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Parabolic Orbits of 2-Nilpotent Elements for Classical Groups

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Abstract. We consider the conjugation-action of the Borel subgroup of the symplectic or the orthogonal group on the variety of nilpotent complex elements of nilpotency degree 2 in its Lie algebra. We translate the setup to a representation-theoretic context in the language of a symmetric quiver algebra. This makes it possible to provide a parametrization of the orbits via a combinatorial tool that we call symplectic/orthogonal oriented link patterns. We deduce information about numerology. We then generalize these classifications to standard parabolic subgroups for all classical groups. Finally, our results are restricted to the nilradical.

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Key Words: B-orbits, symmetric quiver, algebra with self-duality, combinatorial classification, Auslander-Reiten quiver

1. Introduction

Let G be a classical complex group of rank n. Then G is either the general linear group $\mathrm{GL}_n(K)$ or the symplectic group $\mathrm{SP}_{2l}(K)$ or the orthogonal group $\mathrm{O}_n(K)$, where $K = \mathbf{C}$. Let $\mathfrak{g} = \mathrm{Lie}\, G$ be the corresponding Lie algebra.

The study of the adjoint action of (subgroups of) G on $\mathfrak{g}$ and numerous variants thereof is a well-established and much considered task in algebraic Lie theory. Employing methods of geometric invariant theory, a classical topic is the study of orbits and their closures, which is also known as the *vertical* problem [11].

One famous example of a classification problem alike is the study of GL_n-conjugation (or SL_n-conjugation, this doesn't make a difference) on the variety of complex matrices of square-size n. A complete system of representatives up to conjugation is given by the Jordan canonical form [10] which dates back to the 19th century. This system of representatives is given by continuous parameters, the eigenvalues of the matrix, and discrete parameters. In order to determine the latter, it suggests itself to restrict the action to the nilpotent cone, namely to GL_n-conjugation on the set of nilpotent matrices. The number of conjugacy classes of nilpotent matrices is finite and can be described combinatorially by partitions of n.

One generalization of this setup is obtained by restricting the acting group from G to parabolic subgroups $P \subseteq G$. In particular, the Borel subgroup B is considered, then, and the question about a variety admitting only finitely many orbits is closely related to the concept of so-called spherical varieties [5]. One example of a parabolic action can be found in [9], where Hille and Röhrle prove a finiteness criterion for the number of orbits of parabolic conjugation on the unipotent radical of the Lie algebra of P.

Another adaption of the above setup is given by restricting the nilpotent cone $\mathcal{N}$ of nilpotent matrices to certain subvarieties. For example, Melnikov parametrizes the B-orbits in the variety of 2-nilpotent elements in the nilradical $\mathfrak{n}$ of $\mathrm{Lie}(\mathrm{GL}_n(K))$ in [14] which is inspired by the study of orbital varieties. A parametrization in the symplectic setup is published by Barnea and Melnikov in [2]. In [8], Gandini, Maffei, Möseneder Frajria and Papi consider B-stable abelian subalgebras of the nilradical of $\mathfrak{b} = \mathrm{Lie}\, B$ in which they parametrize the B-orbits and describe their closure relations.

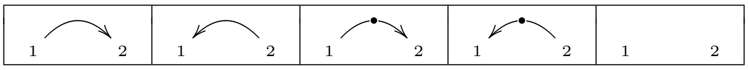
Since the Lie algebra $\mathfrak{g}$ is realized as a space of complex matrices of square size n in a natural way, we can consider potencies of elements in $\mathfrak{g}$ which are given by matrix multiplication. In this article, we consider the algebraic subvariety $\mathcal{N}(2)$ of 2-nilpotent elements of the nilpotent cone of $\mathfrak{g}$, namely

$$\mathcal{N}(2) = \mathcal{N}(2, G) = \{x \in \mathfrak{g} \mid x^2 = 0\}.$$

Every parabolic subgroup P of G acts on $\mathcal{N}(2)$. It is known that the number of orbits is always finite, since Panyushev shows finiteness for the Borel-action in [15]. In case $G = \mathrm{GL}_n(K)$, a parametrization of the P-orbits and a description of their degenerations is given in [4] and [3] for each parabolic subgroup $P \subset G$.

Our first goal in this article is to prove in a different manner that there are only finitely many B-orbits in $\mathcal{N}(2)$ for the remaining classical groups, that is, for types B, C and D. We approach the problem in a way closely related to [4] from a quiver-theoretic point of view – but instead of translating to the representation variety of a quiver with relations of a special dimension vector, we translate the orbits to certain (isomorphism classes of) representations of a *symmetric* quiver with relations of a fixed dimension vector. In this setup we show that there are only finitely many of the latter which are parametrized by combinatorial objects which we call symplectic/orthogonal oriented link patterns, see Definitions 6.2 and 6.7.

For example the Borel-orbits of 2-nilpotent matrices in $\mathfrak{o}_4$ are parametrized by these five patterns:



The following five matrices give a system of representatives of these orbits.

$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Our second goal is to parametrize all orbits explicitly. The approach via a symmetric quiver makes it possible to classify the orbits by representations, and thus, by combinatorial data.

We afterwards generalize these results to parabolic subgroups $P \subset G$. To do so, we consider the isotropic flag corresponding to P and realize it as a representation M_P.

The parabolic P equals the symmetric stabilizer of M_P and we obtain a classification of the P-orbits in $\mathcal{N}(2)$ as in the Borel case.

In the last section, we restrict our results to the action of P on $\mathcal{N}_P(2) := \mathcal{N}(2) \cap \operatorname{Lie} P$ and $\mathfrak{n}_P(2) := \mathcal{N}(2) \cap \mathfrak{n}_P$, that is, on the intersection of $\mathcal{N}(2)$ with the Lie algebra $\operatorname{Lie} P$ of P or with the nilradical $\mathfrak{n}_P$ of $\operatorname{Lie} P$ and obtain parametrizations here. In the symplectic case, for the Borel action, the parametrization coincides with the parametrization by so-called symplectic link patterns of [2], the used methods are different, though.

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2. Classical groups and Lie algebras

Let K be the field of complex numbers $K := \mathbf{C}$ and let n be an integer. We consider the complex classical groups, that is, the general linear group $\mathrm{GL}_n := \mathrm{GL}_n(K)$, the symplectic group $\mathrm{SP}_n := \mathrm{SP}_n(K)$, whenever $n = 2l$ for some integer l, and the orthogonal group $\mathrm{O}_n := \mathrm{O}_n(K)$. The corresponding Lie algebras are denoted by $\mathfrak{gl}_n := \mathfrak{gl}_n(K)$, $\mathfrak{sp}_n := \mathfrak{sp}_n(K)$ and $\mathfrak{o}_n := \mathfrak{o}_n(K)$.

In general, given a vector space V endowed with a non-degenerate bilinear form $\langle -, - \rangle$, let us denote by $\mathrm{Sym}(V)$ the group of symmetries of the vector space V which preserve $\langle -, - \rangle|_{V \times V}$. Then $\mathrm{Sym}(V)$ equals either the symplectic group $\mathrm{SP}(V)$ or the orthogonal group $\mathrm{O}(V)$, depending on whether $(V, \langle -, - \rangle)$ is symplectic or orthogonal. We define $\mathfrak{sym}(V) := \mathrm{Lie}(\mathrm{Sym}(V))$.

Let l be an integer, then we denote by $J = J_l$ the $l \times l$ anti-diagonal matrix with every entry on the anti-diagonal being 1:

$$J_l = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}$$

It is easy to see (and well-known) that $J^{-1} = J$ and that the conjugate $J\,{}^{T}\!AJ$ by J of the *transpose* ${}^{T}\!A$ of a matrix $A \in K^{l \times l}$ is given by "the transpose of A with respect to the anti-diagonal". For example, for $l = 2$, given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$J\,{}^{T}\!AJ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} d & b \\ c & a \end{bmatrix}.$$

We set

$${}^{\mathfrak{T}}\!A := J\,{}^{T}\!AJ.$$

In this notation, it is easy to write down the elements of the symplectic and orthogonal Lie algebras.

2.1. Symplectic group. Let V be an $n = 2l$-dimensional complex vector space. Let us fix a basis of V and a bilinear form $F = F_V\colon V \times V \to K$, $F(v,w) = \langle v, w\rangle$, associated with the matrix (still denoted by F)

$$F = \left[\begin{array}{cc} 0 & J_l \\ -J_l & 0 \end{array}\right]. \tag{1}$$

The symplectic group SP_n consists of those matrices $A \in \mathrm{GL}_n$ which preserve this bilinear form (i.e. $\langle Av, Aw\rangle = \langle v, w\rangle$); in other words A satisfies the equation

$${}^{T}\!AFA = F.$$

The Lie algebra $\mathfrak{sp}_n$ of SP_n consists of those matrices $a \in \mathfrak{gl}_n$ which fulfill

$${}^{T}\!aF + Fa = 0. \tag{2}$$

We write $a = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, where A, B, C, D are $l \times l$-blocks, so that condition (2) translates into the following equations:

$$a = \left[\begin{array}{cc} A & B = {}^{\mathfrak{T}}\!B \\ C = {}^{\mathfrak{T}}\!C & D = -\,{}^{\mathfrak{T}}\!A \end{array}\right]. \tag{3}$$

In particular, $\mathfrak{sp}_n$ has dimension $l^2 + l(l+1) = l(n+1)$. The intersection of $\mathfrak{sp}_n$ with the Borel subalgebra $\mathfrak{b}_n := \mathfrak{b}_n(K)$ of upper-triangular matrices is a solvable subalgebra of $\mathfrak{sp}_n$ of dimension $l(l+1) = l^2 + l$. Since $\mathfrak{sp}_n$ is a Lie algebra of type C_l, the number of positive roots is l^2 and the number of simple roots is l; we hence see that $\mathfrak{b}(\mathfrak{sp}_n) := \mathfrak{sp}_n \cap \mathfrak{b}_n$ is a solvable subalgebra of maximal dimension and hence a Borel subalgebra. This is one of the advantages of working with the form F given by (1).

2.2. Orthogonal group. Let V be an n-dimensional complex vector space (where n can be even or odd). Let us fix a basis of V and let us choose the non-degenerate bilinear form on V associated with the matrix $F = J_n$. The orthogonal group O_n consists of those matrices $A \in \mathrm{GL}_n$ for which ${}^{T}\!AFA = F$ holds true. The Lie algebra $\mathfrak{o}_n$ consists of those matrices $a \in \mathfrak{gl}_n$ satisfying (2) which translates into the relation

$$a = -\,{}^{\mathfrak{T}}\!a. \tag{4}$$

In particular, $\mathfrak{o}_n$ has dimension $\frac{n(n-1)}{2}$. The intersection of $\mathfrak{o}_n$ with the Borel subalgebra $\mathfrak{b}_n$ of upper-triangular matrices, is a solvable subalgebra of $\mathfrak{o}_n$.

- If $n = 2l$, the dimension of such a solvable subalgebra is easily seen to be

  $$\frac{n(n-1)}{2} - l(l-1) = l(2l-1) - l(l-1) = l^2.$$

  Since $\mathfrak{o}_n$ is a Lie algebra of type D_l, the number of positive roots is $l(l-1)$ and the number of simple roots is l; we hence see that $\mathfrak{b}(\mathfrak{o}_n) := \mathfrak{o}_n \cap \mathfrak{b}_n$ is a solvable subalgebra of maximal dimension and hence a Borel subalgebra.

- Similarly, if $n = 2l + 1$, the dimension of $\mathfrak{b}(\mathfrak{o}_n) := \mathfrak{o}_n \cap \mathfrak{b}_n$ is easily seen to be
$$\frac{n(n-1)}{2} - (l(l-1) + l) = (2l+1)l - l^2 = l^2 + l.$$
Since $\mathfrak{o}_n$ is a Lie algebra of type B_l, the number of positive roots is l^2 and the number of simple roots is l; we hence see that $\mathfrak{b}(\mathfrak{o}_n)$ is a solvable subalgebra of maximal dimension and hence a Borel subalgebra.

As before, this is one of the benefits of working with the form F given by J_n.

3. Background on (symmetric) quiver representations.

We include basic knowledge about the representation theory of finite-dimensional algebras via finite quivers [1] before introducing the notion of a symmetric quiver and discussing its representations. This theoretical background will be necessary later on to prove our main results.

A *finite quiver* $\mathcal{Q}$ is a directed graph $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, s, t)$, such that $\mathcal{Q}_0$ is a finite set of *vertices* and $\mathcal{Q}_1$ is a finite set of *arrows*, whose elements are written as $\alpha\colon s(\alpha) \to t(\alpha)$. The *path algebra* $K\mathcal{Q}$ is defined as the K-vector space with a basis consisting of all paths in $\mathcal{Q}$, that is, sequences of arrows $\omega = \alpha_s ... \alpha_1$ with $t(\alpha_k) = s(\alpha_{k+1})$ for all $k \in \{1, ..., s-1\}$; formally included is a path ϵ_i of length zero for each $i \in \mathcal{Q}_0$ starting and ending in i. The multiplication is defined as the concatenation of paths $\omega = \alpha_s ... \alpha_1$ and $\omega' = \beta_t ... \beta_1$, that is,
$$\omega \cdot \omega' = \begin{cases} \alpha_s ... \alpha_1 \beta_t ... \beta_1, & \text{if } t(\beta_t) = s(\alpha_1); \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathrm{rad}(K\mathcal{Q})$ be the *path ideal* of $K\mathcal{Q}$, which is the (two-sided) ideal generated by all paths of positive lengths. An ideal $I \subseteq K\mathcal{Q}$ is called *admissible* if there exists an integer s with $\mathrm{rad}(K\mathcal{Q})^s \subset I \subset \mathrm{rad}(K\mathcal{Q})^2$. If this is the case for an ideal I, then the algebra $\mathcal{A} := K\mathcal{Q}/I$ is finite-dimensional.

