

# POSITIVE SCALAR CURVATURE ON SIMPLY CONNECTED SPIN PSEUDOMANIFOLDS

BORIS BOTVINNIK, PAOLO PIAZZA, AND JONATHAN ROSENBERG

ABSTRACT. Let  $M_\Sigma$  be an  $n$ -dimensional Thom-Mather stratified space of depth 1. We denote by  $\beta M$  the singular locus and by  $L$  the associated link. In this paper we study the problem of when such a space can be endowed with a wedge metric of positive scalar curvature. We relate this problem to recent work on index theory on stratified spaces, giving first an obstruction to the existence of such a metric in terms of a wedge  $\alpha$ -class  $\alpha_w(M_\Sigma) \in KO_n$ . In order to establish a sufficient condition we need to assume additional structure: we assume that the link of  $M_\Sigma$  is a homogeneous space of positive scalar curvature,  $L = G/K$ , where the semisimple compact Lie group  $G$  acts transitively on  $L$  by isometries. Examples of such manifolds include compact semisimple Lie groups and Riemannian symmetric spaces of compact type. Under these assumptions, when  $M_\Sigma$  and  $\beta M$  are spin, we reinterpret our obstruction in terms of two  $\alpha$ -classes associated to the resolution of  $M_\Sigma$ ,  $M$ , and to the singular locus  $\beta M$ . Finally, when  $M_\Sigma$ ,  $\beta M$ ,  $L$ , and  $G$  are simply connected and  $\dim M$  is big enough, and when some other conditions on  $L$  (satisfied in a large number of cases) hold, we establish the main result of this article, showing that the vanishing of these two  $\alpha$ -classes is also sufficient for the existence of a well-adapted wedge metric of positive scalar curvature.

## 1. INTRODUCTION

This paper continues a program begun in [10] and in [12], to understand obstructions to positive scalar curvature (which we will sometime abbreviate as psc) on manifolds with fibered singularities, for metrics that are well adapted to the singularity structure.

In the cases studied in this paper, the stratified spaces or singular manifolds  $M_\Sigma$  that we study are Thom-Mather pseudomanifolds of depth one. For the existence theorem we shall take the two strata to be spin and simply connected; more general situations, involving non-trivial fundamental groups, will be dealt with in a forthcoming article [11]. Topologically,  $M_\Sigma$  is homeomorphic to a quotient space of a compact smooth manifold  $M$  with fibered boundary  $\partial M$ . Then  $M$  is called the *resolution* of  $M_\Sigma$ , and the quotient map  $M \rightarrow M_\Sigma$  is the identity on the interior  $\overset{\circ}{M}$  of  $M$ , and on  $\partial M$ , collapses the fibers of a fiber bundle  $\varphi: \partial M \rightarrow \beta M$ , with fibers all diffeomorphic to a fixed manifold  $L$ , called the *link* of the singularity, and with base  $\beta M$  sometimes called the Bockstein (by analogy with other cases in topology). We briefly refer to these spaces as *manifolds with  $L$ -fibered singularities*.

Note that the structure group of the bundle  $\varphi$  can be an arbitrary subgroup of  $\text{Diff}(L)$ , and for part of our results we do consider this general situation. However, in studying the existence problem

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for wedge metrics of positive scalar curvature we shall need to have more structure in order to relate the topology of the bundle, in particular its bordism theory, with its differential geometric features.

To this end we assume that the fiber bundle  $\varphi: \partial M \rightarrow \beta M$  comes from a principal  $G$ -bundle  $p: P \rightarrow \beta M$ , for some connected compact Lie group  $G$  that acts transitively on  $L$  by isometries for some fixed metric  $g_L$ , and thus  $\partial M = P \times_G L$ . The transitivity of the action of  $G$  on  $L$  means that  $L = G/K$  is a homogeneous space and has constant scalar curvature. We refer to these special singular spaces as *manifolds with  $(L, G)$ -fibered singularities*.

Since the space  $M_\Sigma$  is not a smooth manifold in general (it will be if and only if  $L$  is a standard sphere), we need to explain what kind of metric we want to use. We shall employ *wedge metrics*, also called *iterated conic metrics*, on the regular part of  $M_\Sigma$ . These are defined as follows. First, we identify  $M_\Sigma$  with a union  $M_\Sigma = M \cup_{\partial M} -N(\beta M)$ , where  $M$  is a manifold with fibered boundary and  $N(\beta M)$  is a tubular neighborhood of the singular locus  $\beta M$ . Then on  $M$  we consider a metric  $g_M$  which is a product metric  $dt^2 + g_{\partial M}$  in a small collar neighborhood  $\partial M \times [0, \varepsilon)$  of the boundary  $\partial M$ ; we assume that  $g_{\partial M}$  is a submersion metric for the bundle  $\varphi: \partial M \rightarrow \beta M$ , with horizontal metric  $\phi^*g_{\beta M}$  and vertical metric  $g_{\partial M/\beta M}$ . On the regular part of  $N(\beta M)$  we consider a metric of the following type:

$$g_{N(\beta M)} = dr^2 + r^2 g_{\partial M/\beta M} + \varphi^* g_{\beta M} + O(r).$$

We call such metrics on the regular part of  $M_\Sigma$  *adapted wedge metrics*. Notice that we can consider adapted wedge metrics even if the link is  $S^n$ ; in that case we are considering special metrics on a smooth ambient manifold  $M_\Sigma$  with respect to a specified submanifold  $\beta M$ .

In the case in which  $L$  is a homogeneous space, as above, there is a natural submersion metric on  $\partial M \xrightarrow{\varphi} \beta M$  which is defined as follows. A connection  $\nabla^p$  on the principal bundle  $p: P \rightarrow \beta M$  gives a connection  $\nabla^\varphi$  on the associated fiber bundle  $\varphi: \partial M \rightarrow \beta M$ . Then, since the structure group  $G$  of the bundle  $\varphi: \partial M \rightarrow \beta M$  acts by isometries of the metric  $g_L$ , the connection  $\nabla^\varphi$  provides an orthogonal splitting of the metric  $g_{\partial M}$  into the *horizontal* metric lifted from  $g_{\beta M}$  and the *vertical metric*  $g_L$ . For the details, see Section 3. In other words, when the link  $L$  is a homogeneous space, the wedge metric near the singularity is determined by a metric  $g_{\beta M}$  on the singular locus  $\beta M$  and the standard metric  $g_L$  on  $L$ , together with the connection. To make the problem of existence of an adapted metric of positive scalar curvature maximally interesting, we add one more condition, that  $L$  have constant positive scalar curvature  $\kappa_L$  precisely equal to the scalar curvature  $\kappa_\ell$  of the standard round  $\ell$ -sphere, where  $\ell = \dim L$ . This insures that the cones over  $L$  are actually scalar-flat; see Section 3. An adapted wedge metric  $g$  on  $M_\Sigma$  (with no  $O(r)$  error term) which satisfies those additional conditions is called a *well-adapted wedge metric*.

We note here that since the metric  $g_L$  has positive scalar curvature, it follows that the metric  $g_{\partial M}$  on the boundary  $\partial M$  could be assumed to have positive scalar curvature as well. In the case when all manifolds are spin, this implies that the corresponding Dirac operator on  $M$  gives an invariant  $\alpha_{\text{cyl}}(M) \in KO_n$ , defined by attaching a cylindrical end  $\partial M \times [0, \infty)$  to  $M$  and giving it the product metric  $g_{\partial M} + dt^2$ . Then we take the  $KO_n$ -valued index of the  $C\ell_n$ -linear Dirac operator [21, §II.7]

on the resulting noncompact manifold  $M_\infty := M \cup_{\partial M} (\partial M \times [0, \infty))$ ; the operator is Fredholm since the scalar curvature is bounded away from 0 except on a compact set.

In addition, the Dirac operator on the manifold  $\beta M$  also determines a corresponding class  $\alpha(\beta M) \in KO_{n-1-\ell}$ . For later use, we use the notation  $\kappa_\ell = \ell(\ell - 1)$  for the scalar curvature of the unit sphere  $S^\ell(1)$ . Now we can state our main results on manifolds with  $(L, G)$ -singularities.

**Theorem 1.1** (Obstruction Theorem). *Let  $L = G/K$  be a homogeneous space,  $\dim L = \ell$ , where  $G$  is a connected compact semisimple Lie group, and  $g_L$  be a  $G$ -invariant Riemannian metric on  $L$  of constant scalar curvature equal to  $\kappa_\ell = \ell(\ell - 1)$ . Let  $M_\Sigma = M \cup_{\partial M} -N(\beta M)$ , where  $M$  and  $\beta M$  are compact spin manifolds, and the boundary  $\partial M = P \times_G L$  for some principal  $G$ -bundle  $p: P \rightarrow \beta M$  with classifying map  $\beta M \rightarrow BG$ . Assume that  $M_\Sigma$  admits a well-adapted metric of positive scalar curvature. Then the  $\alpha$ -invariants  $\alpha_{\text{cyl}}(M) \in KO_n$  and  $\alpha(\beta M) \in KO_{n-\ell-1}$  both vanish.*

Some of the background leading to the Obstruction Theorem will be discussed in Section 2. There, building on work of Albin-Gell-Redman, we shall in fact introduce a wedge alpha-class  $\alpha_w(M_\Sigma, g) \in KO_n$  which is defined under much weaker conditions and that yields the most general Obstruction Theorem, Theorem 2.3, for simply connected manifolds with  $L$ -fibered singularities. The special case treated in Theorem 1.1, namely manifolds with  $(L, G)$ -fibered singularities, is treated later, in Section 5. We have singled out this version of the Obstruction Theorem now because it is this theorem for which we will prove a converse, namely an existence result under the assumption that the two obstructions vanish.

To state the existence result, we need one more definition. Let  $X$  be a closed spin manifold endowed with a  $G$ -action. Let  $g_X$  be a psc-metric on  $X$ . We say that a  $(X, g_X)$  is a *spin psc- $G$ -boundary* if there exists a spin  $G$ -manifold  $Z$  bounding  $X$  as a spin  $G$ -manifold and a psc-metric  $g_Z$  on  $Z$  which is a product metric near the boundary with  $g_Z|_X = g_X$ .

**Theorem 1.2** (Existence Theorem). *Let  $L = G/K$  be a homogeneous space,  $\dim L = \ell$ , where  $G$  is a connected compact semisimple Lie group, and  $g_L$  be a  $G$ -invariant Riemannian metric on  $L$  of constant scalar curvature equal to  $\kappa_\ell = \ell(\ell - 1)$ . Let  $M_\Sigma = M \cup_{\partial M} -N(\beta M)$ , where  $M$  and  $\beta M$  are compact spin manifolds, and the boundary  $\partial M = P \times_G L$  for some principal  $G$ -bundle  $p: P \rightarrow \beta M$  with classifying map  $\beta M \rightarrow BG$ . Assume  $M$ ,  $\beta M$ ,  $L$ , and  $G$  are simply connected and  $n \geq \ell + 6$ .*

*Then  $M_\Sigma$  admits a well-adapted psc-metric if and only if the  $\alpha$ -invariants  $\alpha_{\text{cyl}}(M) \in KO_n$  and  $\alpha(\beta M) \in KO_{n-\ell-1}$  both vanish, provided one of the following conditions holds:*

- (i) *the manifold  $(L, g_L)$  is a spin psc- $G$ -boundary; or*
- (ii) *the bordism class  $[\beta M \rightarrow BG]$  vanishes in  $\Omega_{n-\ell-1}^{\text{spin}}(BG)$ .*

**Remark 1.3.**

- (1) The dimensional assumption  $n \geq \ell + 6$  is necessary in order to apply surgery theory to  $M$  and to  $\beta M$ .
- (2) The assumption (i) holds when  $L$  is a sphere, an odd complex projective space, or when  $L = G$ . Other cases where this assumption holds are discussed in Remark 6.2.

- (3) The assumption (ii) holds automatically if  $\partial M = \beta M \times L$  (i.e., the bundle  $\varphi$  is trivial) and if  $L$  is an even quaternionic projective space.

Sections 2, 3, and 4 contain important preliminaries needed for the proofs of these theorems. The proof of Theorem 1.1 is completed in Section 5, and the proof of Theorem 1.2 is in Section 6.

Extensions of these results to the case where  $M$  and  $\partial M$  are not necessarily simply connected will be found in our sequel paper [11]. A quick sketch of the contents of that paper is in Section 7.

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## 2. DIRAC OPERATORS AND ASSOCIATED $KO$ CLASSES

**2.1. Introduction.** In this section we review and revisit necessary constructions and results concerning Dirac operators on smooth spin pseudomanifolds with depth-1 singularities. Our goal here is to describe under what conditions a Dirac operator defines a corresponding  $KO$ -homology class. We shall proceed in some generality first and then specify further assumptions on the link fibration and get sharper results correspondingly.

A particular case of pseudomanifolds with depth-1 singularities is given by manifolds with Baas-Sullivan singularities [7], when a type of singularity (a link  $L$ ) is fixed. Starting with a smooth manifold  $M$  with boundary  $\partial M = \beta M \times L$ , we obtain a manifold with  $L$ -singularity  $M_\Sigma$  by gluing  $M$  to the product  $\beta M \times c(L)$  (where  $c(L)$  is a cone over  $L$ ) along  $\partial M$ , i.e.,  $M_\Sigma = M \cup -\beta M \times c(L)$ . It will be convenient to adapt similar notations for pseudomanifolds with depth-1 singularities.

