



# On the supercritical defocusing NLW outside a ball

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## Abstract

We study a defocusing semilinear wave equation, with a power nonlinearity  $|u|^{p-1}u$ , defined outside the unit ball of  $\mathbb{R}^n$ ,  $n \geq 3$ , with Dirichlet boundary conditions. We prove that if  $p > n + 3$  and the initial data are nonradial perturbations of large radial data, there exists a global smooth solution. The solution is unique among energy class solutions satisfying an energy inequality. The main tools used are the Penrose transform and a Strichartz estimate for the exterior linear wave equation perturbed with a large, time dependent potential.

**Keywords** Supercritical wave equation · Exterior problem · Decay estimates · Global existence

**Mathematics Subject Classification** 35L05 · 58J45 · 35L20

## 1 Introduction

Consider the Cauchy problem for the defocusing wave equation on  $\mathbb{R}_t \times \mathbb{R}_x^n$ :

$$\square u + |u|^{p-1}u = 0, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) \quad (1.1)$$

with Sobolev initial data  $(u_0, u_1) \in H^s \times H^{s-1}$ . The existence of global solutions to this problem has been explored in considerable detail. The *energy critical power* for global smooth solvability is  $p_{cr}(n) = 1 + \frac{4}{n-2}$  for  $n \geq 3$ , while  $p_{cr}(1) = p_{cr}(2) = \infty$ . Global existence in the energy space is known for  $p \leq p_{cr}(n)$  [4,5,8]; regularity for  $p = p_{cr}$  has been explicitly proved up to dimension 7 [5,7] and should hold for all

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dimensions. For *supercritical* nonlinearities  $p > p_{cr}(n)$ , very little is known, and the question of global existence or blow up is still open.

If we restrict to spherically symmetric solutions, the difficulty of the problem is essentially the same. However, if a radial solution blows up, the blow up must occur at the origin. This follows at once from the Sobolev embedding for radial functions

$$|x|^{\frac{n}{2}-1}|u(x)| \lesssim \|\partial u\|_{L^2(\mathbb{R}^n)}, \tag{1.2}$$

which is a well known special case of the family of inequalities

$$|x|^{\frac{n}{p}-\sigma}|u(x)| \lesssim \| |D|^{\sigma} u \|_{L^p_{|x|} L^r_{\omega}}, \quad \frac{n-1}{r} + \frac{1}{p} < \sigma < \frac{n}{p} \tag{1.3}$$

(see [3]). Here the norm  $L^p_{|x|} L^r_{\omega}$  is an  $L^p$  norm in the radial direction of the  $L^r_{\omega}$  norm in the angular direction. We give a precise statement of this blow-up alternative result including its short proof:

**Proposition 1.1** *Let  $u_0, u_1$  be spherically symmetric test functions, and  $T$  the supremum of times  $\tau > 0$  such that a smooth solution  $u(t, x)$  of (1.1) exists on  $[0, \tau] \times \mathbb{R}^n$ . Assume  $T < \infty$ . Then  $u$  blows up at  $x = 0$  as  $t \rightarrow T$ , in the sense that for any  $R > 0$ ,  $u$  is unbounded on the cylinder  $\{|x| < R, t < T\}$ .*

**Proof** By standard local existence results (e.g. [9], Chapter 5) we know that  $T > 0$ , and the solution is spherically symmetric for all  $t \in [0, T)$  by uniqueness of smooth solutions. Denote the energy of the solution  $u$  at time  $t$  by

$$E(t) := \frac{1}{2} \|\partial_{t,x} u(t)\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{p+1} \int |u(t)|^{p+1} dx.$$

By conservation of energy and the radial estimate (1.2) we have

$$|x|^{n-2}|u(t, x)|^2 \leq CE(t) = CE(0), \quad 0 \leq t < T.$$

Assume by contradiction that  $|u(t, x)| \leq M$  for  $(t, x)$  in some cylinder  $\{|x| < R, t < T\}$ , then by the previous bound we get

$$|u(t, x)| \leq M + (CE(0))^{1/2} R^{1-n/2} < \infty \quad \text{for } 0 \leq t < T, \quad x \in \mathbb{R}^n$$

and by standard arguments we can extend  $u$  to a strip  $[0, T'] \times \mathbb{R}^n$  for some  $T' > T$ , against the assumptions. □