We denote by $\mathrm{rep}(K\mathcal{Q})$ the abelian K-linear category of all representations of $\mathcal{Q}$ (which is equivalent to the category of $K\mathcal{Q}$-modules). In more detail, the objects are given as finite-dimensional *(K-)representations* of $\mathcal{Q}$ which, in more detail, are tuples
$$((M_i)_{i \in \mathcal{Q}_0}, (M_\alpha\colon M_i \to M_j)_{(\alpha\colon i \to j) \in \mathcal{Q}_1}),$$
where the M_i are K-vector spaces, and the M_α are K-linear maps. A *morphism of representations* $M = ((M_i)_{i \in \mathcal{Q}_0}, (M_\alpha)_{\alpha \in \mathcal{Q}_1})$ and $M' = ((M'_i)_{i \in \mathcal{Q}_0}, (M'_\alpha)_{\alpha \in \mathcal{Q}_1})$ consists of a tuple of K-linear maps $(f_i\colon M_i \to M'_i)_{i \in \mathcal{Q}_0}$, such that $f_j M_\alpha = M'_\alpha f_i$ for every arrow $\alpha\colon i \to j$ in $\mathcal{Q}_1$.

Let us denote by $\mathrm{rep}(\mathcal{A})$ the category of representations of $\mathcal{Q}$ bound by I: For a representation M and a path ω in $\mathcal{Q}$ as above, we denote $M_\omega = M_{\alpha_s} \cdot ... \cdot M_{\alpha_1}$. A representation M is called *bound by* I, if $\sum_\omega \lambda_\omega M_\omega = 0$ whenever $\sum_\omega \lambda_\omega \omega \in I$. The category rep $\mathcal{A}$) is equivalent to the category of finite-dimensional $\mathcal{A}$-representations.

Let M be an $\mathcal{A}$-representation, let $B_i \subseteq \epsilon_i M$ be a K-basis of $\epsilon_i M$ for every $i \in \mathcal{Q}_0$ and let B be the disjoint union of these sets B_i. We define the *coefficient quiver* $\Gamma(M) := \Gamma(M, B)$ of M with respect to the basis B to be the quiver with exactly one vertex for each element of B, such that for each arrow $\alpha \in \mathcal{Q}_1$ and every element $b \in B_{s(\alpha)}$ we have

$$M_\alpha(b) = \sum_{c \in B_{t(\alpha)}} \lambda^\alpha_{b,c} c$$

with $\lambda^\alpha_{b,c} \in K$. For each $\lambda^\alpha_{b,c} \neq 0$ we draw an arrow $b \to c$ with label α. Thus, the quiver reflects the coefficients corresponding to the representation M with respect to the chosen basis B.

Given a representation $M \in \operatorname{rep}(\mathcal{A})$, its *dimension vector* $\underline{\dim} M \in \mathbf{N}\mathcal{Q}_0$ is defined by $(\underline{\dim} M)_i = \dim_K M_i$ for $i \in \mathcal{Q}_0$. For a fixed dimension vector $\underline{d} \in \mathbf{N}\mathcal{Q}_0$, we denote by $\operatorname{rep}(\mathcal{A}, \underline{d})$ the full subcategory of $\operatorname{rep}(\mathcal{A})$ which consists of representations of dimension vector $\underline{d}$.

Let M and M' be two representations of $\mathcal{A}$. We denote by $\operatorname{Hom}_{\mathcal{A}}(M, M')$ the space of homomorphisms from M to M', by $\operatorname{End}_{\mathcal{A}}(M)$ the set of endomorphisms and by $\operatorname{Aut}_{\mathcal{A}}(M)$ the group of automorphisms of M in $\operatorname{rep}(\mathcal{A})$.

For certain finite-dimensional algebras a convenient tool for the classification of the indecomposable representations (up to isomorphism) and of their homomorphisms is the *Auslander-Reiten quiver* $\Gamma(\mathcal{A})$ of $\operatorname{rep}(\mathcal{A})$. Its vertices $[M]$ are given by the isomorphism classes of indecomposable representations of $\operatorname{rep}(\mathcal{A})$; the arrows between two such vertices $[M]$ and $[M']$ are parametrized by a basis of the space of so-called irreducible maps $f\colon M \to M'$.

By defining the affine space $\mathrm{R}_{\underline{d}}(K\mathcal{Q})\colon = \bigoplus_{\alpha\colon i \to j} \operatorname{Hom}_K(K^{d_i}, K^{d_j})$, one realizes that its points m naturally correspond to representations $M = M_m \in \operatorname{rep}(K\mathcal{Q}, \underline{d})$ with $M_i = K^{d_i}$ for $i \in \mathcal{Q}_0$. Via this correspondence, the set of such representations bound by I corresponds to a closed subvariety $\mathrm{R}_{\underline{d}}(\mathcal{A}) \subset \mathrm{R}_{\underline{d}}(K\mathcal{Q})$.

The algebraic group $\mathrm{GL}_{\underline{d}} = \prod_{i \in \mathcal{Q}_0} \mathrm{GL}_{d_i}$ acts on $\mathrm{R}_{\underline{d}}(K\mathcal{Q})$ and on $\mathrm{R}_{\underline{d}}(\mathcal{A})$ via base change, furthermore the $\mathrm{GL}_{\underline{d}}$-orbits $\mathcal{O}_M$ of this action are in bijection to the isomorphism classes of representations M in $\operatorname{rep}(\mathcal{A}, \underline{d})$.

The notion of symmetry for a finite quiver comes into the picture as follows: A *symmetric quiver* is a pair $(\mathcal{Q}, \sigma)$ where $\mathcal{Q}$ is a finite quiver and $\sigma\colon \mathcal{Q}_0 \cup \mathcal{Q}_1 \to \mathcal{Q}_0 \cup \mathcal{Q}_1$ is an involution, such that $\sigma(\mathcal{Q}_0) = \mathcal{Q}_0$, $\sigma(\mathcal{Q}_1) = \mathcal{Q}_1$ and every arrow $i \xrightarrow{\alpha} j$ is sent to the arrow $\sigma(j) \xrightarrow{\sigma(\alpha)} \sigma(i)$.

In this article, we represent the action of σ by adding the symbol $*$. For example,

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b^*} 2^* \xrightarrow{a^*} 1^*$$

is the symmetric quiver $(\mathcal{Q}, \sigma)$ with underlying quiver $\mathcal{Q}$ being equioriented of type A_5, such that σ acts on $\mathcal{Q}$ by sending an element $x \in \mathcal{Q}_0 \cup \mathcal{Q}_1$ to x^*; the vertex 3 is fixed by σ.

A *symmetric (K-)representation* of a symmetric quiver $(\mathcal{Q}, \sigma)$ is a representation $M = (\{M_p\}_{p \in \mathcal{Q}_0}, \{M_\alpha\}_{\alpha \in \mathcal{Q}_1})$ in $\operatorname{rep}(K\mathcal{Q})$ endowed with a non-degenerate bilinear form

$$\langle -, - \rangle\colon \bigoplus_{p \in \mathcal{Q}_0} M_p \times \bigoplus_{q \in \mathcal{Q}_0} M_q \to K,$$

such that:

(1) The equation $$\langle -,-\rangle|_{M_p\times M_q}=0 \tag{5}$$
holds true, unless $q=\sigma(p)$;

(2) The equation $$\langle M_\alpha(v),w\rangle+\langle v,M_{\sigma(\alpha)}(w)\rangle=0 \tag{6}$$
holds true for every $v\in M_p$, $w\in M_{\sigma(q)}$ and for every arrow $p\xrightarrow{\alpha}q\in\mathcal{Q}_1$.

A representation $(M,\langle -,-\rangle)$ of a symmetric quiver $(\mathcal{Q},\sigma)$ is called *symplectic*, if the bilinear form is skew-symmetric and it is called *orthogonal*, if the bilinear form is symmetric. If the quiver Q is endowed with some relations which give rise to an admissible ideal I of $K\mathcal{Q}$ such that $\sigma\cdot I\subset I$, then the notion of symmetric representation of (Q,σ) extends naturally to the algebra $\mathcal{A}:=K\mathcal{Q}/I$ endowed with the induced involution σ.

We sometimes denote by $\mathrm{srep}(\mathcal{A})$, the full subcategory of $\mathrm{rep}(\mathcal{A})$ whose objects are symmetric (symplectic or orthogonal) representations and make sure that it will always be clear from the context which one is meant.

Given a symmetric dimension vector $\underline{d}$ (i.e. such that $d_{\sigma(i)}=d_i$ for every i) we define the variety of symmetric (symplectic or orthogonal) representations of dimension vector $\underline{d}$ as follows: we consider a graded vector space

$$V_{\underline{d}}=\bigoplus_{i=\sigma(i)\in Q_0}V_i\oplus\bigoplus_{i\neq\sigma(i)}(V_i\oplus V_{\sigma(i)})$$

with $\dim_K V_i=\underline{d}_i$ endowed with a non-degenerate bilinear form $\langle -,-\rangle$ fullfing (5). We define $\mathrm{SR}_{\underline{d}}(\mathcal{A})$ as the closed subvariety of $\mathrm{R}_{\underline{d}}(\mathcal{A})$ consisting of the points x such that M_x fullfills (6) (with respect to the given form $\langle -,-\rangle$ on V). We denote with $\mathrm{srep}(\mathcal{A},\underline{d})$ the full subcategory of $\mathrm{srep}(\mathcal{A})$ whose objects have dimension vector $\underline{d}$.

Given a symmetric representation $(M,\langle -,-\rangle)$ of $(\mathcal{A},\sigma)$, the group of automorphisms of $(M,\langle -,-\rangle)$ is defined naturally as

$$\mathrm{Aut}\,(M,\langle -,-\rangle)=\{g\in\mathrm{Aut}_{\mathcal{A}}(M)|\,\langle gm,gm'\rangle=\langle m,m'\rangle\,\forall m,m'\in M\}.$$

We have

$$\mathrm{Aut}\,(M,\langle -,-\rangle)=\mathrm{Aut}_{K\mathcal{Q}}(M)\bigcap\prod_{p\neq\sigma(p)}\mathrm{Sym}(M_p\oplus M_{\sigma(p)})\times\prod_{p=\sigma(p)}\mathrm{Sym}(M_p).$$

4. B-orbits vs. isoclasses of symmetric representations

Let $G\in\{\mathrm{SP}_n,\mathrm{O}_n\}$ where $n=2l$ in the symplectic case and $n\in\{2l,2l+1\}$ in the orthogonal case for some integer $l\in\mathbf{N}$ and let $\mathfrak{g}$ be the corresponding symplectic or orthogonal Lie algebra. Let B be the standard Borel subgroup of G, that is, the subgroup of G of upper-triangular matrices which is obtained by intersecting the Borel subgroup of GL_n with G.

We consider the algebraic variety $\mathcal{N}(2)$ of 2-nilpotent elements of $\mathfrak{g}$

$$\mathcal{N}(2)=\mathcal{N}(2,G)=\{x\in\mathfrak{g}|\,x^2=0\}.$$

Then B acts on $\mathcal{N}(2)$ via conjugation and our first aim in this article is to prove by means of symmetric quiver representations that the action admits only a finite number of orbits. We thereby specify an explicit parametrization of the orbits.

4.1. Symmetric quiver setup

We define $\mathcal{A}(l)$ to be the algebra given by the quiver

$$\begin{tikzcd} Q_l: \quad 1 \arrow[r, "a_1"] & 2 \arrow[r, "a_2"] & \cdots \arrow[r, "a_{l-1}"] & l \arrow[r, "a_l"] & \omega \arrow[loop above, "\alpha"] \arrow[r, "a_l^*"] & l^* \arrow[r, "a_{l-1}^*"] & \cdots \arrow[r, "a_2^*"] & 2^* \arrow[r, "a_1^*"] & 1^* \end{tikzcd}$$

with relations $\alpha^2 = a_l^* a_l = 0$. Notice that the $2l$ vertices of Q_l are colored; the choice of the color will be clear in a few lines. We consider the dimension vector

$$\mathbf{d} = (d_1, ..., d_l, d_\omega, d_{l^*}, ..., d_{1^*}) = (1, 2, \cdots, l-1, l, n, l, l-1, \cdots, 2, 1)$$

and the graded vector space $V_{\mathbf{d}} = \bigoplus_{i=1}^l (V_i \oplus V_{\sigma(i)}) \oplus V_\omega$. We fix a basis of each homogeneous subspace V_i of V and consider the bilinear form $\langle -, - \rangle$ on V which fulfills (5) and whose restriction to $V_i \oplus V_{\sigma(i)}$ and to V_ω with respect to the given basis is given by a matrix of the form $\begin{pmatrix} 0 & J \\ \varepsilon J & 0 \end{pmatrix}$, where $\varepsilon = 1$ in the orthogonal case, and $\varepsilon = -1$ in the symplectic case. We denote by $\mathrm{SR}_{\mathbf{d}}(\mathcal{A}(l))$ the variety of representations of $\mathcal{A}(l)$ which fullfill (6) with respect to this given form. This variety is acted upon by the group

$$\mathrm{GL}_{\mathrm{sym}} := \mathrm{GL}(d_1) \times \mathrm{GL}(d_2) \times \cdots \times \mathrm{GL}(d_l) \times \mathrm{Sym}(n)$$

where $\mathrm{Sym}(n)$ denotes the isotropy group of $(V_\omega, \langle -, - \rangle|_{V_\omega})$, thus either the symplectic or the orthogonal group on a vector space of dimension n. The stabilizer of a point $x \in \mathrm{SR}_{\mathbf{d}}(\mathcal{A}(l))$ is $\mathrm{Aut}\,(M_x, \langle -, - \rangle)$. Inside the variety $\mathrm{SR}_{\mathbf{d}}(\mathcal{A}(l))$ we consider the open subset $\mathrm{SR}_{\mathbf{d}}(\mathcal{A}(l))^0$ whose points correspond to the representations in the full subcategory $\mathrm{srep}(\mathcal{A}(l), \mathbf{d})^0$ of representations whose linear maps associated with the arrows a_i and a_i^* have maximal rank.