**2.2. Pseudomanifolds of depth 1.** First we recall necessary definitions. For the rest of this section, we fix a closed smooth compact manifold  $L$ , called the *link*. In terms of the notation of [10], the singularity type of our pseudomanifolds is  $\Sigma = (L)$ . We consider a depth-1 Thom-Mather pseudomanifold  $M_\Sigma$  with dense stratum  $M_\Sigma^{\text{reg}}$ , singular stratum  $\beta M$ , and associated link  $L$ . We shall not go into a detailed explanation of the axioms of a Thom-Mather space — see, for example, [13, Section A.1] and [4, Definition 2.1] — but we simply recall that as a consequence of these axioms, we have a locally compact metrizable space  $M_\Sigma$  such that:

- the space  $M_\Sigma$  is the union of two smooth strata,  $M_\Sigma^{\text{reg}}$  and  $\beta M$ ;
- the manifold  $M_\Sigma^{\text{reg}}$  is open and dense in  $M_\Sigma$ ;
- the manifold  $\beta M$  is smooth and compact;
- there is an open neighborhood  $N(\beta M)$  of  $\beta M$  in  $M_\Sigma$ , equipped with a continuous retraction  $\text{re}: N(\beta M) \rightarrow \beta M$  and a continuous map  $\rho: N(\beta M) \rightarrow [0, +\infty)$  which is smooth in  $N(\beta M)^{\text{reg}} := N(\beta M) \cap M_\Sigma^{\text{reg}}$  and such that  $\rho^{-1}(0) = \beta M$ ;

- the neighborhood  $N(\beta M)$  is a (locally trivial) fiber bundle over  $\beta M$  (via the above retraction  $\text{re}$ ) with fiber  $c(L)$ , the cone over  $L$ , and with radial variable along the cones induced by  $\rho$ .

We can associate to  $M_\Sigma$  its *resolution*, which in our case is a manifold  $M := M_\Sigma \setminus \rho^{-1}([0, 1))$  with boundary  $\partial M := \rho^{-1}(1)$ , such that  $\partial M$  is the total space of a smooth fiber bundle  $L \rightarrow \partial M \xrightarrow{\varphi} \beta M$  with fiber  $L$ . This fibration is sometimes called the *link bundle* associated to  $M_\Sigma$ . Clearly, there is a diffeomorphism between the interior  $\overset{\circ}{M}$  of  $M$  and  $M_\Sigma^{\text{reg}}$ . Once a link  $\Sigma = (L)$  is fixed, we will call such a pseudomanifold  $M_\Sigma$  a *pseudomanifold with fibered  $L$ -singularity*. Sometimes we abuse terminology by calling the resolution  $M$  of  $M_\Sigma$  a *manifold with fibered  $L$ -singularity*, in cases where the extra structure is understood.

Thus we have  $M_\Sigma := M \cup_{\partial M} N(\beta M)$ , where  $N(\beta M)$  comes together with a fiber-bundle

$$c(L) \rightarrow N(\beta M) \xrightarrow{\varphi_c} \beta M.$$

(We have used the symbol  $\varphi_c$  here since this bundle is just the result of replacing each fiber  $L$  in the fibration  $\varphi$  by the cone  $c(L)$  over  $L$ .) Moreover if  $v$  is the vertex of  $c(L)$ , then the inclusion  $\{v\} \hookrightarrow c(L)$  induces an embedding:

$$\beta M \hookrightarrow N(\beta M) \subset M_\Sigma.$$

**2.3. Wedge metrics.** Let  $M_\Sigma$  be a depth-1 pseudomanifold as above. A Riemannian metric on  $M_\Sigma$  is, by definition, a Riemannian metric  $g$  on  $M_\Sigma^{\text{reg}}$ . We shall consider special types of metrics. To this end we fix  $g_{\partial M}$ , a Riemannian metric on  $\partial M$  and  $g_{\beta M}$ , a Riemannian metric on  $\beta M$ . We assume that  $\varphi: \partial M \rightarrow \beta M$  is a Riemannian submersion; this means that we have fixed a connection on  $\partial M$ , that is a splitting  $T(\partial M) = T(\partial M/\beta M) \oplus T_H(\partial M)$  with  $T_H(\partial M) \simeq \varphi^*T(\beta M)$  and  $g_{\partial M} = h_{\partial M/\beta M} \oplus \varphi^*g_{\beta M}$ ,<sup>1</sup> with  $h_{\partial M/\beta M}$  a metric on the vertical tangent bundle  $T(\partial M/\beta M)$  of the fibration  $L \rightarrow \partial M \xrightarrow{\varphi} \beta M$ . Let  $r$  now be the radial variable along the cones with  $r = 0$  corresponding to  $\beta M$ .

**Definition 2.1.** We say that  $g$  on  $M_\Sigma^{\text{reg}}$  is a *wedge metric* if on  $N(\beta M)$  it can be written as

$$(1) \quad dr^2 + r^2 h_{\partial M/\beta M} + \varphi_c^* g_{\beta M} + O(r).$$

Equation (1) says that the difference between  $g$  and a metric of the form  $dr^2 + r^2 h_{\partial M/\beta M} + \varphi_c^* g_{\beta M}$  is a smooth section of the symmetric tensor product of the cotangent bundle vanishing at  $r = 0$ . If, in addition,  $g$  is of product type near  $\partial N(\beta M) = \partial M$ , then we call  $g$  an *adapted wedge metric*. We refer to the pair  $(M_\Sigma, g)$  as a *wedge space* (of depth one).

Notice that  $g$  is an incomplete Riemannian metric on the open manifold  $M_\Sigma^{\text{reg}}$ . Using the diffeomorphism between the interior  $\overset{\circ}{M} \subset M$  of the resolved manifold and  $M_\Sigma^{\text{reg}}$ , we can induce a metric on  $\overset{\circ}{M}$ , denoted again  $g$ . We redefine  $r$  to be the boundary defining function for  $\partial M \subset M$ . In connection with a wedge metric  $g$ , we can consider the *wedge tangent bundle*  ${}^w T M$  over the resolved manifold  $M$

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<sup>1</sup>The symbol  $\oplus^\perp$  means orthogonal direct sum, where the pulled-back metric from the base  $\beta M$  is put on the horizontal subspaces as defined by the connection.

(also called the *incomplete edge tangent bundle*). See for example [2, Section 2.1]. The wedge tangent bundle is dual to the wedge cotangent bundle  ${}^wT^*M$ , defined through the Serre-Swan theorem [37] as the vector bundle whose sections are given by the finitely generated projective  $C^\infty(M)$ -module  $\{\omega \in C^\infty(M, T^*M) \mid \iota_\partial^* \omega \in \varphi^* C^\infty(\beta M, T^*(\beta M))\}$ , with  $\iota_\partial$  denoting the inclusion of  $\partial M$  into  $M$ . As explained in detail in [2, Section 2.1], the wedge cotangent bundle has a local basis given by the 1-forms  $\{dr, rd\lambda_1, \dots, rd\lambda_k, dy_1, \dots, dy_\ell\}$  with  $\lambda_1, \dots, \lambda_\ell$  local coordinates on  $L$ , where  $\ell = \dim L$ , and  $y_1, \dots, y_k$  local coordinates on  $\beta M$ ; notice that the differential forms  $\{rd\lambda_1, \dots, rd\lambda_k\}$  vanish on the boundary as sections of the cotangent bundle  $T^*M$  but they do not vanish as sections of  ${}^wT^*M$ . See [2, Section 2.1]. It is easy to show that a wedge metric  $g$  extends from the interior of  $M$  to a smooth metric on  ${}^wTM \rightarrow M$ . We remark that  $M$  can also be given the collared metric  $g|_M$ ; we denote this metric by  $g_M$ ; needless to say,  $g$  and  $g_M$  have different behavior near the boundary.

We also have the *edge tangent bundle*,  ${}^eTM \rightarrow M$ , see [24], defined through the Serre-Swan theorem as the vector bundle whose sections are given by the finitely generated projective  $C^\infty(M)$ -module

$$(2) \quad \mathcal{V}_e = \{V \in C^\infty(M, TM) \text{ such that } V|_{\partial M} \text{ is tangent to the fibers of } \partial M \xrightarrow{\varphi} \beta M\}$$

The vector fields in  $\mathcal{V}_e$  are called *edge vector fields*. The edge tangent bundle  ${}^eTM \rightarrow M$  is locally spanned by the vector fields

$$(3) \quad r\partial_r, \quad \partial_{\lambda_1}, \dots, \partial_{\lambda_\ell}, \quad r\partial_{y_1}, \dots, r\partial_{y_k}.$$

where, as before, the vector fields  $\{\partial_{\lambda_\ell}, r\partial_{y_1}, \dots, r\partial_{y_k}\}$  vanish on the boundary as sections of the tangent bundle  $TM$  but they are non-vanishing as sections of  ${}^eTM \rightarrow M$ . See [25] for more on this general philosophy. What is interesting about  $\mathcal{V}_e$  is that it is a Lie algebra; this means that it generates an algebra of differential operators known as *the algebra of edge differential operators*, denoted  $\text{Diff}_e^*(M)$ . For edge differential operator there is a pseudodifferential calculus developed by Mazzeo [24] and this calculus plays a central role in the analysis of Dirac-type operators associated to wedge metrics.

We say that a wedge metric  $g$  is a psc-metric if it has positive scalar curvature as a Riemannian metric on  $M_\Sigma^{\text{reg}}$ . In this paper we are interested in understanding necessary and (for some particular links  $L$ ) sufficient conditions when a pseudomanifold  $M_\Sigma$  with  $L$ -fibered singularity admits a psc wedge metric. When this space of psc-metrics is non-empty, we shall investigate in the sequel paper [11] its topological properties, for example the number of its connected components or the cardinality of its homotopy groups.

Note that stratified pseudomanifolds can also be endowed with different metrics, for example edge metrics or  $\Phi$ -metrics (in contrast with the wedge case, these are *complete* metrics). For a rather detailed study of the resulting index theoretic obstructions to the existence of  $\Phi$ -metrics with psc, see [29].

**2.4. Spin-stratified pseudomanifolds and Dirac operators:  $KO$ -homology classes.** Let  $(M_\Sigma, g)$  be a wedge space of depth 1, with singular locus  $\beta M$  and link  $L$ . We assume that the

resolved manifold  $M$  admits a spin structure and we fix such a structure. (It would be equivalent to fix a spin structure on  ${}^wTM$ , since this is isomorphic to  $TM$  as bundles.) This fixes a spin structure on  $\partial M$  too. We assume additionally that  $\beta M$  is spin and fix a spin structure for  $\beta M$ . Notice that this also fixes a spin structure for the vertical tangent bundle of the boundary fibration, denoted  $T(\partial M/\beta M)$ , which is endowed with the vertical metric  $g_{\partial M/\beta M}$ . See [21, II, Prop. 1.15].

**Definition 2.2.** Let  $(M_\Sigma, g)$  be a wedge space of depth 1, singular locus  $\beta M$  and link  $L$ . We shall say that  $M_\Sigma$  is **spin-stratified** if both  $M$  and  $\beta M$  are spin.

We denote by  $\mathfrak{S}_g(M) \rightarrow M$  the bundle given by  $P_{\text{spin}} \times_\lambda Cl_n$ , with  $\lambda: \text{Spin}_n \rightarrow \text{Hom}(Cl_n, Cl_n)$  the representation given by left multiplication. See [21, Chapter II, Section 7]. There is a fiberwise right action of  $Cl_n$  on  $\mathfrak{S}_g(M)$  on the right which makes  $\mathfrak{S}_g(M)$  a bundle of rank 1  $Cl_n$ -modules. Notice that  $\mathfrak{S}_g(M)$  is graded:  $\mathfrak{S}_g(M) = \mathfrak{S}_g^0(M) \oplus \mathfrak{S}_g^1(M)$ . Let  $\mathfrak{D}_g$  be the associated  $Cl_n$ -linear Atiyah-Singer operator. The operator  $\mathfrak{D}_g$  is defined in the usual fashion,  $\mathfrak{D}_g := \text{cl} \circ \nabla$ , with  $\nabla$  the Levi-Civita connection on  $\mathfrak{S}_g(M)$ ; it has the local expression

$$\mathfrak{D}_g := \sum_j \text{cl}(e^j) \nabla_{e_j}$$

with  $\{e_j\}$  a local orthonormal frame of vector fields and  $\{e^j\}$  the dual basis defined by the metric. See again [21, Chapter II, Section 7] for the details.  $\mathfrak{D}_g$  is a  $\mathbb{Z}/2$ -graded odd formally self-adjoint operator of Dirac type commuting with the right action of  $Cl_n$ . For this operator the Schrödinger-Lichnerowicz formula holds:

$$(4) \quad \mathfrak{D}_g^2 = \nabla^* \nabla + \frac{1}{4} \kappa_g,$$

with  $\kappa_g$  denoting the scalar curvature of  $g$ . See again [21, Chapter II, Section 7] and also [31] for more details on this crucial point.

**Notation.** Unless absolutely necessary we shall omit the reference to the wedge metric  $g$  in the bundle and the operator, thus denoting the  $Cl_n$ -linear Atiyah-Singer operator simply by  $\mathfrak{D}$ . Moreover, we shall often use the shorter notation  $\mathfrak{S}$  instead of  $\mathfrak{S}(M)$ .

We can regard  $\mathfrak{D}$  as a wedge differential operator of order 1 on  $L^2(M, \mathfrak{S})$ —more on this in a moment—initially with domain equal to  $C_c^\infty(M_\Sigma^{\text{reg}}, \mathfrak{S}) \equiv C_c^\infty(\overset{\circ}{M}, \mathfrak{S}) \subset L^2(M, \mathfrak{S})$ . We are looking for self-adjoint  $Cl_n$ -linear extensions of this differential operator in  $L^2(M, \mathfrak{S})$ . Note, crucially, that the analysis given in [2] and [1] is quite general, since it applies to any Dirac-type operator.