In order to obtain a global radial solution, it would be sufficient to prevent blow-up at 0. This is the case if we replace the whole space with the exterior of a ball:

$$\Omega = \{x \in \mathbb{R}^n : |x| > 1\}$$

and consider the mixed problem on  $\mathbb{R}^+ \times \Omega$  with Dirichlet boundary conditions

$$\square u + |u|^{p-1}u = 0, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad u(t, \cdot)|_{\partial\Omega} = 0. \tag{1.4}$$

In this situation the local solution can not blow up, due to the conservation of  $H^1$  energy and the radial estimate (1.2). Note indeed that estimate (1.2) is valid also for functions in  $H_0^1(\Omega)$ , since it holds on the dense subspace  $C_c^\infty(\Omega)$ . This implies the existence of a global radial solution for arbitrary powers  $p > 1$ . Indeed, we have the following result, where we use the notation

$$C^k X = C^k(\mathbb{R}^+; X), \quad X = L^2(\mathbb{R}^n) \text{ or } H^s(\mathbb{R}^n) \text{ or } H_0^1(\Omega).$$

**Proposition 1.2** *Let  $\Omega = \mathbb{R}^n \setminus \overline{B(0, 1)}$ ,  $n \geq 2$ ,  $p > 1$  and let  $u_0 \in H_0^1(\Omega) \cap H^2(\Omega) \cap L^{p+1}(\Omega)$ ,  $u_1 \in H_0^1(\Omega)$  be two radially symmetric functions.*

*Then the mixed problem (1.4)*

*has a global solution  $u \in C^2L^2 \cap C^1H_0^1 \cap CH^2$ , satisfying the conservation of energy*

$$E(u(t)) = E(u(0)), \quad E(u(t)) := \frac{1}{2} \|\partial_{t,x} u(t)\|_{L^2(\Omega)}^2 + \frac{1}{p+1} \int_{\Omega} |u(t)|^{p+1} dx \tag{1.5}$$

and the uniform bound

$$\|u\|_{L^\infty(\mathbb{R}^+ \times \Omega)} \lesssim C(\|u_0\|_{H^1}, \|u_1\|_{L^2}). \tag{1.6}$$

*If  $v \in C^2L^2 \cap C^1H_0^1 \cap CH^2$  is a second solution of (1.4) with the same data which is either radially symmetric in  $x$ , or locally bounded on  $\mathbb{R}^+ \times \Omega$ , then  $v \equiv u$ .*

The existence of large radial solutions suggests naturally the question of stability: do nonradial perturbations of radial data give rise to global solutions?

The uniform bound (1.6) is not sufficient for a perturbation argument. However, we can prove an actual decay estimate, obtained by reduction to a mixed problem with moving boundary on  $\mathbb{S}^n$  via the Penrose transform. Define for  $M > 1$  the quantity

$$C_M(u_0, u_1) := \|u_0\|_{H^{N_0+1, N_0}(|x| \geq M)} + \|u_1\|_{H^{N_0, N_0+1}(|x| \geq M)}, \quad N_0 = \lfloor \frac{n}{2} \rfloor + 1. \tag{1.7}$$

Then we have

**Theorem 1.3** (Decay of the radial solution) *Let  $u$  be the solution constructed in Proposition 1.2. Assume that for some  $M > 1$  the data satisfy*

$$\|(u_0, u_1)\|_M := C_M(u_0, u_1) + \|u_0\|_{H^2} + \|u_1\|_{H^1} + \|\langle x \rangle^{\frac{n-1}{p+1}-1} u_0\|_{L^{p+1}} < \infty. \tag{1.8}$$

*Then the following decay estimate holds*

$$|u(t, x)| \leq C(\|(u_0, u_1)\|_M) \cdot |x|^{1-\frac{n}{2}} \langle t + |x| \rangle^{-\frac{1}{2}} \langle t - |x| \rangle^{-\frac{1}{2}}. \tag{1.9}$$

