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Unitary and homotopy equivalences: classification of low-dimensional topological phases of quantum matter

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Abstract

Topological insulators have attracted significant attention across physics and mathematics due to their technological potential and their rich geometric and algebraic structure. Their hallmark property is the bulk–boundary correspondence, whereby non-trivial bulk topology enforces the existence of metallic edge states. The mathematical classification of such phases of matter, culminating in the periodic table of topological insulators and superconductors, associates topological invariants to different symmetry classes defined by time-reversal, particle–hole, and chiral symmetries. However, this framework involves several approximations: it replaces projection-valued maps (PVMs) with vector bundles, collapses the torus to a sphere – thus overlooking weak, lower-dimensional invariants –, and employs stable equivalence notions from K-theory that do not always capture the full topological content. This thesis develops a direct homotopy-theoretic approach to the classification of symmetric PVMs, aimed at overcoming these limitations. In particular, it investigates the interplay between unitary equivalence and homotopy, identifies weak invariants absent from the Kitaev table, and establishes a general classification scheme applicable to arbitrary periodic models. The analysis focuses on low-dimensional settings (0, 1, and 2-dimensional systems), which provide a tractable yet non-trivial testing ground. Within this framework, we examine models with a single symmetry present, corresponding to the Altland–Zirnbauer classes A, AI, AII, AIII, C, and D. The results clarify in particular the obstructions to constructing symmetric Wannier bases, refine the understanding of strong versus weak topological invariants, and detail the relation between topological phases of matter and the dimerization choice in discrete models.

Contents

Introduction

I.1 Overview

The field of topological insulators has attracted the attention of both the physics and mathematics community, due to their potential applications for solid state devices and quantum computation as well as their rich mathematical structure. From the physical point of view, the most interesting feature of these materials is that, when they are truncated, a non-trivial topology for the infinite system drives the existence of metallic (i.e. gapless) states which are spatially localized near the cut: this statement is the celebrated principle of bulk-boundary correspondence for topological insulators. For extended reviews of this rich research line, the reader is referred to Refs. [2], [7], and [28]. As will be detailed later on the next section of the Introduction, a topological insulator is a solid described by a Hamilton operator acting on an appropriate Hilbert space, whose spectrum is gapped; this allows to define the occupied energy levels as those below the spectral gap. For translation invariant systems, the classification of topological phases of matter is then tantamount to the classification of projection-valued maps from the momentum space to the vector subspace spanned by these energy states; the first attempt to solve the classification problem was done by considering a different notion of translation invariance that leads to the classification of vector bundles over the torus with the same dimension as the system (*Bloch frame*). Then those were treated as vector bundles over the sphere in order to easily compute the numerical topological invariants that describe their topological nature using K-theory. In particular, different scenarios arise if the system is constrained by certain pseudo-symmetries, which can enrich or trivialize the topology of the manifold of occupied states. The three symmetries usually considered are time reversal symmetry Θ, particle-hole symmetry Ξ and chiral symmetry Π; they are unitary or anti-unitary operators that square to $\pm\mathbb{1}$ (identity operator). A full description of the symmetries will come in the last section of the Introduction. Using techniques from K-theory and Clifford algebras, the topological invariants that describe each symmetry class of vector bundles over the sphere were presented in [37] and later refined in [1] [30] [36] [53] to cite some among many. The following table contains the complete topological invariants classifying symmetric vector bundles over a d-dimensional sphere up to stable equivalences. The table is periodic with period $d = 8$ thanks to Bott periodicity. In the column where the symmetries are present, we denoted with ± 1 the sign of the square of the symmetry and with 0 a broken symmetry.

Classes	Symmetries			Dimension							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

The two ground-breaking results that naturally follow from this table are the presence of topological obstructions in building symmetric frames and a partial classification of phases of matter. For the first, we are interested in finding smooth symmetric frame for the Bloch bundle since in position space they constitute a symmetric Wannier basis as localized as the frame is regular [47]. The latter gives a close description of the shape of the orbital of an electron with energy in the spectral band and can also be used for the development of computational methods that scale linearly with the size of the system [21]. If the Bloch bundle is non trivial it is impossible to find a continuous frame, so, if the topological invariant is non zero, the system cannot admit a localized Wannier basis. However the table does not express how close can we get to the construction of a continuous frame for non trivial Bloch bundles and this will be a recurring problem we will answer in all the considered classes. For the second, if a system undergoes a continuous deformation that maintains the spectral gaps open, the topological nature of these numbers ensures that they remain constant during the transformation. Conversely, the only way in which a system can change a topological invariant is to transition trough a phase without the spectral gap. In fact, among the original motivations of the work of Kitaev was the search for a stable source of qubits to perform quantum computation codes, which he proposed to identify with topologically stable edge states at the edge of a superconducting chain [38]. Those techniques were also used to investigate the phenomenon of piezoelectricity in polyacetylene [56]. Finally, the most important example can be found in [59] in which those techniques were exploited by Thouless, Kohmoto, Nightingale and den Nijs to show a relation between the quantum Hall conductivity and a Chern number, which will be defined later. However, this table only provides a partial classification for projection-valued maps because there are three layers of approximation present in this theory: the first is considering vector bundles instead of projection-valued maps. The difference is that projection-valued maps have a notion of unitary equivalence related to the choice of a periodicity cell and a notion of homotopic equivalence related to the phases of matter. Instead, abstract vector bundles have only a notion of unitary equivalence, and this may lead to misinterpreting the true nature of a topological invariant. The second layer is collapsing the torus to a sphere: in this way the table focuses only on topological invariants that live specifically on the dimension d of the model and overlooks topological invariants that are inherited from lower dimensions. This choice was made for a reason: despite the fact it is possible to compute the K-theory of a torus using the Künneth formula [31], thus possibly recovering the invariants from lower dimensions, such invariants are supposed to vanish when studying periodic models with non-periodic small perturbations (supposition confirmed by [8] for 1-d models). For this reason they are called *weak invariants.* Instead, invariants that live on the proper dimension are supposed to be robust up to small perturbation, and this is why they are referred to as *strong invariants* (some generalizations of topological invariants to non-periodic models can be found in [41], [4] and [8]). The final approximation was the usage of K-theory, which deeply uses the notion of weak stable equivalence that does not ensure that two vector bundles are unitarily equivalent if their invariants are equal, and neither does the notion of isomorphism. So, generally speaking, the invariants obtained are not complete topological invariants. Moreover, there are also some hidden hypotheses that forbid us from using the table in a wide variety of physical models. For example in Classes D and AIII the table consider pairs of projection-valued maps P^+, P^- such that $P^+ + P^- = \mathbb{1}$, but this is not always true. So, our goal is to use a direct approach based on homotopy theory to study symmetric projection-valued maps, study the duality unitary equivalence-homotopy, find the weak topological invariants that may be missing from the Kitaev table, and work in maximum generality in order to develop a theory that can be applied in any periodic model. Due to the difficulty in using homotopy theory in spaces of high dimension, in this thesis we will study models with dimensions 0, 1 and 2, where the dimension zero consists in a single point and is used as a stepping stone toward more complex frameworks. In this thesis we will study models with a single symmetry present, in the Kitaev table they belong to Classes A, AI, AII, AIII, C, and D.

The contents of this thesis will be the following. The Introduction contains the well-known procedure that allows one to study periodic Hamiltonians acting on $L^2(\mathbb{R}^d)$, which represent the energy of a quantum particle moving in a crystalline solid. Thanks to a unitary operator, the Bloch-Floquet-Zak transform, it is possible to define the projection-valued map containing the topological properties we are interested in. Usually this projection-valued map satisfies a pseudoperiodic condition called $\tau-$covariance. However, it is extremely uncomfortable to study the topology of a pseudoperiodic projection-valued map, so in the Introduction we present a small trick that associates a periodic projection-valued map to the original one while preserving all the geometrical properties (already present in [10]). In the Introduction, we also present a similar framework often used by condensed matter physicists to approximate periodic Hamiltonians: discrete models. In a discrete model, the configuration space is approximated with the set of atoms in the solid, which means that it is usually a lattice $\Gamma \simeq \mathbb{Z}^d$. In the Introduction, we show how to study periodic Hamiltonians acting on $l^2(\Gamma)$ using the standard Fourier transform; this method leads to the definition of the periodic projection-valued map we are interested in. The most important aspect of the Introduction is a small discussion that justifies the formalism we present in this thesis. In both cases, continuum in $\mathbb{R}^d$ and discrete in Γ, there is an ambiguity of choice. In fact, the way in which we can relate a pseudoperiodic projection-valued map to a periodic projection-valued map in the continuum case is not unique. Similarly, in the discrete case, we can apply the Fourier transform in several different ways depending on the so-called *dimerization.* This means that there are multiple projection-valued maps that can be associated with a physical model; however, in the Introduction we show that all of them are unitarily equivalent. Different choices of dimerization are related to the "unit cell consistency", argument treated in [60]. The last content of the Introduction is a formal description of the goals of this thesis: given a projection-valued map, how close can we get to obtaining a frame of this projection-valued map that, in some appropriate sense, respects the symmetries present? Moreover, given two different projection-valued maps is it possible to construct a unitary equivalence or an homotopy between the two that respects the symmetries present? And if it is not possible, is there a way to encode the topological obstructions that prevent us from doing so? In the main body of the thesis we answer those questions depending on which type of symmetry is present, every time with the same approach: find an optimal frame, then define the unitary equivalence and finally moving continuously this unitary equivalence to construct an homotopy. In particular, in Chapter 1 about class A we summarize and reinterpret the results presented in [47] about projection-valued maps without any symmetries. We also mixed his result with a different definition of the Chern number present in [46] to better suit our purposes. Similarly, in Chapter 2, which is about Class AI, we apply our formalism to projection-valued maps that enjoy a bosonic time-reversal symmetry (compare [18]). Our original contribution starts at the chapter about class D, where we study projection-valued maps that enjoy an even charge-conjugation symmetry. There our formalism allows to easily treat a difficult class and also shows why it is important to consider separately the unitary equivalence problem and the homotopy problem. In fact the two conditions are not always equivalent and this becomes a crucial aspect when facing with the dimerization ambiguity we mention earlier. Then, in Chapter 4 about class AII, we replicate the studies done in [44] to study projection-valued maps that enjoy a fermionic time-reversal symmetry. However, our formalism allows us to answer several open questions about the structure of an optimal frame and also leads to a new formulation of the Fu-Kane-Mele index that is extremely promising to be applied in non-periodic framework since it is related to a Chern number. Another original chapter is Chapter 5 about class C, where we study projection-valued maps that enjoy an odd charge-conjugation symmetry. Finally, the last original contribution is Chapter 6 about class AIII we studied projection-valued maps that enjoy a chiral symmetry. This, to some extent, is very similar to the chapter about class D because also in this class the difference between unitary equivalence and homotopy becomes clear and crucial. We will also apply our results to several tight-binding models in Chapter 7 about explicit models where we will show how to compute our invariants and give a context in which the differences between unitary equivalences and homotopies can be seen. Some models that will be considered are taken from the physical literature (Kitaev chain [38] and SSH model [56]), others are modifications of those two models invented to break additional symmetries since those two models are actually in Class BDI and we wanted to study proper single symmetric models. Finally, we will collect all results, discuss their originality, and compare them with the existing literature in Chapter 8, there we will also discuss some open problems and perspective on possible further applications of the formalism proposed in this thesis. The Appendix is devoted to some basic tools of algebraic geometry and well known results that will be used extensively. During the entirety of the thesis we will work using only continuity of projection-valued maps, this is because an astonishing amount of existing works (like [10]) proved that it is possible to promote any continuous and symmetric frame to a real-analytic one. So, despite the fact that real-analytic frames are crucial for condensed matter physicists, we preferred to work with the maximum generality. In fact most of the reinterpretations mentioned before consist in defining well-known quantities without asking for additional regularity except continuity.

I.2 From periodic Hamiltonian to projection-valued maps

As stated before, the motivations behind this work are rooted in the study of the dynamic of a quantum particle in the condensed matter regime. In this regime physicists approximate the solid in which the particle is moving as a periodic configuration space $X \subset \mathbb{R}^d$. The periodicity is expressed by the existence of a lattice $\Lambda \simeq \mathbb{Z}^d$ called *Bravais lattice* such that $X = \lambda + X$ for all $\lambda \in \Lambda$. There are two categories of models that are usually used: the *continuum models* in which $X = \mathbb{R}^d$ or the *tight-binding models* in which X is approximated with the set of atoms of the solid and X/Λ is finite. In both cases there is a recurring property: the Hamiltonian H acting on $L^2(X)$, describing the energy of the particle, must commute with all the translation operators $(T_\lambda\phi)(x) = \phi(x-\lambda)$ for all λ in the lattice $\Lambda \simeq \mathbb{Z}^d$. It is also possible to study Hamiltonians that commute with the magnetic translations $T_\lambda(\phi) = \exp\left(-\frac{ie}{\hbar c}A(x)\cdot\lambda\right)\phi(x-\lambda)$ for a magnetic potential A, using the same techniques, but we will not go into the numerous details of this topic.

In order to study the spectral properties of periodic Hamiltonians for continuum models, it is very convenient to use a unitary operator $\mathcal{U}_{\mathrm{BFZ}}$ known as the Bloch-Floquet-Zak transform defined as:

Definition (Bloch-Floquet-Zak transform)**.** Given $\phi \in L^2(\mathbb{R}^d)$, we can define a function depending on $k \in \mathbb{R}^d$ and on $y \in \mathbb{R}^d$ as:

$$(\mathcal{U}_{\mathrm{BFZ}}\phi)(k,y) = \sum_{\lambda\in\Lambda} e^{-ik\cdot(\lambda-y)}(T_\lambda\phi)(y)$$

Proposition I.1. *$\mathcal{U}_{\mathrm{BFZ}}$ is a unitary transformation between $L^2(\mathbb{R}^d)$ and the space of square integrable functions $f(k,y)$ on $y \in \mathbb{R}^d, k \in \mathbb{R}^d$ such that*

$$f(k, y+\lambda) = f(k,y) \quad f(k+\lambda^*, y) = e^{i\lambda^*\cdot y} f(k,y) \quad \forall\lambda \in \Lambda, \lambda^* \in \Lambda^*$$

This Hilbert space can be interpreted as a direct integral of Hilbert spaces $\int_{k\in\mathbb{B}} L_{per}(\mathbb{R}^d)$ where Λ^* is the dual lattice of Λ defined as the set $\Lambda^* = \{\lambda^* \in \mathbb{R}^d | \lambda^* \cdot \lambda' \in 2\pi\mathbb{Z}\ \forall \lambda' \in \Lambda\}$ and $\mathbb{B}$ is the Brillouin zone*, a fundamental cell of Λ^*.*

Proof.

- *y-periodicity:* If $\lambda' \in \Lambda$ then:

$$\begin{aligned}(\mathcal{U}_{\mathrm{BFZ}}\phi)(k, y+\lambda') &= \sum_{\lambda\in\Lambda} e^{-ik\cdot(\lambda-\lambda'-y)}(T_\lambda\phi)(y+\lambda') = \\ &= \sum_{\lambda-\lambda'\in\Lambda} e^{-ik\cdot(\lambda-\lambda'-y)}(T_{\lambda-\lambda'}\phi)(y) = (\mathcal{U}_{\mathrm{BFZ}}\phi)(k,y)\end{aligned}$$

- *k-pseudo-periodicity:* If λ^* is in the dual lattice Λ^*, then:

$$\begin{aligned}(\mathcal{U}_{\mathrm{BFZ}}\phi)(k+\lambda^*, y) &= \sum_{\lambda\in\Lambda} e^{-i(k+\lambda^*)\cdot(\lambda-y)}(T_\lambda\phi)(y) = \\ &= e^{i\lambda^*\cdot y}\sum_{\lambda\in\Lambda} e^{-ik\cdot(\lambda-y)}(T_\lambda\phi)(y) = e^{i\lambda^*\cdot y}(\mathcal{U}_{\mathrm{BFZ}}\phi)(k,y)\end{aligned}$$

- *scalar product-preserving:* Consider $\phi, \psi \in \mathcal{D} \subset L^2(\mathbb{R}^d)$, in the set of smooth and fast decaying functions. Then the scalar product in the arriving Hilbert space is computed in the periodicity cells of both variables. Since the function $\mathcal{U}_{\text{BFZ}}\phi$ is $\Lambda-$periodic in y we can find a subset $\mathbb{W} \subset \mathbb{R}^d$ called *Wigner-Seitz cell* such that $\mathbb{R}^d = \cup_{\lambda\in\Lambda}(\mathbb{W}+\lambda)$ and $\mathbb{W}+\lambda \cap \mathbb{W} = \emptyset$ for all $\lambda \neq 0$. The same can be done for k since it was Λ^*-pseudoperiodic to define the *Brillouin zone* $\mathbb{B} \subset \mathbb{R}^d$. Then the scalar product will be:

$$\langle \mathcal{U}_{\text{BFZ}}\phi, \mathcal{U}_{\text{BFZ}}\psi\rangle = \int_{k\in\mathbb{B}} dk \int_{y\in\mathbb{W}} dy \left(\sum_{\lambda\in\Lambda} e^{ik\cdot(\lambda-y)}\left(T_\lambda\overline{\phi}\right)(y)\right)\left(\sum_{\lambda'\in\Lambda} e^{-ik\cdot(\lambda'-y)}(T_{\lambda'}\psi)(y)\right) =$$
$$= \int_{k\in\mathbb{B}} dk \int_{y\in\mathbb{W}} dy \sum_{\lambda,\lambda'\in\Lambda} e^{ik\cdot(\lambda-\lambda')}\left(T_\lambda\overline{\phi}\right)(y)\left(T_{\lambda'}\psi\right)(y)$$

Since $\int_{k\in\mathbb{B}} e^{ik\cdot(\lambda-\lambda')} = \delta_{\lambda,\lambda'}$ the integral above will be equal to:

$$\int_{y\in\mathbb{W}} dy \sum_{\lambda\in\Lambda}\left(T_\lambda\overline{\phi}\right)(y)\left(T_{\lambda'}\psi\right)(y) = \sum_{\lambda\in\Lambda}\int_{y\in\mathbb{W}+\lambda}\overline{\phi(y)}\psi(y)dy = \int_{\mathbb{R}^d}\overline{\phi(y)}\psi(y)dy$$

This is due to the fast decaying of the functions. Finally, $\mathcal{U}_{\text{BFZ}}$ can be extended uniquely to a unitary operator thanks to the bounded linear transform theorem [52].

□

Using Proposition I.1 we know that $\mathcal{U}_{\text{BFZ}}$ is a linear and unitary operator and that the target function is $\Lambda-$periodic in y and Λ^*-pseudoperiodic in k, where Λ^* is the dual lattice

$$\Lambda^* = \{k \in \mathbb{R}^d | k\cdot\lambda \in 2\pi\mathbb{Z}\ \forall\lambda\in\Lambda\}.$$

If we select a *Wigner-Seitz* cell $\mathbb{W}$, the next proposition states that there is a family of operators $H(k)$ acting on $L^2(\mathbb{W}) \simeq L^2_{per}(\mathbb{R}^d)$ such that:

$$[\mathcal{U}_{\text{BFZ}} H \mathcal{U}_{\text{BFZ}}^{-1}\phi](k,y) = [H(k)\phi(k,\cdot)](y).$$

Proposition I.2. *If A is a linear operator acting on $L^2(\mathbb{R}^d)$, if it commutes with the translation operators $T_\lambda(\phi)(x) = \phi(x-\lambda)$ and if $\tau_\lambda(\phi)(y) = e^{i\lambda\cdot y}\phi(y)$, then there is a family of operators $A(k)$ acting on the space of $\Lambda-$periodic functions $L^2_{per}(\mathbb{R}^d)$ such that for all $\psi \in L^2_{ps.p}(\mathbb{R}^d\times\mathbb{R}^d)$:*

$$\mathcal{U}_{\text{BFZ}} A \mathcal{U}_{\text{BFZ}}^{-1}\psi(k,y) = [A(k)\psi(k,\cdot)](y)$$

that are also $\tau-$covariant in k:

$$A(k+\lambda^*) = \tau_{\lambda^*}A(k)\tau_{\lambda^*}^{-1}.$$

Proof. Given $\phi \in L^2(\mathbb{R}^d)$, we can call Z_k the unitary operator acting on $L^2(\mathbb{R}^d)$ such that $(Z_k\phi)(y) = e^{ik\cdot y}\phi(y)$. It's inverse is clearly Z_{-k} and we can also define $A_k = Z_k A Z_{-k}$ as an operator acting on $L^2(\mathbb{R}^d)$. However, this operator can also act on the space of $\Lambda-$periodic functions $L^2_{per}(\mathbb{R}^d)$ since every periodic function can be posed equal to zero outside a periodicity cell $\mathbb{W}$ to be an element of

$L^2(\mathbb{R}^d)$. We will call $A(k)$ the restriction of A_k over $L^2_{per}(\mathbb{R}^d)$. Now we can write:

$$\begin{aligned}[\mathcal{U}_{\mathrm{BFZ}}A(\phi)](k,y) &= \sum_{\lambda\in\Lambda} e^{-ik\cdot(\lambda-y)}[(T_\lambda A)\phi](y) = \sum_{\lambda\in\Lambda} e^{-ik\cdot\lambda}[(Z_k A T_\lambda)\phi](y) = \\ &= \sum_{\lambda\in\Lambda} e^{-ik\cdot\lambda}[(A_k Z_k T_\lambda)\phi](y) = A(k)\left[\sum_{\lambda\in\Lambda} e^{-ik\cdot(\lambda-y)}(T_\lambda\phi)(y)\right] = \\ &= A(k)(\mathcal{U}_{\mathrm{BFZ}}\phi)(k,y)\end{aligned}$$

Furthermore, since:

$$A_{k+\lambda^*} = Z_{k+\lambda^*}AZ_{-k-\lambda^*} = Z_k Z_{\lambda^*} A Z_{-k} Z_{-\lambda^*} = Z_{\lambda^*}A(k)Z_{-\lambda^*}$$

we can call τ_{λ^*} the restriction of Z_{λ^*} to $L^2_{per}(\mathbb{W})$ for every $\lambda^* \in \Lambda^*$ and have $\tau-$*covariance* of the fibers:

$$A(k+\lambda^*) = \tau_{\lambda^*}A(k)\tau_{\lambda^*}^{-1}.$$

□

This means that $\mathcal{U}_{\mathrm{BFZ}}H\mathcal{U}_{\mathrm{BFZ}}^{-1}$ is a *decomposable operator* in the sense of the following definitions.

Definition I.3 (Decomposable bounded operator)**.** Let $(M, d\mu)$ be a σ-finite measure space and $\mathcal{H}'$ a separable Hilbert space. Let $\hat{\mathcal{H}} = L^2(M, d\mu, \mathcal{H}')$. Then a bounded operator $\hat{A} : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ is decomposable if and only if there exists $A \in L^\infty(M, d\mu, \mathcal{L}(\mathcal{H}'))$ such that for almost every $m \in M$,

$$\left[\hat{A}(\phi)\right](m) = [A(m)][\phi(m)]$$

and the function

$$\langle \phi, A(\cdot)\psi\rangle : M \to \mathbb{R}$$

is measurable $\forall \phi, \psi \in \hat{\mathcal{H}}$. We will call the operator $A(m) : \mathcal{H}' \to \mathcal{H}'$ the fiber of $\hat{A}$ over m.

Definition I.4 (Decomposable unbounded self-adjoint operator)**.** Using the same notation above, a self adjoint and unbounded operator $\hat{A} : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ with domain $\mathcal{D}(A)$ is decomposable if and only if there is a function $A(\cdot)$ from M to the set of self-adjoint (maybe unbounded) operators on $\mathcal{H}'$ such that:

1. For all $\phi, \psi \in \mathcal{H}'$, the function $\left\langle \phi, (A(\cdot)+i)^{-1}\psi\right\rangle : M \to \mathbb{R}$ is measurable.
2. $\mathcal{D}(A) = \{\phi \in \hat{\mathcal{H}} | \phi(m) \in \mathcal{D}(A(m))$ almost everywhere.$\}$
3. $(A\phi)(m) = A(m)\phi(m)$ for every $\phi \in \hat{\mathcal{H}}, m \in M$.

This means that the action of H splits into the action of a family of operators $H(k)$, called *fibers*, each acting on $L^2(\mathbb{W})$ that are continuous in k in the following sense. For example, those techniques can be applied to study Hamiltonians such as:

$$H = \frac{-\hbar^2}{2m}\Delta + V$$

for a $\Lambda-$periodic potential V. Under relatively mild hypotheses about potential V, one can show that the fiber Hamiltonians $H(k)_{k\in\mathbb{R}^d}$ defined previously comprise a family of self-adjoint operators with the following properties:

1. The domain of $H(k)$ within $L^2(\mathbb{W})$ does not depend on $k \in \mathbb{R}^d$.
2. The set $\mathcal{R} = \{(k, z) \in \mathbb{R}^d \times \mathbb{C} | z \in \rho(H(k))\}$ is open and the resolvent map $(k, z) \mapsto (H(k) - z)^{-1}$ is analytic on $\mathcal{R}$, with values in the algebra of compact operators on $L^2(\mathbb{W})$.
3. $\sigma(H) = \cup_{k\in\mathbb{R}^d}\sigma(H(k))$.

The reader is referred to [35] for the proofs of the first and second statements, while for the third statement we can use the following theorem whose proof can be found in [51].

Theorem I.5 (XIII.85-4). *If A is a decomposable self-adjoint operator bounded or unbounded with fiber A', then $\lambda \in \sigma(A)$ if and only if the measure of the set $\{m \in M \mid \sigma(A'(m)) \cap (\lambda-\epsilon, \lambda+\epsilon) \neq \emptyset\}$ is greater then zero for every $\epsilon > 0$.*

Theorem I.6 (XIII-85-5). *If A is a decomposable self-adjoint operator bounded or unbounded with fiber A', then λ is an eigenvalue of A if and only if the measure of the set $\{m \in M \mid \lambda \text{ is an eigenvalue of } A'(m)\}$ is not zero.*

The third property allows us to have a complete description of the spectrum of H; usually it is made up of intervals called *Bloch bands*. However, we can recover additional information, if we select one band Ω and define a complex and smooth closed curve γ that makes just one loop around Ω without containing or intersecting other Bloch bands, then we can define:

$$P_\Omega(k) = \frac{i}{2\pi}\int_\gamma (H(k) - z\mathbb{1})^{-1}\, dz \in \mathcal{L}(L^2(\mathbb{W}))$$

The next proposition tells us that $P_\Omega(k)$ is real-analytic in k and represents the fiber decomposition of $\chi_\Omega(H)$.

Proposition I.7 (Riesz formula). *If $H(k)$ is a family of self-adjoint operators acting on a Hilbert space $\mathcal{H}$ and γ is a smooth and closed complex curve making just one loop around a Bloch band Ω, then*

$$P_\Omega(k) = \frac{i}{2\pi}\int_\gamma (H(k) - z\mathbb{1})^{-1}\, dz$$

is a family of projectors such that, using functional calculus, $P_\Omega(k) = \chi_\Omega(H(k))$. Moreover, $P(k)$ has the same regularity in k as $(H(k) - z)^{-1}$.

Proof. First, we need to notice that if $f(w) = \frac{i}{2\pi}\int_\gamma \frac{1}{(w-z)}dz$, then $P_\Omega(k) = f(H(k))$. Now we just need to notice that f computes the winding number around w of the curve γ, but this is 0 if γ does not contain w or is 1 if w is inside γ. So, if the real numbers inside γ are the interval I, then $f = \chi_I$. So

$$P_\Omega(k) = f(H(k)) = \chi_I(H(k)) = \chi_\Omega(H(k))$$

The regularity is immediate once we observe that the integral form allows every derivative to be moved inside the integral. □

As an easy corollary to this proposition, we find that the rank is fixed because $\mathrm{tr}(P(k)) = \dim \mathrm{Ran}(P(k)) \in \mathbb{N}$ is a continuous map towards a discrete set. Finally, it is immediate to prove that this family is once again $\tau-$covariant:

$$P(k+\lambda^*) = \tau_{\lambda^*} P(k) \tau_{\lambda^*}^{-1}.$$

The geometry of this family of $\tau-$covariant projectors is very interesting from a physical point of view; however, it is very uncomfortable to work with this pseudoperiodicity. To overcome this problem, we can use the following proposition, firstly proven in [10].

Proposition I.8. *If $\mathbb{W} = \mathbb{R}^d/\Lambda$, there is a unitary-valued map $U : [-1/2, 1/2]^d \to \mathcal{U}(L^2(\mathbb{W}))$ such that:*

$$U(k_1^*, \cdots, 1/2_j, \cdots, k_d^*) = \tau_{\lambda_j^*} U(k_1^*, \cdots, -1/2_j, \cdots, k_d^*) \quad \forall j \in \{1, \cdots, d\}. \tag{I.1}$$

where $k_1^*, \cdots, k_d^*$ are the coordinates of k in a lattice basis $\lambda_1^*, \cdots \lambda_d^*$ of Λ^*.

Proof. To prove it, we need to apply the spectral theorem to the unitary operators $\tau_{\lambda_1^*}, \cdots, \tau_{\lambda_d^*}$. This allows us to choose a family of self-adjoint operators (possibly unbounded) $L_1, \cdots, L_d$ such that $\tau_{\lambda_j^*} = e^{iL_j}$. Since the operators $\tau_{\lambda_j^*}$ commute with each other, it is possible to choose L_j such that they also commute with each other. Then we can define

$$U(k_1^*, \cdots, k_d^*) = \prod_{l=1}^{d} e^{ik_l^* L_l}$$

which clearly satisfies the requested property. □

Using this proposition, after we select a lattice basis $\lambda_1^*, \cdots \lambda_d^*$ of Λ^*, we can define

$$\tilde{P}(k) = U(k_1^*, \cdots, k_d^*) P(k) U(k_1^*, \cdots, k_d^*)^{-1}.$$

Where $k_1^*, \cdots, k_d^*$ are the coordinates of k on the chosen lattice basis for k such that $k_j \in [-1/2, 1/2]$ for all $j \in \{1, \cdots, d\}$. It is immediate to notice that this projection-valued map is now Λ^*-periodic in k.

Instead, for a discrete configuration space X, it is very convenient to use the Fourier transform to simplify the study of periodic models because in this framework it allows to avoid the $\tau-$covariance. If the configuration space is a discrete set $X \subset \mathbb{R}^d$ such that there is a lattice $\Lambda \simeq \mathbb{Z}^d$ with $X = X + \lambda$ for all $\lambda \in \Lambda$ and the quotient X/Λ is finite, then we can find a finite number of points $\{x_1, \cdots, x_n\} = \mathbb{W} \subset X$ with $X = \mathbb{W} + \Lambda$. This means that

$$L^2(X) = L^2(\mathbb{W} \times \Lambda) = L^2(\mathbb{W}) \otimes L^2(\Lambda).$$

If we apply the Fourier transform to the second leg of the tensor product, since $\mathbb{R}^d/\Lambda \simeq \mathbb{T}^d$, we get:

$$L^2(X) \simeq L^2(\mathbb{W}) \otimes L^2(\mathbb{T}^d) \simeq \mathbb{C}^n \otimes L^2(\mathbb{T}^d) \simeq L^2(\mathbb{T}^d, dk, \mathbb{C}^n)$$

To be explicit, consider the canonical basis $\{g_j\}_{j\in\{1,\cdots,d\}}$ of $\mathbb{C}^d$ and the basis $\{y_a \otimes \delta_\lambda\}_{a\in\{1,\cdots,n\},\lambda\in\Lambda}$ of $L^2(X)$ such that $y_a \otimes \delta_\lambda(x_i + \lambda') = \delta_{a,i}\delta_{\lambda,\lambda'}$. Then this version of the Fourier transform is a unitary operator because the functions $f_n(k) = e^{in\cdot k}/(2\pi)^{d/2}$ for $n \in \mathbb{Z}^d$ form a basis of $L^2(\mathbb{T}^d, dk, \mathbb{C})$. This is a well-known fact, but a detailed discussion on the Fourier basis as well as the proof of this theorem can be found in the second section of the third chapter of [48]. To be precise $\mathcal{F}$ acts as:

$$\begin{array}{rrcl} \mathcal{F}: & L^2(X) & \to & L^2(\mathbb{T}^d, dk, \mathbb{C}^n) \\ & y_a \otimes \delta_\lambda & \mapsto & [e^{i\lambda\cdot k} g_a]/(2\pi)^{d/2} \\ & f & \mapsto & \left[\sum_{\lambda\in\Lambda} e^{i\lambda\cdot k}(f(x_1+\lambda_1,\cdots,x_d+\lambda_d))\right]/(2\pi)^{d/2} \end{array}$$

An Hamiltonian H acting on $L^2(X)$ will be characterized by a family of matrices $H_{\lambda,\lambda'} \in M_n(\mathbb{C})$ with components $[H_{\lambda,\lambda'}]_{b,a} = \langle y_a \otimes \delta_\lambda, H(y_b \otimes \delta_{\lambda'})\rangle$. If the Hamiltonian commutes with the translations, it is obvious that $H_{\lambda,\lambda'} = H_{\lambda-\lambda',0} = H_\mu$ for $\mu \in \Lambda$. This implies that

$$H(y_b \otimes \delta_\lambda) = \sum_{\mu\in\Lambda}\left(\sum_{a=1}^{n} [H_\mu]_{b,a}\, y_a\right) \otimes \delta_{\lambda+\mu} \quad \forall b \in \{1,\cdots,n\}, \lambda \in \Lambda.$$

So, if we want to compute $\mathcal{F}H\mathcal{F}^{-1}([e^{i\lambda\cdot k} g_a]/(2\pi)^{d/2})$ we get:

$$\begin{aligned} \mathcal{F}H(y_b \otimes \delta_\lambda) &= \mathcal{F}\left(\sum_{\mu\in\Lambda}\sum_{a=1}^{n}[H_\mu]_{b,a} y_a \otimes \delta_{\lambda+\mu}\right) = \\ &= \sum_{\mu\in\Lambda}\sum_{a=1}^{n}[H_\mu]_{b,a}\left[e^{i(\lambda+\mu)\cdot k} g_a\right]/(2\pi)^{d/2} = \\ &= \sum_{a=1}^{n}\left[\sum_{\mu\in\Lambda} H_\mu e^{i\mu\cdot k}\right]_{b,a}\left[e^{i\lambda\cdot k} g_a\right]/(2\pi)^{d/2} \end{aligned}$$

Having enough decay over $||H_\mu||$ as $\mu \to \infty$ exactly means that the operator $\mathcal{F}H\mathcal{F}^{-1}$ admits a fiber decomposition with fibers

$$H(k) = \sum_{\mu\in\Lambda} H_\mu e^{i\mu\cdot k} \quad \text{acting on} \quad \mathbb{C}^n \simeq L^2(\mathbb{W}).$$

Therefore the spectral properties of H can be recovered using Theorems I.5 and I.6. Then we can define a projector-valued map associated with a Bloch band Ω using Proposition I.7 as:

$$P_\Omega(k) = \frac{i}{2\pi}\int_\gamma (H(k) - z\mathbb{1})^{-1}\, dz \in M_{n,n}(\mathbb{C}).$$

Fortunately, this time this family is already Λ^*−periodic in k, so there are no modifications needed.

Since $\mathbb{R}^d/\Lambda^* \simeq \mathbb{T}^d$ we end up with a smooth family of projection-valued map with the dimension of the rank fixed: $P : \mathbb{T}^d \to \mathrm{Proj}_n(L^2(\mathbb{W}))$ where

$$\mathrm{Proj}_n(\mathcal{H}) = \{P \in \mathcal{L}(\mathcal{H}) | P^2 = P, P^* = P \text{ and } \mathrm{Tr}(P) = n\}.$$

In both continuum and discrete settings we end up with a continuous and periodic projection-valued map $P : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ and the topological properties of this map are the ultimate goal of this thesis.

Remark I.9 (Role of unitary equivalences). We can now point out an interesting fact that partially motivated our study: in the discrete models we selected a discrete Wigner-Seitz cell $\mathbb{W} = \{x_1, \cdots, x_n\}$ but this choice was not unique, even the order of the points in the cell could theoretically impact the topological properties of the model. So, it is crucial to check whether a different choice (*dimerization*) could interfere with the geometrical properties of $P(k)$. If we consider another Wigner-Seitz cell $\mathbb{W}' = \{x'_1, \cdots, x'_n\}$, then we can replicate the step done previously to define another version of the Fourier transform $\mathcal{F}'$. This will only be different in the fact that the function $[e^{i\lambda\cdot k} g_a]/(2\pi)^{d/2}$ is now reached by the function with value 1 in $x'_a + \lambda$ and 0 elsewhere instead of the function with value 1 in $x_a + \lambda$ and 0 elsewhere. So we can get two different fiber operators: $\mathcal{F}H\mathcal{F}^{-1} = \int_{\mathbb{T}^d} H(k)dk$ and $\mathcal{F}'H(\mathcal{F}')^{-1} = \int_{\mathbb{T}^d} H'(k)dk$. However the two fiber operators are unitary equivalent thanks to:

$$(\mathcal{F}'\mathcal{F}^{-1}) \left(\int_{\mathbb{T}^d} H(k)dk \right) (\mathcal{F}'\mathcal{F}^{-1})^{-1} = \int_{\mathbb{T}^d} H(k)'dk$$

Moreover it is immediate to check that $\mathcal{F}'\mathcal{F}^{-1}$ is once again a fiber operator since $\mathcal{F}'\mathcal{F}^{-1}$ commutes with the translations, so there is a unitary-valued map $U(k)$ with:

$$\mathcal{F}'\mathcal{F}^{-1} = \int_{\mathbb{T}^d} U(k)dk \Rightarrow H(k) = U(k)H'(k)U(k)^{-1}.$$

Clearly this unitary equivalence gets inherited by the projector, so $P(k) = U(k)P(k)'U(k)^{-1}$. This tells us that every geometrical property of the projector must be independent under unitary equivalences, or it will depend on the choice of a dimerization, and in case it depends, it is crucial to study how.

However, unitary equivalences are also useful in continuum models: in fact in order to obtain full periodicity in k we conjugated $P(k)$ via a unitary operator $U(k)$ such that $\tilde{P}(k) = U(k)P(k)U(k)^{-1}$ is Λ^*-periodic. With another choice $U'(k)$ we would have obtained

$$\tilde{P}(k)' = [U'(k)U(k)^{-1}]\tilde{P}(k)[U'(k)U(k)^{-1}]^{-1}$$

Since $U(k + \lambda_j^*) = \tau_{\lambda_j^*} U(k)$ and same holds for $U'(k)$, we have that $U'(k)U(k)^{-1}$ is Λ^*-periodic in k, so the price we must pay in order to work with full periodicity is to study everything up to unitary equivalences.

Before we move on, it is important to note that dimerization does not play any role in continuum models. In fact, a different connected Wigner-Seitz cell $\mathbb{W}'$ can be moved smoothly to the original Wigner-Seitz cell $\mathbb{W}$, which means that all geometric properties of continuum models are reasonably stable up to any choice of dimerization.

Remark I.10 (Continuous vs Analytic). Since the family of projectors $P(k)_{k\in\mathbb{R}^d}$ that appears in physical models is usually analytic in k, one may ask whether real analytic Bloch frames can be constructed for it. Arguing as in Lemma 2.3 in [10], one can indeed show that whenever a continuous Bloch frame exists, it can be easily modified into a real analytic one; the procedure preserves the symmetries that will be introduced in the following section (periodicity, time reversal symmetry, particle-hole symmetry and chiral symmetry) whenever they are required. So, from now on, we are only interested in finding continuous frames of projection-valued maps.

However, in order to talk about the different topological classes, we need to introduce the main protagonists of this work.

I.3 From symmetrical Hamiltonian to Symmetrical PVM

The three fundamental symmetries considered in the literature on topological phases of matter.

- *time reversal symmetry (TRS)*: An anti-unitary operator Θ that squares to plus or minus identity and commutes with the Hamiltonian. If $\Theta^2 = \mathbb{1}$ it is called bosonic time-reversal symmetry while if $\Theta^2 = -\mathbb{1}$ it is called fermionic time-reversal symmetry
- *particle-hole symmetry (PHS)*: An anti-unitary operator Ξ (often called *Charge-conjugation symmetry*) that squares to plus or minus identity and anti-commutes with the Hamiltonian. If $\Xi^2 = \mathbb{1}$ it is called even particle-hole symmetry while if $\Xi^2 = -\mathbb{1}$ it is called odd particle-hole symmetry. This particle-hole symmetry relates negative energy states ("holes") to positive energy states ("particles"), in analogy to Dirac's view of matter.
- *chiral symmetry (CS)*: A unitary operator Π that squares to the identity and anti-commutes with the Hamiltonian. This chiral symmetry symmetry usually represents a sublattice of periodicity in the configuration space or the composition of a time-reversal symmetry with a chiral symmetry provided that they commute or anti-commute.

Clearly, we expect that any symmetry present should be compatible with the previous construction. This is possible under the reasonable assumption that the symmetries considered must always commute with all the translation operators T_λ. If this happens in the continuous case, we can use Proposition I.2 to obtain a fiber decomposition of symmetries that act linearly, like the chiral symmetry. However, for the symmetries that act anti-linearly, we need the following corollary.

Corollary I.11. *If A is an anti-linear operator acting on $L^2(\mathbb{R}^d)$, if it commutes with the translation operators $T_\lambda(\phi)(x) = \phi(x-\lambda)$ and if $\tau_\lambda(\phi)(y) = e^{i\lambda\cdot y}\phi(y)$, then there is a family of anti-linear operators $A(k)$ acting on the space of $\Lambda-$periodic functions $L^2_{per}(\mathbb{R}^d)$ such that for all $\psi \in L^2_{ps.p}(\mathbb{R}^d \times \mathbb{R}^d)$:*

$$\mathcal{U}_{\mathrm{BFZ}} A \mathcal{U}_{\mathrm{BFZ}}^{-1} \psi(k,y) = [A(k)\psi(-k,\cdot)](y)$$

that are also $\tau-$covariant in k:

$$A(k+\lambda^*) = \tau_{\lambda^*} A(k) \tau_{\lambda^*}.$$

Proof. The proof is almost identical to the proof of Proposition I.2, the only difference is that we need to define $A_k = Z_k A Z_k$, in this case $A(k)$ will be its restriction over $L^2_{per}(\mathbb{R}^d)$. In this way, we obtain:

$$\begin{aligned}
[\mathcal{U}_{\mathrm{BFZ}} A(\phi)](k,y) &= \sum_{\lambda\in\Lambda} e^{-ik\cdot(\lambda-y)}[(T_\lambda A)\phi](y) = \sum_{\lambda\in\Lambda} e^{-ik\cdot\lambda}[(Z_k A T_\lambda)\phi](y) = \\
&= \sum_{\lambda\in\Lambda} e^{-ik\cdot\lambda}[(A_k Z_{-k} T_\lambda)\phi](y) = A(k)\left[\sum_{\lambda\in\Lambda} e^{ik\cdot(\lambda-y)}(T_\lambda\phi)(y)\right] = \\
&= A(k)(\mathcal{U}_{\mathrm{BFZ}}\phi)(-k,y)
\end{aligned}$$

The $\tau-$covariance is easy to prove since:

$$A_{k+\lambda^*} = Z_{k+\lambda^*} A Z_{k+\lambda^*} = Z_k Z_{\lambda^*} A Z_k Z_{\lambda^*} = Z_{\lambda^*} A(k) Z_{\lambda^*}$$

□

The second assumption we make is that any symmetries present must have constant fibers in k. This condition is present in the majority of physical models and allows us to treat easily our problem.

Now we will work separately on each symmetry to understand how they interact with the fiber decomposition, the Riesz formula, and with the $\tau-$covariance, starting from the time reversal symmetry.

Suppose now that the original Hamiltonian H acting on $L^2(\mathbb{R}^d)$ commutes with an anti-unitary operator Θ with $[\Theta, T_\lambda] = 0$ for all $\lambda \in \Lambda$. We can use Corollary I.11 to obtain the fiber decomposition $\Theta(k)$ of the symmetry. It is immediate to notice that each $\Theta(k)$ is an anti-unitary operator acting on $L^2_{per}(\mathbb{R}^d)$ and $[H, \Theta] = 0$ implies that for all $\psi \in L^2_{ps.p}(\mathbb{R}^d \times \mathbb{R}^d)$:

$$[\mathcal{U}_{\text{BFZ}} \Theta H \mathcal{U}_{\text{BFZ}}^{-1} \psi](k, y) = [\mathcal{U}_{\text{BFZ}} H \Theta \mathcal{U}_{\text{BFZ}}^{-1} \psi](k, y) \Rightarrow$$

$$\Rightarrow [\mathcal{U}_{\text{BFZ}} \Theta \mathcal{U}_{\text{BFZ}}^{-1} \mathcal{U}_{\text{BFZ}} H \mathcal{U}_{\text{BFZ}}^{-1} \psi](k, y) = [\mathcal{U}_{\text{BFZ}} H \mathcal{U}_{\text{BFZ}}^{-1} \mathcal{U}_{\text{BFZ}} \Theta \mathcal{U}_{\text{BFZ}}^{-1} \psi](k, y) \Rightarrow$$

$$[\Theta(k) H(-k) \psi(-k, \cdot)](y) = [H(k) \Theta(k) \psi(-k, \cdot)](y).$$

With the additional assumption that the fibers of the symmetry are constant $\Theta(k) \equiv T$ for an anti-unitary operator T acting on $L^2_{per}(\mathbb{R}^d)$, we obtain the crucial relation:

$$H(k)T = TH(-k).$$

This relation survives after the Riesz formula because:

$$\begin{aligned} P_\Omega(k) &= \frac{i}{2\pi} \int_\gamma (H(k) - z\mathbb{1})^{-1} dz = \frac{i}{2\pi} \int_\gamma \left(TH(-k)T^{-1} - T(\overline{z}\mathbb{1})T^{-1}\right)^{-1} dz \\ &= \frac{i}{2\pi} \int_\gamma T(H(-k) - \overline{z}\mathbb{1})^{-1} T^{-1} dz = T \left[\frac{-i}{2\pi} \int_\gamma (H(-k) - \overline{z}\mathbb{1})^{-1} dz \right] T^{-1} = \\ &= T \left[\frac{i}{2\pi} \int_{\overline{\gamma}} (H(-k) - w\mathbb{1})^{-1} dw \right] T^{-1} = TP_\Omega(-k)T^{-1} \end{aligned}$$

So to obtain the object we want to study in this chapter; we only need a way to get rid of the $\tau-$covariance. This can be done using Proposition I.8 with an additional consideration. In the proof of Proposition I.8 we considered the unitary operators $\tau_{\lambda_j^*}(\phi)(y) = e^{i\lambda_j^* \cdot y}\phi(y)$ acting on $L^2(\mathbb{W})$, using the spectral theorem for unitary operators, we saw that there were d self-adjoint operators $L_1, \cdots L_d$ such that $\tau_{\lambda_j^*} = e^{iL_j}$. Since they all commute with each other, we can choose them so that $[L_a, L_b] = 0$ for all $a, b = 1, \cdots, d$. Moreover, the τ-covariance of $\Theta(k)$ implies that $\Theta(k + \lambda_j^*) = \tau_{\lambda_j^*}\Theta(k)\tau_{\lambda_j^*}$, so if the fibers are constant, $Te^{iL_j} = e^{-iL_j}T$ and by functional calculus it is possible to choose L_j such that $TL_j = L_jT$, so that $Te^{iL_jk_j} = e^{-iL_jk_j}T$ $\forall k_j \in \mathbb{R}$. With this small constraint, the unitary-valued map $U : [-1/2, 1/2]^d \to \mathcal{U}(L^2(\mathbb{W}))$ defined as:

$$U(k_1, \cdots, k_d) = \prod_{j=1}^{d} e^{iL_j k_j}$$

is such that $TU(k) = U(-k)T$. Finally, if $k_1^*, \cdots, k_d^*$ are the coordinates of k in the lattice basis $\lambda_1^*, \cdots, \lambda_d^*$ of Λ, we can define:

$$\tilde{P}_\Omega(k) = U(k_1^*, \cdots, k_d^*) P_\Omega(k) U(k_1^*, \cdots, k_d^*)^{-1}$$

and it is immediate to check that this projection-valued map is now Λ^*-periodic in k and still satisfies the symmetry constraint $\tilde{P}_\Omega(k)T = T\tilde{P}_\Omega(-k)$. However, the choices of $L_1, \cdots, L_d$ were not unique! So in general if there is another unitary-valued map $U' : [-1/2, 1/2]^d \to \mathcal{U}(\mathcal{H})$ with $\hat{P}_\Omega(k) = U'(k)P_\Omega(k)U'(k)^{-1}$ still periodic and symmetric with $TU'(k) = U'(-k)T$, then $\tilde{P}_\Omega$ and $\hat{P}_\Omega$ will be intertwined by the (periodic) unitary-valued map $U'(k)U(k)^{-1}$ according to the symmetric relation

$$\hat{P}_\Omega(k) = U'(k)U(k)^{-1}\tilde{P}_\Omega(k) \left(U'(k)U(k)^{-1}\right).$$

However, this time an additional symmetric relation appears because it must be true that

$$TU'(k)U(k)^{-1} = U'(-k)U(-k)^{-1}T \quad \text{for all} \quad k \in \mathbb{T}^d.$$

Instead, if we are dealing with a discrete Hamiltonian H acting on $L^2(X)$ that commutes with an anti-unitary operator Θ squaring to the identity. We can replicate the procedure performed before to create a periodic projection-valued map $P_\Omega(k)$ and characterize the action of Θ using the matrices $\Theta_{\lambda,\lambda'}$ with components $[\Theta_{\lambda,\lambda'}]_{a,b} = \langle y_a \otimes \delta_\lambda, \Theta(y_b \otimes \delta_{\lambda'})\rangle$. Again we have that if $[\Theta, T_\lambda] = 0$ for all $\lambda \in \Lambda$, then $\Theta_{\lambda,\lambda'} = \Theta_{\lambda-\lambda',0} = \Theta_\mu$ for $\mu \in \Lambda$ and Θ is decomposable with

$$\mathcal{F}\Theta\mathcal{F}^{-1} = \Theta(k) = \sum_{\mu\in\Lambda} \Theta_\mu e^{i\mu\cdot k}\mathcal{K}$$

With computations that are identical to those made in the continuum setting, if the fibers are constant in k, namely $\Theta(k) \equiv T$, then the projection-valued map P_Ω satisfies the symmetry constraint:

$$P_\Omega(k)T = TP_\Omega(-k)$$

Once again we must check how a different choice of dimerization could change our study. If we had used a different version of the Fourier transform $\mathcal{F}'$ we would still have a unitary equivalence

$$P'(k) = U(k)P(k)U(k)^{-1}$$

where $U(k)$ are the fibers of $(\mathcal{F}'\mathcal{F}^{-1})$. However, the two models can be related if and only if the symmetry has the same form in both models, which means that it is reasonable to impose:

$$\mathcal{F}\Theta\mathcal{F}^{-1} = \mathcal{F}'\Theta(\mathcal{F}')^{-1} \Leftrightarrow$$

$$\Leftrightarrow T = \mathcal{F}'\mathcal{F}^{-1}T(\mathcal{F}'\mathcal{F}^{-1})^{-1} \Leftrightarrow$$

$$\Leftrightarrow T = U(k)TU(-k)^{-1}.$$

This justifies the object we want to study in case a time reversal symmetry is present.

Definition I.12 (*time-reversal symmetric projection-valued map*)**.** Given an anti-unitary operator T, representing a Time-reversal symmetry, acting on $\mathcal{H}$ with $T^2 = \pm\mathbb{1}$, then a projection-valued map $P : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ is time-reversal symmetric if

$$TP(k) = P(-k)T.$$

Moreover, if a Time-reversal symmetry is present, the unitary equivalences must always satisfy the symmetry constraint:

$$TU(k) = U(-k)T.$$

Instead, if a particle-hole symmetry is present, namely the original Hamiltonian H acting on $L^2(\mathbb{R}^d)$ anti-commutes with an anti-unitary operator Ξ with $\Xi^2 = \pm\mathbb{1}$. Then, under the assumption that $[\Xi, T_\lambda] = 0$ for all $\lambda \in \Lambda$, we can use Corollary I.11 to obtain a fiber decomposition $\Xi(k)$ of the symmetry. It is immediate to notice that each $\Xi(k)$ is an anti-unitary operator acting on $L^2_{per}(\mathbb{R}^d)$ and $\{H, \Xi\} = 0$ implies that for all $\psi \in L^2_{ps.p}(\mathbb{R}^d \times \mathbb{R}^d)$:

$$[\mathcal{U}_{\mathrm{BFZ}}\Xi H\mathcal{U}_{\mathrm{BFZ}}^{-1}\psi](k,y) = -[\mathcal{U}_{\mathrm{BFZ}}H\Xi\mathcal{U}_{\mathrm{BFZ}}^{-1}\psi](k,y) \Rightarrow$$

$$\Rightarrow [\mathcal{U}_{\mathrm{BFZ}}\Xi\mathcal{U}_{\mathrm{BFZ}}^{-1}\mathcal{U}_{\mathrm{BFZ}}H\mathcal{U}_{\mathrm{BFZ}}^{-1}\psi](k,y) = -[\mathcal{U}_{\mathrm{BFZ}}H\mathcal{U}_{\mathrm{BFZ}}^{-1}\mathcal{U}_{\mathrm{BFZ}}\Xi\mathcal{U}_{\mathrm{BFZ}}^{-1}\psi](k,y) \Rightarrow$$

$$[\Xi(k)H(-k)\psi(-k,\cdot)](y) = [-H(k)\Xi(k)\psi(-k,\cdot)](y) \Rightarrow$$

Under the assumption that the fibers of the symmetry are constant $\Xi(k) \equiv C$ for an anti-unitary operator C acting on $L^2_{per}(\mathbb{R}^d)$, we obtain the crucial relation:

$$H(k)C = -CH(-k).$$

In this case something interesting happens after we perform the Riesz formula. If we select a Bloch band Ω such that $0 \in \Omega$, then the relation $H\Xi = -\Xi H$ tells us that the spectrum of H must be symmetric around zero, so it is true that $\Omega = -\Omega$. Then we can choose a complex curve γ around Ω with

$$\begin{aligned}
P_\Omega(k) &= \frac{i}{2\pi}\int_\gamma (H(k) - z\mathbb{1})^{-1}dz = \frac{i}{2\pi}\int_\gamma (-C^{-1}H(-k)C - C^{-1}(\overline{z}\mathbb{1})C)^{-1}dz \\
&= \frac{i}{2\pi}\int_\gamma C^{-1}(-H(-k) - \overline{z}\mathbb{1})^{-1}Cdz = C^{-1}\left[\frac{-i}{2\pi}\int_\gamma (-H(-k) - \overline{z}\mathbb{1})^{-1}dz\right]C = \\
&= C^{-1}\left[\frac{i}{2\pi}\int_{\overline{\gamma}} (-H(-k) - w\mathbb{1})^{-1}dw\right]C = C^{-1}\chi_\Omega(-H(-k))C = \\
&= C^{-1}\chi_{-\Omega}(H(-k))C = C^{-1}\chi_\Omega(H(-k))C = C^{-1}P_\Omega(-k)C
\end{aligned}$$

But this is exactly the relation that a time-reversal symmetric projection-valued map satisfies, so, despite having different names, the topology we are interested in will not differ.