Associated to  $\mathfrak{D}$  there is a well defined boundary operator  $\mathfrak{D}_\partial$ , which is nothing but the Atiyah-Singer operator of the spin manifold  $\partial M$ . We are assuming that  $\partial M$  is a fiber bundle of spin manifolds,  $L \rightarrow \partial M \xrightarrow{\mathcal{L}} \beta M$ ; consequently

$$\mathfrak{S}(\partial M) \simeq \mathfrak{S}(\partial M/\beta M) \hat{\otimes} \varphi^* \mathfrak{S}(\beta M).$$

The careful study made by Albin and Gell-Redman of the Levi-Civita connection near the singular stratum, see Section 2.2 and also Section 3.1 there, implies that

$$(5) \quad \mathfrak{D} = \text{cl}(dr) \left( \partial_r + \frac{\ell}{2r} + \frac{1}{r} \mathfrak{D}_{\partial M/\beta M} \hat{\otimes} \text{Id} + \text{Id} \hat{\otimes} \tilde{\mathfrak{D}}_{\beta M} \right) + \mathfrak{B}$$

with  $\ell = \dim L$ ,  $\mathfrak{D}_{\partial M/\beta M}$  the vertical family of Atiyah-Singer operators on the fibration  $L \rightarrow \partial M \xrightarrow{\varphi} \beta M$ ,  $\tilde{\mathfrak{D}}_{\beta M}$  an explicit horizontal operator, and  $\mathfrak{B}$  a bundle endomorphism which is  $O(r)$ . Formula (5) exhibits  $\mathfrak{D}$  as a wedge differential operator of degree 1. With a small abuse of notation, widely used in family index theory, we denote by  $\mathfrak{D}_L$  the generic operator of this vertical family. The following result holds:

**Theorem 2.3.** *Let  $M_\Sigma$  be spin-stratified and assume that*

$$(6) \quad \text{spec}_{L^2}(\mathfrak{D}_L) \cap (-1/2, 1/2) = \emptyset \quad \text{for each fiber } L.$$

*Then:*

- 1] *The operator  $\mathfrak{D}$  with domain  $C_c^\infty(M_\Sigma^{\text{reg}}, \mathfrak{G}) \subset L^2(M, \mathfrak{G})$  is essentially self-adjoint.*
- 2] *Its unique self-adjoint extension, still denoted by  $\mathfrak{D}$ , defines a  $Cl_n$ -linear Fredholm operator and thus a class  $\alpha_w(M_\Sigma, g)$  in  $KO_n$ , with  $n = \dim M_\Sigma$ .*

*If the vertical metric  $g_{\partial M/\beta M}$  is of psc along the fibers, then there exists  $\epsilon > 0$  such that*

$$\text{spec}_{L^2}(\mathfrak{D}_L) \cap (-\epsilon, \epsilon) = \emptyset \quad \text{for each fiber } L.$$

*Thus, by suitably rescaling the wedge metric  $g$  along the vertical direction of the boundary fibration, we can achieve condition (6) by assuming only that the vertical metric  $g_{\partial M/\beta M}$  is of psc along the fibers.<sup>2</sup>*

*Finally, if  $(M_\Sigma^{\text{reg}}, g)$  has psc everywhere then  $\mathfrak{D}$  is  $L^2$ -invertible; in particular  $\alpha_w(M_\Sigma, g) = 0$  in  $KO_n$ .*

*Proof.* Items 1] and 2] follow directly from the microlocal analysis methods employed by Albin and Gell-Redman, and more specifically from the explicit form of the normal operator associated to  $\mathfrak{D}$  that one obtains from (5). See [2] and [1]. The result relative to the psc assumption along the fibers follows as in Bismut-Cheeger [9, Section 4], using the Schrödinger-Lichnerowicz formula along the fibers. The last statement follows also from the Schrödinger-Lichnerowicz formula.  $\square$

**Definition 2.4.** We shall say that the spin-stratified pseudomanifold  $(M_\Sigma, g)$  is **psc-Witt** if the metric  $g$  is of psc along the links, i.e. if the vertical metric  $g_{\partial M/\beta M}$  induces on each fiber  $L$  a metric of psc.

**Remark 2.5.** In this work we have limited ourselves to the  $\alpha$ -class in  $KO_n$ .  $\alpha$ -classes in  $KO$  of suitable real  $C^*$ -algebras will be treated in the sequel [11] of this article.

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<sup>2</sup>This is analogous to the case of the signature operator on Witt and Cheeger spaces; see [3, 5].



**2.5. Dependence on a metric.** Contrary to the case of closed manifolds, the wedge  $\alpha$ -class  $\alpha_w(M_\Sigma, g)$  of a psc-Witt pseudomanifold can depend on the choice of the adapted wedge metric  $g$  near the singular locus  $\beta M$ . On the other hand, if  $g(t)$  is a smooth 1-parameter family of adapted wedge metrics that are psc-Witt for each  $t$ , then  $\alpha_w(M_\Sigma, g_t)$  in  $KO_n$  does not depend on  $t$ . Indeed, since our assumption ensures that condition (6) is satisfied for all  $t$  (up to scaling), it is possible to construct a continuous family of parametrices. The result then follows from [21, Ch. III, Theorem 10.8].

**2.6. Cylindrical KO-theory classes and a gluing formula.** We decompose

$$M_\Sigma = M \cup_{\partial M} (-N(\beta M)) \quad \text{and} \quad M_\Sigma^{\text{reg}} = M \cup_{\partial M} (-N(\beta M)^{\text{reg}}).$$

Let  $g$  be an adapted wedge metric on  $M_\Sigma$ . Denote by  $g_M$  the Riemannian metric induced by restriction of  $g$  to  $M$ ; similarly, let  $g_{N(\beta M)}$  be the metric induced by restriction of  $g$  to  $N(\beta M)$ , the collar neighborhood of the singular stratum. By assumption

$$g_{N(\beta M)} = dr^2 + r^2 g_{\partial M/\beta M} + \varphi_c^* g_{\beta M} + O(r).$$

Recall that an adapted wedge metric  $g$  is such that  $g_M$  and  $g_{N(\beta M)}$  are of product type in a collar neighborhood of  $\partial M$ . We make the hypothesis that not only does  $g_{\partial M/\beta M}$  have psc along the fibers but that the whole metric  $g_{\partial M}$  is of psc. As explained below, a sufficient condition for this additional property to hold is that the fibers are totally geodesic in  $\partial M$ . Now attach an infinite cylinder to  $M$  along the boundary  $\partial M$  and extend the metric to be constant on the cylinder, obtaining  $(M_\infty, g_\infty)$ , the Riemannian manifold with cylindrical ends associated to  $(M, g_M)$ ; similarly, attach an infinite cylinder to the boundary of  $N(\beta M)$  and extend the metric. Since the metrics near the boundary are of product type and since the metric on the boundary is of psc, we are in the situation where the spin Dirac operator on the manifolds with cylindrical ends is invertible at infinity. By Gromov-Lawson [19, p. 117] (the  $b$ -calculus of Melrose [25] can alternatively be used here), we then have a cylindrical  $\alpha$ -class  $\alpha_{\text{cyl}}(M, g_M)$  in  $KO_n$ <sup>3</sup>. This class only depends on  $M$  and on the metric  $g_{\partial M}$  induced by  $g_M$  on the boundary. See Remark 5.2 for more on this cylindrical class. Combining Gromov-Lawson and the analysis of Albin-Gell-Redman, we also have a mixed class  $\alpha_{\text{cyl},w}(N(\beta M), g_{N(\beta M)})$ , also an element in  $KO_n$ . This gives us the first part of the following Proposition:

**Proposition 2.6.** *Under the above additional assumption, namely that  $g|_{\partial M}$  is of psc, we have KO-classes*

$$\alpha_{\text{cyl}}(M, g_M), \quad \alpha_{\text{cyl},w}(N(\beta M), g_{N(\beta M)}) \quad \text{in} \quad KO_n$$

*and the following gluing formula holds:*

$$(7) \quad \alpha_{\text{cyl}}(M, g_M) + \alpha_{\text{cyl},w}(N(\beta M), g_{N(\beta M)}) = \alpha_w(M_\Sigma, g) \quad \text{in} \quad KO_n.$$

*Proof.* Only the gluing formula needs to be discussed. This follows from a small variation of a well known technique of Bunke, see [15]. □

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<sup>3</sup>We are making a small abuse of notation in that we write  $\alpha_{\text{cyl}}(M, g_M)$  and not  $\alpha_{\text{cyl}}(M_\infty, g_\infty)$ .

### 3. THE CASE WHEN THE LINK IS A HOMOGENEOUS SPACE

**3.1. Geometric setup: motivation.** Here we specify a geometric setup relevant to the existence problem of an adapted wedge psc-metric on a pseudomanifold  $M_\Sigma$  with fibered  $L$ -singularity. Even in our case when the pseudomanifold  $M_\Sigma$  is spin, we have seen that an adapted wedge metric has to be rather special even for an appropriate self-adjoint Dirac operator to exist. This means that pure topological conditions on  $M_\Sigma$  do not give an appropriate setup for existence of a psc-metric.

We will bypass this issue by fixing an appropriate geometrical structure on the pseudomanifolds we would like to study. This will lead to the notion of a *pseudomanifold with fibered  $(L, G)$ -singularity*, where  $G$  is a Lie group acting transitively on a manifold  $L$ .

**3.2. Metric near the singular stratum.** Now we discuss some details of the geometry of the tubular neighborhood  $N = N(\beta M)$  of the singular stratum  $\beta M$ . These will be needed for explaining how we arrived at our definition of a well-adapted metric, and will also be needed for the proof of Theorem 1.1. Since the interior of  $M$  will be irrelevant here, we work simply with a bundle  $\varphi_c: N \rightarrow B$  over an arbitrary base manifold  $B = \beta M$ , where the fibers of  $\varphi_c$  are cones on a fixed manifold  $L$ . Since we want the geometry of  $N$  to be well related to the geometry of  $L$  and of  $B$ , we assume that the bundle  $\varphi_c: N \rightarrow B$  is the associated bundle coming from a principal  $G$ -bundle  $p: P \rightarrow B$ , where  $G$  is a compact Lie group acting transitively on  $L$  by isometries. That means in particular that  $L$  can be identified with a homogeneous space  $G/K$ . The case where  $G$  is a torus has very different behavior than the case where  $G$  is semisimple, so we restrict attention to the latter case in this paper. (The case where  $G = S^1$  was studied in detail in [12].)

Thus in this paper we take  $G$  to be a compact connected semisimple Lie group. In Section 4 we will take  $G$  to be simply connected; this amounts to a kind of “spin” condition on the  $G$ -bundle  $p$  over  $B$  (since it is saying that the structure group of the link bundle lifts to the universal cover), but we won’t need this yet. We fix a bi-invariant metric on  $G$ , or equivalently, an Ad-invariant metric on the Lie algebra  $\mathfrak{g}$  of  $G$  (when  $G$  is simple, this is necessarily a multiple of the Killing form); in practice we will work with a constant multiple of this metric. Then the tangent bundle of  $L$  can be identified with  $G \times_K \mathfrak{p}$ , where  $\mathfrak{p}$  is the orthogonal complement of the Lie algebra  $\mathfrak{k}$  of  $K$  in  $\mathfrak{g}$ . (Note that  $K$  acts on  $\mathfrak{p}$  by the adjoint action.) The space  $\mathfrak{p}$  inherits an inner product from the inner product on  $\mathfrak{g}$ , and thus defines a Riemannian metric on  $L$  which will be fixed once and for all. This metric is  $G$ -invariant and has constant positive scalar curvature, and in fact nonnegative (but not identically zero) sectional curvature given by the formula [27, §5, p. 466]:

$$K(x \wedge y) = \frac{1}{4} \|[x, y]_{\mathfrak{p}}\|^2 + \|[x, y]_{\mathfrak{k}}\|^2,$$

for  $x, y$  orthonormal in  $\mathfrak{p}$ . (This is the only place in this paper where  $K$  denotes sectional curvature, not the isotropy group. The notations  $[x, y]_{\mathfrak{p}}$  and  $[x, y]_{\mathfrak{k}}$  refer to the orthogonal projections of the bracket of  $x$  and  $y$  into  $\mathfrak{p}$  and  $\mathfrak{k}$ , respectively.)

Now suppose we have a principal  $G$ -bundle  $p: P \rightarrow B$ , with base  $B$  a smooth compact manifold. We get an induced associated bundle  $\varphi: \partial N \rightarrow B$ , where  $\partial N = P \times_G (G/K) = P/K$ . The total space

$\partial N$  of this  $L$ -bundle will also be our manifold  $\partial M$ , for  $M$  the resolution of our (pseudo-)manifold with fibered  $(L, G)$ -singularities.

We want to consider a Riemannian structure on  $\partial N$  that is adapted to this fibration structure. We construct this as follows. Fix a connection  $\nabla^p$  on the principal  $G$ -bundle  $p: P \rightarrow B$ . This induces a connection  $\nabla^\varphi$  on the  $L$ -bundle  $\varphi: \partial N \rightarrow B$ . The tangent bundle of  $\partial N$  splits as the *vertical tangent bundle*, or *tangent bundle along the fibers*, which is

$$P \times_G TL = P \times_G (G \times_K \mathfrak{p}) = P \times_K \mathfrak{p},$$

direct sum with the pull-back  $\varphi^*(TB)$  of  $TB$ .

Now we specify a metric on the bundle  $\varphi_c: N \rightarrow B$ , where we replace  $L$  by the cone  $c(L)$  on  $L$ , where  $c(L) = ([0, R] \times L)/(\{0\} \times L)$  (the radius  $R$  of the cone (the distance to the vertex) will be determined later). We put a Riemannian metric  $g_B$  on the base manifold  $B$ . On the complement of the singular stratum (diffeomorphic to  $B$ ) in  $N$ , we put the metric which is  $(dr^2 + r^2 g_L) \oplus \varphi_c^* g_B$ , where  $g_L$  is the metric on  $L$  defined above transported to the vertical tangent bundle by making the vertical fibers totally geodesic. Here the coordinate  $r$  denotes the radial distance from the singular stratum,  $0 < r \leq R$ , and  $\varphi_c^* g_B$  is put on the horizontal fibers with respect to the connection  $\nabla^{\varphi_c}$ . We note that the vertical metric had been previously denoted by  $g_{\partial M/\beta M}$ .