If the initial data are smoother then regularity propagates, and in addition higher Sobolev norms remain bounded as  $t \rightarrow \infty$ . In order to state the regularity result, we introduce briefly the *nonlinear compatibility conditions*, discussed in greater detail in Sect. 3 (see Definition 3.3 and also Definition 2.1). For the mixed problem on  $\mathbb{R}^+ \times \Omega$

$$u_{tt} = \Delta u + f(u), \quad u(0, x) = u_0, \quad u_t(0, x) = u_1, \quad u(t, \cdot)|_{\partial\Omega} = 0$$

we define formally the sequence of functions  $\psi_j$  as follows:

$$\psi_0 = u(0, x) = u_0, \quad \psi_1 = u_t(0, x) = u_1, \quad \psi_j = \partial_t^j u(0, x) = \Delta \psi_{j-2} + \partial_t^{j-2}(f(u))|_{t=0},$$

where in the definition of  $\psi_j$  we set recursively  $\partial_t^k u(0, x) = \psi_k(x)$  for  $0 \leq k \leq j - 2$ . Then we say that the data  $(u_0, u_1, f(s))$  satisfy the *nonlinear compatibility conditions* of order  $N \geq 1$  if

$$\psi_j \in H_0^1(\Omega) \quad \text{for } 0 \leq j \leq N. \tag{1.10}$$

Note that if  $f(s)$  vanishes at  $s = 0$  of sufficient order, to satisfy the nonlinear compatibility conditions it is sufficient to assume that the initial data  $u_0, u_1$  belong to  $H_0^k(\Omega)$  for  $k$  large enough.

We can now state the higher regularity result for the radial solutions. In the statement we use the norm  $\|v\|_{Y^{\infty,2;k}} = \sum_{|\alpha| \leq k} \|\partial_{t,x}^\alpha u\|_{L^2 L^\infty}$ .

**Theorem 1.4** *Let  $n \geq 3, p > N \geq 1, p > \frac{2n}{n-2}$ . Assume the radially symmetric data  $u_0 \in H^{N+1}(\Omega), u_1 \in H^N$  and  $f(u) = |u|^{p-1}u$  satisfy the nonlinear compatibility conditions of order  $N$  and condition (1.8).*

*Then the radial solution constructed in Proposition 1.2 belongs to  $C^k H^{N+1-k} \cap C^N H_0^1$  for all  $0 \leq k \leq N + 1$  and satisfies (besides (1.5)) the uniform bounds*

$$\begin{aligned} \|u\|_{L^\infty L^2} &\leq C(\|(u_0, u_1)\|_M), \\ \|\nabla_{t,x} u\|_{Y^{\infty,2;k}} &\leq C\left(\|(u_0, u_1)\|_M, \|u_0\|_{H^k}, \|u_1\|_{H^{k-1}}\right), \quad 1 \leq k < p - 1. \end{aligned} \tag{1.11}$$

The main result of the paper is the following:

**Theorem 1.5** *Let  $n \geq 3, N \geq \frac{5}{2}n, p > n + 3$ , and  $u_0, u_1$  radial functions satisfying the assumptions in Theorem 1.4. Then there exists  $\epsilon = \epsilon(u_0, u_1) > 0$  such that the following holds.*

*Assume  $v_0 \in H^{N+1}(\Omega), v_1 \in H^N(\Omega)$  and  $f(u) = |u|^{p-1}u$  satisfy the nonlinear compatibility conditions of order  $N$  and  $\|u_0 - v_0\|_{H^{N+1}} < \epsilon, \|u_1 - v_1\|_{H^N} < \epsilon$ . Then Problem (1.4) with data  $v_0, v_1$  has a global solution  $v \in C^k H^{N-k} \cap C^{N-1} H_0^1, 0 \leq k \leq N + 1$ .*