Instead, if the Bloch band does not contain zero, the symmetry tells us that the negative counterpart $-\Omega$ is another Bloch band. So, if γ is a complex curve around Ω, we obtain the relation that characterizes this class:

$$\begin{aligned}
P_\Omega(k) &= \frac{i}{2\pi}\int_\gamma (H(k)-z\mathbb{1})^{-1}dz = \frac{i}{2\pi}\int_\gamma (-C^{-1}H(-k)C - C(\overline{z}\mathbb{1})C)^{-1}dz \\
&= \frac{i}{2\pi}\int_\gamma C^{-1}(-H(-k)-\overline{z}\mathbb{1})^{-1}Cdz = C^{-1}\left[\frac{-i}{2\pi}\int_\gamma (-H(-k)-\overline{z}\mathbb{1})^{-1}dz\right]C = \\
&= C^{-1}\left[\frac{i}{2\pi}\int_{\overline{\gamma}} (-H(-k)-w\mathbb{1})^{-1}dw\right]C = C^{-1}\chi_\Omega(-H(-k))C = \\
&= C^{-1}\chi_{-\Omega}(H(-k))C = C^{-1}P_{-\Omega}(-k)C
\end{aligned}$$

Luckily, we can eliminate the $\tau-$covariance exactly the same way as we did in case a time reversal symmetry is present. The price we need to pay is to study projection-valued maps up to unitary equivalences $U:\mathbb{T}^d\to\mathcal{U}(\mathcal{H})$ such that $CU(k)=U(-k)C$.

Instead, if we are dealing with a discrete Hamiltonian H acting on $L^2(X)$ that anti-commutes with an anti-unitary operator Ξ squaring to $\pm\mathbb{1}$, then we can replicate the procedures done before to create a pair of periodic projection-valued maps $P_\Omega(k)$ and $P_{-\Omega}(k)$. Then, if the particle hole symmetry has constant fibers $\Xi(k)\equiv C$, which is an anti-unitary operator acting on $\mathcal{H}\simeq\mathbb{C}^m$ with $C^2=\mathbb{1}$, we obtain once again the crucial relation of this class $CP_\Omega(k)=P_{-\Omega}(k)C$. Once again we must check how a different choice of dimerization could change our study. If we had used a different version of the Fourier transform $\mathcal{F}'$ we would still have a unitary equivalence with:

$$P'_\Omega(k)=U(k)P_\Omega(k)U(k)^{-1},\quad P'_{-\Omega}(k)=U(k)P_{-\Omega}(k)U(k)^{-1}$$

where $U(k)$ are the fibers of $(\mathcal{F}'\mathcal{F}^{-1})$. However, the two models can be related if and only if the symmetry has the same form in both models, which means that it is reasonable to impose:

$$\mathcal{F}\Xi\mathcal{F}^{-1}=\mathcal{F}'\Xi(\mathcal{F}')^{-1}\Leftrightarrow C=\mathcal{F}'\mathcal{F}^{-1}C(\mathcal{F}'\mathcal{F}^{-1})^{-1}\Leftrightarrow C=U(k)CU(-k)^{-1}.$$

This justifies the object we want when a particle-hole symmetry is present.

Definition I.13 (particle-hole symmetric pair of projection-valued maps)**.** Given an anti-unitary operator C acting on $\mathcal{H}$ with $C^2=\pm\mathbb{1}$, a pair of projector-valued maps $P^\pm:\mathbb{T}^d\to\mathrm{Proj}_n(\mathcal{H})$ is particle-hole symmetric if and only if:

$$CP^+(k)=P^-(-k)C\quad\text{and}\quad P^+(k)P^-(k)=0\quad\forall k\in\mathbb{T}^d.$$

Moreover, if a Particle-hole symmetry is present, the unitary equivalences must always satisfy the symmetry constraint:

$$CU(k)=U(-k)C.$$

Finally if a chiral symmetry is present, namely the original Hamiltonian H acting on $L^2(\mathbb{R}^d)$ anti-commutes with a unitary operator Π with $\Pi^2=\mathbb{1}$. Then, under the assumption that $[\Pi,T_\lambda]=0$ for all $\lambda\in\Lambda$, we can apply Proposition I.2 to obtain a fiber decomposition $\Pi(k)$ of the symmetry. It is immediate to notice that each $\Pi(k)$ is a unitary operator acting on $L^2_{per}(\mathbb{R}^d)$ and $\{H,\Pi\}=0$

implies that for all $\psi \in L^2_{ps.p}(\mathbb{R}^d \times \mathbb{R}^d)$:

$$[\mathcal{U}_{\text{BFZ}} \Pi H \mathcal{U}_{\text{BFZ}}^{-1} \psi](k,y) = -[\mathcal{U}_{\text{BFZ}} H \Pi \mathcal{U}_{\text{BFZ}}^{-1} \psi](k,y) \Rightarrow$$

$$\Rightarrow [\mathcal{U}_{\text{BFZ}} \Pi \mathcal{U}_{\text{BFZ}}^{-1} \mathcal{U}_{\text{BFZ}} H \mathcal{U}_{\text{BFZ}}^{-1} \psi](k,y) = -[\mathcal{U}_{\text{BFZ}} H \mathcal{U}_{\text{BFZ}}^{-1} \mathcal{U}_{\text{BFZ}} \Pi \mathcal{U}_{\text{BFZ}}^{-1} \psi](k,y) \Rightarrow$$

$$[\Pi(k) H(k) \psi(k,\cdot)](y) = [-H(k)\Pi(k)\psi(k,\cdot)](y)$$

As usual, we will suppose that the fibers of the symmetry are constant $\Pi(k) \equiv S$ for a unitary operator S acting on $L^2_{per}(\mathbb{R}^d)$. So we obtain the crucial relation:

$$H(k)S = -SH(k).$$

Now we want to study what happens after we perform the Riesz formula. If we selected a Bloch band Ω such that $0 \in \Omega$, then the relation $H\Pi = -\Pi H$ tells us that the spectrum of H must be symmetric around zero, so it is true that $\Omega = -\Omega$. Then we can choose a complex curve γ around Ω with

$$\begin{aligned} P_\Omega(k) &= \frac{i}{2\pi} \int_\gamma (H(k) - z\mathbb{1})^{-1} dz = \frac{i}{2\pi} \int_\gamma (-SH(k)S - S(z\mathbb{1})S)^{-1} dz \\ &= \frac{i}{2\pi} \int_\gamma S^{-1}(-H(k) - z\mathbb{1})^{-1} S^{-1} dz = S \left[\frac{-i}{2\pi} \int_\gamma (-H(k) - z\mathbb{1})^{-1} dz \right] S = \\ &= S \left[\frac{i}{2\pi} \int_\gamma (-H(k) - z\mathbb{1})^{-1} dz \right] S = S\chi_\Omega(-H(k))S = \\ &= S\chi_{-\Omega}(H(k))S = S\chi_\Omega(H(k))S = SP_\Omega(k)S \end{aligned}$$

Instead, if the Bloch band does not contain zero, the symmetry tells us that the negative counterpart $-\Omega$ is another Bloch band. So, if γ is a complex curve around Ω, we obtain another relation:

$$\begin{aligned} P_\Omega(k) &= \frac{i}{2\pi} \int_\gamma (H(k) - z\mathbb{1})^{-1} dz = \frac{i}{2\pi} \int_\gamma (-SH(k)S - S(z\mathbb{1})S)^{-1} dz \\ &= \frac{i}{2\pi} \int_\gamma S^{-1}(-H(k) - z\mathbb{1})^{-1} S^{-1} dz = S \left[\frac{-i}{2\pi} \int_\gamma (-H(k) - z\mathbb{1})^{-1} dz \right] S = \\ &= S \left[\frac{i}{2\pi} \int_\gamma (-H(k) - z\mathbb{1})^{-1} dz \right] S = S\chi_\Omega(-H(k))s = \\ &= S\chi_{-\Omega}(H(k))S = SP_{-\Omega}(k)S \end{aligned}$$

In order to eliminate the $\tau-$covariance, we notice that $\Pi(k)$ must obey the $\tau-$covariance rule $\Pi(k+\lambda_j^*) = \tau_{\lambda_j^*}\Pi(k)\tau_{\lambda_j^*}^{-1}$. So, if the fibers are constant equal to S, we have the commutation relation $S\tau_{\lambda_j^*} = \tau_{\lambda_j^*}S$. This means that we can use Proposition I.8 with a choice of the self-adjoint operators L_j such that $[L_j, S] = 0 \ \forall j \in \{1, \cdots, d\}$. In this way the unitary-valued map $U(k)$ is such that $[U(k), S] \equiv 0$. Finally, if $k_1^*, \cdots, k_d^*$ are the coordinates of k in the lattice basis $\lambda_1^*, \cdots, \lambda_d^*$ of Λ, we can define:

$$\tilde{P}_\Omega(k) = U(k_1^*, \cdots, k_d^*) P_\Omega(k) U(k_1^*, \cdots, k_d^*)^{-1}$$

and it is immediate to check that this projector-valued map is now Λ^*-periodic in k and still satisfies the symmetry constraints $\tilde{P}_\Omega(k)S = S\tilde{P}_\Omega(k)$ or $\tilde{P}_\Omega(k)S = S\tilde{P}_{-\Omega}(k)$. Once again, the choices of $L_1, \cdots, L_d$ were not unique! So in general if there is another unitary-valued map $U' : [-1/2, 1/2]^d$ with $\hat{P}_{\pm\Omega}(k) = U'(k)P_{\pm\Omega}(k)U'(k)^{-1}$ still periodic and symmetric with $SU'(k) = U'(-k)S$. Then $\tilde{P}_{\pm\Omega}$ and $\hat{P}_{\pm\Omega}$ will be intertwined by the following relation:

$$\hat{P}_{\pm\Omega}(k) = U'(k)U(k)^{-1}\tilde{P}_{\pm\Omega}(k)\left(U'(k)U(k)^{-1}\right)^{-1}$$

where the unitary-valued map $U'(k)U(k)^{-1}$ is actually periodic. However, this time an additional symmetric relation appears because it must be true that

$$SU'(k)U(k)^{-1} = U'(k)U(k)^{-1}S \quad \text{for all} \quad k \in \mathbb{T}^d$$

.

As done previously, if we are dealing with a discrete Hamiltonian H acting on $L^2(X)$ that anti-commutes with a unitary operator Π squaring to the identity. Then we can create a periodic projection-valued map $P_\Omega(k)$, if $0 \in \Omega$, or a pair of periodic projection-valued map $P_\Omega(k)$ and $P_{-\Omega}(k)$ if $0 \notin \Omega$. Then, if the chiral symmetry has constant fibers $\Pi(k) \equiv S$, which is a unitary operator acting on $\mathcal{H} \simeq \mathbb{C}^m$ with $S^2 = \mathbb{1}$, we obtain once again the crucial relations of this class $SP_\Omega(k) = P_\Omega(k)S$ or $SP_\Omega(k) = P_{-\Omega}(k)S$. Once again we must check how a different choice of dimerization could change our study. If we had used a different version of the Fourier transform $\mathcal{F}'$ we would still have a unitary equivalence with:

$$P'_\Omega(k) = U(k)P_\Omega(k)U(k)^{-1}, \quad P'_{-\Omega}(k) = U(k)P_{-\Omega}(k)U(k)^{-1}$$

where $U(k)$ are the fibers of $(\mathcal{F}'\mathcal{F}^{-1})$. However, the two models can be related if and only if the symmetry has the same form in both models, which means that it is reasonable to impose:

$$\mathcal{F}\Pi\mathcal{F}^{-1} = \mathcal{F}'\Pi(\mathcal{F}')^{-1} \Leftrightarrow S = \mathcal{F}'\mathcal{F}^{-1}S(\mathcal{F}'\mathcal{F}^{-1})^{-1} \Leftrightarrow S = U(k)SU(k)^{-1}.$$

.

This justifies the objects we want to study whenever a chiral symmetry is present.

Definition I.14 (chiral-invariant projection-valued map)**.** Given a unitary operator S acting on $\mathcal{H}$ with $S^2 = \mathbb{1}$, a projection-valued map $P : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ is chiral invariant if and only if

$$SP(k) = P(k)S \quad \forall k \in \mathbb{T}^d.$$

Definition I.15 (chiral-symmetric pair of projection-valued maps)**.** Given a unitary operator S acting on $\mathcal{H}$ with $S^2 = \mathbb{1}$, a pair of projection-valued maps $P^\pm : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ is chiral symmetric if and only if:

$$SP^+(k) = P^-(k)S \quad \text{and} \quad P^+(k)P^-(k) = 0 \quad \forall k \in \mathbb{T}^d.$$

To be formal, the goal of this thesis is to study projection-valued maps $P : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$ and for $d = 0, 1, 2$. In particular, we will investigate how close we can get to having a continuous and periodic frame. This is a continuous and periodic collection of orthonormal bases spanning $\mathrm{Ran}(P(k))$. Sometimes it will not be possible to construct a frame, and in those cases we will try to construct pseudoperiodic frames with controlled pseudoperiodic conditions. Using those pseudoperiodic frames, it will be very easy to check whenever there is a unitary equivalence $U(k)$ intertwining $P(k)$ with a different projection-valued map $P'(k)$. Finally, using unitary equivalence, it will be possible to check when two different projection-valued maps P and P' can be connected using a homotopy. We will work in different symmetry classes and we will specify case by case the additional conditions that the symmetries impose to the frame, the unitary equivalences, and the homotopies.

Chapter 1

Class A

In this chapter, we will work in Class A, where there are no symmetries at play, so we are just studying continuous projection-valued maps $P : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$. All of the contents of this chapter are unoriginal results already present in [47], however, they are so useful to study the other symmetry classes that it is worth mentioning them once more. To be precise, given two projection-valued maps $P_0, P_1 : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$, we will try to answer the following questions:

Question 1 (Class A frame)**.** When will there be a continuous frame $\{u_1(k), \cdots, u_n(k)\}$, namely a continuous and periodic collection of orthonormal vectors spanning $\mathrm{Ran}(P_0(k))$? In case there is none, is it possible to have a frame with well-controlled pseudoperiodic conditions? (Like the one present in Theorem 1.10)

Question 2 (Class A unitary equivalence)**.** When will there be a unitary-valued map $U : \mathbb{T}^d \to \mathcal{U}(\mathcal{H})$ such that $P_1(k) = U(k)P_0(k)U(k)^{-1}$ for all $k \in \mathbb{T}^d$?

Question 3 (Class A homotopy)**.** When will there be a continuous map $P : [0,1] \times \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ with $P(0,k) = P_0(k)$ and $P(1,k) = P_1(k)$ for all $k \in \mathbb{T}^d$?

1.1 d=0

Since $\mathbb{T}^0$ is composed of a single point, in this case we just have to study a single projector $P \in \mathrm{Proj}_n(\mathcal{H})$. In this easy setting, the three questions can be answered without issue. Despite being a very simple argument, we want to show the proof, since we will use the same procedure throughout the entire thesis:

Proposition 1.1. *Two projectors $P_0, P_1 \in \mathrm{Proj}_n(\mathcal{H})$ will always have a frame and be unitary equivalent and homotopic.*

Proof. In this setting a frame is just a set of orthonormal vectors $\{v_j^0\}_{j\in\{1,\cdots,n\}}$ spanning $\mathrm{Ran}(P_0)$ and we can find one without any issue. Then we can even complete this set in $\mathrm{Ran}(P_0^\perp)$ to a basis of $\mathcal{H}$ thanks to the fact that $\mathcal{H}$ is separable. We can do the same for P_1 in order to construct $\{v_j^1\}_{j\in\{1,\cdots,n\}}$ and then complete it to a basis of $\mathcal{H}$. Then we can define the unitary equivalence U as the linear operator such that $U(v_j^0) = v_j^1$ for all $0 < j \le \dim(\mathcal{H})$. Then it is obvious that $P_1 = UP_0U^{-1}$. Finally, since $\mathcal{U}(\mathcal{H})$ is always a connected set thanks to Theorems A.3, A.2, we can find a homotopy U_t connecting U with the identity $\mathbb{1}$. Then the projector $P_t = U_tP_0U_t^{-1}$ is a projector-valued homotopy connecting P_0 to P_1. □

1.2 d=1

Things get more interesting when $d = 1$, in this setting, the first step is to find a continuous unitary-valued map $U(k) \in \mathcal{U}(\mathcal{H})$ such that $P(k) = U(k)P(0)U(k)^{-1}$. This can be done using the *Kato-Nagy* construction:

Proposition 1.2 (Class A Kato-Nagy 1-d construction)**.** *If $P : \mathbb{T}^1 \to \mathrm{Proj}_n(\mathcal{H})$ is a continuous projection-valued map, then there is a continuous but not periodic unitary-valued map $U : [0, 2\pi] \to \mathcal{U}(\mathcal{H})$ such that:*

$$P(t) = U(t)P(0)U(t)^{-1}$$

Proof. We need to use a formula developed by Kato and Nagy that appears in [34], telling that if $P, Q \in \mathrm{Proj}_n(\mathcal{H})$ are two projectors with $\|P - Q\| < 1$, then ($\mathbb{1}$ denotes the identity operator)

$$U = [QP + (\mathbb{1} - Q)(\mathbb{1} - P)] \left[\mathbb{1} - (P - Q)^2\right]^{-1/2}$$

is a unitary operator such that $Q = UPU^{-1}$. Now, if we divide $[0, 2\pi]$ into a finite number of intervals $[t_j, t_{j+1}]$ $\forall 0 \leq j \leq J$ with $t_0 = 0, t_J = 2\pi$ such that $\|P(t_j) - P(t)\| < 1/2$ for all $t \in [t_j, t_{j+1}]$, then we can define J unitary-valued maps:

$$U_j(t) = [P(t)P(t_j) + (\mathbb{1} - P(t))(\mathbb{1} - P(t_j))] \left[\mathbb{1} - (P(t_j) - P(t))^2\right]^{-1/2} \quad \forall t \in [t_j, t_{j+1}]$$

Those connect $P(t_j)$ with $P(t)$ for $t \in [t_j, t_{j+1}]$. Then the global unitary we need is:

$$U(t) = U_j(t)U_{j-1}(t_j)U_{j-2}(t_{j-1}) \cdots U_1(t_2)U_0(t_1) \quad \text{for} \quad t \in [t_j, t_{j+1}]$$

and the continuity is immediate because each function is continuous in t since P was continuous and $U_j(t_j) = \mathbb{1}$. □

Lemma 1.3. *If $\mathcal{H}$ is a separable Hilbert space with $\dim(\mathcal{H}) = \infty$ and $P : \mathbb{T}^1 \to \mathrm{Proj}_n(\mathcal{H})$ is a continuous projector-valued map, then there is always a continuous and periodic collection of orthonormal vectors $\{v_j(k)\}_{j\in\mathbb{N}}$ that make up a countable basis of $\ker(P(k))$.*

Proof. Using Proposition 1.2, if we select a countable basis $\{v_j\}_{j\in N}$ of $\ker(P(0))$, then $u_j(t) = U(t)v_j$ constitute a countable basis of $\ker(P(t))$. This in general is not periodic, and to solve this problem we need to define a family of natural isomorphisms:

$$\begin{array}{rccc} I(t): & \ker(P(t)) & \to & l^2(\mathbb{N}) \\ & u_j(t) & \mapsto & \delta_j \end{array} \quad \text{where } \delta_j(n) = \begin{cases} 1 & \text{if } n = j \\ 0 & \text{otherwise} \end{cases}$$

The aperiodicity is expressed by the fact that $I(0)^{-1}I(2\pi) \neq \mathbb{1}_{\mathcal{H}}$. However, thanks to Theorem A.2, we know that there is a homotopy β_t in $\mathcal{U}(l^2(\mathbb{N}))$ with $\beta_0 = \mathbb{1}_{l^2(\mathbb{N})}$ and $\beta_1 = I(2\pi)^{-1}I(0)$. So, the thesis is obtained by putting:

$$v_j(t) = I(t)^{-1}\beta_{t/2\pi}\delta_j \quad \text{for all } j \in \mathbb{N}$$

□

Now we are able to answer our three questions in dimension 1:

Proposition 1.4. *Two projection-valued maps $P_0, P_1 : \mathbb{T}^1 \to \mathrm{Proj}_n(\mathcal{H})$ will always have a frame and be unitary equivalent and homotopic.*

Proof. Given a frame $\{v_j^0\}_{j\in\{1,\cdots,n\}}$ of $P_0(0)$, we can build a continuous orthonormal basis of $P_0(t)$ as $\{U(t)v_j^0 = u_j^0(t)\}_{j\in\{1,\cdots,n\}}$. However, this is not periodic, since $U(2\pi)$ may be different from $\mathbb{1}$. This aperiodicity is expressed as the presence of a *matching matrix*:

$$\alpha \in U(n) \quad \text{such that} \quad u_j^0(2\pi) = \sum_{a=1}^{n} [\alpha]_{j,a} u_a^0(0).$$

Since $U(n)$ is connected due to Theorem A.3, there is a path $\beta(t) \in U(n)$ with $\beta(0) = \mathbb{I}$ and $\beta(2\pi) = \alpha^{-1}$. Using this, we can define

$$v_j^0(t) = \sum_{a=1}^{n} [\beta(t)]_{j,a} u_a^0(t) \quad \forall j \in \{1, \cdots, n\}.$$

Those clearly form an orthonormal basis of $P(t)$ but this time it is periodic because:

$$v_j^0(2\pi) = \sum_{a=1}^{n} [\beta(2\pi)]_{j,a} u_a^0(t) = \sum_{a,b=1}^{n} [\alpha^{-1}]_{j,a} [\alpha]_{a,b} u_b^0(0) = \sum_{b=1}^{n} [\alpha^{-1}\alpha]_{j,b} u_b^0(0) = u_j^0(0) = v_j^0(0).$$

If $\dim(\mathcal{H}) < \infty$ we can replicate this procedure also to $P_0(k)^\perp$, while if $\dim(\mathcal{H}) = \infty$ we can apply Lemma 1.3 in both cases we can always build a continuous frame $\{v_j^0(k)\}_{j\in\{1,\cdots,\dim(\mathcal{H})\}}$ such that the first n vectors span $P_0(k)$ and the others span the orthogonal complement.

So, if $\{v_j^0(k)\}_{j\in\{1,\cdots,\dim(\mathcal{H})\}}$ and $\{v_j^1(k)\}_{j\in\{1,\cdots,\dim(\mathcal{H})\}}$ are two continuous frames of $P_0(k)$ and $P_1(k)$, respectively, then the unitary equivalence is the unitary-valued map such that $V(k)(v_j^0(k)) = v_j^1(k)$ for all $j \in \{1, \cdots, \dim(\mathcal{H})\}$. It is trivial to prove that $P_1(k) = V(k)P_0(k)V(k)^{-1}$.

Now, if $\dim(\mathcal{H}) = \infty$, we know using Theorem A.2 that there is a homotopy $V_t(k) \in \mathcal{U}(\mathcal{H})$ such that $V_0(k) \equiv \mathbb{1}$ and $V_1(k) = V(k)$. We can use it to define the projector-valued map

$$P_t(k) = V_t(k)P_0(k)V_t(k)^{-1}$$

which exactly constitutes the homotopy we were looking for.

Instead, if $\dim(\mathcal{H}) < \infty$ and $[\det(V(k))] = l$, then using Theorem A.3 we know that there is a homotopy $V_t(k)$ such that:

$$V_1(k) = V(k) \quad V_0(k)(v_j(k)) = \begin{cases} e^{ikl} v_1^0(k) & \text{for} \quad j = 1 \\ v_j^0(k) & \text{otherwise} \end{cases}$$

In fact, it is immediately noticeable that $[\det(V_0(k))] = l$ and $V_0(k)P_0(k) = P_0(k)V_0(k)$. So once again

$$P_t(k) = V_t(k)P_0(k)V_t(k)^{-1}$$

is the homotopy we were looking for. □

1.3 d=2

In order to answer our questions when $d = 2$ we want to replicate the construction we did when $d = 1$. However, at some point we will encounter a topological obstruction.

Proposition 1.5 (Class A Kato-Nagy 2-d construction). *If $P : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$ is a continuous projection-valued map, then there is a continuous unitary-valued map $U : [0, 2\pi] \times \mathbb{T}^1 \to \mathcal{U}(\mathcal{H})$, which is periodic in the second argument but not in the first argument, such that:*

$$P(t, k_2) = U(t, k_2)P(0, 0)U(t, k_2)^{-1}$$

Proof. Once again, we can divide $[0, 2\pi]$ into a finite number of intervals $[t_j, t_{j+1}]$ for all $0 \le j < J$ with $t_0 = 0, t_J = 2\pi$ such that $\|P(t_j, k_2) - P(t, k_2)\| < 1/2$ for all $t \in [t_j, t_{j+1}]$. Then we can define J unitary-valued maps:

$$U_j(t, k_2) = \frac{P(t, k_2)P(t_j, k_2) + (\mathbb{1} - P(t, k_2))(\mathbb{1} - P(t_j, k_2))}{\sqrt{\mathbb{1} - (P(t_j, k_2) - P(t, k_2))^2}} \quad \forall t \in [t_j, t_{j+1}]$$

Those connect $P(t_j, k_2)$ with $P(t, k_2)$ for all $k_2 \in \mathbb{T}^1$ and for all $t \in [t_j, t_{j+1}]$. The unitary equivalence for $t \in [0, 2\pi]$ is defined as:

$$U(t, k_2) = U_j(t, k_2)U_{j-1}(t_j, k_2) \cdots U_1(t_2, k_2)U_0(t_1, k_2) \quad \text{if} \quad t \in [t_j, t_{j+1}]$$

Once again this is continuous, periodic in k_2 and such that

$$P(t, k_2) = U(t, k_2)P(0, k_2)U(t, k_2)^{-1}$$

To conclude we need to apply Proposition 1.4. This tells that the projection-valued map $P(0, k_2)$ is unitary equivalent to the constant projection-valued map $P(0, 0)$, so there is a unitary equivalence $V(k_2)$ such that $P(0, k_2) = V(k_2)P(0, 0)V(k_2)^{-1}$, so the unitary we are looking for is:

$$\tilde{U}(t, k_2) = U(t, k_2)V(k_2).$$

□

Definition 1.6 (Chern number). If $P : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$ is a continuous projection-valued map and $U : [0, 2\pi] \times \mathbb{T}^1 \to \mathcal{U}(\mathcal{H})$ is a continuous unitary-valued map periodic on the second argument such that

$$P(t, k_2) = U(t, k_2)P(0, 0)U(t, k_2)^{-1},$$

then $U(0, k_2)^{-1}U(2\pi, k_2)$ commutes with $P(0, 0)$, so after choosing a basis $\{v_j\}_{j \in \{1, \cdots, n\}}$ spanning $\mathrm{Ran}(P(0, 0))$ we can create a periodic unitary-valued map map $\alpha(k_2) \in M_{n,n}(\mathbb{C})$ where

$$\alpha_{a,b}(k_2) = \left\langle v_a, U(0, k_2)^{-1}U(2\pi, k_2)v_b \right\rangle .$$

Then the Chern number is the integer obtained by computing the winding number of the determinant of α defined in Definition A.1:

$$\mathrm{Ch}(P) = [\det(\alpha(\cdot))].$$

Instead, if $\dim(\mathrm{Ran}(P(k))) = \infty$, we just say that $\mathrm{Ch}(P) = 0$.

Proposition 1.7. *The above quantity does not depend on the choice of the unitary operator U, moreover if two projector-valued maps are unitarily equivalent or homotopic, then the Chern numbers are equal.*

Proof.

- (*well posedness*) First we need to show that the Chern number does not depend on the choice of the basis $\{v_j\}_{j\in\{1,\cdots,n\}}$. If we consider a different orthonormal basis $\{v'_j\}_{j\in\{1,\cdots,n\}}$ that leads to a different $\alpha'(k_2)$, then this will be unitarily equivalent to $\alpha(k_2)$ via $[A]_{a,b} = \langle v_a, v'_b\rangle$. This means that $\alpha(k_2) = A^{-1}\alpha'(k_2)A$ and so $\det(\alpha(k_2)) = \det(\alpha'(k_2))$. Now we need to prove that the Chern number does not depend on the choice of the unitary-valued map $U(k)$. So consider two unitary-valued maps $U, V : [0, 2\pi] \times \mathbb{T}^1 \to \mathcal{U}(\mathcal{H})$ such that

$$P(t, k_2) = U(t, k_2)P(0, 0)U(t, k_2)^{-1} = V(t, k_2)P(0, 0)V(t, k_2)^{-1}.$$

Then $V(t, k_2)^{-1}U(t, k_2)$ commutes with $P(0, 0)$ for all t, k_2, so, using a basis $\{v_j\}_{j\in\{1,\cdots,n\}}$ of $\mathrm{Ran}(P(0, 0))$, we can define the matrix

$$[\beta(t, k_2)]_{a,b} = \left\langle v_a, V(t, k_2)^{-1}U(t, k_2)v_b\right\rangle.$$

Thanks to continuity in t and periodicity in k_2, the winding number $[\det \beta(t, \cdot)]$ is constant in t. This means that, using Corollary A.7 and the property $\det(AB) = \det(A)\det(B) = \det(BA)$:

$$[\det(\beta(0, \cdot))] = [\det(\beta(2\pi, \cdot))] \Leftrightarrow [\det(\beta(0, \cdot)^{-1}\beta(2\pi, \cdot))] = 0 \Leftrightarrow$$

$$\Leftrightarrow \left[\det\left(U(0, \cdot)^{-1}V(0, \cdot)V(2\pi, \cdot)^{-1}U(2\pi, \cdot)\mid_{\mathrm{Ran}(P(0,0))}\right)\right] = 0 \Leftrightarrow$$

$$\Leftrightarrow \left[\det\left(U(0, \cdot)^{-1}V(0, \cdot)V(2\pi, \cdot)^{-1}U(0, \cdot)U(0, \cdot)^{-1}U(2\pi, \cdot)\mid_{\mathrm{Ran}(P(0,0))}\right)\right] = 0 \Leftrightarrow$$

$$\Leftrightarrow \left[\det\left(U(0, \cdot)^{-1}V(0, \cdot)V(2\pi, \cdot)^{-1}U(0, \cdot)\mid_{\mathrm{Ran}(P(0,0))}\right)\right] + [\det(\alpha(\cdot))] = 0 \Leftrightarrow$$

$$\Leftrightarrow \left[\det\left(V(2\pi, \cdot)^{-1}U(0, \cdot)U(0, \cdot)^{-1}V(0, \cdot)\mid_{\mathrm{Ran}(P(0,0))}\right)\right] + [\det(\alpha(\cdot))] = 0 \Leftrightarrow$$

$$\Leftrightarrow \left[\det\left(V(0, \cdot)^{-1}V(2\pi, \cdot)\mid_{\mathrm{Ran}(P(0,0))}\right)^{-1}\right] + [\det(\alpha(\cdot))] = 0 \Leftrightarrow$$

$$\Leftrightarrow [\det(\alpha(\cdot))] = \left[\det\left(V(0, \cdot)^{-1}V(2\pi, \cdot)\mid_{P(0,0)}\right)\right]$$

- *(Unitary invariance)* Consider a unitary equivalence $V(k)$ between two projector-valued maps $P_0(k), P_1(k)$, such that

$$P_1(k) = V(k)P_0(k)V(k)^{-1}.$$

Then consider a Kato-Nagy unitary U with

$$P_0(t, k_2) = U(t, k_2)P_0(0, 0)U(t, k_2)^{-1}.$$

Using it we can build a Kato-Nagy unitary also for P_1 as

$$P_1(t,k_2) = V(t,k_2)U(t,k_2)V(0,0)^{-1}P_1(0,0)V(0,0)U(t,k_2)^{-1}V(t,k_2)^{-1}.$$

If $\{v_j^1\}_{j\in\{1,\cdots,n\}}$ is a basis of $\mathrm{Ran}(P_1(0,0))$ we are interested in the unitary-valued map $\alpha_1(k_2) \in M_{n,n}(\mathbb{C})$ with

$$[\alpha_1]_{a,b}(k_2) = \left\langle v_a^1, V(0,0)U(0,k_2)^{-1}V(0,k_2)^{-1}V(2\pi,k_2)U(2\pi,k_2)V(0,0)^{-1}v_b^1 \right\rangle.$$

However, V is $2\pi-$periodic in k_2 and since $v_j^0 := V(0,0)^{-1}v_j^1$ make up an orthonormal basis of $\mathrm{Ran}(P_0(0,0))$, then

$$[\alpha_1]_{a,b} = \left\langle v_a^0, U(0,k_2)^{-1}U(2\pi,k_2)v_b^0 \right\rangle.$$

So we can conclude that

$$\mathrm{Ch}(P_1) = [\det(\alpha_1)] = [\det(\alpha_0)] = \mathrm{Ch}(P_0)$$

- *(Homotopy invariance)* Consider a homotopy of projection-valued maps $P_t(k)$. Then, using compactness of $[0,1]\times\mathbb{T}^2$ and continuity of $P_t(k)$, it is possible to create a partition of $[0,1] = \cup_{j=0}^{J-1}[t_j,t_{j+1}]$ with $t_0 = 0, t_J = 1$ such that

$$\|P_{t_j}(k) - P_t(k)\| \leq 1/2 \quad \forall j \in \{0,\cdots,J-1\}, t \in [t_j,t_{j+1}], k \in \mathbb{T}^2.$$

Then the following is a unitary equivalence between P_{t_j} and $P_{t_{j+1}}$ for all $j = 0,1,\cdots,J-1$:

$$U(k)_j = \left[P_{t_{j+1}}(k)P_{t_j}(k) + (\mathbb{1} - P_{t_{j+1}}(k))(\mathbb{1} - P_{t_j}(k))\right]\left[\mathbb{1} - (P_{t_j}(k) - P_{t_{j+1}}(k))^2\right]^{-1/2}.$$

But we proved previously that unitary equivalent projection-valued maps have equal Chern number, so the thesis follows from the chain of equalities

$$\mathrm{Ch}(P_0) = \mathrm{Ch}(P_{t_1}) = \cdots = \mathrm{Ch}(P_{t_{J-1}}) = \mathrm{Ch}(P_1).$$

□

Lemma 1.8. *The Chern number is additive, meaning that if* $P : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$ *and* $Q : \mathbb{T}^2 \to \mathrm{Proj}_m(\mathcal{H})$ *with* $PQ \equiv 0$*, then* $P + Q : \mathbb{T}^2 \to \mathrm{Proj}_{n+m}(\mathcal{H})$ *and* $\mathrm{Ch}(P) + \mathrm{Ch}(Q) = \mathrm{Ch}(P+Q)$*.*

Proof. If we apply Proposition 1.5 to the three projectors $P, Q, P+Q$, we obtain three unitary equivalences $U, V, Y : [0,2\pi]\times\mathbb{T}^1 \to \mathcal{U}(\mathcal{H})$ such that

$$P(t,k_2) = U(t,k_2)P(0,0)U(t,k_2)^{-1}, \quad Q(t,k_2) = V(t,k_2)Q(0,0)V(t,k_2)^{-1}$$

$$(P+Q)(t,k_2) = Y(t,k_2)(P+Q)(0,0)Y(t,k_2)^{-1}.$$

Then we extend the operator $X : [0, 2\pi] \times \mathbb{T}^1 \to \mathcal{U}(\mathcal{H})$ by linearity with

$$X(u) = \begin{cases} U(u) & \text{if} \quad Pu = u \\ V(u) & \text{if} \quad Qu = u \\ Y(u) & \text{if} \quad Pu = Qu = 0 \end{cases}$$

to obtain a unitary-valued map with

$$(P + Q)(t, k_2) = X(t, k_2)(P + Q)(0, 0)X(t, k_2)^{-1}.$$

Now we compute the Chern number of $P + Q$ using a basis $\{v_j\}_{j\in\{1,\cdots,m+n\}}$ where the first n span $P(0, 0)$ and the other span $Q(0, 0)$. In this way we have

$$\alpha(k_2) = \begin{pmatrix} \beta(k_2) & 0 \\ 0 & \gamma(k_2) \end{pmatrix}$$

where $\beta : \mathbb{T}^2 \to U(n)$ and $\gamma : \mathbb{T}^2 \to U(m)$ are such that:

$$[\beta(k_2)]_{a,b} = \langle v_a, U(0, k_2)^{-1}U(2\pi, k_2)v_b\rangle \quad [\gamma(k_2)]_{a,b} = \langle v_{n+a}, V(0, k_2)^{-1}V(2\pi, k_2)v_{n+b}\rangle$$

So using Corollary A.7 we obtain the thesis

$$\mathrm{Ch}(P + Q) = [\det(\alpha(k_2))] = [\det(\beta(k_2)) \det(\gamma(k_2))] = [\det(\beta(k_2))] + [\det(\gamma(k_2))]$$

□

Lemma 1.9. *If $\mathcal{H}$ is a separable Hilbert space with $\dim(\mathcal{H}) = \infty$ and $P : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$ is a continuous projection-valued map, then there is always a continuous and periodic collection of orthonormal vectors $\{v_j(k)\}_{j\in\mathbb{N}}$ that make up a countable basis of $\ker(P(k))$.*

Proof. Using Proposition 1.5, if we select a countable basis $\{v_j\}_{j\in\mathbb{N}}$ of $\ker(P(0, 0))$, then $u_j(t, k_2) = U(t, k_2)v_j(0, 0)$ constitute a countable basis of $\ker(P(t, k_2))$. This is periodic in the second argument, but not in the first. To solve this problem, we notice that this family defines a family of natural isomorphisms:

$$\begin{array}{rccc} I(t, k_2) : & \ker(P(t, k_2)) & \to & l^2(\mathbb{N}) \\ & u_j(t, k_2) & \mapsto & \delta_j \end{array} \quad \text{where } \delta_j(n) = \begin{cases} 1 & \text{if } n = j \\ 0 & \text{otherwise} \end{cases}$$

The aperiodicity is expressed by the fact that $I(0, k_2)^{-1}I(2\pi, k_2) \neq \mathbb{1}_{\mathcal{H}}$. However, thanks to Theorem A.2, we know that there is a homotopy $\beta_t(k_2)$ in $\mathcal{U}(l^2(\mathbb{N}))$ with $\beta_0(k_2) \equiv \mathbb{1}_{l^2(\mathbb{N})}$ and $\beta_1(k_2) = I(2\pi, k_2)^{-1}I(0, k_2)$. So, the thesis is obtained by putting:

$$v_j(t, k_2) = I(t, k_2)\beta_{t/2\pi}(k_2)\delta_j \quad \text{for all } j \in \mathbb{N}$$

□

Theorem 1.10. *Consider a projection-valued map $P : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$. Then it is always possible to construct a quasiperiodic frame $\{v_j(k)\}_{j\in\{1,\cdots,n\}}$ such that all vectors are periodic in k except the first that satisfies*

$$v_1(2\pi, k_2) = e^{i\,\mathrm{Ch}(P)k_2} v_1(0, k_2), \quad v_1(k_1, 0) = v_1(k_1, 2\pi) \quad \forall k_1, k_2 \in \mathbb{T}^2.$$

Moreover two projection-valued maps $P_0, P_1 : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$ are unitary equivalent and homotopic if and only if $\mathrm{Ch}(P_0) = \mathrm{Ch}(P_1)$.

Proof. Using Proposition 1.5 we can build a unitary-valued map periodic in k_2 such that

$$P(t, k_2) = U(t, k_2)P(0, 0)U(t, k_2)^{-1}.$$

If $\{v_j\}_{j\in\{1,\cdots,n\}}$ is an orthonormal basis of $P(0, 0)$, then $u_j(t, k_2) = U(t, k_2)v_j$ constitute an orthonormal basis of $P(t, k_2)$. This is periodic in k_2 but in general will not be periodic in the first argument. The aperiodicity is once again measured by the matrix $\alpha(k_2) \in U(n)$, with:

$$u_j(2\pi, k_2) = \sum_{a=1}^{n} \alpha_{j,a}(k_2)u_j(0, k_2).$$

Since $[\det(\alpha(k_2))] = \mathrm{Ch}(P)$ the matrix:

$$\beta(k_2) = \begin{pmatrix} e^{(i\,\mathrm{Ch}(P)k_2)} & 0 \\ 0 & \mathbb{1}_{n-1} \end{pmatrix} \alpha(k_2)^{-1}$$

is such that $[\det(\beta(k_2))] = 0$. So Using Theorem A.3, there is a homotopy $\beta_t(k_2)$ with $\beta_1(k_2) = \beta(k_2)$ and $\beta_0(k_2) = \mathbb{1}$. Therefore the vectors

$$v_j(t, k_2) = \sum_{a=1}^{n} \left[\beta_{t/2\pi}(k_2)\right]_{j,a} u_j(t, k_2) \quad \text{for} \quad j \in \{1, \cdots, n\}$$

constitute an orthonormal basis and are such that:

$$\begin{aligned} v_j(2\pi, k_2) &= \sum_{a=1}^{n} [\beta(k_2)]_{j,a} u_a(2\pi, k_2) = \sum_{a,b=1}^{n} [\beta(k_2)]_{j,a} \alpha_{a,b}(k_2) u_b(0, k_2) = \\ &= \sum_{a=1}^{n} [\beta(k_2)\alpha(k_2)]_{j,a} u_a(0, k_2) = \begin{cases} e^{i\,\mathrm{Ch}(P)k_2} v_1(0, k_2) & \text{for} j = 1 \\ v_j(0, k_2) & \text{otherwise} \end{cases} \end{aligned}$$

This concludes the first part of the proof. For the second we should start by noticing that in Proposition 1.7 we already proved that the Chern number is invariant under unitary equivalences and homotopies, so we just need to prove the converse. Consider two projection-valued maps $P_0, P_1 : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$. If their Chern numbers agree ($\mathrm{Ch}(P_0) = \mathrm{Ch}(P_1) = l$), then, using the previous argument, they admit two quasiperiodic frames

$$\{v_j^0(k)\}_{j\in\{1,\cdots,n\}} \subset \mathrm{Ran}(P_0(k)) \quad \{v_j^1(k)\}_{j\in\{1,\cdots,n\}} \subset \mathrm{Ran}(P_1(k)) \quad \text{with}$$

$$v_1^0(2\pi, k_2) = e^{ilk_2} v_1^0(0, 2\pi), \quad v_1^1(2\pi, k_2) = e^{ilk_2} v_1^1(0, k_2)$$

and full periodicity for the other vectors and the other direction. Now, if $\dim(\mathcal{H}) < \infty$, we need to consider the orthogonal projection-valued maps $\mathbb{1} - P_0 = P_0^\perp$ and $\mathbb{1} - P_1 = P_1^\perp$. Those have rank equal to $\dim(\mathcal{H}) - n$. Since $P_0 + P_0^\perp = \mathbb{1}_\mathcal{H} = P_1 + P_1^\perp$ and for obvious reasons $\mathrm{Ch}(\mathbb{1}_\mathcal{H}) = 0$, thanks to Lemma 1.8 $\mathrm{Ch}(P_0^\perp) = \mathrm{Ch}(P_1^\perp) = -l$. Therefore we can repeat the procedure done before to obtain two quasiperiodic frames

$$\{v_j^0(k)\}_{j\in\{n+1,\cdots,\dim(\mathcal{H})\}} \subset \mathrm{Ran}(P_0^\perp(k)), \quad \{v_j^1(k)\}_{j\in\{n+1,\cdots,\dim(\mathcal{H})\}} \subset \mathrm{Ran}(P_1^\perp(k))$$

$$\text{with} \quad v_{n+1}^0(2\pi, k_2) = e^{-ilk_2} v_{n+1}^0(0, k_2), \quad v_{n+1}^1(2\pi, k_2) = e^{-ilk_2} v_{n+1}^1(0, k_2)$$

and full periodicity for the other vectors and the other direction. Then the unitary equivalence we were looking for is the unitary-valued map V such that

$$V(k)(v_j^0(k)) = v_j^1(k) \quad \text{for all} \quad j \in \{1, \cdots, \dim(\mathcal{H})\}.$$

Despite being defined using a quasiperiodic frame this unitary operator is actually periodic since the phase that appears in the first vector when $k_1 = 2\pi$ gets out from the unitary operator leading back to the first vector computed in $k_1 = 0$. For example

$$e^{i\,\mathrm{Ch}(P_1)k_2} v_1^1(0, k_2) = v_1^1(2\pi, k_2) = V(2\pi, k_2) v_1^0(2\pi, k_2) = e^{i\,\mathrm{Ch}(P_0)k_2} V(2\pi, k_2) v_1^0(0, k_2)$$

so if the Chern numbers agree, then the two phases cancel each other and $V(0, k_2)$ acts as $V(2\pi, k_2)$. Instead, if $\mathcal{H}$ is separable with $\dim(\mathcal{H}) = \infty$, we can apply Lemma 1.9 to obtain two continuous, periodic and discrete frames

$$\{v_j^0(k)\}_{j\in\{n+1,\cdots,\infty\}} \subset \mathrm{Ran}(P_0^\perp(k)) \quad \{v_j^1(k)\}_{j\in\{n+1,\cdots,\infty\}} \subset \mathrm{Ran}(P_1^\perp(k)).$$

Then once again the unitary equivalence can be constructed by imposing

$$V(k) v_j^0(k) = v_j^1(k) \quad \text{for all} \quad j \in \mathbb{N}.$$

At last we want to solve the homotopy problem. If $\dim(\mathcal{H}) = \infty$ we know using Theorem A.2 that there is a homotopy $V_t(k)$ with $V_1(k) = V(k)$ and $V_0(k) = \mathbb{1}$, so

$$P_t(k) = V_t(k) P_0(k) V_t(k)^{-1}$$

is the homotopy we were looking for.

Instead, if $\dim(\mathcal{H}) < \infty$, we need to do something trickier. Let

$$[\det(V(0, k_2))] = [\det(V(2\pi, k_2))] = m, \quad [\det(V(k_1, 0))] = [\det(V(k_1, 2\pi))] = r,$$

and define

$$V_0(k_1, k_2) v_j^0(k_1, k_2) = \begin{cases} e^{i(rk_1 + mk_2)} v_1^0(k_1, k_2) & \text{if } j = 1 \\ v_j^0(k_1, k_2) & \text{otherwise} \end{cases}$$

Once again it is immediate to prove that $V_0(k)$ is periodic in both directions with $V(0,0) = \mathbb{1}$ and

$$[\det(V_0(k_1,0))] = [\det(V_0(k_1,2\pi))] = r, \quad [\det(V_0(0,k_2))] = [\det(V_0(2\pi,k_2))] = m.$$

So using Theorem A.3 it is possible to construct a homotopy $V_t(k)$ connecting V_0 with $V_1 = V$ for k in the boundaries of $[0,2\pi]^2$. So we actually defined a unitary-valued map $V_t(k)$ for the values of $t,k \in \partial([0,1]\times[0,2\pi]^2)$. After interpreting it as a map from S^2 to $\mathcal{U}(\mathcal{H})$ we can also extend the definition inside of $[0,1]\times\mathbb{T}^2$ since $\pi_2(\mathcal{U}(\mathcal{H})) = \{0\}$ (Theorem A.3). So the projector-valued map

$$P_t(k) = V_t(k)P_0(k)V_t(k)^{-1}$$

is the homotopy we were looking for since $[V_0(k), P_0(k)] = 0$. □

Chapter 2

Class AI

In class AI, the only symmetry present is a bosonic time-reversal symmetry, so the object we want to study in this chapter is:

Definition 2.1 (bosonic time-reversal symmetric projection-valued map)**.** Given an anti-unitary operator T acting over $\mathcal{H}$ with $T^2 = \mathbb{1}$, a projection-valued map map $P : \mathbb{T}^d \to \text{Proj}_n(\mathcal{H})$ is time-reversal symmetric if and only if

$$TP(k) = P(-k)T.$$

Which is exactly the object defined in Definition I.12 with the additional constraint that $T^2 = \mathbb{1}$. With this in mind, given two bosonic time-reversal symmetric projection-valued maps $P_0, P_1 : \mathbb{T}^d \to \text{Proj}_n(\mathcal{H})$, we will try to answer the following questions:

Question 4 (Class AI frame)**.** When will there be a continuous and bosonic time-reversal symmetric frame $\{u_1(k), \cdots, u_n(k)\}$, namely a continuous and periodic collection of orthonormal vectors spanning $\text{Ran}(P_0(k))$ with $Tu_j(k) = u_j(-k)$ for all $j = 1, \cdots, n$?

Question 5 (Class AI unitary equivalence)**.** When will there be a continuous and time-reversal symmetric unitary-valued map $U : \mathbb{T}^d \to \mathcal{U}(\mathcal{H})$ such that $P_1(k) = V(k)P_0(k)V(k)^{-1}$ and $TV(k) = V(-k)T$ for all $k \in \mathbb{T}^d$? This is the same condition that appeared in the Introduction and in Definition I.12

Question 6 (Class AI homotopy)**.** When will there be a continuous and time-reversal symmetric map $P : [0,1] \times \mathbb{T}^d \to \text{Proj}_n(\mathcal{H})$ with $P(0,k) = P_0(k)$, $P(1,k) = P_1(k)$ and $TP_t(k) = P_t(-k)T$ for all $k \in \mathbb{T}^d$ and $t \in [0,1]$?

In fact, those questions were already answered in [10][18], but these results are so useful that it is worth writing them once more.

2.1 d=0

Since $\mathbb{T}^0$ is made up of a single point, in this case we just have to study a single projector $P \in \text{Proj}_n(\mathcal{H})$ with $TP = PT$. In this small environment, the three questions can be answered without issue. First of all, let us start with a further simplification of the symmetry using the following lemma:

Lemma 2.2. *If T is an antiunitary operator such that $T^2 = \mathbb{1}$, then there is a basis $\{v_j\}_{j=1,\cdots,\dim(\mathcal{H})}$ of $\mathcal{H}$ such that $T(v_j) = v_j$, which means that in this basis T acts as the standard complex conjugation $\mathcal{K}$.*

Proof. Consider $\ker(T - \mathbb{1}) = \mathcal{H}_+$ and $\ker(T + \mathbb{1}) = \mathcal{H}_-$. Those are two real Hilbert subspaces of $\mathcal{H}$ because if $u, v \in \mathcal{H}_\pm$ then

$$\langle u, v\rangle = \overline{\langle T(u), T(v)\rangle} = \overline{\langle \pm u, \pm v\rangle} = \overline{\langle u, v\rangle}.$$

Moreover $v \in \mathcal{H}_+$ if and only if $iv \in \mathcal{H}_-$. Now we can select a real basis $\{v_j\}_{j\in\{1,\cdots,n\}}$ of $\mathcal{H}_+$ so that $\{iv_j\}_{j\in\{1,\cdots,\dim(\mathcal{H})\}}$ is a real basis of $\mathcal{H}_-$. To conclude our argument, we only need to show that as a real vector space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. This is true because for any $v \in \mathcal{H}$ it holds that

$$v = \frac{v + T(v)}{2} + \frac{v - T(v)}{2} \quad \text{where} \quad \frac{v \pm T(v)}{2} \in \mathcal{H}_\pm.$$

□

So, from now on, we will freely swap between T and $\mathcal{K}$.

Proposition 2.3. *Two bosonic time-reversal symmetric projection-valued maps $P_0, P_1 \in \mathrm{Proj}_n(\mathcal{H})$ will always have a symmetric frame and be unitarily equivalent and homotopic with respect to the time-reversal symmetry.*

Proof. First, we notice that it is possible to apply Lemma 2.2 to the four Hilbert subspaces $\mathrm{Ran}(P_0), \mathrm{Ran}(P_1), \ker(P_0), \ker(P_1)$ to obtain four bases on which the symmetry acts as the standard complex conjugation. We can then merge them to obtain two bases $\{v_j^0\}_{j\in\{1,\cdots,\dim(\mathcal{H})\}}$, $\{v_j^1\}_{j\in\{1,\cdots,\dim(\mathcal{H})\}}$ where T acts as $\mathcal{K}$, the first n vectors generate $\mathrm{Ran}(P_0)$ and $\mathrm{Ran}(P_1)$ while the others generate $\ker(P_0)$ and $\ker(P_1)$, respectively. Those will constitute the symmetric frames. Then the symmetric unitary equivalence we need is the operator V such that $V(v_j^0) = v_j^1$ for all $j = 1, \cdots, \dim(\mathcal{H})$ since it is obvious that $P_1 = VP_0V^{-1}$ and that $TV = VT$. Finally, for the homotopy problem, we must consider $\ker(T - \mathbb{1}) = \mathcal{H}_+$ as a real and separable Hilbert space. Then the group of unitary transformations that commute with T are precisely the orthogonal operators $\mathcal{O}(\mathcal{H}) = \mathcal{U}(\mathcal{H}_+)$. Finally, if $\dim(\mathcal{H}) = \infty$, we can use Theorem A.2 to obtain a homotopy $V_t \in \mathcal{U}(\mathcal{H}_+)$ where $V_1 = V$ and $V_0 = \mathbb{1}$ so that $P_t = V_tP_0V_t^{-1}$ is the symmetric homotopy we are looking for because $TV_t = V_tT$. Instead, if $\dim(\mathcal{H}) < \infty$, we can interpret $\mathcal{U}(\mathcal{H}_+)$ as $\mathrm{O}(\dim(\mathcal{H}_+))$ and Theorem A.4 states that we can repeat the same procedure when $\det(U) = 1$. On the other hand, if $\det(U) = -1$, we can find a homotopy $U_t \in \mathrm{O}(\dim(\mathcal{H}_+))$ such that $U_1 = U$ and U_0 is such that $U_0(v_1^0) = -v_1^0$ and $U_0(v_j^0) = v_j^0$ for $j = 2, \cdots, \dim(\mathcal{H}_+)$ because $\det(U_0) = -1 = \det(U)$. So $P_t = U_tP_0U_t^{-1}$ is still a symmetric homotopy because $TU_t = U_tT$ and $U_0P_0 = P_0U_0$. □

2.2 d=1

The first thing we want to do is replicate the construction made in Proposition 1.2 with respect to the time-reversal symmetry.

