Applied Mathematical Sciences, Vol. 16, 2022, no. 12, 709 - 717
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2022.917306

A Numerical Method for Multidimensional Volterra Integral Equations [1]

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Abstract

In this paper, we introduce a new numerical procedure to solve multidimensional Volterra integral equations, based on the weighted mean-value theorem. Our method allows to determine a system of nonlinear equations, where the first one is obtained via the application of the theoretical results, and the remaining ones are built through a Picard-like iterative algorithm. To test the soundness of our proposal, we compare the true and the approximate solutions for several examples.

Mathematics Subject Classification: 45D05, 35B05, 45G15

Keywords: Volterra linear and nonlinear integral equations, weighted mean-value theorem, numerical methods

1 Introduction

Integral equations play a crucial role in a variety of fields, from the demographic analysis of population dynamics models [9], to the evaluation of ruin

[1]The authors were financially supported by Sapienza University of Rome, research project "Actuarial and financial risk management solutions in a pandemic mortality framework", Grant No. RM120172B84AD5EC.

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probability, [10]. Different classifications of integral equations exist, depending on the suitable analytical properties: we distinguish between Fredholm and Volterra integral equations of first and second kind, depending on whether the unknown function appears only under the integral sign or not. However, it is worth pointing out that the former can be transformed to equations of the second kind, using suitable mathematical tricks (such as the regularization method, see e.g. [11]). Although several attempts have been undertaken to find a closed-form solution, the wide variety of these equations does not guarantee the existence of a unified methodology. Therefore, the literature has proposed *ad-hoc* numerical approaches. In [4] the authors propose new algorithms to solve nonlinear Fredholm and Volterra integral equations of the second kind using Haar wavelets, while in [8] a standard Von Neumann expansion of the solution of a Fredholm integral equation of the second kind approximated using Markov chain Monte Carlo methods is considered. Recently, a technique to solving linear and nonlinear Fredholm integral equations using a mean-value theorem approach is proposed in [2]. Concerning the Volterra integral equations, standard numerical methods involve either the Runge-Kutta method, the successive approximations method, the Laplace transform method or the Adomian decomposition method, see e.g. [7, 12]. More recently, in [6] the authors suggest to use Schauder bases to solve two-dimensional Volterra integral equations. In the present paper, we solve Volterra integral equations of the second kind in a multidimensional setting, by applying the weighted mean-value theorem. We calculate the terms by solving related non-linear equations using Brent's method. The obtained values are used to approximate the integral. To further reduce approximation errors, we apply trapezoidal numerical integrations, mimicking the Picard iterative method. Following [2], we assume that the weighted mean-value theorem yields a real constant. Such an assumption, although restrictive, is justified by a large literature, see e.g. [3]. To test the soundness and the flexibility of our numerical proposal, we apply the method to several Volterra integral equations, belonging to different categories, and for which an explicit closed-form solution exists. Specifically, we use this method to solve: a nonlinear Volterra equation of the second univariate type, a linear equation of the second univariate type, a nonlinear Volterra equation of the second bivariate type, and a nonlinear Volterra equation of the second tridimensional type.We compare the exact and the approximate solution in terms of absolute error. The paper is organized as follows. We recall main definitions and results in Section 2. We present our numerical method step-by-step in Section 3. In Section 4 we present numerical results.

2 The weighted mean-value theorems and the Volterra integral equations

Throughout the paper, we consider a nonlinear Volterra integral equation of the second kind of the from

$$\phi(x) = f(x) + \lambda \int_a^x K(x,t)\psi(\phi(t))dt\,, \tag{1}$$

where $\lambda \in \mathbb{R}$, $f(x)$, $x \in [a,b]$, is a continuous function, $\psi(x)$, $x \in [a,b]$, is a unknown nonlinear continuous function, and the kernel function K satisfies the standard assumptions, i.e. $K(x,t) \geq 0$ or $K(x,t) \leq 0$, for any x, $t \in [a,b]$. It is worth noting that Eq. (1) becomes a linear Volterra integral equation by setting ψ as the identity function (i.e., $\psi(\phi(\cdot)) = \phi(\cdot)$).