In other words, nonradial perturbations of large radial initial data in a high Sobolev norm do give rise to global smooth quasiradial solutions. Note that these solutions are unique in the class of locally bounded global solutions, but one can not in principle exclude the existence of other energy class solutions. However, one can prove a weak–strong uniqueness result, which implies in particular that the solution constructed in Theorem 1.5 is the unique energy class solution satisfying an energy inequality:

**Theorem 1.6** *Suppose all the assumptions in Theorem 1.5 are satisfied. Let  $I$  be an open interval containing  $[0, T]$  and  $\underline{v} \in C(I; H_0^1) \cap C^1(I; L^2) \cap L^\infty(I; L^{p+1}(\Omega))$  a distributional solution for  $0 \leq t \leq T$  to Problem (1.4) with the same initial data as  $v$ , which satisfies an energy inequality  $E(\underline{v}(t)) \leq E(\underline{v}(0))$  (see (1.5)). Then we have  $\underline{v}(t) = v(t)$  for  $0 \leq t \leq T$ .*

**Remark 1.1** Similar results can be proved for other dispersive equations, notably for the nonlinear Schrödinger equation. This will be the topic of further work in preparation.

The proof of Theorem 1.5 is based on perturbing the radial solution  $u(t, x)$  with a small term  $w(t, x)$  satisfying the equation

$$\square w + |u + w|^{p-1}(u + w) - |u|^{p-1}u = 0,$$

which can be written in the form

$$\square w + V(t, x)w = w^2 \cdot F[u, w], \quad V(t, x) = p|u|^{p-1}.$$

To solve the last equation, we prove an energy–Strichartz estimate for the exterior problem with potential

$$\square u + V(t, x)u = F, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad u(t, \cdot)|_{\partial\Omega} = 0$$

as an application of the results of [1,6,11]; note that this part of the argument holds also on arbitrary non trapping exterior domains. The Strichartz estimate for the wave equation with potential is proved by reduction to the constant coefficient case, which is possible since from the previous results we know that the potential  $V$  has rapid decay in  $(t, x)$ . Finally, Theorem 1.6 is an adaptation of a weak–strong uniqueness result due to M. Struwe [12].

The plan of the paper is the following. The linear theory and the perturbed energy–Strichartz estimates are developed in Sect. 2. In Sect. 3 the global radial solution is constructed and its regularity and decay properties are studied. In the final Sects. 4, 5 we prove the main results Theorems 1.5 and 1.6.

## 2 Linear theory

Denote by  $-\Delta_D$  the Dirichlet Laplacian, that is the nonnegative selfadjoint operator with domain  $H^2(\Omega) \cap H_0^1(\Omega)$ , and by  $\Lambda$  its nonnegative selfadjoint square root

$$\Lambda = (-\Delta_D)^{1/2}, \quad D(\Lambda) = H_0^1(\Omega).$$

We have  $\Lambda^{2k} = (-\Delta_D)^k$  for integer  $k \geq 0$ , and

$$\begin{aligned} D(\Lambda^{2k}) &= D(\Delta_D^k) = \{f \in H^{2k}(\Omega) : f, \Delta f, \dots, \Delta^{k-1} f \in H_0^1(\Omega)\}, \\ D(\Lambda^{2k+1}) &= \{f \in H^{2k+1}(\Omega) : f, \Delta f, \dots, \Delta^k f \in H_0^1(\Omega)\} \end{aligned}$$

so that

$$D(\Lambda^k) = \{f \in H^k(\Omega) : \Delta^j f \in H_0^1(\Omega), 0 \leq 2j \leq k - 1\}.$$

We shall make repeated use of the equivalence

$$\|\nabla_x g\|_{L^2(\Omega)} = \|\Lambda g\|_{L^2(\Omega)},$$

valid for  $g \in D(\Lambda) = H_0^1(\Omega)$ . The solution of the special mixed problem

$$\square u = 0, \quad u(0, x) = 0, \quad u_t(0, x) = u_1, \quad u(t, \cdot)|_{\partial\Omega} = 0$$

with  $u_1 \in H_0^1(\Omega)$  can be represented in the form

$$S(t)u_1 := \Lambda^{-1} \sin(t\Lambda)u_1 \tag{2.1}$$

Then the solution of the full *linear* mixed problem on  $\mathbb{R}^+ \times \Omega$