In V_ω we fix a complete isotropic flag (with respect to the given $\langle -, - \rangle$):

$$K \subset K^2 \subset \cdots \subset K^l \subset V_\omega$$

where K^l is a Lagrangian subspace. This flag gives rise to the $\mathcal{A}(l)$-representation M_0:

$$\begin{tikzcd} K \arrow[r, hook] & K^2 \arrow[r, hook] & \cdots \arrow[r, hook] & K^l \arrow[r, hook] & V_\omega \arrow[r, two heads] & (K^l)^* \arrow[r, two heads] & \cdots \arrow[r, two heads] & (K^2)^* \arrow[r, two heads] & K^* \end{tikzcd}$$

where the loop acts as zero and the horizontal maps are the natural inclusions and projections (here $*$ denotes the dual with respect to the bilinear form of V_ω). By construction, the stabilizer of M_0 is the standard Borel subgroup of $\mathrm{Sym}(n)$ which, by our choice of the bilinear form, is the intersection of the standard Borel subgroup of GL_n and $\mathrm{Sym}(n)$.

Example 4.1. Let us consider the quiver $\mathcal{Q}_2$

$$\begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[r, "b"] & 3 \arrow[loop above, "\alpha"] \arrow[r, "b^*"] & 2^* \arrow[r, "a^*"] & 1^* \end{tikzcd}$$

and the algebra $\mathcal{A}(2) = K\mathcal{Q}/(\alpha^2, b^* b)$. For $n \in \{4, 5\}$, M_0 has the coefficient quivers of the following table. In detail, every vertex labelled by i stands for a basis element at vertex i and an arrow $i \xrightarrow{x} j$ sends the basis element i to the basis element j via the linear map at arrow x. The linear map at the loop α equals zero.

$1 \xrightarrow{a} 2 \xrightarrow{b} 3$ $2 \xrightarrow{b} 3$ $3 \xrightarrow{b^*} 2^*$ $3 \xrightarrow{b^*} 2^* \xrightarrow{a^*} 1^*$	$1 \xrightarrow{a} 2 \xrightarrow{b} 3$ $2 \xrightarrow{b} 3$ 3 $3 \xrightarrow{b^*} 2^*$ $3 \xrightarrow{b^*} 2^* \xrightarrow{a^*} 1^*$
n=4	n=5

In view of (6), in order for M to be symmetric, the arrows a, b of $\mathcal{Q}_2$ must act by 1 and the arrows a^*, b^* must act as -1. The symmetric structure of M_0 (that is, the choice of a non-degenerate bilinear form) is induced by the symmetric structure on the vector space at vertex 3. In the symplectic case, this bilinear form is given by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix};$$

and for the orthogonal case, it is defined by the anti-diagonal matrix with every entry on the anti-diagonal being 1 (see Section 2).

The Lie algebra of $\mathrm{Aut}(M_0, \langle -, - \rangle)$ has dimension 6 and can be represented by the matrix

$$\left[\begin{array}{cc|cc} a & c & f & g \\ 0 & b & e & f \\ \hline 0 & 0 & -b & -c \\ 0 & 0 & 0 & -a \end{array}\right].$$

In orthogonal type, it is 4-dimensional and represented by

$$\left[\begin{array}{cc|cc} a & c & f & 0 \\ 0 & b & 0 & -f \\ \hline 0 & 0 & -b & -c \\ 0 & 0 & 0 & -a \end{array}\right].$$

For $n = 5$, M_0 has orthogonal structure only and the Lie algebra of its stabilizer is

$$\left[\begin{array}{ccc|cc} a & c & e & f & 0 \\ 0 & b & d & 0 & -f \\ \hline 0 & 0 & 0 & -d & -e \\ 0 & 0 & 0 & -b & -c \\ 0 & 0 & 0 & 0 & -a \end{array}\right]$$

4.2. Translation.

The translation from B-orbits in $\mathcal{N}(2)$ to the representation theory of a symmetric quiver is based on a theorem on associated fibre bundles which we recall for the convenience of the reader. Its origin can be found in [16].

Theorem 4.2. *Let G be an algebraic group, let X and Y be G-varieties, and let $\pi\colon X \to Y$ be a G-equivariant morphism. Assume that Y is a single G-orbit, $Y = Gy_0$. Define $H := \mathit{Stab}_G(y_0) = \{g \in G \,|\, g \cdot y_0 = y_0\}$ and $F := \pi^{-1}(y_0)$. Then X is isomorphic to the associated fibre bundle $G \times^H F$, and the embedding $\iota\colon F \to X$ induces a bijection between H-orbits in F and G-orbits in X preserving orbit closures.*

Remark 4.3. With the notation of Theorem 4.2, given a point $p \in F$, we have $\mathrm{Stab}_H(p) = \mathrm{Stab}_G(p)$: Since H is a subgroup of G, $\mathrm{Stab}_H(p) \subseteq \mathrm{Stab}_G(p)$. Let $g \in \mathrm{Stab}_G(p)$, then $g \cdot y_0 = g \cdot \pi(p) = \pi(g \cdot p) = \pi(p) = y_0$, since π is G-equivariant. Thus, $g \in H$ and the reversed inclusion also holds.

In view of Theorem 4.2, we can now prove the following key lemma, analogous to [3, Lemma 3.1].

Lemma 4.4. *There is a bijection between isoclasses of symplectic/orthogonal $\mathcal{A}(l)$-representations in* $\mathrm{srep}(\mathcal{A}(l), \mathbf{d})^0$ *and symplectic/orthogonal B-orbits in $\mathcal{N}(2)$. This bijection respects orbit closure relations and dimensions of stabilizers.*

Proof. Let $\widetilde{\mathcal{Q}_l}$ be the quiver obtained from $\mathcal{Q}_l$ by removing the loop α and let $\widetilde{\mathcal{A}(l)}$ be the corresponding symmetric algebra (also remove the relation α^2). By defining $\mathrm{SR}_{\mathbf{d}}(\widetilde{\mathcal{A}(l)})^0$ analougously to $\mathrm{SR}_{\mathbf{d}}(\mathcal{A}(l))^0$, we see that this variety is acted upon transitively by $\mathrm{GL}_{\mathrm{sym}}$ and we denote the representation which is given by the complete standard flag by M_0; this is a generating point. The embedding $\widetilde{\mathcal{A}(l)} \subset \mathcal{A}(l)$ induces a $\mathrm{GL}_{\mathrm{sym}}$-equivariant projection

$$\pi : \mathrm{SR}_{\mathbf{d}}(\mathcal{A}(l))^0 \twoheadrightarrow \mathrm{SR}_{\mathbf{d}}(\widetilde{\mathcal{A}(l)})^0$$

which is given by forgetting the linear map associated with the loop α. The fiber of π equals the variety $\mathcal{N}(2)$.

As we have seen before, the stabilizer of the symplectic/orthogonal representation M_0 is isomorphic to the Borel subgroup B of the symplectic/orthogonal group. Thus, Theorem 4.2 proves the claim. ■

We are hence left to classify the isomorphism classes of symplectic/orthogonal representations of $\mathcal{A}(l)$ of dimension vector $\mathbf{d}$ with maximal rank maps, which in view of Krull-Remak-Schmidt's theorem is analogous to classifying the unique decompositions of elements of $\mathrm{srep}(\mathcal{A}(l), \mathbf{d})^0$ into indecomposable symplectic/orthogonal representations (up to symmetric isomorphism). Let M and M' be two points of $\mathrm{srep}(\mathcal{A}(l), \mathbf{d})^0$ which are contained in different orbits. Since $\pi(M) = \pi(M')$ under the morphism of the proof of Lemma 4.4, the only difference beetween them is given by the action of the loop α. This means that the only part of the coefficient quivers of M and M' which differs is the subquiver which represents the loop α.

5. Representation theory of $\mathcal{A}(l)$

In this section, we look at the (symmetric) representation theory of the algebra $\mathcal{A}(l)$ corresponding to the symmetric quiver $\mathcal{Q}_l$. With these considerations, we are able to prove explicit parametrizations of the Borel-orbits in $\mathcal{N}(2)$ in Section 6.

5.1. Indecomposable symmetric $\mathcal{A}(l)$-modules.

The following proposition follows from [6, Section 3] by noticing that there are no band modules.

Proposition 5.1. *The algebra $\mathcal{A}(l)$ is a string algebra of finite representation type. In particular, the indecomposable $\mathcal{A}(l)$-modules are string modules and their isoclasses are parametrized by words with letters in the arrows of $\mathcal{Q}_l$ and their inverses, avoiding relations.*

Let us give names to the indecomposable $\mathcal{A}(l)$-modules (where $l+1 := \omega$).

M_{ij}: For $1 \leq i \leq j \leq l+1$, we denote by M_{ij} the string module associated with the word $a_i \cdots a_{j-1}$, i.e. it is the indecomposable module supported on vertices $i, i+1, \cdots, j$; its coefficient quiver is given by

$$\begin{tikzcd}
i \arrow[r] & i+1 \arrow[r] & \cdots \arrow[r] & j-1 \arrow[r] & j
\end{tikzcd}$$

M_{ij}^*: For $1 \leq i \leq j \leq l+1$, we denote by M_{ij}^* the string module associated with the word $a_{j-1}^* \cdots a_i^*$, i.e. it is the indecomposable module supported on vertices $j^*, (j-1)^*, \cdots, i^*$; its coefficient quiver is given by

$$\begin{tikzcd}
j^* \arrow[r] & (j-1)^* \arrow[r] & \cdots \arrow[r] & (i+1)^* \arrow[r] & i^*
\end{tikzcd}$$

D_{ij}^+: For $1 \leq i \leq j \leq l+1$, we denote by D_{ij}^+ the indecomposable associated with the word $a_i a_{i+1} \cdots a_l \alpha^- a_l^- a_{l-1}^- \cdots a_j^-$; its coefficient quiver has the following form

$$\begin{tikzcd}
i \arrow[r] & i+1 \arrow[r] & \cdots \arrow[r] & j \arrow[r] & j+1 \arrow[r] & \cdots \arrow[r] & l \arrow[r] & \omega \\
 & & & j \arrow[r] & j+1 \arrow[r] & \cdots \arrow[r] & l \arrow[r] & \omega \arrow[u]
\end{tikzcd}$$

D_{ij}^-: For $1 \leq i < j \leq l+1$ we denote by D_{ij}^- the indecomposable associated with the word $a_i a_{i+1} \cdots a_l \alpha a_l^- a_{l-1}^- \cdots a_j^-$; its coefficient quiver is given by

$$\begin{tikzcd}
i \arrow[r] & i+1 \arrow[r] & \cdots \arrow[r] & j \arrow[r] & j+1 \arrow[r] & \cdots \arrow[r] & l \arrow[r] & \omega \arrow[d] \\
 & & & j \arrow[r] & j+1 \arrow[r] & \cdots \arrow[r] & l \arrow[r] & \omega
\end{tikzcd}$$

C_{ij}^+: For $1 \leq i \leq j \leq l+1$ we denote by C_{ij}^+ the indecomposable associated with the word $(a_j^*)^-(a_{j+1}^*)^- \cdots (a_l^*)^- \alpha^- a_l^* a_{l-1}^* \cdots a_i^*$; its coefficient quiver is given by

$$\begin{tikzcd}
\omega \arrow[r] \arrow[d] & l^* \arrow[r] & \cdots \arrow[r] & j^* \arrow[r] & (j-1)^* \arrow[r] & \cdots \arrow[r] & (i+1)^* \arrow[r] & i^* \\
\omega \arrow[r] & l^* \arrow[r] & \cdots \arrow[r] & j^*
\end{tikzcd}$$

C_{ij}^-: For $1 \leq i < j \leq l+1$ we denote by C_{ij}^- the indecomposable module associated with the word $(a_j^*)^-(a_{j+1}^*)^- \cdots (a_l^*)^- \alpha\, a_l^* a_{l-1}^* \cdots a_i^*$; its coefficient quiver is given by

$$\begin{tikzcd}
\omega \arrow[r] & l^* \arrow[r] & \cdots \arrow[r] & j^* \arrow[r] & (j-1)^* \arrow[r] & \cdots \arrow[r] & (i+1)^* \arrow[r] & i^* \\
\omega \arrow[r] \arrow[u] & l^* \arrow[r] & \cdots \arrow[r] & j^*
\end{tikzcd}$$

Z^+_{ij}: For $1 \leq i,j \leq l$ we denote by Z^+_{ij} the indecomposable associated with the word $a_i a_{i+1} \cdots a_l \alpha^- a^*_l \cdots a^*_j$; its coefficient quiver is given by

$$\begin{tikzcd}
i \arrow[r] & i+1 \arrow[r] & \cdots \arrow[r] & l \arrow[r] & \omega & & & \\
& & & & \omega \arrow[u] \arrow[r] & l^* \arrow[r] & \cdots \arrow[r] & j^*
\end{tikzcd}$$

Z^-_{ij}: For $1 \leq i,j \leq l$ we denote by Z^-_{ij} the indecomposable associated with the word $a_i a_{i+1} \cdots a_l \alpha a^*_l \cdots a^*_j$; its coefficient quiver is given by

$$\begin{tikzcd}
i \arrow[r] & i+1 \arrow[r] & \cdots \arrow[r] & l \arrow[r] & \omega \arrow[d] & & & \\
& & & & \omega \arrow[r] & l^* \arrow[r] & \cdots \arrow[r] & j^*
\end{tikzcd}$$

Remark 5.2. All the modules above are non-isomorphic to each other, apart from $D^+_{l+1,l+1} \simeq C^+_{l+1,l+1}$ and $M_{l+1,l+1} \simeq M^*_{l+1,l+1}$.

