$.

Let $ev : \widehat{\mathfrak{h}}^* \rightarrow \mathbb{C} \oplus (\mathfrak{h}^\natural)^*$ be the map defined by

$$(8.5) \quad ev(\widehat{\Lambda}) = -\widehat{\Lambda}(x + D) + \widehat{\Lambda}|_{\mathfrak{h}^\natural}.$$

Set $\Pi^R = ev(\widehat{\Pi})$. By the explicit description of $\widehat{\Pi}$ given above, it is easy to check that $\Pi^R \setminus \{0\}$ is a basis of $\mathbb{C} \oplus (\mathfrak{h}^\natural)^*$. The choice of an ordering of $\Pi^R \setminus \{0\}$ defines a lexicographic total order $\leq_{\Pi^R}$ on $\mathbb{C} \oplus (\mathfrak{h}^\natural)^*$. We use this total order to topologize $\mathcal{R}(\Pi^R)$ by choosing as a fundamental set of open neighborhoods of 0 the sets

$$V_{(z_0, \lambda)} = \{Y \in \mathcal{R}(\Pi^R) \mid z + \mu \leq_{\Pi^R} z_0 + \lambda \text{ for all } z + \mu \in \text{Supp } Y\}.$$

If $\beta \in \mathbb{C} \oplus (\mathfrak{h}^\natural)^*$, $\beta = z + \lambda$, we write e^β for $q^z e^\lambda$, so that

$$(8.6) \quad e^{ev(\widehat{\Lambda})} = q^{-\widehat{\Lambda}(x+D)} e^{\widehat{\Lambda}|_{\mathfrak{h}^\natural}} = q^{-m-\lambda(x)} e^{\lambda|_{\mathfrak{h}^\natural}} \quad \text{if } \widehat{\Lambda} = k\Lambda_0 + m\delta + \lambda, \lambda \in \mathfrak{h}^*.$$

If $\beta \in \mathbb{Z}_+ \Pi^R$ and $a \in \mathbb{Q} \setminus \{0\}$, we note that $(1 - ae^{\pm\beta})$ are both invertible in $\mathcal{R}(\Pi^R)$ with inverses

$$(8.7) \quad (1 - ae^{-\beta})^{-1} = \sum_{n=0}^{\infty} a^n e^{-n\beta},$$

$$(8.8) \quad (1 - ae^{\beta})^{-1} = -\frac{e^{-\beta}}{a} \sum_{n=0}^{\infty} a^{-n} e^{-n\beta}.$$

We would like to extend the map ev to a map from $\mathcal{R}(\widehat{\Pi})$ to $\mathcal{R}(\Pi^R)$ by mapping $e^{\widehat{\Lambda}}$ to $e^{ev(\widehat{\Lambda})}$, but this is not possible: for example $ev(\sum_{n=0}^{\infty} e^{-n(\delta-\theta)})$ does not make sense. To make sense of ev , we must restrict to the following set

$$\mathcal{R}(\widehat{\Pi})_{fin} = \{Y \in \mathcal{R}(\widehat{\Pi}) \mid \text{for all } \mu \in \text{Supp } Y, (\mu + \mathbb{Q}(\delta - \theta)) \cap \text{Supp } Y \text{ is finite}\}.$$

We will need the following special case of the Lemma in § 2.2.8 of [10]:

Lemma 8.3. *Let $\Lambda \in \mathfrak{h}^*$ and set $\widehat{\Lambda} = (k + h^\vee)\Lambda_0 + \Lambda$. Assume that $2\frac{(\widehat{\Lambda}|\alpha)}{(\alpha|\alpha)} \in \mathbb{Z}$ for all real roots in $\widehat{\Delta}^\natural$. Let J be a set of linearly independent isotropic odd roots in $\widehat{\Delta}_+^{\text{tw}}$. Then the series*

$$Y_1 = \sum_{w \in \widehat{W}^\natural} \det(w) e^{w(\widehat{\Lambda})}, \quad Y_2 = \sum_{w \in \widehat{W}^\natural} \det(w) \frac{e^{w(\widehat{\Lambda})}}{\prod_{\beta \in J} (1 + e^{-w(\beta)})}$$

are elements of $\mathcal{R}(\widehat{\Pi})_{fin}$, hence the series

$$ev(Y_1) = \sum_{w \in \widehat{W}^\natural} det(w) e^{ev(w(\widehat{\Lambda}))}, \quad ev(Y_2) = \sum_{w \in \widehat{W}^\natural} det(w) \frac{e^{ev(w(\widehat{\Lambda}))}}{\prod_{\beta \in J} (1 + e^{-ev(w(\beta))})}$$

converge and define elements of $\mathcal{R}(\Pi^R)$.

Proof. We note that, since $k + h^\vee < 0$, $\widehat{\Lambda}$ satisfies the hypothesis of § 2.2.8 of [10] with $W' = \widehat{W}^\natural$, hence Y_1 and Y_2 converge and are elements of $\mathcal{R}(\widehat{\Pi})$.

We need only to show that, if $\mu \in \text{Supp } Y_i$, then the set $\{\mu' \in \text{Supp } Y_i \mid \mu' - \mu \in \mathbb{Q}(\delta - \theta)\}$ is finite. Let $\lambda = \bar{w}(\widehat{\Lambda})$ be the unique maximal element in the $\widehat{W}^\natural$ orbit of $\widehat{\Lambda}$. Since Y_i are skew-invariant for the action of $\widehat{W}^\natural$ and $\delta - \theta$ is fixed by $\widehat{W}^\natural$, we can assume that μ is in the support of $\frac{e^\lambda}{\prod_{\beta \in J} (1 + e^{-\bar{w}(\beta)})}$ so $\mu = \lambda - \sum_i r_i \gamma_i$ with $r_i \in \mathbb{Z}_+$ and $\gamma_i \in \pm \bar{w}(J)$, γ_i positive.

Let $\mu' \in \text{Supp } Y_i$ be such that $\mu - \mu' \in \mathbb{Q}(\delta - \theta)$. Then $\mu' = w(\lambda) - \sum_i r'_i w(\gamma_i)$ with $r'_i \in \mathbb{Z}$ and

$$(8.9) \quad a(\delta - \theta) = \lambda - w(\lambda) - \sum_i r_i \gamma_i + \sum_i r'_i w(\gamma_i).$$

It follows that

$$(8.10) \quad \sum_i r_i \gamma_i(x + D) = (\lambda - w(\lambda) + \sum_i r'_i w(\gamma_i))(x + D).$$

Since $\lambda - w(\lambda)$ is a sum of positive roots in $\widehat{\Delta}^\natural$, we see that

$$(\lambda - w(\lambda))(x + D) = (\lambda - w(\lambda))(D) \in \mathbb{Z}_+.$$

If $w(\gamma_i)$ is a positive root, then $w(\gamma_i)(x + D) \geq 0$ and $r'_i \geq 0$. If $w(\gamma_i)$ is a negative root, then $w(\gamma_i)(x + D) \leq 0$ and $r'_i < 0$. The outcome is that $\sum_i r'_i w(\gamma_i)(x + D) \in \frac{1}{2}\mathbb{Z}_+$. Thus there are only finitely many pairs $(m, n) \in \frac{1}{2}\mathbb{Z}_+ \times \frac{1}{2}\mathbb{Z}_+$ such that $m = (\lambda - w(\lambda))(x + D)$, $n = \sum_i r'_i w(\gamma_i)(x + D)$ and $m + n = \sum_i r_i \gamma_i(x + D)$.

By the combinatorics of reflection groups, for any given m , there are only finitely many $w \in \widehat{W}^\natural$ such that $m = (\lambda - w(\lambda))(D)$. It follows that there is a finite subset X of $\widehat{W}^\natural$ such that (8.10) holds iff $w \in X$. Hence (8.9) can be satisfied only if $w \in X$ and

$$(8.11) \quad 0 = (\lambda - w(\lambda) - \sum_i r_i \gamma_i)|_{\mathfrak{h}^\natural} + \sum_i r'_i w(\gamma_i)|_{\mathfrak{h}^\natural}.$$

Since $\{w(\gamma_i)\}$ is linearly independent, (8.9) has only finitely many solutions. $\square$

Set

$$(8.12) \quad \widehat{F}^{NS} = \prod_{n=1}^{\infty} \frac{(1 - q^n)^{\dim \mathfrak{h}} \prod_{\alpha \in \Delta_+^\natural} (1 - q^{n-1} e^{-\alpha})(1 - q^n e^\alpha)}{\prod_{\alpha \in \Delta_{-1/2}^\natural} (1 + q^{n-\frac{1}{2}} e^{\alpha|_{\mathfrak{h}^\natural}})},$$

$$(8.13) \quad \widehat{F}^R = \prod_{n=1}^{\infty} \frac{(1 - q^n)^{\dim \mathfrak{h}} \prod_{\alpha \in \Delta_+^\natural} (1 - q^{n-1} e^{-\alpha})(1 - q^n e^\alpha)}{\prod_{\eta \in (\overline{\Delta}_{1/2}^+)' } (1 + q^{n-1} e^{-\eta}) \prod_{\eta \in \overline{\Delta}_{1/2}^+} (1 + q^n e^\eta)},$$

where $(\overline{\Delta}_{1/2}^+)' = \overline{\Delta}_{1/2}^+$ if $\theta/2$ is not a root and $(\overline{\Delta}_{1/2}^+)' = \overline{\Delta}_{1/2}^+ \cup \{0\}$ otherwise. Let $\widehat{\Pi}^{NS} = \Pi \cup \{\delta - \theta\}$. If $\Pi^{NS} = ev(\widehat{\Pi}^{NS}) = ev(\Pi) \cup \{0\}$, these infinite products can naturally be seen as invertible elements of $\mathcal{R}(\Pi^{NS})$, $\mathcal{R}(\Pi^R)$ respectively.

The character $ch M$ of a non-twisted highest weight $W_{\min}^k(\mathfrak{g})$ -module M is defined as the trace of $q^{L_0} J_0^{\{h\}}$, $h \in \mathfrak{h}^\natural$, and can naturally be seen as an element of $\mathcal{R}(\Pi^{NS})$. Similarly, the character $ch M$ of a Ramond twisted highest weight $W_{\min}^k(\mathfrak{g})$ -module M is defined as the trace of $q^{L_0^{tw}} J_0^{\{h\}, tw}$, $h \in \mathfrak{h}^\natural$, and can naturally be seen as an element of $\mathcal{R}(\Pi^R)$. Note that $\widehat{F}^{NS}$ and $\widehat{F}^R$ are the denominators of the characters of non-twisted (resp. σ_R -twisted) Verma modules over $W_{\min}^k(\mathfrak{g})$. This is easily seen by computing the character of $M^W(0,0)$ using the basis given in [19, (6.13)] for the NS sector and in (4.20) in the Ramond sector. The formulas in (8.12) and in (8.13) give precisely these characters when expanded according to (8.7) and (8.8).

Let $\widehat{R}$ (resp. $\widehat{R}^{tw}$) be the Weyl denominator for $\widehat{\mathfrak{g}}$ (resp. $\widehat{\mathfrak{g}}^{tw}$). The following result is a special case of Theorem 3.1 of [16] and formula (3.14) of [20]:

Theorem 8.4. (a) *If M is a highest weight $\widehat{\mathfrak{g}}$ -module of highest weight $\widehat{\Lambda}$ and $\widehat{R} ch(M) \in \mathcal{R}(\widehat{\Pi}^{NS})_{fin}$ then*

$$\sum_j (-1)^j \widehat{F}^{NS} ch H_j(M) = q^{\frac{(\widehat{\Lambda}|\widehat{\Lambda}+2\widehat{\rho})}{2(k+h^\vee)}} ev(\widehat{R} ch(M)).$$

(b) *If M is a highest weight $\widehat{\mathfrak{g}}^{tw}$ -module of highest weight $\widehat{\Lambda}$ and $\widehat{R}^{tw} ch(M) \in \mathcal{R}(\widehat{\Pi})_{fin}$ then*

$$\sum_j (-1)^j \widehat{F}^R ch H_j(M) = e^{-\rho_R} q^{\frac{(\widehat{\Lambda}|\widehat{\Lambda}+2\widehat{\rho}^{tw})}{2(k+h^\vee)} + a(k)} ev(\widehat{R}^{tw} ch(M)),$$

where $a(k)$ is given by (8.4).

A weight $\widehat{\Lambda}$ of $\widehat{\mathfrak{g}}$ is called *degenerate* if $\widehat{\Lambda}(\alpha_0^\vee) \in \mathbb{Z}_+$. It is shown in [20] that $H(L(\widehat{\Lambda})) = 0$ if and only if $\widehat{\Lambda}$ is degenerate. Recall that a $V^k(\mathfrak{g})$ -module M is called *integrable* if it is integrable with respect to $\widehat{\mathfrak{g}}_i^\natural$ for all $i = 1, \dots, s$ (see (3.3)).

An easy consequence of Theorem 8.4 is the following Proposition.

Proposition 8.5. *Assume that*

$$(8.14) \quad (\widehat{\Lambda} + \widehat{\rho}^{tw} | \alpha) \neq n \frac{(\alpha | \alpha)}{2} \text{ for all } n \in \mathbb{N} \text{ and } \alpha \in \widehat{\Delta}_+^{tw} \setminus (\widehat{\Delta}^+)^{\natural}$$

and that $L(\widehat{\Lambda})$ is integrable. Then $H(L(\widehat{\Lambda})) \neq 0$ and

$$\begin{aligned} & \widehat{F}^R \sum_j (-1)^j ch H_j(L(\widehat{\Lambda})) \\ &= q^{\frac{(\widehat{\Lambda}|\widehat{\Lambda}+2\widehat{\rho}^{tw})}{2(k+h^\vee)} + a(k)} e^{-\rho_R} \sum_{w \in \widehat{W}^\natural} det(w) q^{-(w(\widehat{\Lambda}+\widehat{\rho}^{tw})-\widehat{\rho}^{tw})(x+D)} e^{w(\widehat{\Lambda}+\widehat{\rho}^{tw})-\widehat{\rho}^{tw}}|_{\mathfrak{h}^\natural}. \end{aligned}$$

Proof. The hypothesis implies in particular that $\widehat{\Lambda}$ is nondegenerate, hence $H(L(\widehat{\Lambda})) \neq 0$. Arguing as in Proposition 11.5 of [15], we see that

$$R^{tw} ch L(\widehat{\Lambda}) = \sum_{w \in \widehat{W}^\natural} det(w) e^{w(\widehat{\Lambda}+\widehat{\rho}^{tw})-\widehat{\rho}^{tw}}.$$

By Lemma 8.3, $\widehat{R}^{tw} ch L(\widehat{\Lambda}) \in \mathcal{R}(\widehat{\Pi})_{fin}$ so, applying Theorem 8.4, we conclude. $\square$

A similar result is the following.

Proposition 8.6. *Assume that $(\hat{\Lambda} + \hat{\rho}^{\text{tw}}|\delta - \theta) \notin \mathbb{N}$ and that $L(\hat{\Lambda})$ is integrable and maximally atypical. Assume that $\hat{\Pi}$ contains a maximal set J of pairwise orthogonal isotropic roots such that $(\hat{\Lambda} + \hat{\rho}^{\text{tw}}|\beta) = 0$ for all $\beta \in J$. Then $H(L(\hat{\Lambda})) \neq 0$ and*

$$\begin{aligned} & \hat{F}^R \sum_j (-1)^j \text{ch } H_j(L(\hat{\Lambda})) \\ &= q^{\frac{(\hat{\Lambda}|\hat{\Lambda} + 2\hat{\rho}^{\text{tw}})}{2(k+h^\vee)} + a(k)} e^{-\rho_R} \sum_{w \in \hat{W}^\natural} \det(w) \frac{q^{-(w(\hat{\Lambda} + \hat{\rho}^{\text{tw}}) - \hat{\rho}^{\text{tw}})(x+D)} e^{w(\hat{\Lambda} + \hat{\rho}^{\text{tw}}) - \hat{\rho}^{\text{tw}}|_{\mathfrak{h}^\natural}}}{\prod_{\beta \in J} (1 + q^{w(\beta)(x+D)} e^{-w(\beta)|_{\mathfrak{h}^\natural}})}. \end{aligned}$$

Proof. The hypothesis implies in particular that $\hat{\Lambda}$ is nondegenerate, hence $H(L(\hat{\Lambda})) \neq 0$.

It is shown in [10] that the hypothesis imply that

$$(8.15) \quad \text{ch } L(\hat{\Lambda}) = \frac{1}{\hat{R}^{\text{tw}}} \sum_{w \in \hat{W}^\natural} \det(w) \frac{e^{w(\hat{\Lambda} + \hat{\rho}^{\text{tw}}) - \hat{\rho}^{\text{tw}}}}{\prod_{\beta \in J} (1 + e^{-w(\beta)})}.$$

Formula (8.15) is a special case of [10, Formula (14)] if $\mathfrak{g} \neq D(2, 1; \frac{m}{n})$ and of [10, Section 6.1] if $\mathfrak{g} = D(2, 1; \frac{m}{n})$ and $\hat{\Lambda}|_{\mathfrak{h}^\natural} = 0$ (for other $\hat{\Lambda}$ (8.15) holds only conjecturally). Here we use that $\hat{\mathfrak{g}}^{\text{tw}} \cong \hat{\mathfrak{g}}$ [13, Remark 8.5], since σ_R is an inner automorphism of $\mathfrak{g}$ in all cases considered. By Lemma 8.3, $\hat{R}^{\text{tw}} \text{ch } L(\hat{\Lambda}) \in \mathcal{R}(\hat{\Pi})_{\text{fin}}$ so, applying Theorem 8.4, we conclude. $\square$

Remark 8.7. The equalities in Propositions 8.5 and 8.6 are to be understood as equalities in $\mathcal{R}(\Pi^{NS})$ or $\mathcal{R}(\Pi^R)$. To be more explicit, observe that $\Pi^{NS} = \{\gamma_0, \gamma_1, \dots, \gamma_r, 0\}$ with $\{\gamma_i\}$ a basis of $\mathbb{C} \oplus (\mathfrak{h}^\natural)^*$, $\gamma_0 = -\frac{1}{2} + \bar{\gamma}_0$, $\bar{\gamma}_0 \in (\mathfrak{h}^\natural)^*$ and $\gamma_i \in (\mathfrak{h}^\natural)^*$ for $i > 0$. Then a series

$$Y = \sum_{(n, \mu) \in \mathbb{Z}_+ \Pi^{NS}} b_{(n, \mu)} q^{z-n} e^{\lambda - \mu} \in \mathcal{R}(\Pi^{NS})$$

can be rewritten as

$$(8.16) \quad Y = \sum_{n \in \mathbb{Z}_+} a_n q^{z + \frac{n}{2}}$$

with

$$a_n = \left(\sum_{n_1, \dots, n_r} b_{(n, \sum_i n_i \gamma_i)} e^{\lambda - n \bar{\gamma}_0 - \sum_i n_i \gamma_i} \right) \in \mathcal{R}(\{\gamma_1, \dots, \gamma_r\}).$$

We refer to the expression in (8.16) as the $|q| < 1$ expansion of Y .

The same argument works verbatim for $\mathcal{R}(\Pi^R)$ except that $\gamma_0 = -1 + \bar{\gamma}_0$ so the $|q| < 1$ expansion reads

$$Y = \sum_n a_n q^{z+n}.$$

with $a_n \in \mathcal{R}(\{\gamma_1, \dots, \gamma_r\})$.

9. UNITARITY BETWEEN $A(k, \nu)$ AND $B(k, \nu, \rho_R)$

For $\nu \in P_k^+$ and $s \in \mathbb{C}$, set

$$(9.1) \quad \hat{\nu}_s = k\Lambda_0 + s\theta + \nu + \rho_R,$$

and set $\ell(s) = \ell(\widehat{\nu}_s)$. An obvious calculation shows that

$$(9.2) \quad \begin{aligned} \ell(s) &= \frac{(\nu - \rho_R | \nu - \rho_R + 2\rho^\natural)}{2(k + h^\vee)} + \frac{s(s - k - 1 + \frac{1}{2}\epsilon(\sigma_R))}{k + h^\vee} + \frac{2}{k + h^\vee}(\rho_R | \rho^\natural - \rho_R) \\ &\quad + \frac{k}{8(k + h^\vee)}(\dim \mathfrak{g}_{1/2} - 2\epsilon(\sigma_R)) - \frac{1}{16} \dim \mathfrak{g}_{1/2}. \end{aligned}$$

Let us call a weight $\nu \in P_k^+$ *Ramond extremal* if

$$(9.3) \quad \nu - \rho_R \notin P_k^+ \text{ or } \nu - \rho_R \text{ is extremal.}$$

We restrict our attention to the irreducible highest weight modules $L^W(\nu, \ell)$ that satisfy the necessary conditions for unitarity proven in Section 6, thus $\nu \in P_k^+$, ℓ is real with $\ell \geq A(k, \nu)$, and, if ν is Ramond extremal, $\ell = A(k, \nu)$.

We want to calculate the difference

$$d(s) = \ell(s) - B(k, \nu, \rho_R).$$

We use the following fact, which is verified by case-wise inspection.

Lemma 9.1.

$$(9.4) \quad (\rho_R | \rho^\natural - \rho_R) = \frac{h^\vee - \frac{1}{2}\epsilon(\sigma_R)}{16}(\dim \mathfrak{g}_{1/2} - \epsilon(\sigma_R)).$$

Lemma 9.2.

$$(9.5) \quad d(s) = \frac{(s - (\frac{k+1}{2} - \frac{\epsilon(\sigma_R)}{4}))^2}{k + h^\vee}.$$

Proof. By (7.25) and (9.2),

$$\begin{aligned} \ell(s) - B(k, \nu, \rho_R) &= \frac{s(s - k - 1 + \frac{1}{2}\epsilon(\sigma_R))}{k + h^\vee} + \frac{2}{k + h^\vee}(\rho_R | \rho^\natural - \rho_R) \\ &\quad + \frac{k}{8(k + h^\vee)}(\dim \mathfrak{g}_{1/2} - 2\epsilon(\sigma_R)) + \frac{(k+1)^2}{4(k + h^\vee)} - \frac{1}{8} \dim \mathfrak{g}_{1/2}. \end{aligned}$$

Using (9.4) and relation $\dim \mathfrak{g}_{1/2} = -2(h^\vee - 2)$, we find

$$\begin{aligned} \ell(s) - B(k, \nu, \rho_R) &= \frac{s(s - k - 1 + \frac{1}{2}\epsilon(\sigma_R))}{k + h^\vee} + \frac{h^\vee - \frac{1}{2}\epsilon(\sigma_R)}{8(k + h^\vee)}(\dim \mathfrak{g}_{1/2} - \epsilon(\sigma_R)) \\ &\quad + \frac{k}{8(k + h^\vee)}(\dim \mathfrak{g}_{1/2} - 2\epsilon(\sigma_R)) + \frac{(k+1)^2}{4(k + h^\vee)} - \frac{1}{8} \dim \mathfrak{g}_{1/2} \\ &= \frac{s(s - k - 1 + \frac{1}{2}\epsilon(\sigma_R))}{k + h^\vee} + \frac{\epsilon(\sigma_R)^2}{16(k + h^\vee)} - \frac{k+1}{4(k + h^\vee)}\epsilon(\sigma_R) + \frac{(k+1)^2}{4(k + h^\vee)} \\ &= \frac{s(s - k - 1 + \frac{1}{2}\epsilon(\sigma_R))}{k + h^\vee} + \frac{(-k - 1 + \frac{1}{2}\epsilon(\sigma_R))^2}{4(k + h^\vee)} \\ &= \frac{(s - (\frac{k+1}{2} - \frac{\epsilon(\sigma_R)}{4}))^2}{k + h^\vee}. \end{aligned}$$

□

We now compute the values of $s \in \mathbb{C}$ such that $\ell(s) = A(k, \nu)$. For this we need the following computation.