Each vertical fiber of  $\varphi_c$  is a totally geodesic metric cone on  $L$ , with metric  $dr^2 + r^2 g_L$ . Away from the cone point where  $r = 0$ , this is a warped product metric on  $(0, R] \times L$ , and so by [16, Lemma 3.1], rederived (apparently independently) in [19, Proposition 7.3], we have:

**Lemma 3.1** ([16, Lemma 3.1] and [19, Proposition 7.3]). *The scalar curvature function  $\kappa$  on each vertical fiber  $c(L)$  of  $\varphi_c$  is  $(\kappa_L - \kappa_\ell)r^{-2}$ , where  $\kappa_\ell = \ell(\ell - 1)$ ,  $\ell = \dim L$ , is the scalar curvature of a standard round sphere  $S^\ell(1)$  of radius 1.*

Note that this is consistent with the fact that the cone on a standard round sphere  $S^{n-1}(1)$ , with metric  $dr^2 + r^2 g_{S^{n-1}(1)}$ , is just flat Euclidean  $n$ -space.

Now we normalize the bi-invariant Riemannian metric on  $G$  (i.e.,  $\text{Ad}(G)$ -invariant inner product on  $\mathfrak{g}$ ) so that the scalar curvature  $\kappa_L$  comes out to be exactly the constant  $\kappa_\ell$ . Because of Lemma 3.1, we obtain:

**Corollary 3.2.** *With the normalization  $\kappa_L = \kappa_\ell$ , the conical vertical fibers of the fibration  $\varphi_c$  have scalar curvature identically 0, i.e., are scalar-flat.*

This normalization is made to cancel out the contribution of the scalar curvature of the fibers to the scalar curvature of an adapted metric. If instead we had taken  $\kappa_L > \kappa_\ell$ , then we could always get an adapted metric of positive scalar curvature on  $N$ , and half of our problem would go away.

Note that we still have at our disposal one more normalization, namely the radius  $R$  of the cones. Since we are taking  $\partial N$  to correspond to  $r = R$ , that means the scalar curvature of each vertical fiber  $L$  of the bundle  $\varphi: \partial N \rightarrow B$  is  $R^{-2}\kappa_\ell$ .

At this point let's summarize the kind of Riemannian metrics we want to deal with. First, we fix a homogeneous space  $L = G/K$  with bi-invariant metric  $g_L$  with constant scalar curvature  $\kappa_L = \ell(\ell - 1)$ . Then we consider only pseudomanifolds  $M_\Sigma$  with  $L$ -fibered singularities whose resolution  $(M, \varphi: \partial M \rightarrow \beta M)$  has bundle  $\varphi$  of the very special form just discussed (i.e., it comes from a principal  $G$ -bundle  $p: P \rightarrow \beta M$ ). We will say that  $M_\Sigma$  has  $(L, G)$ -fibered singularities.

**Definition 3.3.** Assume  $M_\Sigma$  has  $(L, G)$ -fibered singularities. A *well-adapted* Riemannian wedge metric on  $M_\Sigma = M \cup_{\partial M} ([0, \varepsilon] \times \partial M) \cup_{\partial M} N$  is then given by a Riemannian metric  $g_M$  on  $M$ , a transition metric on the small collar  $[0, \varepsilon] \times \partial M$ , and a Riemannian metric  $g_{\beta M}$  on the base of the fibration  $\varphi: \partial M \rightarrow \beta M$ . In addition, we assume the principal  $G$ -bundle  $p: P \rightarrow \beta M$  has been equipped with a connection  $\nabla^p$ , which in turn induces a connection  $\nabla^\varphi$  on  $\varphi$ . On the tubular neighborhood  $N$  of  $\beta M$ , we put the metric  $(dr^2 + r^2 g_L) \oplus \varphi_c^* g_B$  (*orthogonal direct sum*), which is singular along the singular stratum  $\beta M$ .

Furthermore, we require the Riemannian metrics to match to second order along  $\partial M$ , and we require  $g_M$  to be a product metric in a small collar neighborhood of the boundary  $\partial M$ . In the transition region  $[0, \varepsilon] \times \partial M$ , we take the metric to have the form  $(dr^2 + f(r)^2 g_L) \oplus \varphi^* g_B$ , where the  $C^2$  function  $f$  is given by  $R + r$  for  $0 \leq r \leq \frac{\varepsilon}{4}$  and by the constant  $R + \frac{\varepsilon}{2}$  for  $R + \frac{3\varepsilon}{4} \leq r \leq R + \varepsilon$ . In this way we get a  $C^2$  interpolation<sup>4</sup> between the metric on  $N$  and the metric on  $M$ , without affecting positivity of the scalar curvature.

When the Riemannian metrics on  $N$  and  $\partial N$  are of the special form given in Definition 3.3, then the bundles  $\varphi_c: (N \setminus \beta M) \rightarrow \beta M$  and  $\varphi: \partial M \rightarrow \beta M$  are, in fact, Riemannian submersions with fibers  $\mathring{c}(L)$  (the open cone on  $L$ , i.e., the cone with the vertex removed) and  $L$ , respectively.

These submersions have totally geodesic fibers, and translation along the fibers preserves the horizontal spaces since the splitting of horizontal and vertical spaces comes from a connection on the principal bundle  $P$ , so the O'Neill  $T$ -tensor (see [27, p. 460]) vanishes for both of them. We will need the following:

**Proposition 3.4** (O'Neill). *Given a Riemannian submersion  $\varphi: X \rightarrow B$  of Riemannian manifolds, with totally geodesic fibers all isometric to  $F$ , where a Lie group acts transitively on  $F$  and preserves horizontal spaces, the scalar curvatures  $\kappa_X$ ,  $\kappa_F$ , and  $\kappa_B$  are related as follows:*

$$\kappa_X = \kappa_F + \kappa_B + \sum_{j,m} \|A_{x_j} v_m\|^2 - 3 \sum_{j \neq k} \|A_{x_j} x_k\|^2.$$

Here  $\{x_j\}_{j \leq \dim B}$  is an orthonormal frame for the horizontal tangent space and  $\{v_m\}_{m \leq \dim F}$  is an orthonormal frame for the vertical tangent space.

*Proof.* Since the  $T$ -tensor vanishes identically under the conditions of the proposition, this follows immediately from [27, Corollary 1, p. 465] after summing over all pairs of distinct elements of the orthonormal frame  $\{x_j, v_m\}_{j \leq \dim B, m \leq \dim F}$ .  $\square$

<sup>4</sup>The interpolation could be done in a different way without making any essential difference, but it's convenient to make a choice once and for all.

As a consequence of Proposition 3.4, we obtain the following theorem, which is an important part of Theorems 1.1 and 1.2.

**Theorem 3.5.** *Let  $L = G/K$  be a homogeneous space of a compact connected semisimple Lie group  $G$ , equipped with a  $G$ -invariant metric of scalar curvature  $\kappa_L = \kappa_\ell$  as above, and let  $M_\Sigma$  be a compact manifold with  $(L, G)$ -fibered singularities, with resolution  $(M, \varphi: \partial M \rightarrow \beta M)$ , where the  $L$ -bundle comes from a principal  $G$ -bundle  $p: P \rightarrow \beta M$ . Then the following hold.*

- (1)  *$\partial M$  always has a well-adapted Riemannian metric of positive scalar curvature.*
- (2) *If  $\beta M$  has a metric of positive scalar curvature, then its tubular neighborhood  $N$  has a well-adapted metric of positive scalar curvature.*
- (3) *If the tubular neighborhood  $N$  of  $\beta M$  has a well-adapted metric of positive scalar curvature, then  $\beta M$  has a metric of positive scalar curvature.*

*Proof.* We start with an observation about well-adapted metrics, which is that for  $x$  a horizontal vector and  $v$  a vertical vector,  $A_x(v) = 0$ . The reason is that if we choose a geodesic  $\gamma$  with  $\dot{\gamma}(0) = x$ , then by the construction of adapted metrics,  $\dot{\gamma}(t)$  stays horizontal and the parallel transport of  $v$  along  $\gamma$  remains vertical. Thus the horizontal component of  $\nabla_{\dot{\gamma}}(v)$  vanishes, and  $A_x(v) = 0$ . This removes one error term from the formula of Proposition 3.4.

(1). If  $L = G$ , so that  $\partial M$  has a free action of  $G$ , then this is an easy special case of the main result of [22]. In general, this part is codified as the Observation in [35, p. 512]. We can deduce it from Proposition 3.4 by choosing  $R$  very small (recall that in effect we are taking the fibers of  $\varphi$  to be copies of  $L$  with the diameter multiplied by a factor of  $R$ ), and thus with scalar curvature  $R^{-2}\kappa_\ell$ . This then swamps all the other terms on the right in Proposition 3.4.

(2). Let's look at the formula of Proposition 3.4 for the scalar curvature of  $N$ . By our assumption on the metric on  $L$ , the scalar curvature of the  $\hat{c}(L)$  fibers vanishes. Let's rescale the metric on  $\beta M$  by multiplying lengths of vectors by  $t$ . Note that  $\|A_{x_j}x_k\|$  is computed in the metric of the vertical fibers, which we are keeping fixed. However, when we rescale the metric from  $g_{\beta M}$  to  $t^2g_{\beta M}$ , our orthonormal frame changes from  $\{x_j, v_m\}_{j,m}$  to  $\{t^{-1}x_j, v_m\}_{j,m}$ . Thus with the rescaled metric, the scalar curvature of  $N$  becomes

$$t^{-2}\kappa_B - 3 \sum_{j \neq k} t^{-4} \|A_{x_j}x_k\|^2.$$

Now note that if the error term on the right were zero, we would have  $\kappa_B > 0$  and if and only if  $\kappa_N > 0$ . This is not quite true for the original metric if the error term is nonzero, but if we let  $t \rightarrow \infty$ , the negative term (involving  $\|A_{x_j}x_k\|^2$ ) goes to zero fastest. So if  $\kappa_B > 0$ , eventually the rescaled value of  $\kappa_N$  becomes positive. This proves (2).

(3). Suppose  $\kappa_N$  is strictly bigger than 0. In our situation, we have

$$\kappa_N = \kappa_B - (\text{something nonnegative}),$$

so  $\kappa_B > 0$ . □

**Example 3.6.** In this example, we take  $L = G = \mathrm{SU}(2) = S^3$ . Let  $\beta M$  be a K3 surface, a spin simply connected 4-manifold with Betti numbers 1, 0, 22, 0, 1 and  $\widehat{A}(K3) = 2$ . In particular,  $\beta M$  does not admit a metric of positive scalar curvature. However, there is a one-parameter family of principal  $S^3$ -bundles  $P$  over  $\beta M$ , classified by the second Chern class  $c_2$ . (Indeed, since  $[S^4, \mathbb{H}\mathbb{P}^\infty] \cong \pi_3(S^3) \cong \mathbb{Z}$ , we have such a family of bundles over  $S^4$ , and we just pull them back via a map  $\beta M \rightarrow S^4$  of degree 1.) The total space  $\partial M$  of each such bundle is a simply connected spin 7-manifold, and since  $\Omega_7^{\mathrm{spin}} = 0$ , it is a spin boundary, say the boundary of an 8-manifold  $M$ .

The manifold  $M_\Sigma$  obtained by gluing the disk bundle  $D\mathcal{L}$  of the quaternionic line bundle  $\mathcal{L}$  associated to the principal bundle  $P$  to  $M$ , for any choice of  $c_2$ , is a simply connected spin 8-manifold. We can take it to be the double of  $D\mathcal{L}$  along  $\beta M$ , which will have  $\widehat{A}$ -genus 0, and then it admits a metric of positive scalar curvature. However, it does not admit a *well-adapted* metric of positive scalar curvature, since Theorem 3.5(3) would imply then that  $\beta M$  admits a metric of positive scalar curvature, which it does not.

**Example 3.7.** We again take  $L = G = S^3$ . Here is another nontrivial example of a spin manifold with  $S^3$ -fibered singularities. Note that the set of quaternionic line bundles  $\mathcal{L}$  on  $S^n$  is parametrized by  $[S^n, B(\mathrm{Sp}(1))] = \pi_{n-1}(S^3)$ , which is finite for all  $n \geq 5$  and usually nonzero. Take for example  $n = 6$ . Then  $[S^6, B(\mathrm{Sp}(1))] = \pi_5(S^3) \cong \mathbb{Z}/2$ , with the generator represented by  $\eta^2$  ( $\eta$  the generator of the stable 1-stem, as usual). So there are two distinct quaternionic line bundles  $\mathcal{L}$  over  $S^6$ , both of which are rationally trivial (since  $c_2$  must vanish). For each of them,  $\partial(D\mathcal{L})$  is a principal  $S^3 = \mathrm{Sp}(1)$ -bundle over  $S^6$ , rationally equivalent to  $S^3 \times S^6$ . (When  $\mathcal{L}$  is trivial,  $\partial(D\mathcal{L}) = S^3 \times S^6$ , and when  $\mathcal{L}$  is nontrivial,  $\partial(D\mathcal{L}) = (S^3 \vee S^6) \cup e^9$ , with the attaching map of the 9-cell given by  $S^8 \xrightarrow{\eta^2} S^6 \hookrightarrow S^3 \vee S^6$ .) In both cases, this manifold  $\partial M$ , since it carries a free  $S^3$ -action, admits positive scalar curvature [22] and thus has trivial  $\alpha$ -invariant in  $ko_9 \cong \mathbb{Z}/2$ . It is a spin boundary, since we can fill in each  $S^3$  fiber over  $S^6$  with a 4-disk, and we can get a simply connected spin manifold  $M^{10}$  with boundary  $\partial M$ . After gluing in  $D\mathcal{L}$ , we have a closed simply connected singular spin 10-manifold  $M_\Sigma$ , which we can take to be the double of  $D\mathcal{L}$  along its boundary. This admits positive scalar curvature, even if we require our metric to be well adapted in the sense of Definition 3.3, by Theorem 3.5.