We recall the weighted mean-value theorem

Theorem 2.1 *Let $f, g : [a,b] \to \mathbb{R}$ be a continuous on $[a,b]$. If g never changes sign in $[a,b]$, then there exists $\xi \in [a,b]$ such that*

$$\int_a^b f(t)g(t)dt = f(\xi)\int_a^b g(t)dt\,. \tag{2}$$

It is possible to extend the previous theorem to the high-dimensional case.

Theorem 2.2 *Assume that $g \in \mathbb{R}$ and $f \in \mathbb{R}$ on a Jordan-measurable set $S \in \mathbb{R}^n$ and suppose $g(\boldsymbol{x}) \geq 0$ for each $\boldsymbol{x} \in S$. Let $m = \inf f(S)$, $M = \sup f(S)$. Then, there exists a real number α in the interval $m \leq \alpha \leq M$ such that*

$$\int_S f(\boldsymbol{x})g(\boldsymbol{x})d\boldsymbol{x} = \alpha \int_S g(\boldsymbol{x})d\boldsymbol{x}\,. \tag{3}$$

The proofs of the previous Theorems are provided in [1]. By exploiting Theorem 2.1 and Theorem 2.2, we are able to provide a new numerical method to solve some special classes of Volterra integral equations, namely linear unidimensional Volterra equations of the second kind, and nonlinear high-dimensional Volterra equations of the second kind. Inspired by the discussion provided in [2] for Fredholm equations, we focus on the particular case where the weighted mean-value theorem applied to Volterra integral equations produces a number c rather than a function $c(x)$.

3 Description of the proposal

Given the nonlinear Volterra integral equation (1), we describe our proposal according to the following steps.

Step 3.1 *For any positive integer n, we consider a partition Γ of the interval $[a,b]$ into n equally-spaced intervals, with length $\Delta = \frac{(b-a)}{n}$, i.e.,*

$$a = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = b. \tag{4}$$

For each $x_i \in \Gamma$, $i = 0, 1, \ldots, n+1$, we report the discretized version of Eq. (1)

$$\phi(x_i) = f(x_i) + \lambda \int_a^{x_i} K(x_i, t)\psi(\phi(t))dt\,. \tag{5}$$

Step 3.2 *We apply Theorem 2.1, ensuring that we may choose a number ξ_i, $a \leq \xi_i \leq x_i$, for $i = 1, \ldots, n+1$, such that*

$$\phi(x_i) = f(x_i) + \lambda c \int_a^{x_i} K(x_i, t)dt\,, \tag{6}$$

where we have assumed $\psi(\phi(\xi_i)) = c \in \mathbb{R}$, with $\min \phi([a, x_i]) \leq c \leq \max \phi([a, x_i])$.

Step 3.3 *We replace the approximated $\phi(x_i)$ given in* (6) *in both the left and right hand-sides of Eq.* (5)*, obtaining*

$$f(x_i) + \lambda c \int_a^{x_i} K(x_i, t)dt = f(x_i) + \lambda \int_a^{x_i} K(x_i, t)\psi(f(t) + \lambda c \int_a^t K(y, y)dy)dt. \tag{7}$$

Step 3.4 *For $i = 1, \ldots, n+1$, we determine c as a solution to*

$$c \int_a^{x_i} K(x_i, t)dt - \int_a^{x_i} K(x_i, t)\psi(f(t) + \lambda c \int_a^t K(y, y)dy)dt = 0\,. \tag{8}$$

Step 3.5 *For any $x_i \in \Gamma$, we replace the value of c in* (6) *and evaluate the approximate numerical solution, and we have*

$$\phi_0(x_i) = f(x_i) + c\lambda \int_a^{x_i} K(x_i, t)dt\,. \tag{9}$$

Step 3.6 *We apply the Picard iteration method and obtain*

$$\begin{cases} \phi_1(x_i) & = f(x_i) + \lambda \int_a^{x_i} K(x_i,t)\phi_0(t)dt\,, \\ \phi_2(x_i) & = f(x_i) + \lambda \int_a^{x_i} K(x_i,t)\phi_1(t)dt\,, \\ \vdots & \qquad\qquad \vdots \\ \phi_n(x_i) & = f(x_i) + \lambda \int_a^{x_i} K(x_i,t)\phi_{n-1}(t)dt \end{cases}, \tag{10}$$

and the convergence of the iterated approximation is ensured by [12, Theorem 3.1]. We stop the iteration method by introducing a fixed ϵ sufficiently small, such that $|\phi_n(x) - \phi_{n-1}(x)| \leq \epsilon$.