Lemma 9.3. *If $\theta/2$ is not a root of $\mathfrak{g}$, then*

$$(9.6) \quad F_\nu(\eta_{\min}) = 2(\nu|\eta_{\min}) \left((\nu|\eta_{\min}) + 2(\rho^\natural - \rho_R|\eta_{\min}) \right) - (\nu|\rho_R).$$

Proof. We proceed by a case-wise inspection.

$\mathfrak{g} = \mathfrak{psl}(2|2)$. We have $\eta_{\min} = \rho_R = \rho^\natural = \frac{1}{2}(\delta_1 - \delta_2)$. If $\nu = \frac{r}{2}(\delta_1 - \delta_2)$, then $(\nu|\eta_{\min}) = (\nu|\rho_R) = -\frac{r}{2}$, $(\rho^\natural - \rho_R|\eta_{\min}) = 0$ so

$$2(\nu|\eta_{\min})^2 + 4(\nu|\eta_{\min})(\rho^\natural - \rho_R|\eta_{\min}) - 2(\nu|\rho_R) = \frac{1}{2}r^2 + r,$$

which is (6.21).

$\mathfrak{g} = \mathfrak{spo}(2|2r)$. We have $\eta_{\min} = \pm\epsilon_r$, $\rho_R = \frac{1}{2}\sum_{i<r}\epsilon_i \pm \frac{1}{2}\epsilon_r$, and in turn $(\nu|\eta_{\min}) = \mp 1/2m_r$, $(\nu|\rho_R) = -1/4\sum_{i<r}m_i \mp 1/4m_r$, $(\rho_R|\eta_{\min}) = -1/4$, $(\rho^\natural - \rho_R|\eta_{\min}) = 1/4$, so

$$2(\nu|\eta_{\min})^2 + 4(\nu|\eta_{\min})(\rho^\natural - \rho_R|\eta_{\min}) = \frac{1}{2}m_r^2 \mp \frac{1}{2}m_r + \frac{1}{2}\sum_{i<r}m_i \pm \frac{1}{2}m_r = \frac{1}{2}m_r^2 + \frac{1}{2}\sum_{i<r}m_i,$$

which is the expression for $F_\nu(\eta_{\min})$ given in (6.24).

$\mathfrak{g} = F(4)$. We have $\eta_{\min} = \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3)$ or $\eta_{\min} = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3)$ and $\rho_R = \omega_3$ or $\rho_R = \omega_1$ respectively. In the first case $(\nu|\eta_{\min}) = 1/3(m_1 - m_2 - m_3)$, $(\nu|\rho_R) = -1/3(m_1 + m_2 + m_3)$, $(\rho_R|\eta_{\min}) = -1/6$, $(\rho^\natural - \rho_R|\eta_{\min}) = 1/3$, so

$$\begin{aligned} & 2(\nu|\eta_{\min})^2 + 4(\nu|\eta_{\min})(\rho^\natural - \rho_R|\eta_{\min}) - 2(\nu|\rho_R) \\ &= \frac{2}{9}(m_1 - m_2 - m_3)^2 + \frac{4}{9}(m_1 - m_2 - m_3) + \frac{2}{3}(m_1 + m_2 + m_3), \end{aligned}$$

which is (6.28). The same holds in the second case: $(\nu|\eta_{\min}) = 1/3(-m_1 + m_2 + m_3)$, $(\nu|\rho_R) = -2/3m_1$, $(\rho_R|\eta_{\min}) = -1/3$, $(\rho^\natural - \rho_R|\eta_{\min}) = 1/6$ so

$$\begin{aligned} & 2(\nu|\eta_{\min})^2 + 4(\nu|\eta_{\min})(\rho^\natural - \rho_R|\eta_{\min}) - 2(\nu|\rho_R) = \\ & \frac{4}{9}(m_1 - m_2 - m_3)^2 - \frac{2}{9}(m_1 - m_2 - m_3) + \frac{4}{3}m_1 = \frac{2}{9}((-m_1 + m_2 + m_3)^2 + 5m_1 + m_2 + m_3). \end{aligned}$$

$\mathfrak{g} = D(2, 1; a)$. We have $\eta_{\min} = \epsilon_2 - \epsilon_3, -\epsilon_2 + \epsilon_3$, $\rho_R = \omega_1^1, \omega_1^2$. In the first case $(\nu|\eta_{\min}) = \frac{-m_1 + am_2}{2(1+a)}$, $(\nu|\rho_R) = \frac{-m_1}{2(1+a)}$, $(\rho_R|\eta_{\min}) = \frac{-1}{2(1+a)}$, so

$$\begin{aligned} & 2(\nu|\eta_{\min})^2 + 4(\nu|\eta_{\min})(\rho^\natural - \rho_R|\eta_{\min}) - 2(\nu|\rho_R) = \\ & 2\left(\frac{-m_1 + am_2}{2(1+a)}\right)^2 + \frac{a(-m_1 + am_2)}{(1+a)^2} - \frac{-m_1}{(1+a)}, \end{aligned}$$

which is (6.27). The other case is similar. □

Lemma 9.4.

(1) *If $\theta/2$ is a root of $\mathfrak{g}$, then*

$$A(k, \nu) = B(k, \nu, \rho_R),$$

so $\ell(s) = A(k, \nu)$ if and only if $s = \frac{2k+1}{4}$.

(2) *If $\theta/2$ is not a root of $\mathfrak{g}$, then*

$$(9.7) \quad A(k, \nu) = B(k, \nu, \rho_R) + \frac{(\nu - \rho_R + \rho^\natural|\eta_{\min})^2}{k + h^\vee},$$

so $\ell(s) = A(k, \nu)$ if and only if

$$(9.8) \quad s = \frac{k+1}{2} \pm (\nu - \rho_R + \rho^\natural|\eta_{\min}).$$

Proof. If $\theta/2$ is a root of $\mathfrak{g}$, we have

$$A(k, \nu) = \frac{1}{2(k+h^\vee)}((\nu|\nu + 2(\rho^\natural - \rho_R)) - \frac{1}{2}p(k)),$$

while

$$\begin{aligned} B(k, \nu, \rho_R) &= -\frac{(k+1)^2}{4(k+h^\vee)} + \frac{(\nu - \rho_R|\nu - \rho_R + 2\rho^\natural)}{2(k+h^\vee)} + \frac{1}{16} \dim \mathfrak{g}_{1/2} \\ &= -\frac{(k+1)^2}{4(k+h^\vee)} + \frac{(\nu|\nu + 2\rho^\natural)}{2(k+h^\vee)} - \frac{(\rho_R|2\nu - \rho_R + 2\rho^\natural)}{2(k+h^\vee)} + \frac{1}{16} \dim \mathfrak{g}_{1/2} \end{aligned}$$

We need to check that

$$\frac{p(k)}{4(k+h^\vee)} = \frac{(k+1)^2}{4(k+h^\vee)} + \frac{(\rho_R| - \rho_R + 2\rho^\natural)}{2(k+h^\vee)} - \frac{1}{16} \dim \mathfrak{g}_{1/2}.$$

This is easily checked case by case:

- $spo(2|2r+1)$, $r \geq 1$:

$$k^2 + (7/4 - r/2)k + 5/8 - r/4 = (k+1)^2 - r^2/2 + r/4 - (r/2 + 1/4)(k + 3/2 - r).$$

- $G(3)$:

$$k^2 + 1/4k - 3/8 = (k+1)^2 - 4 - 7/4(k - 3/2).$$

If $\theta/2$ is not a root of $\mathfrak{g}$ then

$$A(k, \nu) = \frac{1}{2(k+h^\vee)} \left((\nu|\nu + 2\rho^\natural) - \frac{1}{2}p(k) + F_\nu(\eta_{\min}) \right),$$

so the difference $A(k, \nu) - B(k, \nu, \rho_R)$ is

$$(9.9) \quad \frac{1}{2(k+h^\vee)} \left(-\frac{1}{2}p(k) + F_\nu(\eta_{\min}) \right) + \frac{(k+1)^2}{4(k+h^\vee)} + \frac{(\rho_R|2\nu - \rho_R + 2\rho^\natural)}{2(k+h^\vee)} - \frac{1}{16} \dim \mathfrak{g}_{1/2}.$$

Substituting (9.6) in (9.9) we get

$$\begin{aligned} &\frac{1}{2(k+h^\vee)} \left(-\frac{1}{2}p(k) + 2(\nu|\eta_{\min})^2 + 4(\nu|\eta_{\min})(\rho_R|\eta_{\min}) \right) + \frac{(k+1)^2}{4(k+h^\vee)} \\ &+ \frac{(\rho_R| - \rho_R + 2\rho^\natural)}{2(k+h^\vee)} - \frac{1}{16} \dim \mathfrak{g}_{1/2}, \end{aligned}$$

so we have our claim, provided that

$$\frac{p(k)}{4(k+h^\vee)} = \frac{(k+1)^2}{4(k+h^\vee)} + \frac{(\rho_R| - \rho_R + 2\rho^\natural)}{2(k+h^\vee)} - \frac{1}{16} \dim \mathfrak{g}_{1/2} - \frac{(\rho^\natural - \rho_R|\eta_{\min})^2}{k+h^\vee}.$$

This formula is proved by inspection. $\square$

An immediate consequence of the computation above is the following result.

Corollary 9.5. *If $\theta/2$ is a root of $\mathfrak{g}$ and $\nu \in P_k^+$ is not Ramond extremal, then $L^W(\nu, \ell)$ is unitary if and only if $\ell \geq A(k, \nu)$.*

If $\theta/2$ is not a root of $\mathfrak{g}$, $\nu \in P_k^+$ is not Ramond extremal, and $(\nu - \rho_R + \rho^\natural|\eta_{\min}) = 0$, then $L^W(\nu, \ell)$ is unitary if and only if $\ell \geq A(k, \nu)$.

Proof. In these cases $A(k, \nu) = B(k, \nu, \rho_R)$. $\square$

It remains to discuss the cases when $A(k, \nu) < B(k, \nu, \rho_R)$. For handling these cases we want to compute characters of $H(L(\hat{\nu}_s))$. A first step in this direction is given by Propositions 8.5 and 8.6, therefore we check if $\hat{\nu}_s$ satisfies their hypothesis.

Lemma 9.6. *Assume that $\theta/2$ is not a root of $\mathfrak{g}$. Assume $\nu \in P_k^+$ is not Ramond extremal. If*

$$(9.10) \quad 0 \leq s - \frac{k+1}{2} < |(\nu - \rho_R + \rho^\natural|\eta_{\min})|,$$

then (8.14) holds.

Proof. By [20, Corollary 3.2] and [20, (4.12)], $\hat{\rho}^{\text{tw}} = -\gamma' + h^\vee \Lambda_0$, so, by Lemma 8.1, for $\hat{\nu}_s$, given by (9.1), we have

$$(9.11) \quad \hat{\nu}_s + \hat{\rho}^{\text{tw}} = (k + h^\vee) \Lambda_0 + s\theta + \nu - \rho_R + \rho + \frac{1}{2}\epsilon(\sigma_R)\theta/2.$$

Note that the elements of $\hat{\Delta}_+^{\text{tw}} \setminus (\hat{\Delta}^+)^{\natural}$ are precisely

- (1) $\beta - j\delta$, $\mathfrak{g}_\beta \subset \mathfrak{n}_j(\sigma_R)_+$, $j \neq 0$,
- (2) $\beta + (m - j)\delta$, $\mathfrak{g}_\beta \subset \mathfrak{g}_j$, $m \in \mathbb{N}$, $j \neq 0$.

We check that $(\hat{\nu}_s + \hat{\rho}^{\text{tw}}|\alpha) \neq 0$ for all isotropic roots. If α is such a root, then

$$(9.12) \quad \alpha = \pm(-\frac{1}{2}\delta + \theta/2) + \eta, \quad \eta \in \overline{\Delta}_{1/2}^+,$$

$$(9.13) \quad \alpha = \pm(-\frac{1}{2}\delta + \theta/2) \pm \eta + n\delta, \quad \eta \in \overline{\Delta}_{1/2}^+, n \in \mathbb{N}.$$

We start considering the roots (9.12). We have two cases.

First case: $(\nu - \rho_R + \rho^\natural|\eta_{\min}) < 0$. We compute, using (9.11),

$$\begin{aligned} & ((k + h^\vee) \Lambda_0 + s\theta + \nu - \rho_R + \rho | \pm(-\frac{1}{2}\delta + \theta/2) + \eta) = \\ & = \mp \frac{1}{2}(k + h^\vee) + (\nu - \rho_R + \rho^\natural|\eta) \pm \frac{h^\vee - 1}{2} \pm s \\ & = \pm(s - \frac{k+1}{2}) + (\nu - \rho_R + \rho^\natural|\eta) \\ & = \pm(s - \frac{k+1}{2}) + (\nu - \rho_R + \rho^\natural|\eta_{\min} + \sum_{\beta \in \Delta_+^{\natural}, n_\beta \geq 0} n_\beta \beta) \\ & \leq \pm(s - \frac{k+1}{2}) + (\nu - \rho_R + \rho^\natural|\eta_{\min}) < 0. \end{aligned}$$

The last inequality follows from (9.10).

Second case: $(\nu - \rho_R + \rho^\natural|\eta_{\min}) > 0$. Computing as above

$$\begin{aligned} & ((k + h^\vee) \Lambda_0 + s\theta + \nu - \rho_R + \rho | \pm(-\frac{1}{2}\delta + \theta/2) + \eta) = \\ & = \pm(s - \frac{k+1}{2}) - (\nu - \rho_R + \rho^\natural|\eta) \\ & = \pm(s - \frac{k+1}{2}) - (\nu - \rho_R + \rho^\natural|\eta_{\min} - \sum_{\beta \in \Delta_+^{\natural}, n_\beta \geq 0} n_\beta \beta) \\ & \leq \pm(s - \frac{k+1}{2}) - (\nu - \rho_R + \rho^\natural|\eta_{\min}) < 0. \end{aligned}$$

We now deal with the roots (9.13). The case when $\alpha = \pm(-\frac{1}{2}\delta + \theta/2) + \eta + n\delta$ is handled as above, since $k + h^\vee < 0$. In the remaining case we use the following relations (see [15, (11.23)] for (9.14) and [15, (11.18), (11.19)] for (9.15)):

$$(9.14) \quad k + \frac{h^\vee}{2} \leq -\frac{1}{2} + (\xi|\nu - \rho_R) - (\nu - \rho_R + \rho^\natural|\eta), \quad \eta \in \pm\Delta_{1/2}^+, \quad \eta \neq -\xi,$$

$$(9.15) \quad (\rho^\natural|\xi) = \frac{1}{2}(h^\vee - 1).$$

Formula (9.14) follows from the non-extremality of $\nu - \rho_R$. In the subsequent computation we use (9.14) with $\eta = \pm\eta_{\min}$. When $\mathfrak{g} = \mathfrak{spo}(2|3)$ or $\mathfrak{psl}(2|2)$ (and only in these cases) it

happens that $\eta_{\min} = \pm\xi$. A direct check shows that (9.14) still holds. We have

$$\begin{aligned}
& ((k + h^\vee)\Lambda_0 + s\theta + \nu - \rho_R + \rho | \pm(-\tfrac{1}{2}\delta + \theta/2) - \eta + n\delta) = \\
& = \mp \tfrac{1}{2}(k + h^\vee) - (\nu - \rho_R + \rho^\natural | \eta) \pm \tfrac{h^\vee - 1}{2} \pm s + n(k + h^\vee) \\
& < \pm(s - \tfrac{k+1}{2}) - (\nu - \rho_R + \rho^\natural | \eta) + k + h^\vee \\
& \leq \pm(s - \tfrac{k+1}{2}) - (\nu - \rho_R + \rho^\natural | \eta) + \tfrac{h^\vee}{2} - \tfrac{1}{2} + (\xi | \nu - \rho_R) - (\nu - \rho_R + \rho^\natural | \eta_{\min}) \\
& \leq \pm(s - \tfrac{k+1}{2}) - (\rho^\natural | \eta) + \tfrac{h^\vee}{2} - \tfrac{1}{2} - (\nu - \rho_R + \rho^\natural | \eta_{\min}) \\
& = \pm(s - \tfrac{k+1}{2}) + (\rho^\natural | \xi - \eta) - (\rho^\natural | \xi) + \tfrac{h^\vee}{2} - \tfrac{1}{2} - (\nu - \rho_R + \rho^\natural | \eta_{\min}) \\
& \leq \pm(s - \tfrac{k+1}{2}) - (\nu - \rho_R + \rho^\natural | \eta_{\min})
\end{aligned}$$

and we can conclude if $(\nu - \rho_R + \rho^\natural | \eta_{\min}) > 0$. Otherwise, we repeat the argument with $-\eta_{\min}$.

Finally, we check that $(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | \alpha) \neq \frac{n}{2}(\alpha | \alpha)$ for $n \in \mathbb{N}$ and $\alpha = m\delta + \theta$, $m \in \mathbb{Z}_+$ or $\alpha = m\delta - \theta$, $m \in \mathbb{N}$. Observe that

$$\begin{aligned}
& ((k + h^\vee)\Lambda_0 + s\theta + \nu - \rho_R + \rho | m\delta - \theta) = \\
& \leq k + h^\vee - 2s - h^\vee + 1 \\
& = -2(s - \tfrac{k+1}{2}) \leq 0,
\end{aligned}$$

by our initial assumption. Also, by replacing $\eta_{\min}$ by its opposite, we can assume $(\nu - \rho_R + \rho^\natural | \eta_{\min}) \geq 0$. We have, using again (9.14), (9.15)

$$\begin{aligned}
& ((k + h^\vee)\Lambda_0 + s\theta + \nu - \rho_R + \rho | m\delta + \theta) = \\
& = m(k + h^\vee) + 2s + h^\vee - 1 \\
& \leq 2(s - \tfrac{k+1}{2}) + k + h^\vee \leq 2|(\nu - \rho_R + \rho^\natural | \eta_{\min})| + k + h^\vee \\
& \leq (\nu - \rho_R | \xi - (-\eta_{\min})) + (\rho^\natural | \eta_{\min}) + \tfrac{h^\vee}{2} - \tfrac{1}{2} \\
& \leq (\rho^\natural | \eta_{\min} + \xi) \leq 0.
\end{aligned}$$

□

For calculation of characters of $H(L(\widehat{\nu}_s))$ we will also need

Lemma 9.7. *Assume that $\theta/2$ is a root of $\mathfrak{g}$. Assume $\nu \in P_k^+$ is not Ramond extremal and $s = \frac{2k+1}{4}$. Then (8.14) holds.*

Proof. First of all observe that $(\nu - \rho_R + \rho^\natural | \eta) < 0$ for all $\eta \in \Delta_{-1/2}^+$. This is due to the fact that ν is not Ramond extremal so $\nu - \rho_R + \rho^\natural$ is dominant and regular for $\Delta^\natural$. Moreover η is a short positive root of $\mathfrak{g}^\natural$. Note also that

$$(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | \delta - \theta) = ((k + h^\vee)\Lambda_0 + s\theta + \nu - \rho_R + \rho + \tfrac{\theta}{4} | \delta - \theta) = 0.$$

We check that $(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | \alpha) \neq 0$ for all isotropic roots as described in (9.12) and (9.13)

We start considering the roots (9.12). Since $(\nu - \rho_R + \rho^\natural | \eta_{\min}) < 0$,

$$(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | \pm(-\tfrac{1}{2}\delta + \theta/2) + \eta) = (\nu - \rho_R + \rho^\natural | \eta) < 0.$$

We now deal with the roots (9.13). The case when $\alpha = \pm(-\tfrac{1}{2}\delta + \theta/2) + \eta + n\delta$ is handled as above, since $k + h^\vee < 0$. In the remaining case we argue as in Lemma 9.6 using (9.12) and

(9.13): We have

$$\begin{aligned}
(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | \pm (-\tfrac{1}{2}\delta + \theta/2) - \eta + n\delta) &= -(\nu - \rho_R + \rho^\natural | \eta) + n(k + h^\vee) \\
&\leq -(\nu - \rho_R + \rho^\natural | \eta) + k + h^\vee \\
&\leq -(\nu - \rho_R + \rho^\natural | \eta) + \tfrac{h^\vee}{2} - \tfrac{1}{2} + (\xi | \nu - \rho_R) + (\nu - \rho_R + \rho^\natural | \eta_{\min}) \\
&\leq -(\rho^\natural | \eta) + \tfrac{h^\vee}{2} - \tfrac{1}{2} + (\nu - \rho_R + \rho^\natural | \eta_{\min}) \\
&= (\rho^\natural | \xi - \eta) - (\rho^\natural | \xi) + \tfrac{h^\vee}{2} - \tfrac{1}{2} + (\nu - \rho_R + \rho^\natural | \eta_{\min}) \\
&\leq (\nu - \rho_R + \rho^\natural | \eta_{\min}) < 0.
\end{aligned}$$

Finally, we check that $(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | \alpha) \neq \frac{n}{2}(\alpha | \alpha)$ for $n \in \mathbb{N}$ and $\alpha = m\delta + \theta$, $m \in \mathbb{Z}_+$, $\alpha = m\delta - \theta$, $m \in \mathbb{N}$, $\alpha = m\delta + \frac{1}{2}\delta - \theta/2$, $m \in \mathbb{Z}_+$, $\alpha = m\delta + \frac{1}{2}\delta + \theta/2$, $m \in \mathbb{Z}_+$. Observe that

$$(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | m\delta - \theta) = (m-1)(k + h^\vee) \leq 0,$$

and

$$(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | m\delta + \theta) = m(k + h^\vee) + k + 1 + h^\vee - 1 = (m+1)(k + h^\vee) < 0.$$

Likewise

$$(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | m\delta + \tfrac{1}{2}(\delta - \theta)) = m(k + h^\vee) \leq 0,$$

and

$$(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | m\delta + \tfrac{1}{2}(\delta + \theta)) = m(k + h^\vee) + k + h^\vee = (m+1)(k + h^\vee) < 0.$$

□

We also need to check integrability of $L(\widehat{\nu}_s)$.