The involution σ induces an isomorphism between the algebra $\mathcal{A}(l)$ and its opposite. This gives the structure of an $\mathcal{A}(l)$-module to the linear dual of each $\mathcal{A}(l)$-module. This is done naturally as follows: for every $\mathcal{A}(l)$-module $M = ((M_i),(M_\alpha))$ we consider the dual module $\nabla M := ((\nabla M)_i, (\nabla M)_\alpha)$ where

$$(\nabla M)_i = M^*_{\sigma(i)}, \quad (\nabla M)_\alpha = -M^*_{\sigma(\alpha)}.$$

See [7] for properties of the self-duality ∇.

Remark 5.3. Given an indecomposable $\mathcal{A}(l)$-module M, we need to choose carefully the linear maps. Since we often work with its coefficient quiver, we fix one and for all a convention about these:
The arrows of the coefficient quiver of M colored with $a_1, \cdots, a_l$ act as 1, while the arrows colored with $a^*_l, \cdots, a^*_1$ act as -1.
Furthermore, for every two basis elements ω_i, ω_j at vertex ω (together with their σ-translates $\omega_{i^*}, \omega_{j^*}$) and every pair of two arrows $\omega_i \xrightarrow{\alpha_1} \omega_j$ and $\omega_{j^*} \xrightarrow{\alpha_2} \omega_{i^*}$ colored with ω (if they exist), the following conditions have to be satisfied:

- For V to be orthogonal, α_1 acts as 1 and α_2 as -1.
- For V to be symplectic, if $1 \leq i,j \leq l$ or $1 \leq i^*, j^* \leq l$, then α_1 acts as 1 and α_2 as -1, otherwise α_1 and α_2 both act as 1.

Proposition 5.4. *With the above notation, we have:* $\nabla M_{i,j} \simeq M^*_{i,j}$, $\nabla D^+_{i,j} \simeq C^+_{i,j}$, $\nabla D^-_{i,j} \simeq C^-_{i,j}$, $\nabla Z^+_{i,j} \simeq Z^+_{j,i}$, $\nabla Z^-_{i,j} \simeq Z^-_{j,i}$. *In particular,* $\nabla M_{l+1,l+1} \simeq M_{l+1,l+1}$ *and* $\nabla D^+_{l+1,l+1} \simeq D^+_{l+1,l+1}$

Proof. Let M be an indecomposable module as listed above. The coefficient quiver of the dual ∇M of M is obtained from the coefficient quiver of M by reversing all the arrows, changing their signs and then making a reflection through the middle vertex $\omega = l+1$. ■

Thus, we obtain the following classification lemma.

Lemma 5.5. *The symplectic indecomposable representations of* $\mathcal{A}(l)$ *are* $Z_{ii}^{\pm}$, $M_{ij} \oplus M_{ij}^{*}$, $D_{ij}^{\pm} \oplus C_{ij}^{\pm}$ *(for* $(i,j) \neq (l+1, l+1)$*),* $D_{l+1,l+1}^{+}$ *and* $Z_{ij}^{\pm} \oplus Z_{ji}^{\pm}$ *(for* $i \neq j$*). The orthogonal indecomposable representations of* $\mathcal{A}(l)$ *are* $M_{ij} \oplus M_{ij}^{*}$, $D_{ij}^{\pm} \oplus C_{ij}^{\pm}$, $Z_{ij}^{\pm} \oplus Z_{ji}^{\pm}$ *and* $M_{l+1,l+1}$.

Proof. It follows from [7] that two symmetric representations are isomorphic if and only if they are isomorphic as representations of $\mathcal{A}(l)$. A symmetric $\mathcal{A}(l)$-module is isomorphic to its dual (the isomorphism gives either the symplectic or the orthogonal structure). The result hence follows from Krull-Remak-Schmidt theorem and Proposition 5.4. ■

Remark 5.6. The only indecomposable $\mathcal{A}(l)$-module which can be endowed with an orthogonal structure is one-dimensional and supported at vertex ω. Indeed, let M be an indecomposable (at least two-dimensional) module with orthogonal structure, and let M_α be the linear map associated with the loop α. Such a map is a 2-nilpotent endomorphism of an orthogonal two-dimensional vector space. In order for M to be orthogonal, M_α must lie in the Lie algebra $\mathfrak{o}_2$ of O_2 and hence it must be zero, contradicting the fact that M is indecomposable.

For example, the following representation:

$$\begin{tikzcd}
1 \arrow[r, "1"] & 2 \arrow[r, "1"] & 3 \arrow[r, "1"] & 4 \arrow[r, "1"] & \omega \arrow[dddd, bend left=60, "1"] & & & & \\
& 2 \arrow[r, "1"] & 3 \arrow[r, "1"] & 4 \arrow[r, "1"] & \omega \arrow[d, bend left, "1"] & & & & \\
& & 3 \arrow[r, "1"] & 4 \arrow[r, "1"] & \omega & & & & \\
& & & 4 \arrow[r, "1"] & \omega \arrow[dddd, bend right=60, "b"'] & & & & \\
& & & & \omega \arrow[r, "-1"] & 4^{*} & & & \\
& & & & \omega \arrow[r, "-1"] \arrow[d, bend right, "-1"'] & 4^{*} \arrow[r, "-1"] & 3^{*} & & \\
& & & & \omega \arrow[r, "-1"] & 4^{*} \arrow[r, "-1"] & 3^{*} \arrow[r, "-1"] & 2^{*} & \\
& & & & \omega \arrow[r, "-1"] & 4^{*} \arrow[r, "-1"] & 3^{*} \arrow[r, "-1"] & 2^{*} \arrow[r, "-1"] & 1^{*}
\end{tikzcd}$$

is symplectic if $b = 1$ and orthogonal if $b = -1$.

5.2. Auslander-Reiten quiver of $\mathcal{A}(l)$.

The algebra $\mathcal{A}(l)$ is a string algebra of finite representation-type, that is, it does only admit a finite number of isomorphism classes of indecomposable representations. Its Auslander-Reiten quiver can be obtained in several ways. We prefer to follow the treatment of Butler-Ringel [4] and get the following result.

Proposition 5.7. *The following are the Auslander-Reiten sequences of* $\mathcal{A}(l)$*:*

(1) *Auslander-Reiten sequences starting with* M_{ij}*:*

$$0 \longrightarrow M_{1,\omega} \longrightarrow Z_{1,1}^{+} \longrightarrow M_{1,\omega}^{*} \longrightarrow 0\ ,$$

$$0 \longrightarrow M_{i,\omega} \longrightarrow M_{i-1,\omega} \oplus Z_{i,1}^{+} \longrightarrow Z_{i-1,1}^{+} \longrightarrow 0\ ,\ \textit{if } i > 1,$$

$$0 \longrightarrow M_{i,j} \longrightarrow M_{i,j-1} \oplus M_{i-1,j} \longrightarrow M_{i-1,j-1} \longrightarrow 0\ ,\ \textit{if } i > 1 \textit{ and } j \leq l,$$

$$0 \longrightarrow M_{i,i} = S_i \longrightarrow M_{i-1,i} \longrightarrow M_{i-1,i-1} \longrightarrow 0\ ,\ \textit{if } i = j > 1.$$

(2) *Auslander-Reiten sequences starting with* M^*_{ij}*:*

$$0 \longrightarrow M^*_{1,j} \longrightarrow M^*_{2,j} \oplus M^*_{1,j+1} \longrightarrow M^*_{2,j+1} \longrightarrow 0 \ , \ \textit{if } j \leq l-1 ,$$

$$0 \longrightarrow M^*_{1,l} \longrightarrow M^*_{2,l} \oplus P_\omega \longrightarrow C^-_{1,l} \longrightarrow 0 \ ,$$

$$0 \longrightarrow M^*_{1,\omega} \longrightarrow M^*_{2,\omega} \oplus C^+_{1,1} \longrightarrow C^-_{1,2} \longrightarrow 0 \ ,$$

$$0 \longrightarrow M^*_{i,i} = S_{i^*} \longrightarrow M^*_{i,i+1} \longrightarrow M^*_{i+1,i+1} \longrightarrow 0 \ , \ \textit{if } i < l ,$$

$$0 \longrightarrow M^*_{l,l} = S_{l^*} \longrightarrow C^-_{1,l} \longrightarrow C^-_{1,\omega} \longrightarrow 0 \ ,$$

$$0 \longrightarrow S_\omega \longrightarrow M_{l,\omega} \oplus C^+_{1,\omega} \longrightarrow Z^+_{l,1} \longrightarrow 0 \ ,$$

$$0 \longrightarrow M^*_{i,\omega} \longrightarrow M^*_{i+1,\omega} \oplus C^+_{1,i} \longrightarrow C^+_{1,i+1} \longrightarrow 0 \ , \ \textit{if } 1 < i < l ,$$

$$0 \longrightarrow M^*_{l,\omega} \longrightarrow S_\omega \oplus C^+_{1,l} \longrightarrow C^+_{1,\omega} \longrightarrow 0 \ ,$$

$$0 \longrightarrow M^*_{i,j} \longrightarrow M^*_{i+1,j} \oplus M^*_{i,j+1} \longrightarrow M^*_{i+1,j+1} \longrightarrow 0 \ , \ \textit{if } i > 1 \ \textit{and } j < l .$$

(3) *Auslander-Reiten sequences starting with* D^+_{ij}*:*

$$0 \longrightarrow D^+_{1,\omega} \longrightarrow D^+_{1,l} \oplus S_\omega \longrightarrow M_{l,\omega} \longrightarrow 0 \ ,$$

$$0 \longrightarrow D^+_{i,\omega} \longrightarrow D^+_{i-1,\omega} \oplus D^+_{i,l} \longrightarrow D^+_{i-1,l} \longrightarrow 0 \ , \ \textit{if } 1 < i \leq l ,$$

$$0 \longrightarrow D^+_{1,j} \longrightarrow D^+_{1,j-1} \oplus M_{j,\omega} \longrightarrow M_{j-1,\omega} \longrightarrow 0 \ , \ \textit{if } 1 < j \leq l ,$$

$$0 \longrightarrow D^+_{i,j} \longrightarrow D^+_{i-1,j} \oplus D^+_{i,j-1} \longrightarrow D^+_{i-1,j-1} \longrightarrow 0 \ , \ \textit{if } 1 < i \leq j \leq l .$$

(4) *Auslander-Reiten sequences starting with* D^-_{ij}*:*

$$0 \longrightarrow D^-_{1,\omega} \longrightarrow D^-_{1,l} \longrightarrow M_{l,l} = S_l \longrightarrow 0 \ ,$$

$$0 \longrightarrow D^-_{i,\omega} \longrightarrow D^-_{i-1,\omega} \oplus D^+_{i,l} \longrightarrow D^-_{i-1,l} \longrightarrow 0 \ , \ \textit{if } 1 < i \leq l ,$$

$$0 \longrightarrow D^+_{i,i} \longrightarrow D^-_{i-1,i} \oplus D^+_{i-1,i} \longrightarrow D^+_{i-1,i-1} \longrightarrow 0 \ , \ \textit{if } 1 < i \leq \omega$$

$$0 \longrightarrow D^-_{1,j} \longrightarrow D^-_{1,j-1} \oplus M_{j,l} \longrightarrow M_{j-1,l} \longrightarrow 0 \ , \ \textit{if } 1 < j \leq l ,$$

$$0 \longrightarrow D^-_{i,j} \longrightarrow D^-_{i-1,j} \oplus D^-_{i,j-1} \longrightarrow D^-_{i-1,j-1} \longrightarrow 0 \ , \ \textit{if } 1 < i < j \leq l .$$