$$\square u = F(t, x) \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad u(t, \cdot)|_{\partial\Omega} = 0 \tag{2.2}$$

takes the form

$$u = S(t)u_1 + \partial_t S(t)u_0 + \int_0^t S(t-s)F(s)ds, \tag{2.3}$$

where

$$\partial_t S(t)u_0 = \cos(t\Lambda)u_0.$$

We shall be concerned with several global in time estimates of  $S(t)$  and of the solution to (2.2). We work for positive times  $t > 0$  only, but it is clear that all results are time-reversible. Directly from the spectral theory one gets

$$\|\nabla_x S(t)g\|_{L^2(\Omega)} \leq \|g\|_{L^2(\Omega)}, \quad \|\partial_t S(t)g\|_{L^2(\Omega)} \leq \|g\|_{L^2(\Omega)}, \tag{2.4}$$

$$\|S(t)g\|_{L^2(\Omega)} \leq t\|g\|_{L^2(\Omega)}, \quad \|S(t)g\|_{L^2(\Omega)} \leq \|\Lambda^{-1}g\|_{L^2(\Omega)} \lesssim \|g\|_{L^{\frac{2n}{n+2}}(\Omega)}. \tag{2.5}$$

As a consequence one gets the basic energy estimate for solutions of (2.2):

$$\|\nabla_{t,x} u\|_{L^\infty(0,T;L^2(\Omega))} \lesssim \|\nabla_x u_0\|_{L^2} + \|u_1\|_{L^2} + \|F\|_{L^1(0,T;L^2(\Omega))} \tag{2.6}$$

valid for all  $T > 0$ , with a constant independent of  $T$ .

Higher regularity results require compatibility conditions. Given the data  $(u_0, u_1, F)$  we define recursively the sequence of functions  $h_j$  as follows:

$$h_0 = u_0, \quad h_1 = u_1, \quad h_j = \partial_t^j u(0, x) = \Delta h_{j-2} + \partial_t^{j-2} F(0, x), \quad j \geq 2. \tag{2.7}$$

The function  $h_j$  is obtained by applying  $\partial_t^{j-2}$  to the equation  $u_{tt} = \Delta u + F$ .

**Definition 2.1** (*Linear compatibility conditions*) We say that the data  $(u_0, u_1, F)$  satisfy the *linear compatibility conditions of order*  $N \geq 1$  if  $(u_0, u_1) \in H^{N+1}(\Omega) \times H^N(\Omega)$ ,  $F \in C^k H^{N-k}(\Omega)$  for  $0 \leq k \leq N$ , and

$$h_j \in H_0^1(\Omega) \quad \text{for } 0 \leq j \leq N.$$

To formulate estimates of  $u$  in a compact format we introduce a few notations. We write for short for any interval  $I \subseteq \mathbb{R}$  and  $T \geq 0$

$$L_I^p L^q = L^p(I; L^q(\Omega)), \quad L_T^p L^q = L_{[0,T]}^p L^q, \quad L^p L^q = L_{[0,\infty)}^p L^q.$$

Moreover we denote the  $L^p L^q$  norm of all spacetime derivatives up to the order  $N$  by

$$\|u\|_{Y_I^{p,q;N}} = \sum_{j+|\alpha| \leq N} \|\partial_t^j \partial_x^\alpha u\|_{L^p(I; L^q(\Omega))}. \tag{2.8}$$

When  $I = [0, T]$  or  $I = [0, +\infty)$  we write also

$$Y_T^{q,r;N} = Y_{[0,T]}^{q,r;N} \quad Y^{q,r;N} = Y_{[0,\infty)}^{q,r;N}.$$

The following result is standard, and valid for general domains  $\Omega$  with sufficiently smooth compact boundary. We use the inequality  $\|\Lambda^{-1}g\|_{L^2} \lesssim \|g\|_{L^{\frac{2n}{n+2}}}$  in the formulation of (2.9).