We observe that our proposal works within the Volterra integral equations of the first kind framework. More precisely, we consider the following integral equation

$$f(x) = \lambda \int_a^x K(x,t)\psi(\phi(t))dt\,, \tag{11}$$

where $f(x)$, $K(x,t) \in C^1[a,b]$. By imposing $y(t) = \psi(\phi(t))$ and differentiating Eq. (11) w.r.t. x, we have

$$\lambda K(x,x)y(t) + \lambda \int_a^x K(x,t)y(t)dt = f'(x)\,, \tag{12}$$

provided that $K(x,x) \neq 0$. Rearranging the terms, Eq. (12) can be rewritten as

$$y(t) + \int_a^x K^*(x,t)y(t)dt = f^*(x)\,, \tag{13}$$

where $K^*(x,t) = \frac{1}{K(x,x)}\frac{\partial K(x,t)}{\partial x}$ and $f^*(x) = \frac{f'(x)}{\lambda K(x,x)}$. Thus, we can handle the Volterra integral equation of the second kind given in equation (13) by using the proposed numerical approach.

Finally, we note that also the d-dimensional Volterra integral equations, with $d > 1$, may be solved by using our proposal. For the sake of simplicity, we consider following two-dimensional Volterra integral equation of the second kind

$$\phi(x,t) = f(x,t) + \lambda \int_0^t \int_0^x K(x,t,y,z)\phi(y,z)dydz\,. \tag{14}$$

We define $\Omega := [0,x] \times [0,t]$. By applying Theorem 2.2 together with the proposed algorithm, for any $(x,t) \in \Omega$, the first approximation is

$$\phi_0(x,t) = f(x,t) + c\lambda \int_0^t \int_0^x K(x,t,y,z)dydz\,, \tag{15}$$

where $\inf \phi(\Omega) \leq c \leq \sup \phi(\Omega)$.

4 Numerical Results

In this section, we present some applications of our proposal. For each Volterra integral equation provided, we consider the closed-form solution as a benchmark.

Example 4.1 *Consider the following Volterra integral equations (of second and first kind, respectively)*

$$\phi(x) = \frac{1}{3}x\cos(x^3) + x^3 - \frac{x}{3} + \int_0^x xt^2 \sin(\phi(t))dt\,, \tag{16}$$

$$x = \int_0^x e^{-x+t}\phi(t)dt\,, \tag{17}$$

where $x \in [0,1]$. *The exact solutions are* $\phi(x) = x^3$ *and* $\phi(x) = x+1$, *respectively. The absolute errors between the exact and the approximate solution of Eqs.* (16) *and* (17) *are given in Fig. 1.*

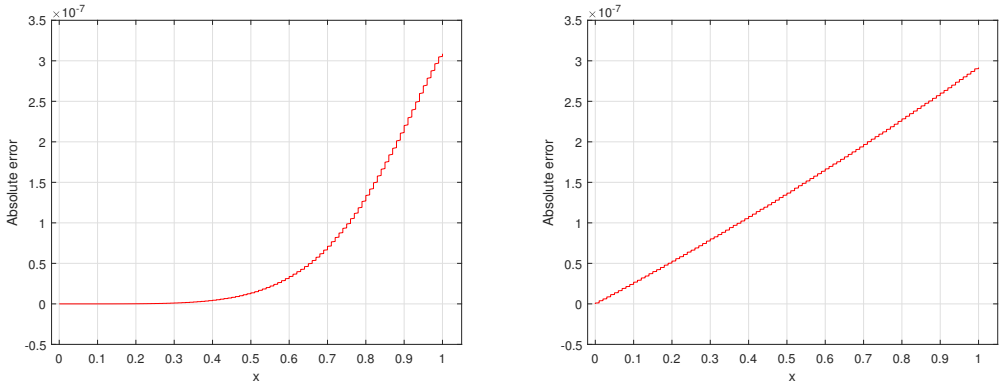

Figure 1: Errors for the Volterra integral equation of the second type, given in Eqs. (16) (left chart) and (17) (right chart). We apply the iterated method with $n = 1000$.