Proposition 2.4 (Class AI Kato-Nagy 1-d construction). *If $P : \mathbb{T}^1 \to \mathrm{Proj}_n(\mathcal{H})$ is a continuous time-reversal symmetric projection-valued map, then there is a continuous but not periodic unitary-valued map $U : [-\pi, \pi] \to \mathcal{U}(\mathcal{H})$ such that:*

$$P(t) = U(t)P(0)U(t)^{-1} \quad \textit{and} \quad TU(t) = U(-t)T$$

Proof. We just need to slightly modify the procedure done in Proposition 1.2, this time we divide $[0, \pi]$ into a finite number of intervals $[t_j, t_{j+1}]$ $\forall 0 \leq j \leq J$ with $t_0 = 0, t_J = \pi$ such that $\|P(t_j) - P(t)\| < 1/2$ for all $t \in [t_j, t_{j+1}]$. Then we can define J unitary-valued maps:

$$U_j(t) = [P(t)P(t_j) + (\mathbb{1} - P(t))(\mathbb{1} - P(t_j))] \left[\mathbb{1} - (P(t_j) - P(t))^2\right]^{-1/2} \quad \forall t \in [t_j, t_{j+1}]$$

Those connect $P(t_j)$ with $P(t)$ for $t \in [t_j, t_{j+1}]$. Then for $t \in [0, \pi]$ the global unitary we need is:

$$U(t) = U_j(t)U_{j-1}(t_j)U_{j-2}(t_{j-1}) \cdots U_1(t_2)U_0(t_1) \quad \text{if} \quad t \in [t_j, t_{j+1}].$$

Finally, we can define $U(t) = T^{-1}U(-t)T$ for $t \in [-\pi, 0]$ to satisfy the symmetry constraint. Once again, the continuity is immediate because $U_j(t_j) = \mathbb{1} \equiv U(0)$ and each function is continuous in t since P was continuous. □

Lemma 2.5. *If $\mathcal{H}$ is a separable Hilbert space with $\dim(\mathcal{H}) = \infty$ and $P : \mathbb{T}^1 \to \mathrm{Proj}_n(\mathcal{H})$ is a continuous bosonic time-reversal symmetric projection-valued map, then there is always a continuous and periodic collection of orthonormal vectors $\{v_j(k)\}_{j\in\mathbb{N}}$ with $T(v_j(k)) = v_j(-k)$ composing a countable basis of $\ker(P(k))$.*

Proof. We can start by using Lemma 2.2 to select a countable basis $\{v_j(0)\}_{j\in N}$ of $\ker(P(0))$, which is also a real basis of $\ker(P(0))_+$, such that $Tv_j(0) = v_j(0)$. Then using Proposition 2.4 $v_j(t) = U(t)v_j(0)$ constitute a countable basis of $\ker(P(t))$ with $Tv_j(t) = v_j(-t)$. This in general is not periodic, and to solve this problem we need to define a family of natural isomorphisms of real Hilbert spaces as done before:

$$\begin{array}{rccc} I(t): & \ker(P(t)) & \to & l^2(\mathbb{N}) \\ & v_j(t) & \mapsto & \delta_j \end{array} \quad \text{where } \delta_j(n) = \begin{cases} 1 & \text{if } n = j \\ 0 & \text{otherwise} \end{cases}$$

The choice of the basis states that $TI(t)^{-1} = I(-t)^{-1}\mathcal{K}$ and $I(t)T = \mathcal{K}I(-t)$ for all $t \in [-\pi, \pi]$. This time, the aperiodicity is expressed by the fact that $I(-\pi)^{-1}I(\pi) \neq \mathbb{1}_{\mathcal{H}}$. Now using the spectral theorem, we know that there is a self-adjoint operator L such that $I(\pi)I(-\pi)^{-1} = e^{iL}$. Moreover, it is true that:

$$\mathcal{K}I(\pi)I(-\pi)^{-1} = I(-\pi)TI(-\pi)^1 = I(-\pi)I(\pi)^{-1}\mathcal{K} = (I(\pi)I(-\pi)^{-1})^{-1}\mathcal{K}.$$

This means that $\mathcal{K}e^{iL} = e^{-iL}$, so $e^{-i\overline{L}} = e^{-iL}$ and therefore we can choose L such that $\overline{L} = L$. Now we can define $\beta_1 = e^{iL/2}$, this is a unitary operator and since $\mathcal{U}(l^2(\mathbb{N}))$ is arcwise connected according to Theorem A.2, there is a homotopy β_t connecting it with $\beta_0 = \mathbb{1}$. For example we can take $\beta_t = e^{itL/2}$. To conclude the proof, we can define:

$$\tilde{v}_j(t) = \begin{cases} I(t)^{-1}\beta_{t/\pi}\delta_j & \text{for} \quad t \in [0,\pi], j \in \mathbb{N} \\ I(t)^{-1}\mathcal{K}\beta_{-t/\pi}\delta_j & \text{for} \quad t \in [-\pi,0], j \in \mathbb{N} \end{cases}$$

This is the collection of vectors we were looking for. In fact it is immediate to show that $T\tilde{v}_j(t) = \tilde{v}_j(-t)$ and it is also periodic since:

$$\begin{aligned} \tilde{v}_j(-\pi) = I(-\pi)^{-1}\mathcal{K}e^{iL/2}\delta_j = I(-\pi)^{-1}e^{-i\overline{L}/2}\delta_j = I(-\pi)^{-1}e^{-iL/2}\delta_j = \\ = I(-\pi)^{-1}e^{-iL}e^{iL/2}\delta_j = I(\pi)^{-1}e^{iL/2}\delta_j = \tilde{v}_j(\pi) \end{aligned}$$

□

Now we are able to answer our three questions in dimension 1:

Proposition 2.6. *Two bosonic time-reversal symmetric projection-valued maps $P_0, P_1 : \mathbb{T}^1 \to \mathrm{Proj}_n(\mathcal{H})$ will always have a symmetric frame and be unitarily equivalent and homotopic with respect to the time reversal symmetry.*

Proof.
- (*symmetric frame*)

Given a symmetric frame $\{v_j^0\}_{j\in\{1,\cdots,n\}}$ of $P_0(0)$ built using Proposition 2.3, we can build a symmetric frame of $P_0(t)$ as $\{U(t)v_j^0 = u_j^0(t)\}_{j\in\{1,\cdots,n\}}$ using Proposition 2.4. However, this may not be periodic and the aperiodicity is expressed as the presence of a matrix:

$$\alpha \in U(n) \quad \text{such that} \quad u_j^0(\pi) = \sum_{a=1}^{n} [\alpha]_{j,a} u_a^0(-\pi).$$

Since $Tu_j(t) = u_j(-t)$ this α has the additional property:

$$\begin{aligned} u_j^0(-\pi) = Tu_j^0(\pi) = \sum_{a=1}^{n} \overline{[\alpha]_{j,a}} Tu_a^0(-\pi) = \sum_{a=1}^{n} \overline{[\alpha]_{j,a}} u_a^0(\pi) = \\ = \sum_{a,b=1}^{n} \overline{[\alpha]_{j,a}}[\alpha]_{a,b}u_b^0(-\pi) = \sum_{b=1}^{n}[\overline{\alpha}\alpha]_{j,b}u_b^0(-\pi) \end{aligned}.$$

Meaning that $\overline{\alpha}\alpha = \mathbb{1}$ or equivalently $\overline{\alpha} = \alpha^* \Leftrightarrow \alpha = \alpha^t$. Using the Sylvester theorem, we know that there is a self-adjoint matrix L with $e^{-iL} = \alpha$ and $\alpha = \alpha^t$ implies that there is a choice of L such that $L = L^t$ and since $L^* = L$ it is also true that $\overline{L} = L$. We can now define $\beta_t = e^{itL/2\pi}$ and $v_j^0(t) = \sum_{b=1}^{n}[\beta_t]_{j,b}u(t)_j^0$ to form the symmetric frame we were looking for. In fact it holds that:

$$\begin{aligned} Tv_j^0(t) &= \sum_{b=1}^{n} \overline{[\beta_t]_{j,b}} Tu_b^0(t) = \sum_{b=1}^{n} \overline{[e^{itL/2\pi}]_{j,b}} u_b^0(-t) = \\ &= \sum_{b=1}^{n}[e^{-it\overline{L}/2\pi}]_{j,b}u_b^0(-t) = \sum_{b=1}^{n}[e^{-itL/2\pi}]_{j,b}u_b^0(-t) = \\ &= \sum_{b=1}^{n}[\beta_{-t}]_{j,b}u_b^0(-t) = v_j^0(-t) \end{aligned}$$

$$\begin{aligned} v_j^0(\pi) &= \sum_{b=1}^{n} [e^{iL/2}]_{j,b} u_b^0(\pi) = \sum_{b,a=1}^{n} [e^{iL/2}]_{j,b} [\alpha]_{b,a} u_a^0(-\pi) = \\ &= \sum_{a,b=1}^{n} [e^{iL/2}]_{j,b} [e^{-iL}]_{b,a} u_a^0(-\pi) = \sum_{a=1}^{n} [e^{iL/2} e^{-iL}]_{j,a} u_a^0(-\pi) = \\ &= \sum_{a=1}^{n} [e^{-iL/2}]_{j,a} u_a^0(-\pi) = v_j^0(-\pi). \end{aligned}$$

- (*unitary equivalence*)

  If $\dim(\mathcal{H}) < \infty$ we can replicate this procedure also to $P_0(k)^\perp$ while, if $\dim(\mathcal{H}) = \infty$, we can apply Lemma 2.5. So, in both cases, we can always build a continuous, periodic and symmetric frame $\{v_j^0(k)\}_{j \in \{1,\cdots,\dim(\mathcal{H})\}}$ such that the first n vectors span $P_0(k)$ and the others span the orthogonal complement.

  So, if $\{v_j^0(k)\}_{j \in \{1,\cdots,\dim(\mathcal{H})\}}$ and $\{v_j^1(k)\}_{j \in \{1,\cdots,\dim(\mathcal{H})\}}$ are two continuous and symmetric frames of $P_0(k)$ and $P_1(k)$ respectively, then the symmetric unitary equivalence is the unitary-valued map such that $V(k)(v_j^0(k)) = v_j^1(k)$ for all $j \in \{1, \cdots, \dim(\mathcal{H})\}$. It is trivial to prove that $P_1(k) = V(k)P_0(k)V(k)^{-1}$ and $TV(k) = V(-k)T$.

- (*homotopy*)

  There is a crucial remark we need to make: the time reversal symmetry endows $\mathcal{H}$ with a real structure. In fact we can see $\mathcal{H} = \ker(T - \mathbb{1}) \oplus_{\mathbb{R}} \ker(T + \mathbb{1}) = \mathcal{H}_+ \oplus \mathcal{H}_-$ and a unitary $V \in \mathcal{U}(\mathcal{H})$ commutes with T if and only if it is compatible with this real structure, namely is an orthogonal operator $V \in \mathcal{O}(\mathcal{H}_+)$. This happens for $V(0)$ and $V(\pi) = V(-\pi)$. So if $\dim(\mathcal{H}) = \infty$, Theorem A.2 states that it is possible to find two homotopies $V_t(0)$ and $V_t(\pi)$ connecting $V(0)$ and $V(\pi)$ with $\mathbb{1}$ inside $\mathcal{O}(\mathcal{H}_+)$. Since $\mathcal{U}(\mathcal{H})$ is contractible, it is possible to extend this pair of homotopies to a homotopy $V_t(k)$ with $V_1(k) = V(k)$ for $k \in [0, \pi]$ and $V_0(k) = \mathbb{1}$, then we can define the homotopy on the other side using the symmetry constraint $V_t(k) = TV(-k)T$ for $k \in [-\pi, 0]$. Now we can define the homotopy of projection-valued map by imposing:

  $$P_t(k) = V_t(k)P_0(k)V_t(k)^{-1}.$$

  Instead if $\dim(\mathcal{H}) = m < \infty$, we can express $\mathcal{O}(\mathcal{H}_+)$ as $\mathrm{O}(m)$ and T as $\mathcal{K}$. Then if $\det(V(k)) = \lambda(k)$ it is obvious that $\overline{\lambda(k)} = \lambda(-k)$, so $\lambda(0) = \pm 1$ as well as $\lambda(\pi) = \pm 1$ by periodicity. In this framework we will start by defining the unitary operator $V_0(k)$ with:

  $$V_0(k)v_j^0(k) = \begin{cases} \lambda(k)v_1^0(k) & \text{if} \quad j = 1 \\ v_j^0(k) & \text{otherwise} \end{cases}$$

  Since $TV(k) = V(-k)T$ it follows easily that $V_0(0), V_0(\pi) \in \mathrm{O}(m)$ and $TV_0(k) = V_0(-k)T$. Then, thanks to Theorem A.4, we can define two homotopies $V_t(0), V_t(\pi)$ with:

  $$V_t(0), V_t(\pi) \in \mathrm{O}(m), \quad V_1(0) = V(0), V_1(\pi) = V(\pi)$$

  $$V_0(0) = \begin{pmatrix} \lambda(0) & 0 \\ 0 & \mathbb{1}_{m-1} \end{pmatrix} \qquad V_0(\pi) = \begin{pmatrix} \lambda(\pi) & 0 \\ 0 & \mathbb{1}_{m-1} \end{pmatrix}$$

At this point we defined $V_t(k)$ for $(t, k) \in \partial([0,1] \times [0,\pi])$ and we need to extend the definition also inside the square. If we interpret this boundary as S^1, this is possible if and only if the degree of the map is zero. Thanks to Theorem A.3 this degree is winding number of the determinant of the map. Luckily the determinant is constant throughout $V_t(0)$ and $V_t(\pi)$ and finally, since $\det(V_0(k)) = \det(V(k)) = \lambda(k)$, the overall degree is zero and it is possible to define $V_t(k)$ for $(t, k) \in [0,1] \times [0,\pi]$. For $k \in [-\pi, 0]$ we can just use the symmetry constraint $V_t(k) = TV_t(-k)T = \mathcal{K}V_t(-k)\mathcal{K}$ and this does not conflict with periodicity or continuity since $V_t(0), V_t(\pi) \in \mathrm{O}(m)$. In the end the symmetric homotopy we were looking for is

$$P_t(k) = V_t(k)P_0(k)V_t(k)^{-1}$$

because $V_0(k)$ commutes with $P_0(k)$.

□

2.3 d=2

In a way similar to the one we did for Class A, we will replicate the construction we did when $d = 1$ to study the case $d = 2$. Fortunately, in this class we will not encounter a topological obstruction. In fact the Chern number that we studied in the previous chapter vanishes when a time reversal symmetry is present thanks to the following remark.

Remark 2.7. As stated in [47], if $P : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$ is a bosonic time-reversal symmetric projection-valued map, then $\mathrm{Ch}(P) = 0$.

Proof. We already proved in 1.10 that $\mathrm{Ch}(P) = 0$ if and only if the projection-valued map admits a continuous and periodic frame, in this case it admits even a symmetric one. However we want to give a proof of this fact without using this information, so we can use Proposition 2.8 to obtain a unitary-valued map $U : [-\pi,\pi] \times \mathbb{T}^1 \to \mathcal{U}(\mathcal{H})$ with $P(t, k_2) = U(t, k_2)P(0,0)U(t, k_2)^{-1}$. In order to obtain a unitary-valued map of the type used in Definition 1.6 we need to define:

$$\tilde{U}(t, k_2) = \begin{cases} U(t, k_2) & \text{for} \quad t \in [0,\pi], \\ U(t-2\pi, k_2)U(-\pi, k_2)^{-1}U(\pi, k_2) & \text{for} \quad t \in [\pi, 2\pi]. \end{cases}$$

Then if $\{v_j\}_{j\in\{1,\cdots,n\}}$ is a symmetric orthonormal basis of $\mathrm{Ran}(P(0,0))$ as in Proposition 2.3, we will have the unitary matrix $\alpha \in U(n)$ with:

$$\begin{aligned}
[\alpha]_{a,b}(k_2) &= \left\langle v_a, \tilde{U}(0, k_2)^{-1}\tilde{U}(2\pi, k_2)v_b \right\rangle = \\
&= \left\langle v_a, U(0, k_2)^{-1}U(0, k_2)U(-\pi, k_2)^{-1}U(\pi, k_2)v_b \right\rangle = \\
&= \langle U(-\pi, k_2)v_a, U(\pi, k_2)v_b \rangle = \\
&= \langle TU(\pi, k_2)v_b, TU(-\pi, k_2)v_a \rangle = \langle U(-\pi, -k_2)v_b, U(\pi, -k_2)v_a \rangle = \\
&= [\alpha]_{b,a}(-k_2).
\end{aligned}$$

This means that $\det(\alpha(k_2)) = \det(\alpha(-k_2))$, but a map with this property has a winding number equal to zero thanks to Lemma A.8. □

Proposition 2.8 (Class AI Kato-Nagy 2-d construction). *If $P : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$ is a continuous time-reversal symmetric projection-valued map, then there is a continuous unitary-valued map $U : [-\pi, \pi] \times \mathbb{T}^1 \to \mathcal{U}(\mathcal{H})$, which is periodic in the second argument but not in the first argument, such that:*

$$P(t, k_2) = U(t, k_2)P(0, 0)U(t, k_2)^{-1}, \quad TU(t, k_2) = U(-t, -k_2)T$$

Proof. Once again we can divide $[0, \pi]$ into a finite number of intervals $[t_j, t_{j+1}]$ for all $0 \leq j \leq J$ with $t_0 = 0, t_J = \pi$ such that $\|P(t_j, k_2) - P(t, k_2)\| < 1/2$ for all $t \in [t_j, t_{j+1}]$. Then we can define J unitary-valued maps:

$$U_j(t, k_2) = [P(t, k_2)P(t_j, k_2) + (\mathbb{1} - P(t, k_2))(\mathbb{1} - P(t_j, k_2))] \left[\mathbb{1} - (P(t_j, k_2) - P(t, k_2))^2\right]^{-1/2}$$

for all $t \in [t_j, t_{j+1}]$. Those connect $P(t_j, k_2)$ with $P(t, k_2)$ for all $k_2 \in \mathbb{T}^1$ and for all $t \in [t_j, t_{j+1}]$. The unitary equivalence for $t \in [0, 2\pi]$ is as:

$$U(t, k_2) = U_j(t, k_2)U_{j-1}(t_j, k_2) \cdots U_1(t_2, k_2)U_0(t_1, k_2) \quad \text{if} \quad t \in [t_j, t_{j+1}]$$

Once again this is continuous, periodic in k_2 and such that

$$P(t, k_2) = U(t, k_2)P(0, k_2)U(t, k_2)^{-1}$$

To conclude we need to apply Proposition 2.6 if $T^2 = \mathbb{1}$, or Proposition 4.9 if $T^2 = -\mathbb{1}$. Those tell us that the projection-valued map $P(0, k_2)$ is symmetrically unitarily equivalent to the constant projection-valued map $P(0, 0)$, so there is a unitary equivalence $V(k_2)$ such that $P(0, k_2) = V(k_2)P(0, 0)V(k_2)^{-1}$ with $TV(k_2) = V(-k_2)T$ so the unitary we are looking for is:

$$\tilde{U}(t, k_2) = \begin{cases} U(t, k_2)V(k_2) & \text{if} \quad t \in [0, \pi] \\ TU(-t, -k_2)V(-k_2)T & \text{if} \quad t \in [-\pi, 0] \end{cases}$$

This is continuous because for $t = 0$ it is true that $TV(k_2) = V(-k_2)T$. ☐

Lemma 2.9. *If $\mathcal{H}$ is a separable Hilbert space with $\dim(\mathcal{H}) = \infty$ and $P : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$ is a continuous bosonic time-reversal symmetric projection-valued map, then there is always a continuous and periodic collection of orthonormal vectors $\{v_j(k)\}_{j \in \mathbb{N}}$ that make up a countable basis of $\ker(P(k))$ with $Tv_j(k) = v_j(-k)$ for all $j \in \mathbb{N}, k \in \mathbb{T}^2$.*

Proof. Using Proposition 2.2 we can select a countable basis $\{v_j\}_{j \in \mathbb{N}}$ of $\ker(P(0, 0))$ with $Tv_j = v_j$, then, using the unitary operator U defined in Proposition 2.8, we have that $u_j(t, k_2) = U(t, k_2)v_j$ constitute a countable basis of $\ker(P(t, k_2))$ with $Tu_j(t, k_2) = u_j(-t, -k_2)$. This is periodic in the second argument, but not in the first. To solve this problem, we notice that this family defines a family of natural isomorphisms:

$$\begin{array}{rccc} I(t, k_2) : & \ker(P(t, k_2)) & \to & l^2(\mathbb{N}) \\ & v_j(t, k_2) & \mapsto & \delta_j \end{array} \quad \text{where } \delta_j(n) = \begin{cases} 1 & \text{if } n = j \\ 0 & \text{otherwise} \end{cases}$$

The aperiodicity is expressed by the fact that $I(-\pi, k_2)^{-1}I(\pi, k_2) \neq \mathbb{1}_{\mathcal{H}}$. We can recover the same properties stated in Lemma 2.5, like the fact that $TI(t,k_2)^{-1} = I(-t,-k_2)^{-1}\mathcal{K}$ or that $I(t,k_2)T = \mathcal{K}I(-t,-k_2)$, to know that there are two self-adjoint operators L, R with $L = \overline{L}, R = \overline{R}$ acting over $l^2(\mathbb{N})$ with

$$I(\pi,0)I(-\pi,0)^{-1} = e^{iL} \quad \text{and} \quad I(\pi,\pi)I(-\pi,\pi)^{-1} = e^{iR}.$$

Using them we can define the unitary operator $\beta(k_2) \in \mathcal{U}(l^2(\mathbb{N}))$ with:

$$\beta(k_2) = \begin{cases} \exp\{\frac{i}{\pi}[(\pi - k_2)\frac{L}{2} + k_2\frac{R}{2}]\} & \text{if} \quad k_2 \in [0,\pi] \\ \mathcal{K}I(-\pi,-k_2)I(\pi,-k_2)^{-1}\beta(-k_2) & \text{if} \quad k_2 \in [-\pi,0] \end{cases}$$

It is very easy to see that this $\beta(k_2)$ is continuous and periodic in k_2 by applying the symmetry relations. Now, thanks to Theorem A.2 we know that there is a homotopy $\beta_t(k_2)$ in $\mathcal{U}(l^2(\mathbb{N}))$ with $\beta_0(k_2) \equiv \mathbb{1}_{l^2(\mathbb{N})}$ and $\beta_1(k_2) = \beta(k_2)$. So, the thesis is obtained by putting:

$$v_j(t,k_2) = \begin{cases} I(t,k_2)^{-1}\beta_{t/\pi}(k_2)\delta_j & \text{if} \quad t \in [0,\pi] \\ TI(-t,-k_2)^{-1}\beta_{-t/\pi}(-k_2)\delta_j & \text{if} \quad t \in [-\pi,0] \end{cases} \quad \text{for all } j \in \mathbb{N}$$

In fact this frame is obviously symmetric, is also continuous in $t = 0$, since $\beta_0 \equiv \mathbb{1}$, and is also periodic in t because for $k_2 \in [0,\pi]$:

$$\begin{aligned} v_j(-\pi,k_2) &= I(-\pi,k_2)^{-1}\mathcal{K}\beta(-k_2)\delta_j = \\ &= I(-\pi,k_2)^{-1}\mathcal{K}\mathcal{K}I(-\pi,k_2)I(\pi,k_2)^{-1}\beta(k_2)\delta_j = \\ &= I(\pi,k_2)\beta(k_2)\delta_j = v_j(\pi,k_2) \end{aligned}$$

while for $k_2 \in [-\pi,0]$:

$$v_j(-\pi,k_2) = Tv_j(\pi,-k_2) = Tv_j(-\pi,-k_2) = v_j(\pi,k_2)$$

□

Theorem 2.10. *Consider a bosonic time-reversal symmetric projection-valued map* $P : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$*, then, it is always possible to construct a symmetric and periodic frame* $\{v_j(k)\}_{j=1,\cdots,n}$ *of* P *with* $Tv_j(k) = v_j(-k)$*. Moreover two bosonic time-reversal symmetric projection-valued maps maps* $P_0, P_1 : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$ *are always unitarily equivalent and homotopic with respect to the symmetry* T*.*

Proof.

- (*Symmetric frame*) Using Proposition 1.5 we can build a unitary-valued map periodic in k_2 such that

$$P(t,k_2) = U(t,k_2)P(0,0)U(t,k_2)^{-1}.$$

If $\{v_j\}_{j\in\{1,\cdots,n\}}$ is an orthonormal basis of $P(0,0)$ then $u_j(t,k_2) = U(t,k_2)v_j$ constitute an orthonormal basis of $P(t,k_2)$. This is periodic in k_2 but in general will not be periodic in the first argument. The aperiodicity is once again measured by the matrix $\alpha(k_2)$, with:

$$u_j(-\pi, k_2) = \sum_{a=1}^{n} [\alpha(k_2)]_{j,a} u_a(\pi, k_2).$$

This α has a symmetry constraint:

$$\begin{aligned} u_j(\pi, -k_2) = Tu_j(-\pi, k_2) = T\sum_{a=1}^{n} [\alpha(k_2)]_{j,a} u_a(\pi, k_2) = \sum_{a=1}^{n} \overline{[\alpha(k_2)]_{j,a}} u_a(-\pi, -k_2) = \\ = \sum_{a,b=1}^{n} \overline{[\alpha(k_2)]_{j,a}} [\alpha(-k_2)]_{a,b} u_b(\pi, -k_2) = \sum_{a=1}^{n} \left[\overline{\alpha(k_2)}\alpha(-k_2)\right]_{j,a} u_a(\pi, -k_2) \end{aligned}$$

that is $\overline{\alpha(k_2)}\alpha(-k_2) = \mathbb{1}_n$ or equivalently $\alpha(-k_2) = \alpha(k_2)^t$. We want to modify the basis $\{u_j(t, k_2)\}$ using a unitary-valued map $\beta : [-\pi, \pi] \times S^1 \to \mathcal{U}(n)$ which is periodic in the second argument. Then we hope that $v_j(t, k_2) = \sum_{a=1}^{n} [\beta_{j,a}(t, k_2)] u_a(t, k_2)$ is a periodic and symmetric frame. If we look closely at what happens when $k_1 = \pm\pi$ we get that:

$$\begin{aligned} v_j(-\pi, k_2) = \sum_{a=1}^{n} [\beta(-\pi, k_2)]_{j,a} u_a(-\pi, k_2) = \sum_{a=1}^{n} [\beta(-\pi, k_2)\alpha(k_2)]_{j,a} u_a(\pi, k_2) = \\ = \sum_{a=1}^{n} [\beta(-\pi, k_2)\alpha(k_2)\beta(\pi, k_2)^{-1}]_{j,a} v_a(\pi, k_2) \end{aligned}$$

So, in order to be periodic it must be true that

$$\beta(-\pi, k_2)\alpha(k_2) = \beta(\pi, k_2) \quad \forall k_2 \in S^1.$$

Instead, if we apply the symmetry we get that:

$$\begin{aligned} Tv_j(t, k_2) = \sum_{a=1}^{n} \overline{[\beta(t, k_2)]_{j,a}} Tu_a(t, k_2) = \sum_{j,a}^{n} \overline{[\beta(t, k_2)]_{j,a}} u_a(-t - k_2) = \\ = \sum_{a,b=1}^{n} \overline{[\beta(t, k_2)]_{j,a}} [\beta(-t, -k_2)^{-1}]_{a,b} v_b(-t, -k_2) = \\ = \sum_{a=1}^{n} [\overline{\beta(t, k_2)}\beta(-t, -k_2)^{-1}]_{j,a} v_a(-t, -k_2) \end{aligned}$$

Meaning that in order to be symmetric we must ask that

$$\overline{\beta(t, k_2)} = \beta(-t, -k_2) \quad \forall t \in (-\pi, \pi), k_2 \in S^1.$$

This means that, in particular, it must be true that:

$$\alpha(k_2) = \beta(\pi, k_2)\beta(\pi, -k_2)^t$$

Now, we can find two self-adjoint matrices L, R with $\alpha(0) = e^{iL}, \alpha(\pi) = e^{iR}$ with $L(0) = L(0)^t$, $L(\pi) = L(\pi)^t$. Then we can define β for $k_1 = \pi$ as the unitary-valued map $\beta : \{\pi\} \times [-\pi, \pi] \to$

$U(n)$:

$$\beta(\pi, k_2) = \begin{cases} \exp \frac{i}{2} \left[(\frac{\pi - k_2}{\pi}) L + \frac{k_2}{\pi} R \right] & \text{if} \quad k_2 \in [0, \pi], \\ \alpha(-k_2)^t \overline{\beta(\pi, -k_2)} & \text{if} \quad k_2 \in [-\pi, 0]. \end{cases}$$

This is perfectly continuous and periodic in k_2 due to the fact that L and R are equal to their transposes and

$$\alpha(0) = \beta(\pi, 0)\beta(\pi, 0)^t, \quad \alpha(\pi) = \beta(\pi, \pi)\beta(\pi, \pi)^t.$$

Now if the winding number of $\det(\beta(\pi, k_2))$ is l, we can define:

$$\beta(0, k_2) = \begin{pmatrix} e^{il\pi k_2} & 0 \\ 0 & \mathbb{I}_{n-1} \end{pmatrix}$$

Thanks to Theorem A.3, we know that $\beta(0, k_2)$ and $\beta(\pi, k_2)$ are homotopic, meaning that there is a continuous unitary-valued map $\gamma : [0, 1] \times S^1 \to U(n)$ such that $\gamma(0, k_2) = \beta(0, k_2)$ and $\gamma(1, k_2) = \beta(1, k_2)$. So we can finally define

$$\beta(t, k_2) = \begin{cases} \gamma(t/\pi, k_2) & \text{for } t \in [0, \pi] \\ \overline{\gamma(-t/\pi, -k_2)} & \text{for } t \in [-\pi, 0] \end{cases}$$

It is easy to check that this β satisfies all the condition we needed, so we got our symmetric frame.

- (*Symmetric unitary equivalence*)

  We can apply the previous argument, if $\dim(\mathcal{H}) < \infty$, or Lemma 2.9, if $\dim(\mathcal{H}) = \infty$, to $\mathbb{1} - P_0(k) = P_0(k)^\perp$ in order to obtain a continuous collection of orthonormal bases $\{v_j^0(k)\}_{j \in \{1, \cdots, \dim(\mathcal{H})\}}$ such that $Tv_j(k) = v_j(-k)$ where the first n vectors span $\mathrm{Ran}(P_0(k))$ while the others span $\ker(P_0(k))$. We can replicate this procedure on $P_1(k)$ to obtain

  $$\{v_j^1(k)\}_{j \in \{=1, \cdots, \dim(\mathcal{H})\}}$$

  with identical properties. Then the symmetric unitary equivalence we need is the unitary operator $V(k)$ such that:

  $$V(k)v_j^0(k) = v_j^1(k) \quad \text{for all} \quad j \in \{1, \cdots, \dim(\mathcal{H})\}, k \in \mathbb{T}^2.$$

- (*Symmetric homotopy*)

  If $\dim(\mathcal{H}) < \infty$, consider the triangle $\triangle$ inside $[-\pi, \pi]^2$ with vertexes

  $$Q_1 = (-\pi, \pi), \quad Q_2 = (\pi, \pi), \quad Q_3 = (\pi, -\pi).$$

  So after renaming $\lambda(k) = \det(V(k))$ we can define the unitary operator $V_0(k)$ with:

  $$V_0(k)v_j^0(k) = \begin{cases} \lambda(k)v_1^0(k) & \text{if} \quad j = 1 \\ v_j^0(k) & \text{otherwise} \end{cases} \qquad \forall k \in [-\pi, \pi]^2$$

Now we want to define a homotopy V_t between V_0 and V and to do so we need to work on $\triangle$. Notice that on the sides of the triangle $\overline{Q_1Q_2}$, $\overline{Q_1Q_3}$ and $\overline{Q_2Q_3}$ the projection-valued maps behave as a one-dimensional bosonic time-reversal symmetric projection-valued maps, in fact:

$$TP(k_1, \pi) = P(-k_1, -\pi)T = P(-k_1, \pi)T,$$

$$TP(\pi, k_2) = P(-\pi, -k_2)T = P(\pi, -k_2),$$

$$TP(k_1, -k_1) = P(-k_1, k_1)T.$$

So we can replicate the proof in Theorem 2.6 to build $V_t(k)$ for $k \in \partial\triangle$ with $TV_t(k) = V_t(-k)$ for all $t \in [0,1], k \in \partial\triangle$. This means that $V_t(k)$ is defined for $(t,k) \in \partial([0,1] \times \triangle)$. This region however is homeomorphic to S^2 and since $\pi_2(U(m)) = \{0\}$, thanks to Theorem A.3, it is always possible to extend the definition inside for $(t,k) \in [0,1] \times \triangle$. As a final adjustment we just need to define the map in the complementary triangle by imposing $V_t(k) = TV_t(-k)T$ for $k \in [-\pi,\pi]^2 \setminus \triangle$. Then $V_t(k)$ is continuous, periodic and symmetric and therefore:

$$P_t(k) = V_t(k)P_0(k)V_t(k)^{-1}$$

is the symmetric homotopy we were looking for since $[P_0(k), V_0(k)] = 0$.

Instead if $\dim(\mathcal{H}) = \infty$ we can replicate all of the previous arguments with $V_0(k) = \mathbb{1}$ since, thanks to Theorem A.2, $\mathcal{U}(\mathcal{H})$ is contractible.

$\square$

Chapter 3

Class D

In class D, the only symmetry present is an even particle-hole symmetry, so the object we want to study in this chapter is the one defined in Definition I.13 with the additional constraint that the symmetry mustt square to $\mathbb{1}$:

Definition 3.1 (even particle-hole symmetric pair of projection-valued maps)**.** Given an anti-unitary operator C acting on $\mathcal{H}$ with $C^2 = \mathbb{1}$, a pair of projection-valued maps $P^{\pm} : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ is particle-hole symmetric if and only if:

$$CP^{+}(k) = P^{-}(-k)C \quad \text{and} \quad P^{+}(k)P^{-}(k) = 0 \quad \forall k \in \mathbb{T}^d.$$

With this in mind, given two even particle-hole symmetric pairs of projection-valued maps $P_0^{\pm}, P_1^{\pm} : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$, we will try to answer the following questions:

Question 7 (Class D frame)**.** When will there be a continuous and even particle-hole symmetric frame $\{v_1(k), \cdots, v_{2n}(k)\}$, namely a continuous and periodic collection of orthonormal vectors where the first half spans $\mathrm{Ran}(P_0^{-}(k))$, the second half spans $\mathrm{Ran}(P_0^{+}(k))$ and $Cu_j(k) = u_{n+j}(-k)$ for all $j \in \{1, \cdots, n\}$? In case there is none, is it possible to have a frame with the properties listed before and regular pseudoperiodic conditions as those in Theorem 3.16?

Question 8 (Class D unitary equivalence)**.** When will there be a particle-hole symmetric unitary-valued map $U : \mathbb{T}^d \to \mathcal{U}(\mathcal{H})$ such that $P_1^{\pm}(k) = U(k)P_0^{\pm}(k)U(k)^{-1}$ and $CU(k) = U(-k)C$ for all $k \in \mathbb{T}^d$? This symmetric condition is the one present in the Introduction in Definition I.13

Question 9 (Class D homotopy)**.** When will there be a continuous and particle-hole symmetric pair of maps map $P^{\pm} : [0,1] \times \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ with $P^{\pm}(0,k) = P_0^{\pm}(k)$, $P^{\pm}(1,k) = P_1^{\pm}(k)$ and $CP_t^{+}(k) = P_t^{-}(-k)C$ for all $k \in \mathbb{T}^d$ and $t \in [0,1]$?

This is the first original part of our work. In this class, something interesting happens; the answers depend on the dimension of the Hilbert space. If $\dim(\mathcal{H}) = 2n$, we will observe a richer topology that is lost in higher dimension. Before moving on, there is a crucial remark to state that we will use extensively:

Remark 3.2. If $P^{\pm} : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ is an even particle-hole symmetric pair of projection-valued maps, then $P^{+}(k) + P^{-}(k)$ can be treated as a bosonic time-reversal symmetric projection-valued map because:

$$C(P^{+}(k) + P^{-}(k)) = (P^{-}(-k) + P^{+}(-k))C$$

3.1 d=0

Before moving to the actual problem, we need to list some properties about Pfaffians

Definition 3.3 (Pfaffian)**.** If $A \in M_{2n,2n}(\mathbb{C})$ with $A = -A^t$, then the Pfaffian of A, $\mathrm{Pf}(A)$, can be computed using the recursive formula:

$$\mathrm{Pf}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a \qquad \mathrm{Pf}(A) = \sum_{j=2}^{2n} (-1)^j a_{1,j} \,\mathrm{Pf}(A_{1,j})$$

where $A_{1,j}$ denotes the matrix A with both the first and j-th rows and columns removed.

Proposition 3.4. *The Pfaffian satisfies the following proprieties:*

1. $\mathrm{Pf}(A^t) = (-1)^n \,\mathrm{Pf}(A)$
2. $\mathrm{Pf}(\lambda A) = \lambda^n \,\mathrm{Pf}(A)$
3. $\mathrm{Pf}(A)^2 = \det(A)$
4. $\mathrm{Pf}(BAB^t) = \det(B)\,\mathrm{Pf}(A)$ *for any matrix* $B \in M_{2n,2n}(\mathbb{C})$
5. $\mathrm{Pf}\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \mathrm{Pf}(A_1)\,\mathrm{Pf}(A_2)$ *for any couple of skew-symmetric matrices* A_1, A_2.
6. $\mathrm{Pf}\begin{pmatrix} 0 & M \\ -M^t & 0 \end{pmatrix} = (-1)^{n(n-1)/2} \det(M)$

Let us see what happens when $\dim(\mathcal{H}) = 2n$: thanks to Lemma 2.2, there is a basis of $\mathcal{H}$ on which C acts as the standard complex conjugation. Using this basis we can see $\mathcal{H} \simeq \mathbb{C}^{2n}$ and $C = \mathcal{K}$. Then, let $P^{\pm} \in \mathrm{Proj}_n(\mathbb{C}^{2n})$ be a pair of projections such that $\mathcal{K}P^+ = P^-\mathcal{K}$. If we call

$$S := i\,(P^+ - P^-)\,,$$

then

$$\begin{cases} P^{-2} = P^- \\ P^{-*} = P^- \\ \mathcal{K}\,P^- = P^+\,\mathcal{K} \end{cases} \Leftrightarrow \begin{cases} S^* = -S \\ S^2 = -\mathbb{I} \\ \mathcal{K}\,S = S\,\mathcal{K} \end{cases} \Leftrightarrow \begin{cases} S^t = -S \\ S\,S^t = \mathbb{I} \\ S \in M_{2n,2n}(\mathbb{R}) \end{cases} \Leftrightarrow \begin{cases} S \in \mathrm{O}(2n) \\ S^t = -S \end{cases}$$

So S is a skew-symmetric, real-valued matrix and admits a *Pfaffian* (Definition 3.3). When $\dim(\mathcal{H}) = 2n$, we will often assume that $\mathcal{H} = \mathbb{C}^{2n}$ and freely swap between C and $\mathcal{K}$.

Definition 3.5 (the $\mathcal{P}$-invariant)**.** If $P^{\pm} : \mathbb{T}^d \to \mathrm{Proj}_n(\mathbb{C}^{2n})$ is a even particle-hole symmetric pairs of projection-valued maps, then, on the points $k_\star$ fixed by the involution $k \mapsto -k$, we can repeat the process done before to define:

$$\mathcal{P}(k_\star) := \mathrm{Pf}\left[i\,(P^+(k_\star) - P^-(k_\star))\right] = \mathrm{Pf}(S(k_\star))$$

Proposition 3.6. *If $k_\star \in \mathbb{T}^d$ is a point fixed by the involution $k \mapsto -k$, then $\mathcal{P}(k_\star) \in \{\pm 1\} = \mathbb{Z}_2$ and this quantity is invariant under homotopies that respect the particle-hole symmetry. However, this quantity is not invariant under symmetric unitary equivalences. In particular, if $V(k)$ is a unitary-valued map such that*

$$P_1^\pm(k) = V(k)P_0^\pm(k)V(k)^{-1} \quad \text{and} \quad CV(k) = V(-k)C \quad \forall k \in \mathbb{T}^d,$$

then

$$\mathcal{P}_1(k_\star) = \det(V(k_\star))\mathcal{P}_0(k_\star).$$

Proof. Using the properties of the Pfaffian detailed in Proposition 3.4, since $S \in \mathrm{O}(2n)$, it must be that $\det(S) = \pm 1$. But it is also true that $\det(S) = \mathrm{Pf}(S)^2$, so the only possibility is that $\det(S) = 1$ and $\mathrm{Pf}(S) = \pm 1$. Instead, if $P_t^\pm(k)$ is a homotopy between two even particle-hole symmetric pairs of projection-valued maps for $t \in [0,1]$, we have that $S_t(k_\star)$ is, for every $t \in [0,1]$ an orthogonal skew-symmetric matrix, so we can compute $\mathrm{Pf}(S_t(k_\star))$ for each t. Thus, we obtain a continuous function from $[0,1]$ to $\{0,1\}$ that must be constant. Finally, if two even particle-hole symmetric pairs of projection-valued maps are symmetrically unitarily equivalent, then it is obvious that $S_1(k_\star) = V(k_\star)S_0(k_\star)V(k_\star)^{-1} = V(k_\star)S_0(k_\star)V(k_\star)^T$, so, using the properties of the Pfaffian, $\mathcal{P}_1(k_\star) = \det(V(k_\star))\mathcal{P}_0(k_\star)$. □

Proposition 3.7. *Two even particle-hole symmetric pairs of projection-valued maps $P_0^\pm, P_1^\pm \in$ $\mathrm{Proj}_n(\mathcal{H})$ will always have a symmetric frame and be symmetrically unitarily equivalent. For the homotopy problem, if $\dim(\mathcal{H}) > 2n$, they are always homotopic with respect to the symmetry. However, if $\dim(\mathcal{H}) = 2n$, they are homotopic if and only if $\mathcal{P}_0 = \mathcal{P}_1$.*

Proof. To obtain a symmetric frame, we can find an orthonormal basis $\{v_j^0\}_{j\in\{1,\cdots,n\}}$ of $\mathrm{Ran}(P_0^-)$. Then we can define $\{v_{n+j} = Cv_j\}_{j\in\{=1,\cdots,n\}}$ and those will constitute an orthonormal basis of $\mathrm{Ran}(P_0^+)$.

For the symmetric unitary equivalence, we can use Proposition 2.3 or Proposition 2.2 to obtain a basis $\{v_j^0\}_{j\in\{2n+1,\cdots,\dim(\mathcal{H})\}}$ of $\ker(P_0^+ + P_0^-)$ such that $Cv_j = v_j$ for $j > 2n$. In the end, we have an orthonormal basis $\{v_j\}_{j\in\{1,\cdots,\dim(\mathcal{H})\}}$ such that the first n vectors span $\mathrm{Ran}(P_0^-)$, the second n vectors span $\mathrm{Ran}(P_0^+)$ and the others span $\ker(P_0^+ + P_0^-)$. They also satisfy the symmetric relations

$$Cv_j^0 = \begin{cases} v_{n+j}^0 & \text{if } 1 \le j \le n \\ v_{j-n}^0 & \text{if } n+1 \le j \le 2n \\ v_j^0 & \text{if } j > 2n \end{cases}$$

If we repeat this procedure to $P_1^\pm$ we will obtain the orthonormal basis $\{v_j^1\}_{j\in\{1,\cdots,\dim(\mathcal{H})\}}$ and the unitary equivalence is the unitary operator $V \in \mathcal{U}(\mathcal{H})$ such that $V(v_j^0) = v_j^1$ for all $j \in \{1,\cdots,\dim(\mathcal{H})\}$. In fact, it is trivial to verify that $CV = VC$ and that $P_1^\pm = VP_0^\pm V^{-1}$.

Instead, for the homotopy problem in $\dim(\mathcal{H}) = 2n$, we can apply the Lemma 2.2 to obtain a basis of $\mathcal{H}$ on which C acts as $\mathcal{K}$, which means that we can interpret $\mathcal{H} = \mathbb{C}^{2n}$, $C = \mathcal{K}$, and $V \in \mathrm{O}(2n)$. Then, since we already proved that $\mathcal{P}$ is a homotopy invariant, we only need to prove that it is a complete topological invariant. So, if $\mathcal{P}_0 = \mathcal{P}_1$, then $\mathrm{Pf}(i(P_0^+ - P_0^-)) = \mathrm{Pf}(i(P_1^+ - P_1^-))$.

However, it is also true that:

$$i(P_1^+ - P_1^-) = V[i(P_0^+ - P_0^-)]V^{-1}$$

and, according to the the properties of the Pfaffian (Proposition 3.4), $\mathcal{P}_0 = \mathcal{P}_1 \det(V)$, so $\det(V) = 1$ and $V \in \mathrm{SO}(2n)$. Since $\mathrm{SO}(2n)$ is connected, there is a homotopy V_t with $V_1 = V$ and $V_0 = \mathbb{1}$. Since we are moving inside $\mathrm{O}(2n)$, $\mathcal{K}V_t = V_t\mathcal{K}$ for every $t \in [0,1]$. Therefore, $P_t^\pm = V_t P_0^\pm V_t^{-1}$ is the symmetric homotopy we were looking for. We can do something similar if $2n < \dim(\mathcal{H}) < \infty$, in this case we can find a homotopy even if $\det(V) = -1$. In fact, in this scenario, we can slightly modify the unitary equivalence by putting:

$$\tilde{V}(v_j^0) = \begin{cases} v_j^1 & \text{if } j \neq 2n+1 \\ -v_{2n+1}^1 & \text{if } j = 2n+1 \end{cases}$$

This new $\tilde{V}$ is once again a symmetric unitary equivalence with $\det(\tilde{V}) = 1$, so we can replicate the proof done before. Finally, if $\dim(\mathcal{H}) = \infty$, we can see $\ker(C - \mathbb{1}) = \mathcal{H}_+$ as a real Hilbert space, then V will be an orthogonal operator acting over $\mathcal{H}_+$. Using Theorem A.2 we can find a homotopy V_t inside $\mathcal{O}(\mathcal{H}_+)$ with $V_1 = V$ and $V_0 = \mathbb{1}$, those can be extend to a unitary-valued map in $\mathcal{U}(\mathcal{H})$. Since this homotopy is inside $\mathcal{O}(\mathcal{H}_+)$ it is true that $CV_t = V_tC$ for every $t \in [0,1]$. Therefore, $P_t^\pm = V_t P_0^\pm V_t^{-1}$ is the symmetric homotopy we were looking for. □

3.2 d=1

Proposition 3.8. *Two even particle-hole symmetric pairs of projection-valued maps* $P_0^\pm, P_1^\pm : \mathbb{T}^1 \to \mathrm{Proj}_n(\mathcal{H})$ *will always have a symmetric frame and be symmetrically unitarily equivalent. For the homotopy problem, if* $\dim(\mathcal{H}) > 2n$*, they are always homotopic with respect to the symmetry. However, if* $\dim(\mathcal{H}) = 2n$*, they are homotopic with respect to the particle-hole symmetry if and only if* $\mathcal{P}_0(0) = \mathcal{P}_1(0)$ *and* $\mathcal{P}_0(\pi) = \mathcal{P}_1(\pi)$.

Proof. The first two statements of the proposition are very easy to prove; in fact, we can apply Proposition 1.4 to construct a periodic and continuous frame $\{v_j^0(k)\}_{j\in\{1,\cdots,n\}}$ of $P_0^-(k)$: Then $\{v_{n+j}(k) = Cv_j^0(-k)\}_{j\in\{1,\cdots,n\}}$ will be a periodic and continuous frame of $P_0^+(k)$ and the union of the two collections will constitute the symmetric frame we needed.

Moreover, we can use Lemma 2.5 or Proposition 2.6 to build a continuous family of orthonormal bases

$$\{v_j^0(k)\}_{j\in\{2n+1,\cdots,\dim(\mathcal{H})\}} \quad \text{of} \quad \ker(P_0^+(k) + P_0^-(k))$$

such that $Cv_j^0(k) = v_j^0(-k)$ for all $j > 2n$. We can repeat this procedure to $P_1^\pm(k)$ in order to obtain $\{v_j^1(k)\}_{j\in\{1,\cdots,\dim(\mathcal{H})\}}$ with identical properties, and afterward the symmetric unitary equivalence we needed is the unitary-valued map $V(k)$ with $V(k)v_j^0(k) = v_j^1(k)$ for all $k \in \mathbb{T}^1$ and $j \in \{1,\cdots,\dim(\mathcal{H})\}$. In fact, it is elementary to check that $CV(k) = V(-k)C$ and that $P_1^\pm(k) = V(k)P_0(k)^\pm V(k)^{-1}$.

Instead, for the homotopy problem in $\dim(\mathcal{H}) = 2n$, we can apply Lemma 2.2 to obtain a basis of $\mathcal{H}$ on which C acts as $\mathcal{K}$, which means that we can interpret $\mathcal{H} = \mathbb{C}^{2n}$, $C = \mathcal{K}$ and $V \in \mathrm{O}(2n)$ as we did in Proposition 3.7. Then, since we already proved that $\mathcal{P}$ is a homotopy invariant, we only need to prove that the couple $(\mathcal{P}(0), \mathcal{P}(\pi))$ is a complete topological invariant. So, if $\mathcal{P}_0(0) = \mathcal{P}_1(0)$ and $\mathcal{P}_0(\pi) = \mathcal{P}_1(\pi)$, then

$$\mathrm{Pf}(i(P_0^+(k_\star) - P_0^-(k_\star))) = \mathrm{Pf}(i(P_1^+(k_\star) - P_1^-(k_\star))) \quad \text{for } k_\star = 0, \pi.$$

However, it is also true that:

$$i(P_1^+(k) - P_1^-(k)) = V(k)[i(P_0^+(k) - P_0^-(k))]V(k)^{-1}$$

and, by the properties of the Pfaffian (Proposition 3.4),

$$\mathcal{P}_0(0) = \mathcal{P}_1(0)\det(V(0)) \quad \mathcal{P}_0(\pi) = \mathcal{P}_1(\pi)\det(V(\pi)),$$

so $\det(V(0)) = 1 = \det(V(\pi))$, which means that $V(0), V(\pi) \in \mathrm{SO}(2n)$. Since $\mathrm{SO}(2n)$ is connected, there are two homotopies $V_t(0)$ and $V_t(\pi)$ with $V_1(k_\star) = V(k_\star)$ and $V_0(k_\star) = \mathbb{1}$ for $k_\star = 0, \pi$. Since we are moving inside $\mathrm{O}(2n)$, $\mathcal{K}V_t(k_\star) = V_t(k_\star)\mathcal{K}$ for every $t \in [0, 1]$. Now consider $\det(V(k))$. This is always a complex number with unitary module such that $\det(V(0)) = \det(V(\pi)) = 1$, so we can define the unitary operator $V_0(k)$ for $k \in [0, \pi]$ as the operator such that:

$$V_0(k)v_j^0(k) = \begin{cases} \det(V(k))v_1^0(k) & \text{if } j = 1, \\ v_j^0(k) & \text{otherwise.} \end{cases}$$

So we defined a homotopy $V_t(k) \in \mathcal{U}(2n)$ for $(t, k) \in \partial([0, 1]\times[0, \pi])$, but this region is homeomorphic to S^1 and it is possible to extend the map inside this region if and only if the winding number of its determinant is zero. However, we build those unitary operators to exactly satisfy this property, in fact $\det(V_t(0)) = \det(V_t(\pi)) \equiv 1$ and $\det(V_0(k)) = \det(V_1(k))$, so it is possible to define $V_t(k)$ for $(t, k) \in [0, 1] \times [0, \pi]$. Now we can extend the definition for $k \in [-\pi, 0]$ by imposing

$$V_t(-k) = CV_t(k)C = \mathcal{K}V_t(k)\mathcal{K} \quad \forall t \in [0, 1],$$

which does not conflict with periodicity and continuity, since $V_t(0), V_t(\pi) \in \mathrm{SO}(2n)$, so they always commute with the complex conjugation. Finally, we can define our symmetric homotopy by imposing:

$$P_t^\pm(k) = V_t(k)P_0^\pm(k)V_t(k)^{-1}$$

in fact $CP_t^+(k) = P_t^-(-k)C$, since $CV_t(k) = V_t(-k)C$ and $P_0^\pm(k)$ are the originals since

$$[V_0(k), P_0(k)] = 0.$$

Instead, if $2n < \dim(\mathcal{H}) < \infty$, we can use the additional dimension to eliminate the uncomfortable winding numbers. To do so, consider $\det(V(k)) = \lambda(k)$, then, the condition $CV(k) = V(-k)C$, tells us that $\lambda(-k) = \overline{\lambda(k)}$. So we can adjust the original unitary equivalence by defining the symmetric unitary equivalence $\tilde{V}(k)$ as the unitary operator such that

$$\tilde{V}(k)v_j^0(k) = \begin{cases} v_j^1(k) & \text{if } j \neq 2n+1 \\ \lambda(k)^{-1}v_{2n+1} & \text{if } j = 2n+1 \end{cases}$$

Now we can replicate every step of the previous argument, but since $\det(\tilde{V}(k)) \equiv 1$ we can have that $V_0(k) \equiv \mathbb{1}$.

Finally, if $\dim(\mathcal{H}) = \infty$, we can identify the operators that commute with C as the orthogonal operators $\mathrm{O}(\mathcal{H}_+)$ of the real Hilbert space $\mathcal{H}_+ = \ker(T - \mathbb{1})$. Since $V(0)$ and $V(k)$ commute with C and $\mathrm{O}(\mathcal{H})$ is contractible, according to Theorem A.2, it is possible to define two homotopies $V_t(0), V_t(\pi)$ inside $\mathrm{O}(\mathcal{H}_+)$ with $V_1(k_\star) = V(k_\star)$ and $V_0(k) = \mathbb{1}$. As we did before, we can use the fact that $\mathcal{U}(\mathcal{H})$ is contractible to define $V_t(k)$ also for $(t,k) \in [0,1]\times[0,\pi]$ and then use the symmetry condition $V_t(k) = CV_t(-k)C$ to extend the definition also for $k \in [-\pi, 0]$. It is obvious to notice that this does not conflict with continuity and periodicity. Then $P_t^\pm(k) = V_t(k)P_0^\pm(k)V_t(k)^{-1}$ is once again the symmetric homotopy we needed. $\square$

Before moving on to the case $d = 2$, it is worth mentioning a different way to interpret the product of two invariants $\mathcal{P}$ that will be useful for further arguments.

Definition 3.9 (The $\mathcal{I}$ invariant)**.** If $P^\pm : \mathbb{T}^1 \to \mathrm{Proj}_n(\mathbb{C}^{2n})$ is an even particle-hole symmetric pair of projection-valued maps with $CP^+(k) = P^-(-k)C$ and $U(k)$ is a periodic and continuous unitary-valued map such that

$$P^\pm(k) = U(k)P^\pm(0)U(k)^{-1} \quad \text{and} \quad CU(k) = U(-k)C,$$

then we define

$$\mathcal{I} = e^{i[\det(U(k))]} \in \{\pm 1\} = \mathbb{Z}_2.$$

Proposition 3.10. *It is true that:*

$$\mathcal{I} = \mathcal{P}(0)\mathcal{P}(\pi) \in \mathbb{Z}_2.$$

Proof. The symmetry relation $CU(k) = U(-k)C$ can be read on a basis on which C acts as $\mathcal{K}$ as the fact that $\overline{U(k)} = U(-k)$, meaning that $U(0), U(\pi) \in \mathrm{O}(2n)$. Moreover, using the relation $P^\pm(k) = U(k)P^\pm(0)U(k)^{-1}$, we get that:

$$i(P^+(\pi) - P^-(\pi)) = U(\pi)(i(P^+(0) - P^-(0))U(\pi)^{-1},$$

$$i(P^+(0) - P^-(0)) = U(0)(i(P^+(0) - P^-(0))U(0)^{-1}.$$

So, using the properties of the Pfaffian (Proposition 3.4) we get:

$$\mathcal{P}(\pi) = \det(U(\pi))\mathcal{P}(0) \quad \mathcal{P}(0) = \det(U(0))\mathcal{P}(0),$$

thus $\det(U(0)) = 1$ and $\det(U(\pi)) = \mathcal{P}(\pi)\mathcal{P}(0)$. Now we can use the standard projection $\pi : \mathbb{R} \to S^1$ with $\pi(t) = e^{i2\pi t}$. Using basic topology we know that every continuous map $\lambda : \mathbb{T}^1 \to S^1$ can be lifted to a continuous map $\mu : \mathbb{T}^1 \to \mathbb{R}$ such that $\lambda(k) = \pi(\mu(k))$ and $[\lambda] = \mu(\pi) - \mu(-\pi)$. If we choose $\lambda(k) = \det(U(k))$, then $\lambda(0) = 1$ and we can choose the lifting μ with $\mu(0) = 0$ so that the property $\lambda(k) = \overline{\lambda(-k)}$ translates into $\mu(k) = -\mu(-k)$.