Lemma 9.8. *If $\nu \in P_k^+$ is not Ramond extremal then the $\widehat{\mathfrak{g}}^{\text{tw}}$ -module $L(\widehat{\nu}_s)$ is integrable for all $s \in \mathbb{C}$.*

Proof. A set of simple roots for $\widehat{\Delta}_+^\natural$ is $\widehat{\Pi}^\natural = \{\delta - \theta_i \mid i = 1, \dots, s\} \cup \Pi^\natural$.

If β is an odd isotropic root we let r_β denote the corresponding odd reflection (see [18] for details on odd reflections). First of all we observe that, if ν is not Ramond extremal, then

$$(9.16) \quad (\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | \alpha^\vee) \in \mathbb{N} \text{ for all } \alpha \in \widehat{\Pi}^\natural.$$

Indeed

$$(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | (\delta - \theta_i)^\vee) = ((k + h^\vee)\Lambda_0 + (s + \epsilon(\sigma_R)/4)\theta + (\nu - \rho_R) + \rho | (\delta - \theta_i)^\vee)$$

and, since $\nu - \rho_R$ is non-extremal, as in the proof of Lemma 11.4 of [15], it follows that

$$(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | (\delta - \theta_i)^\vee) = M_i(k) + \chi_i + 1 - (\nu - \rho_R | \theta_i^\vee) \in \mathbb{N}.$$

If $\alpha \in \Pi^\natural$, then

$$(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | \alpha^\vee) = (\nu - \rho_R + \rho^\natural | \alpha^\vee) \in \mathbb{N}.$$

Assume first that $\theta/2$ is not a root of $\mathfrak{g}$. Both $\alpha_0 = \delta - \theta$ and $\alpha_1 = -\frac{1}{2}\delta + \theta/2 + \eta_{\min}$ are in $\widehat{\Pi}$ and $\alpha_0 + \alpha_1 = \frac{1}{2}\delta - \theta/2 + \eta_{\min}$ is an odd isotropic root. Set $\widehat{\Pi}' = r_{\alpha_0 + \alpha_1} r_{\alpha_1} \widehat{\Pi}$ and let $(\widehat{\nu}_s)'$, $(\widehat{\rho}^{\text{tw}})'$ be the highest weight of $L(\widehat{\nu}_s)$ and the Weyl vector with respect to $\widehat{\Pi}'$.

We note that $\widehat{\Pi}^\natural \subset \widehat{\Pi} \cup \widehat{\Pi}'$, so, to check that $L(\widehat{\nu}_s)$ is integrable, it is enough to prove for each root in $\alpha = \widehat{\Pi}^\natural \cap \widehat{\Pi}$ that $(\widehat{\nu}_s | \alpha^\vee) \in \mathbb{Z}_+$ and, if $\alpha = \widehat{\Pi}^\natural \cap \widehat{\Pi}'$, that $(\widehat{\nu}_s)' | \alpha^\vee \in \mathbb{Z}_+$. If $(\widehat{\nu}_s | \alpha_1) \neq 0$ and $(\widehat{\nu}_s | \alpha_1 + \alpha_0) \neq 0$ this check is equivalent to (9.16).

If $(\widehat{\nu}_s|\alpha_1) = 0$ and $(\widehat{\nu}_s + \alpha_1|\alpha_0 + \alpha_1) \neq 0$ then $(\widehat{\nu}_s)' + (\widehat{\rho}^{\text{tw}})' = \widehat{\nu}_s + \widehat{\rho}^{\text{tw}} + \alpha_1$. If $(\widehat{\nu}_s|\alpha_1) \neq 0$ and $(\widehat{\nu}_s + \alpha_1|\alpha_0 + \alpha_1) = 0$ then $(\widehat{\nu}_s)' + (\widehat{\rho}^{\text{tw}})' = \widehat{\nu}_s + \widehat{\rho}^{\text{tw}} + \alpha_0 + \alpha_1$. If $(\widehat{\nu}_s|\alpha_1) = (\widehat{\nu}_s + \alpha_1|\alpha_0 + \alpha_1) = 0$ then $(\widehat{\nu}_s)' + (\widehat{\rho}^{\text{tw}})' = \widehat{\nu}_s + \widehat{\rho}^{\text{tw}} + 2\eta_{\min}$. In all cases $(\widehat{\nu}_s)' + (\widehat{\rho}^{\text{tw}})' = \widehat{\nu}_s + \widehat{\rho}^{\text{tw}} + \gamma$ with $\gamma|_{\mathfrak{h}^\natural} = p\eta_{\min}$, $p = 1$ or 2 .

If $\alpha \in \widehat{\Pi}^\natural \cap \widehat{\Pi}$ then, as computed above, $(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}}|\alpha^\vee) \in \mathbb{N}$. If $\alpha \in \widehat{\Pi}^\natural \cap \widehat{\Pi}'$, we have to check that

$$(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} + \gamma|\alpha^\vee) \in \mathbb{N}.$$

If $\alpha \in \Pi^\natural \cap \Pi'$, then, one checks that $(\eta_{\min}|\alpha^\vee) \geq 0$ hence

$$(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} + \gamma|\alpha^\vee) = (\nu - \rho_R + \rho^\natural + p\eta_{\min}|\alpha^\vee) \in \mathbb{N}.$$

If $\alpha = \delta - \theta_i \in \widehat{\Pi}' \setminus \widehat{\Pi}$ then $\mathfrak{g} = D(2, 1; a)$ and $(\gamma|(\delta - \theta_i)^\vee) = -p(\eta_{\min}|\theta_i^\vee) = p$ hence

$$(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} + \gamma|(\delta - \theta_i)^\vee) \in \mathbb{N}.$$

We now discuss the cases $\mathfrak{g} = \text{spo}(2|2r+1)$, $r > 1$ and $\mathfrak{g} = G(3)$. Set $\alpha_1 = -\frac{1}{2}\delta + \theta/2 + \eta_{\min}$ and $\widehat{\Pi}' = r_{\alpha_1}\widehat{\Pi}$. We note that $\widehat{\Pi}^\natural \subset \widehat{\Pi} \cup \widehat{\Pi}'$. If $(\widehat{\nu}_s|\alpha_1) \neq 0$, we can conclude using (9.16).

If $(\widehat{\nu}_s|\alpha_1) = 0$, we observe that $\eta_{\min}$ is the only element in $\widehat{\Pi}^\natural \setminus \widehat{\Pi}$ and

$$((\widehat{\nu}_s)' + (\widehat{\rho}^{\text{tw}})'|\eta_{\min}^\vee) = (\widehat{\nu}_s + \widehat{\rho}^{\text{tw}}|\eta_{\min}^\vee) + (\eta_{\min}|\eta_{\min}^\vee) \in \mathbb{N}.$$

It remains to discuss the case $\mathfrak{g} = \text{spo}(2|3)$. In this case $\widehat{\Pi}^\natural = \{\delta - \eta_{\min}, \eta_{\min}\}$. Set $\alpha_1 = -\frac{1}{2}\delta + \delta_1 + \eta_{\min}$, $\alpha_2 = \frac{1}{2}\delta + \delta_1 - \eta_{\min}$ and $\widehat{\Pi}' = r_{\alpha_1}\widehat{\Pi}$, $\widehat{\Pi}'' = r_{\alpha_2}\widehat{\Pi}$. We note that $\widehat{\Pi}^\natural \subset \widehat{\Pi}' \cup \widehat{\Pi}''$. If $(\widehat{\nu}_s|\alpha_i) \neq 0$ for $i = 1, 2$, then we can conclude using (9.16). If $(\widehat{\nu}_s|\alpha_1) = 0$ then $\widehat{\Pi}^\natural \cap \widehat{\Pi}' = \{\eta_{\min}\}$ and

$$((\widehat{\nu}_s)' + (\widehat{\rho}^{\text{tw}})'|\eta_{\min}^\vee) = (\widehat{\nu}_s + \widehat{\rho}^{\text{tw}}|\eta_{\min}^\vee) + (\eta_{\min}|\eta_{\min}^\vee) \in \mathbb{N}.$$

If $(\widehat{\nu}_s|\alpha_2) = 0$ then $\widehat{\Pi}^\natural \cap \widehat{\Pi}'' = \{\delta - \eta_{\min}\}$ and

$$((\widehat{\nu}_s)'' + (\widehat{\rho}^{\text{tw}})''|(\delta - \eta_{\min})^\vee) = (\widehat{\nu}_s + \widehat{\rho}^{\text{tw}}|(\delta - \eta_{\min})^\vee) - (\eta_{\min}|(\delta - \eta_{\min})^\vee) \in \mathbb{N}.$$

□

Lemma 9.9. *If $\nu \in P_k^+$ is Ramond extremal and s is such that $\ell(s) = A(k, \nu)$, then $L(\widehat{\nu}_s)$ is integrable.*

Proof. We argue as in Lemma 9.8, using the same notation. First we prove that

$$(9.17) \quad (\widehat{\nu}_s + \widehat{\rho}^{\text{tw}}|\alpha^\vee) \in \mathbb{N} \text{ for all } \alpha \in \widehat{\Pi}^\natural \cap \widehat{\Pi}.$$

If $\alpha \in \Pi^\natural$, then

$$(\widehat{\nu}_s|\alpha^\vee) = (\nu + \rho_R|\alpha^\vee) \in \mathbb{Z}_+,$$

so, if $\alpha \in \Pi^\natural \cap \widehat{\Pi}$, then (9.17) holds. Moreover,

$$(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}}|(\delta - \theta_i)^\vee) = M_i(k) + \chi_i + 1 - (\nu - \rho_R|\theta_i^\vee) = M_i(k) - (\nu|\theta_i^\vee) + (\rho_R|\theta_i^\vee).$$

Note that, if $(\delta - \theta_i) \in \widehat{\Pi}$, then $(\rho_R|\theta_i^\vee) > 0$ hence (9.17) holds for all $\alpha \in \widehat{\Pi}^\natural \cap \widehat{\Pi}$. Next, we check that

$$(9.18) \quad ((\widehat{\nu}_s)' + (\widehat{\rho}^{\text{tw}})'|\alpha^\vee) \in \mathbb{N} \text{ for all } \alpha \in \widehat{\Pi}^\natural \cap \widehat{\Pi}'.$$

Assume $\theta/2$ is not a root of $\mathfrak{g}$. Since $\ell(s) = A(k, \nu)$, (9.8) holds. Moreover,

$$\begin{aligned}
 (9.19) \quad & (\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | \pm (\tfrac{1}{2}\delta - \theta/2) + \eta_{\min}) \\
 &= ((k + h^\vee)\Lambda_0 + (\tfrac{k+1}{2} \pm (\nu - \rho_R + \rho^\natural | \eta_{\min}))\theta + \nu + \rho - \rho_R | \pm (\tfrac{1}{2}\delta - \theta/2) + \eta_{\min}) \\
 &= \pm (\tfrac{1}{2}(k + h^\vee) - \tfrac{k+1}{2} \pm (\nu - \rho_R + \rho^\natural | \eta_{\min}) - \tfrac{1}{2}(h^\vee - 1)) + (\nu - \rho_R + \rho^\natural | \eta_{\min}) \\
 &= (\nu - \rho_R + \rho^\natural | \eta_{\min}) \pm (\nu - \rho_R + \rho^\natural | \eta_{\min}).
 \end{aligned}$$

The outcome of this computation is that, if $(\nu - \rho_R + \rho^\natural | \eta_{\min}) \neq 0$ and $s = \frac{k+1}{2} \pm (\nu - \rho_R + \rho^\natural | \eta_{\min})$, then

$$(\widehat{\nu}_s)' + (\widehat{\rho}^{\text{tw}})' = \widehat{\nu}_s + \widehat{\rho}^{\text{tw}} \pm (\tfrac{1}{2}\delta - \theta/2) + \eta_{\min},$$

while, if $(\nu - \rho_R + \rho^\natural | \eta_{\min}) = 0$,

$$(\widehat{\nu}_s)' + (\widehat{\rho}^{\text{tw}})' = \widehat{\nu}_s + \widehat{\rho}^{\text{tw}} + 2\eta_{\min}.$$

If $\alpha \in \Pi^\natural \cap \Pi'$, then one checks that $(\eta_{\min} | \alpha^\vee) \geq 0$ so in all cases

$$((\widehat{\nu}_s)' + (\widehat{\rho}^{\text{tw}})' | \alpha^\vee) = (\nu - \rho_R + \rho^\natural + p\eta_{\min} | \alpha^\vee) \geq (\nu - \rho_R + \rho^\natural + \eta_{\min} | \alpha^\vee) = (\widehat{\nu}_s + (\widehat{\rho}^{\text{tw}})' | \alpha^\vee)$$

hence $((\widehat{\nu}_s)' + (\widehat{\rho}^{\text{tw}})' | \alpha^\vee) \geq (\nu + \rho_R | \alpha^\vee) + 1 \in \mathbb{N}$.

If $\delta - \theta_i \in \widehat{\Pi}' \setminus \widehat{\Pi}$ then $\mathfrak{g} = D(2, 1; a)$ and $(\eta_{\min} | \theta_i^\vee) = -1$, so

$$\begin{aligned}
 ((\widehat{\nu}_s)' + (\widehat{\rho}^{\text{tw}})' | (\delta - \theta_i)^\vee) &= M_i(k) + \chi_i + 1 - (\nu - \rho_R + p\eta_{\min} | \theta_i^\vee) \\
 &\geq M_i(k) - (\nu | \theta_i^\vee) + (\rho_R | \theta_i^\vee) + 1 \in \mathbb{N}.
 \end{aligned}$$

This completes the proof in the cases when $\theta/2$ is not a root of $\mathfrak{g}$.

We now discuss the cases $\mathfrak{g} = \text{spo}(2|2r+1)$, $r > 1$, and $\mathfrak{g} = G(3)$. As in the previous cases we need only to check that $((\widehat{\nu}_s)' + (\widehat{\rho}^{\text{tw}})' | \alpha^\vee) \in \mathbb{N}$ for all $\alpha \in \widehat{\Pi}^\natural \cap \widehat{\Pi}'$. Since $\ell(s) = A(k, \nu)$, by Lemma 9.4, $s = \frac{2k+1}{4}$. By Proposition 12.3 below

$$(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | \alpha_1) = 0.$$

It follows that $(\widehat{\nu}_s)' + (\widehat{\rho}^{\text{tw}})' = \widehat{\nu}_s + \widehat{\rho}^{\text{tw}} + \alpha_1$. If $\alpha \in \widehat{\Pi}^\natural \setminus \widehat{\Pi}$, then $\alpha = \eta_{\min}$, so

$$((\widehat{\nu}_s)' + (\widehat{\rho}^{\text{tw}})' | \alpha^\vee) = (\nu - \rho_R + \rho^\natural + \eta_{\min} | \eta_{\min}^\vee) = (\widehat{\nu}_s + (\widehat{\rho}^{\text{tw}})' | \eta_{\min}^\vee) = (\nu + \rho_R | \eta_{\min}^\vee) + 1 \in \mathbb{N}.$$

We finally discuss the case $\mathfrak{g} = \text{spo}(2|3)$. Since $\ell(s) = A(k, \nu)$, by Lemma 9.4, $s = \frac{2k+1}{4}$. If $\nu = 0$, then $(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | \alpha_1) = 0$ and

$$(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | \alpha_2) = ((k + h^\vee)\Lambda_0 + \tfrac{k+1}{2}\theta + \rho - \rho_R | \tfrac{1}{2}\delta + \theta/2 - \eta_{\min}) = k + h^\vee \neq 0,$$

so

$$((\widehat{\nu}_s)' + (\widehat{\rho}^{\text{tw}})' | \eta_{\min}^\vee) = (\eta_{\min} | \eta_{\min}^\vee) = 2,$$

while

$$((\widehat{\nu}_s)'' + (\widehat{\rho}^{\text{tw}})'' | (\delta - \eta_{\min})^\vee) = ((\widehat{\nu}_s) + (\widehat{\rho}^{\text{tw}}) | (\delta - \eta_{\min})^\vee) = M_1(k) - 1 + (\rho_R | \eta_{\min}^\vee) = M_1(k) \in \mathbb{N}.$$

If instead $\nu = \frac{M_1(k)}{2}\eta_{\min}$, then $(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | \alpha_1) = 0$ and

$$(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}} | \alpha_1) = ((k + h^\vee)\Lambda_0 + \tfrac{k+1}{2}\theta + \nu + \rho - \rho_R | -\tfrac{1}{2}\delta + \theta/2 + \eta_{\min}) = k + h^\vee \neq 0,$$

so

$$((\widehat{\nu}_s)' + (\widehat{\rho}^{\text{tw}})' | \eta_{\min}^\vee) = ((\widehat{\nu}_s) + (\widehat{\rho}^{\text{tw}}) | \eta_{\min}^\vee) = M_1(k) \in \mathbb{N},$$

and

$$((\widehat{\nu}_s)'' + (\widehat{\rho}^{\text{tw}})'' | (\delta - \eta_{\min})^\vee) = ((\widehat{\nu}_s) + (\widehat{\rho}^{\text{tw}}) + \alpha_2 | (\delta - \eta_{\min})^\vee) = (\eta_{\min} | \eta_{\min}^\vee) \in \mathbb{N}.$$

□

Corollary 9.10. (a) If ν is not Ramond extremal, then $H(L(\widehat{\nu}_s))$ is a Ramond twisted $W_k^{\min}(\mathfrak{g})$ -module for all $s \in \mathbb{C}$.

(b) If ν is Ramond extremal and $\ell(s) = A(k, \nu)$, then $H(L(\widehat{\nu}_s))$ is a Ramond twisted $W_k^{\min}(\mathfrak{g})$ -module.

Proof. Combining Lemmas 9.8 and 9.9 with [12, Theorem 5.3.1], we see that the modules $L(\widehat{\nu}_s)$, described in the statement, are σ_R -twisted $V_k(\mathfrak{g})$ -modules, so their quantum Hamiltonian reduction gives σ_R -twisted modules for $H(V_k(\mathfrak{g})) = W_k^{\min}(\mathfrak{g})$. $\square$

It was conjectured in [16] that, for an admissible (in the sense of [16]) highest weight $V^k(\mathfrak{g})$ -module $L(\widehat{\Lambda})$, either $H(L(\widehat{\Lambda}))$ is irreducible or 0, and the latter happens iff $\widehat{\Lambda}$ is degenerate. This conjecture was subsequently proved by Arakawa [3]. We used this result in [15] to compute character formulas for irreducible unitary highest weight modules over $W_{\min}^k(\mathfrak{g})$.

We would like to use the same approach for Ramond twisted irreducible highest weight modules but, for that, we need the following Conjecture, which is a “twisted” analogue of that in [16].

Conjecture 9.11.

- a) If M is in category $\mathcal{O}$ of $\widehat{\mathfrak{g}}^{\text{tw}}$ -modules, then $H_j(M) = 0$ if $j \neq 0$.
- b) Assume that $\widehat{\nu}_s$ is nondegenerate so that $H(L(\widehat{\nu}_s))$ is nonzero. Then
 - If $\theta/2$ is not a root of $\mathfrak{g}$, then $H_0(L(\widehat{\nu}_s))$ is irreducible.
 - If $\theta/2$ is a root of $\mathfrak{g}$, then $H_0(L(\widehat{\nu}_s))$ is irreducible, or a direct sum of two irreducible $W_{\min}^k(\mathfrak{g})$ -modules. In the last case $\ell(s) = A(k, \nu)$ and the second module is isomorphic to the first with the opposite parity.

Some evidence towards this conjecture is provided in Section 13: see Remark 13.2. Consequences of Conjecture 9.11 are the following results.

Proposition 9.12. Assume Conjecture 9.11 and that $\widehat{\nu}_s$ is non-degenerate.

If $\ell(s) \neq A(k, \nu)$, then $H_0(L(\widehat{\nu}_s)) = L^W(\nu, \ell(s))$ and, if $\ell(s) = A(k, \nu)$, then $H_0(L(\widehat{\nu}_s)) = L^W(\nu, \ell(s)) \oplus \epsilon(\sigma_R)\Pi L^W(\nu, \ell(s))$, where Π is the reversal of parity functor.

Proof. By Conjecture 9.11 a), the functor H is equal to H_0 and it is exact. It follows that $H(L(\widehat{\nu}_s))$ is a quotient of $H(M(\widehat{\nu}_s))$, hence, by Proposition 4.3, $H(L(\widehat{\nu}_s))$ is a quotient of $M^W(\nu, \ell(s))$. Since, by hypothesis, $\widehat{\nu}_s$ is non-degenerate, $H(L(\widehat{\nu}_s))$ is non-zero. By Conjecture 9.11 b) any direct summand of $H(L(\widehat{\nu}_s))$ is a highest weight module of highest weight $(\nu, \ell(s))$. $\square$

Now we can prove character formulas in the Ramond sector.

Theorem 9.13. Assume Conjecture 9.11. Let k be in the unitary range and let $\nu \in P_k^+$ be a weight which is not Ramond extremal. Fix $\ell \geq A(k, \nu)$ and, if $\theta/2$ is not a root of $\mathfrak{g}$, assume $\ell \neq A(k, \nu)$. Choose s such that $\ell = \ell(s)$ and, if $\ell < B(k, \nu, \rho_R)$, assume that s satisfies (9.10). Then, if $\ell > A(k, \nu)$, we have

$$\begin{aligned}
 & \widehat{F}^R \text{ch } L^W(\nu, \ell) \\
 (9.20) \quad &= q^{\frac{(\widehat{\nu}_s | \widehat{\nu}_s + 2\widehat{\rho}^{\text{tw}})}{2(k+h^\vee)} + a(k)} e^{-\rho_R} \sum_{w \in \widehat{W}^{\natural}} \det(w) q^{-(w(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}}) - \widehat{\rho}^{\text{tw}})(x+D)} e^{(w(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}}) - \widehat{\rho}^{\text{tw}})_{|\mathfrak{h}^{\natural}}},
 \end{aligned}$$

while, if $\theta/2$ is a root of $\mathfrak{g}$ and $\ell = A(k, \nu)$, we have

$$(9.21) \quad \begin{aligned} & \widehat{F}^{Rch} L^W(\nu, \ell) \\ &= \frac{1}{2} q^{\frac{(\widehat{\nu}_s | \widehat{\nu}_s + 2\widehat{\rho}^{tw})}{2(k+h^\vee)} + a(k)} e^{-\rho_R} \sum_{w \in \widehat{W}^\natural} \det(w) q^{-(w(\widehat{\nu}_s + \widehat{\rho}^{tw}) - \widehat{\rho}^{tw})(x+D)} e^{(w(\widehat{\nu}_s + \widehat{\rho}^{tw}) - \widehat{\rho}^{tw})|_{\mathfrak{h}^\natural}}. \end{aligned}$$

where $a(k)$ is given by (8.4).