Part of the reason why this example is interesting is that if the bundle  $p$  is nontrivial, then the classifying map  $S^6 \rightarrow \mathbb{H}\mathbb{P}^\infty$  represents a nontrivial class in  $\widetilde{ko}_6(\mathbb{H}\mathbb{P}^\infty) \cong ko_2 \cong \mathbb{Z}/2$  (see the following Theorem 3.8), since the generator of  $ko_2$  is represented by  $\eta^2$ .

We will need as a technical tool the following theorem pointed out to us by Bob Bruner:

**Theorem 3.8** (Bruner–Greenlees).  *$ko_*(\mathbb{H}\mathbb{P}^\infty)$  is a free  $ko_*$ -module on generators  $z_j$  in dimensions  $4j$ ,  $j \in \mathbb{N}$ . The action of  $ko^*(\mathbb{H}\mathbb{P}^\infty) \cong ko^*[[z]]$ ,  $\dim z = 4$ , is by  $z \cdot z_j = z_{j-1}$ .*

*Proof.* There are a few possible proofs. One is to show by induction on  $n$  that the Atiyah–Hirzebruch spectral sequence (AHSS)

$$H_p(\mathbb{H}\mathbb{P}^n, ko_q) \implies ko_{p+q}(\mathbb{H}\mathbb{P}^n)$$

collapses at  $E_2$ , which is proved in [20, Lemmas 2.4 and 2.5], or alternatively, that the boundary map in the exact cofiber sequence

$$\cdots \rightarrow ko_j(\mathbb{H}\mathbb{P}^{n-1}) \rightarrow ko_j(\mathbb{H}\mathbb{P}^n) \rightarrow \widetilde{ko}_j(S^{4n}) \rightarrow \cdots$$

vanishes, which is proved in [26, Corollary 3.1].

The proof suggested to us by Bruner is a bit slicker; it is shown in [14, p. 86, Theorem 5.3.1] that  $ko^*(\mathbb{H}\mathbb{P}^\infty) = ko^*[[z]]$  for  $z$  in dimension 4. The additive result follows by the universal coefficient theorem and the  $ko^*[[z]]$ -module structure follows from the local cohomology spectral sequence [17].  $\square$

#### 4. RELEVANT BORDISM THEORY

**4.1. Bordism of pseudomanifolds.** It is known that a meaningful bordism theory in the framework of stratified pseudomanifolds requires some restrictions on their stratification and the equivalence relation, see, say, [8]. On the other hand, under some natural restrictions, such bordism groups could be highly interesting. For instance, the Witt-bordism groups are such, and they emerged naturally in the contexts related to the signature and to the signature operator [34]. In our geometrical context, we will consider the following two bordism groups:  $\Omega_*^{\text{spin}, L\text{-fb}}$  and  $\Omega_*^{\text{spin}, (L, G)\text{-fb}}$ .

We start with  $\Omega_*^{\text{spin}, L\text{-fb}}$ . Let  $\Sigma = (L)$ , and let  $M_\Sigma$  and  $M'_\Sigma$  be two pseudomanifolds with fibered  $L$ -singularities. Then a bordism  $W_\Sigma: M_\Sigma \rightsquigarrow M'_\Sigma$  is a pseudomanifold  $W_\Sigma$  with *boundary* and with fibered  $L$ -singularities. This means that  $W_\Sigma = W \cup -N(\beta W)$ , where the resolving manifold  $W$  is a spin manifold with corners and the boundary  $\partial W$  of the resolution is given a splitting  $\partial W = \partial^{(0)}W \cup \partial^{(1)}W$ , where

$$\partial^{(0)}W = M \sqcup -M', \quad \partial(\partial^{(1)}W) = \partial M \sqcup -\partial M', \quad \partial(\beta W) = \beta M \sqcup -\beta M',$$

i.e.,  $\partial^{(1)}W: \partial M \rightsquigarrow \partial M'$  and  $\beta W: \beta M \rightsquigarrow \beta M'$  are usual bordisms between closed spin manifolds, and  $\partial^{(0)}W \cap \partial^{(1)}W = \partial M \sqcup -\partial M'$ . Furthermore, it is assumed that the fiber bundle  $F: \partial^{(1)}W \rightarrow \beta W$  restricts to the fiber bundles  $f: \partial M \rightarrow \beta M$  and  $f': \partial M' \rightarrow \beta M'$  respectively. This gives a well-defined bordism group  $\Omega_*^{\text{spin}, L\text{-fb}}$  and, in fact, a bordism theory  $\Omega_*^{\text{spin}, L\text{-fb}}(-)$ .

**Remark 4.1.** It is worth noticing that, in the above setting, the bordism group  $\Omega_*^{\text{spin}, L\text{-fb}}$  is rather complicated. Indeed, according to our definition, an  $L$ -fibration  $f: \partial M \rightarrow \beta M$  is just a smooth fiber bundle, and thus such a fibration is classified by a map  $\beta M \rightarrow \text{BDiff}(L)$ . Then the correspondence  $M_\Sigma \mapsto (\beta M \rightarrow \text{BDiff}(L))$  defines a Bockstein homomorphism  $\beta: \Omega_*^{\text{spin}, L\text{-fb}} \rightarrow \Omega_{*-\ell-1}^{\text{spin}}(\text{BDiff}(L))$ . There is also a transfer homomorphism  $\tau: \Omega_*^{\text{spin}}(\text{BDiff}(L)) \rightarrow \Omega_{*+\ell}^{\text{spin}}$  which takes a bordism class  $B \rightarrow \text{BDiff}(L)$  to the corresponding smooth  $L$ -fiber bundle  $E \rightarrow B$ . All together, these fit into an exact triangle of homology theories

$$(8) \quad \begin{array}{ccc} \Omega_*^{\text{spin}}(-) & \xrightarrow{i} & \Omega_*^{\text{spin}, L\text{-fb}}(-) \\ & \swarrow \tau & \searrow \beta \\ & \Omega_*^{\text{spin}}(\text{BDiff}(L) \wedge -) & \end{array}$$

where  $i$  is a transformation considering each spin manifold as a pseudomanifold with empty singularities. Thus the complexity of the bordism group  $\Omega_*^{\text{spin}, L\text{-fb}}$  is determined by the classifying space  $\text{BDiff}(L)$ , which is known to be very complicated for almost every smooth manifold  $L$ .

**4.2. Spin manifolds with fibered  $(L, G)$ -singularities.** We recall our definition of a closed manifold with fibered  $(L, G)$ -singularities. For this we need to fix the following topological data:

- (i) a closed spin smooth manifold  $L$  (a link, as above),  $\dim L = \ell$ ;
- (ii) a Lie group  $G$  mapping to  $\text{Diff}(L)$  with a lifting of  $G$  to an action on the principal spin bundle. (This is necessary for some things we will do later involving the Dirac operator.)

These data are good enough to construct a relevant bordism theory; however, we would also like to fix some geometrical data, namely a Riemannian metric  $g_L$  on the link  $L$  such that

- (iii) the scalar curvature  $\kappa_L$  of the metric  $g_L$  is the positive constant  $\kappa_\ell$  and the Lie group  $G$  acts transitively by a subgroup of the isometry group  $\text{Isom}(L, g_L) \subset \text{Diff}(L)$ .

We denote by  $BG$  the corresponding classifying space and by  $EG \rightarrow BG$  the universal principal bundle. We say that  $\pi: E \rightarrow B$  is an  $(L, G)$ -fiber bundle if it is a smooth fiber bundle with fiber  $L$  and structure group  $G$ . There is a universal  $(L, G)$ -fiber bundle  $E(L) \rightarrow BG$ , where  $E(L) := EG \times_G L$ . Then for any  $(L, G)$ -fiber bundle  $\pi: E \rightarrow B$ , there is a classifying map  $f: B \rightarrow BG$  such that  $E = f^*E(L)$ .

As above, we denote by  $c(L)$  the cone over  $L$  and let  $G$  act on  $c(L)$  slice-wise. We obtain an associated  $(c(L), G)$ -fiber bundle  $\pi_c: N(B) \rightarrow B$ , where  $N(B) = f^*(EG \times_G c(L))$ . Note that if  $B$  is a closed manifold, then  $N(B)$  is a singular manifold with boundary  $\partial N(B) = E$  and singular set  $B$ . We obtain a commutative diagram of fiber bundles:

$$(9) \quad \begin{array}{ccc} E = \partial N(B) & \xrightarrow{i} & N(B) \\ \pi \downarrow & & \downarrow \pi_c \\ B & \xrightarrow{Id} & B \end{array}$$

where the fiber  $L$  is identified with the boundary of the cone  $c(L)$ .

Let  $M_\Sigma = M \cup_{\partial M} N(\beta M)$ , where  $M$  is a spin manifold with boundary  $\partial M$ , where  $\partial M$  is the total space of an  $(L, G)$ -fiber bundle  $\varphi: \partial M \rightarrow \beta M$  given by its structure map  $f: \beta M \rightarrow BG$ .  $M_\Sigma$  is a *closed*  $(L, G)$ -singular spin pseudomanifold.

**Definition 4.2.** We say that a closed  $(L, G)$ -singular spin pseudomanifold  $M_\Sigma = M \cup_{\partial M} N(\beta M)$  as above is an  $(L, G)$ -boundary if there is a spin manifold  $\bar{M}$  (with corners) such that

- (1)  $\partial \bar{M} = \partial^{(0)} \bar{M} \cup \partial^{(1)} \bar{M}$ , where  $\partial^{(1)} \bar{M} = M$  (i.e., we have chosen a diffeomorphism  $\partial^{(1)} \bar{M} \cong M$ );
- (2)  $\partial^{(0)} \bar{M}$  is a total space of an  $(L, G)$ -fiber bundle  $\bar{\varphi}: \partial^{(0)} \bar{M} \rightarrow \beta \bar{M}$  given by a structure map  $\bar{f}: \beta \bar{M} \rightarrow BG$  with the restriction  $\partial(\beta \bar{M}) = \beta M$  and  $\bar{f}|_{\beta \bar{M}} = f$ ;
- (3)  $\partial(\partial^{(0)} \bar{M}) = \partial M$  and the restriction  $\bar{\varphi}|_{\partial(\partial^{(0)} \bar{M})} = \phi$ .



Then the space  $\bar{M}_\Sigma = \bar{M} \cup_{\partial \bar{M}} N(\beta \bar{M})$  is a *spin manifold with fibered  $(L, G)$ -singularities with boundary  $\delta \bar{M}_\Sigma = M_\Sigma$* . For short, we say that  $\bar{M}_\Sigma$  is an  *$(L, G)$ -singular spin pseudomanifold with boundary  $\delta \bar{M}_\Sigma = M_\Sigma$* .

Now we say that two  $(L, G)$ -singular spin pseudomanifolds  $M_\Sigma, M'_\Sigma$  are bordant if there exists an  $(L, G)$ -singular spin pseudomanifold  $W_\Sigma$  with boundary  $\delta W_\Sigma = M_\Sigma \sqcup -M'_\Sigma$ . Given a bordism as above, we use the notation  $W_\Sigma: M_\Sigma \rightsquigarrow M'_\Sigma$ . This determines corresponding bordism groups  $\Omega_n^{\text{spin}, (L, G)\text{-fb}}$  and a bordism theory  $\Omega_*^{\text{spin}, (L, G)\text{-fb}}(-)$ . We now give a few more details in this direction.

**Definition 4.3.** Let  $M_\Sigma = M \cup_{\partial M} N(\beta M)$  and let  $f: \beta M \rightarrow BG$  be the corresponding structure map, so that we have a commutative diagram

$$(10) \quad \begin{array}{ccc} \partial M & \xrightarrow{\hat{f}} & E(L) \\ \phi \downarrow & & \phi_0 \downarrow \\ \beta M & \xrightarrow{f} & BG \end{array}$$

A map  $\xi: M \rightarrow X$  determines an element in  $\Omega_*^{\text{spin}, (L, G)\text{-fb}}(X)$  if the restriction  $\xi|_{\partial M}$  coincides with the composition  $pr \circ (\hat{f} \times \xi_\beta)$ , where  $\xi_\beta: \beta M \rightarrow X$  is some map and the map  $(\hat{f} \times \xi_\beta)$  is given by the following commutative diagram:

$$(11) \quad \begin{array}{ccccc} \partial M & \xrightarrow{\hat{f} \times \xi_\beta} & E(L) \times X & \xrightarrow{pr} & X \\ \phi \downarrow & & \downarrow \phi_0 \times Id & & \\ \beta M & \xrightarrow{f \times \xi_\beta} & BG \times X & & \end{array}$$

Here  $pr$  is a projection on the second factor. We note that, by definition, there is a canonical extension of such a map  $\xi: M \rightarrow X$  to a map  $\xi_\Sigma: M_\Sigma \rightarrow X$ .

Let  $i: \Omega_*^{\text{spin}}(-) \rightarrow \Omega_*^{\text{spin}, (L, G)\text{-fb}}(-)$  be the natural transformation given by considering a closed manifold as a pseudomanifold with empty  $(L, G)$ -singularity. Then the natural transformation  $\beta: \Omega_*^{\text{spin}, (L, G)\text{-fb}}(-) \rightarrow \Omega_{*-\ell-1}^{\text{spin}}(BG \wedge -)$  is given by the correspondence  $M_\Sigma \mapsto (f: \beta M \rightarrow BG)$ . Finally, the natural transformation  $T: \Omega_*^{\text{spin}}(BG \wedge -) \rightarrow \Omega_{*+\ell}^{\text{spin}}(-)$  is a transfer map which takes a map  $f: B \rightarrow BG$  to the induced  $(L, G)$ -bundle  $E \rightarrow B$ .