Finally, if $\det(U(\pi)) = -1$, then $\mu(\pi) \in \mathbb{N} + 1/2$. So, the winding number $[\lambda(k)] = \mu(\pi) - \mu(-\pi) = 2\mu(\pi)$ is odd. Instead, if $\det(U(\pi)) = 1$, then $\mu(\pi) \in \mathbb{N}$. So, $[\lambda(k)] = \mu(\pi) - \mu(-\pi) = 2\mu(\pi)$ is even. In the end, we explicitly get $\mathcal{I} = e^{i\pi[\det(U(k))]} = \mathcal{P}(0)\mathcal{P}(\pi)$. □

Remark 3.11. Since $\mathcal{P}(0), \mathcal{P}(\pi)$ are homotopic invariants of the particle-hole symmetry pair of projector-valued maps $P^{\pm}(k)$, the quantity $\mathcal{I}$ is once again a homotopic invariant of the pair and does not depend on the choice of the unitary-valued map U provided that the defining properties hold.

Moreover, the statement of Proposition 3.8 can be reformulated by saying that the complete set of topological invariants that describe the homotopy class of particle-hole symmetry pairs of projector-valued maps is $(\mathcal{P}(0), \mathcal{I})$ instead of $(\mathcal{P}(0), \mathcal{P}(\pi))$.

Finally, it may be interesting to point out the fact that there is a way to write $\mathcal{I}$ as a Berry phase; in fact, if $\{v_j(k)\}_{j\in\{1,\cdots,n\}}$ is a periodic and smooth frame of $P^-(k)$, we can use the arguments discussed in [45] to obtain the following expression:

$$\mathcal{I} = \exp\left\{ i\pi \left[\frac{1}{i\pi} \int_{S^1} dk \sum_{j=1}^{n} \langle u_j(k), \partial_k u_j(k)\rangle \right] \right\}$$

3.3 d=2

We want to start this section with a small curious property that relates the Pfaffian with the Chern number when $\dim(\mathcal{H}) = 2n$.

Proposition 3.12 (Class D Kato-Nagy 2-d construction)**.** *If $P^{\pm} : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathbb{C}^{2n})$ is bosonic time-reversal symmetric projection-valued map, then there is a continuous unitary-valued map $U : [-\pi, \pi] \times \mathbb{T}^1 \to \mathcal{U}(\mathcal{H})$, which is periodic in the second argument but not in the first argument, such that:*

$$P^{\pm}(t, k_2) = U(t, k_2)P^{\pm}(0, 0)U(t, k_2)^{-1}, \quad CU(t, k_2) = U(-t, -k_2)C$$

Proof. We want to follow the step of Proposition 2.8. Once again we can divide $[0, \pi]$ into a finite number of intervals $[t_j, t_{j+1}]$ for all $0 \le j \le J$ with $t_0 = 0, t_J = \pi$ such that

$$\|P^-(t_j, k_2) - P^-(t, k_2)\| = \|P^+(t_j, k_2) - P^+(t, k_2)\| < 1/2$$

for all $t \in [t_j, t_{j+1}]$. Then we can define J unitary-valued maps:

$$\begin{aligned} U_j(t, k_2) = &\left[P^-(t, k_2)P^-(t_j, k_2) + (\mathbb{1} - P^-(t, k_2))(\mathbb{1} - P^-(t_j, k_2))\right] \cdot \\ &\cdot \left[\mathbb{1} - (P^-(t_j, k_2) - P^-(t, k_2))^2\right]^{-1/2} \end{aligned} \qquad \forall t \in [t_j, t_{j+1}]$$

Those connect $P^-(t_j, k_2)$ with $P^-(t, k_2)$. However, since $\mathcal{H} = \mathbb{C}^{2n}$ and by hypothesis $P^-(k)P^+(k) \equiv 0$, it is true that $P^-(k) + P^+(k) = \mathbb{1}$. So, $U_j(t, k_2)$ also connects $P^+(t_j, k_2)$ with $P^+(t, k_2)$ for all $k_2 \in \mathbb{T}^1$ and for all $t \in [t_j, t_{j+1}]$. The unitary equivalence for $t \in [0, 2\pi]$ is as:

$$U(t, k_2) = U_j(t, k_2)U_{j-1}(t_j, k_2) \cdots U_1(t_2, k_2)U_0(t_1, k_2) \quad \text{if} \quad t \in [t_j, t_{j+1}]$$

Once again this is continuous, periodic in k_2 and such that

$$P^{\pm}(t, k_2) = U(t, k_2)P^{\pm}(0, k_2)U(t, k_2)^{-1}$$

To conclude we need to apply Proposition 3.8, this tells that the projection-valued maps $P^{\pm}(0, k_2)$ are symmetrically unitarily equivalent to the constant pair $P^{\pm}(0, 0)$, so there is a unitary equivalence $V(k_2)$ such that $P^{\pm}(0, k_2) = V(k_2)P \pm (0, 0)V(k_2)^{-1}$ with $CV(k_2) = V(-k_2)C$ so the unitary we are looking for is:

$$\tilde{U}(t, k_2) = \begin{cases} U(t, k_2)V(k_2) & \text{if} \quad t \in [0, \pi] \\ CU(-t, -k_2)V(-k_2)C & \text{if} \quad t \in [-\pi, 0] \end{cases}$$

This is continuous because for $t = 0$ it is true that $CV(k_2) = V(-k_2)C$. □

Proposition 3.13. *Given a even particle-hole symmetric pair of projection-valued maps $P^{\pm} : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathbb{C}^{2n})$, there are four points fixed by the involution $k \mapsto -k$. They are:*

$$k_\star = (0, 0), (\pi, 0), (0, \pi), (\pi, \pi).$$

For those points, it is true that:

$$\mathcal{P}(0, 0)\mathcal{P}(\pi, 0)\mathcal{P}(0, \pi)\mathcal{P}(\pi, \pi) = e^{i\,\mathrm{Ch}(P^-)}.$$

Equivalently, since every Pfaffian is in $\mathbb{Z}_2$, if $U_0(k_2), U_\pi(k_2)$ are two unitary-valued maps such that:

$$P^{\pm}(0, k_2) = U_0(k_2)P^{\pm}(0, 0)U_0(k_2)^{-1}, \quad CU_0(k_2) = U_0(-k_2)C$$

$$P^{\pm}(\pi, k_2) = U_\pi(k_2)P^{\pm}(\pi, 0)U_\pi(k_2)^{-1}, \quad CU_\pi(k_2) = U_\pi(-k_2)C,$$

then:

$$e^{i\{[\det(U_\pi(k_2))] - [\det(U_0(k_2))]\}} = e^{-i\,\mathrm{Ch}(P^-)}.$$

Proof. We can use Proposition 3.12 to construct a unitary-valuedmap $U(t, k_2)$ such that

$$P^{\pm}(t, k_2) = U(t, k_2)P^{\pm}(0, 0)U(t, k_2)^{-1} \quad \text{and} \quad CU(t, k_2) = U(-t, -k_2)C.$$

Now we can define $U_0(k_2) = U(0, k_2)$, but we need to be careful with $U_\pi(k_2)$, in fact, we must define it as:

$$U_\pi(k_2) = \begin{cases} U(\pi, k_2)U(\pi, 0)^{-1} & \text{on} \quad \mathrm{Ran}(P^+(\pi, 0)) = \mathrm{Ran}(P^+(-\pi, 0)) = \mathrm{Ran}(P^-(\pi, 0))^\perp \\ U(-\pi, k_2)U(-\pi, 0)^{-1} & \text{on} \quad \mathrm{Ran}(P^-(\pi, 0)) = \mathrm{Ran}(P^-(-\pi, 0)) = \mathrm{Ran}(P^+(\pi, 0))^\perp \end{cases}$$

in fact only in this way $CU_\pi(k_2) = U_\pi(-k_2)C$. Now, thanks to the continuity in t, it is true that

$$[\det(U_0(k_2))] = [\det(U(0, k_2))] = [\det(U(\pi, k_2))] = [\det(U(\pi, k_2)U(\pi, 0)^{-1})]$$

Therefore:

$$[\det(U_\pi(k_2))] - [\det(U_0(k_2))] = [\det\left(U(\pi, 0)U(\pi, k_2)^{-1}U_\pi(k_2)\right)]$$

However, it is also true that:

$$U(\pi,0)U(\pi,k_2)^{-1}U_\pi(k_2) = \begin{cases} \mathbb{1} & \text{on} \quad P^+(\pi,0) \\ U(\pi,0)U(\pi,k_2)^{-1}U(-\pi,k_2)U(-\pi,0)^{-1} & \text{on} \quad P^-(\pi,0) \end{cases}$$

So its determinant is equal to the determinant of $U(\pi,0)U(\pi,k_2)^{-1}U(-\pi,k_2)U(-\pi,0)^{-1}$ seen as an operator acting on $\mathrm{Ran}(P^-(\pi,0))$. Since constant matrices do not contribute to the winding number, we can define $\gamma(k_2) \in U(n)$, for an orthonormal basis $\{v_j\}_{j\in\{1,\cdots,n\}}$ of $\mathrm{Ran}(P^-(0,0))$, as

$$[\gamma(k_2)]_{a,b} = \left\langle v_a, U(\pi,k_2)^{-1}U(-\pi,k_2)v_b\right\rangle = \langle U(\pi,k_2)v_a, U(-\pi,k_2)v_b\rangle$$

and conclude that:

$$[\det\left(U(\pi,0)U(\pi,k_2)^{-1}U_\pi(k_2)\right)] = [\det(\gamma(k_2))]$$

We are close to the conclusion, but we need a unitary-valued map like the one in Definition 1.6 to compute the Chern number. To obtain it, we can use the procedure done in Remark 2.7. Let $\tilde{U}(k_2) \in \mathcal{U}(\mathcal{H})$ be the operator such that:

$$\begin{cases} U(t,k_2) & \text{for} \quad t \in [0,\pi], \\ U(t-2\pi,k_2)U(-\pi,k_2)^{-1}U(\pi,k_2) & \text{for} \quad t \in [\pi,2\pi]. \end{cases}$$

This is such that $P^-(t,k_2) = U(t,k_2)P^-(0,0)U(t,k_2)^{-1}$ so we can compute the Chern number using:

$$\begin{aligned} [\alpha]_{a,b}(k_2) &= \left\langle v_a, \tilde{U}(0,k_2)^{-1}\tilde{U}(2\pi,k_2)v_b\right\rangle = \\ &= \left\langle v_a, U(0,k_2)^{-1}U(0,k_2)U(-\pi,k_2)^{-1}U(\pi,k_2)v_b\right\rangle = \\ &= \langle U(-\pi,k_2)v_a, U(\pi,k_2)v_b\rangle = \overline{\langle U(\pi,k_2)v_b, U(-\pi,k_2)v_a\rangle} = \\ &= \overline{[\gamma(k_2)]}_{b,a}. \end{aligned}$$

To obtain that

$$\mathrm{Ch}(P^-) = [\det(\alpha(k_2))] = [\det(\gamma(k_2)^*)] = -[\det(\gamma(k_2))]$$

and this, plus Proposition 3.10, concludes the proof. □

Example 3.14. Although it is relatively easy to show examples of symmetric projectors with arbitrary Chern numbers, it is slightly more difficult to exhibit an even particle-hole symmetric pair of projection-valued maps with

$$e^{i\pi\,\mathrm{Ch}(P^-)} = \mathcal{P}(0,0)\mathcal{P}(\pi,0)\mathcal{P}(0,\pi)\mathcal{P}(\pi,\pi) = -1.$$

so we give an example to show that the topological class of projectors with $e^{i\pi\,\mathrm{Ch}(P^-)} = -1$ is not empty. Consider for $k_1, k_2 \in [-\pi,\pi]$ the following matrices:

$$P^-(k_1,k_2) = \begin{pmatrix} \frac{1}{2} + \left|\cos\frac{k_1}{2}\right|\frac{\sin k_2}{2} & \frac{(\cos k_2 - 1)}{2}\left(-i\cos^2\frac{k_1}{2} - \frac{1}{2}\sin k_1\right) - \frac{i}{2} \\ \frac{(\cos k_2 - 1)}{2}\left(i\cos^2\frac{k_1}{2} - \frac{1}{2}\sin k_1\right) + \frac{i}{2} & \frac{1}{2} - \left|\cos\frac{k_1}{2}\right|\frac{\sin k_2}{2} \end{pmatrix}$$

$$P^+(k_1,k_2) = \begin{pmatrix} \frac{1}{2} - \left|\cos\frac{k_1}{2}\right| \frac{\sin k_2}{2} & \frac{(\cos k_2 - 1)}{2}\left(i\cos^2\frac{k_1}{2} - \frac{1}{2}\sin k_1\right) + \frac{i}{2} \\ \frac{(\cos k_2 - 1)}{2}\left(-i\cos^2\frac{k_1}{2} - \frac{1}{2}\sin k_1\right) - \frac{i}{2} & \frac{1}{2} + \left|\cos\frac{k_1}{2}\right| \frac{\sin k_2}{2} \end{pmatrix}$$

Then it is easy to see (after some straightforward computations) that the following proprieties are true:

1. $P^+(k_1,k_2) + P^-(k_1,k_2) = \mathbb{1}$.
2. $\mathcal{K}P^-(k_1,k_2) = P^+(-k_1,-k_2)\mathcal{K}$.
3. $P^\pm(k_1,k_2)^* = P^\pm(k_1,k_2)$.
4. $(P^\pm(k_1,k_2))^2 = P^\pm(k_1,k_2)$.
5. $P^-(k_1,k_2)P^+(k_1,k_2) \equiv 0$.
6. $P^\pm(k_1,k_2) = P^\pm(k_1+2l\pi, k_2+2m\pi) \quad \forall l,m \in \mathbb{Z}$.

This means that they are an even particle-hole symmetric pair of projection-valued maps. If we compute then $S(k_1,k_2) = i[P^+(k_1,k_2) + P^-(k_1,k_2)]$ we get:

$$S(k_1,k_2) = \begin{pmatrix} -i\left|\cos\frac{k_1}{2}\right|\sin k_2 & (\cos k_2 - 1)(-\cos^2\frac{k_1}{2} + \frac{i}{2}\sin k_1) - 1 \\ (\cos k_2 - 1)(\cos^2\frac{k_1}{2} + \frac{i}{2}\sin k_1) + 1 & i\left|\cos\frac{k_1}{2}\right|\sin k_2 \end{pmatrix}$$

Now it is easy to compute:

$$S(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad S(\pi,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad S(0,\pi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad S(\pi,\pi) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So only one Pfaffian will be different from the others, therefore $\mathcal{L} = -1$.

Remark 3.15. Given a even particle-hole symmetric pair of projection-valued maps $P^\pm : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$, it is true that $\mathrm{Ch}(P^-) = -\mathrm{Ch}(P^+)$, in fact, in view of Remark 3.2, $P^+ + P^-$ is a time-reversal symmetric projection-valued map. So, thanks to Remark 2.7, $\mathrm{Ch}(P^+ + P^-) = 0$. Finally, the thesis is obtained by using the additivity of the Chern number proved in the Lemma 1.8.

Theorem 3.16. *Consider a even particle-hole symmetric pair of projection-valued maps* $P^\pm$: $\mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$*, then it is always possible to construct a symmetric and quasiperiodic frame* $\{v_j(k)\}_{j\in\{1,\cdots,2n\}}$ *such that:*

- *The first* n *vectors span* $\mathrm{Ran}(P^-(k))$*, and the others span* $\mathrm{Ran}(P^+(k))$*.*
- $Cv_j(k) = v_{n+j}(-k)$ *for* $1 \le j \le n$*.*
- *All of them are periodic in* k *except* v_1 *and* v_{n+1} *which are only periodic in* k_2 *with pseudo-periodicity in* k_1*:*

$$v_1(\pi,k_2) = e^{i\,\mathrm{Ch}(P^-)k_2}v_1(-\pi,k_2), \quad v_{n+1}(\pi,k_2) = e^{i\,\mathrm{Ch}(P^+)k_2}v_{n+1}(-\pi,k_2), \quad \forall k_2 \in \mathbb{T}^1.$$

Moreover, two even particle-hole symmetric pairs of projection-valued maps $P_0^\pm, P_1^\pm : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$ *are symmetrically unitarily equivalent if and only if* $\mathrm{Ch}(P_0^-) = \mathrm{Ch}(P_1^-)$*. Finally, if* $\dim(\mathcal{H}) > 2n$*,* *they are also symmetrically homotopic if and only if* $\mathrm{Ch}(P_0^-) = \mathrm{Ch}(P_1^-)$. *However, if* $\dim(\mathcal{H}) = 2n$, *they are symmetrically homotopic if and only if*

$$\mathrm{Ch}(P_0^-) = \mathrm{Ch}(P_1^-) \quad \textit{and} \quad \mathcal{P}_0(k_\star) = \mathcal{P}_1(k_\star) \quad \textit{for} \quad k_\star = (0,0), (\pi, 0), (0, \pi).$$

Remark 3.17. Before the proof, we want to address a small fact: Why does $\mathcal{P}(\pi,\pi)$ not appear? The answer is pretty simple; thanks to Proposition 3.13, we have that

$$\mathcal{P}(\pi,\pi) = \mathcal{P}(0,0)\mathcal{P}(\pi,0)\mathcal{P}(0,\pi)e^{i\pi\,\mathrm{Ch}(P^-)}$$

So knowing the four terms on the right side is enough to determine $\mathcal{P}(\pi,\pi)$.

Proof.

- (*Symmetric frame*)

  We can apply Theorem 1.10 to the projection-valued map $\tilde{P}(k_1,k_2) = P^-(k_1-\pi,k_2)$ to obtain a frame $\{\tilde{v}_j(k)\}_{j\in\{1,\cdots,n\}}$ of $\tilde{P}(k)$ with pseudo-periodicity $\tilde{v}_1(2\pi,k_2) = e^{i\,\mathrm{Ch}(\tilde{P})k_2}v_1(0,k_2)$. This means that $\{v_j(k_1,k_2) = \tilde{v}_j(k_1+\pi,k_2)\}_{j\in\{1,\cdots,n\}}$ will be a pseudo-periodic frame of $P^-(k)$ and, since P^- and $\tilde{P}$ are homotopic through $P_t(k_1,k_2) = P^-(k_1-t\pi,k_2)$, $\mathrm{Ch}(P^-) = \mathrm{Ch}(\tilde{P})$. Then $\{v_{n+j}(k) = Cv_j(-k)\}_{j\in\{1,\cdots,n\}}$ will be a frame of $P^+(k)$. The union of the two frames will constitute the symmetric quasi-periodic frame we were looking for because $\mathrm{Ch}(P^-) = -\mathrm{Ch}(P^+)$ thanks to Remark 3.15.

- (*Symmetric unitary equivalence*)

  We can start slowly by noticing that if two even particle-hole symmetric pairs of projection-valued maps are symmetrically unitarily equivalent, then the projection-valued maps P_0^-, P_1^- must be unitarily equivalent, therefore $\mathrm{Ch}(P_0^-) = \mathrm{Ch}(P_1^-)$. To prove the converse we can apply the previous argument to $P_0^\pm(k)$ and $P_1^\pm(k)$ and Lemma 2.9, if $\dim(\mathcal{H}) = \infty$, or Proposition 2.10, if $\dim(\mathcal{H}) < \infty$ to obtain two collections of orthonormal bases $\{v_j^0(k)\}_{j\in\{1,\cdots,\dim(\mathcal{H})\}}$, $\{v_j^1(k)\}_{j\in\{1,\cdots,\dim(\mathcal{H})\}}$ such that:

  1. All $v_j^0(k)$ and $v_j^1(k)$ are continuous in k.
  2. All $v_j^0(k)$ and $v_j^1(k)$ are periodic in k with the only exceptions being:

  $$v_1^0(\pi,k_2) = e^{i\,\mathrm{Ch}(P_0^-)k_2}v_1^0(-\pi,k_2), \quad v_{n+1}^0(\pi,k_2) = e^{i\,\mathrm{Ch}(P_0^+)k_2}v_{n+1}^0(-\pi,k_2), \quad \forall k_2\in\mathbb{T}^1$$

  $$v_1^1(\pi,k_2) = e^{i\,\mathrm{Ch}(P_1^-)k_2}v_1^1(-\pi,k_2), \quad v_{n+1}^1(\pi,k_2) = e^{i\,\mathrm{Ch}(P_1^+)k_2}v_{n+1}^1(-\pi,k_2), \quad \forall k_2\in\mathbb{T}^1.$$

  3. The vectors are linked by the symmetric relations:

  $$Cv_j^0(k) = v_{n+j}^0(-k) \quad Cv_j^1(k) = v_{n+j}^1(-k) \quad \text{for all } k\in\mathbb{T}^2, j\in\{1,\cdots,n\}$$

  $$Cv_j^0(k) = v_j^0(-k) \quad Cv_j^1(k) = v_j^1(-k) \quad \text{for all } k\in\mathbb{T}^2, j > 2n.$$

  4. The first n vectors span respectively $\mathrm{Ran}(P_0^-(k))$ and $\mathrm{Ran}(P_1^-(k))$, the second n vectors span respectively $\mathrm{Ran}(P_0^+(k))$ and $\mathrm{Ran}(P_1^+(k))$ while the remaining span respectively $\ker(P_0^-(k)+P_0^+(k))$ and $\ker(P_1^-(k)+P_1^+(k))$ because by hypothesis $P_0^-(k)P_0^+(k) \equiv 0 \equiv P_1^-(k)P_1^+(k)$.

Then the symmetric unitary equivalence we need is the unitary operator $V(k)$ such that:

$$V(k)v_j^0(k) = v_j^1(k) \quad \text{for all} \quad j \in \{1, \cdots, \dim(\mathcal{H})\}, k \in \mathbb{T}^2.$$

In fact, it is obvious that this respects the symmetry constraint $CV(k) = V(-k)C$ and that $P_1^{\pm}(k) = V(k)P_0^{\pm}(k)V(k)^{-1}$. However it may not be so obvious that this is periodic in k_1 since it was defined over a pseudo-periodic frame. Luckily it is immediate to check that:

$$\begin{aligned}
V(\pi, k_2)v_1^0(\pi, k_2) &= v_1^1(\pi, k_2) = e^{i\,\mathrm{Ch}(P_1^-)k_2}v_1^1(-\pi, k_2) = \\
&= e^{i\,\mathrm{Ch}(P_1^-)k_2}V(-\pi, k_2)v_1^0(-\pi, k_2) = \\
&= V(-\pi, k_2)v_1^0(\pi, k_2) \\
V(\pi, k_2)v_{n+1}^0(\pi, k_2) &= v_{n+1}^1(\pi, k_2) = e^{i\,\mathrm{Ch}(P_1^+)k_2}v_{n+1}^1(-\pi, k_2) = \\
&= e^{i\,\mathrm{Ch}(P_1^+)k_2}V(-\pi, k_2)v_{n+1}^0(-\pi, k_2) = \\
&= V(-\pi, k_2)v_{n+1}^0(\pi, k_2)
\end{aligned}$$

So $V(-\pi, k_2)$ acts as $V(\pi, k_2)$ and this prove the periodicity of the unitary-valued map V.

- (*Symmetric homotopy*)

  We want to start with the topologically richer case. When $\dim(\mathcal{H}) = 2n$, we already proved in Proposition 3.6 that the Pfaffian is invariant under homotopic transformations, so we need to prove that, together with the Chern number, they form complete homotopy invariants. If $\mathrm{Ch}(P_0^-) = \mathrm{Ch}(P_1^-)$ and the three Pfaffians coincide, namely if

$$\mathcal{P}_0(0,0) = \mathcal{P}_1(0,0) \quad \mathcal{P}_0(\pi,0) = \mathcal{P}_1(\pi,0) \quad \mathcal{P}_0(0,\pi) = \mathcal{P}_1(0,\pi),$$

  then it is possible to construct a continuous and periodic symmetric unitary equivalence $V(k)$ between the two pairs $P_0^{\pm}(k), P_1^{\pm}(k)$ such that

$$P_1^{\pm}(k) = V(k)P_0^{\pm}(k)V(k)^{-1} \quad CV(k) = V(-k)C \quad \forall k \in \mathbb{T}^2$$

  as we did previously. So, if $\lambda(k) = \det(V(k))$, the periodicity tells us that λ is periodic and the symmetric constraint tells us that $\lambda(-k) = \overline{\lambda(k)}$. Finally, after selecting a basis on which C acts as $\mathcal{K}$ (using Lemma 2.2), we obtain that $V(k_\star) \in \mathrm{O}(2n)$ for $k_\star = (0,0), (\pi,0), (0,\pi)$ and the properties of the Pfaffian tells us that $\lambda(0,0) = \lambda(\pi,0) = \lambda(0,\pi) = 1$ since:

$$i(P_1^+(k_\star) - P_1^-(k_\star)) = V(k_\star)i(P_0^+(k_\star) - P_0^-(k_\star))V(k_\star)^t \Rightarrow$$

$$\Rightarrow \mathcal{P}_1(k_\star) = \det(V(k_\star))\mathcal{P}_0(k_\star) \Rightarrow \det(V(k_\star)) = 1.$$

  Now using the canonical covering $\pi : \mathbb{R} \to S^1$ with $\pi(t) = e^{i2\pi t}$ we can lift the map λ to a continuous map $f : [-\pi,\pi]^2 \to \mathbb{R}$ such that $f(0,0) = 0$ and $\lambda(t_1,t_2) = e^{i2\pi f(t_1,t_2)}$. Moreover, since $\lambda(-k) = \overline{\lambda(k)}$, we can choose f such that $f(-t_1,-t_2) = -f(t_1,t_2)$ and the periodicity of λ tells us that

$$f(\pi,t_2) - f(-\pi,t_2), \quad f(t_1,\pi) - f(t_1,-\pi) \in \mathbb{Z} \quad \forall t_1,t_2 \in [-\pi,\pi].$$

Another crucial property is that, since $\lambda(\pi,0)=\lambda(0,\pi)=1$, it must be true that

$$f(\pi,0),f(0,\pi)\in\mathbb{Z}.$$

We can combine all of those properties to have that the functions

$$\frac{f(\pi,t_2)-f(-\pi,t_2)}{2},\quad \frac{f(t_1,\pi)-f(t_1,-\pi)}{2}$$

have constant value in $\mathbb{Z}/2$ thanks to continuity. This means that we compute the actual value using $t_2=0$ for the first and $t_1=0$ for the second, obtaining:

$$\begin{gathered}\frac{f(\pi,t_2)-f(-\pi,t_2)}{2}\equiv\frac{f(\pi,0)-f(-\pi,0)}{2}=f(\pi,0)\in\mathbb{Z},\\ \frac{f(t_1,\pi)-f(t_1,-\pi)}{2}\equiv\frac{f(0,\pi)-f(0,-\pi)}{2}=f(0,\pi)\in\mathbb{Z}\Rightarrow\\ \Rightarrow e^{i\pi f(\pi,t_2)}\equiv e^{i\pi f(-\pi,t_2)}\quad\text{and}\quad e^{i\pi f(t_1,\pi)}\equiv e^{i\pi f(t_1,-\pi)}\end{gathered}$$

At this point we can use the symmetric frame $\{v_j^0\}_{j\in\{1,\cdots,2n\}}$ of $P_0^{\pm}(k)$ to define the unitary-valued map $V_0:[-\pi,\pi]^2\to\mathcal{U}(\mathbb{C}^{2n})$ such that:

$$V_0(t_1,t_2)v_j^0(t_1,t_2)=\begin{cases}e^{i\pi f(t_1,t_2)}v_j(t_1,t_2) & \text{for } j=1 \text{ or } j=n+1\\ v_j(t_1,t_2) & \text{otherwise}\end{cases}$$

Clearly, since $f(-t)=-f(t)$, this unitary-valued map is such that

$$P_0(t)=V_0(t)P_0(t)V_0(t)^{-1},\quad CV_0(t)=V_0(-t)C\quad\forall t\in[-\pi,\pi]^2$$

But if we look closely we notice that this map is also periodic, in fact

$$\begin{aligned}V_0(\pi,t_2)v_j^0(\pi,t_2)&=\begin{cases}e^{i\pi f(\pi,t_2)}v_j(\pi,t_2) & \text{for } j=1 \text{ or } j=n+1\\ v_j(\pi,t_2) & \text{otherwise}\end{cases}\\ &=\begin{cases}e^{i\pi f(-\pi,t_2)}e^{i\,\mathrm{Ch}(P_0^-)t_2}v_j(-\pi,t_2) & \text{for } j=1\\ e^{i\pi f(-\pi,t_2)}e^{i\,\mathrm{Ch}(P_0^+)t_2}v_j(-\pi,t_2) & \text{for } j=n+1\\ v_j(-\pi,t_2) & \text{otherwise}\end{cases}\\ &=\begin{cases}e^{i\,\mathrm{Ch}(P_0^-)t_2}V_0(-\pi,t_2)v_j(-\pi,t_2) & \text{for } j=1\\ e^{i\,\mathrm{Ch}(P_0^+)t_2}V_0(-\pi,t_2)v_j(-\pi,t_2) & \text{for } j=n+1\\ V_0(-\pi,t_2)v_j(-\pi,t_2) & \text{otherwise}\end{cases}\\ &=\begin{cases}V_0(-\pi,t_2)v_j(\pi,t_2) & \text{for } j=1\\ V_0(-\pi,t_2)v_j(\pi,t_2) & \text{for } j=n+1\\ V_0(-\pi,t_2)v_j(\pi,t_2) & \text{otherwise}\end{cases}\\ &=V_0(-\pi,t_2)v_j^0(\pi,t_2)\quad\forall j=1,\cdots,2n\end{aligned}$$

This means that $V_0(\pi, t_2)$ and $V_0(-\pi, t_2)$ acts identically on the same basis of $\mathcal{H}$, so they are equal. We can repeat the same procedure to $V_0(t_1, \pi)$ and $V_0(t_1, -\pi)$ and in the end we will get full periodicity in both directions, so we can use the notation $V_0(k)$ for $k \in \mathbb{T}^2$.

Now we want to construct a homotopy $V_t(k)$ connecting $V_1(k) = V(k)$ to $V_0(k)$ such that $CV_t(k) = V_t(-k)C$ for every $t \in [0,1], k \in \mathbb{T}^2$. On the points $k_\star$ fixed by the involution $k \mapsto -k$ this symmetry relation forces the homotopy $V_t(k_\star)$ to move inside $\mathrm{O}(2n)$, provided that we represent everything on a basis on which C acts as $\mathcal{K}$. By construction $\det(V_0(k)) \equiv \det(V(k))$, in particular $\det(V(k_\star)) \equiv \det(V_0(k_\star)) \equiv 1$. This means that $V(k_\star), V_0(k_\star) \in \mathrm{SO}(2n)$. Then, we can use Theorem A.4 to connect those unitaries inside $\mathrm{SO}(2n)$, obtaining $V_t(k_\star) \in \mathrm{SO}(2n)$. Now we can consider the segment $\overline{OQ_1} = (-l, l), l \in [0, \pi]$, connecting $O = (0,0)$ with $Q_1 = (-\pi, \pi)$. The map $V_t(k)$ is defined over $\partial(\overline{OQ_1} \times [0,1])$ which is homeomorphic to S^1. It is possible to extend this map inside the square if and only if this map is contractible to a point. Thanks to Theorem A.3, this happens if and only if the winding number of the determinant is zero. By construction the determinant is always 1 on the boundaries that depend on t, while on the other boundaries it is true that $\det(V_0(k)) = \det(V(k))$, so after some manipulations we get a map that satisfies the hypotheses of Lemma A.8, therefore it is contractible. We can then choose an extension of $V_t(k)$ inside $\overline{OQ_1} \times [0,1]$ and mirror it on $\overline{OQ_2} \times [0,1]$ for $Q_2 = (\pi, -\pi)$ using the symmetric constraint $V_t(-k) = CV_t(k)C$. This procedure can be replicated on the segments $\overline{Q_1Q_3}$ and $\overline{Q_2Q_3}$ for $Q_3 = (\pi, \pi)$. Every time we define the extension up to the middle points $(\pi, 0), (0, \pi)$ and then mirror the extension using the symmetry constraint. So if $\triangle \subset [-\pi, \pi]^2$ is the triangle with vertices Q_1, Q_2 and Q_3, our homotopy $V_t(k)$ is defined for $t, k \in \partial([0,1] \times \triangle)$. This region is homeomorphic to S^2 and it is possible to extend the map inside the region if and only if this map is contractible. Luckily this is always possible since $\pi_2(\mathcal{U}(2n)) = \{0\}$ (Theorem A.3), so after choosing an extension of $V_t(k)$ inside $[0,1] \times \triangle$ we can mirror it in $[-\pi, \pi]^2 \setminus \triangle$ using the symmetry constraint $V_t(k) = CV_t(-k)C$. Using this, we can finally define the symmetric homotopy between $P_1^\pm(k)$ and $P_0^\pm(k)$ as

$$P_t^\pm(k) = V(k) P_0^\pm(k) V(k)^{-1}.$$

Instead, if $2n < \dim(\mathcal{H}) < \infty$, we can once again define $\lambda(k) = \det(V(k))$, this time however we can define the unitary-valued map $V_0(k)$ as:

$$V_0(k) v_j^0(k) = \begin{cases} \lambda(k) v_{2n+1}^0(k) & \text{if} \quad j = 2n+1 \\ v_j^0(k) & \text{otherwise} \end{cases}$$

This map satisfies the same properties of $V_0(k)$ from the previous argument and we can replicate the proof of the case $\dim(\mathcal{H}) = 2n$ in the exact same way to obtain the symmetric homotopy needed.

Instead, if $\dim(\mathcal{H}) = \infty$, we can impose $V_0(k) = \mathbb{1}$ and replicate all of the previous arguments. In fact thanks to Lemma 2.2, there is a basis of $\mathcal{H}$ on which C acts as $\mathcal{K}$, meaning that a unitary operator U commutes with C if and only if it is an orthogonal $U \in \mathrm{O}(\ker(C - \mathbb{1}))$. Thanks to Theorem A.2, $\mathcal{U}(\mathcal{H})$ and $\mathrm{O}(\ker(C - \mathbb{1}))$ are contractible, so all of the steps done in lower dimension can be replicated without obstructions.

□

Chapter 4

Class AII

In class AII, the only symmetry present is a fermionic time-reversal symmetry, so the object we want to study in this chapter is the one defined in Definition I.12 with the additional constraint that $T^2 = -\mathbb{1}$:

Definition 4.1 (fermionic time-reversal symmetric projection-valued map)**.** Given an anti-unitary operator T acting over $\mathcal{H}$ with $T^2 = -\mathbb{1}$, a projection-valued map $P : \mathbb{T}^d \to \mathrm{Proj}_{2n}(\mathcal{H})$ is time-reversal symmetric if

$$TP(k) = P(-k)T.$$

The fact that the dimension of the rank must be even is a mandatory condition that will be investigated in Lemma 4.4. Moreover, since $T^2 = -\mathbb{1}$ we can never ask for fixed vectors in view of the same result, so we will need a different notion of a fixed frame. With this in mind, given two fermionic time-reversal symmetric projection-valued maps $P_0, P_1 : \mathbb{T}^d \to \mathrm{Proj}_{2n}(\mathcal{H})$, we will try to answer the following questions:

Question 10 (Class AII frame)**.** When will there be a smooth and fermionic time-reversal symmetric frame $\{v_1(k), \cdots, v_{2n}(k)\}$, namely a continuous and periodic collection of orthonormal vectors spanning $\mathrm{Ran}(P(k))$ with $Tv_j(k) = -v_{n+j}(-k)$ for all $j \in \{1, \cdots, n\}$? In case there is none, is it possible to have a frame with the properties listed before and regular pseudoperiodic conditions like the ones in Theorem 4.16?

Question 11 (Class AII unitary equivalence)**.** When will there be a continuous and time-reversal symmetric unitary-valued map $U : \mathbb{T}^d \to \mathcal{U}(\mathcal{H})$ such that $P_1(k) = V(k)P_0(k)V(k)^{-1}$ and $TV(k) = V(-k)T$ for all $k \in \mathbb{T}^d$? This is the symmetric constraint that appeared in Definition I.12

Question 12 (Class AII homotopy)**.** When will there be a continuous and time-reversal symmetric map $P : [0,1] \times \mathbb{T}^d \to \mathrm{Proj}_{2n}(\mathcal{H})$ with $P(0,k) = P_0(k)$, $P(1,k) = P_1(k)$ and $TP_t(k) = P_t(-k)T$ for all $k \in \mathbb{T}^d$ and $t \in [0,1]$?

In fact, those questions were already answered in [19], but here we provide a different interpretation of these results. Our most important contribution is a new interpretation of the Fu-Kane-Mele index presented in Theorem 4.14. This new interpretation is tied to the notion of decomposability introduced in the following definition.

Definition 4.2. A fermionic time-reversal symmetric projection-valued map $P : \mathbb{T}^d \to \mathrm{Proj}_{2n}(\mathcal{H})$ is decomposable if there is a pair of projection-valued maps $P^{\pm} : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ such that

$$P^+(k)P^-(k) = 0, \quad P(k) = P^+(k) + P^-(k), \text{ and } \quad TP^+(k) = P^-(-k)T \quad \forall k \in \mathbb{T}^d$$

We will see that such a pair is identical to a odd particle-hole symmetric pair of projection-valued maps and if the pair admits a symmetric frame in the sense of the chapter about Class C, this decomposition can be done by imposing:

$$P^-(k) = \mathrm{Span}\{v_j(k)\}_{j\in\{1,\cdots,n\}} \quad P^+(k) = \mathrm{Span}\{v_j(k)\}_{j\in\{n+1,\cdots,2n\}} \quad \forall k \in \mathbb{T}^d.$$

Although this information appears to be less relevant for $d \leq 1$, this idea is crucial to understand the topology of this class for $d = 2$.

4.1 d=0

Before trying to answer the previous questions, it is worth recalling some notions about quaternionic Hilbert spaces.

Definition 4.3 (Quaternionic Hilbert space, bases and unitaries)**.** Consider a complex Hilbert $\mathcal{H}$ with an anti-unitary operator $T : \mathcal{H} \to \mathcal{H}$ such that $T^2 = -\mathbb{1}$. Then the couple $(\mathcal{H}, T)$ constitutes a quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$.

A collection of vectors $\{v_j\}_{j\in J}$ for a set of indexes J is a quaternionic orthonormal basis of $\mathcal{H}_{\mathbb{H}}$ if the vectors $\{v_j\}_{j\in J} \cup \{Tv_j\}_{j\in J}$ constitute an orthonormal basis of $\mathcal{H}$ in the canonical sense.

A unitary operator $U \in \mathcal{U}(\mathcal{H})$ is quaternionic if $UT = TU$. The set of quaternionic unitaries is called $\mathcal{U}(\mathcal{H}_{\mathbb{H}})$.

To better explain the nature behind the quaternionic structure that T creates, it is worth making a small digression. Suppose that we are studying $\mathbb{R}^2$ as a real Hilbert space with the natural product $\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ that sends $(t, (x, y))$ to (tx, ty). Then the two vectors $(1, 0), (0, 1)$ constitute a basis of $\mathbb{R}^2$, in particular they are linearly independent because there are no ways in which we can move the first vector into the second using the natural product. Now we can consider the rotation of 90 degrees described by the unitary operator R and this is such that $R^2 = -\mathbb{1}$. If we want to see this operation in the set of natural operations of $\mathbb{R}^2$, then we are interested in the structure of the couple $(\mathbb{R}^2, R)$. The first interesting aspect is that the vectors $(1, 0), (0, 1)$ will not be linearly independent because $R(1, 0) = (0, 1)$. So we will need a new notion for an orthonormal basis and a new notion for orthogonal operators that respect the structure that R creates. This is the standard procedure that allows us to define complex Hilbert space using an orthogonal operator that squares to $-\mathbb{1}$. In particular, a complex orthonormal basis will be a collection of orthonormal vectors that are linearly independent and generate the Hilbert space using multiplications of real elements and multiplications of R. Moreover, an orthogonal operator is compatible with the complex structure if it commutes with R, leading to the canonical definition of a unitary operator. The step from complex to quaternionic follows the same rule. In the end, the pair $(\mathcal{H}, T)$ will be considered a quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$ where $\mathbb{H}$ is the set of natural elements that can be multiplied to the vectors of $\mathcal{H}$, this is often called the quaternionic group:

$$\mathbb{H} = \{a + bi + cT + d(iT) | a, b, c, d \in \mathbb{R}\} \simeq \mathbb{R}^4.$$

This group has a standard commutative sum and also a non-commutative product that it is distributive with respect to the sum and such that $Ti = -iT$. The elements $i, T, (iT)$ are often interpreted as three different imaginary units because they all square to $-\mathbb{1}$. Now we can state some properties of quaternionic Hilbert spaces.

Lemma 4.4. *If $\mathcal{H}_{\mathbb{H}} = (\mathcal{H}, T)$ is a quaternionic Hilbert space and $V \subset \mathcal{H}$ is a quaternionic Hilbert subspace, namely an Hilbert subspace with $TV = V$, then the following statements are true:*

1. *V admits a quaternionic basis.*
2. *As a complex Hilbert subspace,* $\dim(V)$ *is infinite or even.*
3. *If* $\dim(V) = 2L$ *as a complex Hilbert space, it is always possible to build a complex basis of the form*

$$\{v_1, \cdots, v_{\dim(V)/2}, -Tv_1, \cdots, -Tv_{\dim(V)/2}\}.$$

This means that on this basis T acts as $\mathcal{K}J$, for $\mathcal{K}$ standard complex conjugation and $J = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}$.

This argument is already present in several works, for example in [24].

Proof. We can define the set composed of the collection of orthonormal vectors that are orthogonal to their multiplied with T. Namely, we consider the set:

$$Z = \{\{v_a\}_{a\in J} \subset \mathcal{H} | \langle v_a, v_b \rangle = \delta_{b,a}, \langle v_a, Tv_b \rangle = 0 \ \forall a, b \in J\}$$

This set has a natural partial order given by the inclusion, and every increasing sequence

$$\{v_j\}_{j\in J_1} \subset \cdots \subset \{v_j\}_{j\in J_n} \subset \cdots$$

can be viewed up to changing the indexes as a sequence $J_1 \subset \cdots \subset J_n \subset \cdots$. Then we can consider $J = \cup_{i=1}^{\infty} J_i$ and look at $\{v_j\}_{j\in J}$, which is clearly an element of Z. The fact that $T^2 = -\mathbb{1}$ implies that

$$\langle v, Tv \rangle = \overline{\langle Tv, T^2 v \rangle} = -\overline{\langle Tv, v \rangle} = -\langle v, Tv \rangle$$

so Z is not empty because a single vector consists of an element of it, and we can apply the Zorn lemma to obtain that there is also a maximal element $\{v_j\}_{j\in J}$. This maximal element is such that

$$\mathcal{H} = \mathrm{Span}_{\mathbb{C}}\left(\{v_j\}_{j\in\{1,\cdots,\dim(\mathcal{H})/2\}}\right) \oplus \mathrm{Span}_{\mathbb{C}}\left(\{Tv_j\}_{j\in\{1,\cdots,\dim(\mathcal{H})/2\}}\right).$$

In fact, by definition, the two spans intersect only at 0 and if, by contradiction, there is a vector v that cannot be obtained by adding elements of the two spans. Then, up to renormalization, the vector

$$\tilde{v} = v - \sum_{j\in J} \langle v_j, v \rangle v_j - \sum_{j\in J} \langle Tv_j, v \rangle Tv_j$$

is non-zero and orthogonal to both spans. Moreover:

$$T\tilde{v} = Tv - \sum_{j\in J} \overline{\langle v_j, v\rangle\, Tv_j} + \sum_{j\in J} \overline{\langle Tv_j, v\rangle} v_j =$$
$$= Tv - \sum_{j\in J} \langle Tv_j, Tv\rangle\, Tv_j - \sum_{j\in J} \langle v_j, Tv\rangle\, v_j$$

So also Tv is non-trivial and orthogonal to both spans, meaning that we can add v to the collection $\{v_j\}_{j\in J}$ and contradict maximality. □

So, from now on, we will freely swap between T and $J\mathcal{K}$.

Remark 4.5. Given two a quaternionic bases $\{v_j^0\}_{j\in J}, \{v_j^1\}_{j\in J}$ of $\mathcal{H}_{\mathbb{H}}$, we can define the unitary operator $V \in \mathcal{U}(\mathcal{H})$ such that $V(v_j^0) = v_j^1$ and $V(Tv_j^0) = Tv_j^1$ for all $j \in J$. This is a unitary operator in the classical sense, but it is also true that $[V, T] = 0$, so this is also an element of $\mathcal{U}(\mathcal{H}_{\mathbb{H}})$, so it is a quaternionic unitary. In a finite-dimensional quaternionic Hilbert space, on a basis on which $T = J\mathcal{K}$, the condition $VT = TV$ is equivalent to:

$$V \in \mathcal{U}(\mathcal{H}_{\mathbb{H}}) \Leftrightarrow TV = VT \Leftrightarrow \mathcal{K}JV = V\mathcal{K}J \Leftrightarrow J\overline{V} = VJ \Leftrightarrow J = VJV^T \Leftrightarrow V \in \mathrm{Sp}(\dim(\mathcal{H})).$$

Where $\mathrm{Sp}(2L)$ denotes the set of symplectic and unitaries $2L \times 2L$.

Since $\mathbb{T}^0$ is made up of a single point, in this case we just have to study a single projection $P \in \mathrm{Proj}_{2n}(\mathcal{H})$ with $PT = TP$. In this small environment, the three questions can be answered without issue.

Proposition 4.6. *Two fermionic time-reversal symmetric projection-valued maps $P_0, P_1 \in \mathrm{Proj}_{2n}(\mathcal{H})$ will always have a symmetric frame and be unitarily equivalent and homotopic with respect to the time-reversal symmetry.*

Proof. First of all, we notice that it is possible to apply Lemma 4.4 to the four Hilbert subspaces $\mathrm{Ran}(P_0), \mathrm{Ran}(P_1), \ker(P_0), \ker(P_1)$ to obtain four quaternionic bases that we can merge into $\{v_j^0\}_{j\in\{1,\cdots,\dim(\mathcal{H})/2\}}$, $\{v_j^1\}_{j\in\{1,\cdots,\dim(\mathcal{H})/2\}}$ where the first $n/2$ vectors generate $\mathrm{Ran}(P_0)$ and $\mathrm{Ran}(P_1)$ while the others generate $\ker(P_0)$ and $\ker(P_1)$ in as quaternionic bases. This means that the symmetric frame we are looking for is:

$$\{v_1^0, \cdots, v_{n/2}^0, -Tv_1^0, \cdots, -Tv_{n/2}^0\}.$$

Then the symmetric unitary equivalence we need is the quaternionic operator V such that $V(v_j^0) = v_j^1$ for all $j \in \{1, \cdots, \dim(\mathcal{H})/2\}$. Thanks to Remark 4.5 it is such that $TV = VT$ and it is obvious that $P_1 = VP_0V^{-1}$. Finally, for the homotopy problem, we must consider $\mathcal{H}$ as a quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$. If $\dim(\mathcal{H}) < \infty$, we can choose a basis on which T acts as $J\mathcal{K}$ and the unitary transformations that commute with T are precisely the operators in $\mathrm{Sp}(\dim(\mathcal{H}))$. Since it is connected due to Theorem A.5, we can find a homotopy $V_t \in \mathrm{Sp}(\dim(\mathcal{H}))$ such that $V_1 = V$ and $V_0 = \mathbb{1}$. Something similar can be done if $\dim(\mathcal{H}) = \infty$. In this case the unitary operators that commute with T are those who preserve the quaternionic structure and are the elements of $\mathcal{U}(\mathcal{H}_{\mathbb{H}})$. So, we can use Theorem A.2 to obtain a homotopy $V_t \in \mathcal{U}(\mathcal{H}_{\mathbb{H}})$ where $V_1 = V$ and $V_0 = \mathbb{1}$. In both cases $P_t = V_tP_0V_t^{-1}$ is the symmetric homotopy we are looking for because $TV_t = V_tT$. □

Remark 4.7. In particular, a fermionic time-reversal symmetry projection P can always be decomposed in the sense of Definition 4.2. This can be done by defining

$$P^- = \text{Span}\left\{v_1, \cdots, v_{n/2}\right\} \quad P^+ = \text{Span}\{Tv_1, \cdots, Tv_{n/2}\}$$

4.2 d=1

The first thing we want to notice is that we can replicate exactly the construction performed in Proposition 2.4. In fact, in the proof of the proposition we did not use the fact that $T^2 = \mathbb{1}$ and those results were valid for a generic time-reversal symmetric projection-valued map.

Lemma 4.8. *If $\mathcal{H}$ is a separable Hilbert space with $\dim(\mathcal{H}) = \infty$ and $P : \mathbb{T}^1 \to \text{Proj}_{2n}(\mathcal{H})$ is a continuous fermionic time-reversal symmetric projection-valued map with $TP(k) = P(-k)T$, then there is always a continuous and periodic collection of orthonormal vectors $\{v_j(k)\}_{j\in\mathbb{N}}$ such that*

$$\{v_1(k), -Tv_1(-k), \cdots, v_j(k), -Tv_j(-k), \cdots\}$$

is an orthonormal and countable basis of $\ker(P(k))$.