Proof. By Lemma 9.8, $L(\widehat{\nu}_s)$ is integrable. We want to apply Proposition 8.5, so we need only to check that (8.14) holds.

We start by checking that (8.14) holds. If $\ell > B(k, \nu, \rho_R)$, then $\ell = \ell(s)$ with $s = \frac{k+1}{2} + \sqrt{-1}t_0$ with $t_0 \neq 0$. It is then obvious that (8.14) holds since the left hand side of (8.14) is not real. If instead $\theta/2$ is not a root of $\mathfrak{g}$ and $A(k, \nu) < \ell \leq B(k, \nu, \rho_R)$, then (8.14) holds in this case by Lemma 9.6, and, if $\theta/2$ is a root and $\ell = A(k, \nu)$, (8.14) holds by Lemma 9.7.

By Proposition 9.12 and Conjecture 9.11, we have that, if $\ell > A(k, \nu)$,

$$(9.22) \quad \sum_j (-1)^j H_j(L(\widehat{\nu}_s)) = H_0(L(\widehat{\nu}_s)) = L^W(\nu, \ell(s)),$$

while, if $\theta/2$ is a root of $\mathfrak{g}$ and $\ell = A(k, \nu)$,

$$(9.23) \quad \sum_j (-1)^j H_j(L(\widehat{\nu}_s)) = H_0(L(\widehat{\nu}_s)) = L^W(\nu, \ell(s)) \oplus L^W(\nu, \ell(s)).$$

Applying Proposition 8.5, equalities (9.20) and (9.21) now follow from (9.22), (9.23). $\square$

We finish this section explaining how Theorem 9.13 implies unitarity of $L^W(\nu, \ell)$ for ν that is not Ramond extremal and $\ell \geq A(k, \nu)$. For this we need a free version of the modules $N(\mu, \nu)$. This is constructed as follows. Let y be an indeterminate. Define an action of the abelian Lie algebra $\mathcal{H}_0 = \mathbb{C}a + \mathbb{C}K$ on $\mathbb{C}[y]$ by letting K act as the identity and a (cf. (7.11)) act by multiplication by y . Extend this action to $\mathbb{C}[t] \otimes \mathbb{C}a$ by letting at^j act trivially. Let $M(y)$ be the corresponding induced module to the affine algebra $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}a \oplus \mathbb{C}K$. This module can be regarded as a $V^1(\mathbb{C}a)$ -module by means of the field $Y(a, z)$ defined by setting, for $m \in M(y)$,

$$Y(a, z)m = \sum_{j \in \mathbb{Z}} (t^j \otimes a) \cdot m z^{-j-1}.$$

Set $M := M(y) \otimes L(\nu) \otimes F(\mathfrak{g}_{1/2}, \sigma_R)$ and define

$$(9.24) \quad \widetilde{N}(y, \nu) = \Psi(W_{\min}^k(\mathfrak{g})) \cdot (1 \otimes \mathbb{C}[y] \otimes v_\nu \otimes 1) \hookrightarrow M.$$

Since $M(y) \otimes L(\nu) \otimes F(\mathfrak{g}_{1/2}, \sigma_R)$ is free as a $\mathbb{C}[y]$ -module, $\widetilde{N}(y, \nu)$ is also free. If $\mu \in \mathbb{C}$, set also

$$(9.25) \quad \widetilde{N}(\mu, \nu) = (\mathbb{C}[y]/(y - \mu)) \otimes_{\mathbb{C}[y]} \widetilde{N}(y, \nu).$$

By construction $\widetilde{N}(\mu, \nu)$ is clearly a σ_R -twisted highest weight module for $W_{\min}^k(\mathfrak{g})$. Recall from (7.14), (7.15) the definitions of $N(\mu, \nu)$ and $\ell_0(\mu, \nu)$. The embedding (9.24) defines a map $\psi : \widetilde{N}(\mu, \nu) \rightarrow (\mathbb{C}[y]/(y - \mu)) \otimes_{\mathbb{C}[y]} M = M(1, \mu) \otimes L(\nu) \otimes F(\mathfrak{g}_{1/2}, \sigma_R)$ whose image is $N(\mu, \nu)$.

Proposition 9.14. *There is a countable set $\mathcal{M} \subset \mathbb{C}$ such that, for $\mu \in \mathbb{C} \setminus \mathcal{M}$, the map ψ defines an isomorphism between $\widetilde{N}(\mu, \nu)$ and $N(\mu, \nu)$.*

Proof. Given $\lambda(y) \in \mathbb{C}[y]$, set $M_{\lambda(y)} = \{m \in M \mid L_0 m = \lambda(y)m\}$ and $M_n = M_{\ell_0(y, \nu) + n}$. We observe that

$$(1) \quad M = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_+} M_n;$$

$$(2) \quad M_n = M_{\ell_0(y, \nu) + n} \text{ is a } \mathbb{C}[y]\text{-module of finite rank.}$$

Let $\{m_1^{(n)}, \dots, m_{i_n}^{(n)}\}$ be a basis of M_n over $\mathbb{C}[y]$. Consider $\tilde{N}(y, \mu) \cap M_n$: it is a free module over $\mathbb{C}[y]$. Fix a basis $\mathcal{V} = \{v_1, \dots, v_r\}$, so that $v_i = \sum_{j=1}^{i_n} p_{ij}(y) m_j^{(n)}$. Then the $r \times i_n$ matrix $(p_{ij}(\mu))$ has rank less than r for a finite number of values of μ : otherwise the matrix with polynomial entries $(p_{ij}(y))$ has rank less than r , against the fact that $\mathcal{V}$ is a basis. Let E_n be the set of these values, and define $\mathcal{M} = \bigcup_n E_n$. We now show that ψ is an isomorphism outside $\mathcal{M}$. Set $\tilde{N}(\mu, \nu)_n = \{v \in N(\mu, \nu) \mid L_0 v = (\ell_0(\mu, \nu) + n)v\}$. Then $\tilde{N}(\mu, \nu) = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_+} \tilde{N}(\mu, \nu)_n$, and the previous argument shows that if $\mu \notin \mathcal{M}$, then $\psi|_{\tilde{N}(\mu, \nu)_n}$ is injective, hence an isomorphism onto its image. $\square$

Lemma 9.15. *Set $\mu(s) = \sqrt{-1} \frac{\sqrt{2}s}{\sqrt{|k+h^\vee|}}$. Then*

$$\ell_0(\mu(s), \nu - \rho_R) = \ell(s),$$

where $\ell(s)$ is defined by (9.2).

In particular, the module $L^W(\nu, \ell(s))$ is the irreducible quotient of $N(\mu(s), \nu - \rho_R)$.

Proof. It follows from (7.15) that

$$\ell_0(\mu, \nu - \rho_R) - B(k, \nu, \rho_R) = \frac{\mu^2}{2} - s_k \mu + \frac{(k+1)^2}{4(k+h^\vee)} = \frac{1}{2}(\mu - s_k)^2 = \frac{1}{2}(\mu - \sqrt{-1} \frac{(k+1)}{\sqrt{2|k+h^\vee|}})^2.$$

On the other hand

$$\ell(s) - B(k, \nu, \rho_R) = \frac{(s - \frac{k+1}{2})^2}{k + h^\vee},$$

so that

$$\frac{1}{\sqrt{2}}\mu = \sqrt{-1} \frac{(k+1)}{2\sqrt{|k+h^\vee|}} \pm (\sqrt{-1} \frac{s - \frac{k+1}{2}}{\sqrt{|k+h^\vee|}}).$$

Choosing the plus sign in the previous formula we have the claim. $\square$

We note that

$$(9.26) \quad \tilde{N}(\mu, \nu) = \bigoplus_{\lambda \in \mathfrak{h}^\natural, n \in \frac{1}{2}\mathbb{Z}_+} \tilde{N}(\mu, \nu)_{(\lambda, \ell_0(\mu, \nu) + n)},$$

where $\tilde{N}(\mu, \nu)_{(\lambda, \ell)} = \{m \in \tilde{N}(\mu, \nu) \mid h_{(0)} m = \lambda(h)m, h \in \mathfrak{h}^\natural, L_0 m = \ell m\}$. From (9.26) we deduce that

$$ch \tilde{N}(\mu, \nu) = \sum_{\lambda \in \mathfrak{h}^\natural, n \in \frac{1}{2}\mathbb{Z}_+} \dim \tilde{N}(\mu, \nu)_{(\lambda, \ell_0(\mu, \nu) + n)} q^{\ell_0(\mu, \nu) + n} e^\lambda.$$

We have already noticed that $\tilde{N}(\mu, \nu)$ is free in the variable μ , i.e. $\dim \tilde{N}(\mu, \nu)_{(\lambda, \ell_0(\mu, \nu) + n)}$ does not depend on μ . Then

$$(9.27) \quad q^{-\ell_0(\mu, \nu)} ch \tilde{N}(\mu, \nu) = \sum_{\lambda \in \mathfrak{h}^\natural, n \in \frac{1}{2}\mathbb{Z}_+} (\dim \tilde{N}(\mu, \nu)_{(\lambda, \ell_0(\mu, \nu) + n)}) q^n e^\lambda$$

does not depend on μ .

Proposition 9.16. *Assume Conjecture 9.11. Given $\ell > A(k, \nu)$ choose $s \in \mathbb{C}$ such that $\ell = \ell(s)$ and, if $A(k, \nu) < \ell \leq B(k, \nu, \rho_R)$, choose s in the interval (9.10).*

Then

$$\tilde{N}(\mu(s), \nu - \rho_R) = L^W(\nu, \ell(s)).$$

Proof. Put $s_0 = \frac{k+1}{2} + \sqrt{-1}t_0$, $t_0 \in \mathbb{R}$. By Proposition 9.14, there exists $t_0 \neq 0$ such that $N(\mu(s_0), \nu - \rho_R) = \tilde{N}(\mu(s_0), \nu - \rho_R)$. Since $N(\mu(s_0), \nu - \rho_R)$ is unitary, it is irreducible. Hence $\tilde{N}(\mu(s_0), \nu - \rho_R) = L^W(\nu, \ell(s_0))$. In particular, by Theorem 9.13 (which uses Conjecture 9.11),

$$\widehat{F}^R ch \tilde{N}(\mu(s_0), \nu - \rho_R) = q^{\frac{(\widehat{\nu}_{s_0}|\widehat{\nu}_{s_0} + 2\widehat{\rho}^{tw})}{2(k+h^\vee)} + a(k)} e^{-\rho_R} \sum_{w \in \widehat{W}^\natural} \det(w) e^{ev(w(\widehat{\nu}_{s_0} + \widehat{\rho}^{tw}) - \widehat{\rho}^{tw})}.$$

We now prove that

$$(9.28) \quad \widehat{F}^R ch \tilde{N}(\mu(s), \nu - \rho_R) = q^{\frac{(\widehat{\nu}_s|\widehat{\nu}_s + 2\widehat{\rho}^{tw})}{2(k+h^\vee)} + a(k)} e^{-\rho_R} \sum_{w \in \widehat{W}^\natural} \det(w) e^{ev(w(\widehat{\nu}_s + \widehat{\rho}^{tw}) - \widehat{\rho}^{tw})}$$

for any $s \in \mathbb{C}$.

Since $w(x) = x$ for all $w \in \widehat{W}^\natural$,

$$e^{ev(w(\widehat{\nu}_s + \widehat{\rho}^{tw}) - \widehat{\rho}^{tw})} = q^{s_0 - s} e^{ev(w(\widehat{\nu}_{s_0} + \widehat{\rho}^{tw}) - \widehat{\rho}^{tw})},$$

so the RHS of (9.28) is

$$q^{s_0 - s} q^{\frac{(\widehat{\nu}_{s_0}|\widehat{\nu}_{s_0} + 2\widehat{\rho}^{tw})}{2(k+h^\vee)} + a(k)} e^{-\rho_R} \sum_{w \in \widehat{W}^\natural} \det(w) \frac{e^{ev(w(\widehat{\nu}_{s_0} + \widehat{\rho}^{tw}) - \widehat{\rho}^{tw})}}{\widehat{F}^R} = q^{\ell(s) - \ell(s_0)} ch \tilde{N}(\mu(s_0), \nu - \rho_R).$$

Since, by (9.27), $q^{-\ell_0(\mu, \nu - \rho_R)} ch \tilde{N}(\mu, \nu - \rho_R)$ does not depend on μ , we obtain that

$$\begin{aligned} q^{\ell(s) - \ell(s_0)} ch \tilde{N}(\mu(s_0), \nu - \rho_R) &= q^{\ell_0(\mu(s), \nu - \rho_R)} q^{-\ell_0(\mu(s_0), \nu - \rho_R)} ch \tilde{N}(\mu(s_0), \nu - \rho_R) \\ &= ch \tilde{N}(\mu(s), \nu - \rho_R), \end{aligned}$$

as claimed. Now, if s lies in (9.10), by Theorem 9.13 (here we use Conjecture 9.11 again), we have

$$(9.29) \quad ch \tilde{N}(\mu(s), \nu - \rho_R) = ch L^W(\nu, \ell(s)).$$

Since $L^W(\nu, \ell(s))$ is a quotient of $\tilde{N}(\mu(s), \nu - \rho_R)$, (9.29) implies that they are isomorphic. $\square$

The same proof of Theorem 11.1 of [15] provides the following extension of Corollary 9.5:

Theorem 9.17. *Assume Conjecture 9.11. If $\ell \geq A(k, \nu)$, k is in the unitary range, and $\nu \in P_k^+$ is not Ramond extremal, then $L^W(\nu, \ell)$ is a unitary σ_R -twisted $W_{\min}^k(\mathfrak{g})$ -module.*

Proof. We can assume that $\theta/2$ is not a root of $\mathfrak{g}$. For each weight (λ, m) , fix a basis of $\tilde{N}(\mu, \nu - \rho_R)_{(\lambda, m)}$ independent from μ . Define $\det_{(\lambda, m)}(\mu)$ to be the determinant of the matrix of the Hermitian invariant form H in this basis. We have seen in the proof of Proposition 9.16 that there is μ_0 with $\ell_0(\mu_0) > A(k, \nu)$ such that $\tilde{N}(\mu_0, \nu - \rho_R)$ is unitary. By Proposition 9.16, which uses Conjecture 9.11, $\tilde{N}(\mu, \nu - \rho_R) = L^W(\ell_0(\mu), \nu)$ if $\ell_0(\mu) > A(k, \nu)$, hence $\det_{(\lambda, m)}(\mu) \neq 0$ if $\ell_0(\mu) > A(k, \nu)$, thus the Hermitian invariant form H remains positive definite for $\ell_0(\mu) > A(k, \nu)$ and it is positive semidefinite if $\ell_0(\mu) = A(k, \nu)$. $\square$

10. EXPLICIT CONDITIONS FOR UNITARITY FOR RAMOND TWISTED MODULES $L^W(\nu, \ell)$

In Section 6 we found necessary conditions of unitarity of σ_R -twisted $W_{\min}^k(\mathfrak{g})$ -modules $L^W(\nu, \ell)$, where k is in the unitarity range, see Theorem 6.10.

We conjecture that all these modules are unitary. There are three types of these modules satisfying the necessary conditions of unitarity:

- (1) the modules $L^W(\nu, \ell)$ with $\nu \in P_k^+$ not Ramond extremal and $\ell \geq B(k, \nu, \rho_R)$; we proved in Section 7 that these modules are indeed unitary;
- (2) if $\theta/2$ is not a root of $\mathfrak{g}$, the modules $L^W(\nu, \ell)$ with $\nu \in P_k^+$ not Ramond extremal and $A(k, \nu) \leq \ell < B(k, \nu, \rho_R)$; we proved in Section 9 that these modules are unitary assuming Conjecture 9.11;
- (3) the modules $L^W(\nu, \ell)$ with the weight $\nu \in P_k^+$ Ramond extremal in which case $\ell = A(k, \nu)$ (by Lemma 6.9); we don't know how to establish unitarity in this case.

Below, for each $\mathfrak{g}$ from Table 4, we make explicit the necessary conditions of Theorem 6.10 and the sufficient conditions of Section 9.

10.1. $\mathfrak{g} = psl(2|2)$. In this case

$$\mathfrak{g}^\natural = sl(2), \quad M_1(k) = -k - 1, \quad P_k^+ = \{\nu = r\theta_1/2 \mid r \in \mathbb{Z}_+, \quad 0 \leq r \leq M_1(k)\}.$$

- The weight $\nu = 0$ is the only Ramond extremal weight. The necessary conditions for unitarity for $L^W(0, \ell)$ are $M_1(k) \in \mathbb{Z}_+$ and $\ell = -\frac{k+1}{4}$.
- If ν is not Ramond extremal then $\nu = \frac{r}{2}\theta_1$, $1 \leq r \leq M_1(k)$ and the necessary conditions for unitarity are $M_1(k) \in \mathbb{Z}_+$ and

$$(10.1) \quad \ell \geq -\frac{k+1}{4}.$$

The sufficient conditions for unitarity are $M_1(k) \in \mathbb{Z}_+$ and

$$\ell \geq -\frac{k^2 + k + r^2}{4k}.$$

Condition (10.1) is also sufficient assuming Conjecture 9.11.

The inequality $\ell \geq \frac{M_1(k)}{4}$ is precisely the bound stated in [9].

10.2. $\mathfrak{g} = spo(2|3)$. In this case

$$\mathfrak{g}^\natural = sl(2), \quad M_1(k) = -4k - 2, \quad P_k^+ = \{\nu = r\theta_1/2 \mid r \in \mathbb{Z}_+, \quad 0 \leq r \leq M_1(k)\}.$$

The Ramond extremal weights are $\nu = 0$ and $\nu = \frac{M_1(k)}{2}\theta_1$.

- The necessary conditions for unitarity for $L^W(\nu, \ell)$ with $\nu = r\theta_1/2$ Ramond extremal weight are $M_1(k) \in \mathbb{Z}_+$ and

$$\ell = -\frac{8k^2 + 10k + 2r^2 + 3}{32k + 16}.$$

- If ν is not Ramond extremal, then $\nu = \frac{r}{2}\theta_1$, $1 \leq r < M_1(k)$ and the necessary and sufficient conditions for unitarity are $M_1(k) \in \mathbb{N}$ and

$$\ell \geq -\frac{8k^2 + 10k + 2r^2 + 3}{32k + 16}.$$

In terms of $M_1(k)$, the inequality reads

$$(10.2) \quad \ell \geq \frac{(M_1(k) - 1)}{16} + \frac{r^2}{4M_1(k)},$$

which is precisely the bound stated in [23, 2.3.11].

10.3. $\mathfrak{g} = \mathfrak{spo}(2|2r)$, $r > 2$. In this case

$$\mathfrak{g}^\natural = \mathfrak{so}(2r), \quad M_1(k) = -2k - 1,$$

$$P_k^+ = \{\nu = \sum_i m_i \epsilon_i, m_i \in \frac{1}{2} + \mathbb{Z} \text{ or } m_i \in \mathbb{Z}, m_1 \geq \dots \geq m_{r-1} \geq |m_r|, m_1 + m_2 \leq M_1(k)\},$$

and

$$\begin{aligned} A(k, \nu) &= \frac{-4 \left(\sum_{i=1}^{r-1} (2(r-i) - 1) m_i + m_i^2 \right) - 4k^2 + 2(r-4)k + r - 3}{16(k + 2 - r)} \\ &= - \frac{\left(\sum_{i=1}^{r-1} (2(r-i) - 1) m_i + m_i^2 \right) + p(k)}{4(k + h^\vee)}. \end{aligned}$$

If $\eta_{\min} = \epsilon_r$, the Ramond extremal weights are the weights ν such that $m_r = -m_{r-1}$

- If ν is Ramond extremal, the necessary conditions for unitarity are $M_1(k) \in \mathbb{Z}_+$ and $\ell = A(k, \nu)$.
- If ν is not Ramond extremal, the necessary condition for unitarity are $M_1(k) \in \mathbb{Z}_+$ and

$$(10.3) \quad \ell \geq A(k, \nu).$$

The sufficient conditions are $M_1(k) \in \mathbb{Z}_+$ and

$$\ell \geq B(k, \nu, \rho_R) = - \frac{(\sum_{i=1}^r (2(r-i) - 1) m_i + m_i^2) + p(k)}{4(k + h^\vee)}.$$

The conditions (10.3) are also sufficient assuming Conjecture 9.11.

If $\eta_{\min} = -\epsilon_r$, the Ramond extremal weights are the weights ν such that $m_r = m_{r-1}$.

- If ν is Ramond extremal, the necessary conditions for unitarity are $M_1(k) \in \mathbb{Z}_+$ and $\ell = A(k, \nu)$.
- If ν is not Ramond extremal, the necessary conditions for unitarity are $M_1(k) \in \mathbb{Z}_+$ and

$$(10.4) \quad \ell \geq A(k, \nu).$$

The sufficient conditions are

$$\ell \geq B(k, \nu, \rho_R) = - \frac{(\sum_{i=1}^r (|2(r-i) - 1|) m_i + m_i^2) + p(k)}{4(k + h^\vee)}.$$

The conditions (10.4) are also sufficient assuming Conjecture 9.11.

10.4. $\mathfrak{g} = \mathfrak{spo}(2|2r+1)$, $r > 1$. In this case

$$\mathfrak{g}^{\natural} = \mathfrak{so}(2r+1), \quad M_1(k) = -2k-1,$$

$$P_k^+ = \{\nu = \sum_i m_i \epsilon_i, \quad m_i \in \frac{1}{2}\mathbb{Z} \text{ or } m_i \in \mathbb{Z}, \quad m_1 \geq \dots \geq m_{r-1} \geq m_r \geq 0, \quad m_1 + m_2 \leq M_1(k)\}.$$

and

$$\begin{aligned} A(k, \nu) &= \frac{-8(\sum_{i=1}^r (2(r-i+1)m_i + m_i^2) - 8k^2 + (4r-14)k + 2r-5)}{32(k+3/2-r)} \\ &= -\frac{(\sum_{i=1}^r (2(r-i)+1)m_i + m_i^2) + p(k)}{4(k+h^\vee)}. \end{aligned}$$

The Ramond extremal weights are the weights ν such that $m_r = 0$.