(5) *Auslander-Reiten sequences starting with* C^+_{ij}*:*

$$0 \longrightarrow C^+_{i,\omega} \longrightarrow C^+_{i+1,\omega} \oplus Z^+_{l,i} \longrightarrow Z^+_{l,i+1} \longrightarrow 0 \ , \ \textit{if } 1 \leq i < l ,$$

$$0 \longrightarrow C^+_{l,\omega} \longrightarrow C^+_{\omega,\omega} \oplus Z^+_{l,l} \longrightarrow D^+_{l,\omega} \longrightarrow 0 \ ,$$

$$0 \longrightarrow C^+_{i,j} \longrightarrow C^+_{i+1,j} \oplus C^+_{i,j+1} \longrightarrow C^+_{i+1,j+1} \longrightarrow 0 \ , \ \textit{if } 1 < i \leq j \leq l .$$

(6) *Auslander-Reiten sequences starting with* C^-_{ij}*:*

$$0 \longrightarrow C^-_{1,\omega} \longrightarrow Z^-_{l,1} = P_l \oplus C^-_{2,\omega} \longrightarrow Z^-_{l,2} \longrightarrow 0 \ ,$$

$$0 \longrightarrow C^-_{i,\omega} \longrightarrow Z^-_{l,i} \oplus C^-_{i+1,\omega} \longrightarrow Z^-_{l,i+1} \longrightarrow 0 \ , \ \textit{if } 1 \leq i \leq l ,$$

$$0 \longrightarrow C^-_{1,1} = P_\omega \longrightarrow C^-_{1,2} \oplus C^+_{1,2} \longrightarrow C^+_{2,2} \longrightarrow 0 \ ,$$

$$0 \longrightarrow C^-_{i,j} \longrightarrow C^-_{i+1,j} \oplus C^-_{i,j+1} \longrightarrow C^-_{i+1,j+1} \longrightarrow 0 \ , \ \textit{if } 1 < i \leq j \leq n .$$

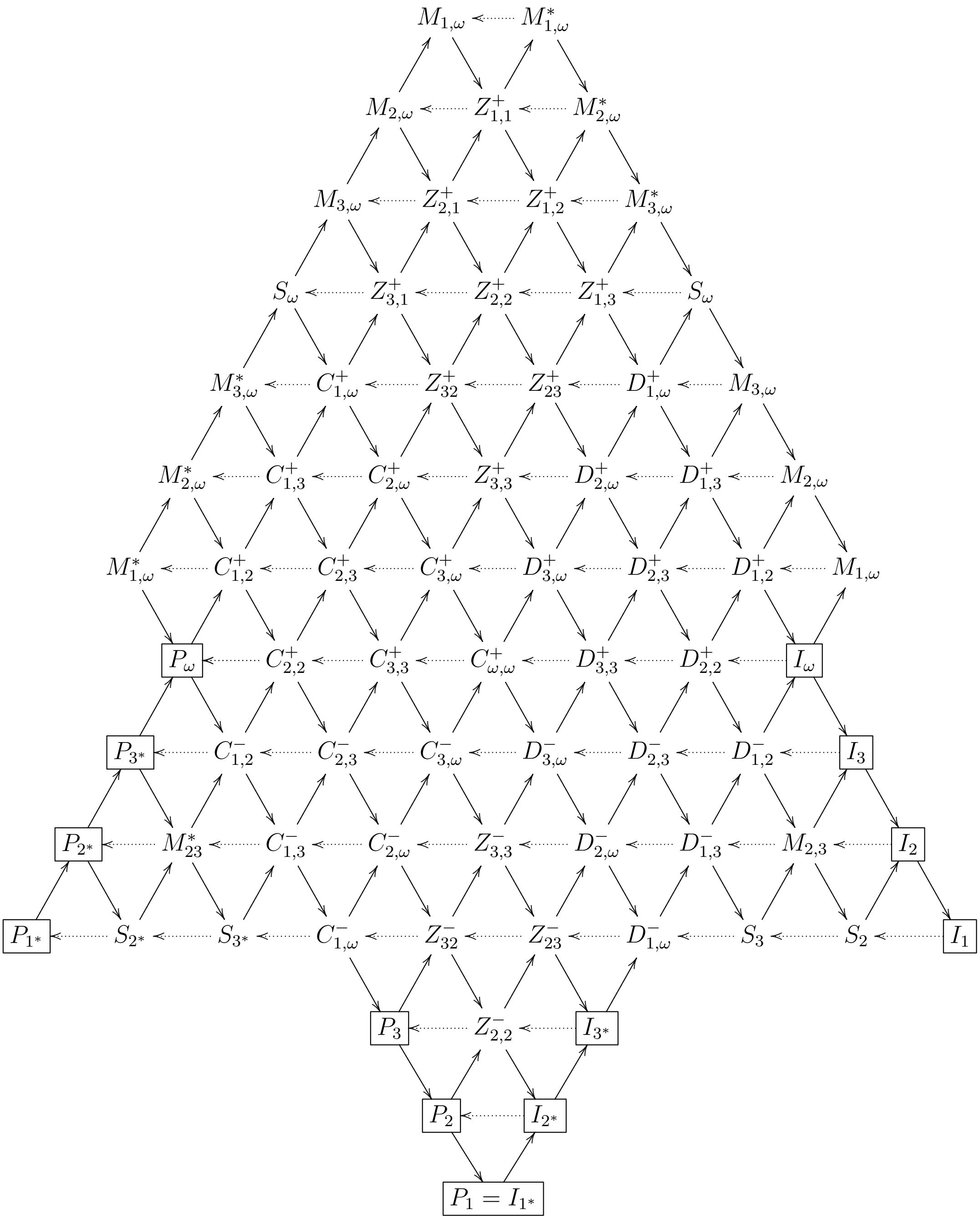


Figure 1: Auslander-Reiten quiver of $\mathcal{A}(3)$

(7) *Auslander-Reiten sequences starting with* Z^+_{ij} *(note that* $Z^+_{i,\omega} = D^+_{i,\omega}$*):*

$$0 \longrightarrow Z^+_{1,j} \longrightarrow Z^+_{1,j+1} \oplus M^*_{j,\omega} \longrightarrow M^*_{j+1,\omega} \longrightarrow 0\ ,$$

$$0 \longrightarrow Z^+_{i,j} \longrightarrow Z^+_{i,j+1} \oplus Z^+_{i-1,j} \longrightarrow Z^+_{i-1,j+1} \longrightarrow 0\ ,\ \textit{if } i > 1.$$

(8) *Auslander-Reiten sequences starting with* Z^-_{ij} *(note that* $Z^-_{1j} = I_{j^*}$*):*

$$0 \longrightarrow Z^-_{i,1} = P_i \longrightarrow Z^-_{i-1,1} \oplus Z^-_{i,2} \longrightarrow Z^-_{i-1,2} \longrightarrow 0\ ,\ \textit{if } i > 1,$$

$$0 \longrightarrow Z^-_{i,j} \longrightarrow Z^-_{i-1,j} \oplus Z^-_{i,j+1} \longrightarrow Z^-_{i-1,j+1} \longrightarrow 0\ ,\ \textit{if } 1 < i, j \leq l,$$

The resulting Auslander-Reiten quiver of $\mathcal{A}(l)$ has the shape of a "christmas tree"; its bottom part consists of pre-projective modules and its top consists of $l+1$ periodic τ-orbits. The duality ∇ acts as a reflection through the vertical line formed by the self-dual $\mathcal{A}(l)$-modules $Z_{ii}^{\pm}$ and $D_{l+1,l+1}^{+}$. Figure 1 shows the Auslander-Reiten quiver of $\mathcal{A}(3)$.

6. Parametrization of orbits

By Lemma 4.4, the B-orbits of $\mathcal{N}(2)$ are parametrized by certain symmetric representations of the algebra $\mathcal{A}(l)$. In this section we provide the explicit parametrization and reformulate it in terms of a combinatorial gadget that we call symplectic/orthogonal oriented link pattern. We begin by discussing symplectic orbits and deduce orthogonal orbits, then. In each type, we generalize the results to parabolic orbits in Section 7.

6.1. Orbits in type C

Let $G = \mathrm{SP}_n$, where $n = 2l$ for some integer l. We denote by B the standard Borel subgroup of G and consider the algebra $\mathcal{A}(l)$ and its symmetric representations as discussed in Section 5. Due to Lemma 4.4, we are interested in symplectic representations of dimension vector $\mathbf{d} = (1, 2, ..., l, 2l, l, ..., 2, 1)$.
Let us begin with an example.

Example 6.1. Figure 2 shows the complete list of isomorphism classes of symplectic representations in $\mathrm{srep}(\mathcal{A}(2), \mathbf{d})^0$, where $n = 4 = 2l$. In more detail, the indecomposables are displayed by their coefficient quiver and then interpreted combinatorially by graphs on two vertices.

This observation leads us to the following definition.

Definition 6.2. A *symplectic oriented link pattern* (solp for short) of size l consists of a set of l vertices $\{1, 2, \cdots, l\}$ together with a collection of oriented arrows between them, such that

(1) every arrow is either dotted or plain;
(2) every vertex is touched by at most one arrow;
(3) every loop is dotted.

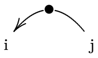
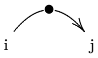
We denote by Solp_l the set of solps of size l.

Note that the loops with dots have an orientation. The definition of symplectic link patterns leads to a combinatorial parametrization of symplectic Borel-orbits in $\mathcal{N}(2)$.

Theorem 6.3. *The B-orbits in the variety $\mathcal{N}(2) \subseteq \mathfrak{sp}_n$ are in bijection with the set Solp_l of solps of size l.*

Proof. By Krull-Remak-Schmidt, there is a quite obvious bijection between the set of solps of size l and the set of symplectic representations in $\mathrm{srep}(\mathcal{A}(l), \mathbf{d})^0$ up to isomorphism which maps an isomorphism class $[M]$ of a symplectic representation M to the subquiver of the coefficient quiver of M induced by M_α. Since this subquiver is completely determined by the vertices $1, \cdots, l$ and the arrows between them, it can unambiguously be shrunk to the first l vertices. By Lemma 4.4, the claim follows.

We state in detail, how the bijection works and how we can read off the representative 2-nilpotent matrix of a particular solp. For this, the table of the next page gives a recipe; let $i < j$ and translate i^* and j^* via $(k^* = n - k + 1)$.

(Part of) solp	Indecomposable	(Part of) Matrix
i	$M_{i,l+1} \oplus M^*_{i,l+1}$	0
	$D^+_{i,j} \oplus C^+_{i,j}$	$E_{i,j} - E_{j^*,i^*}$
	$D^-_{i,j} \oplus C^-_{i,j}$	$E_{j,i} - E_{i^*,j^*}$
	$Z^+_{i,j} \oplus Z^+_{j,i}$	$E_{i,j^*} + E_{j,i^*}$
	$Z^-_{i,j} \oplus Z^-_{j,i}$	$E_{j^*,i} + E_{i^*,j}$
	$Z^+_{i,i}$	E_{i,i^*}
	$Z^-_{i,i}$	$E_{i^*,i}$

■

Since we have a combinatorial description, we can count the number of orbits.

Proposition 6.4. *Let s_l be the cardinality of* Solp_l*. Then the sequence $\{s_l\}$ is determined by*

$$s_0 = 1, \quad s_1 = 3, \quad s_l = 3s_{l-1} + 4(l-1)s_{l-2}.$$

Proof. We divide the set Solp_l into the subset of symmetric link patterns where vertex 1 is not touched by any arrow and its complement. ■

The sequence $1, 3, 13, 63, 345, 2043, ...$ of numbers of slps is classified in OEIS as A202837 [18].

Remark 6.5. For GL_n, the oriented link patterns considered in [4] only have to satisfy the first condition of Definition 6.2 that is, the 2-nilpotency condition. We hence see that solps are special oriented link patterns as defined in [4]. This is not surprising; indeed the following fact is known by [12, Proposition 2.1]: if two symplectic elements are conjugate under the Borel of GL_n, then they are conjugate under the Borel of SP_n, as well.

The following example shows the classification of B-orbits in $\mathcal{N}(2)$ for SP_4.

Example 6.6. The B-orbits in $\mathcal{N}(2) \subseteq \mathfrak{sp}_4$ are classified by the collection of solps of size 2. These are explicitly listed in the following table and the corresponding representative matrices in $\mathcal{N}(2)$ are displayed. This completes the example which we already considered in Figure 2.