Proof. We can start by using Lemma 4.4 to select a quaternionic basis $\{v_j(0)\}_{j\in N}$ of $\ker(P(0))$. This basis, together with $\{Tv_j(0)\}_{j\in\mathbb{N}}$ is a complex basis of $\ker(P(0))$. Then we can use Proposition 2.4 to construct a unitary-valued map $U(t)$ such that $P(t) = U(t)P(0)U(t)^{-1}$ and $TU(t) = U(-t)T$. We can compose a basis of $\ker(P(t))$ using the vectors

$$\{U(t)v_1(0), U(t)Tv_1(0), \cdots, U(t)v_j(0), U(t)Tv_j(0), \cdots\}.$$

This in general is not periodic. To solve this problem we need to define a family of natural isomorphisms of complex Hilbert spaces as:

$$\begin{array}{rrcc} I(t): & \ker(P(t)) & \to & l^2(\mathbb{N}) \times l^2(\mathbb{N}) \\ & U(t)v_j(0) & \mapsto & (\delta_j, 0) \\ & U(t)Tv_j(0) & \mapsto & (0, \delta_j) \end{array} \quad \text{where } \delta_j(n) = \begin{cases} 1 & \text{if } n = j \\ 0 & \text{otherwise} \end{cases}$$

The choice of the basis and the fact that $U(t)T = TU(-t)$ imply that

$$TI(t)^{-1}(\delta_j, 0) = TU(t)v_j(0) = U(-t)Tv_j(0) = I(-t)^{-1}(0, \delta_j)$$

$$TI(t)^{-1}(0, \delta_j) = TU(t)Tv_j(0) = -U(-t)v_j(0) = I(-t)^{-1}(-\delta_j, 0)$$

so by anti-linearity

$$TI(t)^{-1} = I(-t)^{-1}\mathcal{K}J \quad \text{where } J = \begin{pmatrix} 0 & \mathbb{1}_{l^2(\mathbb{N})} \\ -\mathbb{1}_{l^2(\mathbb{N})} & 0 \end{pmatrix} \quad \forall t \in [-\pi, \pi]$$

and $\mathcal{K}$ is the standard complex conjugation. This time, the aperiodicity is expressed by the fact that $I(-\pi)^{-1}I(\pi) \neq \mathbb{1}_{\mathcal{H}}$. Now using the spectral theorem, we know that there is a self-adjoint operator

L such that $I(\pi)I(-\pi)^{-1} = e^{iL}$. Moreover, it is true that:

$$\mathcal{K}J^{-1}I(\pi)I(-\pi)^{-1} = I(-\pi)T^{-1}I(-\pi)^{1} = I(-\pi)I(\pi)^{-1}\mathcal{K}J^{-1} = (I(\pi)I(-\pi)^{-1})^{-1}\mathcal{K}J^{-1}.$$

This means that $\mathcal{K}J^{-1}e^{iL}\mathcal{K}J = e^{-iL}$, so $e^{-iJ^{-1}\overline{L}J} = e^{-iL}$ and therefore we can choose L such that $J^{-1}\overline{L}J = L$. Now we can define $\beta_1 = e^{iL/2}$. This is a unitary operator and since $\mathcal{U}(l^2(\mathbb{N})^2)$ is arcwise connected owing to Theorem A.2, there is a homotopy β_t connecting it with $\beta_0 = \mathbb{1}$. For example we can take $\beta_t = e^{itL/2}$. To conclude the proof we can define $\tilde{I}^{-1}(t) : l^2(\mathbb{N})^2 \to \ker(P(k))$ as:

$$\tilde{I}(t)^{-1} = \begin{cases} I(t)^{-1}\beta_{t/\pi} & \text{for } t \in [0,\pi], j \in \mathbb{N} \\ I(t)^{-1}\mathcal{K}J^{-1}\beta_{-t/\pi}\mathcal{K}J & \text{for } t \in [-\pi,0], j \in \mathbb{N} \end{cases}$$

Then $\{\tilde{v}_j(t) = \tilde{I}(t)^{-1}(\delta_j, 0)\}_{j\in\mathbb{N}}$ is the collection of vectors we were looking for. In fact, it is periodic and continuous because $\beta_0 = \mathbb{1}$ and

$$\begin{aligned} I(-\pi)^{-1}\mathcal{K}J^{-1}\beta_1\mathcal{K}J &= I(\pi)^{-1}I(\pi)I(-\pi)^{-1}\mathcal{K}J^{-1}e^{iL/2}\mathcal{K}J = \\ &= I(\pi)^{-1}e^{iL}J^{-1}e^{-i\overline{L}/2}J = I(\pi)^{-1}e^{-iJ^{-1}\overline{L}J/2} = \\ &= I(\pi)^{-1}e^{iL}e^{-iL/2} = I(\pi)^{-1}e^{iL/2} \end{aligned}$$

Moreover elements like $(0,\delta_j)$ together with $(\delta_j, 0)$ form an orthonormal basis of $l^2(\mathbb{N})^2$, so their images $\{\tilde{v}_j(t) = \tilde{I}(t)^{-1}(\delta_j,0)\}_{j\in\mathbb{N}}$ together with $\{\tilde{u}_j(t) = \tilde{I}(t)^{-1}(0,\delta_j)\}_{j\in\mathbb{N}}$ form an orthonormal basis of $\ker(P(t))$. Most importantly, the relation $(\delta_j, 0) = \mathcal{K}J(0,\delta_j)$ implies that $\tilde{I}(t)(\delta_j,0) = \tilde{I}(t)\mathcal{K}J(0,\delta_j)$, so, for $t \geq 0$ this means that:

$$\begin{aligned} \tilde{v}_j(t) &= I(t)^{-1}e^{itL/2\pi}\mathcal{K}J(0,\delta_j) = I(t)^{-1}\mathcal{K}J\mathcal{K}J^{-1}e^{itL/2\pi}\mathcal{K}J(0,\delta_j) = \\ &= TI(-t)^{-1}\mathcal{K}J^{-1}e^{itL/2\pi}\mathcal{K}J(0,\delta_j) = T\tilde{I}(-t)(0,\delta_j) = \\ &= T\tilde{u}_j(-t) \end{aligned}$$

and similarly, for $t \leq 0$, using that $J^{-1} = -J$ we have that:

$$\begin{aligned} \tilde{v}_j(t) &= I(t)^{-1}\mathcal{K}J^{-1}e^{itL/2\pi}\mathcal{K}J\mathcal{K}J(0,\delta_j) = I(t)^{-1}\mathcal{K}(-J^{-1})e^{itL/2\pi}(0,\delta_j) = \\ &= TI(-t)^{-1}e^{itL/2\pi}(0,\delta_j) = T\tilde{I}(-t)(0,\delta_j) = \\ &= T\tilde{u}_j(-t) \end{aligned}$$

So the vectors

$$\{\tilde{v}_1(t), \tilde{u}_1(t), \cdots, \tilde{v}_j(t), \tilde{u}_j(t), \cdots\}$$

is exactly the collection of orthonormal vectors we needed to complete the proof. □

Now we are able to answer our three questions in dimension 1:

Proposition 4.9. *Two fermionic time-reversal symmetric projection-valued maps $P_0, P_1 : \mathbb{T}^1 \to \mathrm{Proj}_{2n}(\mathcal{H})$ will always have a symmetric frame and be unitarily equivalent and homotopic with respect to the fermionic time-reversal symmetry.*

Proof.
- (*symmetric frame*)

Given a symmetric frame $\{v_j^0\}_{j\in\{1,\cdots,2n\}}$ of $P_0(0)$ built using Proposition 2.3, we can build a symmetric frame of $P_0(t)$ as $\{U(t)v_j^0 = u_j^0(t)\}_{j\in\{1,\cdots,2n\}}$ using Proposition 2.4. However, this may not be periodic and aperiodicity is expressed as the presence of a matrix:

$$\alpha \in U(n) \quad \text{such that} \quad u_j^0(\pi) = \sum_{a=1}^{n} [\alpha]_{j,a} u_a^0(-\pi).$$

Since

$$Tu_j(t) = \begin{cases} -u_{n+j}(-t) & \text{for } 1 \leq j \leq n \\ u_{j-n}(-t) & \text{for } n < j \leq 2n \end{cases}$$

We can say that

$$T\left(\sum_{a=1}^{2n} x_a u_a(t)\right) = \sum_{a,b=1}^{2n} \left(\overline{x}_a [J]_{a,b} u_b(-t)\right) \quad \text{for } x_a \in \mathbb{C}, J = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}$$

So this α has the additional property:

$$\begin{aligned} \sum_{c=1}^{2n} [J]_{j,c} u_c^0(-\pi) = Tu_j^0(\pi) = T\left(\sum_{a=1}^{2n}[\alpha]_{j,a} u_a^0(-\pi)\right) = \sum_{a,b=1}^{n} \overline{[\alpha]_{j,a}} [J]_{a,b} u_b^0(\pi) = \\ = \sum_{a,b,c=1}^{n} \overline{[\alpha]_{j,a}} [J]_{a,b} [\alpha]_{b,c} u_c^0(-\pi) = \sum_{b=1}^{n} [\overline{\alpha}(J)\alpha]_{j,c} u_c^0(-\pi). \end{aligned}$$

This means that $\overline{\alpha} J \alpha = J$ or equivalently $\alpha^t = J\alpha J^{-1}$ since $\alpha^{-1} = \overline{\alpha}^t \Rightarrow (\overline{\alpha})^{-1} = \alpha^t$. Using the Sylvester theorem, we know that there is a self-adjoint matrix L with $e^{-iL} = \alpha$ and $J\alpha J^{-1} = \alpha^t$ implies that we can choose L such that $JLJ^{-1} = L^t$. Since $L^* = L$, it is also true that $\overline{L} = L^t$, so we also have $JLJ^{-1} = \overline{L}$. We can now define $\beta_t = e^{itL/2\pi}$ and $v_j^0(t) = \sum_{b=1}^{2n} [\beta_t]_{j,b} u_j^0(t)$ to form the symmetric frame we were looking for. In fact it holds that:

$$\begin{aligned} Tv_j^0(t) &= T\left(\sum_{b=1}^{2n} [\beta_t]_{j,b} u_b^0(t)\right) = \sum_{b=1}^{2n} \overline{[e^{itL/2\pi}]_{j,b}} Tu_b^0(t) = \\ &= \sum_{b,c=1}^{2n} [e^{-it\overline{L}/2\pi}]_{j,b} [J]_{b,c} u_c^0(-t) = \sum_{b=1}^{2n} [e^{-it\overline{L}/2\pi} J]_{j,b} u_b^0(-t) = \\ &= \sum_{b=1}^{2n} [Je^{-itL/2\pi}]_{j,b} u_n^0(-t) = \sum_{b,c=1}^{2n} [J]_{j,b} [\beta_{-t}]_{b,c} u_c^0(-t) = \\ &= \sum_{b=1}^{2n} [J]_{j,b} v_b^0(-t) = \begin{cases} -v_{n+j}(-t) & \text{for } 1 \leq j \leq n \\ v_{j-n}(-t) & \text{for } n < j \leq 2n \end{cases} \end{aligned}$$

and we also have that:

$$\begin{aligned} v_j^0(\pi) &= \sum_{b=1}^{n}[e^{iL/2}]_{j,b}u_b^0(\pi) = \sum_{b,a=1}^{n}[e^{iL/2}]_{j,b}[\alpha]_{b,a}u_a^0(-\pi) = \\ &= \sum_{a,b=1}^{n}[e^{iL/2}]_{j,b}[e^{-iL}]_{b,a}u_a^0(-\pi) = \sum_{a=1}^{n}[e^{iL/2}e^{-iL}]_{j,a}u_a^0(-\pi) = \\ &= \sum_{a=1}^{n}[e^{-iL/2}]_{j,a}u_a^0(-\pi) = v_j^0(-\pi). \end{aligned}$$

- (*unitary equivalence*)

  If $\dim(\mathcal{H}) < \infty$, we can also apply this procedure to $P_0(k)^\perp$ to obtain a continuous, periodic, and symmetric frame $\{v_j^0(k)\}_{j\in\{1,\cdots,\dim(\mathcal{H})\}}$ such that the first $2n$ vectors span $P_0(k)$ and the others span the orthogonal. Then we can repeat everything for $P_1(k)$ and $P_1(k)^\perp$ to construct another continuous, periodic, and symmetric frame $\{v_j^1(k)\}_{j\in\{1,\cdots,\dim(\mathcal{H})\}}$ and then the unitary equivalence we need is the unitary-valued map $V : \mathbb{T}^1 \to \mathcal{U}(\mathcal{H})$ such that $V(v_j^0(k)) = v_j^1(k)$ for every $j \in \{1,\cdots,\dim(\mathcal{H})\}, k \in \mathbb{T}^1$. In fact, it is trivial to prove that $P_1(k) = V(k)P_0(k)V(k)^{-1}$ and $TV(k) = V(-k)T$.

  Instead, if $\dim(\mathcal{H}) = \infty$, we can apply the Lemma 4.8 to $\ker(P_0(k))$ and $\ker(P_1(k))$ to obtain two continuous and periodic orthonormal bases of $\mathcal{H}$:

$$\begin{Bmatrix} v_1^0(k),\cdots,v_n^0(k),-Tv_1^0(-k),\cdots,-Tv_n^0(-k), \\ v_{n+1}^0(k),-Tv_{n+1}^0(-k),\cdots,v_N^0(k),-Tv_N^0(-k),\cdots \end{Bmatrix}$$

$$\begin{Bmatrix} v_1^1(k),\cdots,v_n^1(k),-Tv_1^1(-k),\cdots,-Tv_n^1(-k), \\ v_{n+1}^1(k),-Tv_{n+1}^1(-k),\cdots,v_N^1(k),-Tv_N^1(-k),\cdots \end{Bmatrix}$$

  where the first $2n$ vectors span $P_0(k)$ and $P_1(k)$, while the others span $\ker(P_0(k))$ and $\ker(P_1(k))$, respectively. Then the symmetric unitary equivalence is the unitary-valued map $V : \mathbb{T}^1 \to \mathcal{U}(\mathcal{H})$ such that $Vv_j^0(k) = v_j^1(k)$ and $VTv_j^0(k) = Tv_j^1(k)$ for every $j \in \mathbb{N}$. It is trivial to prove that $P_1(k) = V(k)P_0(k)V(k)^{-1}$ and $TV(k) = V(-k)T$.

- (*homotopy*)

  Once again we remark that the fermionic time-reversal symmetry endowss $\mathcal{H}$ with a quaternionic structure and a unitary $V \in \mathcal{U}(\mathcal{H})$ commutes with T if and only if it is compatible with this quaternionic structure, namely is an operator $V \in \mathcal{U}(\mathcal{H}_\mathbb{H})$, as said in Remark 4.5. This happens for $V(0)$ and $V(\pi) = V(-\pi)$. So, if $\dim(\mathcal{H}) = \infty$, Theorem A.2 states that it is possible to find two homotopies $V_t(0)$ and $V_t(\pi)$ connecting $V(0)$ and $V(\pi)$ with $\mathbb{1}$ inside $\mathcal{U}(\mathcal{H}_\mathbb{H})$. Since $\mathcal{U}(\mathcal{H})$ is contractible, it is possible to extend this couple of homotopies to a homotopy $V_t(k)$ with $V_1(k) = V(k)$ for $k \in [0,\pi]$, we can then define the homotopy on the other side $k \in [-\pi, 0]$ using the symmetry constraint $V_t(k) = T^{-1}V(-k)T$. Now we can define the homotopy of fermionic time-reversal symmetric projection-valued map by imposing:

$$P_t(k) = V_t(k)P_0(k)V_t(k)^{-1}.$$

Instead, if $\dim(\mathcal{H}) = 2m < \infty$, we can use Lemma 4.4 to find a basis of $\mathcal{H}$ on which T acts as $\mathcal{K}J$. Then, on this basis, $V(k)$ is such that $\mathcal{K}JV(k) = V(-k)\mathcal{K}J$. In particular on the points $k_\star = 0, \pi$ fixed by the involution $k \mapsto -k$ we have that:

$$J\overline{V(k_\star)} = V(k_\star)J \Leftrightarrow V(k_\star)^t J V(k_\star) = J \Leftrightarrow V(k_\star) \in \mathrm{Sp}(2m).$$

Since $J^t = -J$ it follows easily that $V(k_\star)^t JV(k_\star)$ is skew-symmetric, and using the properties of the Pfaffian listed in Proposition 3.4 leads to:

$$\mathrm{Pf}(J) = \mathrm{Pf}(V(k_\star)^t JV(k_\star)) = \det(V(k_\star))\,\mathrm{Pf}(J) \Rightarrow \det(V(k_\star)) \equiv 1$$

So every matrix in $\mathrm{Sp}(2m)$ has determinant equal to one. Now we can consider the well-known covering $f : \mathbb{R} \to S^1$ with $f(t) = e^{2\pi it}$ and, if $\lambda(k) = \det(V(k))$ is a continuous function, it is possible to find $\mu : [-\pi, \pi] \to \mathbb{R}$ such that $\lambda(k) = e^{2\pi i\mu(k)}$. The fact that $\lambda(0) = \lambda(\pi) = \lambda(-\pi) = 1$ implies that it is possible to choose μ such that $\mu(0) = 0$ and $\mu(\pi), \mu(-\pi) \in \mathbb{Z}$. Moreover, the condition $\mathcal{K}JV(k) = V(-k)\mathcal{K}J$ implies that $\lambda(k) = \overline{\lambda(-k)}$, so the condition $\mu(0) = 0$ implies that $\mu(k) = -\mu(-k)$. Afterwards, we can define the unitary-valued map $V_0 : \Pi^1 \to U(2m)$ as the unitary operator such that:

$$V_0(k)v_j^0(k) = \begin{cases} e^{i\pi\mu(k)}v_j^0(k) & \text{for } j = 1 \\ v_j^0(k) & \text{otherwise} \end{cases}$$

$$V_0(k)(-Tv_j^0(-k)) = \begin{cases} e^{i\pi\mu(k)}(-Tv_j^0(-k)) & \text{for } j = 1 \\ -Tv_j^0(-k) & \text{otherwise} \end{cases}$$

It is immediate to check that $TV_0(k) = V_0(-k)T$ and that $V_0(k)P_0(k)V_0(k)^{-1} = P_0(k)$, but most importantly $\det(V_0(k)) = e^{i\pi\mu(k)}e^{i\pi\mu(k)} = \lambda(k)$. Now we can define our homotopy $V_t(k)$ connecting $V(k) = V_1(k)$ to $V_0(k)$ by first connecting the points $k_\star$ fixed by the involution $k \mapsto -k$. In fact $V_0(k_\star), V(k_\star) \in \mathrm{Sp}(2n)$ and the Theorem A.5 states that $\mathrm{Sp}(2m)$ is simply connected, so we can define two homotopies $V_t(k_\star) \in \mathrm{Sp}(2m)$ for $k_\star = 0, \pi$ connecting the two pairs of unitaries. The previous argument ensures that $\det(V_t(k_\star)) \equiv 1$. Now, our homotopy $V_t(k)$ is defined for $(t, k) \in \partial([0, 1] \times [0, \pi])$ and we need to extend the definition also inside the square. If we interpret this boundary as S^1, this is possible if and only if the degree of the map is zero. Thanks to Theorem A.3 this degree is winding number of the determinant of the map. Luckily, the determinant is constant equal to one throughout $V_t(0)$ and $V_t(\pi)$ and finally, since $\det(V_0(k)) = \det(V(k)) = \lambda(k)$, the overall degree is zero and it's possible to define $V_t(k)$ for $t, k \in [0, 1] \times [0, \pi]$. For negative k we can just use the symmetry constraint $V_t(k) = T^{-1}V_t(-k)T = \mathcal{K}JV_t(k)\mathcal{K}J$ for $k \in [-\pi, 0]$ and this does not conflict with periodicity or continuity since $V_t(0), V_t(\pi) \in \mathrm{Sp}(2m)$. In the end the symmetric homotopy we were looking for is

$$P_t(k) = V_t(k)P_0(k)V_t(k)^{-1}$$

because $V_0(k)$ commutes with $P_0(k)$.

□

Remark 4.10. In particular, if $d = 1$, every fermionic time-reversal symmetric projection-valued map map is decomposable in the sense of Definition 4.2. In particular given a continuous, symmetric and periodic frame for P like the one of the previous Proposition, we can define

$$P^-(k) = \mathrm{Span}\{v_1(k), \cdots, v_{n/2}(k)\}, \quad P^+(k) = \mathrm{Span}\{Tv_1(k), \cdots, Tv_{n/2}(k)\}.$$

4.3 d=2

In the previous case, the fact that a fermionic time-reversal symmetric projection-valued map is decomposable was a useful remark. However, in this section, it becomes crucial to investigate under which conditions a decomposition occurs. Luckily, we can replicate exactly the construction done in Proposition 2.8. In fact, in the proof we did not use the fact that $T^2 = -\mathbb{1}$.

Lemma 4.11. *If $\mathcal{H}$ is a separable Hilbert space with $\dim(\mathcal{H}) = \infty$ and $P : \mathbb{T}^2 \to \mathrm{Proj}_{2n}(\mathcal{H})$ is a continuous fermionic time-reversal symmetric projection-valued map with $TP(k) = P(-k)T$, then there is always a continuous and periodic collection of orthonormal vectors $\{v_j(k)\}_{j\in\mathbb{N}}$ such that*

$$\{v_1(k), -Tv_1(-k), \cdots, v_j(k), -Tv_j(-k), \cdots\}$$

is always an orthonormal and countable basis of $\ker(P(k))$.

Proof. We can start by using Lemma 4.4 to select a discrete quaternionic basis $\{v_j\}_{j\in\mathbb{N}}$ of $\ker(P(0,0))$. This basis, together with $\{Tv_j\}_{j\in\mathbb{N}}$, is a complex basis of $\ker(P(0,0))$. Then we can use Proposition 2.8 to construct a unitary-valued map $U(t, k_2)$ such that

$$P(t, k_2) = U(t, k_2)P(0,0)U(t, k_2)^{-1} \quad \text{and} \quad TU(t, k_2) = U(-t, -k_2)T.$$

We can compose a discrete orthonormal basis of $\ker(P(t, k_2))$ using the vectors

$$\{U(t, k_2)v_1, U(t, k_2)Tv_1, \cdots, U(t, k_2)v_j, U(t, k_2)Tv_j, \cdots\}.$$

This is periodic in the second argument but not in the first. To solve this problem, we notice that this family defines a family of natural isomorphisms:

$$\begin{array}{rccc} I(t, k_2): & \ker(P(t, k_2)) & \to & l^2(\mathbb{N}) \times l^2(\mathbb{N}) \\ & U(t, k_2)v_j & \mapsto & (\delta_j, 0) \\ & U(t, k_2)Tv_j & \mapsto & (0, \delta_j) \end{array} \qquad \text{where } \delta_j(n) = \begin{cases} 1 & \text{if } n = j \\ 0 & \text{otherwise} \end{cases}$$

We can recover the same properties present in Lemma 4.8 like:

$$TI(t, k_2)^{-1}(\delta_j, 0) = TU(t, k_2)v_j(0) = U(-t, -k_2)Tv_j(0) = I(-t, -k_2)^{-1}(0, \delta_j)$$

$$TI(t, k_2)^{-1}(0, \delta_j) = TU(t, k_2)Tv_j = -U(-t, -k_2)v_j = I(-t, -k_2)^{-1}(-\delta_j, 0)$$

so by anti-linearity

$$TI(t, k_2)^{-1} = I(-t, -k_2)^{-1}\mathcal{K}J \quad \text{where } J = \begin{pmatrix} 0 & \mathbb{1}_{l^2(\mathbb{N})} \\ -\mathbb{1}_{l^2(\mathbb{N})} & 0 \end{pmatrix} \quad \forall t \in [-\pi, \pi]$$

and $\mathcal{K}$ is the standard complex conjugation. The aperiodicity is expressed by the fact that

$$I(-\pi,k_2)^{-1}I(\pi,k_2)\neq \mathbb{1}_{\mathcal{H}}.$$

Now using the spectral theorem, we know that there are two self-adjoin operators L,R acting over $l^2(\mathbb{N})^2$ with

$$I(\pi,0)I(-\pi,0)^{-1}=e^{iL}\quad\text{and}\quad I(\pi,\pi)I(-\pi,\pi)^{-1}=e^{iR}.$$

Moreover, it is true that:

$$\begin{aligned}\mathcal{K}J^{-1}I(\pi,0)I(-\pi,0)^{-1}&=I(-\pi,0)T^{-1}I(-\pi,0)^{1}=I(-\pi,0)I(\pi,0)^{-1}\mathcal{K}J^{-1}=\\&=(I(\pi,0)I(-\pi,0)^{-1})^{-1}\mathcal{K}J^{-1}.\end{aligned}$$

This means that we can $\mathcal{K}J^{-1}e^{iL}\mathcal{K}J=e^{-iL}$, so $e^{-iJ^{-1}\overline{L}J}=e^{-iL}$ and therefore L can be taken such that $J^{-1}\overline{L}J=L$. We can do the same for R to obtain that $J^{-1}\overline{R}J=R$. Using them, we can define the unitary operator $\beta(k_2)\in\mathcal{U}(l^2(\mathbb{N})^2)$ with:

$$\beta(k_2)=\begin{cases}\exp\{\frac{i}{\pi}[(\pi-k_2)\frac{L}{2}+k_2\frac{R}{2}]\} & \text{if}\quad k_2\in[0,\pi]\\ \mathcal{K}J^{-1}I(-\pi,-k_2)I(\pi,-k_2)^{-1}\beta(-k_2)\mathcal{K}J & \text{if}\quad k_2\in[-\pi,0]\end{cases}$$

It is very easy to see that this $\beta(k_2)$ is continuous and periodic in k_2 by applying the symmetry relations. Now, thanks to Theorem A.2 we know that there is a homotopy $\beta_t(k_2)$ in $\mathcal{U}(l^2(\mathbb{N})^2)$ with $\beta_0(k_2)\equiv\mathbb{1}_{l^2(\mathbb{N})^2}$ and $\beta_1(k_2)=\beta(k_2)$. To conclude the proof we can define $\tilde{I}^{-1}(t,k_2):l^2(\mathbb{N})^2\to\ker(P(t,k_2))$ as:

$$\tilde{I}(t,k_2)^{-1}=\begin{cases}I(t,k_2)^{-1}\beta_{t/\pi}(k_2) & \text{for } t\in[0,\pi],k_2\in\mathbb{T}^1\\ I(t,k_2)^{-1}\mathcal{K}J^{-1}\beta_{-t/\pi}(-k_2)\mathcal{K}J & \text{for } t\in[-\pi,0],k_2\in\mathbb{T}^1\end{cases}$$

This $\tilde{I}$ is continuous for $t=0$ because $\beta_0(k_2)\equiv\mathbb{1}$ and is also periodic because, for $k_2\in[0,\pi]$, we have that

$$\begin{aligned}\tilde{I}(-\pi,k_2)&=I(-\pi,k_2)^{-1}\mathcal{K}J^{-1}\beta_1(-k_2)\mathcal{K}J=\\&=I(-\pi,k_2)^{-1}\mathcal{K}J^{-1}\mathcal{K}J^{-1}I(-\pi,k_2)I(\pi,k_2)^{-1}\beta(k_2)\mathcal{K}J\mathcal{K}J=\\&=I(\pi,k_2)^{-1}\beta(k_2)=\tilde{I}(\pi,k_2)\end{aligned}$$

And for $k_2\in[-\pi,0]$ we have that

$$\begin{aligned}\tilde{I}(-\pi,k_2)&=I(-\pi,k_2)^{-1}\mathcal{K}J^{-1}\beta_1(-k_2)\mathcal{K}J=\\&=I(\pi,k_2)^{-1}I(\pi,k_2)I(-\pi,k_2)^{-1}\mathcal{K}J^{-1}\beta(-k_2)\mathcal{K}J=\\&=I(\pi,k_2)^{-1}\mathcal{K}J^{-1}I(-\pi,-k_2)I(\pi,-k_2)^{-1}\beta(-k_2)\mathcal{K}J=\\&=I(\pi,k_2)^{-1}\beta(k_2)=\tilde{I}(\pi,k_2)\end{aligned}$$

Moreover, $\tilde{I}$ also satisfies a symmetry constraint:

$$\begin{aligned} T\tilde{I}(t,k_2)^{-1} &= \begin{cases} I(-t,-k_2)^{-1}\mathcal{K}J\beta_{t/\pi}(k_2) & \text{for } t\in[0,\pi] \\ I(-t,-k_2)^{-1}\mathcal{K}J\mathcal{K}J^{-1}\beta_{-t/\pi}(-k_2)\mathcal{K}J & \text{for } t\in[-\pi,0] \end{cases} \\ &= \begin{cases} I(-t,-k_2)^{-1}\mathcal{K}J^{-1}\beta_{t/\pi}(k_2)\mathcal{K}J\mathcal{K}J & \text{for } t\in[0,\pi] \\ I(-t,-k_2)^{-1}\beta_{-t/\pi}(-k_2)\mathcal{K}J & \text{for } t\in[-\pi,0] \end{cases} \\ &= \tilde{I}(-t,-k_2)^{-1}\mathcal{K}J \end{aligned}$$

Then $\{\tilde{v}_j(t,k_2)=\tilde{I}(t,k_2)^{-1}(\delta_j,0)\}_{j\in\mathbb{N}}$ is the collection of vectors we were looking for. In fact, it is periodic and continuous because so is $\tilde{I}$. Moreover, elements such as $(0,\delta_j)$ together with $(\delta_j,0)$ form a discrete orthonormal basis of $l^2(\mathbb{N})^2$, so their images $\{\tilde{v}_j(t,k_2)=\tilde{I}(t,k_2)^{-1}(\delta_j,0)\}_{j\in\mathbb{N}}$ together with $\{\tilde{u}_j(t,k_2)=\tilde{I}(t,k_2)^{-1}(0,\delta_j)\}_{j\in\mathbb{N}}$ form a discrete orthonormal basis of $\ker(P(t,k_2))$. Most importantly, the relation $(\delta_j,0)=\mathcal{K}J(0,\delta_j)$ implies that $\tilde{v}_j(t,k_2)=T\tilde{u}_j(-t,-k_2)$ for all $j\in\mathbb{N}, t\in[-\pi,\pi], k_2\in\mathbb{T}^1$. So, the vectors

$$\{\tilde{v}_1(t),\tilde{u}_1(t),\cdots,\tilde{v}_j(t),\tilde{u}_j(t),\cdots\}$$

are exactly the collection of orthonormal vectors we needed to complete the proof. □

Now we want to replicate the construction done in [19] to define a a $\mathbb{Z}_2$-valued topological invariant that can be associated to a fermionic time-reversal symmetric projection-valued map $P:\mathbb{T}^2\to\mathrm{Proj}_{2n}(\mathcal{H})$. Consider a unitary-valued map $U:\mathbb{T}^2\to\mathcal{U}(\mathcal{H})$, obtained, for example, by using Proposition 2.8, such that

$$P(t,k_2)U(t,k_2)=U(t,k_2)P(0,0) \quad TU(t,k_2)=U(-t,-k_2)T \quad \forall t\in[-\pi,\pi], k_2\in\mathbb{T}^1.$$

Then we can fix a symmetric orthonormal basis $\{u_j\}_{j\in\{1,\cdots,2n\}}$ of $P(0,0)$ to obtain a continuous collection of orthonormal bases of $P(t,k_2)$:

$$\{u_j(t,k_2)=U(t,k_2)u_j\}_{j\in\{1,\cdots,2n\}}.$$

This collection is periodic in k_2 and is such that:

$$Tu_j(t,k_2)=\sum_{a=1}^{2n}[J]_{j,a}u_a(-t,-k_2).$$

The aperiodicity in the first direction is expressed by a continuous family of unitary matrices $\alpha(k_2)\in U(2n)$ such that

$$u_j(-\pi,k_2)=\sum_{a=1}^{2n}[\alpha(k_2)]_{j,a}u_a(\pi,k_2).$$

This family satisfies the symmetry constraint $J = \overline{\alpha(k_2)} J \alpha(-k_2)$ because:

$$\begin{aligned}\sum_{a=1}^{2n} [J]_{j,a} u_a(\pi, -k_2) = T u_j(-\pi, k_2) = T \sum_{a=1}^{2n} [\alpha(k_2)]_{j,a} u(\pi, k_2)\\ = \sum_{a,b=1}^{2n} \overline{[\alpha(k_2)]_{j,a}} [J]_{a,b} u_b(-\pi, -k_2) =\\ = \sum_{a,c=1}^{2n} \left[\overline{\alpha(k_2)}(J)\right]_{j,a} [\alpha(-k_2)]_{a,c} u_c(\pi, -k_2) =\\ = \sum_{a=1}^{2n} \left[\overline{\alpha(k_2)}(J)\alpha(-k_2)\right]_{j,a} u_a(\pi, -k_2)\end{aligned}$$

Using this symmetry constraint we can find two matrices γ_0, γ_π such that $\alpha(0) = J\gamma_0^t J^{-1}\gamma_0$ and same for γ_π. For example, since $J\alpha(0)^t J^{-1} = \alpha(0)$, we can find a self-adjoint matrix L such that $\alpha(0) = e^{iL}$ and $JL^t J^{-1} = L$, then $\gamma_0 = e^{iL/2}$ will work. It is obvious that $\det(\alpha(0)) = \det(\gamma_0)^2$ and $\det(\alpha(\pi)) = \det(\gamma_\pi)^2$. If $\det(\gamma_0) = e^{i\lambda_0}$ and $\det(\gamma_\pi) = e^{i\lambda_\pi}$ for two real numbers $\lambda_0, \lambda_\pi \in \mathbb{R}$, we can choose a function $\mu : [0, \pi] \to \mathbb{R}$ such that $\det(\alpha(k_2)) = e^{i\mu(k_2)}$ and $\mu(0) = 2\lambda_0$, then it holds that $e^{i\mu(\pi)} = e^{i2\lambda_\pi}$, so $\mu(\pi) - 2\lambda_\pi \in 2\pi\mathbb{Z}$.

Definition 4.12 ($\delta(P)$)**.** Under the hypotheses listed before, we will define:

$$\delta(P) = e^{i\frac{(2\lambda_\pi - \mu(\pi))}{2}} \in \{\pm 1\} = \mathbb{Z}_2.$$

Proposition 4.13. *The quantity previously defined does not depend on the choices of* $\lambda_0, \lambda_\pi, \gamma_0, \gamma_\pi$*.*

Proof. If we took λ_0', λ_π' such that $e^{i\lambda_0} = e^{i\lambda_0'}$ and $e^{i\lambda_\pi} = e^{i\lambda_\pi'}$, we would have that $\lambda(0) = \lambda_0' + 2\pi l$, $\lambda_\pi = \lambda_\pi' + 2\pi m$ for some $l, m \in \mathbb{Z}$. This means that we need to impose $\mu'(k) = \mu(k) + 4\pi l$ and obtain:

$$\delta' = e^{i\frac{2\lambda_\pi' - \mu'(\pi)}{2}} = e^{i\frac{2\lambda_\pi + 4\pi m - \mu(\pi) - 4\pi l}{2}} = \delta \cdot e^{i2\pi(m-l)} = \delta$$

Instead, if we took another γ_0' such that $\alpha(0) = J\gamma_0^t J^{-1}\gamma_0 = J(\gamma_0')^t J^{-1}\gamma_0'$, then it is true that:

$$(\gamma_0')^t J \gamma_0' = \gamma_0^t J \gamma_0 \Leftrightarrow J = (\gamma_0(\gamma_0')^{-1})^t J (\gamma_0(\gamma_0')^{-1})$$

This means that $\gamma_0(\gamma_0')^{-1} \in \mathrm{Sp}(2n)$, so $\det(\gamma_0(\gamma_0')^{-1}) = 1$ and therefore $\det(\gamma_0) = \det(\gamma_0')$. Since the same holds for γ_π, we have that $\det(\gamma_0), \det(\gamma_\pi)$ depend only on $\alpha(0), \alpha(\pi)$. □

Theorem 4.14. *Given a time-reversal symmetric projection-valued map* $P : \mathbb{T}^2 \to \mathrm{Proj}_{2n}(\mathcal{H})$ *and an integer* $h \in \mathbb{Z}$*, if* $e^{i\pi h} = \delta$*, it is always possible to decompose* P *as* $P(k) = P^+(k) \oplus P^-(k)$ *in the sense of Definition 4.2 with* $\mathrm{Ch}(P^-) = h$*. Vice versa, if* P *can be decomposed as* $P(k) = P^+(k) \oplus P^-(k)$*, then* $e^{i\pi\,\mathrm{Ch}(P^-)} = \delta$*.*

Proof. We want to start in the same setting as the Definition 4.12, which means that $P(t, k_2)$ is spanned by $\{u_j(t, k_2)\}_{j\in\{1,\cdots,2n\}}$ with constraints:

$$Tu_j(t, k_2) = \sum_{a=1}^{2n} [-J]_{j,a} u_a(-t, -k_2), \quad u_j(-\pi, k_2) = \sum_{a=1}^{2n} [\alpha(k_2)]_{j,a} u_a(\pi, k_2)$$

where α is a periodic unitary-valued map such that $J = \overline{\alpha(k_2)} J \alpha(-k_2)$.

- *General conditions:*

  We notice that the existence of a decomposition is tantamount to the existence of a symmetric frame with appropriate pseudo-periodic conditions. In fact, given a decomposition $P(k) = P^+(k) \oplus P^-(k)$, we can apply Theorem 1.10 to $P(k_1 - \pi, k_2)$ to construct a frame $\{v_j(k_1, k_2)\}_{j\in\{1,\cdots,n\}}$ of $P^-(k_1, k_2)$ with complete periodicity except for the first vector that must satisfy

$$v_1(\pi, k_2) = e^{i\,\mathrm{Ch}(P^-)k_2} v_1(-\pi, k_2) = e^{ihk_2} v_1(-\pi, k_2)$$

  since $\mathrm{Ch}(P^-(k_1 - \pi, k_2)) = \mathrm{Ch}(P^-(k_1, k_2))$. Then

$$\{v_{n+j}(t, k_2) = -T v_j(-t, -k_2)\}_{j\in\{1,\cdots,n\}}$$

  is a frame of $P^+(t, k_2) = T^{-1} P^-(-t, -k_2) T$ with complete periodicity except for the first vector that satisfies

$$v_{n+1}(\pi, k_2) = e^{-i\,\mathrm{Ch}(P^-)k_2} v_{n+1}(-\pi, k_2) = e^{-ihk_2} v_{n+1}(-\pi, k_2).$$

  The union of the two will constitute a symmetric frame for P and this also implies that $\mathrm{Ch}(P^+) = -\,\mathrm{Ch}(P^-)$. Vice versa, if we have a frame with this shape, we can take $P^-(k)$ as the span of the first n vectors and $P^+(k)$ as the span of the remaining n vectors.

  At the same time, the existence of a frame of this type is tantamount to the existence of a unitary-valued map $\beta : [-\pi, \pi] \times \mathbb{T}^1 \to U(2n)$ such that the following conditions are met:

  1. $v_j(t, k_2) = \sum_{a=1}^{2n} [\beta(t, k_2)]_{j,a} u_a(t, k_2)$.
  2. $J\beta(t, k_2) = \overline{\beta(-t, -k_2)} J$.
  3. $\beta(\pi, k_2) = \Lambda(k_2)\beta(-\pi, k_2)\alpha(k_2)$ where $\Lambda(k)$ is a diagonal matrix with

$$\Lambda(k) = \mathrm{diag}(e^{ihk_2}, \underbrace{1, \cdots, 1}_{n-1}, e^{-ihk_2}, \underbrace{1, \cdots, 1}_{n-1}).$$

  This equivalence is due to the fact that if we want to build the frame $\{v_j(t, k_2)\}_{j\in\{1,\cdots,n\}}$ using the frame $\{u_j(t, k_2)\}_{j\in\{1,\cdots,n\}}$ we need to mix the vectors u using the unitary-valued map β. Then the frame $\{v_j(t, k_2)\}_{j\in\{1,\cdots,n\}}$ is symmetric if and only if $J\beta(t, k_2) = \overline{\beta(-t, -k_2)} J$ because:

$$T v_j(t, k_2) = \sum_{a=1}^{2n} [J]_{j,a} v_a(-t, -k_2) \Leftrightarrow$$

$$\Leftrightarrow \sum_{a=1}^{2n} \overline{[\beta(t, k_2)]_{j,a}} T u_a(t, k_2) = \sum_{a,b=1}^{2n} [J]_{j,a} [\beta(-t, -k_2)]_{a,b} u_b(-t, -k_2) \Leftrightarrow$$

$$\Leftrightarrow \sum_{b=1}^{2n} \left[\overline{\beta(t, k_2)}(J)\right]_{j,b} u_b(-t, -k_2) = \sum_{b=1}^{2n} [J\beta(-t, -k_2)]_{j,b}\, u_b(-t, -k_2).$$

Moreover, the frame $\{v_j(t,k_2)\}_{j\in\{1,\cdots,n\}}$ satisfies the pseudoperiodic condition if and only if

$$\beta(\pi,k_2)=\Lambda(k_2)\beta(-\pi,k_2)\alpha(k_2)$$

because:

$$\begin{aligned} v_j(\pi,k_2) &= \sum_{a=1}^{2n}[\Lambda(k_2)]_{j,a}v_a(-\pi,k_2)\Leftrightarrow \\ \sum_{a=1}^{2n}[\beta(\pi,k_2)]_{j,a}u_a(\pi,k_2) &= \sum_{a=1}^{2n}[\Lambda(k_2)\beta(-\pi,k_2)]_{j,a}u_a(-\pi,k_2)\Leftrightarrow \\ \sum_{a=1}^{2n}[\beta(\pi,k_2)]_{j,a}u_a(\pi,k_2) &= \sum_{a=1}^{2n}[\Lambda(k_2)\beta(-\pi,k_2)\alpha(k_2)]_{j,a}u_a(\pi,k_2) \end{aligned}$$

However, the symmetry condition forces a constraint over β. In fact if $J\beta(0,k_2)=\overline{\beta(0,-k_2)}J$, then $\beta(0,0),\beta(0,\pi)\in \mathrm{Sp}(2n)$. This means that $\det(\beta(0,0))=\det(\beta(0,\pi))=1$. Moreover, $\det(\beta(0,k_2))=\overline{\det(\beta(0,-k_2)}$, so we can use the Lemma A.9 to conclude that the winding number in the second direction $[\det(\beta(t,\cdot))]$ must be even.

- *Decomposition when* $\delta=1$

  Now we want to investigate under which assumption it is possible to find such a β when h even. First of all, we can look at points $(\pi,0),(\pi,\pi)$. There it must be true that:

  $$\beta(\pi,0)=\Lambda(0)\beta(-\pi,0)\alpha(0),\quad \beta(\pi,\pi)=\Lambda(\pi)\beta(-\pi,\pi)\alpha(\pi)$$

  But it is immediately noticeable that, if $h\in 2\mathbb{Z}$, then $\Lambda(0)=\Lambda(\pi)=\mathbb{I}_{2n}$. If we use the periodicity in k_2 and the symmetry constraint, we get that β must satisfy:

  $$\beta(\pi,k_2)=\Lambda(k_2)J\overline{\beta(\pi,-k_2)}J^{-1}\alpha(k_2)$$

  in particular:

  $$J\beta(\pi,0)^tJ^{-1}\beta(\pi,0)=\alpha(0),\quad J\beta(\pi,\pi)^tJ^{-1}\beta(\pi,\pi)=\alpha(\pi).$$

  This means that, since $\det(\Lambda(k_2))\equiv 1$, it is true that

  $$\begin{cases}\det(\beta(\pi,k_2))=\overline{\det(\beta(\pi,-k_2))}\det(\alpha(k_2))\\ \det(\beta(\pi,0))^2=\det(\alpha(0))\\ \det(\beta(\pi,\pi))^2=\det(\alpha(\pi))\end{cases}$$

  Since the function β is continuous, we can choose a continuous function $\lambda:[0,\pi]\to\mathbb{R}$ such that $\det(\beta(\pi,k_2))=e^{i\lambda(k_2)}$, afterward we can choose a continuous function $\mu:[-\pi,\pi]\to\mathbb{R}$ such that $\det(\alpha(k_2))=e^{i\mu(k_2)}$ and with $\mu(0)=2\lambda(0)$. This means that:

  $$\det(\beta(\pi,k_2))=\begin{cases}e^{i\lambda(k_2)} & \text{for } k_2\in[0,\pi]\\ e^{-i\lambda(-k_2)+i\mu(-k_2)} & \text{for } k_2\in[-\pi,0]\end{cases}$$

So we can write $\det(\beta(\pi, k_2)) = e^{i\phi(k_2)}$ where $\phi : [-\pi, \pi] \to \mathbb{R}$ is a continuous but not periodic function such that:

$$\phi(k_2) = \begin{cases} \lambda(k_2) & \text{for } k_2 \in [0, \pi] \\ \lambda(-k_2) + \mu(-k_2) & \text{for } k_2 \in [-\pi, 0] \end{cases}$$

This implies that

$$[\det(\beta(\pi, \cdot))] = \frac{\lambda(\pi) - [-\lambda(\pi) + \mu(\pi)]}{2\pi} = \frac{2\lambda(\pi) - \mu(\pi)}{2\pi}$$

So, if we interpret $\beta(\pi, 0)$ as γ_0 and $\beta(\pi, \pi)$ as γ_π, then we can say that $\lambda(0) = \lambda_0, \lambda(\pi) = \lambda_\pi$ and obtain that

$$e^{i\pi[\det(\pi, k_2)]} = e^{i\pi \frac{2\lambda(\pi) - \mu(\pi)}{2\pi}} = \delta(P)$$

This means that if $\delta(P) = -1$, it is impossible to find β because δ does not depend on the choices of $\beta(\pi, 0)$ and $\beta(\pi, \pi)$ (Proposition 4.13) and the symmetry constraint imposed that $[\det(\beta(t, \cdot))]$ must be even for every t. Instead if $\delta(P) = 1$ we can use the Sylvester theorem to find two self-adjoint operators L, R with $JLJ^{-1} = L^t, JRJ^{-1} = R^t$ and such that $\alpha(0) = e^{iL}, \alpha(\pi) = e^{iR}$, then we can define:

$$\beta(\pi, k_2) = \begin{cases} \exp\left[i\frac{\pi - k_2}{2\pi} L + i\frac{k_2}{2\pi} R\right] & \text{for } k_2 \in [0, \pi] \\ \Lambda(k_2) J \overline{\beta(\pi, -k_2)} J^{-1} \alpha(k_2) & \text{for } k_2 \in [-\pi, 0] \end{cases}$$

This is continuous and periodic in k_2 because when $k_2 = 0$ we get:

$$e^{iL/2} = \Lambda(0) J e^{-i\overline{L}/2} J^{-1} e^{iL} = e^{-iJL^tJ^{-1}/2} e^{iL} = e^{iL/2}$$

and similar computations hold for $k_2 = \pi$. The previous argument states that $[\det(\beta(\pi, k_2))]$ is even equal to $2r$. Then we can also define

$$\beta(0, k_2) = \text{diag}(e^{irk_2}, \underbrace{1, \cdots, 1}_{n-1}, e^{irk_2}, \underbrace{1, \cdots, 1}_{n-1})$$

and it is trivial to check that $\beta(0, k_2)$ satisfies the symmetry constraint and has $[\det(\beta(0, k_2))] = 2r = [\det(\beta(\pi, k_2))]$. Finally we can use Theorem A.3 to find a homotopy $\beta_t(k_2)$ in $U(2n)$ with $\beta_0(k_2) = \beta(0, k_2), \beta_1(k_2) = \beta(\pi, k_2)$ and then we can define:

$$\beta(t, k_2) = \begin{cases} \beta_{t/\pi}(k_2) & \text{for } t \in [0, \pi] \\ J^{-1} \overline{\beta_{-t/\pi}(-k_2)} J & \text{for } t \in [-\pi, 0] \end{cases}$$

This satisfies all required conditions, so if $\delta(P) = 1$, we can build a decomposition with $\text{Ch}(P^-) = h$ for any even number h. While if $\delta(P) = -1$ this construction is impossible.

- *Decomposition when* $\delta = -1$

Now we want to investigate under which assumption it is possible to find a decomposition with $\text{Ch}(P^-)$ odd. It is immediate to check that the conditions are identical to the even case with a small but crucial difference. If $\mathrm{Ch}(P^-)$ is odd, it holds that:

$$\Lambda(0) = \mathbb{I}_{2n}, \quad \Lambda(\pi) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \mathbb{I}_{n-1} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \mathbb{I}_{n-1} \end{pmatrix}$$

Once again, if we use the periodicity in k_2 and the symmetry constraint, we get that β must satisfy:

$$\beta(\pi, k_2) = \Lambda(k_2) J \overline{\beta(\pi, -k_2)} J^{-1} \alpha(k_2)$$

in particular:

$$J\beta(\pi, 0)^t J^{-1} \beta(\pi, 0) = \alpha(0), \quad J\beta(\pi,\pi)^t J^{-1} \Lambda(\pi)\beta(\pi,\pi) = \alpha(\pi).$$

With some computation, it is easy to check that

$$\begin{pmatrix} 0 & -\mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \mathbb{I}_{n-1} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \mathbb{I}_{n-1} \end{pmatrix} =$$

$$= \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & \mathbb{I}_{n-1} & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & \mathbb{I}_{n-1} \end{pmatrix}^t \begin{pmatrix} 0 & -\mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix} \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & \mathbb{I}_{n-1} & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & \mathbb{I}_{n-1} \end{pmatrix}$$

meaning that if

$$Y = \mathrm{diag}(i, \underbrace{1, \cdots, 1}_{n-1}, i, \underbrace{1, \cdots, 1}_{n-1})$$

we have that $\det(Y) = -1$ and the last condition can be rewritten as:

$$J(Y\beta(\pi,\pi))^t J^{-1}(Y\beta(\pi,\pi)) = \alpha(\pi).$$

We can replicate the previous step of the proof to have that it is possible to define β if and only if $\frac{2\lambda(\pi)-\mu(\pi)}{2\pi} \in 2\mathbb{Z}$. In this case we can interpret $\beta(\pi, 0)$ as γ_0 and $Y\beta(\pi,\pi)$ as γ_π. Then we can impose $\lambda_0 = -i\ln(\det(\pi, 0)) = \lambda(0)$, so that μ does not need to be changed, and $\lambda_\pi = -i\ln(\det(Y\beta(\pi,\pi))) = \lambda(\pi) + \pi$ because

$$\det(Y\beta(\pi,\pi)) = -\det(\beta(\pi,\pi)) = e^{i(\lambda(\pi)+\pi)}.$$

We conclude that, in this case, we have that:

$$\exp\left(i\frac{2\lambda(\pi) - \mu(\pi)}{2}\right) = \exp\left(i\frac{2\lambda_\pi - \mu(\pi) - \pi}{2}\right) = -\delta(P)$$

Since we need an even winding number of $[\det(\beta(\pi, k_2))]$, if $-\delta(P) = -1$ it is impossible to construct a frame with h odd, instead if $-\delta(P) = 1$ we can proceed exactly as before and obtain the thesis. Namely, it is possible to decompose $P = P^+ \oplus P^-$ with $\mathrm{Ch}(P^-)$ odd if and only if $\delta(P) = -1$.

□

Lemma 4.15. *Consider two fermionic time-reversal symmetric projection-valued maps $P, Q : \mathbb{T}^2 \mapsto \mathrm{Proj}_{2n}(\mathcal{H})$ such that $P(k)Q(k) \equiv 0$. Then $\delta(P)\delta(Q) = \delta(P \oplus Q)$.*

Proof. We can use the previous theorem to find two decompositions $P(k) = P^+(k) \oplus P^-(k)$, $Q(k) = Q^+(k) \oplus Q^-(k)$. Then it is obvious that we can find a decomposition of $P(k) \oplus Q(k)$ as:

$$P(k) \oplus Q(k) = [P^-(k) \oplus Q^-(k)] \oplus [P^+(k) \oplus Q^+(k)]$$

Now we can use the Lemma 1.8 to know that $\mathrm{Ch}(P^- \oplus Q^-) = \mathrm{Ch}(P^-) + \mathrm{Ch}(Q^-)$. So we can conclude using the previous theorem

$$\delta(P \oplus Q) = e^{i\pi\,\mathrm{Ch}(P^- \oplus Q^-)} = e^{i\pi\,\mathrm{Ch}(P^-)} \cdot e^{i\pi\,\mathrm{Ch}(Q^-)} = \delta(P) \cdot \delta(Q).$$

□

Theorem 4.16. *Consider a fermionic time-reversal symmetric projection-valued map $P : \mathbb{T}^2 \to \mathrm{Proj}_{2n}(\mathcal{H})$. Then, it is always possible to construct a symmetric and quasiperiodic frame*

$$\{v_1, \cdots, v_n(k), -Tv_1(-k), \cdots, -Tv_n(-k)\}$$

of $P(k)$ such that, if $\delta(P) = 1$, the frame has full periodicity, while if $\delta(P) = -1$, the frame is periodic with the only exceptions being:

$$v_1(\pi, k_2) = e^{ik_2} v_1(-\pi, k_2) \quad v_{n+1}(\pi, k_2) = e^{-ik_2} v_{n+1}(-\pi, k_2) \forall k_2 \in \mathbb{T}^1.$$

Moreover two fermionic time-reversal symmetric projection-valued maps $P_0, P_1 : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$ are symmetrically unitarily equivalent and homotopic if and only if $\delta(P_0) = \delta(P_1)$.

Proof.
- (*Symmetric frame*) The construction of the symmetric frame can be done exactly as in the proof of Theorem 4.14, in particular if $\delta(P) = 1$ we can build a decomposition with any even Chern number. So we can build a decomposition with $\mathrm{Ch}(P^-) = 0$ and obtain full periodicity. Instead, if $\delta(P) = -1$, we can only have odd Chern numbers, so we can build a decomposition with $\mathrm{Ch}(P^-) = 1$ and obtain the thesis.

- (*Symmetric unitary equivalence*)

  First, we need to prove that if two projectors are symmetrically unitarily equivalent, then the δ invariant is preserved. This is obvious because if there is a unitary-valued map $U : \mathbb{T}^2 \to \mathcal{U}(\mathcal{H})$ such that $P_1(k) = U(k)P_0(k)U(k)^{-1}$ with $TU(k) = U(-k)T$. Then we can choose a decomposition $P_0 = P_0^-(k) \oplus P_0^+(k)$ where $\oplus$ indicates that the two elements we are summing are orthogonal. Clearly

$$P_1(k) = U(k)P_0^-(k)U(k)^{-1} \oplus U(k)P_0^+(k)U(k)^{-1}$$

is a decomposition of $P_1(k)$ and, since the Chern number is stable under unitary equivalences (Proposition 1.7) it holds that $\delta(P_1) = e^{i\pi \operatorname{Ch}(P_0^-)} = \delta(P_0)$.

Conversely, suppose that $\delta(P_0) = \delta(P_1)$. Then we can construct two frames

$$\{v_1^0(k), \cdots, v_n^0(k), -Tv_1^0(-k), \cdots, -Tv_n^0(-k)\}$$

$$\{v_1^1(k), \cdots, v_n^1(k), -Tv_1^1(-k), \cdots, -Tv_n^1(-k)\}$$

of $P_0(k)$ and $P_1(k)$ respectively, such that

$$\begin{array}{l} v_1^0(\pi, k_2) = e^{i\pi h(\delta) k_2} v_1^0(-\pi, k_2) \\ v_1^1(\pi, k_2) = e^{i\pi h(\delta) k_2} v_1^1(-\pi, k_2) \end{array} \quad \text{where } h(\delta) = \begin{cases} 0 & \text{if } \delta(P) = 1 \\ 1 & \text{if } \delta(P) = -1 \end{cases}$$

Now we need to look at $\mathbb{1} - P_0(k)$ and $\mathbb{1} - P_1(k)$. In fact, those are once again fermionic time-reversal symmetric projection-valued maps and, if $\dim(\mathcal{H}) < \infty$, thanks to Lemma 4.15 it holds that

$$\delta(\mathbb{1} - P_0) = \delta(\mathbb{1}) \cdot \delta(P_0) = \delta(P_0) = \delta(P_1) = \delta(\mathbb{1}) \cdot \delta(P_1) = \delta(\mathbb{1} - P_1).$$

So we can apply the previous point to extend the previous frames obtaining:

$$\{v_1^0(k), \cdots, -Tv_n^0(-k), v_{n+1}^0(k), \cdots, v_{\dim(\mathcal{H})/2}^0(k), -Tv_{n+1}^0(-k), \cdots - Tv_{\dim(\mathcal{H})/2}^1(-k)\}$$

$$\{v_1^1(k), \cdots, -Tv_n^1(-k), v_{n+1}^1(k), \cdots, v_{\dim(\mathcal{H})/2}^1(k), -Tv_{n+1}^1(-k), \cdots - Tv_{\dim(\mathcal{H})/2}^1(-k)\}.$$

where the added vectors span $\ker(P_0(k))$ and $\ker(P_1(k))$ respectively and have full periodicity with exceptions

$$v_{n+1}^0(\pi, k_2) = e^{i\pi h(\delta) k_2} v_{n+1}^0(-\pi, k_2) \quad v_{n+1}^1(\pi, k_2) = e^{i\pi h(\delta) k_2} v_{n+1}^1(-\pi, k_2).$$

Instead, if $\dim(\mathcal{H}) = \infty$ we can use Lemma 4.11 to extend the previous frames to

$$\{v_1^0(k), \cdots, -Tv_n^0(-k), v_{n+1}^0(k), -Tv_{n+1}^0(-k), \cdots, v_N^0(k), -Tv_N^0(-k), \cdots\}$$

$$\{v_1^1(k), \cdots, -Tv_n^1(-k), v_{n+1}^1(k), -Tv_{n+1}^1(-k), \cdots, v_N^1(k), -Tv_N^1(-k), \cdots\}$$

where the added vectors have full periodicity and are an orthonormal and countable basis of $\ker(P_0(k))$ and $\ker(P_1(k))$, respectively.

In both cases, the unitary-valued map $V(k)$ such that $V(k)v_j^0(k) = v_j^1(k)$ and $V(k)Tv_j^0(-k) = Tv_j^1(-k)$ for all $j \in \{1, \cdots, \dim(\mathcal{H})/2\}$ is the symmetric unitary equivalence we needed. In fact, it is trivial to prove that $P_1(k) = V(k)P_0(k)V(k)^{-1}$ and that $TV(k) = V(-k)T$. It is a bit less trivial to prove that this map is periodic despite the fact that it is defined over a non-periodic orthonormal basis. However, we can check, for example, that:

$$\begin{aligned} v_1^1(\pi, k_2) &= V(\pi, k_2)v_1^0(\pi, k_2) = V(\pi, k_2)e^{ih(\delta)k_2}v_1^0(-\pi, k_2) = e^{ih(\delta)k_2}V(\pi, k_2)v_1^0(-\pi, k_2) \\ &= e^{ih(\delta)k_2}v_1^1(-\pi, k_2) = e^{ih(\delta)k_2}V(-\pi, k_2)v_1^0(-\pi, k_2) \end{aligned}$$

So we have $V(-\pi, k_2)v_1^0(-\pi, k_2) = V(\pi, k_2)v_1^0(\pi, k_2)$ and similar arguments can be done for all the non-periodic vectors to prove that $V(-\pi, k_2)$ acts as $V(\pi, k_2)$.

- (*Symmetric homotopy*)

  First, we need to prove that, if two maps are homotopic, δ is preserved. This is easy to do because if $P_t(k)$ is the homotopy with $TP_t(k) = P_t(-k)T$ for $t \in [0, 1]$, then we can divide $[0, 1]$ into a partition $[t_j, t_{j+1}]_{j\in\{0,\cdots,J\}}$ such that $t_0 = 0, t_J = 1$ such that $\|P_t(k) - P_{t_j}(k)\| < 1$ for all $t \in [t_j, t_{j+1}], k \in \mathbb{T}^2$. Then we can use the Kato-Nagy formula to define the unitary-valued maps

$$U_j(k) = \left[P_{t_{j+1}}(k)P_{t_j}(k) + (\mathbb{1} - P_{t_{j+1}}(k))(\mathbb{1} - P_{t_j}(k))\right]\left[\mathbb{1} - (P_{t_j}(k) - P_{t_{j+1}}(k))^2\right]^{-1/2}$$

  for all $j \in \{0, \cdots, J-1\}$. It is easy to see that $U_j(k)$ is a symmetric unitary equivalence between $P_{t_j}(k)$ and $P_{t_{j+1}}(k)$ for all $j \in \{0, \cdots, J-1\}$, so, using the previous point, we see that δ is preserved at each step and therefore $\delta(P_0) = \delta(P_1)$.