**Proposition 2.2** *Let  $N \geq 1$ , and assume  $u_0 \in H^{N+1}(\Omega)$ ,  $u_1 \in H^N(\Omega)$  and  $F \in C^k H^{N-k}(\Omega)$  for  $0 \leq k \leq N$  satisfy the linear compatibility conditions of order  $N$ . Then Problem (2.2) has a unique solution belonging to*

$$u \in C^k H^{N+1-k} \cap C^N H_0^1 \quad \text{for all } 0 \leq k \leq N + 1.$$

The solution satisfies for all  $T > 0$  the energy estimates

$$\|u\|_{L_T^\infty L^2} \lesssim \|u_0\|_{L^2} + \|u_1\|_{L^{\frac{2n}{n+2}}} + \|F\|_{L_T^1 L^{\frac{2n}{n+2}}}, \tag{2.9}$$

$$\|\nabla_{t,x} u\|_{Y_T^{\infty,2;N}} \lesssim \|u_0\|_{H^{N+1}} + \|u_1\|_{H^N} + \|F\|_{Y_T^{1,2;N}} \tag{2.10}$$

with a constant independent of  $T$ .

We next recall Strichartz estimates for the exterior problem, following [1,6,11]. These estimates are valid on the exterior of any strictly convex obstacle with smooth boundary in  $\mathbb{R}^n$ ,  $n \geq 2$ . With our notations one has

$$\|S(t)g\|_{L^q L^r} \lesssim \|g\|_{\dot{H}^s}, \quad s = \frac{1}{2} - \frac{1}{r} + \frac{1}{q} \tag{2.11}$$

provided  $n \geq 3$  and

$$\frac{2}{q} + \frac{n-1}{r} = \frac{n-1}{2}, \quad 2 < q \leq \infty, \quad 2 \leq r < \frac{2(n-1)}{n-3}. \tag{2.12}$$

A couple  $(q, r)$  as in (2.12) is called *admissible*; note that the endpoint  $(q, r) = (2, \frac{2(n-1)}{n-3})$  is not included and it is not known if the estimate holds also in this case.

One can further extend the range of indices by combining (2.11) with Sobolev embedding. We shall focus on the following special case:

$$\|S(t)g\|_{L^q L^r} \lesssim \|g\|_{\dot{H}^1} \tag{2.13}$$

valid for  $n \geq 3$  and for couples of indices of the form

$$q = \frac{2}{\delta}, \quad r = \frac{2n}{n-2-\delta}, \quad 0 \leq \delta < 1. \tag{2.14}$$

Then we have:

**Proposition 2.3** *The solution  $u$  to (2.2) satisfies, for any interval  $I$  containing 0, the Sobolev–Strichartz estimate*

$$\|u\|_{L^q L^r} \lesssim \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2} + \|F\|_{L^1 L^2} \tag{2.15}$$

provided  $n \geq 3$  and  $(q, r)$  satisfy (2.14).

**Proof** The proof is a standard application of the Christ–Kiselev Lemma to the representation (2.3) of the solution. □

Differentiating (2.2) with respect to  $t$  and applying the usual recursive procedure one gets, more generally, the following higher order estimates;

**Proposition 2.4** *Assume the data  $(u_0, u_1, F)$  of (2.2) satisfy the compatibility conditions of order  $N \geq 1$ . Then, for all  $(q, r)$  as in (2.14) and for any interval  $I$  of length  $|I| \gtrsim 1$  containing 0, the solution satisfies the estimates*

$$\|u\|_{Y^{q,r;N}} \lesssim \|u_0\|_{H^{N+1}} + \|u_1\|_{H^N} + \|F\|_{Y^{1,2;N}}. \tag{2.16}$$