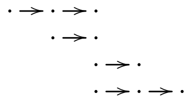
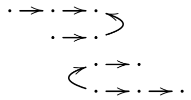
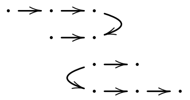
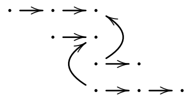
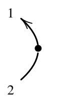
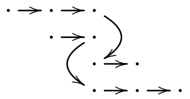
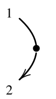
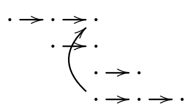
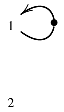
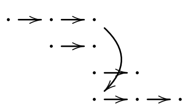
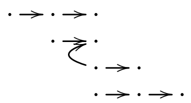
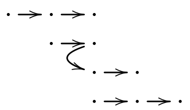
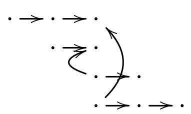
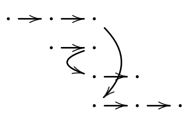
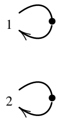
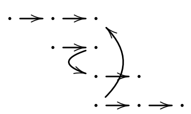
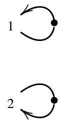
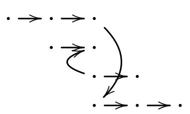
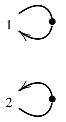
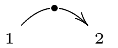
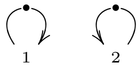
$M_0 = (M_{13} \oplus M_{13}^*)$ $\oplus(M_{23} \oplus M_{23}^*)$		1 2			
$D_{12}^+ \oplus \nabla D_{12}^+$			$D_{12}^- \oplus \nabla D_{12}^-$		
$Z_{21}^+ \oplus \nabla Z_{21}^+$			$Z_{12}^- \oplus \nabla Z_{12}^-$		
$Z_{11}^+ \oplus (M_{23} \oplus M_{23}^*)$			$Z_{11}^- \oplus (M_{23} \oplus M_{23}^*)$		
$Z_{22}^+ \oplus (M_{13} \oplus M_{13}^*)$			$Z_{22}^- \oplus (M_{13} \oplus M_{13}^*)$		
$Z_{11}^+ \oplus Z_{22}^+$			$Z_{11}^- \oplus Z_{22}^-$		
$Z_{11}^+ \oplus Z_{22}^-$			$Z_{11}^- \oplus Z_{22}^+$		

Figure 2: Isomorphism classes of symplectic representations in $\mathrm{srep}(\mathcal{A}(2), \mathbf{d})^0$ and their combinatorial interpretation

				1 2
$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
				
$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	
				
$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	

6.2. Orbits in types B and D

Let $G = \mathrm{O}_n$, where $n \in \{2l, 2l+1\}$. We denote by B the standard Borel subgroup of G and consider the algebra $\mathcal{A}(l)$ as discussed in Section 5. We are interested in orthogonal representations of dimension vector $\mathbf{d} = (1, 2, ..., l, n, l, ..., 2, 1)$ by Lemma 4.4.

Definition 6.7. An *orthogonal oriented link pattern* (oolp for short) of size l consists of a set of l vertices $\{1, 2, \cdots, l\}$ together with a collection of oriented arrows between them, such that

(1) every arrow is either dotted or plain;
(2) every vertex is touched by at most one arrow;
(3) there are no loops.

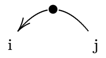
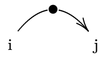
We denote by Oolp_l the set of oolps of size l.

Thus, an oolp is a solp without loops.

Example 6.8. Figure 3 shows the complete list of isomorphism classes of orthogonal representations in $\mathrm{srep}(\mathcal{A}(2), \mathbf{d})^0$. The first table shows the isomorphism classes of orthogonal representations where $\mathbf{d} = (1, 2, 4, 2, 1)$, i.e. it corresponds to O_4. The second table shows the isomorphism classes of orthogonal representations where $\mathbf{d} = (1, 2, 5, 2, 1)$, i.e. it corresponds to O_5.

As in the symplectic case, the parametrization of the Borel-orbits in $\mathcal{N}(2)$ follows straight away.

Theorem 6.9. *The B-orbits in the variety $\mathcal{N}(2) \subseteq \mathfrak{o}_n$, where $n \in \{2l, 2l+1\}$, are in bijection with the set Oolp_l of oolps of size l.*

(Part of) oolp	Indecomposable	(Part of) Matrix
i	$M_{i,l+1} \oplus M^*_{i,l+1}$	0
	$D^+_{i,j} \oplus C^+_{i,j}$	$E_{i,j} - E_{j^*,i^*}$
	$D^-_{i,j} \oplus C^-_{i,j}$	$E_{j,i} - E_{i^*,j^*}$
	$Z^+_{i,j} \oplus Z^+_{j,i}$	$E_{i,j^*} - E_{j,i^*}$
	$Z^-_{i,j} \oplus Z^-_{j,i}$	$E_{j^*,i} - E_{i^*,j}$

Proof. In a similar manner to Theorem 6.3, there is a quite obvious bijection between the set Oolp_l of oolps of size l and the set of isoclasses of orthogonal representations of $\mathcal{A}(l)$ in $\mathrm{srep}(\mathcal{A}(l), \mathbf{d})^0$ which maps an isomorphism class $[M]$ of an orthogonal representation to a particular subquiver of the coefficient quiver of M induced by M_α.
In case n is even, this is done analogously to the symplectic case.
In case n is odd, the middle vertex of this particular subquiver is always determined as a fixed point as visualized in Figure 3. This is due to the fact that it corresponds to the direct summand $M_{\omega,\omega}$. The diagram representing M_α can thus be restricted to $2l$ vertices and we obtain the sought subquiver. This can be translated to an oolp

as in the symplectic case again. In more detail, the translation of indecomposables to vertices / arrows in the oolp can be read off the following table - where also the translation to matrices in $\mathcal{N}(2)$ can be found.
As before, the claim follows from Lemma 4.4. ■

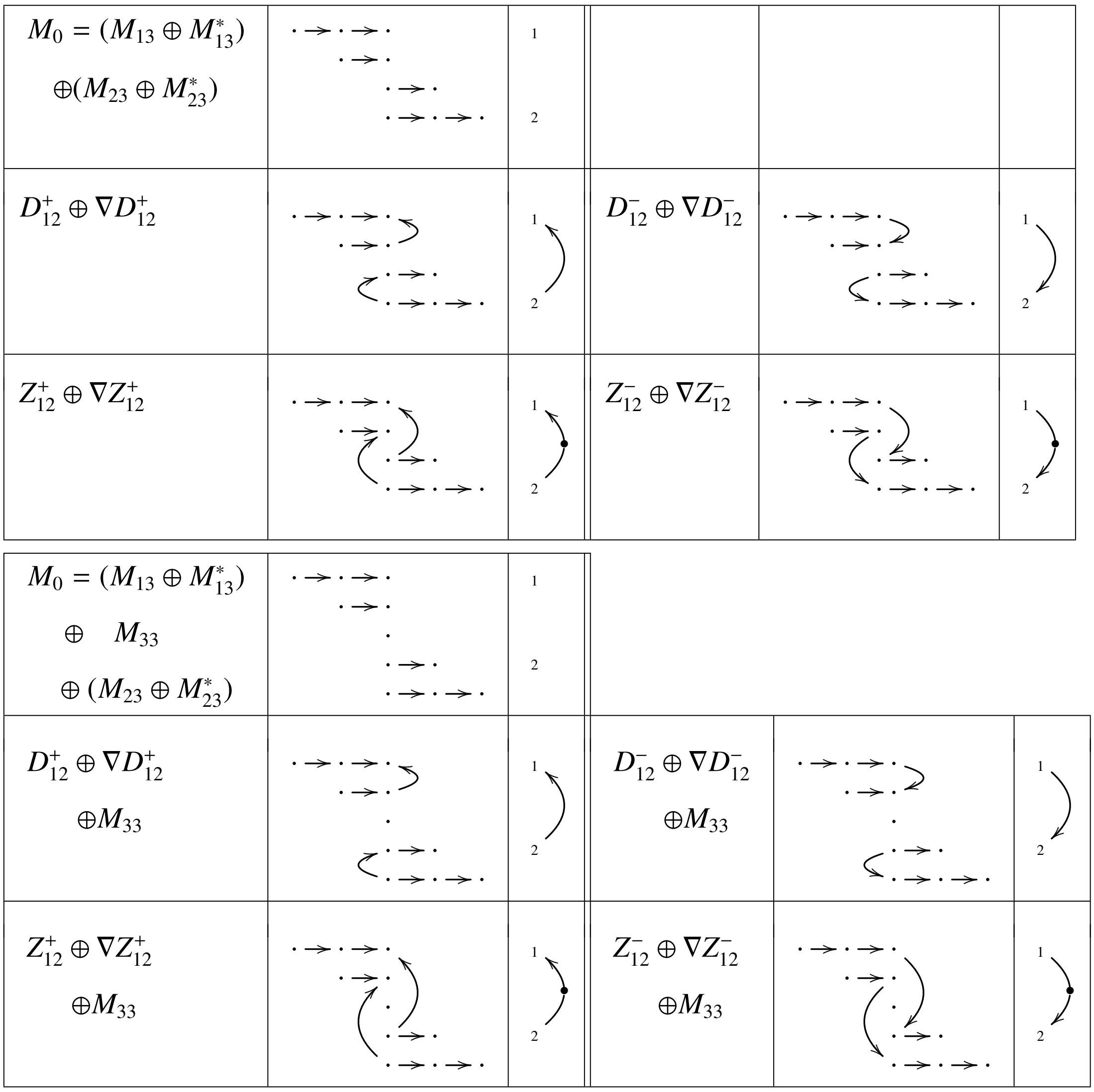


Figure 3: Isomorphism classes of orthogonal representations in $\mathrm{srep}(\mathcal{A}(2), \mathbf{d})^0$ and their combinatorial interpretation

Proposition 6.10. *Let* o_l *be the cardinality of* Oolp_l. *Then the sequence* $\{o_l\}$ *is determined by* $o_0 = 1,\ o_1 = 1,\ o_l = o_{l-1} + 4(l-1)o_{l-2}$.

Proof. We divide the set Oolp_l into the subset of oolps where vertex 1 is not touched by any arrow and its complement. ■

The sequence $\{o_l\}$ of integers $1, 1, 5, 13, 73, 281, 1741, ...$ is classified as A115329 [17] in OEIS.

Remark 6.11. As before, we see that oolps are special oriented link patterns. As in the symplectic case, this fact also follows from [12]: if two orthogonal elements are conjugate under the Borel of GL_n, then they are conjugate under the Borel of O_n, as well.

Again, we end the subsection by writing down the explicit classification of B-orbits in $\mathcal{N}(2)$ for the orthogonal group O_4.

Example 6.12. Let $B \subset \mathrm{O}_4$, then the collection of oolps of size 2 and the corresponding matrices in $\mathcal{N}(2) \subset \mathfrak{o}_4$ are given in the following table:

				1 2
$\begin{pmatrix} 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&-1&0 \end{pmatrix}$	$\begin{pmatrix} 0&1&0&0\\ 0&0&0&0\\ 0&0&0&-1\\ 0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&-1&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&1&0\\ 0&0&0&-1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$

7. Generalization to parabolic actions

Since there are only finitely many Borel orbits in $\mathcal{N}(2)$, we know that every parabolic P acts with finitely many orbits on the variety $\mathcal{N}(2)$, too.

7.1. Parabolic subgroups

We have seen before that the standard Borel subgroups of G equal the intersection of G with the standard Borel subgroup of GL_n. For standard parabolic subgroups, this is only true for types B and C; even for the special orthogonal group SO_n, in type D, there are standard parabolic subgroups P which are not given as the intersection of a standard parabolic subgroup of GL_n with SO_n.
Following Malle-Testermann in [13, Chapter 12], the parabolic subgroups of SP_n and SO_n are in bijection with the so-called totally isotropic flags. These are flags

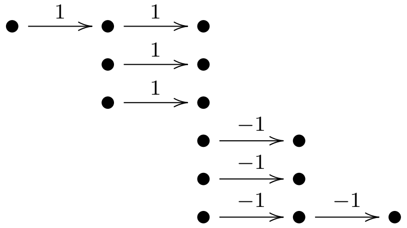
$$V_1 \subset ... \subset V_k,$$

such that every two elements $v, w \in V_i$ vanish according to the bilinear form of G, namely $\langle v, w\rangle = 0$. Given such totally isotropic flag $F = (V_1 \subset ... \subset V_k)$, the stabilizer $P := \mathrm{stab}_G(F)$ includes the standard Borel B in both the symplectic and orthogonal case and is, thus, a parabolic. In the same way, the parabolic subgroups of the orthogonal groups O_n arise as stabilizers of totally isotropic flags. We define $d_i := \dim_K V_i$, and endow the flag F with a natural structure of an $\mathcal{A}(k)$-module of dimension vector $\mathbf{d}_F = \mathbf{d}_P = (d_1, ..., d_k, n, d_k, ..., d_1)$, that we denote by $M_F = M_P$.