  For the converse, suppose that $\delta(P_0) = \delta(P_1)$, then from the previous point we can create a symmetric unitary equivalence $P_1(k) = V(k)P_0(k)V(k)^{-1}$ with $TV(k) = V(-k)T$. The first step is to define a unitary-valued map $V_0 : \mathbb{T}^2 \to \mathcal{U}(\mathcal{H})$ such that $TV_0(k) = V_0(-k)T$ and $P_0(k) = V_0(k)P_0(k)V_0(k)^{-1}$. If $\dim(\mathcal{H}) = \infty$ we can define $V_0(k) \equiv \mathbb{1}$. Instead, if $\dim(\mathcal{H}) = 2N < \infty$, we need to do something more. First, we need to use Lemma 4.4 to select a basis on which T acts as $J\mathcal{K}$. Then we need to observe that the condition $TV(k) = V(-k)T$ means that $J\overline{V(k)} = V(-k)J$. So, if we consider the triangle $\triangle$ inside $[-\pi, \pi]^2$ with vertexes

$$Q_1 = (-\pi, \pi), \quad Q_2 = (\pi, \pi), \quad Q_3 = (\pi, -\pi)$$

  we can rename $\lambda(k) = \det(V(k))$. Then on the points $k_\star$ fixed by the involution $k \mapsto -k$ we have that $J = V(k_\star)^t JV(k_\star)$, so $V(k_\star) \in \mathrm{Sp}(2n)$. Moreover, on the segments $\overline{Q_1Q_2}, \overline{Q_1, Q_3}, \overline{Q_2Q_3}$, we have that $\lambda(k) = \overline{\lambda(-k)}$. Therefore, in those segments, we can apply Lemma A.9 to have that

$$[\lambda(t, \pi)]_{t\in[-\pi,\pi]} = 2m \quad [\lambda(\pi, t)]_{t\in[-\pi,-\pi]} = 2l \quad [\lambda(t, -t)]_{t\in[-\pi,\pi]} = 2h.$$

  Since the map is defined also inside the triangle, the winding number of λ in $\triangle$ must be zero, so $2m = 2l + 2h$. Now we can use the quasi-periodic frame $\{v_1^0(k), \cdots, -Tv_N^0(-k)\}$ to define the unitary operator $V_0(k)$ with:

$$V_0(k)v_j^0(k) = \begin{cases} e^{i(mk_1+lk_2)}v_1^0(k) & \text{if} \quad j = 1 \\ v_j^0(k) & \text{otherwise} \end{cases}$$

$$V_0(k)(-Tv_j^0(-k)) = \begin{cases} e^{i(mk_1+lk_2)}(-Tv_1^0(-k)) & \text{if} \quad j = 1 \\ -Tv_j^0(-k) & \text{otherwise} \end{cases}$$

This is actually periodic in k because, for example,

$$\begin{aligned}V_0(-\pi,k_2)v_1^0(\pi,k_2) &= V_0(-\pi,k_2)e^{i\pi h(\delta)}v_1^0(-\pi,k_2) = e^{i\pi h(\delta)}e^{i(-m\pi+lk_2)}v_1^0(-\pi,k_2) = \\ &= e^{-2im\pi}e^{i(m\pi+lk_2)}e^{i\pi h(\delta)}v_1^0(-\pi,k_2) = e^{i(m\pi+lk_2)}v_1^0(\pi,k_2) = \\ &= V_0(\pi,k_2)v_1^0(\pi,k_2)\end{aligned}$$

In both cases, $\dim(\mathcal{H}) = \infty, \dim(\mathcal{H}) = 2N$ we want to define a homotopy V_t between V_0 and V. To do so we need to work on $\triangle$. Notice that on the sides of the triangle $\overline{Q_1Q_2}$, $\overline{Q_1Q_3}$ and $\overline{Q_2Q_3}$ the projection-valued maps behave as a one-dimensional fermionic time-reversal symmetric projection-valued maps, in fact:

$$TP(k_1,\pi) = P(-k_1,-\pi)T = P(-k_1,\pi)T,$$

$$TP(\pi,k_2) = P(-\pi,-k_2)T = P(\pi,-k_2)T,$$

$$TP(k_1,-k_1) = P(-k_1,k_1)T.$$

So we can replicate the proof in Theorem 4.9 to build $V_t(k)$ for $k \in \partial\triangle$ with $TV_t(k) = V_t(-k)T$ for all $t \in [0,1], k \in \partial\triangle$. This means that $V_t(k)$ is defined for $(t,k) \in \partial([0,1]\times\triangle)$. This region however, is homeomorphic to S^2 and since $\pi_2(U(2N)) = \{0\} = \pi_2(\mathcal{U}(\mathcal{H}))$, thanks to Theorems A.3 A.2, it is always possible to extend the definition inside for $(t,k) \in [0,1]\times\triangle$. As a final adjustment we just need to define the map in the complementary triangle by imposing $V_t(k) = T^{-1}V_t(-k)T$ for $k \in [-\pi,\pi]^2 \setminus \triangle$. Then $V_t(k)$ is continuous, periodic and symmetric and therefore:

$$P_t(k) = V_t(k)P_0(k)V_t(k)^{-1}$$

is the symmetric homotopy we were looking for since $[P_0(k), V_0(k)] = 0$.

□

Chapter 5

Class C

In class C, the only symmetry present is an odd particle-hole symmetry, so the object we want to study in this chapter is the one defined in Definition I.13 with the additional constraint that $C^2 = -\mathbb{1}$:

Definition 5.1 (odd particle-hole symmetric pair of projection-valued maps)**.** Given an anti-unitary operator C acting over $\mathcal{H}$ with $C^2 = -\mathbb{1}$, a pair of projection-valued maps $P^{\pm} : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ is particle-hole symmetric if and only if:

$$CP^+(k) = P^-(-k)C \quad \text{and} \quad P^+(k)P^-(k) = 0 \quad \forall k \in \mathbb{T}^d.$$

With this in mind, given two odd particle-hole symmetric pairs of projection-valued maps $P_0^{\pm}, P_1^{\pm} : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$, we will try to answer the following questions:

Question 13 (Class C frame)**.** When will there be a continuous and particle-hole symmetric frame $\{v_1(k), \cdots, v_{2n}(k)\}$, namely a continuous and periodic collection of orthonormals vectors where the first half spans $\mathrm{Ran}(P_0^-(k))$, the second half spans $\mathrm{Ran}(P_0^+(k))$ and $Cu_j(k) = u_{n+j}(-k)$ for all $j \in \{1, \cdots, n\}$? In case there is none, it is possible to have a frame with the properties listed before and regular pseudoperiodic conditions like those obtained in Theorem 5.5?

Question 14 (Class C unitary equivalence)**.** When will there be a particle-hole symmetric unitary-valued map $U : \mathbb{T}^d \to \mathcal{U}(\mathcal{H})$ such that $P_1^{\pm}(k) = U(k)P_0^{\pm}(k)U(k)^{-1}$ and $CU(k) = U(-k)C$ for all $k \in \mathbb{T}^d$? This symmetric condition is the one that appears in Definition I.13

Question 15 (Class C homotopy)**.** When will there be a continuous and particle-hole symmetric pair of maps map $P^{\pm} : [0,1] \times \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ with $P^{\pm}(0,k) = P_0^{\pm}(k)$, $P^{\pm}(1,k) = P_1^{\pm}(k)$ and $CP_t^+(k) = P_t^-(-k)C$ for all $k \in \mathbb{T}^d$ and $t \in [0,1]$?

This is the second original part of our work (to the best of our knowledge); in this case, we formally prove that the results present in the Kitaev table are complete topological indices. Before we move on, there is an interesting remark worth noticing:

Remark 5.2. If $P^{\pm} : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ is a odd particle-hole symmetric pair of projection-valued maps, then $P^+(k) \oplus P^-(k)$ can be treated as a fermionic time-reversal symmetric projection-valued map because:

$$C(P^+(k) + P^-(k)) = (P^-(-k) + P^+(-k))C$$

5.1 d=0

Thanks to Lemma 4.4, we can interpret $\mathcal{H}$ as a quaternionic Hilbert space where C acts as a secondary imaginary unit. Moreover, if $\dim(\mathcal{H}) < \infty$, then the dimension is even $\dim(\mathcal{H}) = 2N$ and there is a basis of $\mathcal{H}$ on which C acts as $\mathcal{K}J$where $\mathcal{K}$ is the standard complex conjugation and $J = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$.

Proposition 5.3. *Two odd particle-hole symmetric pairs of projection-valued maps $P_0^\pm, P_1^\pm \in \mathrm{Proj}_n(\mathcal{H})$ will always have a symmetric frame and be symmetrically unitarily equivalent and homotopic.*

Proof. In order to obtain a symmetric frame, we can find an orthonormal basis $\{v_j^0\}_{j\in\{1,\cdots,n\}}$ of $\mathrm{Ran}(P_0^-)$. Then we can define $\{v_{n+j} = Cv_j\}_{j\in\{1,\cdots,n\}}$ and those will constitute an orthonormal basis of $\mathrm{Ran}(P_0^+)$. The union of the two will constitute the symmetric frame we needed.

For the symmetric unitary equivalence, we can use Proposition 4.6 or Proposition 4.4 to obtain a basis $\{v_j^0\}_{j\in\{2n+1,\cdots,\dim(\mathcal{H})\}}$ of $\ker(P_0^+ + P_0^-)$. For simplicity we do not apply these theorems faithfully, but we want to change the order of the vectors so that the following holds:

$$Cv_j^0 = \begin{cases} -v_{n+j}^0 & \text{if } 1 \le j \le n \\ v_{j-n}^0 & \text{if } n+1 \le j \le 2n \\ -v_{j+1}^0 & \text{if } j > 2n \text{ odd} \\ v_{j-1}^0 & \text{if } j > 2n \text{ even} \end{cases}$$

If we repeat this procedure to $P_1^\pm$, we will obtain the orthonormal basis $\{v_j^1\}_{j\in\{1,\cdots,\dim(\mathcal{H})\}}$ with identical properties and the unitary equivalence is the unitary operator $V \in \mathcal{U}(\mathcal{H})$ such that $V(v_j^0) = v_j^1$ for all $j \in \{1,\cdots,\dim(\mathcal{H})\}$. In fact, it is trivial to verify that $CV = VC$ and that $P_1^\pm = VP_0^\pm V^{-1}$.

Instead, for the homotopy problem, we can interpret V as an element of $\mathrm{Sp}(2N)$ if $\dim(\mathcal{H}) = 2N$, or as an element of $\mathcal{U}(\mathcal{H}_\mathbb{H})$ if $\dim(\mathcal{H}) = \infty$. In both cases, these sets are the sets of unitary operators that commute with C. Since both are path-connected thanks to Theorems A.5, A.2, we can find a homotopy V_t between V and $\mathbb{1}$. Therefore, it is elementary to see that $P_t^\pm = V_t P_0^\pm V_t^{-1}$ is the symmetric homotopy we were looking for. □

5.2 d=1

Proposition 5.4. *Two odd particle-hole symmetric pairs of projection-valued maps $P_0^\pm, P_1^\pm : \mathbb{T}^1 \to \mathrm{Proj}_n(\mathcal{H})$ will always have a symmetric frame and be symmetrically unitarily equivalent and homotopic.*

Proof. The first two statements of the proposition are very easy to prove; in fact, we can apply Proposition 1.4 to construct a periodic and continuous frame $\{v_j^0(k)\}_{j\in\{1,\cdots,n\}}$ of $P_0^-(k)$: Then $\{v_{n+j}(k) = Cv_j^0(-k)\}_{j\in\{1,\cdots,n\}}$ will be a periodic and continuous frame of $P_0^+(k)$ and the union of the two collections will constitute the symmetric frame we needed.

Moreover, we can use the Lemma 4.8 or Proposition 4.9 to build a continuous orthonormal basis

$$\{v_j^0(k)\}_{j\in\{2n+1,\cdots,\dim(\mathcal{H})\}} \quad \text{of} \quad \ker(P_0^+(k) + P_0^-(k))$$

such that

$$Cv_j^0(k) = \begin{cases} -v_{n+j}^0(-k) & \text{if } 1 \le j \le n \\ v_{j-n}^0(-k) & \text{if } n+1 \le j \le 2n \\ -v_{j+1}^0(-k) & \text{if } j > 2n \text{ odd} \\ v_{j-1}^0(-k) & \text{if } j > 2n \text{ even} \end{cases}$$

We can repeat this procedure to $P_1^{\pm}(k)$ in order to obtain $\{v_j^1(k)\}_{j\in\{1,\cdots,\dim(\mathcal{H})\}}$ with identical properties, and then the symmetric unitary equivalence we needed is the unitary-valued map $V(k)$ with $V(k)v_j^0(k) = v_j^1(k)$ for all $k \in \mathbb{T}^1$ and $j \in \{1, \cdots, \dim(\mathcal{H})\}$. In fact, it is elementary to check that $CV(k) = V(-k)C$ and that $P_1^{\pm}(k) = V(k)P_0(k)^{\pm}V(k)^{-1}$.

Instead, for the homotopy problem, we want to define $V_0(k)$ such that $CV_0(k) = V_0(-k)C$, $P_0^{\pm}(k) = V_0(k)P_0^{\pm}(k)V_0(k)^{-1}$ and such that the unitary-valued maps $V_0(k), V(k)$ are homotopic. If $\dim(\mathcal{H}) = \infty$, we can impose $V_0(k) \equiv \mathbb{1}$. Instead, if $\dim(\mathcal{H}) = 2N < \infty$, we can apply the Lemma 4.4 to obtain a basis on which C acts as $\mathcal{K}J$, therefore, the symmetric relation $CV(k) = V(-k)C$ implies that $V(0), V(\pi) \in \mathrm{Sp}(2N)$ and $\det(V(k)) = \overline{\det(V(-k))}$. This means that we can apply the Lemma A.9 to conclude that $[\det(V(k))] = 2l$. Then we can define $V_0(k)$ as the operator such that:

$$V_0(k)v_j^0(k) = \begin{cases} e^{ilk_2}v_j^0(k) & \text{for } j \in \{1, n+1\} \\ v_j^0(k) & \text{otherwise} \end{cases}$$

It is immediate to notice that $[P_0(k), V_0(k)] = 0$, $CV(k) = V(-k)C$ and that $[\det(V_0(k))] = 2l = [\det(V(k))]$. Actually, we know something more: it is possible to compute $[\det(V(k))]_{k\in[0,\pi]}$ because $\det(V(0)) = \det(V(\pi)) = 1$ since $V(0), V(\pi)$ are symplectic matrices. This partial winding number is exactly equal to l as well as $[\det(V_0(k))]_{k\in[0,\pi]} = l$. Now we want to define a homotopy $V_t(k)$ with $V_1(k) = V(k)$ such that $CV_t(k) = V_t(-k)C$ for all $t \in [0,1], k \in \mathbb{T}^1$. This means that $V_t(0), V_t(\pi)$ must commute with C for all $t \in [0,1]$. If $\dim(\mathcal{H}) = \infty$, this happens if and only if $V_t(0), V_t(\pi) \in \mathcal{U}(\mathcal{H})_{\mathbb{H}}$ which is contractible thanks to Theorem A.2. Instead, if $\dim(\mathcal{H}) = 2N < \infty$, this happens if and only if $V_t(0), V_t(\pi) \in \mathrm{Sp}(2N)$, which is connected owing to Theorem A.5. This means that $V_t(k)$ is defined for $(t,k) \in \partial([0,1] \times [0,\pi])$. Now we want to extend the definition inside the square, this is always possible if $\dim(\mathcal{H}) = \infty$ because $\mathcal{U}(\mathcal{H})$ is contractible due to Theorem A.2. When $\dim(\mathcal{H}) = 2N < \infty$, this is possible if the winding number of the determinant is zero. However, we built $V_0(k)$ exactly to make this possible, because the determinant stays constant as $V_t(k)$ travels inside $\mathrm{Sp}(2N)$ and

$$[\det(V(k))]_{k\in[0,\pi]} = [\det(V_0(k))]_{k\in[0,\pi]} = l.$$

This means that it is possible to define $V_t(k)$ also for $(t,k) \in [0,1] \times [0,\pi]$. To conclude, we only need to use the symmetry constraint to define $V_t(k) = C^{-1}V_t(-k)C$ for $k \in [-\pi, 0]$. This is perfectly symmetric and periodic because when $k \in \{0, \pi\}$ the map takes values inside the space of unitary operators that commute with C. This means that

$$P_t^{\pm}(k) = V_t(k)P_0^{\pm}(k)V_t(k)^{-1}$$

is the symmetric homotopy we were looking for. □

5.3 d=2

Theorem 5.5. *Consider a odd particle-hole symmetric pair of projection-valued maps* $P^{\pm} : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$*, then it is always possible to construct a symmetric and quasiperiodic frame* $\{v_j(k)\}$ *for* $j \in \{1, \cdots, 2n\}$ *such that:*

- *The first* n *vectors span* $\mathrm{Ran}(P^-(k))$*, and the others span* $\mathrm{Ran}(P^+(k))$*.*
- $Cv_j(k) = -v_{n+j}(-k)$ *for* $1 \leq j \leq n$*.*
- *All of them are periodic in* k *except* v_1 *and* v_{n+1} *which are only periodic in* k_2 *with pseudo-periodicity in* k_1*:*

$$v_1(\pi, k_2) = e^{i\,\mathrm{Ch}(P^-)k_2} v_1(-\pi, k_2), \quad v_{n+1}(\pi, k_2) = e^{i\,\mathrm{Ch}(P^+)k_2} v_{n+1}(-\pi, k_2), \quad \forall k_2 \in \mathbb{T}^1.$$

Moreover, two odd particle-hole symmetric pairs of projection-valued maps $P_0^{\pm}, P_1^{\pm} : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$ *are symmetrically unitarily equivalent and homotopic if and only if* $\mathrm{Ch}(P_0^-) = \mathrm{Ch}(P_1^-)$*.*

Proof.

- (*Symmetric frame*)

  We can apply Theorem 1.10 to the projection-valued map $\tilde{P}(k_1, k_2) = P^-(k_1 - \pi, k_2)$ to obtain a frame $\{\tilde{v}_j(k)\}_{j\in\{1,\cdots,n\}}$ of $\tilde{P}(k)$ with pseudoperiodicity $\tilde{v}_1(2\pi, k_2) = e^{i\,\mathrm{Ch}(\tilde{P})k_2} v_1(0, k_2)$. This means that $\{v_j(k_1, k_2) = \tilde{v}_j(k_1 - \pi, k_2)\}_{j\in\{1,\cdots,n\}}$ will be a pseudoperiodic frame of $P^-(k)$ and, since P^- and $\tilde{P}$ are homotopic thanks to $P_t(k_1, k_2) = P^-(k_1 - t\pi, k_2)$, $\mathrm{Ch}(P^-) = \mathrm{Ch}(\tilde{P})$. Then $\{v_{n+j}(k) = Cv_j(-k)\}_{j\in\{1,\cdots,n\}}$ will be a frame of $P^+(k)$. The union of the two frames will constitute the symmetric quasiperiodic frame we were looking for because $\mathrm{Ch}(P^-) = -\mathrm{Ch}(P^+)$ thanks to Remark 3.15.

- (*Symmetric unitary equivalence*)

  We can start slowly by noticing that if two odd particle-hole symmetric pairs of projection-valued maps are symmetrically unitarily equivalent, then the projection-valued maps P_0^-, P_1^- must be unitarily equivalent, therefore $\mathrm{Ch}(P_0^-) = \mathrm{Ch}(P_1^-)$. To prove the converse we can apply the previous argument to $P_0^{\pm}(k)$ and $P_1^{\pm}(k)$ in conjunction with Lemma 4.11, if $\dim(\mathcal{H}) = \infty$, or Proposition 4.16, if $\dim(\mathcal{H}) < \infty$ to obtain two collections of orthonormal bases $\{v_j^0(k)\}_{j\in\{1,\cdots,\dim(\mathcal{H})\}}$, $\{v_j^1(k)\}_{j\in\{1,\cdots,\dim(\mathcal{H})\}}$ such that:

  1. All $v_j^0(k)$ and $v_j^1(k)$ are continuous in k.
  2. All $v_j^0(k)$ and $v_j^1(k)$ are periodic in k with the only exceptions being:

$$v_1^0(\pi, k_2) = e^{i\,\mathrm{Ch}(P_0^-)k_2} v_1^0(-\pi, k_2), \quad v_{n+1}^0(\pi, k_2) = e^{i\,\mathrm{Ch}(P_0^+)k_2} v_{n+1}^0(-\pi, k_2), \quad \forall k_2 \in \mathbb{T}^1$$

$$v_1^1(\pi, k_2) = e^{i\,\mathrm{Ch}(P_1^-)k_2} v_1^1(-\pi, k_2), \quad v_{n+1}^1(\pi, k_2) = e^{i\,\mathrm{Ch}(P_1^+)k_2} v_{n+1}^1(-\pi, k_2), \quad \forall k_2 \in \mathbb{T}^1$$

$$v_{2n+1}^0(\pi, k_2) = e^{ih(\delta_0)k_2} v_{2n+1}^0(-\pi, k_2), \quad v_{2n+2}^0(\pi, k_2) = e^{ih(\delta_0)k_2} v_{2n+2}^0(-\pi, k_2) \quad \forall k_2 \in \mathbb{T}^1$$

$$v_{2n+1}^1(\pi, k_2) = e^{ih(\delta_1)k_2} v_{2n+1}^1(-\pi, k_2), \quad v_{2n+2}^1(\pi, k_2) = e^{ih(\delta_1)k_2} v_{2n+2}^1(-\pi, k_2) \quad \forall k_2 \in \mathbb{T}^1$$

Where

$$h(\delta) = \begin{cases} 0 & \text{if } \dim(\mathcal{H}) = \infty \\ 0 & \text{if } \dim(\mathcal{H}) < \infty \text{ and } \delta(\mathbb{1} - P^+(k) - P^-(k)) = 1 \\ 1 & \text{if } \dim(\mathcal{H}) < \infty \text{ and } \delta(\mathbb{1} - P^+(k) - P^-(k)) = -1 \end{cases}$$

for δ defined in Definition 4.12.

3. The vectors are linked by the symmetric relations:

$$Cv_j^0(k) = \begin{cases} -v_{n+j}^0(-k) & \text{if } 1 \leq j \leq n \\ v_{j-n}^0(-k) & \text{if } n+1 \leq j \leq 2n \\ -v_{j+1}^0(-k) & \text{if } j > 2n \text{ odd} \\ v_{j-1}^0(-k) & \text{if } j > 2n \text{ even} \end{cases}$$

4. The first n vectors span respectively $\mathrm{Ran}(P_0^-(k))$ and $\mathrm{Ran}(P_1^-(k))$, the second n vectors span respectively $\mathrm{Ran}(P_0^+(k))$ and $\mathrm{Ran}(P_1^+(k))$ while the remaining span respectively $\ker(P_0^- + P_0^+(k))$ and $\ker(P_1^-(k) + P_1^+(k))$.

Then the symmetric unitary equivalence we need is the unitary operator $V(k)$ such that:

$$V(k)v_j^0(k) = v_j^1(k) \quad \text{for all} \quad j \in \{1, \cdots, \dim(\mathcal{H})\}, k \in \mathbb{T}^2.$$

In fact, it is obvious that this respects the symmetry constraint $CV(k) = V(-k)C$ and that $P_1^\pm(k) = V(k)P_0^\pm(k)V(k)^{-1}$. However it may not be so obvious that this is periodic in k_1 since it was defined over a pseudoperiodic frame. Luckily it is immediate to check that:

$$\begin{aligned} V(\pi, k_2)v_1^0(\pi, k_2) &= v_1^1(\pi, k_2) = e^{i\,\mathrm{Ch}(P_1^-)k_2}v_1^1(-\pi, k_2) = \\ &= e^{i\,\mathrm{Ch}(P_1^-)k_2}V(-\pi, k_2)v_1^0(-\pi, k_2) = \\ &= V(-\pi, k_2)v_1^0(\pi, k_2) \\ V(\pi, k_2)v_{n+1}^0(\pi, k_2) &= v_{n+1}^1(\pi, k_2) = e^{i\,\mathrm{Ch}(P_1^+)k_2}v_{n+1}^1(-\pi, k_2) = \\ &= e^{i\,\mathrm{Ch}(P_1^+)k_2}V(-\pi, k_2)v_{n+1}^0(-\pi, k_2) = \\ &= V(-\pi, k_2)v_{n+1}^0(\pi, k_2) \end{aligned}$$

Moreover, if we interpret $P_0^+(k) \oplus P_0^-(k)$ as an element of class AII we have that $\delta(P_0^+ \oplus P_1^-) = e^{i\pi\,\mathrm{Ch}(P_0^-)}$, this also means that

$$\delta(P_1^+ \oplus P_1^-) = e^{i\pi\,\mathrm{Ch}(P_1^-)} = e^{i\pi\,\mathrm{Ch}(P_0^-)} = \delta(P_0^+ \oplus P_1^-)$$

So, using Lemma 4.15 we know that

$$1 = \delta(\mathbb{1}) = \delta(\mathbb{1} - (P_0^+ \oplus P_0^-))\delta(P_0^+ \oplus P_0^-) = \delta(\mathbb{1} - (P_1^+ \oplus P_1^-))\delta(P_1^+ \oplus P_1^-)$$

In particular we have that $h(\delta_0) = h(\delta_1)$, so v_{2n+1}^0 and v_{2n+1}^1 satisfy the same pseudoperiodic conditions as well as v_{2n+2}^0 and v_{2n+2}^1. So it is very easy to prove that $V(-\pi, k_2)$ acts as $V(\pi, k_2)$ on $\mathcal{H}$ and this proves the periodicity of the unitary-valued map V.

- (*Symmetric homotopy*) First of all we need to notice that a homotopy of pairs of projection-valued maps includes having a homotopy between P_0^- and P_1^-, so the Chern number must be preserved thanks to Theorem 1.10. Conversely, suppose that $\mathrm{Ch}(P_0^-) = \mathrm{Ch}(P_1^-)$. Then from the previous point we can create a symmetric unitary equivalence $P_1(k) = V(k)P_0(k)V(k)^{-1}$ with $CV(k) = V(-k)C$. The first step is to define a unitary-valued map $V_0 : \mathbb{T}^2 \to \mathcal{U}(\mathcal{H})$ such that $TV_0(k) = V_0(-k)T$, $P_0^\pm(k) = V_0(k)P_0^\pm(k)V_0(k)^{-1}$ and $V_0(k)$ is homotopic to $V(k)$. If $\dim(\mathcal{H}) = \infty$ we can define $V_0(k) \equiv \mathbb{1}$. Instead, if $\dim(\mathcal{H}) = 2N < \infty$, we need to do something more. First, we need to use Lemma 4.4 to select a basis on which C acts as $J\mathcal{K}$. Then we need to observe that the condition $CV(k) = V(-k)C$ means that $J\overline{V(k)} = V(-k)J$. So, if we consider the triangle $\triangle$ inside $[-\pi,\pi]^2$ with vertexes

$$Q_1 = (-\pi,\pi), \quad Q_2 = (\pi,\pi), \quad Q_3 = (\pi,-\pi)$$

we can name $\lambda(k) = \det(V(k))$, then on the points $k_\star$ fixed by the involution $k \mapsto -k$ we have that $J = V(k_\star)^t JV(k_\star)$, so $V(k_\star) \in \mathrm{Sp}(2n)$. Moreover, on the segments $\overline{Q_1Q_2}, \overline{Q_1,Q_3}, \overline{Q_2Q_3}$, we have that $\lambda(k) = \overline{\lambda(-k)}$. Therefore, on those segments, we can apply Lemma A.9 to have that

$$[\lambda(t,\pi)]_{t\in[-\pi,\pi]} = 2m, \quad [\lambda(\pi,t)]_{t\in[-\pi,-\pi]} = 2l, \quad [\lambda(t,-t)]_{t\in[-\pi,\pi]} = 2h.$$

Since the map is defined also inside the triangle, the winding number of λ in $\triangle$ must be zero, so $2m = 2l + 2h$. Now we can use the quasiperiodic frame $\{v_1^0(k), \cdots, -Tv_N^0(-k)\}$ to define the unitary operator $V_0(k)$ with:

$$V_0(k)v_j^0(k) = \begin{cases} e^{i(mk_1+lk_2)}v_j^0(k) & \text{if} \quad j \in \{1, n+1\} \\ v_j^0(k) & \text{otherwise} \end{cases}$$

This is actually periodic in k because, for example,

$$\begin{aligned} V_0(-\pi,k_2)v_1^0(\pi,k_2) &= V_0(-\pi,k_2)e^{i\pi h(\delta)}v_1^0(-\pi,k_2) = e^{i\pi h(\delta)}e^{i(-m\pi+lk_2)}v_1^0(-\pi,k_2) = \\ &= e^{-2im\pi}e^{i(m\pi+lk_2)}e^{i\pi h(\delta)}v_1^0(-\pi,k_2) = e^{i(m\pi+lk_2)}v_1^0(\pi,k_2) = \\ &= V_0(\pi,k_2)v_1^0(\pi,k_2) \end{aligned}$$

In both cases, $\dim(\mathcal{H}) = \infty, \dim(\mathcal{H}) = 2N$ we want to define a homotopy V_t between V_0 and V and to do so we need to work on $\triangle$. Notice that on the sides of the triangle $\overline{Q_1Q_2}$, $\overline{Q_1Q_3}$ and $\overline{Q_2Q_3}$ the projector-valued maps behave as a one-dimensional time reversal symmetry projector-valued maps: in fact

$$CP(k_1,\pi) = P(-k_1,-\pi)C = P(-k_1,\pi)C,$$

$$CP(\pi,k_2) = P(-\pi,-k_2)C = P(\pi,-k_2)C,$$

$$CP(k_1,-k_1) = P(-k_1,k_1)C.$$

So we can replicate the proof in Theorem 5.4 to build $V_t(k)$ for $k \in \partial\triangle$ with $CV_t(k) = V_t(-k)C$ for all $t \in [0,1], k \in \partial\triangle$. This means that $V_t(k)$ is defined for $(t,k) \in \partial([0,1]\times\triangle)$. This region however, is homeomorphic to S^2 and since $\pi_2(U(2N)) = \{0\} = \pi_2(\mathcal{U}(\mathcal{H}))$, thanks to Theorems

A.3 and A.2, it is always possible to extend the definition inside for $(t,k) \in [0,1] \times \triangle$. As a final adjustment we just need to define the map in the complementary triangle by imposing $V_t(k) = C^{-1}V_t(-k)C$ for $k \in [-\pi,\pi]^2 \setminus \triangle$. Then $V_t(k)$ is continuous, periodic and symmetric and therefore

$$P_t(k) = V_t(k)P_0(k)V_t(k)^{-1}$$

is the symmetric homotopy we were looking for since $[P_0(k), V_0(k)] = 0$.

□

Chapter 6

Class AIII

In class AIII, the only symmetry present is a chiral symmetry, so the objects we want to study in this chapter are defined in Definitions I.15 I.14 that we write once more for the sake of completeness:

Definition 6.1 (chiral-invariant projection-valued map)**.** Given a unitary operator S acting on $\mathcal{H}$ with $S^2 = \mathbb{1}$, a projection-valued map $P : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ is chiral invariant if and only if

$$SP(k) = P(k)S \quad \forall k \in \mathbb{T}^d.$$

Definition 6.2 (chiral-symmetric pair of projection-valued maps)**.** Given a unitary operator S acting on $\mathcal{H}$ with $S^2 = \mathbb{1}$, a pair of projection-valued maps $P^{\pm} : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ is chiral symmetric if and only if:

$$SP^+(k) = P^-(k)S \quad \text{and} \quad P^+(k)P^-(k) = 0 \quad \forall k \in \mathbb{T}^d.$$

Definition 6.3. We can decompose $\mathcal{H} = \ker(S - \mathbb{1}) \oplus \ker(S + \mathbb{1})$ and these are two orthogonal Hilbert subspaces. In this decomposition, S acts as $\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$. Now any chiral-invariant projection-valued map $P : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ can be written in this decomposition as $\begin{pmatrix} Q_1(k) & Q_2(k) \\ Q_3(k) & Q_4(k) \end{pmatrix}$ and it is immediate to prove that the following relations are equivalent:

$$\begin{cases} P(k)^2 = P(k) \\ P(k) = P(k)^* \\ SP(k) = P(k)S \end{cases} \Leftrightarrow \begin{cases} Q_1(k)^2 = Q_1(k) \\ Q_4(k)^2 = Q_4(k) \\ Q_1(k)^* = Q_1(k) \\ Q_4(k)^* = Q_4(k) \\ Q_2(k) = 0 \\ Q_3(k) = 0 \end{cases} \quad \text{for all } k \in \mathbb{T}^d$$

This means that $P(k)$ is uniquely determined by two projection-valued maps $Q_1(k)$ and $Q_4(k)$ that we will rename $P^{\uparrow}(k) \in \mathrm{Proj}(\ker(S - \mathbb{1}))$ and $P^{\downarrow}(k) \in \mathrm{Proj}(\ker(S + \mathbb{1}))$.

In general, even if we do not know the ranks of $P^{\updownarrow}(k)$, it is easy to prove that the ranks are constant in k. In fact, we can write the projection on $\ker(S \mp \mathbb{1})$ as $(\mathbb{1} \pm S)/2$, and therefore:

$$P^{\updownarrow}(k) = P(k)\frac{\mathbb{1} \pm S}{2} \Rightarrow \dim(\mathrm{Ran}(P^{\updownarrow}(k))) = \mathrm{tr}\left[P(k)\frac{\mathbb{1} \pm S}{2}\right]$$

This must be constant in k since the right term is a continuous function with values in $\mathbb{Z}$. From a physical point of view this is related to a fiber Hamiltonian $H(k)$ that anti-commutes with S. If there is $\tilde{k} \in \mathbb{T}^d$ such that the spectrum of $H(\tilde{k})$ does not contain zero, then we can find all eigenvectors $\{v_j(\tilde{k})\}_{j\in\{1,\cdots,l\}}$ with positive eigenvalues in the Bloch band Ω and $\{Sv_j(\tilde{k})\}_{j\in\{1,\cdots,l\}}$ will be all eigenvectors with negative eigenvalues in Ω. Therefore, all of them will make up a basis of $\mathrm{Ran}(P(\tilde{k}))$, so $2l = n$ and it is immediate to prove that $\{v_j(\tilde{k}) + Sv_j(\tilde{k})\}_{j\in\{1,\cdots,l\}}$ will be a basis of $P^{\uparrow}(\tilde{k})$ while $\{v_j(\tilde{k}) - Sv_j(\tilde{k})\}_{j\in\{1,\cdots,l\}}$ will be a basis of $P^{\downarrow}(\tilde{k})$. In the great majority of physical model it happens that such $\tilde{k}$, so in general $n = 2l$ and $\dim(\mathrm{Ran}(P^{\updownarrow}(k))) \equiv l$. However, to maintain maximum generality, we will not assume that they are equal.

This means that two chiral-invariant projection-valued maps $P_0, P_1 : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ are tantamount to two pairs of projection-valued maps $P_0^{\updownarrow}, P_1^{\updownarrow} : \mathbb{T}^d \to \mathrm{Proj}(\ker(S \mp \mathbb{1}))$ with fixed ranks. Those pairs are not intertwined by any symmetry relation, so they can be treated as pairs of unrelated elements in Class A; this means that it will be very easy to answer the following questions.

Question 16 (Class AIII fixed frame)**.** When will there be a continuous and chiral-fixed frame $\{v_1(k), \cdots, v_n(k)\}$, namely a continuous and periodic collection of orthonormal vectors that span $P_0(k)$ such that $Su_j(k) = \pm u_j(k)$ for all $j \in \{1, \cdots, n\}$? In case there is none, is it possible to have a frame with the properties listed before and regular pseudoperiodic conditions like the ones obtained in Theorem 6.9?

Question 17 (Class AIII fixed unitary equivalence)**.** When will there be a chiral-fixed unitary-valued map $V : \mathbb{T}^d \to \mathcal{U}(\mathcal{H})$ such that $P_1^{\pm}(k) = V(k)P_0^{\pm}(k)SV(k)^{-1}$ and $SV(k) = V(k)S$ for all $k \in \mathbb{T}^d$?

Question 18 (Class AIII fixed homotopy)**.** When will there be a continuous and chiral- fixed map $P : [0,1] \times \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ with $P^{\pm}(0,k) = P_0^{\pm}(k)$, $P^{\pm}(1,k) = P_1^{\pm}(k)$ and $SP_t(k) = P_t(k)S$ for all $k \in \mathbb{T}^d$ and $t \in [0,1]$?

Since we have all the instruments to answer these questions now, we write the answers here.

Theorem 6.4. *Consider a pair of chiral-invariant projection-valued maps $P_0, P_1 : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$. Consider the decompositions*

$$P_0^{\updownarrow}(k) = P_0(k)\frac{\mathbb{1} \pm S}{2}, \quad P_1^{\updownarrow}(k) = P_1(k)\frac{\mathbb{1} \pm S}{2}$$

and set

$$m_0^{\updownarrow} = \dim(\mathrm{Ran}(P_0^{\updownarrow}(k))), \quad m_1^{\updownarrow} = \dim(\mathrm{Ran}(P_1^{\updownarrow}(k))).$$

If $d < 2$, it is always possible to construct two continuous and periodic fixed frames, and two chiral-invariant projection-valued maps are unitarily equivalent and homotopic with respect to chiral symmetry if and only if $m_0^{\updownarrow} = m_1^{\updownarrow}$. Instead, if $d = 2$, it is always possible to construct a pseudoperiodic fixed frame $\{v_j(k)\}_{j\in\{1,\cdots,n\}}$ such that:

1. *The first* $m_0^\uparrow$ *vectors span* $P_0^\uparrow(k)$ *and the other span* $P_0^\downarrow(k)$.

2. *They are continuous and orthonormal.*

3. *They are all periodic in both arguments with the only exceptions being:*

$$v_1(2\pi, k_2) = e^{i\,\mathrm{Ch}(P_0^\uparrow)k_2} v_1(0, k_2) \quad v_{m_0^\uparrow+1}(2\pi, k_2) = e^{i\,\mathrm{Ch}(P_0^\downarrow)k_2} v_{m_0^\downarrow}(0, k_2) \quad \forall k_2 \in \mathbb{T}^1$$

4. $Sv_j(k) = \begin{cases} v_j(k) & \text{if } j \in \{1, \cdots, m_0^\uparrow\} \\ -v_j(k) & \text{if } j \in \{m_0^\uparrow, \cdots, n\} \end{cases}$

Finally, the two chiral-invariant projection-valued maps are unitarily equivalent and homotopic if and only if $m_0^\updownarrow = m_1^\updownarrow$ *and* $\mathrm{Ch}(P_0^\updownarrow) = \mathrm{Ch}(P_1^\updownarrow)$.

Proof. The proof is obtained by applying Proposition 1.1, Proposition 1.4 and Theorem 1.10 to $P_0^\uparrow$, $P_1^\uparrow$ and $P_0^\downarrow, P_1^\downarrow$ on the two Hilbert subspaces $\ker(S - \mathbb{1})$ and $\ker(S - \mathbb{1})$. □

Instead, if $P_0^\pm, P_1^\pm : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ are two chiral-symmetric pairs of projection-valued maps, we will try to answer the following questions.

Question 19 (Class AIII symmetric frame)**.** When will there be a continuous and chiral symmetric frame $\{v_1(k), \cdots, v_{2n}(k)\}$, namely a continuous and periodic collection of orthonormal vectors such that the first n span $P_0^-(k)$, the other n span $P_0^+(k)$, and with $Sv_j(k) = v_{n+j}(k)$ for all $j \in \{1, \cdots, n\}, k \in \mathbb{T}^d$? In case there is none, is it possible to have a frame with the properties listed before and regular pseudoperiodic conditions?

Question 20 (Class AIII symmetric unitary equivalence)**.** When will there be a chiral symmetric unitary-valued map $V : \mathbb{T}^d \to \mathcal{U}(\mathcal{H})$ such that $P_1^\pm(k) = V(k)P_0^\pm(k)V(k)^{-1}$ and $SV(k) = V(k)SC$ for all $k \in \mathbb{T}^d$?

Question 21 (Class AIII symmetric homotopy)**.** When will there be a continuous and chiral symmetric pair of maps $P^\pm : [0,1] \times \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ with $P^\pm(0,k) = P_0^\pm(k)$, $P^\pm(1,k) = P_1^\pm(k)$ and $SP^+(t,k) = P^-(t,k)S$ for all $k \in \mathbb{T}^d$ and $t \in [0,1]$?

This is the third original part of our work to the best of our knowledge, and, similarly to what happens in class D, the answers depend on the dimension of the Hilbert space.

Remark 6.5. If $P^\pm : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ is a chiral-symmetric pair of projection-valued maps, then $P^+(k) + P^-(k)$ and $\mathbb{1} - P^+(k) - P^-(k)$ are two chiral-invariant projection-valued map because:

$$S(P^+(k) + P^-(k)) = (P^-(k) + P^+(k))S.$$

Furthermore, for a fixed $\tilde{k} \in \mathbb{T}^d$, we can find an orthonormal basis $\{v_j\}_{j\in\{1,\cdots,n\}}$ of $P^-(\tilde{k})$, so that $\{Sv_j\}_{j\in\{1,\cdots,l\}}$ is an orthonormal basis of $P^+(\tilde{k})$. Then $\{v_j(k) \pm Sv_j\}_{j\in\{1,\cdots,l\}}$ will be an orthonormal basis of $\mathrm{Ran}(P^-(\tilde{k}) + P^+(\tilde{k})) \cap \ker(S \mp \mathbb{1})$. In particular $\dim(\ker(S \pm \mathbb{1})) \geq n$.

6.1 d=0

Proposition 6.6. *Two chiral-symmetric pairs of projection-valued maps $P_0^\pm, P_1^\pm \in \mathrm{Proj}_n(\mathcal{H})$ will always have a symmetric frame and be symmetrically unitarily equivalent and homotopic.*

Proof. In order to obtain a symmetric frame, we can find an orthonormal basis $\{v_j^0\}_{j\in\{1,\cdots,n\}}$ of $\mathrm{Ran}(P_0^-)$. Then we can define $\{v_{n+j} = Sv_j\}_{j\in\{1,\cdots,n\}}$ and those will constitute an orthonormal basis of $\mathrm{Ran}(P_0^+)$. The union of the two sets will constitute the symmetric frame we needed.

For the symmetric unitary equivalence, we can see $\mathbb{1} - P_0^- - P_0^+$ as a chiral-invariant projection-valued map. Then in the sense of Definition I.14 we can decompose it as:

$$\mathbb{1} - P_0^- - P_0^+ = P^\uparrow + P^\downarrow.$$

Thanks to Remark 6.5 we can define:

$$m_\updownarrow = \dim(\mathrm{Ran}(P_0^\updownarrow)) = \dim(\ker(S \mp \mathbb{1})) - n,$$

and if we choose an orthonormal basis for both, it is easy to build an orthonormal basis

$$\{v_j^0\}_{j\in\{-m_\downarrow+1,\cdots,2n+m_\uparrow\}}$$

such that:

1. Those vectors span $P_0^\downarrow$ for $-m_\downarrow < j \le 0$.
2. They span $P_0^\uparrow$ for $2n < j \le m_\uparrow$.
3. They span P_0^- for $0 < j \le n$.
4. They span P_0^+ for $n < j \le 2n$.
5. $$Sv_j^0 = \begin{cases} -v_j^0 & \text{for } m_\downarrow < j \le 0 \\ v_{n+j}^0 & \text{for } 0 < j \le n \\ v_{j-n}^0 & \text{for } n < j \le 2n \\ v_j & \text{for } 2n < j \le 2n + m_\uparrow \end{cases}$$

If we replicate the procedure with $P_1^\pm$ we can build $\{v_j^1\}_{j\in\{-m_\downarrow+1,\cdots,2n+m_\uparrow\}}$ with identical properties, and it is immediate to check that the unitary operator V such that $Vv_j^0 = v_j^1$ for all $m_\downarrow < j \le 2n+m_\uparrow$ satisfies

$$P_1^\pm = VP_0^\pm V^{-1} \quad SV = VS.$$

Instead, for the homotopy problem, we notice that the condition $SV = VS$ is met if and only if V is a diagonal operator $\begin{pmatrix} V^\uparrow & 0 \\ 0 & V^\downarrow \end{pmatrix}$ in the decomposition $\mathcal{H} = \ker(S - \mathbb{1}) \oplus \ker(S + \mathbb{1})$, where $V^\updownarrow \in \mathcal{U}(\ker(S \mp \mathbb{1}))$. Since both $\mathcal{U}(\ker(S \mp \mathbb{1}))$ are connected, thanks to Theorems A.2 or A.3, we can find two homotopies $V_t^\updownarrow$ with $V_0^\updownarrow = \mathbb{1}$ and $V_1^\updownarrow = V^\updownarrow$. Then the operator $V_t = \begin{pmatrix} V_t^\uparrow & 0 \\ 0 & V_t^\downarrow \end{pmatrix}$

commutes with S and $P_t^{\pm} = V_t P_0^{\pm} V_t^{-1}$ is the homotopy we are looking for since the symmetry constraint over V_t implies that $SP_t^+ = P_t^- S$ for all $t \in [0,1]$. □

6.2 d=1

We want to start this section with the definition of a topological invariant that exists only when $\dim(\mathcal{H}) = 2n$. In this setting, we can identify $\mathcal{H} = \mathbb{C}^{2n}$ and decompose all operators on a basis on which S acts as $\begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & -\mathbb{I}_n \end{pmatrix}$.

Definition 6.7 (The $\mathcal{Z}$ invariant)**.** If $P^{\pm} : \mathbb{T}^1 \to \mathrm{Proj}_n(\mathbb{C}^{2n})$ is a chiral-symmetric pair of projection-valued maps with $SP^+(k) = P^-(k)S$, we can define $R(k) = P^+(k) - P^-(k)$ and it is true that:

$$R(k)^2 = \mathbb{I}_{2n}, \quad R(k) = R(k)^*, \quad SR(k) = -R(k)S.$$

So it follows immediately that, on the basis where S is written as $S = \begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & \mathbb{I}_n \end{pmatrix}$, $R(k)$ is of the form $R(k) = \begin{pmatrix} 0 & U(k) \\ U(k)^* & 0 \end{pmatrix}$, for a unitary-valued map $U : \mathbb{T}^1 \to U(n)$. Now we can define

$$\mathcal{Z} = [\det(U(k))] \in \mathbb{Z}.$$

Proposition 6.8. *The quantity $\mathcal{Z}$ is a homotopy invariant of the pair; however, it is not invariant under symmetric unitary equivalences. Moreover, if two chiral-symmetric pairs of projection-valued maps are symmetrically unitarily equivalent, $P_1^{\pm}(k) = V(k)P_0^{\pm}(k)V(k)^{-1}$ and*

$$S = \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & -\mathbb{1}_n \end{pmatrix}, V(k) = \begin{pmatrix} V^{\uparrow}(k) & 0 \\ 0 & V^{\downarrow}(k) \end{pmatrix}$$

We have that the invariant will change according to the relation:

$$\mathcal{Z}_1 = [\det(V^{\uparrow}(k))] - [\det(V^{\downarrow}(k))] + \mathcal{Z}_0$$

Proof. If $P_t^{\pm}(k)$ is a chiral symmetric homotopy, then, on a suitable basis,

$$R_t(k) = P_t^+(k) - P_t^-(k) = \begin{pmatrix} 0 & U_t(k) \\ U_t(k)^* & 0 \end{pmatrix}$$

is still a continuous map, and so it is $U_t(k)$. So, the map $t \mapsto [\det(U_t(k))]$ is continuous and, since it can only assume values in $\mathbb{Z}$, is constant. Instead, if $P_0^{\pm}$ and $P_1^{\pm}$ are symmetrically unitarily equivalent, the definition tells us that there is a unitary-valued map $V(k)$ with $P_1^{\pm}(k) = V(k)P_0^{\pm}(k)V(k)^{-1}$ and $SV(k) = V(k)S$. This means that on the same suitable basis V can be decomposed as

$$V(k) = \begin{pmatrix} V^{\uparrow}(k) & 0 \\ 0 & V^{\downarrow}(k) \end{pmatrix} \quad \text{where } V^{\updownarrow} : \mathbb{T}^1 \to U(n).$$

This implies that $R_1(k) = V(k)R_0(k)V(k)^{-1}$, so:

$$\begin{pmatrix} 0 & U_1(k) \\ U_1(k)^* & 0 \end{pmatrix} = \begin{pmatrix} V^{\uparrow}(k) & 0 \\ 0 & V^{\downarrow}(k) \end{pmatrix} \begin{pmatrix} 0 & U_0(k) \\ U_0(k)^* & 0 \end{pmatrix} \begin{pmatrix} V^{\uparrow}(k)^* & 0 \\ 0 & V^{\downarrow}(k)^* \end{pmatrix} =$$
$$= \begin{pmatrix} 0 & V^{\uparrow}(k)U_0(k) \\ V^{\downarrow}(k)U_0(k)^* & 0 \end{pmatrix} \begin{pmatrix} V^{\uparrow}(k)^* & 0 \\ 0 & V^{\downarrow}(k)^* \end{pmatrix} =$$
$$= \begin{pmatrix} 0 & V^{\uparrow}(k)U_0(k)V^{\downarrow}(k)^* \\ (V^{\uparrow}(k)U_0(k)V^{\downarrow}(k)^*)^* & 0 \end{pmatrix}$$

Therefore we can measure how the invariant changes after unitary equivalences:

$$\mathscr{Z}_1 = [\det(U_1(k))] = [\det(V^{\uparrow}(k))] - [\det(V^{\downarrow}(k))] + \mathscr{Z}_0$$

□

Proposition 6.9. *Two chiral-symmetric pairs of projection-valued maps $P_0^{\pm}, P_1^{\pm} : \mathbb{T}^1 \to \mathrm{Proj}_n(\mathcal{H})$ will always have a symmetric frame and be symmetrically unitarily equivalent. For the homotopy problem, if $\dim(\mathcal{H}) > 2n$, they are always homotopic with respect to the symmetry. However, if $\dim(\mathcal{H}) = 2n$, they are homotopic with respect to the chiral symmetry if and only if $\mathscr{Z}_0 = \mathscr{Z}_1$.*

Proof. The first two statements of the proposition are very easy to prove; in fact, we can apply Proposition 1.4 to construct a periodic and continuous frame $\{v_j^0(k)\}_{j\in\{1,\cdots,n\}}$ of $P_0^-(k)$: Then $\{v_{n+j}(k) = Sv_j^0(k)\}_{j\in\{1,\cdots,n\}}$ will be a periodic and continuous frame of $P_0^+(k)$ and the union of the two collections will constitute the symmetric frame we needed.

Moreover, we can call

$$P_0^{\updownarrow}(k) = (\mathbb{1} - P_0^+(k) - P_0^-(k))(S \mp \mathbb{1})/2$$

according to Definition 6.3. Then we can apply Lemma 1.3 or Proposition 1.4 to both of them to build two continuous orthonormal basis

$$\{v_j^0(k)\}_{j\in\{-m_\downarrow+1,\cdots,0\}} \quad \{v_j^0(k)\}_{j\in\{2n+1,\cdots,2n+m_\uparrow\}}$$

spanning $P_0^{\downarrow}(k)$ and $P_0^{\uparrow}(k)$, respectively. This means that:

$$Sv_j^0(k) = \begin{cases} -v_j^0(k) & \text{for } m_\downarrow < j \le 0 \\ v_{n+j}^0(k) & \text{for } 0 < j \le n \\ v_{j-n}^0(k) & \text{for } n < j \le 2n \\ v_j(k) & \text{for } 2n < j \le 2n + m_\uparrow \end{cases} \qquad \forall k \in \mathbb{T}^1.$$

We can repeat this procedure to $P_1^{\pm}(k)$ to obtain $\{v_j^1(k)\}_{j\in\{m_\downarrow-1,\cdots,2n+m_\uparrow\}}$ with identical properties, and afterward the symmetric unitary equivalence we needed is the unitary-valued map $V(k)$ with $V(k)v_j^0(k) = v_j^1(k)$ for all $k \in \mathbb{T}^1$ and $j \in \{1, \cdots, \dim(\mathcal{H})\}$. In fact, it is elementary to check that $SV(k) = V(k)S$ and that $P_1^{\pm}(k) = V(k)P_0(k)^{\pm}V(k)^{-1}$.

Instead, for the homotopy problem in $\dim(\mathcal{H}) = 2n$, we notice that the classification of chiral symmetry pairs of projector-valued maps is tantamount to the classification of unitary-valued maps. In fact, on a suitable basis on which S is diagonal, we can build $U(k)$ starting from $P^{\pm}(k)$ as we did in Definition I.14. But we can do the converse because in $\mathbb{C}^{2n}$ it holds that $P^{-}(k) + P^{+}(k) = \mathbb{I}_{2n}$ So, given a unitary-valued map $U(k)$, we can define

$$P^{\pm}(k) = \frac{1}{2}\begin{pmatrix} \mathbb{I}_n & \pm U(k) \\ \pm U(k)^* & \mathbb{I}_n \end{pmatrix}$$

and it is very easy to check that this $P^{\pm}(k)$ is a pair of chiral symmetric projection-valued maps. Since these constructions preserve the continuity in k we find that two chiral-symmetric pairs of projection-valued maps $P_0^{\pm}, P_1^{\pm}$ are homotopic if and only if their off-diagonal components are homotopic. Due to Theorem A.3, this happens if and only if $\mathcal{Z}_0 = \mathcal{Z}_1$.

Instead, if $2n < \dim(\mathcal{H}) < \infty$, we can use the additional dimensions to eliminate the winding numbers. Since $\dim(\mathcal{H}) > 2n$ we have that one between $m_{\uparrow}$, $m_{\downarrow}$ must be greater than zero. Suppose for simplicity that $m_{\downarrow} > 0$, the other case is equivalent. Then, given a symmetric unitary equivalence $V(k)$ between $P_1^{\pm}(k)$ and $P_0^{\pm}(k)$, the symmetry constraint $SV(k) = V(k)S$ forces $V(k)$ to be decomposed as

$$V(k) = \begin{pmatrix} V^{\uparrow}(k) & 0 \\ 0 & V^{\downarrow}(k) \end{pmatrix} \quad \text{where } V^{\updownarrow} : \mathbb{T}^1 \to \mathcal{U}(\ker(S \mp \mathbb{1})).$$

Now, if we call $l^{\updownarrow} = [\det(V^{\updownarrow}(k))] \in \mathbb{Z}$, we can define the unitary-valued map $V_0(k)$ such that:

$$V_0(k)v_j^0(k) = \begin{cases} e^{ikl^{\uparrow}} v_j^0(k) & \text{for } j = 1, n+1 \\ e^{ik(l^{\downarrow} - l^{\uparrow})} v_j^0(k) & \text{for } j = 0 \\ v_j^0(k) & \text{otherwise} \end{cases}$$

This $V_0(k)$ commutes with S, so it can be decomposed into $V_0^{\updownarrow}(k)$. We can notice that the vectors $v_j^0(k) + v_{n+j}^0(k)$ for $1 \leq j \leq n$ together with the vectors $v_j^0(k)$ for $j > 2n$ form an orthonormal basis of $\ker(S - \mathbb{1})$ and $V_0^{\uparrow}(k)$ acts diagonally on this basis with $\det(V_0^{\uparrow}(k)) = e^{ikl^{\uparrow}}$, so $[\det(V_0^{\uparrow}(k))] = l^{\uparrow}$. Similarly, the vectors $v_j^0(k) - v_{n+j}^0(k)$ for $1 \leq j \leq n$ with the vectors $v_j^0(k)$ for $j \leq 0$ form an orthonormal basis of $\ker(S+\mathbb{1})$ and once more, $V_0^{\downarrow}(k)$ acts diagonally on this basis with $\det(V_0^{\downarrow}(k)) = e^{ikl^{\uparrow}} e^{ik(l^{\downarrow} - l^{\uparrow})} = e^{ikl^{\downarrow}}$, so $[\det(V_0^{\downarrow}(k))] = l^{\downarrow}$. Since the winding numbers agree, we can use Theorem A.3 to build two homotopies $V_t^{\updownarrow}(k) \in \mathcal{U}(\ker(S \mp \mathbb{1}))$ connecting $V^{\updownarrow}(k)$ with $V_0^{\updownarrow}(k)$. In the end, we can merge the two homotopies into $V_t(k) = \begin{pmatrix} V_t^{\uparrow}(k) & 0 \\ 0 & V_t^{\downarrow}(k) \end{pmatrix}$ which clearly commutes with S for every $(t, k) \in [0, 1] \times \mathbb{T}^1$ and finally our symmetric homotopy will be:

$$P_t^{\pm}(k) = V_t(k) P_0^{\pm}(k) V_t(k)^{-1}$$

because $SP^{+}(k) = P^{-}(k)S$ and $V_0(k)$ commutes with both $P_0^{\pm}(k)$ for every $k \in \mathbb{T}^1$.