**Proposition 4.4.** *There is an exact triangle of bordism theories:*

$$(12) \quad \begin{array}{ccc} \Omega_*^{\text{spin}}(-) & \xrightarrow{i} & \Omega_*^{\text{spin}, (L, G)\text{-fb}}(-) \\ & \swarrow T & \searrow \beta \\ & \Omega_*^{\text{spin}}(BG \wedge -) & \end{array} .$$

*Proof.* As usual, that the composites  $i \circ T$ ,  $\beta \circ i$ , and  $T \circ \beta$  vanish is clear. To prove exactness, the coefficients  $(-)$  “come along for the ride,” so we just give the proof without them for simplicity of notation. If  $T([B \rightarrow BG]) = 0$ , that means that the total space  $E$  of the induced  $(L, G)$ -bundle  $E \rightarrow B$  is a spin boundary, so we can write  $E = \partial M$  for some spin manifold  $M$ , which means that

$[B = \beta M \rightarrow BG]$  is in the image of  $\beta$ . If  $\beta([M_\Sigma]) = 0$ , that means we have  $(M, \partial M)$  with an  $(L, G)$ -bundle  $\partial M \rightarrow \beta M$  so that the underlying principal  $G$ -bundle  $G \rightarrow P \rightarrow \beta M$  bounds in  $\Omega_*^{\text{spin}}(BG)$ . Suppose we have a spin manifold  $N$  with boundary, mapping to  $BG$ , so that  $\partial N = \beta M$  and the principal  $G$ -bundle  $P' \rightarrow N$  extends  $P \rightarrow \beta M$ . Then  $P' \times_G L = M'$  bounds  $\partial M$ , and  $M_\Sigma$  is bordant as a spin manifold with  $(L, G)$ -singularities to the closed manifold  $M \cup_{\partial M} -M'$ , with  $M' \rightarrow N$  providing the bordism, and hence  $[M_\Sigma]$  hence comes from the image of  $i$ . Finally, suppose we have a closed spin manifold  $M$  with  $i([M]) = 0$ . That means there is a spin  $(L, G)$ -pseudomanifold  $W_\Sigma$  with  $\delta W_\Sigma = M$ . Resolving, we have a spin manifold  $W$  (with boundary<sup>5</sup>) with  $\partial W = \partial^{(0)}W \sqcup \partial^{(1)}W$ , where  $\partial^{(0)}W$  is closed (since  $\partial M = \emptyset$ ),  $\partial^{(1)}W = M$ , and  $\partial^{(0)}W$  is the total space of an  $(L, G)$ -bundle  $\partial^{(0)}W \rightarrow \beta W$ , with  $\beta W$  also closed since  $\beta M = \emptyset$ . This shows  $M$  is spin bordant to  $-\partial^{(0)}W$ , and thus  $[M] = T(-[\beta W \rightarrow BG])$ .  $\square$

**4.3. Bordism theorem.** One of our goals is to push a well-adapted metric of positive scalar curvature through an  $(L, G)$ -singular bordism. Here is the key result:

**Theorem 4.5** (Bordism Theorem). *Assume  $G$  is a simply connected Lie group and  $L$  is spin. Let  $M_\Sigma, M'_\Sigma$  be two  $(L, G)$ -singular spin pseudomanifolds of dimension  $n \geq 6 + \ell$  representing the same class  $x \in \Omega_n^{\text{spin}, (L, G)\text{-fb}}$ , with  $M$  and  $\beta M$  simply connected. Assume  $M'_\Sigma$  has a well-adapted psc-metric  $g'$ . Then there exists an  $(L, G)$ -bordism  $W_\Sigma: M_\Sigma \rightsquigarrow M'_\Sigma$  together with a well-adapted psc-metric  $\bar{g}$  which is a product metric near the boundary  $\delta W_\Sigma = M_\Sigma \sqcup -M'_\Sigma$  such that  $\bar{g}|_{M'_\Sigma} = g'$ . In particular,  $M_\Sigma$  admits a well-adapted psc-metric  $g$ .*

This result will follow from a purely topological result which is just surgery-theoretic:

**Theorem 4.6** (Surgery Theorem). *Assume  $G$  is a simply connected Lie group and  $L$  is spin. Let  $M_\Sigma, M'_\Sigma$  be two  $(L, G)$ -singular spin pseudomanifolds of dimension  $n \geq 6 + \ell$  representing the same class  $x \in \Omega_n^{\text{spin}, (L, G)\text{-fb}}$ . Assume that  $M_\Sigma = M \cup_{\partial M} N(\beta M)$ ,  $M'_\Sigma = M' \cup_{\partial M'} N(\beta M')$  with corresponding structure maps  $f: \beta M \rightarrow BG$  and  $f': \beta M' \rightarrow BG$ . Also assume that  $M$  and  $\beta M$  are spin and simply connected.<sup>6</sup> Then there exists an  $(L, G)$ -bordism  $W_\Sigma: M_\Sigma \rightsquigarrow M'_\Sigma$ ,  $W_\Sigma = W \cup_{\partial W} N(\beta W)$ , with a structure map  $\bar{f}: \beta W \rightarrow BG$ , such that  $(\beta W, \beta M)$  and  $(W, M)$  are 2-connected.*

*Proof of Theorem 4.6.* Start with any  $(L, G)$ -bordism  $W_\Sigma: M_\Sigma \rightsquigarrow M'_\Sigma$ . Recall that this comes with a map  $F: \beta W \rightarrow BG$  restricting to the given bundle data  $f: \beta M \rightarrow BG$  and  $f': \beta M' \rightarrow BG$ . First we modify  $W$  by surgery to reduce to the case where  $(\beta W, \beta M)$  is 2-connected. Recall that we are assuming  $\beta M$  and  $M$  are simply connected, though we make no such assumption on  $\beta M'$  and  $M'$ . We begin by killing  $\pi_1(\beta W)$  through surgery. Recall that we are assuming that  $\dim(\beta M) \geq 5$ , so  $\dim(\beta W) \geq 6$ . Given any class in  $\pi_1(\beta W)$ , we can represent it by an embedded circle, which will have trivial normal bundle. Since  $G$  is simply connected,  $BG$  will actually be 3-connected (any Lie group has vanishing  $\pi_2$ , and we are assuming  $\pi_1(G) = 0$ ). So we can do surgery on this circle so that

<sup>5</sup>Usually there would be corners, but here the corner  $\partial^{(0)}W \cap \partial^{(1)}W$  is empty.

<sup>6</sup>This implies that  $M_\Sigma$  is 1-connected, since  $\beta M$  and  $L$  1-connected imply that  $\partial M$  is 1-connected, and by Van Kampen's Theorem,  $\pi_1(M_\Sigma) = \pi_1(M) *_{\pi_1(\partial M)} \pi_1(\beta M)$ .

$F$  extends over the trace of the surgery, which will be a manifold  $V$  with boundary which we can attach to  $\beta W$ . Thus we can suppose that  $\beta W$  is simply connected. Next, look at the exact sequence  $\pi_2(\beta W) \rightarrow \pi_2(\beta W, \beta M) \rightarrow \pi_1(\beta M) = 0$ . If  $\pi_2(\beta W, \beta M) \neq 0$ , we can represent any generator of this group by an embedded  $S^2$  in  $\beta W$ . Since everything is spin, this 2-sphere has trivial normal bundle. Again, the map  $F$  from this 2-sphere to  $BG$  is null-homotopic since  $BG$  is 2-connected (even 3-connected). So again we can do surgery so that  $F$  extends over the trace of the surgery. After attaching the traces of all surgeries needed to  $\beta W$ , we have reduced to the case where  $(\beta W, \beta M)$  is 2-connected.

The next step is to do something similar on the interior of  $W$  to make  $(W, M)$  2-connected. The argument is exactly the same. Note by the way that whatever changes we make in  $W$  can automatically be lifted up to changes in  $W_\Sigma$  by lifting surgeries on  $\beta W$  to modifications of  $N(\beta W)$ .  $\square$

*Proof of Theorem 4.5 from Theorem 4.6.* Apply Theorem 4.6 and assume we have a bordism  $W$  with  $(W, M)$ ,  $(\beta W, \beta M)$  2-connected. That means that we can decompose the bordism into a sequence of surgeries on  $(M', \partial M' \rightarrow \beta M')$  (compatible with the map to  $BG$  on the Bockstein) which are always in codimension 3 or more. Then we apply the Gromov-Lawson surgery theorem [18], first on the Bocksteins, to push a psc metric on  $\beta M'$  to one on  $\beta M$ . Since the bordism is compatible with the maps to  $BG$ , we get a well-adapted metric of positive scalar curvature on the tubular neighborhood of  $\beta W$ . The next step is to push the psc metric on the interior of  $M'$  to one on the interior of  $M$ . For this we use the Gromov-Lawson surgery theorem again, possible since  $(W, M)$  is 2-connected.  $\square$

## 5. $KO$ -OBSTRUCTIONS ON $(L, G)$ -FIBERED PSEUDOMANIFOLDS

Let  $M_\Sigma$  be a pseudomanifold with  $(L, G)$ -fibered singularities. Let  $f: \beta M \rightarrow BG$  be the associated classifying map and let  $g$  be a well-adapted wedge metric on  $M_\Sigma$ . Recall that by definition, the restriction  $g_{\partial M} = g|_{\partial M}$  is consistent with the natural vertical metric  $g_{\partial M/\beta M}$  on  $\partial M$  induced by the psc-metric  $g_L$  on the link  $L$ . See Definition 3.3 for details.

Recall from Theorem 3.5 that  $g_{\partial M}$  is a psc-metric: indeed we can rescale the fiber metric  $g_L$  to achieve that. Moreover, if  $g_{\beta M}$  is a psc-metric on  $\beta M$ , then, up to rescaling,  $g_{N(\beta M)}$  is also a psc-metric on  $N(\beta M)$ . Vice versa, if  $g_{N(\beta M)}$  is a psc-metric, then  $g_{\beta M}$  is also a psc-metric.

Denote, as above,  $g_M = g|_M$ . We know that in this setting the  $KO$ -classes  $\alpha_{\text{cyl}}(M, g_M)$  and  $\alpha_w(M_\Sigma, g)$  are well defined. Moreover, Proposition 2.6 implies that these classes coincide for any pseudomanifold  $M_\Sigma$  with  $(L, G)$ -fibered singularities, provided  $g_{N(\beta M)}$  is a psc-metric.

Assume now that  $g$  on  $M_\Sigma$  is a psc-metric. Then, obviously, the metric  $g_{N(\beta M)}$  is also psc and so is  $g_{\beta(M)}$ . This implies that

$$(13) \quad \alpha(\beta(M), g_{\beta M}) = 0 \quad \text{in} \quad KO_{n-\ell-1}.$$

and

$$(14) \quad \alpha_{\text{cyl}}(M, g_M) = \alpha_w(M_\Sigma, g) = 0 \quad \text{in} \quad KO_n.$$

with the first equality in (14) following from Proposition 2.6, as we have already remarked, and the second one from the classic results of Gromov-Lawson or from Theorem 2.3, item (4).

Formulae (13) and (14) prove the obstruction theorem (Theorem 1.1 stated in the Introduction).

Note, however, that the class  $f_*[\mathfrak{P}_{\beta M}] \in KO_{n-\ell-1}(BG)$  is *not* an obstruction. Example 3.7 is a counterexample.

These obstructions are in fact obtained from suitable group homomorphisms, as we shall now explain.

**Proposition 5.1.** *Let  $\Omega_*^{\text{spin},(L,G)\text{-fb}}$  be the bordism group as above. Then we have well defined homomorphisms:*

$$(15) \quad \alpha_{\text{cyl}}: \Omega_*^{\text{spin},(L,G)\text{-fb}} \rightarrow KO_* \quad \text{and}$$

$$(16) \quad \alpha_{\beta M}: \Omega_*^{\text{spin},(L,G)\text{-fb}} \rightarrow KO_{*-\ell-1}.$$

*Proof.* First of all, we have to define  $\alpha_{\text{cyl}}[M_\Sigma]$ , with  $[M_\Sigma] \in \Omega_*^{\text{spin},(L,G)\text{-fb}}$ . We recall that  $M_\Sigma = M \cup_{\partial M} N(\beta M)$  and set

$$\alpha_{\text{cyl}}[M_\Sigma] := \alpha_{\text{cyl}}(M, g_M)$$

with  $g$  a well-adapted wedge metric on the regular part of  $M_\Sigma$  and  $g_M := g|_M$ , as usual. Here, because of the very definition of well-adapted wedge metric on a manifold with  $(L, G)$ -fibered singularities, the homomorphism is well defined, independent of the choice of  $g$ . Indeed, we know that if  $g$  and  $g'$  are wedge metrics, then  $g_{\partial M}$  and  $g'_{\partial M}$  are of psc and with the same vertical metric, the one induced by the natural metric on  $L$ . Consider an arbitrary path of wedge metrics joining  $g$  and  $g'$ , call it  $\{g(t)\}_{t \in [0,1]}$ . Remark that the family  $\{g(t)\}_{t \in [0,1]}$  restricts to a family of submersion metrics  $\{g(t)|_{\partial M}\}_{t \in [0,1]}$  on  $\partial M$  and the latter fixes a Riemannian metric  $g_{\partial M \times [0,1]}$  on  $\partial M \times [0,1]$  that we can assume to be of product-type near the boundary. Then, always from Bunke [15], we have:

$$\alpha_{\text{cyl}}(M, g_M) - \alpha_{\text{cyl}}(M, g'_M) = \alpha_{\text{cyl}}(\partial M \times [0,1], g_{\partial M \times [0,1]}).$$

(The right hand side is in fact the relative index of  $g_{\partial M}$  and  $g'_{\partial M}$ .) However, as before, the vertical part of the metrics  $\{g(t)|_{\partial M}\}_{t \in [0,1]}$  is fixed and equal to the metric induced by the natural one on  $L$ ; in particular each  $\{g(t)|_{\partial M}\}$  is a psc-metric; see again Theorem 3.5. We conclude that  $g_{\partial M \times [0,1]}$  is a metric of psc and so  $\alpha_{\text{cyl}}(\partial M \times [0,1], g_{\partial M \times [0,1]}) = 0$ , giving that  $\alpha_{\text{cyl}}(M, g_M) = \alpha_{\text{cyl}}(M, g'_M)$ , as required. This result is of course also a consequence of the argument below, with the bordism equal to a cylinder; however, we think it is worthwhile to see it first and separately as a warm-up case.