Example 4.2 *Consider the following nonlinear two-dimensional Volterra integral equation of second kind (see e.g. [6])*

$$\phi(x,t) = f(x,t) + \int_0^t \int_0^x (xy^2 + \cos(z))\phi(y,z)^2 dydz\,, \tag{18}$$

where $f(x,t) = x\sin(t)(1 - \frac{1}{9}x^2\sin^2(t)) + \frac{1}{10}x^6(\frac{1}{2}\sin(2t) - t)$ *and* $0 \le x,t \le 1$. *The exact solution is given by* $\phi(x,t) = x\sin(t)$. *The absolute error between the exact and the approximate solution of Eq.* (18) *is given in Figure 2.*

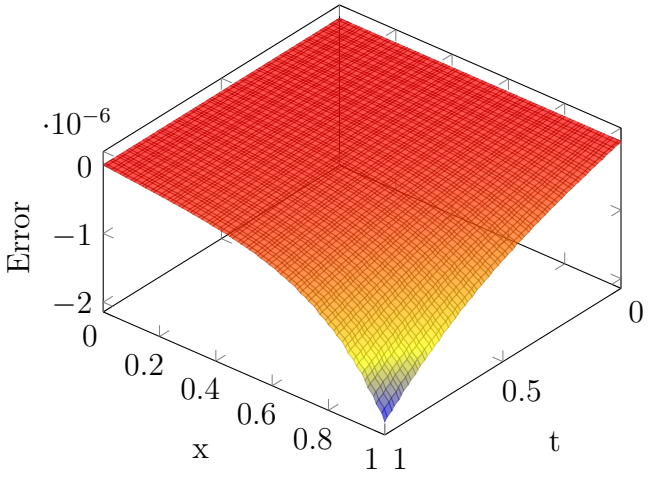

Figure 2: Absolute error for the two-dimension Volterra integral equation given by (18). We apply the iterated method with $n = 100$.

Example 4.3 *Consider the following three-dimensional linear Volterra integral equation of the second kind (see e.g. [5])*

$$\phi(x,y,t) = x\cos(t) - \frac{x^3y^3}{9}\sin(t) + \int_0^t \int_0^y \int_0^x rs^2\phi(r,s,z)drdsdz\,, \qquad (19)$$

with $0 \le x, y, t \le 1$. *The exact solution is given by* $\phi(x,y,t) = x\cos(t)$. *The absolute error between the exact and the approximate solution of Eq.* (19) *is provided in Table 1.*

(x,y,t)	Absolute Error
$(0,0,0)$	0
$(0.1,0.1,0.1)$	$-2.943547433575944e-09$
$(0.2,0.2,0.2)$	$-8.936410778037640e-08$
$(0.3,0.3,0.3)$	$-6.644113785814199e-07$
$(0.4,0.4,0.4)$	$-2.742513994513640e-06$
$(0.5,0.5,0.5)$	$-8.169220481357709e-06$
$(0.6,0.6,0.6)$	$-1.975452180008608e-05$
$(0.7,0.7,0.7)$	$-4.136983347724499e-05$
$(0.8,0.8,0.8)$	$-7.873166081395766e-05$
$(0.9,0.9,0.9)$	$-1.462932846576503e-04$
$(1,1,1)$	$-3.124093832777008e-04$

Table 1: Absolute errors for the three-dimensional linear Volterra equation in (19). We apply the iterated method with $n = 20$.

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Received: December 21, 2022; Published: December 23, 2022