- If ν is Ramond extremal, the necessary conditions for unitarity are $M_1(k) \in \mathbb{Z}_+$ and $\ell = A(k, \nu)$.
- If ν is not Ramond extremal, the necessary and sufficient conditions for unitarity are $M_1(k) \in \mathbb{Z}_+$ and $\ell \geq A(k, \nu)$.

10.5. $\mathfrak{g} = D(2, 1; \frac{m}{n})$, $m, n \in \mathbb{N}$, m, n coprime. In this case

$$\mathfrak{g}^{\natural} = \mathfrak{g}_1^{\natural} \oplus \mathfrak{g}_2^{\natural}, \quad \mathfrak{g}_i^{\natural} \simeq \mathfrak{sl}(2), \quad M_1(k) = -\frac{m+n}{n}k-1, \quad M_2(k) = -\frac{m+n}{m}k-1,$$

$$P_k^+ = \{\nu = \frac{r_1}{2}\theta_1 + \frac{r_2}{2}\theta_2 \mid r_i \in \mathbb{Z}_+, r_i \leq M_i(k)\},$$

and

$$\begin{aligned} A(k, \nu) &= -\frac{(\frac{m}{n}+1)^2 k(k+1) + \frac{m}{n}((r_1+r_2+1)^2)}{4(\frac{m}{n}+1)^2 k} \\ &= -\frac{\frac{mn}{(m+n)^2}((r_1+r_2)^2 + 2(r_1+r_2)) + p(k)}{4(k+h^\vee)}. \end{aligned}$$

If $\eta_{\min} = \epsilon_2 - \epsilon_3$, the Ramond extremal weights are the weights ν such that $r_1 = 0$ or $r_2 = M_2(k)$.

- If ν is Ramond extremal, the necessary conditions for unitarity are

$$(M_1(k), M_2(k)) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \text{ and } \ell = A(k, \nu).$$

- If ν is not Ramond extremal, the necessary conditions for unitarity are

$$(M_1(k), M_2(k)) \in \mathbb{Z}_+ \times \mathbb{Z}_+$$

and

$$(10.5) \quad \ell \geq A(k, \nu).$$

The sufficient conditions are

$$\ell \geq B(k, \nu, \rho_R) = -\frac{k+1}{4} + \frac{m(r_2+1)^2 + nr_1^2}{4(m+n)k}.$$

The conditions (10.5) are also sufficient assuming Conjecture 9.11.

If $\eta_{\min} = -\epsilon_2 + \epsilon_3$, the Ramond extremal weights are the weights ν such that $r_1 = M_1(k)$ or $r_2 = 0$.

- If ν is Ramond extremal, the necessary conditions for unitarity are

$$(M_1(k), M_2(k)) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \text{ and } \ell = A(k, \nu).$$

- If ν is not Ramond extremal, the necessary conditions for unitarity are

$$(M_1(k), M_2(k)) \in \mathbb{Z}_+ \times \mathbb{Z}_+$$

and

$$(10.6) \quad \ell \geq A(k, \nu).$$

The sufficient conditions are

$$\ell \geq B(k, \nu; \rho_R) = -\frac{k+1}{4} + \frac{mr_2^2 + n(r_1+1)^2}{4(m+n)k}.$$

The conditions (10.6) are also sufficient assuming Conjecture 9.11.

10.6. $\mathfrak{g} = F(4)$. In this case

$$\mathfrak{g}^\natural = so(7), \quad M_1(k) = -\frac{3}{2}k - 1,$$

$$P_k^+ = \{\nu = r_1\epsilon_1 + r_2\epsilon_2 + r_3\epsilon_3, \quad r_1 \geq r_2 \geq r_3 \geq 0, \quad r_i \in \frac{1}{2} + \mathbb{Z} \text{ or } r_i \in \mathbb{Z}, \quad r_1 + r_2 \leq M_1(k)\},$$

and

$$\begin{aligned} A(k, \nu) &= -\frac{9k^2 + 8r_1^2 + 8r_1(r_2 + r_3 + 5) + 8r_2^2 - 8r_2r_3 + 32r_2 + 8r_3^2 + 8r_3 - 4}{36(k-2)} \\ &= -\frac{\frac{8}{9}(r_1^2 + r_1(r_2 + r_3 + 5) + r_2^2 - r_2r_3 + 4r_2 + r_3^2 + r_3) + p(k)}{4(k+h^\vee)}. \end{aligned}$$

If $\eta_{\min} = \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3)$, the Ramond extremal weights are the weights ν such that $r_3 = 0$.

- If ν is Ramond extremal, the necessary conditions for unitarity are $M_1(k) \in \mathbb{Z}_+$ and $\ell = A(k, \nu)$.
- If ν is not Ramond extremal, the necessary conditions for unitarity are $M_1(k) \in \mathbb{Z}_+$ and

$$(10.7) \quad \ell \geq A(k, \nu).$$

The sufficient conditions are

$$\ell \geq B(k, \nu, \rho_R) = -\frac{3k^2 + 4(r_1^2 + 4r_1 + r_2^2 + 2r_2 + r_3^2)}{12(k-2)}.$$

The conditions (10.7) are also sufficient assuming Conjecture 9.11.

If $\eta_{\min} = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3)$, the Ramond extremal weights are the weights ν such that $r_1 = r_2$.

- If ν is Ramond extremal, the necessary conditions for unitarity are $M_1(k) \in \mathbb{Z}_+$ and $\ell = A(k, \nu)$.
- If ν is not Ramond extremal, the necessary conditions for unitarity are $M_1(k) \in \mathbb{Z}_+$ and

$$(10.8) \quad \ell \geq A(k, \nu).$$

The sufficient conditions are

$$\ell \geq B(k, \nu, \rho_R) = -\frac{3k^2 + 4r_1^2 + 12r_1 + 4r_2^2 + 12r_2 + 4r_3^2 + 4r_3 - 1}{12(k-2)}.$$

The conditions (10.8) are also sufficient assuming Conjecture 9.11.

10.7. $\mathfrak{g} = G(3)$. In this case

$$\mathfrak{g}^{\natural} = G_2, \quad M_1(k) = -\frac{4}{3}k - 1,$$

$$P_k^+ = \{\nu = r_1\epsilon_1 + r_2\epsilon_2, \quad 2r_1 \geq r_2 \geq r_1, \quad r_i \in \mathbb{Z}_+, \quad r_1 + r_2 \leq M_1(k)\},$$

and

$$\begin{aligned} A(k, \nu) &= \frac{8k^2 + 2k + 8r_1^2 - 8r_1r_2 + 8r_2^2 + 24r_2 - 3}{48 - 32k} \\ &= -\frac{r_1^2 - r_1r_2 + r_2^2 + 3r_2 + p(k)}{4(k + h^\vee)}. \end{aligned}$$

The Ramond extremal are the weights ν such that $2r_1 = r_2$.

- If ν is Ramond extremal, the necessary conditions for unitarity are $M_1(k) \in \mathbb{Z}_+$ and $\ell = A(k, \nu)$.
- If ν is not Ramond extremal, the necessary and sufficient conditions for unitarity are $M_1(k) \in \mathbb{Z}_+$ and $\ell \geq A(k, \nu)$.

11. UNITARITY FOR RAMOND EXTREMAL MODULES OF THE $N = 3$ AND $N = 4$ SUPERCONFORMAL ALGEBRAS

11.1. $N = 3$. Let R be the Lie conformal superalgebra with basis

$$\{\tilde{L}, \tilde{G}^\pm, \tilde{G}^0, J^\pm, J^0, \Phi, K\}$$

and commutation relations given in [19, § 8.5]. The $N = 3$ superconformal vertex algebra is

$$\mathcal{W}_{N=3}^k = V(R)/(K - (k + \frac{1}{2})|0\rangle).$$

Recall that there is a conformal vertex algebra isomorphism

$$(11.1) \quad \mathcal{W}_{N=3}^k \rightarrow W_{\min}^k(\mathfrak{spo}(2|3)) \otimes F_\Phi,$$

where F_Φ is the fermionic vertex algebra $F(\mathbb{C}\Phi)$ constructed in Example 2.2.

By Lemma 7.3, F_Φ admits a Ramond twisted unitary module F_Φ^{tw} generated by 1.

It follows that, if M is Ramond twisted module for $W_{\min}^k(\mathfrak{spo}(2|3))$ then $M \otimes F_\Phi^{\text{tw}}$ admits a $W_{\min}^k(\mathfrak{spo}(2|3)) \otimes F_\Phi$ -invariant form. Since the isomorphism in (11.1) is conformal, a Hermitian form that is invariant for $W_{\min}^k(\mathfrak{spo}(2|3)) \otimes F_\Phi$ is also invariant for $\mathcal{W}_{N=3}^k$.

As explained in Section 2, the Ramond twisted modules for $V(R)$ are the same as the restricted $\text{Lie}(R, \sigma_R)$ -modules, hence a Ramond twisted $\mathcal{W}_{N=3}^k$ -module M is the same as a restricted $\text{Lie}(R, \sigma_R)$ -modules such that K acts by $(k + \frac{1}{2})I_M$. In particular, if M (resp. M') are Ramond twisted modules for $\mathcal{W}_{N=3}^k$ (resp. $\mathcal{W}_{N=3}^{k'}$), then $M \otimes M'$ is a Ramond twisted $\mathcal{W}_{N=3}^{k+k'+\frac{1}{2}}$ -module. Clearly, if both M, M' are unitary, then $M \otimes M'$ is unitary.

Proposition 11.1. *Let $M_1 = -4k - 2 \in \mathbb{N}$. Then the Ramond extremal $W_{\min}^k(\mathfrak{spo}(2|3))$ -modules $L^W(0, \frac{M_1-1}{16})$, $L^W(\frac{M_1}{2}\theta_1, \frac{M_1-1}{16} + \frac{M_1}{4})$ are both unitary.*

Proof. To make the argument more transparent we make explicit the dependence on k , so we write $L(k, \nu, \ell_0)$ for the $W_{\min}^k(\mathfrak{spo}(2|3))$ -module $L^W(\nu, \ell_0)$.

We proceed by induction on M_1 . The base case $M_1 = 1$ corresponds to the collapsing level $k = -3/4$, when $W_{-3/4}^{\min}(\mathfrak{spo}(2|3)) = V_1(\mathfrak{sl}_2)$. Recall that $V_1(\mathfrak{sl}_2)$ has only two irreducible modules N_1 and N_2 , which are both unitary and have highest weights $\nu = 0$ and $\nu = \frac{\theta_1}{2}$. The necessary condition for unitarity (given explicitly in § 10.2) imply that $N_1 = L(-3/4, 0, 0)$ and $N_2 = L(-3/4, \frac{\theta_1}{2}, 1/4)$.

Assume now $M_1 > 1$, $k = -\frac{M_1+2}{4}$, and set $k_1 = -\frac{M_1+1}{4}$. Assume by induction that $L(k_1, 0, \frac{M_1-2}{16})$ and $L(k_1, \frac{M_1-1}{2}\theta_1, \frac{M_1-2}{16} + \frac{M_1-1}{4})$ are unitary. Then $M = L(k_1, 0, \frac{M_1-2}{16}) \otimes F_\Phi^{\text{tw}}$ is unitary for $\mathcal{W}_{N=3}^{k_1}$ and $M' = L(-3/4, 0, 0) \otimes F_\Phi^{\text{tw}}$ is unitary for $\mathcal{W}_{N=3}^{-3/4}$. Therefore $M \otimes M'$ is unitary for $\mathcal{W}_{N=3}^{k_2}$, $k_2 = k_1 - \frac{3}{4} + \frac{1}{2} = -\frac{M_1+1}{4} - \frac{3}{4} + \frac{1}{2} = -\frac{M_1}{4} - \frac{1}{2} = k$, hence $M \otimes M'$ is a unitary Ramond twisted $W_{\min}^k(\mathfrak{spo}(2|3))$ -module. In particular, the $W_{\min}^k(\mathfrak{spo}(2|3))$ -module generated by $v_{0, \frac{M_1-2}{16}} \otimes 1 \otimes v_{0,0} \otimes 1$ is a unitary highest weight module $L(k, \nu, \ell_0)$. Clearly $\nu = 0$ and, by the necessary conditions of § 10.2, $\ell_0 = \frac{M_1-1}{16}$, as required.

Repeating the same argument with $M = L(k_1, \frac{M_1-1}{2}\theta_1, \frac{M_1-2}{16} + \frac{M_1-1}{4}) \otimes F_\Phi^{\text{tw}}$ and $M' = L(-3/4, \frac{\theta_1}{2}, \frac{1}{4}) \otimes F_\Phi^{\text{tw}}$ we prove the unitarity of $L(k, \frac{M_1-1}{2}\theta_1, \frac{M_1-1}{16} + \frac{M_1}{4})$. $\square$

Our results match [23, (2.3.ii)].

11.2. $N = 4$. In this subsection we recover results of Eguchi-Taormina (cf. [9, (5),(6)]) using their free field realization. The $N = 4$ superconformal algebra is $W_{\min}^k(\mathfrak{psl}(2|2))$. We choose strong generators $J^0, J^\pm, G^\pm, \bar{G}^\pm, L$ for $W_{\min}^k(\mathfrak{psl}(2|2))$ as in [19, § 8.4]. The λ -brackets among these generators are linear. It is therefore enough to prove unitarity of the Ramond extremal module $L^W(0, 1/4)$ at level $k = -2$ (see § 10.1). Arguing as in the $N = 3$ case, the Ramond extremal modules at level $k < -2$ are obtained by iterated tensor product of $L^W(0, 1/4)$.

The unitarity of $L^W(0, 1/4)$ is proved by constructing this module as a submodule of a manifestly unitary module. This is achieved by using the free field realization $FFR : W_{\min}^{-2}(\mathfrak{psl}(2|2)) \rightarrow \mathcal{F}$, given in [8] (see also § 13.2 of [15]), where $\mathcal{F} = V^1(\mathbb{C}^4) \otimes F(\mathbb{C}_1^4)$. Here $\mathbb{C}^4$ is viewed as the four-dimensional abelian Lie algebra and $\mathbb{C}_1^4$ is the four-dimensional totally odd space.

According to [14, § 5.2] and Lemma 7.3 above, $\mathcal{F}^{\text{tw}} = V^1(\mathbb{C}^4) \otimes F(\mathbb{C}_1^4, \sigma_R)$ is a unitary Ramond twisted $\mathcal{F}$ -module. Since FFR is conformal and preserves the $\mathbb{Z}_2$ -gradation, $\mathcal{F}^{\text{tw}}$ is also a unitary Ramond twisted module for $W_{\min}^k(\mathfrak{psl}(2|2))$. It is clear that the $W_{\min}^k(\mathfrak{psl}(2|2))$ -submodule of $\mathcal{F}^{\text{tw}}$ generated by $|0\rangle \otimes 1$ is a unitary highest weight representation $L^W(0, \ell_0)$ of $W_{\min}^{-2}(\mathfrak{psl}(2|2))$. By the necessary conditions for unitarity $\ell_0 = \frac{1}{4}$ and we are done.

12. THE CHARACTERS OF MASSLESS RAMOND TWISTED MODULES FOR MINIMAL W -ALGEBRAS

Definition 12.1. We say that a σ_R -twisted irreducible highest weight $W_{\min}^k(\mathfrak{g})$ -module $L^W(\nu, \ell)$ is *massless* if there exists $s \in \mathbb{C}$ such that $\ell = \ell(s)$ with $L(\hat{\nu}_s)$ an atypical representation of $\hat{\mathfrak{g}}^{\text{tw}}$, where $\hat{\nu}_s$ is defined by (9.1).

It can be easily proved that this definition yields the representations called massless in [9] and [23] for $\mathfrak{psl}(2|2)$ and $\mathfrak{spo}(2|3)$, respectively.

Remark 12.2. Recall that, by Lemma 9.2, $\ell(s) = \ell(s')$ if and only if $s' = k + 1 + \epsilon(\sigma_R) - s$. Using this relation, it is easy to check that, if $\eta \in \bar{\Delta}_{1/2}$, we have

$$(\hat{\nu}_s + \hat{\rho}^{\text{tw}} | \frac{p}{2}\delta + \frac{\theta}{2} + \eta) = (\hat{\nu}_{s'} + \hat{\rho}^{\text{tw}} | -\frac{p+2}{2}\delta + \frac{\theta}{2} - \eta).$$

In particular $\hat{\nu}_s$ is atypical if and only if $\hat{\nu}_{s'}$ is atypical.

Define

$$\Pi_1^\nu = \begin{cases} \{-\frac{1}{2}\delta + \theta/2 + \eta_{\min}\} & \text{if } \mathfrak{g} \neq spo(2|3), psl(2|2), \\ \{-\frac{\delta}{2} + \delta_1 + \epsilon_1\} & \text{if } \mathfrak{g} = spo(2|3) \text{ and } \nu = 0, \\ \{\frac{\delta}{2} + \delta_1 - \epsilon_1\} & \text{if } \mathfrak{g} = spo(2|3) \text{ and } \nu = \frac{M_1(k)}{2}\epsilon_1, \\ \{-\frac{\delta}{2} + \delta_1 - \epsilon_2, -\frac{\delta}{2} + \epsilon_1 - \delta_2\} & \text{if } \mathfrak{g} = psl(2|2). \end{cases}$$

Tote that if $\mathfrak{g} \neq spo(2|3)$, then Π_1^ν is the set of odd isotropic simple roots in $\widehat{\Delta}_+^{\text{tw}}$.

If $\theta/2$ is not a root of $\mathfrak{g}$, set $s_0 = \frac{k+1}{2} - (\nu - \rho_R + \rho^\natural|\eta_{\min})$. If $\theta/2$ is a root of $\mathfrak{g}$, set $s_0 = \frac{2k+1}{4}$. In all cases $\ell(s_0) = A(k, \nu)$.

Proposition 12.3. 1) If $L^W(\nu, \ell)$ is massless, then $\ell = A(k, \nu)$. If $\theta/2$ is not a root of $\mathfrak{g}$, then the converse holds.

2) If $\theta/2$ is a root of $\mathfrak{g}$, then $L^W(\nu, \ell)$ is massless if and only if $\ell = A(k, \nu)$ and ν is Ramond extremal.

3) In all cases Π_1^ν is the set of simple isotropic roots orthogonal to $\widehat{\nu}_{s_0} + \widehat{\rho}^{\text{tw}}$.

Proof. If $\ell > A(k, \nu)$, then $\ell = \ell(s)$ with either $s = \frac{k+1}{2} + \frac{\epsilon(\sigma_R)}{4} + \sqrt{-1}t$, $t \in \mathbb{R}$, or $s \in \mathbb{R}$. In the former case, the claim is obvious if $t \neq 0$. If $t = 0$ and $\epsilon(\sigma_R) = 1$ or $\epsilon(\sigma_R) = 0$ and $(\nu - \rho_R + \rho^\natural|\eta_{\min}) = 0$, then $\ell(s) = A(k, \nu)$. It remains only to check the case where $s \in \mathbb{R}$, $\epsilon(\sigma_R) = 0$, $(\nu - \rho_R + \rho^\natural|\eta_{\min}) \neq 0$, and

$$|s - \frac{k+1}{2}| < |(\nu - \rho_R + \rho^\natural|\eta_{\min})|.$$

By Remark 12.2, we can assume that s belongs to (9.10). With this assumption, we have shown already in Lemma 9.6 that $(\widehat{\nu}_s + \widehat{\rho}^{\text{tw}}|\alpha) \neq 0$ for all odd isotropic roots α .

Assume now that $\theta/2$ is not a root and $\ell = A(k, \nu) = \ell(s_0)$. In this case

$$\begin{aligned} (12.1) \quad & (\widehat{\nu}_{s_0} + \widehat{\rho}^{\text{tw}}| -\frac{1}{2}\delta + \theta/2 + \eta_{\min}) \\ &= ((k + h^\vee)\Lambda_0 + (\frac{k+1}{2} - (\nu - \rho_R + \rho^\natural|\eta_{\min}))\theta + \nu + \rho - \rho_R| -\frac{1}{2}\delta + \theta/2 + \eta_{\min}) \\ &= -\frac{1}{2}(k + h^\vee) + \frac{k+1}{2} - (\nu - \rho_R + \rho^\natural|\eta_{\min}) + \frac{1}{2}(h^\vee - 1) + (\nu - \rho_R + \rho^\natural|\eta_{\min}) = 0. \end{aligned}$$

Hence $L^W(\nu, \ell)$ is massless. This proves 1) and the fact that the simple isotropic root orthogonal to $\widehat{\nu}_{s_0} + \widehat{\rho}^{\text{tw}}$ is precisely the root in Π_1^ν . Statement 2) is proved via a case-wise analysis. Consider the case $\mathfrak{g} = G(3)$. The set of positive odd isotropic roots of $\widehat{\mathfrak{g}}^{\text{tw}}$ is

$$\{\gamma_i(p) \mid 1 \leq i \leq 9, p \text{ odd}, p \geq -1 \text{ for } i = 1, 2, 3, p \geq 1 \text{ otherwise}\},$$

where $\gamma_i(p)$ are displayed in Table 5.

We prove that if there exist p and i such that $(\widehat{\nu}_{\frac{2k+1}{4}} + \widehat{\rho}^{\text{tw}}|\gamma_i(p)) = 0$, then $\nu - \rho_R \notin P^+$.

Recall that $\nu = a\epsilon_1 + b\epsilon_2$ with $a, b \in \mathbb{Z}_+$, $2a \geq b \geq a$, $b \leq m := M_1(k)$. The condition on a, b implied by $(\widehat{\nu}_{s_0} + \widehat{\rho}^{\text{tw}}|\gamma_i(p)) = 0$ is listed in Table 5.

	$\gamma_i(p)$	$(\hat{\nu}_{s_0} + \hat{\rho}^{\text{tw}} \gamma_i(p)) = 0$
1	$\frac{p}{2}\delta + \delta_1 + \epsilon_1$	$-4a + 2b = 3(3+m)(1+p)$
2	$\frac{p}{2}\delta + \delta_1 + \epsilon_2$	$2a - 4b = 9 + 3(3+m)(1+p)$
3	$\frac{p}{2}\delta + \delta_1 + \epsilon_1 + \epsilon_2$	$2a + 2b = 9 + 3(3+m)(1+p)$
4	$\frac{p}{2}\delta - \delta_1 + \epsilon_1$	$4a - 2b = 3(3+m)(-1+p)$
5	$\frac{p}{2}\delta - \delta_1 + \epsilon_2$	$2a - 4b = 6 + 3(3+m)(p-1)$
6	$\frac{p}{2}\delta - \delta_1 + \epsilon_1 + \epsilon_2$	$2a + 2b = -6 - 3(p-1)(m+3)$
7	$\frac{p}{2}\delta + \delta_1 - \epsilon_1$	$4a - 2b = 3(3+m)(1+p)$
8	$\frac{p}{2}\delta + \delta_1 - \epsilon_2$	$-2a + 4b = -6 + 3(3+m)(p+1)$
9	$\frac{p}{2}\delta + \delta_1 - \epsilon_1 - \epsilon_2$	$2a + 2b = -6 + 3(3+m)(p+1)$

TABLE 5. $G(3)$

Case 1. Since $-4a + 2b \leq 0$, and the r.h.s is non-negative, the only possibility is $p = -1$, which forces $b = 2a$ and the only simple isotropic root orthogonal to $\hat{\nu}_s + \hat{\rho}^{\text{tw}}$ is precisely the root in Π_1^ν . In this case $\nu - \rho_R = (a-1)\epsilon_1 + (2a-1)\epsilon_2 \notin P^+$.