Let now  $W_\Sigma: M_\Sigma \rightsquigarrow M'_\Sigma$  be a bordism between two spin pseudomanifolds with fibered  $(L, G)$ -singularities. We endow  $M'_\Sigma$  with a well-adapted wedge metric  $g'$ . Recall that  $W_\Sigma = W \cup -N(\beta W)$ , where the resolution  $W$  is a manifold with corners, and its boundary  $\partial W$  is given a splitting  $\partial W = \partial^{(0)}W \cup \partial^{(1)}W$ , where

$$\partial^{(0)}W = M \sqcup -M', \quad \partial(\partial^{(1)}W) = \partial M \cup -\partial M', \quad \partial(\beta W) = \beta M \sqcup -\beta M',$$

i.e.,  $\partial^{(1)}W: \partial M \rightsquigarrow \partial M'$  and  $\beta W: \beta M \rightsquigarrow \beta M'$  are usual bordisms between closed spin manifolds. Also we have that the  $(L, G)$ -fiber bundle  $F: \partial^{(1)}W \rightarrow \beta W$  restricts to the  $(L, G)$ -fiber bundles  $f: \partial M \rightarrow \beta M$  and  $f': \partial M' \rightarrow \beta M'$ , respectively. We must show that  $\alpha_{\text{cyl}}(M, g_M) = \alpha_{\text{cyl}}(M', g'_{M'})$ . By smoothing the corners we can assume that the resolution  $W$  is a manifold with boundary equipped with a splitting  $\partial W = \partial^{(0)}W \cup \partial^{(1)}W$  as above. We can endow  $\partial W$  with a metric  $g_{\partial W}$  which is equal to the metric  $g_M \sqcup (-g'_{M'}) =: g^{(0)}$  on the manifold with boundary  $M \sqcup -M' \equiv \partial^{(0)}W$  and is equal to an extension  $g^{(1)}$  of the submersion metric  $g^{(0)}|_{\partial^{(0)}W} \equiv g^{(0)}|_{\partial^{(1)}W}$  on  $\partial^{(1)}W$ . As we have anticipated, since  $\partial^{(1)}W$  is a fiber bundle with boundary, with fiber  $L$  and base  $\beta W$ , we can and we shall choose the submersion metric  $g^{(1)}$  to be the natural one in the vertical  $L$ -direction, rescaled (by choosing the radius  $R$  as in the comments following Corollary 3.2) so that the scalar curvature of the fibers is sufficiently large. Notice that the Riemannian manifolds with boundary  $(\partial^{(0)}W, g^{(0)})$  and  $(\partial^{(1)}W, g^{(1)})$  are collared near the boundary. We extend the metric  $g_{\partial W}$  on  $\partial W$  to a collared metric  $g_W$  on  $W$ . Then, by well known bordism invariance, we have that  $\alpha(\partial W, g_{\partial W}) = 0$ . On the other hand, by the gluing formula of Bunke, see [15], we have that

$$\begin{aligned} 0 &= \alpha(\partial W, g_{\partial W}) = \alpha_{\text{cyl}}(\partial^{(0)}W, g^{(0)}) + \alpha_{\text{cyl}}(\partial^{(1)}W, g^{(1)}) \\ &= \alpha_{\text{cyl}}(M, g_M) - \alpha_{\text{cyl}}(M', g'_{M'}) + \alpha_{\text{cyl}}(\partial^{(1)}W, g^{(1)}). \end{aligned}$$

But  $(\partial^{(1)}W, g^{(1)})$  is a Riemannian manifold with boundary with a psc-metric. This means that  $\alpha_{\text{cyl}}(\partial^{(1)}W, g^{(1)}) = 0$  and so

$$0 = \alpha_{\text{cyl}}(M, g_M) - \alpha_{\text{cyl}}(M', g'_{M'}),$$

as required.

Finally, the homomorphism  $\alpha_{\beta M}$  associates to  $[M_\Sigma]$  the  $\alpha$ -invariant of  $\beta M$ . This is well-defined because it is the composition of the group homomorphism

$$\beta: \Omega_*^{\text{spin}, (L, G)\text{-fb}} \rightarrow \Omega_{*-\ell-1}^{\text{spin}}$$

with the well-known  $\alpha$ -homomorphism. □

Refinements of these results will be given in [11].

**Remark 5.2.** The cylindrical class associated to a spin manifold with boundary, endowed with a psc metric on the boundary, together with its relationship with bordism, has been also considered in previous work related to Stolz'  $R$ -groups; see [15, 23]. More recent results are given in [28], where the whole Stolz' surgery sequence is mapped to a suitable exact  $K$ -theory sequence via index theory.

## 6. EXISTENCE THEOREMS

**6.1. Existence when  $L$  is a spin psc- $G$ -boundary.** In this subsection we deal with a special case of the existence problem for well adapted positive scalar curvature metrics, which covers the cases where  $L = S^n$  ( $n \geq 2$ ) or  $L = G$ . This is already a large class of situations. Namely, we assume that our link manifold  $L$  is a spin psc- $G$ -boundary in the sense of Section 1, the boundary of a manifold  $\bar{L}$  of positive scalar curvature, so that the metric  $g_L$  on  $L$  extends nicely over  $\bar{L}$ , and the  $G$ -action

on  $L$  extends to a  $G$ -action on  $\bar{L}$ . This is clearly the case when  $L = S^n$  ( $n \geq 2$ ),  $G = SO(n+1)$ , and we take  $\bar{L}$  to be the upper hemisphere in  $S^{n+1}$ . This case also applies to the case of  $G = L$  a simply connected compact Lie group, as we shall now explain.

**Theorem 6.1.** *Let  $G$  be a simple simply connected Lie group. Then  $G$  is a spin boundary and there is a spin manifold with boundary  $\bar{G}$  such that  $\bar{G}$  admits a positive scalar curvature metric extending the bi-invariant metric on  $G$  and the  $G$ -action on  $G$  (by left translation) extends to a  $G$ -action on  $\bar{G}$ .*

*Proof.* If  $G = \mathrm{SU}(2) \cong \mathrm{Spin}(3) \cong \mathrm{Sp}(1) \cong S^3$  has rank 1, then view  $G = S^3$  as the boundary of the upper hemisphere  $D^4$  in  $S^4$ . Thus  $G$  is a spin boundary and we can put on  $D^4$  a metric which in polar coordinates around the north pole is a warped product  $dr^2 + f(r)^2 g_G$ , where  $f(r) = \sin(r)$  for  $0 \leq r \leq \frac{\pi}{2} - \frac{\varepsilon}{2}$  and  $f(r) = 1$  for  $\frac{\pi}{2} + \frac{\varepsilon}{2} \leq r \leq \frac{\pi}{2} + \varepsilon$ , which gives a nice interpolation, without changing positivity of the scalar curvature, between the usual round metric on  $S^4$  and the cylinder metric on the product of  $S^3$  with an interval. There is a  $G$ -action on  $D^4$  extending the  $G$ -action on  $G$  itself if we think of  $G$  as the unit quaternions and  $D^4$  as the unit disk in  $\mathbb{H}$ .

For  $G$  of higher rank,  $G$  contains a copy of  $\mathrm{SU}(2)$  such that the inclusion  $\mathrm{SU}(2) \rightarrow G$  is an isomorphism on  $\pi_3$ . Thus we get a fibration  $\mathrm{SU}(2) \rightarrow G \rightarrow Y$ , where  $Y = G/\mathrm{SU}(2)$ . Replacing  $\mathrm{SU}(2)$  in this fibration with  $D^4$  with the above metric gives a  $\bar{G}$  with boundary  $G$ .

Now we need to show that  $\bar{G}$  carries a  $G$ -action extending the action of  $G$  on itself. This can be shown as follows. Note that  $\bar{G}$  as we just defined it is a quotient of  $G \times [0, 1]$  with  $G \times \{0\}$  collapsed to  $Y$ . More precisely,

$$\bar{G} = \{(g, y, t) : g \in G, y \in Y, t \in [0, 1], g \mapsto y\} / \sim,$$

where  $(g, y, 0) \sim (g', y, 0)$ .

Note that, as required, the fiber of  $\bar{G}$  over  $y \in Y$  is just the cone on the fiber of  $G$  over  $y$ . The space  $\bar{G}$  clearly carries a left  $G$ -action via  $g_1 \cdot [(g, y, t)] = [(g_1 g, g_1 \cdot y, t)]$ , and this action extends the left  $G$ -action on  $G$ .  $\square$

**Remark 6.2.** It is easy to modify the proof to apply to a simply connected compact Lie group that is semisimple but not simple. We leave details to the reader.

The proof of Theorem 6.1 also applies for any  $L$  which comes with a  $G$ -equivariant spherical fibration  $S^k \rightarrow L \rightarrow Y$ , and also to some other similar situations which we won't outline here for lack of compelling applications. Since  $\mathbb{H}\mathbb{P}^1 \cong S^4$ , this covers the case of examples such as the quaternionic flag manifold  $\mathrm{Sp}(n)/\mathrm{Sp}(1)^n$ ,  $n \geq 3$ , since this fibers as

$$S^4 = \mathbb{H}\mathbb{P}^1 \rightarrow \mathrm{Sp}(n)/\mathrm{Sp}(1)^n \rightarrow \mathrm{Sp}(n)/(\mathrm{Sp}(2) \times \mathrm{Sp}(1)^{n-2}).$$

In the literature one can find a simpler but less explicit result than Theorem 6.1, namely that  $G$  is a spin boundary. The proof is that  $G$  is parallelizable, but the image of the natural map  $\Omega^{\mathrm{fr}} \rightarrow \Omega^{\mathrm{spin}}$  is detected by the  $\alpha$ -invariant [6, Corollary 2.7], and  $G$  has a positive scalar curvature metric, so  $G$  is trivial in  $\Omega^{\mathrm{spin}}$ . But for our purposes we need to keep track of the metric and the  $G$ -action as well.

Still another case where  $L$  is a spin psc boundary is the case  $L = \mathbb{C}\mathbb{P}^{2n+1}$  of an odd complex projective space. This can be viewed as the space of complex lines in  $\mathbb{C}^{2n+2} = \mathbb{H}^{n+1}$ , so it fibers over  $\mathbb{H}\mathbb{P}^n$  with fiber the space of complex lines in a quaternionic line, or  $\mathbb{C}\mathbb{P}^1 = S^2$ . Filling in the  $S^2$  with a disk shows that the standard metric on  $L = \mathbb{C}\mathbb{P}^{2n+1}$  extends over an explicit spin boundary  $\bar{L}$  with positive scalar curvature. (We found this proof in [38].) Since  $\mathbb{H}\mathbb{P}^n = \mathrm{Sp}(n+1)/(\mathrm{Sp}(1) \times \mathrm{Sp}(n))$ , we can write  $\mathbb{C}\mathbb{P}^{2n+1}$  as  $\mathrm{Sp}(n+1) \times_{\mathrm{Sp}(1) \times \mathrm{Sp}(n)} S^2$ , where  $\mathrm{Sp}(1)$  acts transitively on  $S^2$  and  $\mathrm{Sp}(n)$  acts trivially on it, and then write  $\bar{L}$  as  $\mathrm{Sp}(n+1) \times_{\mathrm{Sp}(1) \times \mathrm{Sp}(n)} D^3$ . We are not sure if there is a choice for  $\bar{L}$  bounding  $\mathbb{C}\mathbb{P}^{2n+1}$  and carrying an  $\mathrm{SU}(2n+2)$ -action, as  $\mathrm{Sp}(n+1)$  is a smaller group than  $\mathrm{SU}(2n+2)$ . However, taking  $G = \mathrm{Sp}(n+1)$  is still good enough to apply Theorem 6.3 in this context.

The case of even complex projective spaces is totally different; these are not spin and do not bound even as non-oriented manifolds since they have odd Euler characteristic.

**Theorem 6.3.** *Let  $M_\Sigma \equiv (M, \partial M \rightarrow \beta M)$  be a closed  $(L, G)$ -singular spin manifold. Assume that  $M$ ,  $\beta M$ , and  $G$  are all simply connected, that  $n - \ell \geq 6$ , and suppose that  $L$  is a spin boundary, say  $L = \partial \bar{L}$ , with the standard metric  $g_L$  on  $L$  extending to a positive scalar curvature metric on  $\bar{L}$ , and with the  $G$ -action on  $L$  extending to a  $G$ -action on  $\bar{L}$ . Assume that the two obstructions  $\alpha(\beta M) \in KO_{n-\ell-1}$  and  $\alpha_{\mathrm{cyl}}(M) \in KO_n$  both vanish. Then  $M_\Sigma$  admits a well-adapted metric of positive scalar curvature.*

*Proof.* We use the bordism exact sequence (12) as well as the Bordism Theorem, Theorem 4.5.