Finally, if $\dim(\mathcal{H}) = \infty$, one between $m_{\uparrow}$ and $m_{\downarrow}$ must be infinite. If we suppose for simplicity that $m_{\downarrow} = \infty$, we can replicate the proof done before with $l^{\downarrow} = 0$. Then we will obtain the same result after applying Theorem A.2 to $V^{\downarrow}(k)$ instead of Theorem A.3. Instead, if $m_{\uparrow} = m_{\downarrow} = \infty$ we can impose $V_0 = \mathbb{1}$ and apply Theorem A.2 to both $V^{\updownarrow}(k)$, thus obtaining the thesis. □

6.3 d=2

In this symmetry class the answers will depend deeply on the dimension of the Hilbert space, so the main theorem will be split into two different statements.

Remark 6.10. If $\dim(\mathcal{H}) = 2n$ and $P^{\pm} : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$ is a chiral-symmetric pair of projection-valued maps, we can replicate the steps of Definition 6.7 to obtain the off-diagonal decomposition

$$P^+(k) - P^-(k) = \begin{pmatrix} 0 & U(k) \\ U(k)^* & 0 \end{pmatrix}.$$

This time U is a map $U : \mathbb{T}^2 \to \mathcal{U}(n)$, which means that we can compute the winding number in two different directions:

$$\mathcal{Z}(P^{\pm}) = ([\det(U(k_1, 0)], [\det(U(0, k_2))]) \in \mathbb{Z}^2$$

Theorem 6.11. *If* $\dim(\mathcal{H}) = 2n$*, two chiral-symmetric pairs of projection-valued maps* $P_0^{\pm}, P_1^{\pm} : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$ *will always admit a symmetric and periodic frame and be symmetrically unitarily equivalent. However, they will be symmetrically homotopic if and only if* $\mathcal{Z}_0 = \mathcal{Z}_1$*.*

Instead, if $\dim(\mathcal{H}) > 2n$*, given a chiral-symmetric pair of projection-valued maps* $P^{\pm} : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$*, then it is always possible to construct a symmetric and quasiperiodic frame* $\{v_j(k)\}$ *for* $j \in \{1, \cdots, 2n\}$ *such that:*

- *The first* n *vectors span* $\mathrm{Ran}(P^-(k))$*, and the others span* $\mathrm{Ran}(P^+(k))$*.*
- $Sv_j(k) = v_{n+j}(k)$ *for* $1 \leq j \leq n$*.*
- *All of them are periodic in* k *except* v_1 *and* v_{n+1} *which are only periodic in* k_2 *with pseudo-periodicity in* k_1*:*

$$v_1(2\pi, k_2) = e^{i\,\mathrm{Ch}(P^-)k_2} v_1(0, k_2), \quad v_{n+1}(2\pi, k_2) = e^{i\,\mathrm{Ch}(P^-)k_2} v_{n+1}(0, k_2), \quad \forall k_2 \in \mathbb{T}^1.$$

Moreover, two chiral-symmetric pairs of projection-valued maps $P_0^{\pm}, P_1^{\pm} : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$ *are symmetrically unitarily equivalent and homotopic if and only if:*

$$\mathrm{Ch}(P_0^-) = \mathrm{Ch}(P_1^-), \quad \mathrm{Ch}(P_0^{\uparrow}) = \mathrm{Ch}(P_1^{\uparrow}) \quad \textit{and} \quad \mathrm{Ch}(P_0^{\downarrow}) = \mathrm{Ch}(P_1^{\downarrow}).$$

Remark 6.12. We want to emphasize some aspects of this theorem before its proof. First of all, why does $\mathrm{Ch}(P^+)$ not appear? The answer is pretty simple; the symmetry constraint tells us that $P^+(k)$ is unitarily equivalent to $P^-(k)$, so they will have the same Chern number. This also means that if $\dim(\mathcal{H}) = 2n$, then $P^-(k) + P^+(k) = \mathbb{I}_{2n}$. So $\mathrm{Ch}(P^+ + P^-) = 0$ and, thanks to the Lemma 1.8, we can conclude that if $\dim(\mathcal{H}) = 2n$, then $\mathrm{Ch}(P^-) = \mathrm{Ch}(P^+) = 0$. Moreover, is there any relation between the different $\mathbb{Z}$ invariants presented? Clearly, the two integer numbers that compose $\mathcal{Z}$ cannot be related as it is very easy to define a unitary-valued map with arbitrary winding numbers in the two directions. Then one may ask if there is a relation between $\mathcal{Z}$ and $\mathrm{Ch}(P^-)$, but once again the answer is pretty simple, because when the first exists, the other must be zero. In the end, there may be a constraint between all the Chern numbers. This can be found because, if $2n < \dim(\mathcal{H}) < \infty$, we can apply Lemma 1.8 to

$$\mathbb{I}_{\dim(\mathcal{H})} = P^-(k) + P^+(k) + P^\uparrow(k) + P^\downarrow(k)$$

obtaining that:

$$0 = \mathrm{Ch}\left(\mathbb{I}_{\dim(\mathcal{H})}\right) = 2\,\mathrm{Ch}(P^-) + \mathrm{Ch}(P^\uparrow) + \mathrm{Ch}(P^\downarrow).$$

Instead, if $\dim(\mathcal{H}) = \infty$, we lose this relation, but one or both between $\mathrm{Ch}(P^\uparrow), \mathrm{Ch}(P^\downarrow)$ must be zero.

Proof.

- ($\dim(\mathcal{H}) = 2n$)

  We can apply Theorem 1.10 to the projector-valued map $P_0^-(k_1, k_2)$ to obtain a frame $\{v_j^0(k)\}$ for $j \in \{1, \cdots, n\}$ of $P_0^-(k)$ which is continuous and periodic since $\mathrm{Ch}(P^-) = 0$ thanks to the previous remark. Then $\{v_{n+j}^0(k) = Sv_j^0(k)\}_{j\in\{1,\cdots,n\}}$ will be a frame of $P^+(k)$ and the union of the two frames will constitute the symmetric periodic frame we were looking for. We can then replicate the procedure with $P_1^\pm(k)$ to build $\{v_j^1(k)\}_{j\in\{1,\cdots,2n\}}$ with identical properties and the unitary-valued map $V(k)$ such that $V(k)v_j^0(k) = v_j^1(k)$ will be the unitary equivalence we needed.

  For the homotopy problem we can apply Proposition 6.8 to obtain that, if two chiral-symmetric pairs of projection-valued maps are homotopic, then $\mathcal{Z}_1 = \mathcal{Z}_0$. For the converse, we recall a statement in the proof of Proposition 6.9 that ensures us that the classification of chiral-symmetric pairs of projection-valued maps is tantamount to the classification of unitary-valued maps. In fact, on a suitable basis on which S is diagonal, we can build $U(k)$ starting from $P^\pm(k)$ as we did in Definition I.14. But we can do the converse because in $\mathcal{H} = \mathbb{C}^{2n}$ it holds that $P^-(k) + P^+(k) = \mathbb{I}_{2n}$. So, given a unitary-valued map $U(k)$, we can define the chiral-symmetric pair of projection-valued maps:

$$P^\pm(k) = \frac{1}{2}\begin{pmatrix} \mathbb{I}_n & \pm U(k) \\ \pm U(k)^* & \mathbb{I}_n \end{pmatrix}$$

  So we want to create a homotopy between $U_0(k)$ and $U_1(k)$ moving in $\mathrm{U}(n)$. Since $\mathcal{Z}_0 = \mathcal{Z}_1$, we know that

$$[\det(U_0(k_1, 0))] = [\det(U_1(k_1, 0))], \quad [\det(U_0(0, k_2))] = [\det(U_1(0, k_2))]$$

  Then we can apply Theorem A.3 to build the homotopy $U_t(k)$ for $k = (k_1, 0)$ and for $k = (0, k_2)$. Thanks to periodicity this means that we also defined the homotopy for $k = (k_1, 2\pi)$ and $k = (2\pi, k_2)$. This means that our homotopy $U_t(k)$ is defined up to this point in the region $(t, k_1, k_2) \in \partial([0, 1] \times [0, 2\pi] \times [0, 2\pi])$ which is homeomorphic to S^2. Thanks to Theorem A.3, it is possible to extend the map continuously also inside this region, obtaining the homotopy we needed.

- ($\dim(\mathcal{H}) > 2n$)

We can start slowly by noticing that if two chiral-symmetric pairs of projection-valued maps are unitarily equivalent or homotopic with respect to the symmetry, then the projection-valued maps P_0^-, P_1^- must be unitarily equivalent, therefore $\mathrm{Ch}(P_0^-) = \mathrm{Ch}(P_1^-)$. The same also happens with the projection-valued maps $P_0^{\updownarrow}, P_1^{\updownarrow}$ because if $P_t^{\pm}(k)$ is a symmetric homotopy, then

$$P_t^{\updownarrow}(k) = (\mathbb{1} - P_t^+(k) - P_t^-(k))(S \pm \mathbb{1})/2$$

is a homotopy connecting them, so $\mathrm{Ch}(P_0^{\updownarrow}) = \mathrm{Ch}(P_1^{\updownarrow})$. To prove the converse suppose that the three Chern numbers of both families agree (recall that if the rank of the projection has infinite dimension we imposed a null Chern number in Definition 1.6).

We can start by applying Theorem 1.10 to $P_0^-(k)$ in order to obtain a continuous frame $\{v_j^0(k)\}_{j\in\{1,\cdots,n\}}$ with periodicity in k except for the first that satisfies

$$v_1^0(2\pi, k_2) = e^{i\,\mathrm{Ch}(P_0^-)k_2} v_1^0(0, k_2)$$

Then we can impose $v_{n+j}^0(k) = Sv_j^0(k)$ for $0 < j \leq n$ to obtain the symmetric frame depicted in the hypothesis. Now we can apply Theorem 1.10 to both $P_0^{\updownarrow}(k)$ (or Lemma 1.9 if some range has infinite dimension) to obtain two continuous collections of orthonormal bases

$$\{v_j^0(k)\}_{j\in\{-m_\downarrow+1,\cdots,0\}} \quad \{v_j^0(k)\}_{j\in\{2n+1,\cdots,2n+m_\uparrow\}}$$

spanning $P_0^{\downarrow}(k)$ and $P_0^{\uparrow}(k)$ respectively, those collections are taken with full periodicity in k with the only exception being:

$$v_0^0(2\pi, k_2) = e^{i\,\mathrm{Ch}(P_0^{\downarrow})k_2} v_0^0(0, k_2), \quad v_{2n+1}^0(2\pi, k_2) = e^{i\,\mathrm{Ch}(P_0^{\uparrow})k_2} v_{2n+1}(0, k_2) \quad \forall k_2 \in \mathbb{T}^1.$$

This means that:

$$Sv_j^0(k) = \begin{cases} -v_j^0(k) & \text{for } m_\downarrow < j \leq 0 \\ v_{n+j}^0(k) & \text{for } 0 < j \leq n \\ v_{j-n}^0(k) & \text{for } n < j \leq 2n \\ v_j(k) & \text{for } 2n < j \leq 2n + m_\uparrow \end{cases} \qquad \forall k \in \mathbb{T}^2.$$

We can repeat this procedure to $P_1^{\pm}(k)$ in order to obtain $\{v_j^1(k)\}_{j\in\{m_\downarrow-1,\cdots,2n+m_\uparrow\}}$ with identical properties and afterward the symmetric unitary equivalence we needed is the unitary-valued map $V(k)$ with $V(k)v_j^0(k) = v_j^1(k)$ for all $k \in \mathbb{T}^1$ and $j \in \{1, \cdots, \dim(\mathcal{H})\}$. In fact it is elementary to check that $SV(k) = V(k)S$ and that $P_1^{\pm}(k) = V(k)P_0(k)^{\pm}V(k)^{-1}$. However, it may not be clear that this V is periodic in k. Luckily it is immediate to check on a non-periodic vector ($v_1(k)$ for simplicity) that:

$$\begin{aligned} V(2\pi, k_2)v_1^0(2\pi, k_2) &= v_1^1(2\pi, k_2) = e^{i\,\mathrm{Ch}(P_1^-)k_2} v_1^1(-\pi, k_2) = \\ &= e^{i\,\mathrm{Ch}(P_0^-)k_2} V(0, k_2)v_1^0(0, k_2) = \\ &= V(0, k_2)v_1^0(2\pi, k_2) \end{aligned}$$

so $V(2\pi, k_2)$ acts on the basis in the same way as $V(0, k_2)$. We can replicate this argument with all the non-periodic vectors and obtain that if all the Chern numbers agree, then V is periodic.

As for the homotopy problem, since $\dim(\mathcal{H}) > 2n$, we have that one between $m_\uparrow$ and $m_\downarrow$ must be greater than zero. Suppose for simplicity that $m_\downarrow > 0$, the other case is equivalent. Then, given a symmetric unitary equivalence $V(k)$ between $P_1^\pm(k)$ and $P_0^\pm(k)$, the symmetry constraint $SV(k) = V(k)S$ forces $V(k)$ to be decomposed as

$$V(k) = \begin{pmatrix} V^\uparrow(k) & 0 \\ 0 & V^\downarrow(k) \end{pmatrix} \quad \text{where } V^\updownarrow : \mathbb{T}^2 \to \mathcal{U}(\ker(S \mp \mathbb{1})).$$

Suppose now that $\dim(\mathcal{H}) < \infty$, then we can call

$$l^\updownarrow = \Big([\det(V^\updownarrow(k_1, 0))], [\det(V^\updownarrow(0, k_2))] \Big) \in \mathbb{Z}^2$$

and we can define the unitary-valued map $V_0(k)$ such that:

$$V_0(k) v_j^0(k) = \begin{cases} e^{ik\cdot l^\uparrow} v_j^0(k) & \text{for } j = 1, n+1 \\ e^{ik\cdot(l^\downarrow - l^\uparrow)} v_j^0(k) & \text{for } j = 0 \\ v_j^0(k) & \text{otherwise} \end{cases}$$

This $V_0(k)$ commutes with S, so it can be decomposed into $V_0^\updownarrow(k)$. We can notice that the vectors $v_j^0(k) + v_{n+j}^0(k)$ for $1 \le j \le n$ with the vectors $v_j^0(k)$ for$j > 2n$ form an orthonormal basis of $\ker(S - \mathbb{1})$ and $V_0^\uparrow(k)$ acts diagonally on this basis with $\det(V_0^\uparrow(k)) = e^{ik\cdot l^\uparrow}$, so

$$\Big([\det(V_0^\uparrow(k_1, 0))], [\det(V_0^\uparrow(0, k_2))] \Big) = l^\uparrow.$$

Similarly, the vectors $v_j^0(k) - v_{n+j}^0(k)$ for $1 \le j \le n$ with the vectors $v_j^0(k)$ for $j \le 0$ form an orthonormal basis of $\ker(S + \mathbb{1})$ and once more, $V_0^\downarrow(k)$ acts diagonally on this basis with $\det(V_0^\downarrow(k)) = e^{ik\cdot l^\uparrow} e^{ik\cdot(l^\downarrow - l^\uparrow)} = e^{ik\cdot l^\downarrow}$, so

$$\Big([\det(V_0^\downarrow(k_1, 0))], [\det(V_0^\downarrow(0, k_2))] \Big) = l^\downarrow.$$

Since the winding numbers agree, we can use Theorem A.3 to build four homotopies

$$V_t^\updownarrow(k_1, 0), V_t^\updownarrow(0, k_2) \in \mathcal{U}(\ker(S \mp \mathbb{1}))$$

connecting $V^\updownarrow(k_1, 0)$ with $V_0^\updownarrow(k_1, 0)$ and $V^\updownarrow(0, k_2)$ with $V_0^\updownarrow(0, k_2)$. Using periodicity we can impose $V_t^\updownarrow(k_1, 2\pi) = V_t^\updownarrow(k_1, 0)$ and $V_t^\updownarrow(2\pi, k_2) = V_t^\updownarrow(0, k_2)$. This means that $V_t^\updownarrow(k)$ are defined for $(t, k_1, k_2) \in \partial([0,1] \times [0, 2\pi] \times [0, 2\pi])$ and this region is homeomorphic to S^2. So, thanks to Theorem A.3, it is always possible to extend continuously those homotopies inside the cuboid, obtaining $V_t^\updownarrow(k) \in \mathcal{U}(\ker(S \mp \mathbb{1}))$ for every $(t, k_1, k_2) \in [0,1] \times [0, 2\pi] \times [0, 2\pi]$. In the end, we can merge the two homotopies into $V_t(k) = \begin{pmatrix} V_t^\uparrow(k) & 0 \\ 0 & V_t^\downarrow(k) \end{pmatrix}$ which clearly commutes with

S for every $(t,k) \in [0,1] \times \mathbb{T}^2$ and finally our symmetric homotopy will be:

$$P_t^{\pm}(k) = V_t(k)P_0^{\pm}(k)V_t(k)^{-1}$$

because $SP^+(k) = P^-(k)S$ and $V_0(k)$ commutes with both $P_0^{\pm}(k)$ for every $k \in \mathbb{T}^1$.

Finally, if $\dim(\mathcal{H}) = \infty$, one between $m_\uparrow$ and $m_\downarrow$ must be infinite. If we suppose for simplicity that $m_\downarrow = \infty$, we can replicate the proof done before with $l^\downarrow = 0$, then we can obtain the same result after applying Theorem A.2 to $V^\downarrow(k)$ instead of Theorem A.3. Instead, if $m_\uparrow = m_\downarrow = \infty$ we can impose $V_0 = \mathbb{1}$ and apply Theorem A.2 to both $V^\updownarrow(k)$ obtaining the thesis.

□

Chapter 7

Explicit models

Now we want to analyze one-dimensional models studied in the physical literature and propose some variations that are useful to highlight the role of dimerization in tight-binding (discrete) models. The two famous models that we want to analyze are the SSH model and the Kitaev chain proposed in [56] and [38].

A one-dimensional discrete model is described by a self-adjoint operator H acting on $l^2(\Gamma)$ where Γ is a discrete and periodic subset of $\mathbb{R}$ given by a finite number of points $X = \{x_1, \cdots, x_n\}$ plus all its translates $X + \mathbb{Z} = \Gamma$. If the Hamiltonian commutes with integer translations $T_m(f)(x) = f(x - m)$ for all $m \in \mathbb{Z}$, then the Hamiltonian is said to be periodic, and there is a standard way to study the spectrum and its properties. As said in the Introduction, we need to choose a dimerization, which means that we choose a periodicity cell $\mathbb{W} = \{x'_1, \cdots, x'_n\} \subset \Gamma$ such that $\Gamma = \cup_{n\in\mathbb{Z}} \mathbb{W} + n$ and with $\mathbb{W} \cap \mathbb{W} + n = \emptyset$ for all $n \neq 0$, a natural choice is $\mathbb{W} = X$. Then we can define a unitary transformation: $U : l^2(\Gamma) \to l^2(\mathbb{Z}) \otimes \mathbb{C}^n$ using the canonical basis $\{e_1, \cdots, e_n\}$ of $\mathbb{C}^n$ as follows:

$$\begin{array}{rccc} U: & l^2(\Gamma) & \to & l^2(\mathbb{Z}) \otimes \mathbb{C}^n \\ & \delta_{m+x'_j} & \mapsto & \delta_m \otimes e_j \end{array} \quad \text{where} \quad \delta_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

It is immediate to check that, if H is a periodic operator, UHU^{-1} is still a periodic operator acting on $l^2(\mathbb{Z}) \otimes \mathbb{C}^n$. Then it is customary to use the Fourier transform:

$$\begin{array}{rccc} \mathcal{F}: & l^2(\mathbb{Z}) \otimes \mathbb{C}^n & \to & L^2(S^1, dk, \mathbb{C}^n) \\ & \delta_m \otimes e_j & \mapsto & \frac{e^{ikm}}{\sqrt{2\pi}} \cdot e_j \end{array}$$

So that we can study the operator $\hat{H} = \mathcal{F}UHU^{-1}\mathcal{F}^{-1}$. Since UHU^{-1} was translation invariant, we have that $\hat{H}$ is a fiber operator in the sense of Definition I.3, meaning that its action over a function splits into the action of a family of matrices, i.e. there is a family of matrices $H(k)$ such that:

$$\left(\hat{H}f\right)(k) = H(k) \cdot f(k).$$

This family of matrices contains all the topological properties useful for the study of the original H.

Suppose that we have two different choices of dimerization, that is, two unitary operators $U_{1,2} : l^2(\Gamma) \to l^2(\mathbb{Z}) \otimes \mathbb{C}^n$ built as before. The first thing we need to notice is that the operator $U_1 U_2^{-1}$

must commute with the integer translations T_m, so the operator $\mathcal{F}U_1U_2^{-1}\mathcal{F}^{-1}$ will admit a fiber decomposition:

$$\left(\mathcal{F}U_1U_2^{-1}\mathcal{F}^{-1}f\right)(k) = V(k)f(k).$$

Then we can use those two unitaries to repeat the process described before to obtain $H_{1,2} = U_{1,2}HU_{1,2}^{-1}$ and $\hat{H}_{1,2} = \mathcal{F}H_{1,2}\mathcal{F}^{-1}$ with fibers $H_{1,2}(k)$. However, it is clear that the relation between the two is:

$$\hat{H}_1 = U_1U_2^{-1}H_2U_2U_1^{-1} \Rightarrow H_1(k) = V(k)H_2(k)V(k)^{-1}.$$

Now, if a symmetry A is present, there is a reasonable constraint that we need to place. In fact we will have two ways to dimerize the symmetry: $A_{1,2} = U_{1,2}AU_{1,2}^{-1}$, meaning that the two Hamiltonians $H_{1,2}$ will act on the same space but will respect two different symmetries, and since all the topological invariants make large use of the symmetries, we need to find a common ground between them. In fact, it is very productive to study the couples of choices that lead to the same symmetry, meaning that for every possible symmetry we will ask that:

$$A_1 = A_2 \Leftrightarrow U_1AU_1^{-1} = U_2AU_2^{-1} \Leftrightarrow \left[U_2^{-1}U_1, A\right] = 0 \tag{7.1}$$

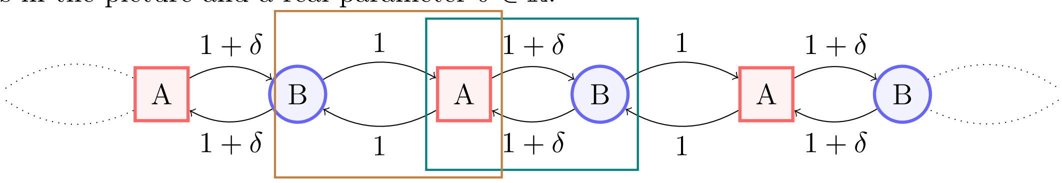
7.1 SSH Model

7.1.1 Original model

This model was originally proposed in [56] to describe the motion of a particle inside a chain of polyacetylene. In this model the particle can occupy a discrete set of positions, the sites labeled A and B; for simplicity we assume that the lattice of the sites A coincides with $\mathbb{Z}$ while the lattice of the sites B is its translate $d + \mathbb{Z}$. Then the Hamiltonian acting on $l^2(X) = l^2(\mathbb{Z} \cup (\mathbb{Z} + d))$ is described using the functions

$$\delta_x(x') = \begin{cases} 0 & \text{if } x \neq x' \\ 1 & \text{if } x = x' \end{cases}$$

In particular the term $H_{x,x'} = \langle \delta_x, H\delta_{x'} \rangle$ is called hopping term and is depicted in the following picture as an arrow going from site x to site x' with value $H_{x,x'}$. We will indeed represent discrete models by oriented graphs with weights. In this case, the SSH model is defined using the hopping terms in the picture and a real parameter $\delta \in \mathbb{R}$:



In order to obtain an Hamiltonian acting on $l^2(\mathbb{Z}) \times \mathbb{C}^n$, we need to replicate the procedure described before and choose a dimerization. This can be done in multiple ways, but for simplicity, in this section we just study the two commonly adopted dimerizations. Using the teal dimerization, as represented in the above figure, means using the unitary operator U_1 such that:

$$\begin{array}{rccc} U_1: & l_2(\mathbb{Z}\cup\mathbb{Z}+d) & \to & l^2(\mathbb{Z})\otimes\mathbb{C}^2 \\ & \delta_n & \mapsto & \delta_n\otimes e_1 \\ & \delta_{n+d} & \mapsto & \delta_n\otimes e_2 \end{array}$$

Instead, using the brown dimerization means using the unitary operator U_2 such that:

$$\begin{array}{rccc} U_2: & l_2(\mathbb{Z}\cup\mathbb{Z}+d) & \to & l^2(\mathbb{Z})\otimes\mathbb{C}^2 \\ & \delta_n & \mapsto & \delta_n\otimes e_1 \\ & \delta_{n-1+d} & \mapsto & \delta_n\otimes e_2 \end{array}$$

The two different Hamiltonians acting over the basis $\delta_n\otimes e_j$ of $l^2(\mathbb{Z})\otimes\mathbb{C}^2$ will be:

$$H_1 = U_1HU_1^{-1}(\delta_n\otimes e_j) = \delta_{n-1}\otimes\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}e_j + \delta_n\otimes\begin{pmatrix}0 & 1+\delta\\ 1+\delta & 0\end{pmatrix}e_j + \delta_{n+1}\otimes\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}e_j$$

$$H_2 = U_2HU_2^{-1}(\delta_n\otimes e_j) = \delta_{n-1}\otimes\begin{pmatrix}0 & 1+\delta\\ 0 & 0\end{pmatrix}e_j + \delta_n\otimes\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}e_j + \delta_{n+1}\otimes\begin{pmatrix}0 & 0\\ 1+\delta & 0\end{pmatrix}e_j$$

Moreover, it is true that:

$$U_1U_2^{-1} = V \text{ acts as } \begin{array}{ccc} \delta_n\otimes e_1 & \mapsto & \delta_n\otimes e_1 \\ \delta_n\otimes e_2 & \mapsto & \delta_{n-1}\otimes e_2 \end{array}$$

and by construction $H_1 = VH_2V^{-1}$.

The fundamental properties of these models are:

- *Near hopping terms*: The particle can jump only between neighborhood sites.
- *Chiral symmetry*: The system enjoys the chiral symmetry $S = \mathbb{1}\otimes\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ with $SH = -HS$. Moreover the representation of S is independent of the chosen dimerization has required in Equation 7.1.
- *Periodicity*: The system is translation invariant, meaning that the Hamiltonian commutes with the integer translations $T_m(\delta_n\otimes v) = \delta_{n+m}\otimes v$.

If we replicate the procedure illustrated in the introduction, we obtain the two fiber Hamiltonians:

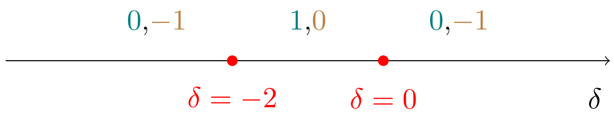
$$H_1(k) = \begin{pmatrix}0 & e^{ik}+1+\delta\\ e^{-ik}+1+\delta & 0\end{pmatrix}, \quad H_2(k) = \begin{pmatrix}0 & 1+(1+\delta)e^{-ik}\\ 1+(1+\delta)e^{ik} & 0\end{pmatrix}.$$

Also $\mathcal{F}V\mathcal{F}^{-1}$ is a fiber operator with

$$(\mathcal{F}V\mathcal{F}^{-1}f)(k) = \begin{pmatrix}1 & 0\\ 0 & e^{-ik}\end{pmatrix}\cdot f(k) = V(k)f(k)$$

This means that $H_1(k)$ and $H_2(k)$ are unitarily equivalent because $H_1(k) = V(k)H_2(k)V(k)^{-1}$.

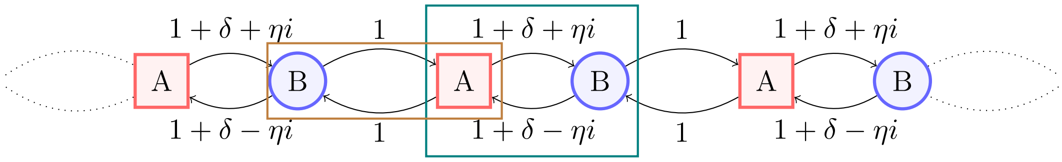
Using Theorem I.5 we know that the spectrum of H is the union of the spectra of $H(k)$ for $k \in \mathbb{T}^1$. In particular, if $0 \in \sigma(H)$, we will obtain a chiral-invariant projection-valued map by considering the spectral projection around zero. However, Proposition 6.4 tells us that this case is topologically trivial, so, to obtain something nontrivial, we need to impose $0 \notin \sigma(H)$ in order to have a spectral gap. This means that we need to impose $\det(H_1(k)) \neq 0 \neq \det(H_2(k))$ for all $k \in \mathbb{T}^1$. Since the matrices are always off-diagonal, this happens if and only if $1+\delta+e^{ik} \neq 0 \neq 1+(1+\delta)e^{ik}$ for all $k \in \mathbb{T}^1$. The first locus $1+\delta+e^{ik}$, for ranging $k \in \mathbb{T}^1$, describes a circumference in $\mathbb{C}$ with center in $1+\delta$ and radius 1 that crosses zero when $\delta = 0, -2$. The second locus $1+(1+\delta)e^{ik}$ is a circumference in $\mathbb{C}$ with radius $|1+\delta|$ and center in 1 that crosses zero when $\delta = 0, -2$. This means that we have a spectral gap when $\delta \neq 0, -2$. Proposition 6.9 tells us that the topological information of this class is encoded in $P^+(k) - P^-(k)$, which is exactly equal to $\operatorname{sgn}(H(k)) = H(k)/|H(k)|$. In particular, the proposition states that the topological phases are classified using $\mathcal{Z}$, which is the winding number around zero of the determinant of the off-diagonal component of $\operatorname{sgn}(H(k))$. In this case, the absolute value does not contribute to the winding number, so to compute $\mathcal{Z}$ it is sufficient to check the winding number of the off-diagonal component of $H(k)$ directly. For $H_1(k)$, we consider the circumference $1+\delta+e^{ik}$: since the center is the point $1+\delta$, the radius is 1 and the parameter moves through the circumference anticlockwise, the winding number is clearly 1 if $\delta \in (-2, 0)$ and zero otherwise. Instead, for $H_2(k)$, we consider the circumference $1+(1+\delta)e^{-ik}$. Since the center is the point 1, the radius is $|1+\delta|$ and the parameter moves through the circumference clockwise, the winding number is zero if $\delta \in (0, 2)$ and -1 otherwise. This is consistent with the statement of Proposition 6.8: in fact, after a different dimerization, we must have that $\mathcal{Z}_1 = \mathcal{Z}_2 + [\det(V(k)^{\uparrow})] - [\det(V(k)^{\downarrow})]$. In this case $V(k)^{\uparrow} \equiv 1, V(k)^{\downarrow} = e^{-ik}$, so $[\det(V(k)^{\uparrow})] = 0, [\det(V(k)^{\downarrow})] = -1$. In the end, we can summarize the topological information in the following picture, where the red part denotes the points in which the spectrum is not gapped and the colored numbers denote the values of $\mathcal{Z}$ depending on the two different dimerizations (teal and brown).



From a physical point of view, the system has an accidental time reversal symmetry: in fact, the Hamiltonian and the chiral symmetry commute with the standard complex conjugation, which acts as an even time reversal symmetry. Considering also the even particle-hole symmetry given by the composition of chiral and time-reversal symmetry, we realize that the model is actually an element in the BDI class.

7.1.2 Complexified model

In this part we want to propose a refined SSH-like model for the AIII class that does not have additional symmetries. To do so, we need to break the time reversal symmetry: our first attempt is to change the parameter δ with a complex one $\delta + \eta i$ such that the model is no longer invariant under complex conjugation. Therefore, the model becomes as follows:



So, the Hamiltonian after dimerization in teal is:

$$H_1(\delta_n \otimes e_j) = \delta_{n-1} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} e_j + \delta_n \otimes \begin{pmatrix} 0 & 1+\delta+\eta i \\ 1+\delta-\eta i & 0 \end{pmatrix} e_j + \delta_{n+1} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} e_j.$$

While, after the dimerization in brown, the Hamiltonian is:

$$H_2(\delta_n \otimes e_j) = \delta_{n-1} \otimes \begin{pmatrix} 0 & 1+\delta-\eta i \\ 0 & 0 \end{pmatrix} e_j + \delta_n \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e_j + \delta_{n+1} \otimes \begin{pmatrix} 0 & 0 \\ 1+\delta+\eta i & 0 \end{pmatrix} e_j.$$

Using the standard Fourier theory we can obtain the fiber Hamiltonians:

$$H_1(k) = \begin{pmatrix} 0 & 1+\delta+\eta i+e^{ik} \\ 1+\delta-\eta i+e^{-ik} & 0 \end{pmatrix}$$

$$H_2(k) = \begin{pmatrix} 0 & (1+\delta-\eta i)e^{-ik}+1 \\ (1+\delta+\eta i)e^{ik}+1 & 0 \end{pmatrix}$$

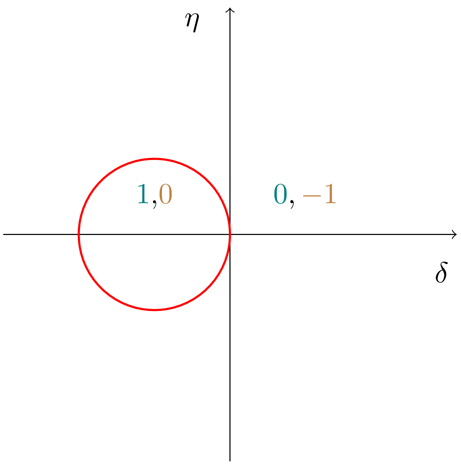
This model is not time-reversal symmetric for $\eta \neq 0$ also because the spectrum of the fibers is $\lambda_\pm(k) = \pm\left|1+\delta+\eta i+e^{ik}\right|$ and it's clear that $\eta \neq 0 \Rightarrow \lambda_+(k) \neq \lambda_-(k)$ and this conflicts with the fact that when a time reversal symmetry is present the condition on the spectrum is $\lambda(k) = \lambda(-k)$. In fact, every reasonable time reversal symmetry should commute with translation operators, which leads to the fact that $H(k)$ is antiunitary equivalent to $H(-k)$, which is not the case since their eigenvalues are different.

Before we move on, we need to check the values of the parameters that do not produce a spectral gap. The admissible values need to satisfy:

$$\lambda(k) \neq 0 \ \forall k \in S^1 \Leftrightarrow \begin{cases} \eta \neq \sin(k) \\ 1+\delta \neq -\cos(k) \end{cases} \Leftrightarrow \begin{cases} \eta \neq \sin(k) \\ \delta \neq 1-\cos(k) \end{cases}$$

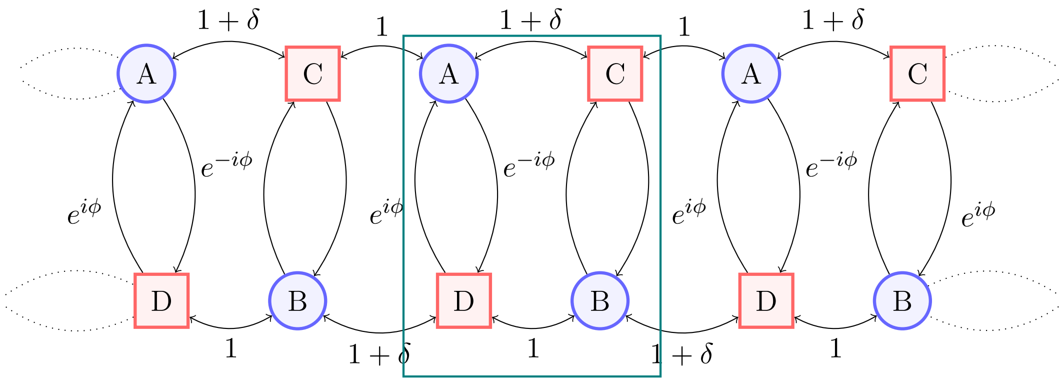
Therefore we need to exclude the points in the plane $(\delta, \eta) \in \mathbb{R}^2$ lying in the circumference with center $(-1, 0)$ and radius 1.

Finally, we can study the topological invariants of the model. As before, we need to compute the winding number around zero of the determinant of the off-diagonal term. In this case, it is simply the winding number around zero of the function $z_1(k) = 1+\delta+\eta i+e^{ik}$ or $z_2(k) = (1+\delta-\eta i)e^{-ik}+1$. The first is clearly 1 if the parameters (δ, η) are within the circle with center $(-1, 0)$ and radius 1 and zero otherwise. The second is 0 within the circle and -1 otherwise. If we depict the points that close the spectral gap in red and the values of the invariant in teal and brown, we obtain the picture:



7.1.3 Double model

In order to give physical meaning to the proposed complex hoppings, one could interpret them as the effect of an external magnetic field. However, in a pure one-dimensional model every magnetic field can be gauged away and reduced to a setting which is again time-reversal invariant. So in order to definitely break the time reversal symmetry we propose a different model. Since there are no magnetic fields in dimension one, we need to move in dimension two. In fact if we join two SSH models and allow complex hoppings between the two, the model can still be treated as a near-hopping 1-d model, but there will be a closed loop with non-zero magnetic flux so the time-reversal symmetry is broken beyond any further objections. We can describe this model using nodes and hopping terms in the following way:



Here δ is a real parameter representing the difference of potential between two consecutive carbon atoms and ϕ another real parameter representing the strength of the magnetic field.

Notice that the loop inside the teal box has non zero magnetic flux inside of it, but that the slightly larger cycle $A \to C \to A \to D \to B \to D \to A$ has zero magnetic flux, so the average magnetic flux of the system is zero.

If we choose the teal box as a dimerization (to be concise we will only study a single dimerization) and enumerate the sites in the box using the alphabetical order we have the Hamiltonian $H : l^2(\mathbb{Z}) \otimes \mathbb{C}^4 \to l^2(\mathbb{Z}) \otimes \mathbb{C}^4$ and the chiral symmetry S such that:

$$H(\delta_n \otimes v) = \delta_{n-1} \otimes A_1^* v + \delta_n \otimes A_0 v + \delta_{n+1} \otimes A_1 v \quad S(\delta_n \otimes v) = Mv$$

with:

$$A_0 = \begin{pmatrix} 0 & 0 & 1+\delta & e^{i\phi} \\ 0 & 0 & e^{i\phi} & 1 \\ 1+\delta & e^{-i\phi} & 0 & 0 \\ e^{-i\phi} & 1 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1+\delta & 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Using the canonical techniques we can write the fiber Hamiltonian:

$$H(k) = \begin{pmatrix} 0 & 0 & 1+\delta+e^{ik} & e^{i\phi} \\ 0 & 0 & e^{i\phi} & 1+(1+\delta)e^{-ik} \\ 1+\delta+e^{-ik} & e^{-i\phi} & 0 & 0 \\ e^{-i\phi} & 1+(1+\delta)e^{ik} & 0 & 0 \end{pmatrix}$$

Now the topological study can be done by looking only at the off diagonal block

$$z(k) = \begin{pmatrix} 1+\delta+e^{ik} & e^{i\phi} \\ e^{i\phi} & 1+(1+\delta)e^{-ik} \end{pmatrix}.$$

First of all, the values of the parameters that do not close the spectral gap are those for which $\det(z(k)) \neq 0 \ \forall k \in S^1$. So we need to exclude the points (ϕ, δ) such that exists $k \in S^1$ with:

$$\det(z(k)) = 2 + 2\delta - \cos(2\phi) + \cos(k)(2 + 2\delta + \delta^2) - i\left[\sin(k)(2\delta + \delta^2) + \sin(2\phi)\right] = 0$$

In particular at $\delta = 0$ the equation reduces to the system:

$$\begin{cases} 2 - \cos(2\phi) = -2\cos(k) \\ \sin(2\phi) = 0 \end{cases} \Rightarrow \begin{cases} 1 - \frac{\cos(2\phi)}{2} = -\cos(k) \\ \phi \in \frac{\pi}{2}\mathbb{Z} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \cos(k) = -1 + \frac{(-1)^{2\phi/\pi}}{2} \\ \phi \in \frac{\pi}{2}\mathbb{Z} \end{cases} \Rightarrow \begin{cases} \cos(k) = -1/2 \\ \phi \in \pi\mathbb{Z} \end{cases}$$

meaning that the points $(\pi\mathbb{Z}, 0)$ must be excluded. While at $\delta = -2$ the equation becomes:

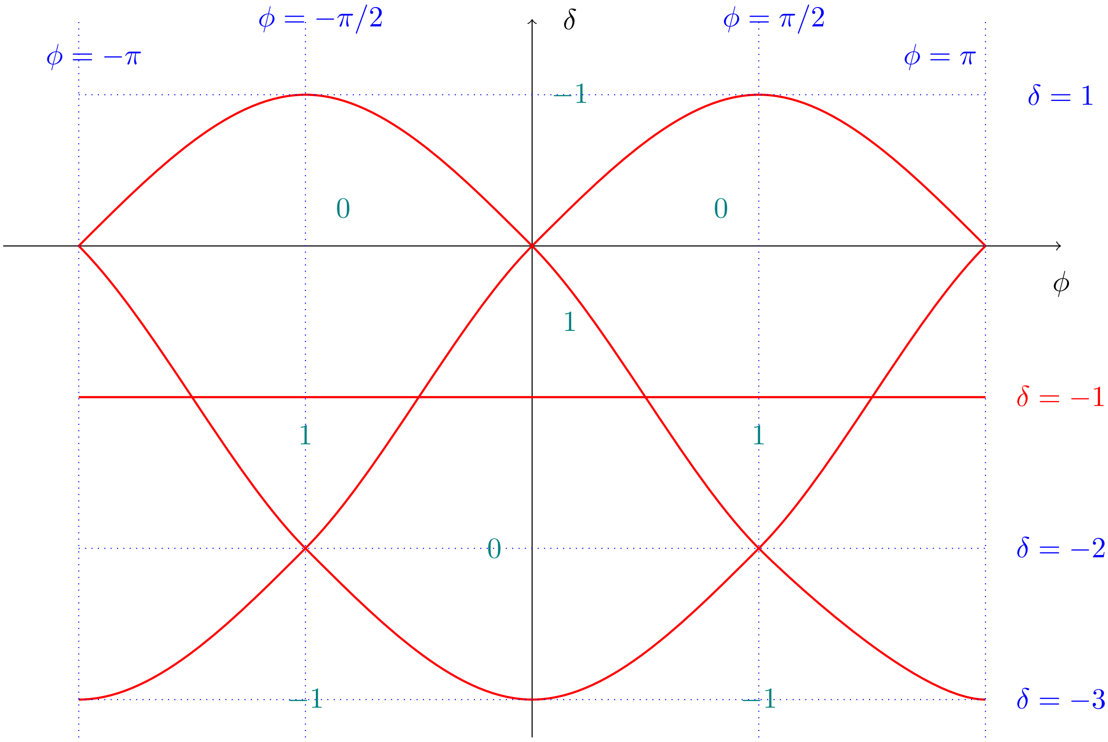
$$\begin{cases} -2 - \cos(2\phi) = -2\cos(k) \\ \sin(2\phi) = 0 \end{cases} \Rightarrow \begin{cases} 1 + \frac{\cos(2\phi)}{2} = \cos(k) \\ \phi \in \frac{\pi}{2}\mathbb{Z} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \cos(k) = 1 + \frac{(-1)^{2\phi/\pi}}{2} \\ \phi \in \frac{\pi}{2}\mathbb{Z} \end{cases} \Rightarrow \begin{cases} \cos(k) = 1/2 \\ \phi \in \frac{\pi}{2} + \pi\mathbb{Z} \end{cases}$$

So also the points $(\frac{\pi}{2} + \pi\mathbb{Z}, -2)$ must be excluded. Instead, for δ different from those two values we notice that a couple of parameters (ϕ, δ) must be excluded if and only if:

$$\left[\frac{2 + 2\delta - \cos(2\phi)}{2 + 2\delta + \delta^2}\right]^2 + \left[\frac{\sin(2\phi)}{2\delta + \delta^2}\right]^2 = 1$$

because if this equation is true, then we can rewrite the first term as $\cos(k)$ and the second term as $\sin(k)$ for some $k \in \mathbb{T}^1$. By plotting this equation, we get the curve of forbidden points (in red) which is clearly π-periodical in ϕ:



In the picture the teal numbers are the values of the topological invariant $\mathcal{Z}$ equal to the winding number of the determinant of $z(k)$ in the connected components of the parameters space. In fact the invariant reduces to the following winding numbers denoted $[f(k)]$ for the following values of the parameters:

$$(\phi, \delta) = (\pi/2, 3) \Rightarrow \mathcal{Z} = [9 + 17\cos(k) + -15i\sin(k)] = -1$$

$$(\phi, \delta) = (\pi/2, 0) \Rightarrow \mathcal{Z} = [3 + 2\cos(k)] = 0$$

$$(\phi, \delta) = (\pi/2, -3/2) \Rightarrow \mathcal{Z} = \left[\frac{5}{4}\cos(k) + \frac{3i}{4}\sin(k)\right] = 1$$

$$(\phi, \delta) = (\pi/2, -3) \Rightarrow \mathcal{Z} = [-3 + 5\cos(k) - 3i\sin(k)] = -1$$

$$(\phi, \delta) = (0, -1/2) \Rightarrow \mathcal{Z} = \left[\frac{5}{4}\cos(k) + \frac{3i}{4}\sin(k)\right] = 1$$

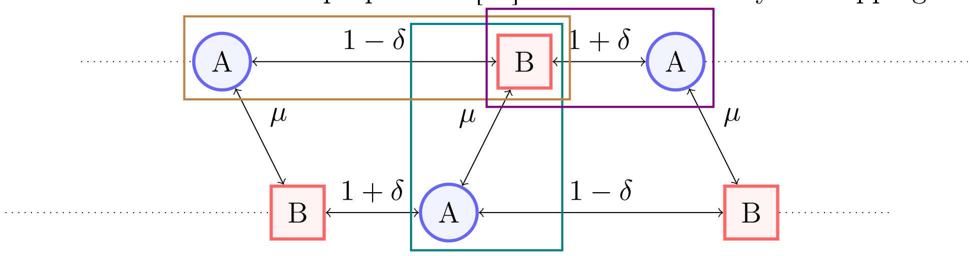
$$(\phi, \delta) = (\pi/2, -2) \Rightarrow \mathcal{Z} = [-3 + 2\cos(k)] = 0$$

Since $\mathcal{Z}$ is invariant under continuous changes of the parameters (ϕ, δ) which do not cross the curves describing spectrally gapless Hamiltonians, the values computed above are sufficient to label all connected components of the parameter space. In particular, we find several regions showing non-trivial topological phases.

7.2 Kitaev Chain

7.2.1 Original Model

The Hamiltonian of the model proposed in [38] can be described by the hopping terms in the picture:



We can use three different dimerizations: teal, brown, and violet. They lead to three Hamiltonians H_1, H_2, H_3 acting on the basis $\delta_n \otimes e_j$ of $l^2(\mathbb{Z}) \otimes \mathbb{C}^2$ after we identify the sites A with e_1 and the sites B with e_2:

$$H_1(\delta_n \otimes e_j) = \delta_{n-1} \otimes \begin{pmatrix} 0 & 1-\delta \\ 1+\delta & 0 \end{pmatrix} e_j + \delta_n \otimes \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix} e_j + \delta_{n+1} \otimes \begin{pmatrix} 0 & 1+\delta \\ 1-\delta & 0 \end{pmatrix} e_j$$

$$H_2(\delta_n \otimes e_j) = \delta_{n-2} \otimes \begin{pmatrix} 0 & 0 \\ 1+\delta & 0 \end{pmatrix} e_j + \delta_{n-1} \otimes \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix} e_j + \delta_n \otimes \begin{pmatrix} 0 & 1-\delta \\ 1-\delta & 0 \end{pmatrix} e_j +$$
$$+ \delta_{n+1} \otimes \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix} e_j + \delta_{n+2} \otimes \begin{pmatrix} 0 & 1+\delta \\ 0 & 0 \end{pmatrix}$$

$$H_3(\delta_n \otimes e_j) = \delta_{n-2} \otimes \begin{pmatrix} 0 & 1-\delta \\ 0 & 0 \end{pmatrix} e_j + \delta_{n-1} \otimes \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix} e_j + \delta_n \otimes \begin{pmatrix} 0 & 1+\delta \\ 1+\delta & 0 \end{pmatrix} e_j +$$
$$+ \delta_{n+1} \otimes \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix} e_j + \delta_{n+2} \otimes \begin{pmatrix} 0 & 0 \\ 1-\delta & 0 \end{pmatrix}$$

where δ, μ are real parameters. The fundamental properties of this model are:

- *Near hopping terms*: The particle can jump only between neighborhood sites.
- *Particle-hole symmetry*: The model enjoys the particle-hole symmetry $C = \mathbb{1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathcal{K}$, where $\mathcal{K}$ is the standard complex conjugation on $\mathbb{C}^2$: $CH = -HC$.
- *Periodicity*: The system is translation invariant, meaning that the Hamiltonian commutes with the integer translations $T_m(\delta_n \otimes v) = \delta_{n+m} \otimes v$.

After the Fourier transform, we have the three fiber Hamiltonians:

$$H_1(k) = \begin{pmatrix} 0 & \mu + 2\cos(k) + 2i\delta\sin(k) \\ \mu + 2\cos(k) - 2i\delta\sin(k) & 0 \end{pmatrix}$$
$$H_2(k) = \begin{pmatrix} 0 & (1-\delta) + \mu e^{ik} + (1+\delta)e^{2ik} \\ (1-\delta) + \mu e^{-ik} + (1+\delta)e^{-2ik} & 0 \end{pmatrix}$$
$$H_3(k) = \begin{pmatrix} 0 & (1-\delta)e^{-2ik} + \mu e^{-ik} + (1+\delta) \\ (1-\delta)e^{2ik} + \mu e^{ik} + (1+\delta) & 0 \end{pmatrix}$$

As we said in the beginning of the chapter about Class D, if $0 \in \sigma(H)$ the spectral projection around it defines an element in class AI. However, this class is topologically trivial owning to Proposition 2.6. In order to study something topologically interesting, we need to impose that $0 \notin \sigma(H)$, which leads to the condition $\det(H_1(k)) \neq 0 \ \forall k \in \mathbb{T}^1$. So we need to impose $(\mu + 2\cos(k))^2 + (2\delta \sin(k))^2 \neq 0 \ \forall k \in \mathbb{T}^1$. This curve is an ellipse in $\mathbb{C}$ with center on μ, real axis with length 4 and imaginary axis with length 4δ. With some calculations, we have that the solutions of the previous inequality require $\mu \neq \pm 2$ and if $\delta = 0$ we need to exclude $\mu \notin (-2, 2)$. Proposition 3.8 tells us that the topological classification depends on

$$\mathrm{Pf}(i\,\mathrm{sgn}(H(0))), \quad \mathrm{Pf}(i\,\mathrm{sgn}(H(\pi))),$$

provided that everything is written on a basis on which C acts as $\mathcal{K}$. We can overcome this problem by observing that $Ce_1 = e_1$ and $C(ie_2) = ie_2$, so we can change the basis of $\mathbb{C}^2$ using $U = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ and study:

$$\begin{aligned}
UH_1(k)U^* &= \begin{pmatrix} 0 & -i(\mu + 2\cos(k) + 2i\delta \sin(k)) \\ i(\mu + 2\cos(k) - 2i\delta \sin(k)) & 0 \end{pmatrix} \\
UH_2(k)U^* &= \begin{pmatrix} 0 & -i((1-\delta) + \mu e^{ik} + (1+\delta)e^{2ik}) \\ i((1-\delta) + \mu e^{-ik} + (1+\delta)e^{-2ik}) & 0 \end{pmatrix} \\
UH_3(k)U^* &= \begin{pmatrix} 0 & -i((1-\delta)e^{-2ik} + \mu e^{-ik} + (1+\delta)) \\ i(1-\delta)e^{2ik} + \mu e^{ik} + (1+\delta)) & 0 \end{pmatrix} \\
UCU^* &= U \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathcal{K}U^* = \mathcal{K}
\end{aligned}$$

This means that the topological indexes we are looking for are:

$$\mathcal{P}_1 = \left(\mathrm{Pf}\left(i\,\mathrm{sgn}\begin{pmatrix} 0 & -i(\mu+2) \\ i(\mu+2) & 0 \end{pmatrix}\right), \mathrm{Pf}\left(i\,\mathrm{sgn}\begin{pmatrix} 0 & -i(\mu-2) \\ i(\mu-2) & 0 \end{pmatrix}\right)\right)$$

$$\mathcal{P}_2 = \left(\mathrm{Pf}\left(i\,\mathrm{sgn}\begin{pmatrix} 0 & -i(\mu+2) \\ i(\mu+2) & 0 \end{pmatrix}\right), \mathrm{Pf}\left(i\,\mathrm{sgn}\begin{pmatrix} 0 & -i(2-\mu) \\ i(2-\mu) & 0 \end{pmatrix}\right)\right)$$

$$\mathcal{P}_3 = \left(\mathrm{Pf}\left(i\,\mathrm{sgn}\begin{pmatrix} 0 & -i(\mu+2) \\ i(\mu+2) & 0 \end{pmatrix}\right), \mathrm{Pf}\left(i\,\mathrm{sgn}\begin{pmatrix} 0 & -i(2-\mu) \\ i(2-\mu) & 0 \end{pmatrix}\right)\right)$$

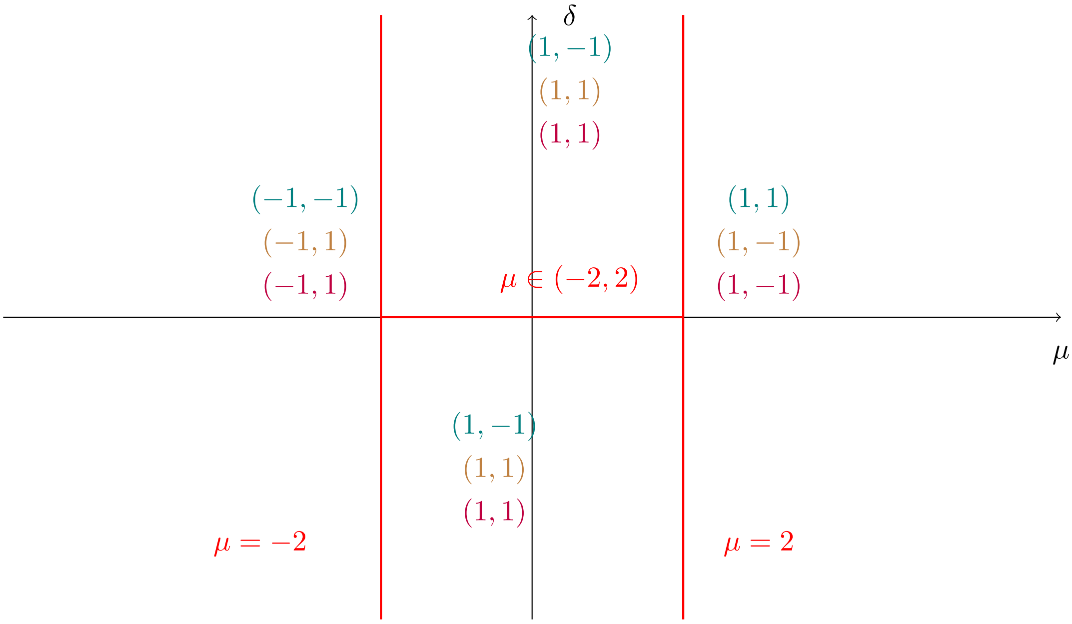
In every case, if $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is the second Pauli matrix, we have that $\mathrm{sgn}(\sigma_2) = \sigma_2$, so it holds that

$$\mathrm{Pf}(i\,\mathrm{sgn}(a\sigma_2)) = \mathrm{Pf}(i\,\mathrm{sgn}(a)\,\mathrm{sgn}(\sigma_2)) = \mathrm{Pf}(i\,\mathrm{sgn}(a)\sigma_2) = \mathrm{Pf}\begin{pmatrix} 0 & \mathrm{sgn}(a) \\ -\mathrm{sgn}(a) & 0 \end{pmatrix} = \mathrm{sgn}(a)$$

Therefore we obtain:

$$\mathcal{P}_1 = (\mathrm{sgn}(\mu+2), \mathrm{sgn}(\mu-2)) \quad \mathcal{P}_2 = \mathcal{P}_3 = (\mathrm{sgn}(\mu+2), \mathrm{sgn}(2-\mu))$$

So we can depict the topological phases and their topological indexes in the following picture. We will denote the points that make the system gapless in red and the values of the topological phases using the color of the dimerization they refer to.