Cases 2, 5. Since $2a - 4b \leq 0$, equality cannot hold.

Case 3. If $p \geq 1$, then equality cannot occur since $2a + 2b \leq 4m$. If $p = -1$, the equality becomes $2a + 2b = 9$, which is impossible.

Case 4. If $p = 1$ we are back to Case 1. If $p \geq 3$, equality cannot occur since $4a - 2b \leq 4m$ whereas the r.h.s is greater than $6m$.

Case 6. The left hand side is non-negative and right hand side is negative.

Case 7. Similar to Case 4.

Cases 8, 9. In both cases the left hand side is less or equal than $4m$, whereas $-6 + 3(3+m)(p+1) \geq 6m + 3$, hence equality cannot hold.

Let now $\mathfrak{g} = \text{spo}(2|2r+1)$. The set of odd isotropic roots of $\hat{\mathfrak{g}}^{\text{tw}}$ is $\{p\delta/2 \pm \epsilon_1 \pm \delta_1 \mid 1 \leq i \leq r, p \text{ odd integer}\}$. Set $m := M_1(k)$. Recall that $\nu = \sum_{j=1}^r a_i \epsilon_i$ with $a_1 \geq a_2 \geq \dots \geq a_r \geq 0$ and $a_i \in \mathbb{Z}_+$ for all i or $a_i \in \frac{1}{2} + \mathbb{Z}_+$ for all i , and finally $a_1 + a_2 \leq m$. Relation $(\hat{\nu}_{s_0} + \hat{\rho}^{\text{tw}} | \gamma_i(p)) = 0$ implies

$$\frac{p \pm 1}{2}(-m + 2 - 2r) = \pm(a_i + r - i),$$

which in turn implies

$$(12.2) \quad \left| \frac{p \pm 1}{2} \right| (m + r + r - 2) = (a_i + r - i) \leq m + r - 1.$$

If $r > 1$, (12.2) implies $p = \pm 1$, $i = r$ and $a_r = 0$ and the only simple isotropic root orthogonal to $\hat{\nu}_{s_0} + \hat{\rho}^{\text{tw}}$ is precisely the root in Π_1^ν .

Finally consider the case $r = 1$, i.e. $\mathfrak{g} = \text{spo}(2|3)$. Then, if $\nu = \frac{a}{2}\epsilon_1$, we have

$$(12.3) \quad (\hat{\nu}_{s_0} + \hat{\rho}^{\text{tw}} | \pm \epsilon_1 \pm \delta_1 + \frac{p}{2}\delta) = \frac{1}{8}(\mp 2a + m(\mp 1 - p)).$$

If (12.3) vanishes, then m divides a , and since $0 \leq a \leq m$, we have either $a = 0$ or $a = m$. In the former case the $\nu - \rho_R \notin P^+$, in the latter $\nu - \rho_R$ is extremal. In both cases the only simple isotropic root orthogonal to $\widehat{\nu}_{s_0} + \widehat{\rho}^{\text{tw}}$ is precisely the root in Π_1^ν . $\square$

Theorem 12.4. *Assume Conjecture 9.11. Let k be in the unitary range, $\nu \in P_k^+$, and assume that $\widehat{\nu}_{s_0}$ is non-degenerate. Then*

$$(12.4) \quad (1 + \epsilon(\sigma_R)) \widehat{F}^R \text{ch } L^W(\nu, A(k, \nu)) = \\ q^{\frac{(\widehat{\nu}_{s_0} | \widehat{\nu}_{s_0} + 2\widehat{\rho}^{\text{tw}})}{2(k+h^\vee)} + a(k)} e^{-\rho_R} \sum_{w \in \widehat{W}^\natural} \det(w) \frac{q^{(w(\widehat{\nu}_{s_0} + \widehat{\rho}^{\text{tw}}) - \widehat{\rho}^{\text{tw}})(x+D)} e^{(w(\widehat{\nu}_{s_0} + \widehat{\rho}^{\text{tw}}) - \widehat{\rho}^{\text{tw}})_{|\mathfrak{h}^\natural}}}{\prod_{\beta \in \Pi_1^\nu} (1 + q^{w(\beta)(x+D)} e^{-w(\beta)_{|\mathfrak{h}^\natural}})},$$

where $a(k)$ is given by (8.4).

Proof. Combining Lemmas 9.8 and 9.9 with Proposition 12.3, we have proved that the hypothesis of Proposition 8.6 are satisfied. Formula (12.4) follows from Proposition 8.6 using Conjectures 9.11 in the same way as in the proof of Theorem 9.13. $\square$

Remark 12.5. In the NS sector a formula similar to (12.4) holds. More precisely

$$(12.5) \quad \widehat{F}^{NS} \text{ch } L^W(\nu, A(k, \nu)) = q^{\frac{(\widehat{\nu}_{t_0} | \widehat{\nu}_{t_0} + 2\widehat{\rho})}{2(k+h^\vee)}} \sum_{w \in \widehat{W}^\natural} \det(w) \frac{q^{(w(\widehat{\nu}_{t_0} + \widehat{\rho}) - \widehat{\rho})(x+D)} e^{(w(\widehat{\nu}_{t_0} + \widehat{\rho}) - \widehat{\rho})_{|\mathfrak{h}^\natural}}}{\prod_{\beta \in \Pi_1} (1 + q^{w(\beta)(x+D)} e^{-w(\beta)_{|\mathfrak{h}^\natural}})},$$

where t_0 is either $(\nu|\xi)$ or $k+1 - (\nu|\xi)$ (we have shown in [15] that at least one of the two values yields a non-degenerate ν_{t_0}). This is essentially formula (14.6) from [15]. Note that if $\nu = 0$ then $(k\Lambda_0 + \widehat{\rho}|\delta - \theta) = k+1$, which is never a positive integer, hence we can choose $t_0 = 0$.

13. DENOMINATOR IDENTITIES

Let k_0 be non-critical and such that $W_{k_0}^{\min}(\mathfrak{g}) = \mathbb{C}[0]$. Since, as shown in [2], this happens if and only if $M_i(k_0) = 0$ for all i , it follows from Table 4 that this happens in the following cases, where $v = u_i$ from Table 4.

$\mathfrak{g}$	$psl(2 2)$	$spo(2 m), m \neq 3$	$F(4)$	$G(3)$
k_0	-1	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{3}{4}$
u	-2	-1	$-\frac{4}{3}$	$-\frac{3}{2}$
$h^\vee$	0	$2 - \frac{m}{2}$	-2	$-\frac{3}{2}$
$\bar{h}^\vee$	-2	$1 - \frac{m}{2}$	$-\frac{10}{3}$	-3
$b = \frac{h^\vee + \bar{h}^\vee}{u}$	1	$m - 3$	4	3

TABLE 6

Recall from [13, §6] the decomposition

$$(13.1) \quad \widehat{W}^\natural = W^\natural \ltimes T^\natural,$$

where $T^\natural = \{t_\alpha \mid \alpha \in M^\natural\}$, $M^\natural$ is the $\mathbb{Z}$ -span of the long roots of $\mathfrak{g}^\natural$, and

$$(13.2) \quad t_\alpha(\lambda) = \lambda + \frac{2}{v}\lambda(K)\alpha - \frac{2}{v}((\lambda|\alpha) + \frac{1}{v}(\alpha|\alpha)\lambda(K))\delta$$

(cf. [13, (6.5.3)], where this formula is given using the normalized invariant bilinear form).

Let $b = \frac{h^\vee + h^\vee}{u}$ (see Table 6 for its values).

Theorem 13.1. *We have for all $\mathfrak{g}$ listed in Table 6, except for $\mathfrak{g} = \mathfrak{spo}(2|N)$, $0 \leq N \leq 3$:*

$$(13.3) \quad \widehat{F}^{NS} = e^{-\rho^\natural} \sum_{\bar{w} \in W^\natural} \sum_{\alpha \in M^\natural} \det(\bar{w}) \frac{e^{\bar{w}(\rho^\natural + b\alpha)}}{\prod_{\beta \in \Pi_1^\vee} (1 + e^{-\bar{w}(\beta|_{\mathfrak{h}^\natural})} q^{-\frac{2}{u}(\alpha|\beta) + \frac{1}{2}})} q^{\frac{b}{v}(\alpha|\alpha) + \frac{2}{v}(\rho^\natural|\alpha)}.$$

If Conjecture 9.11 holds, then

$$(13.4) \quad \widehat{F}^R = \frac{e^{\rho_R - \rho^\natural}}{1 + \epsilon(\sigma_R)} \sum_{\bar{w} \in W^\natural} \sum_{\alpha \in M^\natural} \det(\bar{w}) \frac{e^{\bar{w}(\rho^\natural - \rho_R + b\alpha)}}{\prod_{\beta \in \Pi_1^\vee} (1 + e^{-\bar{w}(\beta|_{\mathfrak{h}^\natural})} q^{-\frac{2}{u}(\alpha|\beta)})} q^{\frac{b}{v}(\alpha|\alpha) + \frac{2}{v}(\rho^\natural - \rho_R|\alpha)}.$$

Proof. We apply Remark 12.5 to the (untwisted) $\widehat{\mathfrak{g}}$ -module $L(k_0\Lambda_0) \simeq V_{k_0}(\mathfrak{g})$ and $t_0 = 0$. Since $H_0(L(k_0\Lambda_0)) = W_{k_0}^{\min}(\mathfrak{g}) = \mathbb{C}|0\rangle$, we obtain

$$(13.5) \quad \widehat{F}^{NS} = \sum_{w \in \widehat{W}^\natural} \det(w) \frac{q^{(w(k_0\Lambda_0 + \widehat{\rho}) - \widehat{\rho})(x+D)} e^{(w(k_0\Lambda_0 + \widehat{\rho}) - \widehat{\rho})|_{\mathfrak{h}^\natural}}}{\prod_{\beta \in \Pi_1^\vee} (1 + q^{w(\beta)(x+D)} e^{-w(\beta)|_{\mathfrak{h}^\natural}})}.$$

To compute the R.H.S. of (13.5), write $w = \bar{w}t_\alpha$, given by the decomposition (13.1) and formula (13.2), to obtain

$$\begin{aligned} w(k_0\Lambda_0 + \widehat{\rho}) - \widehat{\rho} &= \bar{w}t_\alpha((k_0 + h^\vee)\Lambda_0 + \rho) - \widehat{\rho} \\ &= k_0\Lambda_0 + \bar{w}(\rho) - \rho + \frac{2}{v}(k_0 + h^\vee)\bar{w}(\alpha) - \frac{2}{v}(\alpha|\rho + \frac{1}{v}(k_0 + h^\vee)\alpha) \\ &= k_0\Lambda_0 + \bar{w}(\rho) - \rho + b\bar{w}(\alpha) - \frac{2}{v}(\alpha|\rho + \frac{b}{2}\alpha)\delta, \end{aligned}$$

so that, since $(\alpha|\rho) = (\alpha|\rho^\natural)$, we have

$$q^{(w(k_0\Lambda_0 + \widehat{\rho}) - \widehat{\rho})(x+D)} e^{(w(k_0\Lambda_0 + \widehat{\rho}) - \widehat{\rho})|_{\mathfrak{h}^\natural}} = e^{\bar{w}(\rho^\natural + b\alpha) - \rho^\natural} q^{\frac{2}{v}(\alpha|\rho^\natural + \frac{b}{2}\alpha)}.$$

Similarly

$$w(\beta) = \bar{w}t_\alpha(\beta) = \bar{w}(\beta) - \frac{2}{v}(\beta|\alpha)\delta,$$

so that, since $\bar{w}(\beta)(D+x) = \bar{w}(\beta)(x) = \beta(x) = \frac{1}{2}$,

$$q^{w(\beta)(x+D)} e^{-w(\beta)|_{\mathfrak{h}^\natural}} = e^{-\bar{w}(\beta|_{\mathfrak{h}^\natural})} q^{-\frac{2}{v}(\beta|\alpha) + \frac{1}{2}}.$$

Substituting, we find (13.3).

Now we prove (13.4). Recall that

$$s_0 = \begin{cases} \frac{k_0+1}{2} + (\rho_R - \rho^\natural|\eta_{\min}) & \text{if } \theta/2 \text{ is not a root of } \mathfrak{g}, \\ \frac{2k_0+1}{4} & \text{if } \theta/2 \text{ is a root of } \mathfrak{g}. \end{cases}$$

By Lemma 9.4, $\ell(s_0) = A(k_0, 0)$. Recall the polynomial $p(k)$, mentioned at the beginning of Section 5. Recall [2, Theorem 3.3] that $p(k_0) = 0$. Combining this observation with (6.31) and (6.19), we find that $A(k_0, 0) = 0$, so $\ell(s_0) = 0$. If $\nu = 0$, then (see (9.1)) $\widehat{\nu}_{s_0} = k_0\Lambda_0 + s_0\theta + \rho_R$. Since

$$(\widehat{\nu}_{s_0} + \widehat{\rho}^{\text{tw}}|\delta - \theta) = \begin{cases} 2(\rho_R - \rho^\natural|\eta_{\min}) & \text{if } \theta/2 \text{ is not a root of } \mathfrak{g}, \\ 0 & \text{if } \theta/2 \text{ is a root of } \mathfrak{g}, \end{cases}$$

we see by a case-wise inspection that $\widehat{\nu}_{s_0}$ is non-degenerate. Assuming Conjecture 9.11, by Proposition 9.12 and the fact that $\nu = 0$ and $\ell(s_0) = 0$, we have

$$H_0(L(\widehat{\nu}_{s_0})) = (1 + \epsilon(\sigma_R))L^W(0, 0).$$

Since $W_{k_0}^{\min}(\mathfrak{g}) = \mathbb{C}|0\rangle$ and the maximal proper ideal of $W_{\min}^{k_0}(\mathfrak{g})$ is σ_R -stable, $\mathbb{C}|0\rangle$ is a one-dimensional σ_R -twisted representation of $W_{\min}^{k_0}(\mathfrak{g})$, hence $L^W(0,0) = \mathbb{C}|0\rangle$. Apply now Theorem 12.4 to $L^W(0,0)$. As above, we compute:

$$\begin{aligned} w(k_0\Lambda_0 + s_0\theta + \rho_R + \hat{\rho}^{\text{tw}}) - \hat{\rho}^{\text{tw}} &= \bar{w}t_\alpha((k_0 + h^\vee)\Lambda_0 + s_0\theta - \rho_R + \rho) - \hat{\rho}^{\text{tw}} = \\ &= k_0\Lambda_0 + s_0\theta + \bar{w}(\rho - \rho_R) + 2\rho_R - \rho + b\bar{w}(\alpha) + \frac{2}{v}(\alpha|\rho - \rho_R + \frac{b}{2}\alpha)\delta, \end{aligned}$$

so

$$e^{ev(w(k_0\Lambda_0 + s_0\theta + \rho_R + \hat{\rho}^{\text{tw}}) - \hat{\rho}^{\text{tw}})} = e^{\bar{w}(\rho^\natural - \rho_R + b\alpha)} e^{2\rho_R - \rho^\natural} q^{-s_0} q^{\frac{2}{v}(\alpha|\rho^\natural - \rho_R + \frac{b}{2}\alpha)}.$$

Similarly

$$w(\beta) = \bar{w}t_\alpha(\beta) = \bar{w}(\beta) - \frac{2}{v}(\beta|\alpha)\delta,$$

so that, since $\bar{w}(\beta)(D+x) = \beta(D+x) = 0$,

$$e^{-ev(w(\beta))} = e^{-\bar{w}(\beta|_{\mathfrak{h}^\natural})} q^{-\frac{2}{v}(\beta|\alpha)}.$$

Substituting in (12.4) we find (13.4), since $s_0 = \hat{\nu}_{s_0}(x+D)$ and

$$\frac{(\hat{\nu}_{s_0}|\hat{\nu}_{s_0} + 2\hat{\rho}^{\text{tw}})}{2(k+h^\vee)} + a(k) - \hat{\nu}_{s_0}(x+D) = \ell(s_0) = 0.$$

□

In the subsequent subsections we write down the denominator identities (13.3) and (13.4) explicitly. To simplify notation we set

(13.6)

$$\vartheta_0(x) = \prod_{j=1}^{\infty} (1 - xq^{j-1})(1 - x^{-1}q^j), \quad \vartheta_1(x) = \vartheta_0(-xq^{\frac{1}{2}}) = \prod_{j=1}^{\infty} (1 + xq^{j-\frac{1}{2}})(1 + x^{-1}q^{j-\frac{1}{2}}),$$

(13.7)

$$\varphi(q) = \prod_{j=1}^{\infty} (1 - q^j), \quad \varphi_1(q) = \prod_{j=1}^{\infty} (1 + q^{j-\frac{1}{2}}), \quad \varphi_2(q) = \frac{\varphi(q^2)}{\varphi(q)} = \prod_{j=1}^{\infty} (1 + q^j).$$

If $\mathfrak{g} = \mathfrak{spo}(2|2r)$, $\mathfrak{g} = D(2, 1; a)$, or $\mathfrak{g} = F(4)$, the denominator formulas in the Ramond sector depend on the choice of the set $\hat{\Pi}$ of simple roots for $\hat{\mathfrak{g}}^{\text{tw}}$, which ultimately depends on the choice of $\eta_{\min}$. We now explain how to obtain one formula from the other one. Choose $\eta_{\min}$ and write $F^R(\eta_{\min})$ for the corresponding denominator and $\rho_R(\eta_{\min})$ for the corresponding ρ_R . We observe that $F^R(-\eta_{\min}) = e^{-\eta_{\min}} F^R(\eta_{\min})$ and that $\rho_R(-\eta_{\min}) = \rho_R(\eta_{\min}) - \eta_{\min}$. It follows that the denominator formula for $-\eta_{\min}$ is obtained from the formula for $\eta_{\min}$ by multiplying both sides by $e^{-\eta_{\min}}$. In these cases we make only one choice for $\eta_{\min}$ and write down only the corresponding formula.

13.1. $\mathfrak{g} = \mathfrak{spo}(2|N)$, $N = 0, 1, 2$, $k_0 = -\frac{1}{2}$. These cases are not covered by (13.3) and (13.4).

If $N = 0$, $W_{\min}^k(\mathfrak{g})$ is the universal Virasoro vertex algebra of central charge $c(k) = \frac{3k}{k+2} - 6k - 2$, so, since $c(k_0) = 0$, $W_k^{\min}(\mathfrak{g}) = H_0(L(-\frac{1}{2}\Lambda_0)) = \mathbb{C}$. Next note that $-\frac{1}{2}$ is an admissible level so, by [17, Theorem 1 and Example 1],

$$(13.8) \quad \hat{R} \, ch \, L(-\frac{1}{2}\Lambda_0) = \sum_{w \in \hat{W}_{int}} \det(w) e^{w(-\frac{1}{2}\Lambda_0 + \hat{\rho}) - \hat{\rho}},$$

where $\widehat{W}_{int}$ is the Weyl group of the set of roots corresponding to the set of simple roots $\widehat{\Pi}_{int} = \{\theta, 2\delta - \theta\}$. Applying the functor H , it follows, by Arakawa theorem, that

$$(13.9) \quad \prod_{n \geq 1} (1 - q^n) = \sum_{w \in \widehat{W}_{int}} \det(w) q^{-(w(-\frac{1}{2}\Lambda_0 + \widehat{\rho}) - \widehat{\rho})(x+D)}.$$

We argue as in Theorem 13.1: write $w \in \widehat{W}_{int}$ as $\bar{w}t_{n\theta}$ with $\bar{w} \in \{1, s_\theta\}$ and

$$t_{n\theta}(\widehat{\Lambda}) = \widehat{\Lambda} + 2n\widehat{\Lambda}(K)\theta - 4n(n\widehat{\Lambda}(K) + \widehat{\Lambda}(x))\delta.$$

In our special case we obtain

$$t_{n\theta}(-\frac{1}{2}\Lambda_0 + \widehat{\rho}) - \widehat{\rho} = -\frac{1}{2}\Lambda_0 + 3n\theta - 2n(1 + 3n)\delta.$$

while

$$s_\theta t_{n\theta}(-\frac{1}{2}\Lambda_0 + \widehat{\rho}) - \widehat{\rho} = -\frac{1}{2}\Lambda_0 - (3n + 1)\theta - 2n(1 + 3n)\delta$$

so (13.9) becomes

$$\prod_{n \geq 1} (1 - q^n) = \sum_{n \in \mathbb{Z}} (q^{6n^2 - n} - q^{6n^2 + 5n + 1}) = \sum_{m \in \mathbb{Z}} (-1)^m q^{\frac{3m^2 + m}{2}},$$

which is the Euler identity for the classical partition function.