**Proof** We give a sketch of the proof. Applying  $\partial_t$  to the equation, by (2.15) we get

$$\|u_t\|_{L^q L^r} \lesssim \|u_0\|_{\dot{H}^2} + \|u_1\|_{\dot{H}^1} + \|F(0, \cdot)\|_{L^2} + \|F_t\|_{L^1 L^2}$$

since  $u_{tt}(0) = \Delta u_0 + F(0, x)$  from the equation. We note that  $\|F(0, \cdot)\|_{L^2} \lesssim \|F\|_{Y^{1,2;1}}$  provided the interval has length  $|I| \gtrsim 1$ , and this gives the estimate for  $u_t$ . In a similar way one can estimate all derivatives  $\partial_t^j u$ . We next estimate  $\Delta u = u_{tt} - F$ :

$$\|\Delta u\|_{L^q L^r} \leq \|u_{tt}\|_{L^q L^r} + \|F\|_{L^q L^r}.$$

The  $u_{tt}$  term has already been estimated. As to the second term, we note that

$$\|F\|_{Y^{1,2;2}} \gtrsim \|F\|_{L^{\infty} H^1} \gtrsim \|F\|_{L^{\infty} L^{\frac{2n}{n-2}}}$$



which is the endpoint  $\delta = 0$  in (2.14). Moreover,

$$\|F\|_{Y_t^{1,2;2}} \gtrsim \|F\|_{L_t^1 H^2} + \|F\|_{L_t^\infty H^1}$$

and by interpolation and Sobolev embedding

$$\|F\|_{Y_t^{1,2;2}} \gtrsim \|F\|_{L_t^2 H^{3/2}} \gtrsim \|F\|_{L_t^2 L^{\frac{2n}{n-3}}}$$

which is the other endpoint  $\delta = 1$  in (2.14). This argument can be modified in the case  $n = 3$  by using the Sobolev embedding into *BMO* instead of  $L^\infty$ . Again by interpolation we get

$$\|F\|_{Y_t^{1,2;2}} \gtrsim \|F\|_{L_t^q L^r}$$

for all  $(q, r)$  as in (2.14), and we conclude

$$\|\Delta u\|_{L_t^q L^r} \lesssim \|u_0\|_{H^3} + \|u_1\|_{H^2} + \|F\|_{Y^{1,2;2}}.$$

Also by interpolation this covers the case  $N = 1$  of (2.16). For larger values of  $N$  one proceeds in a similar way by recursion, using the embedding  $Y^{1,2;N} \hookrightarrow Y^{q,r;m-2}$  just proved for all  $(q, r)$  as in (2.14).  $\square$

We shall also need estimates for the exterior wave equation with a time dependent potential. We denote by  $u = S_V(t; t_0)g$  the solution of the mixed problem with initial data at time  $t = t_0$

$$\square u + V(t, x)u = 0, \quad u(t_0, x) = 0, \quad u_t(t_0, x) = g, \quad u(t, \cdot)|_{\partial\Omega} = 0$$

and by  $u = S'_V(t; t_0)f$  (which is not  $\partial_t S_V g$ ) the solution of

$$\square u + V(t, x)u = 0, \quad u(t_0, x) = f, \quad u_t(t_0, x) = 0, \quad u(t, \cdot)|_{\partial\Omega} = 0.$$

If  $V$  is a sufficiently smooth potential with good behaviour at infinity, the existence and uniqueness of a solution is standard. The solution of the full problem

$$\square u + V(t, x)u = F(t, x), \quad u(t_0, x) = u_0, \quad u_t(t_0, x) = u_1, \quad u(t, \cdot)|_{\partial\Omega} = 0. \tag{2.17}$$

can be represented by Duhamel as

$$u = S'_V(t; t_0)u_0 + S_V(t; t_0)u_1 + \int_{t_0}^t S_V(t; s)F(s)ds. \tag{2.18}$$

Note also that it is not necessary to modify the compatibility conditions, provided the potential  $V$  has a sufficient regularity. Indeed the correct condition would require

$$h_j = \partial_t^j u(0, x) = \Delta h_{j-2} + \sum_{\ell=0}^{j-2} \partial_t^{j-2-\ell} V(0, x)h_\ell + \partial_t^{j-2} F(0, x) \in H_0^1$$

but the term  $\partial_t^{j-2-\ell} V(0, x)h_\ell$  is already in  $H_0^1$  since  $h_\ell \in H_0^1$ , and can be omitted. By Duhamel we can write  $S_V(t; s)$ ,  $S'_V(t, s)$  as perturbations of  $S(t)$ ,  $\partial_t S(t)$ :

$$S_V(t; t_0) = S(t - t_0) - \int_{t_0}^t S(t - s)V(s, x)S_V(s; t_0)ds, \tag{2.19}$$

$$S'_V(t; t_0) = \partial_t S(t - t_0) - \int_{t_0}^t S(t - s)V(s, x)S'_V(s; t_0)ds. \tag{2.20}$$