First observe that since  $L$  is a spin  $G$ -boundary, the transfer map  $\Omega_{n-\ell-1}^{\mathrm{spin}}(BG) \rightarrow \Omega_{n-1}^{\mathrm{spin}}$  vanishes identically. Indeed, given any  $(L, G)$ -fiber bundle  $\varphi: X \rightarrow B$ , since the  $G$ -action on  $L$  extends to a  $G$ -action on  $\bar{L}$ ,  $X$  is the spin boundary of another fiber bundle over  $B$  which is the result of replacing each fiber  $L$  with  $\bar{L}$ . So the long exact sequence of bordism groups becomes a short exact sequence

$$(17) \quad 0 \rightarrow \Omega_n^{\mathrm{spin}} \xrightarrow{\iota} \Omega_n^{\mathrm{spin}, (L, G)\text{-fb}} \xrightarrow{\beta} \Omega_{n-\ell-1}^{\mathrm{spin}}(BG) \rightarrow 0.$$

Now suppose that  $(M, \partial M \rightarrow \beta M)$  is as in the theorem. We will construct another  $(L, G)$ -singular spin manifold in the same bordism class with a well-adapted metric of positive scalar curvature. Then  $M_\Sigma$  will admit a well-adapted metric of positive scalar curvature by Theorem 4.5. By assumption,  $\alpha(\beta M) = 0$ ,  $\beta M$  is simply connected, and  $\dim \beta M \geq 5$ . So by Stolz's Theorem, [35],  $\beta M$  has a Riemannian metric of positive scalar curvature. Use a connection on the  $G$ -bundle over  $\beta M$  associated to  $\partial M$  in order to construct a well-adapted metric of positive scalar curvature on the tubular neighborhood  $N$  of  $\beta M$  in  $M_\Sigma$ . The boundary of  $N$  is an  $(L, G)$ -fiber bundle over  $\beta M$  with a positive scalar curvature metric with a Riemannian submersion to  $\beta M$ . Let  $M' = P \times_G \bar{L}$  be the  $\bar{L}$ -bundle over  $\beta M$  associated to the corresponding principal  $G$ -bundle  $P \rightarrow \beta M$ . Then  $M'$  has a bundle metric of positive scalar curvature, and joining  $M'$  to  $N$ , we get an  $(L, G)$ -singular spin manifold  $M'_\Sigma$  with a well-adapted metric of positive scalar curvature. Since  $M'_\Sigma$  and  $M_\Sigma$  coincide near  $\beta M$ , by (17), their bordism classes differ by a class in the image of  $\Omega_n^{\mathrm{spin}}$ ; that is, there exists a

closed spin manifold  $N$  such that

$$[M'_\Sigma] - [M_\Sigma] = \iota[N].$$

Consider now the following diagram:

$$\begin{array}{ccccc} \Omega_n^{\text{spin}} & \xrightarrow{\iota} & \Omega_n^{\text{spin},(L,G)\text{-fb}} & \xrightarrow{\beta} & \Omega_{n-\ell-1}^{\text{spin}}(BG) \\ \downarrow \alpha & & \downarrow \alpha_{\text{cyl}} & & \\ KO_n & \xrightarrow{\text{Id}} & KO_n & & \end{array}$$

which is obviously commutative. By assumption  $\alpha_{\text{cyl}}(M) = 0$  in  $KO_n$ , and also  $\alpha_{\text{cyl}}(M') = 0$  since  $M'$  has positive scalar curvature. Thus from the above diagram we infer that  $\alpha(N) = 0$ . By doing surgery we can assume that  $N$  is spin-bordant to a closed spin manifold  $M''$  which is simply connected. Thus  $\alpha(M'') = 0$ ,  $M''$  is simply connected and  $[M'_\Sigma] - [M_\Sigma] = \iota[M'']$ . Summarizing,  $M_\Sigma$  is in the same bordism class in  $\Omega_n^{\text{spin},(L,G)\text{-fb}}$  as  $M'_\Sigma \amalg M''$ , where  $M''$  is a closed spin  $n$ -manifold with positive scalar curvature and where  $M'_\Sigma$  is a  $(L, G)$ -pseudomanifold with psc. Now we can apply Theorem 4.5 to get the conclusion.  $\square$

**6.2. Existence when the map to  $BG$  is null-bordant.** In this subsection we consider a different case of the existence problem, the case where  $\beta M \rightarrow BG$  represents a trivial element in  $\Omega_*^{\text{spin}}(BG)$ . This case will also be easy to deal with, using the bordism exact sequence (12) and the Bordism Theorem, Theorem 4.5.

**Theorem 6.4.** *Let  $M_\Sigma \equiv (M, \partial M \rightarrow \beta M)$  be a closed  $(L, G)$ -singular spin manifold  $M_\Sigma$ . Assume that  $M$ ,  $\beta M$ , and  $G$  are all simply connected and that  $n - \ell \geq 6$ . We make no additional assumptions on  $G$  and  $L$ , but we assume that the class of  $\beta M \rightarrow BG$  represents 0 in  $\Omega_{n-\ell-1}^{\text{spin}}(BG)$  and that  $\alpha_{\text{cyl}}(M) \in KO_n$  vanishes. Then  $M_\Sigma$  admits a well-adapted metric of positive scalar curvature.*

**Remark 6.5.** The assumption of this theorem is admittedly somewhat special, but is satisfied more often than one might expect. First of all, vanishing of the bordism class of  $\beta M \rightarrow BG$  is weaker than assuming both that  $\beta M$  is a spin boundary and that the  $G$ -bundle over  $\beta M$  is trivial (i.e.,  $\beta M \rightarrow BG$  is null-homotopic). For example, if  $G = \text{SU}(2)$ , then  $BG = \mathbb{H}\mathbb{P}^\infty$  and all torsion in  $\Omega_*^{\text{spin}}(\mathbb{H}\mathbb{P}^\infty)$  is 2-primary. But if  $\beta M$  is a sphere, the homotopy class of  $\beta M \rightarrow \mathbb{H}\mathbb{P}^\infty$  lies in  $\pi_{n-\ell-1}(\mathbb{H}\mathbb{P}^\infty) = \pi_{n-\ell-2}(S^3)$ . It is a simple well-known fact [30, Corollary 1.2.4] that the homotopy groups of  $S^3$  contain torsion of order  $p$  for any prime  $p$ . However, none of the odd torsion shows up in  $\Omega_*^{\text{spin}}(\mathbb{H}\mathbb{P}^\infty)$ .

Secondly, there are some cases where the main assumption of the theorem is automatic, namely cases where the kernel of the  $(L, G)$ -transfer map  $\Omega_{n-\ell-1}^{\text{spin}}(BG) \rightarrow \Omega_{n-1}^{\text{spin}}$  is trivial. Just as an example, if  $G$  is a symplectic group and  $L$  is a quaternionic projective space, then  $\Omega_*^{\text{spin}}(BG)$  is 2-primary torsion except in dimensions divisible by 4. So if  $n - \ell$  is not 1 mod 4, then the class of  $\beta M \rightarrow BG$  is at most 2-primary torsion in the bordism group, and some finite multiple of it satisfies the hypotheses of the theorem.



*Proof of Theorem 6.4.* By assumption, there is a spin manifold  $W$  with boundary and a principal  $G$ -bundle over  $W$  with  $\partial W = \beta M$  and the bundle on  $W$  extending the principal  $G$ -bundle over  $\beta M$ . Let  $M'$  be the associated  $L$ -bundle over  $W$ . Then  $\partial M' = \partial M$ . Choose a metric of positive scalar curvature on  $W$  restricting to a product metric of positive scalar curvature in a neighborhood of  $\beta M$ . Using this metric and the bundle structure over  $\beta M$ , we get a manifold  $M'_\Sigma$  with  $(L, G)$ -fibered singularities with a well-adapted metric of positive scalar curvature. By the bordism exact sequence (12), the difference between the bordism class of  $M_\Sigma$  and the bordism class of  $M'_\Sigma$  lies in the image of  $\Omega_n^{\text{spin}}$ . So  $M_\Sigma$  is spin bordant to  $M'_\Sigma \amalg M''$ , where  $M''$  is a closed spin manifold. By additivity of the  $\alpha$ -invariant and the assumption that  $\alpha_{\text{cyl}}(M) \in KO_n$  vanishes,  $\alpha(M'') = 0$ . So the result follows from the Bordism Theorem, Theorem 4.5.  $\square$

We proceed to give another application of Theorem 6.4. Suppose we look only at manifolds with Baas-Sullivan singularities, i.e., we require that  $\partial M = \beta M \times L$ , and suppose  $L = \mathbb{H}\mathbb{P}^{2k}$ ,  $k \geq 1$ ,  $G = \text{Sp}(2k + 1)$ . (This is one of the key examples where  $L$  is *not* a spin boundary, so that Theorem 6.3 doesn't apply.)

**Lemma 6.6.** *The class of  $\mathbb{H}\mathbb{P}^{2k}$ ,  $k \geq 1$ , is not a zero-divisor in the spin bordism ring  $\Omega_*^{\text{spin}}$ .*

*Proof.* Note that  $\mathbb{H}\mathbb{P}^{2k}$  has nonzero signature and odd Euler characteristic, so it represents nontrivial elements of  $\Omega_*^{\text{spin}} \otimes \mathbb{Q}$  and of  $\mathfrak{N}_*$ , the non-oriented bordism ring, which are polynomial rings (over  $\mathbb{Q}$  and  $\mathbb{F}_2$ , respectively). So if  $x \in \Omega_*^{\text{spin}}$  is nonzero either in  $\Omega_*^{\text{spin}} \otimes \mathbb{Q}$  or in  $\mathfrak{N}_*$ , its product with  $\mathbb{H}\mathbb{P}^{2k}$  can't be a spin boundary, and hence the image of  $\mathbb{H}\mathbb{P}^{2k}$  in  $\Omega_*$  is not a zero-divisor. So for  $\mathbb{H}\mathbb{P}^{2k}$  to be a zero-divisor in  $\Omega_*^{\text{spin}}$ , it would have to annihilate a non-zero element  $x$  in the kernel of the forgetful map  $\Omega_*^{\text{spin}} \rightarrow \Omega_*$ . Now apply [6, Corollaries 2.3 and 2.6]. This element  $x$  would have to live in dimension 1 or 2 mod 8 and be of the form (torsion-free element)  $\times \eta^j$ ,  $j = 1$  or 2. (Here  $\eta$  is the usual generator of  $\Omega_1^{\text{spin}}$ .) But multiplying such an element by  $\mathbb{H}\mathbb{P}^{2k}$  would give another element of the same form (in dimension  $8k$  higher) which would be non-zero again. So  $\mathbb{H}\mathbb{P}^{2k}$  cannot be a zero-divisor.  $\square$

**Theorem 6.7.** *Let  $M_\Sigma \equiv (M^n, \partial M \rightarrow \beta M)$  be a closed  $(L, G)$ -singular spin manifold  $M_\Sigma$ , with  $L = \mathbb{H}\mathbb{P}^{2k}$  and  $G = \text{Sp}(2k + 1)$ ,  $n \geq 1$ . Assume that  $\partial M = \beta M \times L$ , i.e., the  $L$ -bundle over  $\beta M$  is trivial, or in other words that the singularities are of Baas-Sullivan type. Then if  $M$  and  $\beta M$  are both simply connected and  $n - 8k \geq 6$ ,  $(M, \partial M)$  has an adapted metric of positive scalar curvature if and only if the  $\alpha$ -invariants  $\alpha(\beta M) \in KO_{n-8k-1}$  and  $\alpha_{\text{cyl}}(M) \in KO_n$  both vanish.*

*Proof.* In the case where the  $L$ -bundle over  $\beta M$  is trivial, the  $(L, G)$ -transfer map  $\Omega_{n-\ell-1}^{\text{spin}} \rightarrow \Omega_{n-1}^{\text{spin}}$  is just multiplication by the class of  $L$  in  $\Omega_\ell^{\text{spin}}$ . When  $L = \mathbb{H}\mathbb{P}^{2k}$ , by Lemma 6.7, this map is injective, and thus  $[\beta M] = 0$  in  $\Omega_{n-\ell-1}^{\text{spin}}$ . So we can apply Theorem 6.4.  $\square$

**Remark 6.8.** Note that Lemma 6.6 fails for odd quaternionic projective spaces, since these annihilate torsion classes in the kernel of the forgetful map  $\Omega_*^{\text{spin}} \rightarrow \Omega_*$ . Nevertheless, we expect that the case of Baas-Sullivan singularities with  $L = \mathbb{H}\mathbb{P}^{2k+1}$  is treatable, but will require a more complicated argument. We leave this to future work.

## 7. PREVIEW OF THE NON-SIMPLY CONNECTED CASE

We conclude by mentioning some “unfinished business” that will be treated in a continuation of this paper [11]. We start by extending the obstruction theory to the cases where  $M_\Sigma$  and/or  $\beta M$  are not simply connected. As in the theory of psc on general closed manifolds (see [32] for a survey), this involves obstructions in the  $KO$ -theory of the group  $C^*$ -algebras of the relevant fundamental groups. Then we generalize the Surgery Theorem and Bordism Theorem (Theorems 4.6 and 4.5) to this situation. As a result we are able to generalize the Existence Theorem (Theorem 1.2) to the case where the relevant groups satisfy the Gromov-Lawson-Rosenberg conjecture. If the groups are in the class where the Baum-Connes assembly map is injective, then by a theorem of Stolz [36, §3], we can at least prove a “stable” existence theorem in the sense of [33].

Other problems to be discussed in [11] involve the topology of the space of well-adapted psc-metrics if this space is non-empty. In some cases where  $M_\Sigma$  is not simply connected, rho-invariants on manifolds with  $L$ -fibered singularities can be used to show that this space has infinitely many components. We will also see that the topology of the space of well-adapted psc-metrics on  $M_\Sigma$  is at least as complicated as that of the space of psc-metrics on  $\beta M$ .

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE OR 97403-1222, USA

*Email address*, Boris Botvinnik: [botvinn@uoregon.edu](mailto:botvinn@uoregon.edu)

*URL*: <http://pages.uoregon.edu/botvinn/>

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA LA SAPIENZA, PIAZZALE ALDO MORO, 00185 ROMA, ITALY

*Email address*, Paolo Piazza: [piazza@mat.uniroma1.it](mailto:piazza@mat.uniroma1.it)

*URL*: <http://www1.mat.uniroma1.it/people/piazza/>

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742-4015, USA

*Email address*, Jonathan Rosenberg: [jmr@math.umd.edu](mailto:jmr@math.umd.edu)

*URL*: <http://www2.math.umd.edu/~jmr/>