Next we discuss $N = 1$. In the NS sector we have $\mathbb{C}|0\rangle = W_{-1/2}^{\min}(spo(2|1)) = H(L(-\frac{1}{2}\Lambda_0))$. It follows from [17, Example 2 and Theorem 1], that $-\frac{1}{2}\Lambda_0$ is an admissible weight for $spo(2|1)^\wedge$, hence (13.8) holds with $\widehat{W}_{int}$ the Weyl group of the root subsystem generated by the set of simple roots

$$\Pi_{int} = \{\theta/2, \delta - \theta/2\}.$$

Applying the functor H and using Arakawa theorem, we obtain

$$(13.10) \quad \prod_{n \geq 1} \frac{1 - q^n}{1 + q^{n - \frac{1}{2}}} = \sum_{w \in \widehat{W}_{int}} \det(w) q^{-(w(-\frac{1}{2}\Lambda_0 + \widehat{\rho}) - \widehat{\rho})(x+D)}.$$

Note that the group $\widehat{W}_{int}$ is the same as in the $N = 0$ case. In our special case we obtain

$$t_{n\theta}(-\frac{1}{2}\Lambda_0 + \widehat{\rho}) - \widehat{\rho} = -\frac{1}{2}\Lambda_0 - 2n\theta - n(1 + 4n)\delta,$$

while

$$s_\theta t_{n\theta}(-\frac{1}{2}\Lambda_0 + \widehat{\rho}) - \widehat{\rho} = -\frac{1}{2}\Lambda_0 - (2n + \frac{1}{2})\theta - n(1 + 4n)\delta,$$

so (13.10) becomes

$$\begin{aligned} \prod_{n \geq 1} \frac{1 - q^n}{1 + q^{n - \frac{1}{2}}} &= \sum_{n \in \mathbb{Z}} (q^{4n^2 - n} - q^{4n^2 + 3n + \frac{1}{2}}) \\ &= \sum_{n=0}^{\infty} (q^{4n^2 - n} - q^{4n^2 + 3n + \frac{1}{2}}) + \sum_{n=1}^{\infty} (q^{4n^2 + n} - q^{4n^2 - 3n + \frac{1}{2}}) \\ &= \sum_{m \in 4\mathbb{Z}_+ - 1} q^{\frac{1}{4}m(m+1)} - \sum_{m \in 4\mathbb{Z}_+ + 1} q^{\frac{1}{4}m(m+1)} + \sum_{m \in 4\mathbb{N}} q^{\frac{1}{4}m(m+1)} - \sum_{m \in 4\mathbb{N} - 2} q^{\frac{1}{4}m(m+1)} \\ &= \sum_{m=0}^{\infty} (-q^{\frac{1}{2}})^{m(m+1)/2}. \end{aligned}$$

Replacing q by q^2 and then changing the sign of q , we obtain the Gauss identity for the generating series of triangular numbers:

$$\prod_{n \geq 1} \frac{1 - q^{2n}}{1 - q^{2n+1}} = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}.$$

In the Ramond sector, using Conjecture 9.11, we have

$$(13.11) \quad H(L(-\tfrac{1}{2}\Lambda_0)) = L^W(0,0) \oplus L^W(0,0) = \mathbb{C}^2.$$

The character of the $\widehat{\mathfrak{g}}^{\text{tw}}$ -module $L(-\tfrac{1}{2}\Lambda_0)$ is given by

$$ch L(-\tfrac{1}{2}\Lambda_0) = \sum_{w \in \widehat{W}_{int}} \det(w) \frac{e^{w(-\frac{1}{2}\Lambda_0 + \widehat{\rho}^{\text{tw}}) - \widehat{\rho}^{\text{tw}}}}{\widehat{R}^{\text{tw}}}$$

so, applying the twisted quantum Hamiltonian reduction functor, the identity becomes

$$(13.12) \quad 2 \prod_{n \geq 1} \frac{1 - q^n}{1 + q^{n-1}} = \prod_{n \geq 1} \frac{1 - q^n}{1 + q^n} = \sum_{w \in \widehat{W}_{int}} \det(w) q^{-(w(-\frac{1}{2}\Lambda_0 + \widehat{\rho}^{\text{tw}}) - \widehat{\rho}^{\text{tw}})(x+D)}.$$

The Weyl group $\widehat{W}_{int}$ is again the same as in the $N = 0$ case. We obtain

$$t_{n\theta}(-\tfrac{1}{2}\Lambda_0 + \widehat{\rho}^{\text{tw}}) - \widehat{\rho}^{\text{tw}} = -\tfrac{1}{2}\Lambda_0 + 2n\theta - 2n(1 + 2n)\delta,$$

while

$$s_{\theta} t_{n\theta}(-\tfrac{1}{2}\Lambda_0 + \widehat{\rho}^{\text{tw}}) - \widehat{\rho}^{\text{tw}} = -\tfrac{1}{2}\Lambda_0 - (2n + 1)\theta - 2n(1 + 2n)\delta,$$

so (13.12) becomes

$$\prod_{n \geq 1} \frac{1 - q^n}{1 + q^n} = \sum_{n \in \mathbb{Z}} (q^{(2n)^2} - q^{(2n+1)^2}) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2},$$

which is Gauss identity for the generating series of square numbers.

Remark 13.2. This gives some evidence for Conjecture 9.11, which has been used in a crucial way in (13.11).

In the $N = 2$ case we have $W_{-1/2}^{\min}(spo(2|2)) = \mathbb{C}$. By [10, Corollary 11.2.4] and the remark thereafter, we can apply formula (35) of [loc. cit.]. Following [10], we choose the set of simple roots for $\mathfrak{g} = spo(2|2)$ to be $\Pi = \{\alpha_1, \alpha_2\}$ with both simple roots isotropic so that we can compute the character of $L(-\tfrac{1}{2}\Lambda_0)$ explicitly, using [10, (14)]. By applying the quantum Hamiltonian reduction functor as in Proposition 8.6, we derive the character formula of $H(L(-\tfrac{1}{2}\Lambda_0))$:

$$ch H(L(-\tfrac{1}{2}\Lambda_0)) = \frac{1}{\widehat{F}^{NS}} \sum_{w \in \widehat{W}_{int}} \det(w) \frac{q^{-(w(-\frac{1}{2}\Lambda_0 + \widehat{\rho}) - \widehat{\rho})(x+D)} e^{(w(-\frac{1}{2}\Lambda_0 + \widehat{\rho}) - \widehat{\rho})|_{\mathfrak{h}^{\natural}}}}{\prod_{\beta \in \Pi_1} (1 + q^{(w\beta)(x+D)} e^{-w(\beta)|_{\mathfrak{h}^{\natural}}})}$$

In this case $\Pi_1 = \{\alpha_1\}$ and $\widehat{W}_{int}$ is, once again, the group as in the $N = 0$ case.

Arguing as in the proof of Theorem 13.1, we find

$$(13.13) \quad \begin{aligned} & \prod_{n \geq 1} \frac{(1 - q^n)^2}{\left(1 + e^{-(\alpha_1)|_{\mathfrak{h}^{\natural}}} q^{n - \frac{1}{2}}\right) \left(1 + e^{-(\alpha_2)|_{\mathfrak{h}^{\natural}}} q^{n - \frac{1}{2}}\right)} \\ &= \sum_{w \in \widehat{W}_{int}} \det(w) \frac{q^{-(w(-\frac{1}{2}\Lambda_0 + \widehat{\rho}) - \widehat{\rho})(x+D)} e^{(w(-\frac{1}{2}\Lambda_0 + \widehat{\rho}) - \widehat{\rho})|_{\mathfrak{h}^{\natural}}}}{1 + q^{(w\alpha_1)(x+D)} e^{-w(\alpha_1)|_{\mathfrak{h}^{\natural}}}}. \end{aligned}$$

More explicitly, write $\alpha_1 = \delta_1 + \epsilon_1$, so that $\alpha_2 = \delta_1 - \epsilon_1$, and set $z = e^{\epsilon_1}$. Since $\hat{\rho} = \Lambda_0$ in this case, we have

$$t_{n\theta}(-\tfrac{1}{2}\Lambda_0 + \hat{\rho}) - \hat{\rho} = -\tfrac{1}{2}\Lambda_0 + n\theta - 2n^2\delta, \quad t_{n\theta}(\alpha_1) = -2n\delta + \alpha_1,$$

while

$$s_{\theta}t_{n\theta}(-\tfrac{1}{2}\Lambda_0 + \hat{\rho}) - \hat{\rho} = -\tfrac{1}{2}\Lambda_0 - n\theta - 2n^2\delta, \quad s_{\theta}t_{n\theta}(\alpha_1) = -2n\delta - \alpha_2,$$

so (13.13) becomes

$$(13.14) \quad \frac{\varphi(q)^2}{\vartheta_1(z^{-1})} = \sum_{n \in \mathbb{Z}} \left(\frac{q^{2n^2-n}}{1 + z^{-1}q^{-2n+\frac{1}{2}}} - \frac{q^{2n^2+n}}{1 + z^{-1}q^{-2n-\frac{1}{2}}} \right).$$

Here and further this is viewed as an identity of formal power series in q with functions in z as coefficients, using the $|q| < 1$ expansion of the two series (see Remark 8.7).

In the Ramond sector choose $\eta_{\min} = \epsilon_1$, so that $s_0 = 0$ and $\rho_R = \epsilon_1/2$. Using Conjecture 9.11 b), we have

$$H(L(-\tfrac{1}{2}\Lambda_0 + \tfrac{1}{2}\epsilon_1)) = L^W(0, 0) = \mathbb{C}.$$

As in the NS sector, we apply formula (35) of [10], using (14) of [loc. cit.] to compute its right hand side, and then apply the quantum Hamiltonian reduction functor. This gives the character formula:

$$\begin{aligned} & ch H(L(-\tfrac{1}{2}\Lambda_0 + \tfrac{1}{2}\epsilon_1)) \\ &= \frac{e^{-\rho_R}}{\widehat{F}^R} \sum_{w \in \widehat{W}_{int}} \det(w) \frac{q^{-(w(-\frac{1}{2}\Lambda_0 + \frac{1}{2}\epsilon_1 + \hat{\rho}^{tw}) - \hat{\rho}^{tw})(x+D)} e^{(w(-\frac{1}{2}\Lambda_0 + \frac{1}{2}\epsilon_1 + \hat{\rho}^{tw}) - \hat{\rho}^{tw})|_{\mathfrak{h}^{\natural}}}}{\prod_{\beta \in \Pi_1} (1 + q^{(w\beta)(x+D)} e^{-w(\beta)|_{\mathfrak{h}^{\natural}}})}, \end{aligned}$$

where $\Pi_1 = \{\beta\}$ with $\beta = -\frac{1}{2}\delta + \alpha_1$ and $\widehat{W}_{int}$ is the group as in the $N = 0$ case. The denominator identity becomes

$$\begin{aligned} & \prod_{n \geq 1} \frac{(1 - q^n)^2}{(1 + z^{-1}q^{n-1})(1 + zq^n)} \\ (13.15) \quad &= z^{-\frac{1}{2}} \sum_{w \in \widehat{W}_{int}} \det(w) \frac{q^{-(w(-\frac{1}{2}\Lambda_0 + \epsilon_1/2 + \hat{\rho}^{tw}) - \hat{\rho}^{tw})(x+D)} e^{(w(-\frac{1}{2}\Lambda_0 + \epsilon_1/2 + \hat{\rho}^{tw}) - \hat{\rho}^{tw})|_{\mathfrak{h}^{\natural}}}}{1 + q^{(w\beta)(x+D)} e^{-w(\beta)|_{\mathfrak{h}^{\natural}}}}}. \end{aligned}$$

In this case, we have

$$t_{n\theta}(-\tfrac{1}{2}\Lambda_0 + \epsilon/2 + \hat{\rho}^{tw}) - \hat{\rho}^{tw} = -\tfrac{1}{2}\Lambda_0 + \epsilon/2 + n\theta - 2n^2\delta, \quad t_{n\theta}(\beta) = -(\tfrac{1}{2} + 2n)\delta + \alpha_1,$$

while

$$s_{\theta}t_{n\theta}(-\tfrac{1}{2}\Lambda_0 + \hat{\rho}) - \hat{\rho} = -\tfrac{1}{2}\Lambda_0 + \epsilon/2 - n\theta - 2n^2\delta, \quad s_{\theta}t_{n\theta}(\beta) = -(\tfrac{1}{2} + 2n)\delta - \alpha_2,$$

so (13.15) becomes

$$(13.16) \quad \frac{\varphi(q)^2}{\vartheta_0(-z^{-1})} = \sum_{n \in \mathbb{Z}} \left(\frac{q^{2n^2-n}}{1 + z^{-1}q^{-2n}} - \frac{q^{2n^2+n}}{1 + z^{-1}q^{-2n-1}} \right) = \sum_{r \in \mathbb{Z}} (-1)^r \frac{q^{\frac{r(r+1)}{2}}}{1 + z^{-1}q^r}.$$

This last identity is proven in [18] by specializing a denominator identity for $sl(2|1)^{\wedge}$ (see [18], formula (4.8)), which is the celebrated Ramanujan identity:

$$(13.17) \quad \frac{\varphi(q)^2 \vartheta_0(xy)}{\vartheta_0(-x) \vartheta_0(-y)} = \left(\sum_{m,n=0}^{\infty} - \sum_{m,n=-1}^{-\infty} \right) (-1)^{m+n} x^m y^n q^{mn},$$

therefore (13.16) can be seen as another piece of evidence for Conjecture 9.11. Actually (13.17) follows from (13.14) by replacing z by $zq^{\frac{1}{2}}$. This is not surprising due to the spectral flow.

13.2. $\mathfrak{g} = \mathfrak{psl}(2|2)$, $k_0 = -1$. In the NS sector, (13.3) gives, letting $e^{\delta_1} = x$ and $e^{-\delta_2} = y$, we obtain

$$(13.18) \quad \frac{\varphi(q)^2 \vartheta_0(x^{-1}y^{-1})}{\theta_1(x)\theta_1(y)} = \sum_{n \in \mathbb{Z}} \left(\frac{x^n y^n}{(1+xq^{n+\frac{1}{2}})(1+yq^{n+\frac{1}{2}})} - \frac{x^{-n-1}y^{-n-1}}{(1+x^{-1}q^{n+\frac{1}{2}})(1+y^{-1}q^{n+\frac{1}{2}})} \right) q^{n^2+n}.$$

In the Ramond sector, recalling that $\Pi_1^\nu = \{-\frac{1}{2}\delta + \delta_1 - \epsilon_2, -\frac{1}{2}\delta + \epsilon_1 - \delta_2\}$, (13.4) gives, letting $e^{\delta_1} = x$ and $e^{-\delta_2} = y$,

$$(13.19) \quad \frac{\varphi(q)^2 \vartheta_0(x^{-1}y^{-1})}{\vartheta_0(-x^{-1})\vartheta_0(-y^{-1})} = \sum_{n \in \mathbb{Z}} \left(\frac{x^n y^n}{(1+x^{-1}q^{-n})(1+y^{-1}q^{-n})} - \frac{x^{-n}y^{-n}}{(1+xq^{-n})(1+yq^{-n})} \right) q^{n^2}.$$

Note that this identity follows from (13.17) by replacing x with x^{-1} and y by y^{-1} .

13.3. $\mathfrak{g} = \mathfrak{spo}(2|3)$, $k_0 = -\frac{1}{2}$. In the $N = 3$ case k_0 is critical for $\mathfrak{g}$, hence our previous approach does not apply. Nevertheless we are able to prove a denominator formula by replacing in (13.17) q by q^2 and then setting $x = qz$, $y = q$. We obtain

$$\prod_{n \geq 1} \frac{(1-q^{2n})^2(1-z^{-1}q^{2n-2})(1-zq^{2n})}{(1+q^{2n-1})^2(1+zq^{2n-1})(1+z^{-1}q^{2n-1})} = \left(\sum_{m,n=0}^{\infty} - \sum_{m,n=-1}^{-\infty} \right) (-1)^{m+n} z^m q^{2mn+m+n},$$

or, after replacing q by $q^{\frac{1}{2}}$ and z by z^{-1} ,

$$\frac{\varphi(q)^2 \vartheta_0(z)}{\vartheta_1(z)} = \varphi_1(q) \left(\sum_{m,n=0}^{\infty} - \sum_{m,n=-1}^{-\infty} \right) (-1)^{m+n} z^{-m} q^{mn+\frac{1}{2}(m+n)}.$$

which, setting $z = e^{-\epsilon_1} = e^{\alpha_1|_{\mathfrak{h}}}$, is the denominator identity for $W_{\min}^k(\mathfrak{spo}(2|3))$ in NS sector.

In the Ramond sector we rewrite (13.17) as

$$\begin{aligned} \prod_{n \geq 1} \frac{(1-q^n)^2(1-x^{-1}y^{-1}q^{n-1})(1-xyq^n)}{(1+x^{-1}q^{n-1})(1+xq^n)(1+y^{-1}q^{n-1})(1+yq^n)} &= \left(\sum_{n=1}^{\infty} (-1)^n y^{-n} + \sum_{m=0}^{\infty} (-1)^m x^{-m} \right) \\ &+ \left(\sum_{m,n=1}^{\infty} - \sum_{m,n=-1}^{-\infty} \right) (-1)^{m+n} x^{-m} y^{-n} q^{mn} \end{aligned}$$

which, in $\mathcal{R}(\Pi^R)$, is equivalent to

$$(13.20) \quad \begin{aligned} &\prod_{n \geq 1} \frac{(1-q^n)^2(1-x^{-1}y^{-1}q^{n-1})(1-xyq^n)}{(1+x^{-1}q^n)(1+xq^n)(1+y^{-1}q^{n-1})(1+yq^n)} = 1 + \left(\sum_{n=1}^{\infty} (-1)^n (1+x)y^{-n} \right) \\ &+ (1+x^{-1}) \left(\sum_{m,n=1}^{\infty} - \sum_{m,n=-1}^{-\infty} \right) (-1)^{m+n} x^{-m} y^{-n} q^{mn}. \end{aligned}$$

We note that we can specialize $x = 1$ in both sides so we obtain the identity (setting $z = y^{-1}$)

$$\frac{\varphi(q)^2 \vartheta_0(z)}{\varphi_2(q) \vartheta_0(-z)} = \varphi_2(q) \left(1 + 2 \sum_{n=1}^{\infty} (-1)^n z^n + 2 \left(\sum_{m,n=1}^{\infty} - \sum_{m,n=-1}^{-\infty} \right) (-1)^{m+n} z^n q^{mn} \right).$$

13.4. $\mathfrak{g} = D(2, 1; 1) = \mathfrak{spo}(2|4)$, $k_0 = -\frac{1}{2}$. In the NS sector, (13.3) gives, letting $x = e^{\theta_1/2}$, $y = e^{\theta_2/2}$:

$$\begin{aligned} & \frac{\varphi(q)^3 \vartheta_0(x^{-2}) \vartheta_0(y^{-2})}{\vartheta_1(xy) \vartheta_1(xy^{-1})} = \\ & \sum_{m,n \in \mathbb{Z}} \left(\frac{x^{2m} y^{2n}}{1 + xy q^{m+n+\frac{1}{2}}} - \frac{x^{-2m-2} y^{2n}}{1 + x^{-1} y q^{m+n+\frac{1}{2}}} - \frac{x^{2m} y^{-2n-2}}{1 + xy^{-1} q^{m+n+\frac{1}{2}}} + \frac{x^{-2m-2} y^{-2n-2}}{1 + x^{-1} y^{-1} q^{m+n+\frac{1}{2}}} \right) q^{m^2+n^2+m+n}. \end{aligned}$$

In the Ramond sector, we have two choices for $\eta_{\min} = \pm(\theta_1/2 - \theta_2/2)$. Choosing the + sign, (13.4) gives, letting $e^{\theta_1/2} = x$, $e^{\theta_2/2} = y$:

$$\begin{aligned} & \frac{\varphi(q)^3 \vartheta_0(x^{-2}) \vartheta_0(y^{-2})}{\vartheta_0(-y^{-1} x^{-1}) \vartheta_0(-x^{-1} y)} = \\ & \sum_{m,n \in \mathbb{Z}} \left(\frac{x^{-2m} y^{2n}}{1 + x^{-1} y q^{n+m}} - \frac{x^{2m} y^{2n}}{1 + xy q^{n+m}} - \frac{x^{-2m} y^{-2n-2}}{1 + x^{-1} y^{-1} q^{n+m}} + \frac{x^{2m} y^{-2n-2}}{1 + xy^{-1} q^{n+m}} \right) q^{m^2+n^2+n}. \end{aligned}$$

13.5. $\mathfrak{g} = \mathfrak{spo}(2|2r)$, $r > 2$, $k_0 = -\frac{1}{2}$. Set $y_i = e^{\epsilon_i}$, $i = 1, \dots, r$. Recall that in this case $W^\natural$ is the subgroup of $\{\pm 1\}^r \rtimes \mathfrak{S}_r$ consisting of elements $(i_1, \dots, i_r)\sigma$ with an even number of -1 in $(i_1, \dots, i_r)$. Moreover,

$$M^\natural = \left\{ \sum_{i=1}^r m_i \epsilon_i \mid m_i \in \mathbb{Z}, 1 \leq i \leq r, \sum_{i=1}^r m_i \in 2\mathbb{Z} \right\}.$$

The denominator identity (13.3) is

$$\begin{aligned} & \frac{\varphi(q)^{r+1} \prod_{1 \leq i < j \leq r} \vartheta_0(y_i^{-1} y_j) \vartheta_0(y_i^{-1} y_j^{-1})}{\prod_{1 \leq i \leq r} \vartheta_1(y_i)} \\ & = \prod_{i=1}^r y_i^{i-r} \sum_{\bar{w} \in W^\natural} \sum_{m_1, \dots, m_r} \det(\bar{w}) \frac{\bar{w}(\prod_{i=1}^r y_i^{(2r-3)m_i + r-i})}{1 + \bar{w}(y_1) q^{m_1 + \frac{1}{2}}} q^{(r-\frac{3}{2}) \sum_i m_i^2 + \sum_i (r-i)m_i}. \end{aligned}$$

Choosing $\eta_{\min} = \epsilon_r$, the denominator identity (13.4) is

$$\begin{aligned} & \prod_{n=1}^{\infty} \frac{\varphi(q)^{r+1} \prod_{1 \leq i < j \leq r} \vartheta_0(y_i^{-1} y_j) \vartheta_0(y_i^{-1} y_j^{-1})}{\prod_{1 \leq i \leq r} \vartheta_0(-y_i^{-1})} \\ & = \frac{1}{2} \prod_{i=1}^r y_i^{\frac{1}{2}+i-r} \sum_{\bar{w} \in W^\natural} \sum_{m_1, \dots, m_r} \det(\bar{w}) \frac{\bar{w}(\prod_{i=1}^r y_i^{(2r-3)m_i + r-i-\frac{1}{2}})}{1 + \bar{w}(y_r^{-1}) q^{-m_r}} q^{(r-\frac{3}{2}) \sum_i m_i^2 + \sum_i (r-i-\frac{1}{2})m_i}. \end{aligned}$$

13.6. $\mathfrak{g} = \mathfrak{spo}(2|2r+1)$, $r \geq 2$, $k_0 = -\frac{1}{2}$. Set $y_i = e^{\epsilon_i}$, $i = 1, \dots, r$. Recall that in this case $W^\natural \cong \{\pm 1\}^r \rtimes \mathfrak{S}_r$; if the isomorphism is $\bar{w} \leftrightarrow (i_1, \dots, i_r)\sigma$, $i_j \in \{\pm 1\}$, the action of σ on the y_i is just the permutation action, whereas $(i_1, \dots, i_r)(y_j) = y_{i_j}^{i_j}$. The lattice $M^\natural$ is the same

as in the even case. The denominator identity (13.3) is

$$\begin{aligned} & \frac{\varphi(q)^{r+1} \prod_{1 \leq i < j \leq r} \vartheta_0(y_i^{-1} y_j) \vartheta_0(y_i^{-1} y_j^{-1}) \prod_{1 \leq i \leq r} \vartheta_0(y_i^{-1})}{\varphi_1(q) \prod_{1 \leq i \leq r} \vartheta_1(y_i)} \\ &= \prod_{i=1}^r y_i^{\frac{2i-2r-1}{2}} \sum_{\bar{w} \in W^{\natural}} \sum_{m_1, \dots, m_r} \det(\bar{w}) \frac{\bar{w}(\prod_{i=1}^r y_i^{2(r-1)m_i + \frac{2r+1-2i}{2}})}{1 + \bar{w}(y_1) q^{m_1 + \frac{1}{2}}} q^{(r-1) \sum_i m_i^2 + \sum_i (r-i)m_i}. \end{aligned}$$

The denominator identity (13.4) is

$$\begin{aligned} & \frac{\varphi(q)^{r+1} \prod_{1 \leq i < j \leq r} \vartheta_0(y_i^{-1} y_j) \vartheta_0(y_i^{-1} y_j^{-1}) \prod_{1 \leq i \leq r} \vartheta_0(y_i^{-1})}{\varphi_2(q) \prod_{1 \leq i \leq r} \vartheta_0(-y_i^{-1})} \\ &= \prod_{i=1}^r y_i^{i-r} \sum_{\bar{w} \in W^{\natural}} \sum_{m_1, \dots, m_r} \det(\bar{w}) \frac{\bar{w}(\prod_{i=1}^r y_i^{2(r-1)m_i + r-i})}{1 + \bar{w}(y_r^{-1}) q^{-m_r}} q^{(r-1) \sum_i m_i^2 + \sum_i (r-i + \frac{1}{2})m_i}. \end{aligned}$$

13.7. $\mathfrak{g} = F(4), k_0 = -\frac{2}{3}$. Set $y_i = e^{\epsilon_i/2}$, $i = 1, 2, 3$. Since $\mathfrak{g}^{\natural} = so(7)$ we identify $W^{\natural}$ with $\{\pm 1\}^3 \rtimes \mathfrak{S}_3$ as in the $spo(2|2r+1)$ case.