Example 7.1. Let $n = 6$. We look at the totally isotropic flag

$$F_1 := (V_1 = \langle e_1\rangle \subset V_2 = \langle e_1, e_2, e_3\rangle)$$

Then F_1 corresponds to the $\mathcal{A}(2)$-representation M_{F_1} of dimension vector $(1, 3, 6, 3, 1)$ with coefficient quiver



(7)

Its stabilizer in G is given as the upper-block standard parabolic subgroup of GL_6 of block sizes $(1,2,2,1)$, intersected with G. Let now $n=4$ and set

$$F_2 := (V_1 = \langle e_1, e_3 \rangle)$$

Then F_2 corresponds to the $\mathcal{A}(1)$-representation M_{F_2} of dimension vector $(2,4,2)$ with coefficient quiver

$$\begin{tikzcd}
\bullet \arrow[r, "1"] & \bullet & \\
& \bullet \arrow[r, "-1"] & \bullet \\
\bullet \arrow[r, "1"] & \bullet & \\
& \bullet \arrow[r, "-1"] & \bullet
\end{tikzcd} \tag{8}$$

Its stabilizer in O_4 is not given as the intersection of O_4 with an upper-block standard parabolic subgroup of GL_4. In fact,

$$\mathrm{Lie}\,\mathrm{stab}_{\mathrm{O}_4}(F_2) \cong \left\{ \begin{pmatrix} a & c & d & 0 \\ 0 & b & 0 & -d \\ e & 0 & -b & -c \\ 0 & -e & 0 & -a \end{pmatrix} \mid a,b,c,d,e \right\} \supset \mathfrak{b}.$$

Let us fix a standard parabolic P now, together with the corresponding totally isotropic flag F_P and the representation $M_P := M_{F_P}$ of dimension vector $\mathbf{d}_P$. Our notions from Section 4 generalize to this setup by looking at the variety $\mathrm{SR}_{\mathbf{d}_P}(\mathcal{A}(k))$ which is acted upon by the group

$$\mathrm{GL}_{\mathrm{sym}}(P) := \mathrm{GL}(d_1) \times \mathrm{GL}(d_2) \times \cdots \times \mathrm{GL}(d_k) \times \mathrm{Sym}(n)$$

where $\mathrm{Sym}(n)$ denotes either the symplectic or the orthogonal group on a vector space of dimension n, as before. Again, the open subset $\mathrm{SR}_{\mathbf{d}_P}(\mathcal{A}(k))^0$ corresponds to the full subcategory $\mathrm{srep}(\mathcal{A}(k), \mathbf{d}_P)^0$ of $\mathrm{srep}(\mathcal{A}(k), \mathbf{d}_P)$ of maximal rank representations.

We define the block vector $\mathbf{b}_P = (b_1, ..., b_k, b_{k+1})$ as follows: $b_1 = d_1$, $b_i = d_i - d_{i-1}$ (for $i = 2, \cdots k$) and $b_{k+1} = b_\omega := n/2 - d_k$ in types C and D, $b_{k+1} = b_\omega := (n-1)/2 - d_k$ in type B. This vector is closely related to the explicit structure of the parabolic (especially, if it is induced by a parabolic in GL_n) as can be seen in Example 7.1.

We obtain a translation as in Lemma 4.4 which can be proven in the same way.

Lemma 7.2. *There is a bijection between the isomorphism classes of symplectic/orthogonal $\mathcal{A}(k)$-representations in* $\mathrm{srep}(\mathcal{A}(k), \mathbf{d}_P)^0$ *and symplectic/orthogonal P-orbits in $\mathcal{N}(2) \subseteq \mathfrak{g}$.*

We aim to classify the P-orbits in $\mathcal{N}(2) \subset \mathfrak{g}$ by combinatorial objects and define the latter now; as before we denote $\omega = k+1$.

Definition 7.3. An *enhanced symmetric oriented link pattern* of type $(b_1, ..., b_k, b_{k+1})$ consists of a set of $k+1$ vertices $1, 2, \cdots, k+1$ together with a collection of oriented arrows which can be either dotted or plain (with or without dots) between different vertices, unoriented loops and oriented loops with dots. For every vertex i we put:

(1) x_i= the number of arrows (non-loops) touching i,

(2) y_i= the number of unoriented loops touching i,

(3) z_i= the number of loops with dots touching i.

The pattern is called *enhanced symplectic oriented link pattern* (esolp for short), if

(Sp) $x_i + 2y_i + z_i \leq b_i$ holds for each vertex $i < k+1$,

(Sp$_\omega$) $z_{k+1} = 0$ and vertex $k+1$ is not incident with an arrow with a dot. Furthermore, $x_{k+1} + y_{k+1} \leq b_{k+1}$.

It is called *enhanced orthogonal oriented link pattern* (eoolp for short), if

(Or) $x_i + 2y_i + 2z_i \leq b_i$ holds for each vertex $i \leq k+1$,

(Or$_\omega$) $z_{k+1} = 0$ and vertex $k+1$ is not incident with an arrow with a dot.

7.2. Orbits in type C

We generalize the results of Subsection 6.1 in a straightforward manner.

Theorem 7.4. *There is a natural bijection between the set of P-orbits in $\mathcal{N}(2) \subseteq \mathfrak{sp}_n$ and the set of esolps of type $\mathbf{b}_P = (b_1, ..., b_k, b_{k+1})$.*

Proof. By Lemma 7.2, there is a bijection between the P-orbits in $\mathcal{N}(2)$ and the isomorphism classes of representations in $\mathrm{srep}(\mathcal{A}(k), \mathbf{d}_P)^0$. Every such representation M decomposes into a direct sum of indecomposable symplectic representations

$$M \cong \bigoplus_{1 \leq s \leq k+1} (M_{s,\omega} \oplus M^*_{s,\omega})^{c_s} \oplus \bigoplus_{\substack{1\,\leq\, s\,\leq\, t\,\leq\, k+1\\ (s,\,t)\,\neq\,(k+1,\,k+1)}} (D^+_{s,t} \oplus C^+_{s,t})^{d^+_{s,t}} \oplus \bigoplus_{1 \leq s < t \leq k+1} (D^-_{s,t} \oplus C^-_{s,t})^{d^-_{s,t}}$$

$$\oplus\ (D_{k+1,k+1})^{d_{k+1}} \oplus \bigoplus_{1 \leq s < t \leq k} (Z^\pm_{s,t} \oplus Z^\pm_{t,s})^{e^\pm_{s,t}} \ \oplus \bigoplus_{s=1}^{k} (Z^\pm_{s,s})^{e^\pm_{s,s}}$$

for certain integers $c_s, d^\pm_{s,t}, d_{k+1}, e^\pm_{s,t}$ which are unique by Krull-Remak-Schmidt. Since $\underline{\mathrm{dim}} M = \mathbf{d}_P$, we know that (for $i < k+1$):

$$b_i = d_i - d_{i-1} = c_i + \sum_{j=1}^{i-1} (d^+_{j,i} + d^-_{j,i}) + \sum_{j=i+1}^{k+1} (d^+_{i,j} + d^-_{i,j}) + 2d^+_{i,i} + \sum_{j=1}^{k} (e^+_{i,j} + e^-_{i,j})$$

$$\omega = n/2 - d_k = c_{k+1} + \sum_{j=1}^{k} (d^+_{j,k+1} + d^-_{j,k+1}) + d_{k+1}$$

We obtain an esolp of type $(b_1, ..., b_{k+1})$ by drawing arrows and loops according to the algorithm depicted in Figure 4. By definition, we then have

$$x_i = \sum_{j=1}^{i-1} (d^+_{j,i} + d^-_{j,i}) + \sum_{j=i+1}^{k+1} (d^+_{i,j} + d^-_{i,j}) + \sum_{j=1}^{i-1} (e^+_{i,j} + e^-_{i,j}) + \sum_{j=i+1}^{k} (e^+_{i,j} + e^-_{i,j})$$

$$y_i = d^+_{i,i},\ z_i = e^+_{i,i} + e^-_{i,i},\ x_{k+1} = \sum_{j=1}^{k} (d^+_{j,k+1} + d^-_{j,k+1}),\ y_{k+1} = d_{k+1},\ z_{k+1} = 0$$

In particular, non-oriented loops appear, since $D^+_{s,s} \oplus C^+_{s,s} \cong D^-_{s,s} \oplus C^-_{s,s}$. Thus, $x_i + 2y_i + z_i = b_i - c_i \leq b_i$ for all $i < k+1$, no arrow with a dot starts or ends in $k+1$ and $x_{k+1} + y_{k+1} = b_{k+1} - c_{k+1} \leq b_{k+1}$.

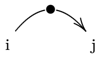
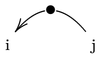
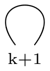
Indecomposable	Multiplicity	Count in esolp/eoolp
$M_{s,k+1} \oplus M^*_{s,k+1} \quad (s \leq k+1)$	c_s	$b_s - x_s - 2y_s - z_s$ (symp.) $b_s - x_s - 2y_s$ (orth.)
$D^-_{i,j} \oplus C^-_{i,j}$	$d^-_{i,j}$	number of 
$D^+_{s,s} \oplus C^+_{s,s} \cong D^-_{s,s} \oplus C^-_{s,s}$	$d^+_{s,s}$	number of 
$D^+_{i,j} \oplus C^+_{i,j}$	$d^+_{i,j}$	number of 
$Z^-_{i,j} \oplus Z^-_{j,i} \quad (j < k+1)$	$e^-_{i,j}$	number of 
$Z^+_{i,j} \oplus Z^+_{j,i} \quad (j < k+1)$	$e^+_{i,j}$	number of 
$Z^-_{s,s}$ (symp.) $Z^-_{s,s} \oplus Z^-_{s,s}$ (orth.)	$e^-_{s,s}$	number of 
$Z^+_{s,s}$ (symp.) $Z^+_{s,s} \oplus Z^+_{s,s}$ (orth.)	$e^+_{s,s}$	number of 
$D^+_{k+1,k+1}$ (symp.) $D^+_{k+1,k+1} \oplus D^+_{k+1,k+1}$ (orth.)	d_{k+1} $d^+_{k+1,k+1}$	number of 
$M_{k+1,k+1}$ (odd orth.)	1	

Figure 4: Correspondence between indecomposables and esolps/eoolps for values $i < j \leq k+1$ and $s < k+1$

Thus, there is a uniquely determined esolp of type $(b_1, ..., b_{k+1})$. In the same way, for every esolp of type $(b_1, ..., b_k, b_{k+1})$, we get a unique (up to isomorphism) symplectic representation $M \in \operatorname{srep}(\mathcal{A}(k), \mathbf{d}_P)^0$ by interpreting the arrows and loops as depicted in Figure 4. Pictorially speaking, an esolp is obtained by

(1) restricting the coefficient quiver of M to the subdiagram corresponding to M_α,

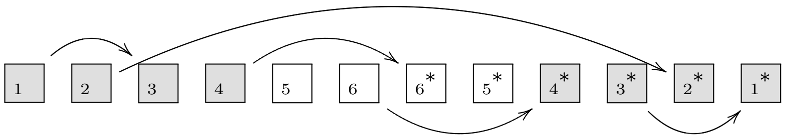
(2) by symmetry restricting it further to half of the vertices (as in the Borel case)

(3) by then glueing all vertices $i \in \{1, 2, ..., l+1\}$ together which correspond to the same step in the flag F, say to step $V_s \setminus V_{s-1}$.

By Lemma 7.2, the claim follows. ■

Clearly, solps are special esolps: they are of type $(1, ..., 1, 0) \in \mathbf{N}^{l+1}$, such that we obtain the classification of Borel orbits.

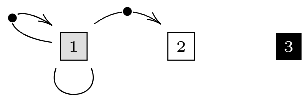
Example 7.5. Let P be the symplectic parabolic subgroup of block sizes $(4, 2, 2, 4)$, thus, $b_1 = 4$, $b_2 = 2$ and $b_3 = b_\omega = 0$. Then a symplectic representation of dimension vector $(4, 6, 12, 6, 4)$ is represented by a pattern of 12 coloured vertices which represents the map M_α of the representation, for example by



This pattern corresponds to the indecomposable direct sum decomposition

$$(D^+_{1,1} \oplus C^+_{1,1}) \oplus Z^-_{1,1} \oplus (Z^-_{1,2} \oplus Z^-_{2,1}) \oplus (M_{2,3} \oplus M^*_{2,3})$$

of a representation of the quiver $\mathcal{Q}_2$. The corresponding esolp is given by



We see that $x_1 + 2y_1 + z_1 = 1 + 2 + 1 \leq 4 = b_1$, $x_2 + 2y_2 + z_2 = 1 + 0 + 0 \leq 2 = b_2$ and $x_3 + y_3 = 0 + 0 = 6 - 6 = n/2 - d_k = b_3$; these equations show that the multiplicity of the indecomposable $M_{1,3} \oplus M^*_{1,3}$ is zero and the multiplicity of the indecomposable $M_{2,3} \oplus M^*_{2,3}$ is one.

7.3. Orbits in type B and D

Let us consider a parabolic subgroup P of O_{2l+1} or O_{2l} together with its totally isotropic flag F now. Then the classification of orbits is done analogously to the symplectic case.