This is perfectly compatible with the statement of Proposition 3.6: in fact one can easily verify that the fiber Hamiltonians $UH_j(k)U^*$ are unitarily equivalent using the unitary-valued maps

$$V_{2,1}(k) = \begin{pmatrix} 1 & 0 \\ 0 & e^{ik} \end{pmatrix}, \quad V_{3,1}(k) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-ik} \end{pmatrix}$$

in fact

$$UH_1(k)U^* = V_{2,1}(k)UH_2(k)U^*V_{2,1}(k)^* \quad UH_1(k)U^* = V_{3,1}(k)UH_3(k)U^*V_{3,1}(k)^*$$

Since $\det(V_{2,1}(0)) = \det(V_{3,1}(0)) = 1$ and $\det(V_{2,1}(\pi)) = \det(V_{3,1}(\pi)) = -1$ we explicitly recover the changes of the topological phases present in the picture.

The counterintuitive features of this model are two hidden symmetries: a time reversal symmetry given by the standard complex conjugation (on the original basis), and a chiral symmetry $S = \mathbb{1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with $SH = -HS$. So we have an element in the BDI class once again and we would love to modify the Kitaev chain in order to have a model which is strictly in Class D. To do so, we need to investigate some crucial properties of the model.

First of all, the chiral symmetry is present independently of the values of the hopping terms. In fact if every closed hopping loop in the model has an even number of jumps, then we can paint the nodes in the model in a chessboard style and we can create a chiral symmetry as the operator that changes the signs of the functions valued at the blue nodes while maintaining the signs of the functions valued at the red nodes. So, chiral symmetry is a property that depends on which jumps are present and which are not, regardless of the hopping amplitude.

Moreover, a system can be particle-hole symmetric and chiral symmetric without being time-reversal symmetric. This is due to the fact then in Class BDI one needs to assume specific (anti)commutation relation among two of the symmetries to obtain the third as their composition; in fact, if they do not commute or anti-commute their composition does not square to plus or minus one, so it does not qualify as a symmetry according to our definition.

Now, our goal is to propose a Kitaev-like model for the D class that does not have additional symmetries. In order to achieve that, we can try to break the time-reversal symmetry. To do so we can proceed in three different ways: We can try to break directly the time-reversal symmetry, we can try to break the chiral symmetry, or we can try to break the commutation rule between the chiral symmetry and the particle-hole symmetry.

The last, however, is not so useful since the chiral symmetry gives birth to the richest topological structure, and there are no gains in looking at a chiral model just to study the topology given by a particle-hole symmetry.

7.2.2 Failure of the complexified model

It is reasonable to replicate the process done for the SSH model and try to replace the parameters with complex parameters; however, this approach will inevitably fail. Whatever parameters we may place in the Kitaev chain, it will always have a time reversal symmetry.

Let us see what happens with this general Hamiltonian acting over $l^2(\mathbb{Z}) \otimes \mathbb{C}^2$ of the form:

$$H(\delta_n \otimes e_j) = \delta_{n-1} \otimes A_1^* e_j + \delta_n \otimes A_0 e_j + \delta_{n+1} \otimes A_1 e_j$$

with $A_1, A_0 = A_0^* \in M_{2,2}(\mathbb{C})$. If the system also has a particle-hole symmetry written in the form $C = Id \otimes M$ with $M : \mathbb{C}^2 \to \mathbb{C}^2$ an anti-unitary operator such that $HC = -CH$, then there is always a basis of $\mathbb{C}^2$ such that M acts on this basis as the standard complex conjugation $\mathcal{K}$, so without losing generality we just assume that this basis is already the standard basis e_1, e_2. Then, if we compute the anti-commutation relation over $\delta_n \otimes e_j$, we get:

$$\delta_{n-1} \otimes \mathcal{K} A_1^* e_j + \delta_n \otimes \mathcal{K} A_0 e_j + \delta_{n+1} \otimes \mathcal{K} A_1 e_j = -\delta_{n-1} \otimes A_1^* \mathcal{K} e_j - \delta_n \otimes A_0 \mathcal{K} e_j - \delta_{n+1} \otimes A_1 \mathcal{K} e_j$$

so that we end up with the three relations:

$$\overline{A_1^*} = -A_1^*, \quad \overline{A_0} = -A_0, \quad \overline{A_1} = -A_1.$$

This means that A_0, A_1 are purely imaginary matrices and since A_0 is self-adjoint it must be in the form $A_0 = \mu \cdot \sigma_2$ with $\mu \in \mathbb{R}$. Instead, the real vector space of purely imaginary matrices is generated by the set $\{i\mathbb{I}, i\sigma_1, \sigma_2, i\sigma_3\}$, meaning that $A_1 = a \cdot i\mathbb{I} + b \cdot i\sigma_1 + c \cdot \sigma_2 + d \cdot i\sigma_3$ with $a, b, c, d \in \mathbb{R}$. Now to create a proper chiral symmetry S we need a unitary matrix squaring to the identity that anti-commutes with both A_0 and A_1 while commuting with the complex conjugation. Since $\{\sigma_j, \sigma_k\} = \mathbb{I}\delta_{j,k}$, the set of matrices that anti-commute with A_0 is a complex vector space generated by σ_1 and σ_3, but we need to take only the real linear combinations to ensure commutation with the complex conjugation. Finally, for the chiral symmetry to square to the identity we need to have:

$$(x\sigma_1 + y\sigma_3)^2 = x^2\mathbb{I} + xy\sigma_1\sigma_3 + yx\sigma_3\sigma_1 + y^2\mathbb{I} = (x^2 + y^2)\mathbb{I}$$

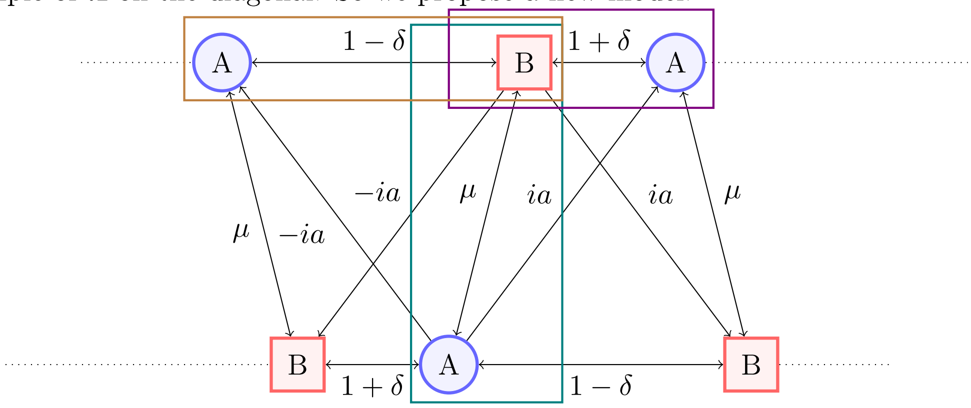
meaning that we need to take only linear combinations with $S = \cos(t)\sigma_1 + \sin(t)\sigma_2$. Finally we need to check when this anti-commutes with A_1:

$$\{A_1, S\} = ia\cos(t)\{\mathbb{I}, \sigma_1\} + ib\cos(t)\{\sigma_1, \sigma_1\} + ai\sin(t)\{\mathbb{I}, \sigma_3\} + id\sin(t)\{\sigma_3, \sigma_3\} =$$
$$= 2ia\cos(t)\sigma_1 + 2ia\sin(t)\sigma_3 + (2ib\cos(t) + 2id\sin(t))\mathbb{I}$$

Whenever $a = 0$ the equation above can be set to zero by taking $t = \arctan(-b/d)$, meaning that we end up with a perfectly functioning chiral symmetry that commutes with the particle-hole symmetry, giving us also a time reversal symmetry. Moreover $ia = \mathrm{tr}(A_1)$, so whenever the hopping matrix is without trace we can find a time reversal symmetry and this is precisely what happens in the Kitaev chain, regardless of the possible values of the parameters.

7.2.3 Additional hopping

Now we can try to break the chiral symmetry by adding additional hopping terms to the model. As suggested by the arguments above, we will try to implement these hoppings such that A_1 has a multiple of $i\mathbb{I}$ on the diagonal. So we propose a new model:



Since showing all the possible jumps would have led to a very clumped picture, we omitted showing the reverse hoppings with conjugate amplitude. The Hamiltonians H_1, H_2, H_3 after the dimerizations teal, brown and violet (respectively) are in the form:

$$H_1(\delta_n \otimes e_j) = \delta_{n-1} \otimes \begin{pmatrix} -ia & 1-\delta \\ 1+\delta & -ia \end{pmatrix} e_j + \delta_n \otimes \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix} e_j + \delta_{n+1} \otimes \begin{pmatrix} ia & 1+\delta \\ 1-\delta & ia \end{pmatrix} e_j$$

$$H_2(\delta_n \otimes e_j) = \delta_{n-2} \otimes \begin{pmatrix} 0 & 1+\delta \\ 0 & 0 \end{pmatrix} e_j + \delta_{n-1} \otimes \begin{pmatrix} -ia & \mu \\ 0 & -ia \end{pmatrix} e_j + \delta_n \otimes \begin{pmatrix} 0 & 1-\delta \\ 1-\delta & 0 \end{pmatrix} e_j +$$
$$+ \delta_{n+1} \otimes \begin{pmatrix} ia & 0 \\ \mu & ia \end{pmatrix} e_j + \delta_{n+2} \otimes \begin{pmatrix} 0 & 0 \\ 1+\delta & 0 \end{pmatrix} e_j$$

$$H_3(\delta_n \otimes e_j) = \delta_{n-2} \otimes \begin{pmatrix} 0 & 0 \\ 1-\delta & 0 \end{pmatrix} e_j + \delta_{n-1} \otimes \begin{pmatrix} -ia & 0 \\ \mu & -ia \end{pmatrix} e_j + \delta_n \otimes \begin{pmatrix} 0 & 1+\delta \\ 1+\delta & 0 \end{pmatrix} e_j +$$
$$+ \delta_{n+1} \otimes \begin{pmatrix} ia & \mu \\ 0 & ia \end{pmatrix} e_j + \delta_{n+2} \otimes \begin{pmatrix} 0 & 1-\delta \\ 0 & 0 \end{pmatrix} e_j$$

After the Fourier transform we get the fiber Hamiltonians:

$$H_1(k) = \begin{pmatrix} iae^{ik} - iae^{-ik} & \mu + (1+\delta)e^{ik} + (1-\delta)e^{-ik} \\ \mu + (1+\delta)e^{-ik} + (1-\delta)e^{ik} & iae^{ik} - iae^{-ik} \end{pmatrix}$$
$$H_2(k) = \begin{pmatrix} iae^{ik} - iae^{-ik} & (1+\delta)e^{-2ik} + \mu e^{-ik} + (1-\delta) \\ (1-\delta) + \mu e^{ik} + (1+\delta)e^{2ik} & iae^{ik} - iae^{-ik} \end{pmatrix}$$
$$H_3(k) = \begin{pmatrix} iae^{ik} - iae^{-ik} & (1-\delta)e^{-2ik} + \mu e^{-ik} + (1+\delta) \\ (1+\delta) + \mu e^{ik} + (1-\delta)e^{2ik} & iae^{ik} - iae^{-ik} \end{pmatrix}$$

The particle-hole symmetry is once again $\mathbb{I} \otimes \sigma_3 \mathcal{K}$. Using the simple form $\lambda^2 - \mathrm{tr}(H(k))\lambda + \det(H(k))$ of the characteristic polynomial of a 2×2 matrix we get that the eigenvalues of the fiber are:

$$\lambda_\pm(k) = \frac{-4a\sin(k) \pm \sqrt{16a^2\sin(k)^2 - 4 \cdot [4a^2\sin(k)^2 + (\mu + 2\cos(k))^2 + 4\delta^2\sin(k)^2]}}{2} =$$
$$= -2a\sin(k) \pm |\mu + 2\cos(k) + 2i\delta\sin(k)|$$

Notice that as $a \neq 0$ it true that $\lambda_+(k) \neq \lambda_+(-k)$ and $\lambda_+(k) \neq \lambda_-(k)$ so the model cannot have a chiral symmetry nor a time-reversal symmetry.

In order for the model to be an insulator the eigenvalues must always be different from zero, meaning that the product of the two shall always be smaller then zero, so we study the condition:

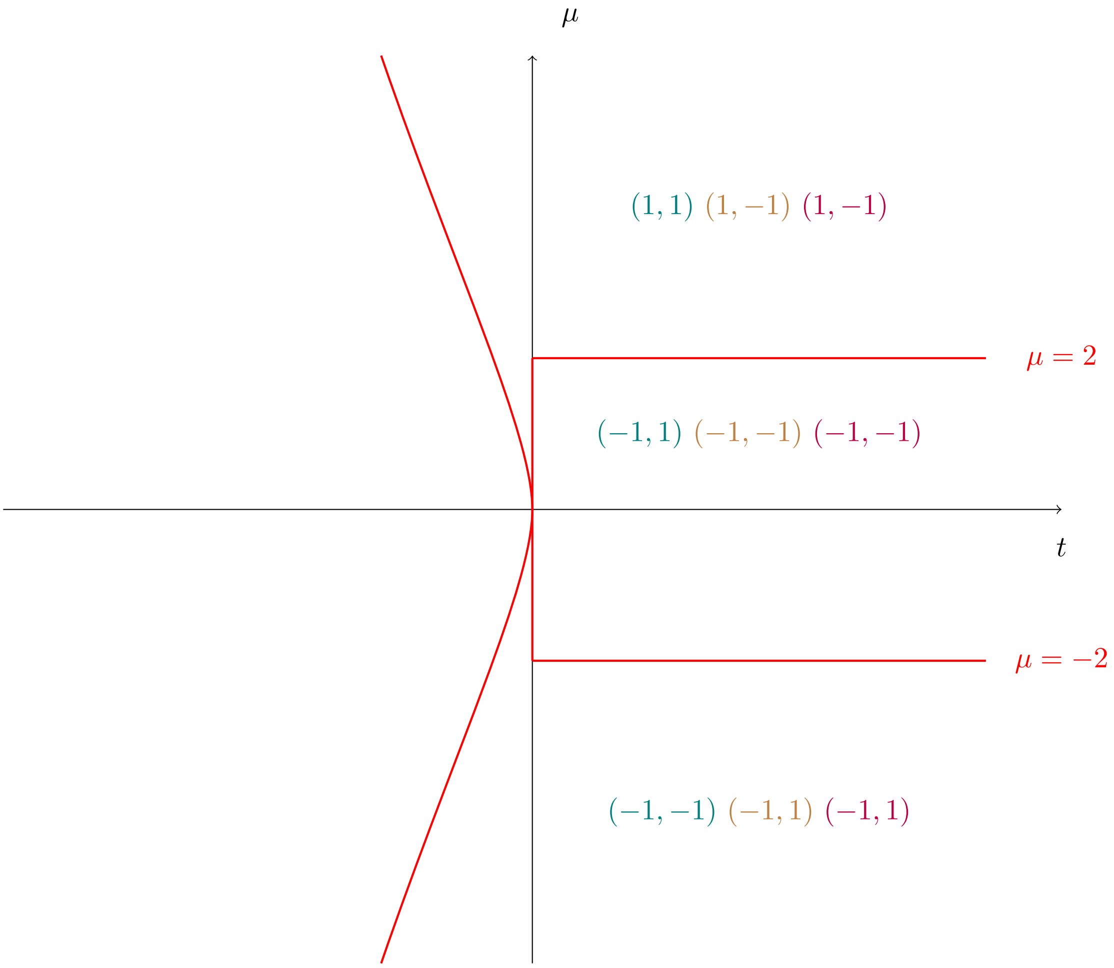
$$\lambda_+(k)\lambda_-(k) < 0 \Leftrightarrow 4a^2\sin(k)^2 - (\mu + 2\cos(k))^2 - 4\delta\sin(k)^2 < 0 \Leftrightarrow$$
$$\Leftrightarrow (\mu + 2\cos(k))^2 + 4\sin(k)^2(\delta^2 - a^2) > 0$$

Now just replace $\delta^2 - a^2 = t$ so we get that as in the normal Kitaev Chain $\mu \neq \pm 2$ otherwise for $k = n\pi$ the spectral gap closes. It is clear that for $t > 0$ and $\mu \neq \pm 2$ the inequality is always true because we are adding two positive terms, while for $t \leq 0$ we need to exclude all the μ such that $\mu = -2\cos(k) \pm 2\sin(k)\sqrt{-t}$ this means that we need to exclude the range of the function $-2\cos(k) - 2\sin(k)\sqrt{-t}$ which is $[-2\cos\arctan(\sqrt{-t})(1-t), 2\cos\arctan(\sqrt{-t})(1-t)]$.

Now we can use the change of basis used before $U = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ and study:

$$UH_1(0)U^* = \begin{pmatrix} 0 & -i(\mu+2) \\ i(\mu+2) & 0 \end{pmatrix} \quad UH_1(\pi)U^* = \begin{pmatrix} 0 & -i(\mu-2) \\ i(\mu-2) & 0 \end{pmatrix}$$
$$UH_2(0)U^* = \begin{pmatrix} 0 & -i(\mu+2) \\ i(\mu+2) & 0 \end{pmatrix} \quad UH_2(\pi)U^* = \begin{pmatrix} 0 & -i(2-\mu) \\ i(2-\mu) & 0 \end{pmatrix}$$
$$UH_3(0)U^* = \begin{pmatrix} 0 & -i(\mu+2) \\ i(\mu+2) & 0 \end{pmatrix} \quad UH_3(\pi)U^* = \begin{pmatrix} 0 & -i(2-\mu) \\ i(2-\mu) & 0 \end{pmatrix}$$

This is exactly the case studied before, so we can just conclude the study of the topological phases of this model by drawing the Pfaffians $(\mathcal{P}(0), \mathcal{P}(\pi))$ in the following picture:



μ
$(1, 1)$ $(1, -1)$ $(1, -1)$
$\mu = 2$
$(-1, 1)$ $(-1, -1)$ $(-1, -1)$
t
$\mu = -2$
$(-1, -1)$ $(-1, 1)$ $(-1, 1)$

Chapter 8

Conclusions

In this final chapter we want to summarize the results obtained in this thesis and better explain their contribution to the existing literature.

8.1 Summary

We recall that the we were interested in continuous projection-valued maps $P : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ or in pairs of continuous projection-valued maps $P^{\pm} : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ with a fixed rank n in a separable Hilbert space. Those maps may satisfy one of the symmetry conditions

$$TP(k) = P(-k)T, \quad CP^{+}(k) = P^{-}(-k)C, \quad \text{or} \quad SP^{+}(k) = P^{-}(k)S$$

for anti-unitary operators T, C or a unitary operator S with $T^2 = \pm\mathbb{1}$, $C^2 = \pm\mathbb{1}$ and $S^2 = \mathbb{1}$. The details of the symmetry conditions are given in Definitions I.14, I.15, 2.1, 3.1, 4.1, 5.1. Since in this thesis we studied models with a single symmetry, we assume that only one symmetry can be present at any given time. This means that we studied Classes A, AI, AII, AIII, D and C of Kitaev's table I.1 present in the Introduction. Our first goal was to find for every symmetric projection-valued map a continuous and symmetric frame with the best pseudoperiodic conditions we could get. When $d = 1$, we never encounter a topological obstruction in the construction of frames with full periodicity. The details of the symmetric conditions and the proofs are presented in Propositions 1.4, 2.6, 3.8, 4.9, 5.4, and 6.9. Instead, when $d = 2$, we encountered a topological obstruction in the construction of a periodic and symmetric frame. In particular, for a projection-valued map without symmetries (in Class A), all vectors can be constructed with full periodicity except one with pseudoperiodicity $v(2\pi, k_2) = e^{i\,\mathrm{Ch}(P) k_2} v(0, k_2)$ due to Theorem 1.10 ($\mathrm{Ch}(P)$ is the Chern number in Definition 1.6). This topological obstruction vanished in Class AI, where it was always possible to construct a continuous, periodic, and symmetric frame as stated in Theorem 2.10. Something interesting happens in Class AII, where we needed Theorem 4.14 to decompose $P = P^{+} \oplus P^{-}$ where the pair $P^{\pm}$ is essentially an element in Class C and $\delta(P) = e^{i\pi\,\mathrm{Ch}(P^{-})}$ is the invariant defined in Definition 4.12. Then we found a pseudoperiodic frame for P^{-} and took the symmetric counterparts of all vectors to obtain a frame for P^{+}. The union of the two frames constituted a frame of P. Since one element of the frame of P^{-} was generally pseudoperiodic, the symmetric counterpart had a conjugated pseudoperiodicity, and the frame for P had two pseudoperiodic vectors as stated in Theorem 4.16. The procedure was replicated to all classes with pairs of projection-valued maps

(AIII,C,D) as stated in Theorems 3.16, 5.5, and 6.11 .

Then we were interested in intertwining two projection-valued maps or pairs of projection-valued maps in the same class using a unitary-valued map $V(k)$ such that

$$TV(k) = V(-k)T, \quad CV(k) = V(-k)C, \quad \text{or} \quad SV(k) = V(k)S$$

depending on the symmetry present. This divides each symmetry class into subclasses of unitarily equivalent elements. Propositions 1.4, 2.6, 3.8, 4.9, 3.8, 5.4, 6.9 and Theorems 1.10, 2.10, 3.16, 4.16, 5.5, 6.11 stated that, when $d < 2$, every element is unitarily equivalent to any other element of the same symmetry class. Instead, when $d = 2$, there may be pairs of elements that are not unitarily equivalent. In particular, the subclasses of unitarily equivalent elements are in bijection with the possible values of the topological quantities present in the right column of the following table ($P^{\updownarrow}$ are defined in Theorem 6.11).

Classes	Symmetries			Topological labels of unitarily equivalent classes
AZ	T	C	S	$d = 2$
A	0	0	0	$\text{Ch}(P) \in \mathbb{Z}$
AIII	0	0	1	$(\text{Ch}(P^-), \text{Ch}(P^\uparrow), \text{Ch}(P^\downarrow)) \in \mathbb{Z}^3$
AI	1	0	0	$\{0\}$
D	0	1	0	$\text{Ch}(P^-) \in \mathbb{Z}$
AII	-1	0	0	$\delta(P) \in \mathbb{Z}_2$
C	0	-1	0	$\text{Ch}(P^-) \in \mathbb{Z}$

Finally, we were interested in connecting two projection-valued maps or pairs of projection-valued maps using an homotopy that stays within the symmetry class throughout the homotopy. In other words, we were interested in the connected components of the symmetry classes. The propositions and theorems mentioned before stated that the connected components coincide precisely with the subclasses of unitarily equivalent elements except when $\dim(\mathcal{H}) = 2n$ for Class AIII and class D. In those cases, we can express the number of connected components using the following table:

Classes	Symmetries			Connected components ($\dim(\mathcal{H}) = 2n$)		
AZ	T	C	S	$d = 0$	$d = 1$	$d = 2$
AIII	0	0	1	$\{0\}$	$\mathcal{Z}(P^\pm) \in \mathbb{Z}$	$\mathcal{Z}(P^\pm) \in \mathbb{Z}^2$
D	0	1	0	$\mathcal{P} \in \mathbb{Z}_2$	$\mathcal{P}(0), \mathcal{P}(\pi) \in \mathbb{Z}_2^2$	$\mathcal{P}(0,0), \mathcal{P}(\pi,0), \mathcal{P}(0,\pi), \text{Ch}(P^-) \in \mathbb{Z}_2^3 \times \mathbb{Z}$

The explicit definitions of $\mathcal{P}$ and $\mathcal{Z}$ can be found respectively in Definition 3.5 and 6.7.

8.2 Originality and comparison with existing literature

As stated in the beginning of this thesis, some results are not original, and we want to compare them with already existing theorems.

Before moving to a class by class discussion, we want to comment on some general aspects of the literature that are present in all classes. We discussed in the Introduction the original topological classification made using K-theory [1, 7, 37] and the physical details it overlooks. Although some issued can still be addressed using K-theory [58], or homotopy theory of states [36] (for $d \leq 3$), we preferred to build on the research line of [23] because we believe it yields to easier proofs, clearer depiction of the dymerization ambiguity, and more computable invariants than the K-theoretic counterparts [40]. However, our approach is less applicable to disordered models than the K-theoretic approach or the Fredholm index approach. The former was used in [26] to derive a topological classification for projection-valued maps using indexes of pairs of projections and then used in [8, 9] to obtain a topological classification of local disordered symmetric Hamiltonians in $d \leq 2$.

In Chapter 1, our formulation of the Chern number in Definition 1.6 may look different from existing formulas, so we want to prove that our formulation is a generalization of the usual Chern number that can be computed for continuous projection-valued maps while the other formulations usually require differentiable projection-valued maps. The easiest way to prove this is to use the parallel transport. So in the following, we will briefly recall some results present in [10].

Theorem 8.1 (Parallel transport). *If $P : \mathbb{T}^1 \to \mathrm{Proj}_n(\mathcal{H})$ is a real-analytic projection-valued map, then the following Cauchy problem has a unique unitary solution $T(t) \in \mathcal{U}(\mathcal{H})$ for all $t \in \mathbb{R}$*

$$\begin{cases} \partial_t T(t) = [\partial_t P(t) P(t)] T(t) \\ T(0) = \mathbb{1} \end{cases}$$

Moreover, this solution is as regular as P and it satisfies the property:

$$P(t) = T(t) P(0) T(t)^{-1}.$$

Corollary 8.2. *If $T(2\pi) = e^{iL}$ for a self-adjoint operator L, then it is possible to make parallel transport periodic by defining $U(t) = T(t)e^{itL/2\pi}$. This U is now periodic with the same intertwining property:*

$$P(t) = U(t) P(0) U(t)^{-1}.$$

Using those two results, we can consider a projection-valued map $P : \mathbb{T}^2 \to \mathrm{Proj}_n(\mathcal{H})$), define $U(k_2)$ for $P(0, k_2)$, and then construct a k_2−family of parallel transports

$$\begin{cases} \partial_t T_{k_2}(t) = [\partial_t P(t, k_2), P(t, k_2)] T_{k_2}(t) \\ T_{k_2}(0) = \mathbb{1} \end{cases}$$

This means that the operator $T_{k_2}(t)U(k_2)$ is continuous, periodic in k_2 and with the nice property:

$$P(t, k_2) = T_{k_2}(t) U(k_2) P(0, 0) [T_{k_2}(t) U(k_2)]^{-1}.$$

So we can compute our Chern number using an orthonormal basis $\{v_j\}_{j \in \{1, \cdots, n\}}$ of $P(0, 0)$:

$$\mathrm{Ch}(P) = \left[\det\left[\langle v_a, U(k_2)^{-1} T_{k_2}(2\pi) U(k_2) v_b \rangle\right]\right] = [\det[\langle v_a(k_2), T_{k_2}(2\pi) v_b(k_2) \rangle]]$$

where $\{v_j(k_2)\} = \{U(k_2)v_j\}$ is a frame of $P(0, k_2)$. However, this coincides exactly with the definition of the Chern number given in [46], precisely in Theorem III.2.4.

Of course, it would be impossible to conclude this section without mentioning the double commutator formula

$$\mathrm{Ch}(P) = \int_{\mathbb{T}^2} \frac{1}{2\pi i} \mathrm{Tr}\{P(k)[\partial_{k_1} P(k), \partial_{k_2} P(k)] P(k)\} dk$$

that allows us to explicitly compute the Chern number using only projectors.

Therefore, our results on Class A are perfectly compatible with the existing literature, such as the results obtained in: [47], where the Chern number was interpreted as a topological obstruction to the construction of a smooth frame; [49], where the K-theoretic approach was used to define a bulk-edge correspondence; [12], where the homotopy theory of states is used as an attempt to treat the case of interacting Hamiltonian; [11] and [46], where the authors investigated the problem of constructing Parseval frames in $d \leq 4$. This means that our original contributions for Class A are minimal; the completeness of the invariant with respect to homotopy, the extension of existing result in less regularity, continuous instead of C^1 and a faster way to prove existing theorems.

The same happens for Class AI, where our methods give the same results present in: [10, 18], where the problem of constructing a symmetric frame was solved in $d \leq 3$; [13], the topological classification is achieved using K-theory ; and [15], where the topological classification is done using differential geometric invariants. Since our homotopy classification coincides with the K-theoretic one, our contributions for this class are minimal and identical to those of Class A.

Something more interesting happens for Class AII, where our contributions are moderate because in Theorem 4.2 we discovered that any fermionic time-reversal symmetric projection-valued map can be decomposed as a pair of odd particle-hole symmetric pair of projection-valued maps. This is compatible with the fact that mathematical models in Class AII usually descend from quantum systems of spin-$\frac{1}{2}$ particles, and the decomposition is given using the spin operator S_z. If it commutes with the original Hamiltonian and anti-commutes with T, then it commutes with P and the canonical decomposition is given in terms of spin up and spin down $P^{\pm}(k) = P^{\updownarrow}(k) = P(k)\frac{S_z \pm \mathbb{1}}{2}$. Under these hypotheses, it was already known that the $\mathbb{Z}_2$ invariant can be expressed as the parity of the Chern number of $P^{\uparrow} = P^{+}$ as well as the parity of $P^{\downarrow} = P^{-}$ ([3, 17, 26, 54]). However, if S_z does not commute with H, this easy decomposition is impossible to perform. Luckily, our decomposition theorem gives a way to decompose any fermionic time-reversal symmetric projection-valued map, and the two projection-valued maps could be interpreted as the first step to obtain a pseudo-spin decomposition of P, parallel to the approach presented in [42]. What we lack in this thesis is a way to explicitly construct a pseudo-spin operator $\tilde{S}_z$ acting on the original Hilbert space $L^2(\mathbb{R}^d)$ or $l^2(\Gamma)$ such that it commutes with H, it anti-commutes with T and such that $P^{\pm} = P\frac{\tilde{S}_z \pm \mathbb{1}}{2}$. Moreover, there are several ways to compute the $\mathbb{Z}_2$ invariant present in this class, and in principle it could be possible to find a direct equality between our formulation and the ones present in [18] or in [33]. However, Theorem 4.16 states that if our invariant is zero, it is possible to construct a symmetric, periodic, and continuous frame. It also states that our invariant is a complete topological invariant of this class, so there are only two connected components to study. Since the results present in [18] explicitly state that their $\mathbb{Z}_2$ invariant is constant on the connected components of the class, coincides with all the previous formulations of the invariant, and measures the obstruction in building continuous, symmetric, and periodic frames, we have that our invariant coincides with the others. As for the homotopy part, our classification does not obtain different results then the classification made using vector bundle theory [14], differential geometry invariants [15], so our contributions in this topic are similar to those of Class A

Our noticeable contributions appear when comparing our results for Classes D, AIII with the existing literature. Propositions 6.9, 3.8 and Theorems 6.11, 3.16 expand the results present in [23] for $d = 2$ and state that, when the dimension of the fiber Hilbert space is minimal ($\dim(\mathcal{H}) = 2n$), the complete topological invariants that describe these classes are more than the one predicted in the Kitaev table (see the Introduction). This is not a surprise since the K-theory used to compute the Kitaev table leads to the topological invariants that depend specifically on the dimension d considered. So, it is natural that the $\mathbb{Z}^2$-invariant $\mathcal{Z}$ (Definition 6.7) we obtained for Class AIII when $d = 2$ does not appear in the table since it comes from the $\mathcal{Z}$ invariant in dimension $d = 1$. The same can be said for the additional copies of $\mathbb{Z}_2$ that appear in the classification of Class D. In fact, they all descend from the zero-dimensional invariant $\mathcal{P}$ (Definition 3.5) and do not appear in the Kitaev table. The only exception occurs for $d = 1$ when the product between $\mathcal{P}(0)$ and $\mathcal{P}(\pi)$ can be expressed as the 1-dimensional invariant $\mathcal{I}$ (see Definition 3.9 and Proposition 3.10). However, there is an extremely interesting fact that we noticed: some invariants in these classes are not preserved under unitary equivalences, and in the chapter about explicit models we gave several examples of this. This means that the notion of topological invariant needs to be refined to take into account the difference between a homotopic invariant and a unitary invariant. Up to this point, the topological invariants were divided into strong invariants and weak invariants: as said before, the first were those that depend specifically on the dimension d and the latter were those inherited from lower dimensions. Since in the Introduction we establish different choices of a dimerization as an example where it is relevant to study unitary equivalences, we propose to call homotopic invariants that are not unitarily invariant *relative invariants* to remind that their values depend on the choice of a dimerization. On the other hand, we propose to call invariants that are unitarily invariant *absolute invariants* to remind that they do not depend on a choice of a dimerization. We summarize the strong/weak vs absolute /relative invariants in the following table.

	Strong	Weak
Absolute	$\mathrm{Ch}(P), \delta(P)$ when $d = 2$	$\mathrm{Ch}(P(k_1, k_2, 0)), \delta(P(k_1, k_2, 0)$ when $d > 2$
Relative	$\mathcal{I}, \mathcal{Z}$ when $d = 1$	$\mathcal{P}(0)$ if $d = 1$, $\mathcal{I}, \mathcal{Z}$ when $d = 2$

What is truly surprising is the fact that our results are in complete contrast with the Kitaev table as soon as $\dim(\mathcal{H}) > 2n$. In particular, we lose one-dimensional invariants in both classes and gain at least one invariant in Class AIII for $d = 2$. We will give an explanation to this phenomenon on Class D, but something similar can be done for Class AIII. From a mathematical point of view, this can be explained by the fact that there is a hidden hypothesis that leads to the Kitaev table. This hypothesis is that the pair of projection-valued maps must sum to the identity $P^+(k) + P^-(k) \equiv \mathbb{1}$. This hypothesis is also used in the K-theory approach to the classification problem. In fact, the K-theory classification depends on the notion of stable equivalence which states that two vector bundles $\mathcal{E}, \mathcal{D}$ are stably equivalent if and only if there exist two trivial vector bundles $\mathcal{I}_1, \mathcal{I}_2$ such that $\mathcal{E} \oplus \mathcal{I}_1$ is isomorphic to $\mathcal{D} \oplus \mathcal{I}_2$. This condition can be read in the projection-valued map formalism as the fact that two projection-valued maps P_0, P_1 are equivalent according to K-theory if and only if there are two projection-valued maps I_0, I_1 such that $P_0 \oplus I_0$ is homotopic to $P_1 \oplus I_1$. When this notion is used to study pairs of particle-hole symmetric pairs of projection-valued maps, we have that two pairs $P_0^\pm, P_1^\pm$ are equivalent according to K-theory if there are two particle-hole symmetric pairs of projection-valued maps $I_0^\pm, I_1^\pm$ such that $(P_0 \oplus I_0)^\pm$ are homotopic to $(P_1 \oplus I_1)^\pm$. According to K-theory, every invariant obtained is preserved after stable equivalences. So, it would be reasonable to suppose that adding additional dimensions to the Hilbert space $\mathcal{H}$ cannot break any invariant that comes from K-theory. This seems to contradict the fact that, when $\dim(\mathcal{H}) > 2n$, we lose the topological invariants $\mathcal{P}$. However, the dimensions added using the K-theory approach come from the particle-hole symmetric pair of projection-valued maps $I^{\pm}$. This means that, in order to proper apply the K-theory approach, one needs to split the fiber Hilbert space $\mathcal{H}$ into subspaces

$$\mathcal{H} = \mathcal{H}^{-} \oplus \mathcal{H}^{+} = \mathrm{Ran}(P^{-} \oplus I^{-}) \oplus \mathrm{Ran}(P^{+} \oplus I^{+})$$

From a physical point of view this means that the original Hamiltonian must have a spectral gap around zero and that we are actually studying the particle-hole symmetric pair of projection-valued maps $\tilde{P}^{\pm} = P^{\pm} \oplus I^{\pm}$. For this pair, it is true that $\dim(\mathrm{Ran}(\tilde{P}^{\pm})) = \dim(\mathcal{H})/2$ and that $\tilde{P}^{+} \oplus \tilde{P}^{-} = \mathbb{1}$. This implies that if $\dim(\mathcal{H}) < \infty$, the K-theory essentially studies models in which $\dim(\mathcal{H}) = 2\dim(\mathrm{Ran}(\tilde{P}^{\pm}))$. So, applying the K-theory approach to Class D consist of considering periodic Hamiltonians with a spectral gap around zero, focusing on the Bloch bands immediately adjacent to zero and computing the invariants of the particle-hole symmetric pair of projection-valued maps that come from the two Bloch bands. Those invariants are supposed to be stable up to summing the particle-hole symmetric pairs of projection-valued maps that descends from other Bloch bands. Instead, the homotopy approach does not require a spectral gap around zero and studies the properties of the specific pair of symmetric Bloch bands selected. It can also perfectly treat the dimerization ambiguity which is impossible to treat using K-theory. So, if $\dim(\mathcal{H}) > 2n$, our results can be interpreted as a complete topological classification that cannot be derived from K-theory. We believe this is a major achievement of this formalism, in fact K-theoretic classification of Class AIII where present in the literature [16] as well as classification of disordered materials through index theorems [50] and bulk-edge correspondence [25], but they all avoid to address the continuum setting and the dymerization ambiguity. One attempt to reconcile the K-theory with the dymerization ambiguity is present in [57] that also contains the notion of *relative invariant*. But in the end the difficulties present in the study of continuum models where never fully addressed except for the SSH model [55] and for one-dimensional models that are also periodic in time [22]. We hope that our results for the case $\dim(\mathcal{H}) > 2n$ can give meaningful insight on the difference between continuum and tight-binding models.

Finally, for Class C we rigorously proved in Proposition 5.4 and in Theorem 5.5 that the K-theory classification predicted in the Kitaev table coincides with our complete topological classification and there are no exotic phenomena occurring in this class like those occurring in Class D.

To conclude, we believe that the true original contribution is the overall formalism presented, in fact, the three step procedure consisting in:

- describing topological invariants as characteristic pseudo-periodic conditions;
- construct symmetric unitary equivalences using pseudoperiodic frames;
- construct symmetric homotopies from unitary equivalences,

allows for a complete description of topological phases of matter that takes into account the different physical information one is interested in. It also allows for easy computations of the invariants that are otherwise harder to derive using K-theory. Moreover, it solves in a natural way the dymerization ambiguity mentioned in the Introduction and explain the vanishing of topological invariants in continuum models observed in the literature.

8.3 Future developments

- (*Extension to the other entries of the Kitaev table*) The techniques developed in this thesis can be used to replicate the study we did for the classes with a single symmetry to the four remaining classes with three symmetries and also for $d \leq 3$. The obstacle in studying the remaining four classes is that they have three symmetries instead of one, and one needs to carefully describe all the possible commuting or anti-commuting relations between the symmetries before attempting a topological classification. Instead, the difficulty present in the study of the $d = 3$ case is that $\pi_3(U(n))$ is not trivial for $n > 1$, making the topological study much harder.

- (*Non-periodic framework*) In case the original Hamiltonian is slightly perturbed, it is expected that the strong invariants will survive also in absence of translation invariance. Some important results in this direction are obtained in [8] for $d = 1$ and in [20] and [4] for Class AII in $d = 2$. However, in $d = 2$, there is an ongoing debate on the proper local perturbations that can be considered to obtain a class of local Hamiltonians with topological properties. For Class AII in particular, we hope that our interpretation of the δ invariant as a parity of a Chern number can be generalized in non-periodic frameworks as a parity of a Chern marker (defined in [41]). For the other classes in $d = 2$ the problem of finding the proper local conditions such that the homotopy classes of local Hamiltonians is the one predicted by the Kitaev table remains open (to the best of our knowledge).

- (*Bulk-edge correspondence*) Since the models studied in this thesis were fully periodic, they are used to approximate the motion of a particle in the bulk of the material, where the presence of edges is negligible. For this reason, the topological invariants obtained by studying periodic models are called *bulk invariants*. It is also possible to study Hamiltonians acting on the half-line or the half-plane and those models represent in particular the motion of a particle near the edges. Those Hamiltonians admit a topological classification in terms of Fredholm operators and the topological invariants are usually Fredholm indices and Atiyah–Singer indices ([4], [8]). Since they take into account the behavior of the system near the edge, those invariants are referred to as *edge invariants*. An important result would be to topologically classify the edge models and establish a *bulk-edge correspondence* between the bulk invariants of a periodic model and the edge invariants of its truncation. Some important results have already been achieved. For example in [59] the Chern number is shown to be the coefficient of the quantum Hall conductivity in a proper two-dimensional model. Instead in [8] is present a topological edge classification for one-dimensional models. Other great contributions are present in [49], [5], or [6].

- (*Recovering the lost topological invariants*) Our results about relative topological invariants are extremely concerning. In fact, the relative invariants we obtained are only present when $\dim(\mathcal{H}) = 2n$ and this means that they are only present in tight-binding models. However, discrete models are essentially approximations of continuum models and it is disturbing that this thesis states that there is a complete loss of topological information in continuum models. We want to present a conjecture that describes what could be really going on. Suppose that a continuum system undergoes a change of the physical parameters while maintaining periodicity and possible symmetric constraint. Suppose for simplicity that we are studying a model in Class D and that all Bloch bands of the system remain separated with just one exception: the two bands near zero (Ω_t and $-\Omega_t$) are separated at the start of the process ($t = 0$), then at some point ($0 < \tilde{t} < 1$) they become united in a bigger Bloch band that contains zero and at the end of the process ($t = 1$) they separate once again. If this happens Ω_t and $-\Omega_t$ will have the two associated projection-valued maps $P_t^+, P_t^- : \mathbb{T}^d \to \mathrm{Proj}_n(\mathcal{H})$ only for the values of t for which the two Bloch bands are separated. However it is always possible to define the projection-valued map $P_t : \mathbb{T}^d \to \mathrm{Proj}_{2n}(\mathcal{H})$ associated to $\Omega_t \cup -\Omega_t$. On the values of t for which the bands are separated we have that $P_t = P_t^+ \oplus P_t^-$. If the dependence on t is continuous it is possible to define a unitary-valued map $U : \mathbb{T}^d \to \mathcal{U}(\mathcal{H})$ such that $P_1(k)U(k) = U(k)P_0(k)$ using iteratively the Kato-Nagy construction (as we did in Propositions 1.2, 1.5, 2.4 and 2.8). Then we can choose a symmetric frame (due to Proposition 3.8) $\{v_j(k)\}_{j\in\{1,\cdots,2n\}}$ of $P_0^\pm(k)$ such that the matrix representation

$$[\tilde{P}_0^\pm(k)]_{a,b} = \left\langle v_a(k), P_0^\pm v_b(k) \right\rangle$$

as a $2n \times 2n$ matrix is constant. Then we have that $\{U(k)v_j(k)\}_{j\in\{1,\cdots,2n\}}$ is a frame of $P_1(k)$ but is not a symmetric frame for the pair $P_1^\pm(k)$. This means that the matrix representation

$$[\tilde{P}_1^\pm(k)]_{a,b} = \left\langle U(k)v_a(k), P_0^\pm U(k)v_b(k) \right\rangle$$

is not necessarily constant. Now the two matrix-valued maps $\tilde{P}_0^\pm(k)$ and $\tilde{P}_1^\pm(k)$ can be treated as two projection-valued maps from $\mathbb{T}^d$ to $\mathbb{C}^{2n}$ and, provided that U respects the symmetry constraint $CU(k) = U(k)C$, they constitute a symmetric pair in the sense of Definition 3.1. Now we can compute the topological invariants of this class for both maps and usually the invariants of the first will be trivial because the map is constant, while the others may be non-trivial. In order for this to have some meaning, it is important to control how the invariants change depending on the choices of U and the frame $\{v_j(k)\}$. A different frame $\{u_j(k)\}$ will lead to a unitary-valued map $[V(k)]_{a,b} = \langle v_a(k), u_b(k) \rangle$ and the new matrix representations will be:

$$V(k)^{-1}\tilde{P}_0^\pm(k)V(k), \quad V(k)^{-1}\tilde{P}_1^\pm V(k).$$

Therefore the topological invariants may change according to Proposition 3.6 but the changes are equal on both sides, meaning that the topological information that should remain constant is the change of topological phase between the starting point of the transformation and the arriving point. Instead the dependence on U can really change everything beyond control because it depends on the specific transformation considered and we conjecture that it is possible to have a unique U for any given perturbation of the Hamiltonian that passes through a gapless phase. If this is possible, then the relative topological invariants can even be interpreted in continuum models as changes of topological invariants as the model moves through a gapless phase. Hence, they are *relative* with respect to the specific transformation considered.

Appendix A

Tools from algebraic geometry

Definition A.1 (Winding number)**.** Given a map $f : S^1 \to S^1$, the winding number, often denoted $[f]$, is an integer number exhibited by the natural isomorphism $\pi_1(S^1) \simeq \mathbb{Z}$. It can be computed by considering the natural covering $g : \mathbb{R} \to S^1$ such that $g(t) = e^{i2\pi t}$. Then, the map f can be lifted to a continuous map $\mu : [-\pi, \pi] \to \mathbb{R}$ such that $g(\mu(t)) = f(\pi(t))$. Despite the numerous lifting options, the quantity $[f] = \mu(\pi) - \mu(-\pi) \in \mathbb{Z}$ is well defined and depends only on the homotopy class of f.

Theorem A.2 (Kuiper's theorem)**.** *If $\mathcal{H}$ is a real, complex, or quaternionic separable Hilbert space with infinite dimension, then $\mathcal{U}(\mathcal{H})$ is weakly contractible.*

The proof can be found in [39].

Theorem A.3. *If $\mathcal{H}$ is a complex Hilbert space with finite dimension, then $\pi_0(\mathcal{U}(\mathcal{H})) \simeq \{0\}$, $\pi_1(\mathcal{U}(\mathcal{H})) \simeq \mathbb{Z}$ using the homomorphism that computes the winding number of the determinant and $\pi_2(\mathcal{U}(\mathcal{H})) \simeq \{0\}$.*

Proof. We can express $\mathcal{U}(\mathcal{H})$ as $\mathrm{U}(n)$, set of $n \times n$ unitary and complex matrices. Then we can express it as a Cartesian product using the bijective map:

$$\begin{array}{ccc} \mathrm{U}(n) & \to & S^1 \times \mathrm{SU}(n) \\ U & \mapsto & \left(\det(U), \begin{pmatrix} \det(U)^{-1} & 0 \\ 0 & \mathbb{I}_{n-1} \end{pmatrix} U \right) \end{array}$$

This means that there is a short exact sequence of groups

$$\{\mathbb{I}\} \to \mathrm{SU}(n) \to \mathrm{U}(n) \to S^1 \to \{\mathbb{I}\}.$$

A sequence like this induces a long exact sequence

$$\cdots \pi_{j+1}(S^1) \to \pi_j(\mathrm{SU}(n)) \to \pi_j(\mathrm{U}(n)) \to \pi_j(S^1) \to \pi_{j-1}(\mathrm{SU}(n)) \to \cdots$$

Since S^1 admits a universal covering using $\mathbb{R}$, we have that $\pi_j(S^1)$ is trivial for $j < 1$. Moreover, using the table containing the values of $\pi_j(\mathrm{SU}(n))$ in Appendix A, Section 6, Part VII of [32] we have that:

$$\pi_0(S^1) \simeq \pi_2(S^1) \simeq \pi_0(\mathrm{SU}(n)) \simeq \pi_1(\mathrm{SU}(n)) \simeq \pi_2(\mathrm{SU}(n)) \simeq \{0\},$$

and also that $\pi_1(S^1) \simeq \mathbb{Z}$ using the homomorphism that computes the winding number of the map. Additional details on this topic can be found in [29] on page 344. $\square$

Theorem A.4. *The set of* $n \times n$ *real orthogonal matrices* $\mathrm{O}(n)$ *has two arc-wise connected components* $O_{\pm}(n) = \{M \in \mathrm{O}(n) | \det(M) = \pm 1\}$.

The detailed proof of this well-known fact can be found in Chapter 10 of [43].

Theorem A.5. *If* $J = \begin{pmatrix} 0 & -\mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}$, $\mathcal{K}$ *is the standard complex conjugation and* $\mathrm{Sp}(2n) = \{U \in U(2n) | J\mathcal{K}U = UJ\mathcal{K}\}$, *then* $\mathrm{Sp}(2n)$ *is simply connected.*

Proof. A matrix U is in $\mathrm{Sp}(2n)$ if and only if it can be written in exponential form using $U = e^{iH}$, where H is self-adjoint such that $(JH)^t = JH$ thanks to Proposition 3.5 and Corollary 11.10 in [27]. Since this relation is linear, we can move U toward the identity with the continuous transformation $U_t = e^{itH}$ for $t \in [0, 1]$. $\square$

Proposition A.6. *Let* $(X, \circ)$ *be a topological group and* $\pi_1(X, Id)$ *its fundamental group centered at the identity. If* $f, g \in \pi_1(X, Id)$ *then the function* $f \circ g(t) = f(t) \circ g(t)$ *is homotopic to* $[f * g](t)$ *which is the concatenation of the loops* f *and* g.

Proof. The homotopy we are looking for is the following:

$$F(t,s) = \begin{cases} f\left(\frac{2t}{2-s}\right) & \text{if} \quad t \leq \frac{s}{2} \\ f\left(\frac{2t}{2-s}\right) \circ g\left(\frac{2t-s}{2-s}\right) & \text{if} \quad \frac{s}{2} \leq t \leq 1 - \frac{s}{2} \\ g\left(\frac{2t-s}{2-s}\right) & \text{if} \quad 1 - \frac{s}{2} \leq t \end{cases}$$

In fact it is continuous since f, g start and end at the identity so $f(0) = f(1) = g(0) = g(1) = Id$ and $F(t, 0) = f(t) \circ g(t) = f \circ g(t)$ while

$$F(t,1) = \begin{cases} f(2t) & \text{if} \quad t \leq 1/2 \\ g(2t-1) & \text{if} \quad t \geq 1/2 \end{cases} = [f * g](t).$$

$\square$

Corollary A.7. *Consider* $S^1 = \left\{e^{i\theta} \in \mathbb{C}\right\}$ *as a topological group with the standard product of* $\mathbb{C}$. *If* $f, g : S^1 \to S^1$ *are two smooth maps with* $f(1) = g(1) = 1$ *then the winding number of* $f \cdot g$ *is the sum of the winding number of* f *and the one of* g.

Proof. If we apply the previous theorem we know that $f \cdot g$ is homotopic to $f * g$ and it is well known that $\pi_1(S^1, 1)$ is isomorphic as a group to $\mathbb{Z}$ with the group isomorphism that maps every function to its winding number: again this fact can be checked in section 5 of the first chapter in [43]. So, if we denote by $[h]$ the winding number of the function h, we have that:

$$[f \cdot g] = [f * g] = [f] + [g].$$

$\square$

Lemma A.8. *If $\lambda : \mathbb{T}^1 \to \mathbb{T}^1$ is a continuous map with $\lambda(k) = \lambda(-k)$, then $[\lambda] = 0$.*

Proof. We can construct an explicit homotopy λ_t for $t \in [0,1]$ between λ and the constant function $\lambda_0 \equiv \lambda(\pi) = \lambda(-\pi)$:

$$\lambda_t(k) = \begin{cases} \lambda(k) & \text{for} \quad k \in [-\pi, (t-1)\pi] \cup [(1-t)\pi, \pi] \\ \lambda[(t-1)\pi] = \lambda[(1-t)\pi] & \text{for} \quad k \in [(t-1)\pi, (1-t)\pi] \end{cases}$$

In fact, the two branches of the function connect for $k = (t-1)\pi, (1-t)\pi$ and also connect for $k = \pi = -\pi$ since $\lambda(\pi) = \lambda(-\pi)$. This means that $\lambda_t(k)$ is continuous. □

Lemma A.9. *If $\lambda : \mathbb{T}^1 \to S^1$ is a map with $\lambda(\pi) = \lambda(-\pi) = \lambda(0) = 1$ and such that $\lambda(-k) = \overline{\lambda(k)}$, then $[\lambda] \in 2\mathbb{Z}$ is even.*

Proof. We can choose a lift $\mu : [-\pi, \pi] \to \mathbb{R}$ such that $e^{i\mu(k)} = \lambda(k)$ with $\mu(0) = 0$. Then the proprieties translate to the fact that $\mu(k) = -\mu(-k)$ and $\mu(\pi), \mu(-\pi) \in 2\pi\mathbb{Z}$. So, the winding number will be:

$$[\lambda] = \frac{\mu(\pi) - \mu(-\pi)}{2\pi} = \frac{2\mu(\pi)}{2\pi} \in 2\mathbb{Z}$$

□

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Nomenclature

χ_Ω Characteristic function over the set Ω

Π Chiral symmetry (CS)

$\mathbb{1}$ Identity operator on an Hilbert space

$\mathbb{I}_n$ $n \times n$ identity matrix

∂X Boundary of the topological space X

Ξ Particle-hole symmetry (PHS)

$\pi_n(X)$ Group of the maps from S^n to X modulo homotopies

$\mathrm{Proj}_n(\mathcal{H})$ Set of the projections of $\mathcal{H}$ with rank n

$\mathrm{SU}(n)$ Group of $n \times n$ complex and unitary matrices with unitary determinant

$\mathrm{U}(n)$ Group of $n \times n$ complex and unitary matrices

Θ Time reversal symmetry (TRS)

$\mathcal{U}(\mathcal{H})$ Set of unitary operators from $\mathcal{H}$ to itself

$\mathcal{U}_{\mathrm{BFZ}}$ Bloch-Floquet-Zak transform

$L^2(X)$ Space of square-integrable functions over X

$M_n(\mathbb{C})$ $n \times n$ matrices with complex coefficients

$\mathrm{Sp}(2n)$ Unitary complex matrices that are also symplectic matrices.

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