In the NS sector, (13.3) reads

$$\begin{aligned} & \frac{\varphi(q)^4 \prod_{1 \leq i \leq 3} \vartheta_0(y_i^{-2}) \prod_{1 \leq i \leq j \leq 3} \vartheta_0(y_i^{-2} y_j^{-2}) \prod_{1 \leq i \leq j \leq 3} \vartheta_0(y_i^{-2} y_j^2)}{\vartheta_1(y_1 y_2 y_3) \vartheta_1(y_1 y_2 y_3^{-1}) \vartheta_1(y_1 y_2^{-1} y_3) \vartheta_1(y_1^{-1} y_2 y_3)} \\ &= y_1^{-5} y_2^{-3} y_3^{-1} \sum_{\substack{m, r, t \in \mathbb{Z} \\ m+t+r \equiv 0 \pmod{2}}} q^{2m^2 + 2r^2 + 2t^2 + \frac{5m+3r+t}{2}} \left(\sum_{\bar{w} \in W^{\natural}} \det(\bar{w}) \frac{\bar{w}(y_1^{8m+5} y_2^{8r+3} y_3^{8t+1})}{1 + q^{b_{m,r,t}} \bar{w}(y_1 y_2 y_3)} \right), \end{aligned}$$

where $b_{m,r,t} = (m + t + r + 1)/2$. In the Ramond sector, we choose $\eta_{\min} = \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3)$. Then (13.4) becomes

$$\begin{aligned} & \frac{\varphi(q)^4 \prod_{1 \leq i \leq 3} \vartheta_0(y_i^{-2}) \prod_{1 \leq i \leq j \leq 3} \vartheta_0(y_i^{-2} y_j^{-2}) \prod_{1 \leq i \leq j \leq 3} \vartheta_0(y_i^{-2} y_j^2)}{\vartheta_0(-y_1^{-1} y_2^{-1} y_3^{-1}) \vartheta_0(-y_1^{-1} y_2^{-1} y_3) \vartheta_0(-y_1^{-1} y_2 y_3^{-1}) \vartheta_0(-y_1 y_2^{-1} y_3^{-1})} \\ &= y_1^{-4} y_2^{-2} \sum_{\substack{m, r, t \in \mathbb{Z} \\ m+t+r \equiv 0 \pmod{2}}} q^{2m^2 + 2r^2 + 2t^2 + 2m+r} \left(\sum_{\bar{w} \in W^{\natural}} \det(\bar{w}) \frac{\bar{w}(y_1^{8m+4} y_2^{8r+2} y_3^{8t})}{1 + q^{b_{m,r,t}} \bar{w}(y_1 y_2^{-1} y_3^{-1})} \right), \end{aligned}$$

where $\bar{b}_{m,r,t} = (m - r - t)/2$.

13.8. $\mathfrak{g} = G(3), k_0 = -\frac{3}{4}$. Set $y_i = e^{\epsilon_i}$, $i = 1, 2$. Since $\mathfrak{g}^{\natural} = G_2$, $W^{\natural}$ is the dihedral group of order 12 with Coxeter generators s_1, s_2 acting as

$$s_1(y_1) = y_1^{-1}, \quad s_2(y_1) = y_2, \quad s_1(y_2) = y_1 y_2, \quad s_2(y_2) = y_1.$$

In the NS sector, (13.3) becomes

$$\begin{aligned} & \frac{\varphi(q)^3 \vartheta_0(y_1^{-1}) \vartheta_0(y_2^{-1}) \vartheta_0(y_1 y_2^{-1}) \vartheta_0(y_1^{-1} y_2^{-1}) \vartheta_0(y_1^{-2} y_2^{-1}) \vartheta_0(y_1^{-1} y_2^{-2})}{\varphi_1(q) \vartheta_1(y_1) \vartheta_1(y_2) \vartheta_1(y_1 y_2)} \\ &= y_1^{-2} y_2^{-3} \sum_{\substack{m, n \in \mathbb{Z} \\ m+n \equiv 0 \pmod{3}}} q^{m^2 + n^2 + \frac{m-3mn+4n}{3}} \left(\sum_{\bar{w} \in W^{\natural}} \det(\bar{w}) \frac{\bar{w}(y_1^{3m+2} y_2^{3n+3})}{1 + \bar{w}(y_1 y_2) q^{a_{m,n}}} \right), \end{aligned}$$

where $a_{m,n} = \frac{m+n}{3} + \frac{1}{2}$. In the Ramond sector, (13.4) becomes

$$\frac{\varphi(q)^3 \vartheta_0(y_1^{-1}) \vartheta_0(y_2^{-1}) \vartheta_0(y_1 y_2^{-1}) \vartheta_0(y_1^{-1} y_2^{-1}) \vartheta_0(y_1^{-2} y_2^{-1}) \vartheta_0(y_1^{-1} y_2^{-2})}{\varphi_2(q) \vartheta_0(-y_1^{-1}) \vartheta_0(-y_2^{-1}) \vartheta_0(-y_1^{-1} y_2^{-1})} =$$

$$y_1^{-1} y_2^{-2} \sum_{\substack{m,n \in \mathbb{Z} \\ m+n \equiv 0 \pmod{3}}} \left(\sum_{\bar{w} \in W^{\natural}} \det(\bar{w}) \frac{\bar{w}(y_1^{3m+1} y_2^{3m+2})}{1 + \bar{w}((y_1 y_2)^{-1}) q^{a_{m,n}}} \right) q^{m^2 + n^2 + m - mn},$$

where $\bar{a}_{m,n} = \frac{n-2m}{3}$.

14. APPENDIX. DENOMINATOR IDENTITY FOR MINIMAL W -ALGEBRAS OF DELIGNE SERIES

Let $\mathfrak{g}$ be the simple Lie algebra D_4 , E_6 , E_7 , or E_8 , and let $a = \frac{h^\vee}{6} + 1$. Then, for any integer j such that $0 < j < a$, there exist unique simple roots $\alpha_1, \dots, \alpha_j$ such that $\theta - \sum_{i=1}^j \alpha_i$ is a root. Set $\alpha = \theta - \sum_{i=1}^{a-1} \alpha_i$. Then $(\rho|\alpha) = h^\vee - a$. Let $k_0 = -a$, then $(k_0 \Lambda_0 + \hat{\rho}|\alpha) = k_0 + h^\vee = b = \frac{h^\vee + h^\vee}{2}$ (in our cases $b = 4, 9, 14, 24$ respectively). By [4, Theorem 7.2] or [2, Proposition 3.4], $\dim W_{k_0}^{\min}(\mathfrak{g}) = 1$. A character formula, for certain $\mathfrak{g}$ -modules $L(\Lambda)$ of negative integer level $k \geq k_0$, has been conjectured in [21, (3.1)] and proved in [5], formulas (5) and (6). A special case of this formula is

$$(14.1) \quad \widehat{R}ch L(k_0 \Lambda_0) = \frac{1}{2} \sum_{w \in W} \det(w) \sum_{\gamma \in Q} ((\gamma|\alpha) + 1) e^{wt_\gamma(k_0 \Lambda_0 + \hat{\rho}) - \hat{\rho}},$$

where Q is the root lattice of $\mathfrak{g}$ and W is its Weyl group. Let

$$\widehat{F}^{NS} = \prod_{n=1}^{\infty} (1 - q^n)^{\dim \mathfrak{h}} \prod_{\alpha \in \Delta_+^{\natural}} (1 - q^{n-1} e^{-\alpha}) (1 - q^n e^{\alpha}) \prod_{\beta \in \Delta_{\frac{1}{2}}} (1 - q^{\frac{1}{2}+n} e^{\beta|_{\mathfrak{h}^{\natural}}}).$$

be the denominator for the W -algebra $W_{\min}^k(\mathfrak{g})$.

Recall that any element $w \in W$ can be uniquely written as $w = w^{\natural} \bar{w}$, where $w^{\natural} \in W^{\natural}$ and $\bar{w}$ is a right coset representative of $W^{\natural}$ in W of minimal length, and that $\rho - \bar{w}(\rho) = \sum_{\eta \in N(\bar{w})} \eta$, where $N(\bar{w}) = \{\eta \in \Delta^+ \mid -\bar{w}^{-1}(\eta) \in \Delta^+\}$. If $\bar{w}^{-1}(\theta) \in \Delta^+$ then $(\eta|\theta) = 1$ for each $\eta \in N(\bar{w})$, so $(\rho - \bar{w}(\rho)|\theta) = \ell(\bar{w})$. Observe that the map $\bar{w} \mapsto \bar{w}^{-1}(\theta)$ is a bijection between $W^{\natural} \backslash W$ and Δ . Let $\eta \mapsto \bar{w}_\eta$ be its inverse.

Theorem 14.1.

$$(14.2) \quad \widehat{F}^{NS} =$$

$$\frac{e^{-\rho|_{\mathfrak{h}^{\natural}}}}{2} \sum_{\substack{\eta \in \Delta^+ \\ \gamma \in Q \\ w^{\natural} \in W^{\natural}}} \det(\bar{w}_\eta w^{\natural}) (\gamma|\alpha) e^{w^{\natural} \bar{w}_\eta(b\gamma + \rho)|_{\mathfrak{h}^{\natural}}} q^{(\rho|\gamma) + \frac{b(\gamma|\gamma) + h^\vee - 1}{2}} \left(q^{\frac{c_{\gamma,\eta}}{2}} + q^{-\frac{c_{\gamma,\eta}}{2}} \right),$$

where $c_{\gamma,\eta} = b(\gamma|\eta) - \ell(\bar{w}_\eta) + h^\vee - 1$.

Proof. Since $H_0(L(k_0 \Lambda_0)) = W_{k_0}^{\min}(\mathfrak{g}) = \mathbb{C}|0\rangle$, from (14.1) we obtain

$$(14.3) \quad \widehat{F}^{NS} = \frac{1}{2} \sum_{w \in W} \det(w) \sum_{\gamma \in Q} ((\gamma|\alpha) + 1) e^{ev(wt_\gamma(k_0 \Lambda_0 + \hat{\rho}) - \hat{\rho})}$$

Note that the coefficient $(\gamma|\alpha) + 1$ can be replaced by $(\gamma|\alpha)$ since the term corresponding to 1 in the R.H.S. of (14.1), multiplied by $2e^{\hat{\rho}}$, is $\sum_{w \in \widehat{W}} \det(w) e^{w(k_0 \Lambda_0 + \hat{\rho})}$, which is 0 since $k_0 \Lambda_0 + \hat{\rho}$ is orthogonal to $\delta - \alpha$.

By the definition (8.5) of ev we have

$$(14.4) \quad ev(wt_\gamma(k_0\Lambda_0 + \hat{\rho}) - \hat{\rho}) = ((wt_\gamma(k_0\Lambda_0 + \hat{\rho}) - \hat{\rho})(-x - D), (wt_\gamma(k_0\Lambda_0 + \hat{\rho}) - \hat{\rho})|_{\mathfrak{h}^\natural}).$$

To compute the R.H.S. of (14.4), use formula (13.2), noting that $u_i = 2$ for simply laced Lie algebras:

$$wt_\gamma(k_0\Lambda_0 + \hat{\rho}) - \hat{\rho} = k_0\Lambda_0 + w(b\gamma + \rho) - \rho - ((\rho|\gamma) + b\frac{(\gamma|\gamma)}{2})\delta,$$

so (14.3) becomes

$$\hat{F}^{NS} = \frac{1}{2} \sum_{w \in W} \det(w) \sum_{\gamma \in Q} (\gamma|\alpha) e^{(w(b\gamma + \rho) - \rho)|_{\mathfrak{h}^\natural}} q^{(\rho|\gamma) + b\frac{(\gamma|\gamma)}{2}} q^{(w(b\gamma + \rho) - \rho)(-x - D)}$$

To compute

$$(w(b\gamma + \rho) - \rho)(-x - D) = (w(b\gamma + \rho) - \rho)(-x),$$

observe that $\bar{w}_{-\eta}$ and $s_\theta \bar{w}_\eta$ are in the same right coset mod $W^\natural$, so the set $\{\bar{w}_\eta \mid \eta \in \Delta^+\} \cup \{s_\theta \bar{w}_\eta \mid \eta \in \Delta^+\}$ is a set of right coset representatives for $W^\natural \backslash W$. If $w = w^\natural \bar{w}_\eta$, $\eta \in \Delta^+$, then

$$(14.5) \quad \begin{aligned} (w(b\gamma + \rho) - \rho)(-x) &= (w(b\gamma + \rho) - \rho| - \theta/2) = (\bar{w}_\eta(b\gamma + \rho) - \rho| - \theta/2) \\ &= -b(\bar{w}_\eta \gamma | \theta/2) + (\rho - \bar{w}_\eta(\rho) | \theta/2) = -\frac{b}{2}(\gamma | \bar{w}_\eta^{-1}(\theta)) + \frac{1}{2}(\rho - \bar{w}_\eta(\rho) | \theta). \end{aligned}$$

On the other hand, if $w = w^\natural s_\theta \bar{w}_\eta$, then

$$(14.6) \quad \begin{aligned} (w(b\gamma + \rho) - \rho)(-x) &= (w(b\gamma + \rho) - \rho| - \theta/2) = (\bar{w}_\eta(b\gamma + \rho) + \rho) | \theta/2 \\ &= b(\bar{w}_\eta \gamma | \theta/2) + (\rho | \theta) + (\bar{w}_\eta(\rho) - \rho | \theta/2) = \frac{b}{2}(\gamma | \bar{w}_\eta^{-1}(\theta)) + (\rho | \theta) - \frac{1}{2}(\rho - \bar{w}_\eta(\rho) | \theta). \end{aligned}$$

Since $w^\natural s_\theta \bar{w}_\eta(b\gamma + \rho)|_{\mathfrak{h}^\natural} = s_\theta w^\natural \bar{w}_\eta(b\gamma + \rho)|_{\mathfrak{h}^\natural} = w^\natural \bar{w}_\eta(b\gamma + \rho)|_{\mathfrak{h}^\natural}$, plugging (14.5) and (14.6) into (14.3), by the discussion preceding the statement of the theorem, we obtain

$$\begin{aligned} \hat{F}^{NS} &= \frac{1}{2} \sum_{\eta \in \Delta^+} \det(\bar{w}_\eta) \sum_{\gamma \in Q} (\gamma|\alpha) \sum_{w^\natural \in W^\natural} \det(w^\natural) e^{w^\natural \bar{w}_\eta(b\gamma + \rho)|_{\mathfrak{h}^\natural} - \rho|_{\mathfrak{h}^\natural}} q^{(\rho|\gamma) + \frac{b(\gamma|\gamma - \eta) + \ell(\bar{w}_\eta)}{2}} \\ &\quad + \frac{1}{2} \sum_{\eta \in \Delta^+} \det(\bar{w}_\eta) \sum_{\gamma \in Q} (\gamma|\alpha) \sum_{w^\natural \in W^\natural} \det(w^\natural) e^{w^\natural \bar{w}_\eta(b\gamma + \rho)|_{\mathfrak{h}^\natural} - \rho|_{\mathfrak{h}^\natural}} q^{(\rho|\gamma + \theta) + \frac{b(\gamma|\gamma + \eta) - \ell(\bar{w}_\eta)}{2}}. \end{aligned}$$

which, observing that $\rho|_{\mathfrak{h}^\natural} = \rho^\natural$, is (14.2). $\square$

Acknowledgements. The authors are grateful to T. Arakawa for correspondence and to D. Adamović for useful discussions. P.P. wishes to thank C. Krattenthaler for useful discussions. P. M-F. wishes to thank Indam for its hospitality during the workshop held in Rome from 11 to 15 December 2023. P.M-F. and P.P. are partially supported by the PRIN project 2022S8SSW2 - Algebraic and geometric aspects of Lie theory - CUP B53D2300942 0006, a project cofinanced by European Union - Next Generation EU fund. V.K. is partially supported by the Simons Collaboration grant.

REFERENCES

- [1] D. ADAMOVIĆ, V. G. KAC, P. MÖSENER FRAJRIA, AND P. PAPI, *Defining relations for minimal unitary quantum affine W -algebras*, Commun. Math. Phys., **405**, No. 2 (2024), 33 pp.
- [2] D. ADAMOVIĆ, V. G. KAC, P. MÖSENER FRAJRIA, P. PAPI, AND O. PERŠE, *Conformal embeddings of affine vertex algebras in minimal W -algebras I: structural results*, J. Algebra, **500** (2018), pp. 117–152.
- [3] T. ARAKAWA, *Representation theory of superconformal algebras and the Kac-Roan-Wakimoto conjecture*, Duke Math. J., **130**, No. 3 (2005), pp. 435–478.

- [4] T. ARAKAWA AND A. MOREAU, *Joseph ideals and lisse minimal W -algebras*, J. Inst. Math. Jussieu, **17**, No. 2 (2018), pp. 397–417.
- [5] R. BEZRUKAVNIKOV, V. G. KAC, AND V. KRYLOV, *Subregular nilpotent orbits and explicit character formulas for modules over affine Lie algebras*, Pure and Applied Mathematics Quarterly, **20**, No. 1 (2024), pp. 81–138.
- [6] A. DE SOLE AND V. G. KAC, *Finite vs affine W -algebras*, Jpn. J. Math., **1**, No. 1 (2006), pp. 137–261.
- [7] C. DONG AND X. LIN, *Unitary vertex operator superalgebras*, J. Algebra, **397** (2014), pp. 252–277.
- [8] T. EGUCHI AND A. TAORMINA, *Unitary representations of the $N = 4$ superconformal algebra*, Phys. Lett. B, **196**, No. 1 (1987), pp. 75–81.
- [9] ———, *On the unitary representations of $N = 2$ and $N = 4$ superconformal algebras*, Phys. Lett. B, **210**, No. 1-2 (1988), pp. 125–132.
- [10] M. GORELIK AND V. G. KAC, *Characters of (relatively) integrable modules over affine Lie superalgebras*, Jpn. J. Math., **10**, No. 2 (2015), pp. 135–235.
- [11] M. GORELIK, V. G. KAC, P. MÖSENER FRAJRIA, AND P. PAPI, *Denominator identities for finite-dimensional Lie superalgebras and Howe duality for compact dual pairs*, Jpn. J. Math., **7**, No. 1 (2012), pp. 41–134.
- [12] M. GORELIK AND V. SERGANOVA, *Integrable modules over affine Lie superalgebras $\mathfrak{sl}(1|n)^{(1)}$* , Commun. Math. Phys., **364**, No. 2 (2018), pp. 635–654.
- [13] V. G. KAC, *Infinite-dimensional Lie algebras*, Cambridge University Press, Cambridge, third ed., 1990.
- [14] V. G. KAC, P. MÖSENER FRAJRIA, AND P. PAPI, *Invariant Hermitian forms on vertex algebras*, Commun. Contemp. Math., **24**, No. 5 (2022), 41 pp.
- [15] ———, *Unitarity of minimal W -algebras and their representations I*, Commun. Math. Phys., **401**, No. 1 (2023), pp. 79–145.
- [16] V. G. KAC, S.-S. ROAN, AND M. WAKIMOTO, *Quantum reduction for affine superalgebras*, Commun. Math. Phys., **241**, No. 2-3 (2003), pp. 307–342.
- [17] V. G. KAC AND M. WAKIMOTO, *Modular invariant representations of infinite-dimensional Lie algebras and superalgebras*, Proc. Nat. Acad. Sci. U.S.A., **85**, No. 14 (1988), pp. 4956–4960.
- [18] ———, *Integrable highest weight modules over affine superalgebras and number theory*, in Lie theory and geometry, vol. 123 of Progr. Math., Birkhäuser Boston, MA, 1994, pp. 415–456.
- [19] ———, *Quantum reduction and representation theory of superconformal algebras*, Adv. Math., **185**, No. 2 (2004), pp. 400–458.
- [20] ———, *Quantum reduction in the twisted case*, in Infinite dimensional algebras and quantum integrable systems, vol. 237 of Progr. Math., Birkhäuser, Basel, 2005, pp. 89–131.
- [21] ———, *On characters of irreducible highest weight modules of negative integer level over affine Lie algebras*, in Lie groups, geometry, and representation theory, vol. 326 of Progr. Math., Birkhäuser/Springer, Cham, 2018, pp. 235–252.
- [22] H.-S. LI, *Local systems of twisted vertex operators, vertex operator superalgebras and twisted modules*, in Moonshine, the Monster, and related topics (South Hadley, MA, 1994), vol. 193 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1996, pp. 203–236.
- [23] K. MIKI, *The representation theory of the $SO(3)$ invariant superconformal algebra*, Internat. J. Modern Phys. A, **5**, No. 7 (1990), pp. 1293–1318.
- [24] A. PREMÉT, *Enveloping algebras of Slodowy slices and the Joseph ideal*, J. Eur. Math. Soc. (JEMS), **9**, No. 3 (2007), pp. 487–543.