Theorem 7.6. *There is a natural bijection between the set of P-orbits in $\mathcal{N}(2) \subseteq \mathfrak{g}$ and the set of eoolps of type $\mathbf{b}_P = (b_1, ..., b_k, b_{k+1})$.*

Proof. Let $M \in \mathrm{srep}(\mathcal{A}(k), \mathbf{d}_P)^0$, then M decomposes into a direct sum of indecomposable orthogonal representations

$$M \cong \bigoplus_{s=1}^{k+1} (M_{s,\omega} \oplus M^*_{s,\omega})^{c_s} \oplus \bigoplus_{1 \leq s \leq t \leq k+1} (D^+_{s,t} \oplus C^+_{s,t})^{d^+_{s,t}} \oplus \bigoplus_{1 \leq s < t \leq k+1} (D^-_{s,t} \oplus C^-_{s,t})^{d^-_{s,t}}$$

$$\oplus \bigoplus_{1 \leq s \leq t \leq k} (Z^+_{s,t} \oplus Z^+_{t,s})^{e^+_{s,t}} \oplus \bigoplus_{1 \leq s \leq t \leq k} (Z^-_{s,t} \oplus Z^-_{t,s})^{e^-_{s,t}} \oplus M^h_{k+1,k+1}$$

for certain integers $c_s, d^\pm_{s,t}, e^\pm_{s,t}$ and $h = 1$ in type B, $h = 0$ in type D. These are unique by Krull-Remak-Schmidt. Since $\underline{\dim} M = \mathbf{d}_P$, we know that (for $i < k + 1$):

$$b_i = c_i + \sum_{j=1}^{i-1} (d^+_{j,i} + d^-_{j,i}) + \sum_{j=i+1}^{k+1} (d^+_{i,j} + d^-_{i,j}) + 2d^+_{i,i} + \sum_{j=1}^{k} (e^+_{i,j} + e^-_{i,j}) + e^+_{i,i} + e^-_{i,i}$$

$$b_{k+1} = c_{k+1} + \sum_{j=1}^{k} (d^+_{j,k+1} + d^-_{j,k+1}) + 2d^+_{k+1,k+1}$$

For this representation, we obtain an eoolp of type $(b_1, ..., b_{k+1})$ by drawing arrows and loops according to the algorithm depicted in Figure 4. By definition, we then have

$$x_i = \sum_{j=1}^{i-1} (d^+_{j,i} + d^-_{j,i}) + \sum_{j=i+1}^{k+1} (d^+_{i,j} + d^-_{i,j}) + \sum_{j=1}^{i-1} (e^+_{i,j} + e^-_{i,j}) + \sum_{j=i+1}^{k} (e^+_{i,j} + e^-_{i,j})$$

$$y_i = d^+_{i,i}, \ z_i = e^+_{i,i} + e^-_{i,i}, \ x_{k+1} = \sum_{j=1}^{k} (d^+_{j,k+1} + d^-_{j,k+1}), \ y_{k+1} = d_{k+1}, \ z_{k+1} = 0,$$

and thus $x_i + 2y_i + 2z_i = b_i - c_i \leq b_i$ for all $i < k + 1$, no arrow or loop with a dot starts or ends in $k + 1$ and $x_{k+1} + 2y_{k+1} = b_{k+1} - c_{k+1} \leq b_{k+1}$.

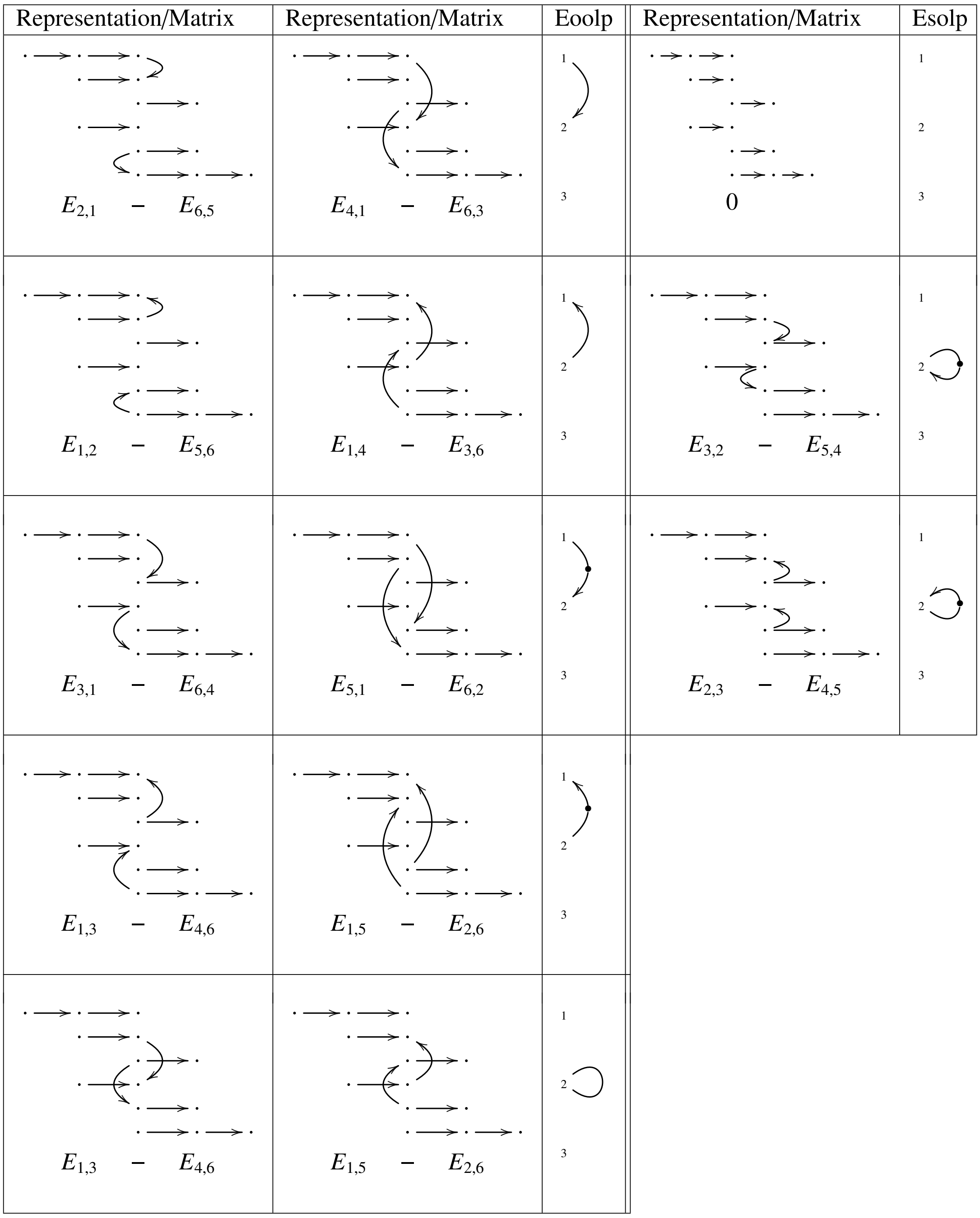


Figure 5: Isomorphism classes of orthogonal representations in $\mathrm{srep}(\mathcal{A}(2), \mathbf{d}_P)^0$ and their combinatorial interpretation

Thus, there is a uniquely determined eoolp of type $(b_1, ..., b_{k+1})$. In the same way, for every eoolp of type $(b_1, ..., b_{k+1})$, we get a unique (up to isomorphism) orthogonal representation $M \in \mathrm{srep}(\mathcal{A}(k), \mathbf{d}_P)^0$ by interpreting the arrows and loops as depicted in Figure 4. ■

Clearly, oolps are special eoolps, they are of type $(1, ..., 1, 0) \in \mathbf{N}^{l+1}$, such that we obtain the classification of Borel-orbits.

Example 7.7. Let us consider $n = 6$ and the totally isotropic flag

$$F = (V_1 = \langle e_1 \rangle \subset V_2 = \langle e_1, e_2, e_4 \rangle).$$

Then the coefficient quiver of M_F is

$$M_F = \begin{tikzcd}[row sep=tiny]
\bullet \arrow[r, "1"] & \bullet \arrow[r, "1"] & \bullet & & \\
& \bullet \arrow[r, "1"] & \bullet & & \\
& & \bullet \arrow[r, "-1"] & \bullet & \\
& \bullet \arrow[r, "1"] & \bullet & & \\
& & \bullet \arrow[r, "-1"] & \bullet & \\
& & \bullet \arrow[r, "-1"] & \bullet \arrow[r, "-1"] & \bullet
\end{tikzcd} \tag{9}$$

and the isomorphism classes of indecomposables are listed in the table in Figure 5.

8. Restriction to the nilradical

Let P be a parabolic subgroup of G which is given as the stabilizer of a totally isotropic standard flag F, that is, $P = \mathrm{stab}_{\mathrm{GL}_n}(F) \cap G$. Let $\mathfrak{n}_P(2) := \mathcal{N}(2) \cap \mathfrak{n}_P$ be the variety of 2-nilpotent matrices in the nilradical $\mathfrak{n}_P$ of $\operatorname{Lie} P$ and let $\mathcal{N}_P(2) := \mathcal{N}(2) \cap \operatorname{Lie} P$. Then the P-action on $\mathcal{N}(2)$ restricts to both of these varieties. The parametrization of the orbits can be obtained from our parametrizations of Sections 6 and 7 straight away.

Note that the Borel subgroup B thus acts on $\mathcal{N}_B(2) = \mathfrak{n}_B(2)$; this action is parametrized in [2] in the symplectic case, where Barnea and Melnikov also derive a description of the orbit closures and look at applications to orbital varieties in detail. In [8], Gandini, Maffei, Möseneder Frajria and Papi consider B-stable abelian subalgebras of the nilradical of $\mathfrak{b} = \operatorname{Lie} B$ in which they parametrize the B-orbits and describe their closure relations.

Definition 8.1. An *enhanced symplectic link pattern* (eslp) (*enhanced orthogonal link pattern* (eolp), resp.) of type $(b_1, ..., b_{k+1})$ is an esolp (eoolp, resp.) without orientation.

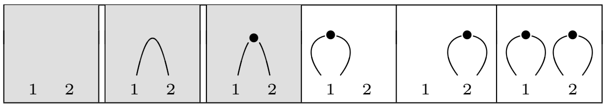
Lemma 8.2. *There is a bijection between the set of P-orbits in $\mathcal{N}_P(2) \subseteq \mathfrak{g}$ and the set of eslps (symplectic setup)/eolps (orthogonal setup). The P-orbits in $\mathfrak{n}_P(2)$ correspond bijectively to the set of eslps (symplectic setup)/eolps (orthogonal setup) without non-dotted loops.*

Proof. Via Lemma 7.2, restriction to the Lie algebra of P corresponds to the fact that the representation in $\mathrm{srep}(\mathcal{A}(k), \mathbf{d}_P)^0$ cannot have a direct summand isomorphic to the irreducible symplectic/orthogonal representations $D^-_{i,j} \oplus C^-_{i,j}$ for $i \neq j$ and $Z^-_{i,j} \oplus Z^-_{j,i}$ for arbitrary i, j. Thus, every oriented arrows of the esolp/eoolp points to the left and we can delete the orientation of the arrows in a unique way.

Restricting to $\mathfrak{n}_P(2)$ additionally means that the indecomposables $D^+_{i,i} \oplus C^+_{i,i} \cong D^-_{i,i} \oplus C^-_{i,i}$ for arbitrary i cannot appear as direct summands, such that we obtain the claimed bijection. ∎

We end the section with giving an example.

Example 8.3. For $l = 2$, we consider the Borel-action. Figures 2 and 3 show the possible patterns. Taking away the orientation is equivalent to only considering upper-triangular matrices. Thus, the following table gives a complete list of representatives of orbits for type C. The patterns which are marked gray give a complete list of orthogonal B-orbits. Note that we delete vertex 3, since $b_3 = 0$.



References

- [1] I. Assem, D. Simson, A. Skowroński: *Elements of the Representation Theory of Associative Algebras*, Vol. 1, Cambridge University Press, Cambridge (2006).
- [2] N. Barnea, A. Melnikov: *B-orbits of square zero in nilradical of the symplectic algebra*, Transformation Groups 22 (2017) 885–910.
- [3] M. Boos: *Finite parabolic conjugation on varieties of nilpotent matrices*, Algebras and Representation Theory 17 (2014) 1657–1682.
- [4] M. Boos, M. Reineke: *B-orbits of 2-nilpotent matrices and generalizations*, Progress in Mathematics 295 (2011) 147–166.
- [5] M. Brion: *Spherical varieties: an introduction*, Progress in Mathematics 80 (1989) 11–26.
- [6] M. C. R. Butler, C. M. Ringel: *Auslander-Reiten sequences with few middle terms and applications to string algebras*, Comm. Algebra 15 (1987) 145–179.
- [7] H. Derksen, J. Weyman: *Generalized quivers associated to reductive groups*, Colloquium Mathematicum 94 (2002) 151–173.
- [8] J. Gandini, A. Maffei, P. Möseneder Frajria, P. Papi: *The Bruhat order on abelian ideals of Borel subalgebras*, preprint (2018).
- [9] L. Hille, G. Röhrle: *A classification of parabolic subgroups of classical groups with a finite number of orbits on the unipotent radical*, Transformation Groups 4 (1999) 35–52.
- [10] M. E. C. Jordan: *Sur la résolution des équations différentielles linéaires*, Oeuvres 4 (1871) 313–317.
- [11] H. Kraft: *Geometrische Methoden in der Invariantentheorie*, Vieweg Verlag, Braunschweig (1984).
- [12] P. Magyar, J. Weyman, A. Zelevinsky: *Symplectic multiple flag varieties of finite type*, J. Algebra 230 (2000) 245–265.
- [13] G. Malle, D. Testerman: *Linear Algebraic Groups and Finite Groups of Lie Type*, Cambridge University Press, Cambridge (2011).
- [14] A. Melnikov: *B-orbits in solutions to the equation $X^2 = 0$ in triangular matrices*, J. Algebra 223 (2000) 101–108.
- [15] D. I. Panyushev: *Some amazing properties of spherical nilpotent orbits*, Math. Zeitschrift 245 (2003) 557–580.
- [16] J.-P. Serre: *Espaces fibrés algébriques*, Séminaire Claude Chevalley 3 (1958) 1–37.
- [17] The On-Line Encyclopedia of Integer Sequences, Sequence a115329, published electronically at http://oeis.org (2010).
- [18] The On-Line Encyclopedia of Integer Sequences, Sequence a202837, published electronically at http://oeis.org (2010).

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