**Proposition 2.5** (Perturbed energy–Strichartz estimate) *Let  $n \geq 3$ ,  $m \geq 1$ . Assume the data  $(u_0, u_1, F)$  satisfy the compatibility conditions of order  $m$ , and that*

$$\|V\|_{Y^{1,n;m}} < \infty. \tag{2.21}$$

Then for any interval  $I$  containing  $t_0$  the solution of Problem (2.17) satisfies

$$\|u\|_{L_I^\infty L^2} \lesssim \|u_0\|_{L^2} + \|u_1\|_{L^{\frac{2n}{n+2}}} + \|F\|_{L_I^1 L^{\frac{2n}{n+2}}}, \tag{2.22}$$

$$\|\nabla_{t,x} u\|_{Y_I^{\infty,2;m}} + \|u\|_{Y_I^{q,r;m}} \lesssim \|u_0\|_{H^{m+1}} + \|u_1\|_{H^m} + \|F\|_{Y_I^{1,2;m}} \tag{2.23}$$

provided the couple  $(q, r)$  is of the form (2.14).

**Proof** We can assume  $I = I(t) = [t_0, t]$ ; the proof for  $t < t_0$  is identical. Since  $u$  solves  $\square u = F - Vu$ , by (2.9) we get

$$\|u(t)\|_{L^2} \lesssim \|u_0\|_{L^2} + \|u_1\|_{L^{\frac{2n}{n+2}}} + \|F\|_{L_I^1 L^{\frac{2n}{n+2}}} + \int_{t_0}^t \|Vu\|_{L^{\frac{2n}{n+2}}} ds.$$

Noting that

$$\int_{t_0}^t \|Vu\|_{L^{\frac{2n}{n+2}}} ds \leq \int_{t_0}^t \|V(s)\|_{L^n} \|u(s)\|_{L^2} ds$$

and  $a(s) = \|V(s)\|_{L^n}$  is integrable by (2.21), by Gronwall’s Lemma we deduce (2.22).

In a similar way, by (2.10) and (2.16) we can write

$$\|\nabla_{t,x} u\|_{Y_{I(t)}^{\infty,2;m}} + \|u\|_{Y_{I(t)}^{q,r;m}} \lesssim \|u_0\|_{H^{m+1}} + \|u_1\|_{H^m} + \|F\|_{Y_{I(t)}^{1,2;m}} + \|Vu\|_{Y_{I(t)}^{1,2;m}}.$$

We have

$$\|Vu\|_{Y_{I(t)}^{1,2;m}} \lesssim \int_{t_0}^t b(s) \|u\|_{Y_{I(s)}^{\infty, \frac{2n}{n-2}; m}} ds \lesssim \int_{t_0}^t b(s) \|\nabla_x u\|_{Y_{I(s)}^{\infty, 2; m}} ds$$

where

$$b(s) = \sum_{j+|\alpha| \leq m} \|\partial_t^j \partial_x^\alpha V(s)\|_{L^n}$$

is integrable on  $\mathbb{R}$  by (2.21). Using again Gronwall’s inequality we obtain (2.23).  $\square$

### 3 The global radial solution

This section is devoted to the proof of Proposition 1.2 and Theorems 1.3, 1.4. We begin with a few preliminary results on the mixed problem

$$\square u + f(u) = 0, \quad u(0, x) = u_0, \quad u_t(0, x) = u_1, \quad u(t, \cdot)|_{\partial\Omega} = 0. \quad (3.1)$$

For data of low regularity, a solution to (3.1) is intended to be a solution of the integral equation

$$u = S(t)u_1 + \partial_t S(t)u_0 - \int_0^t S(t-s)f(u(s))ds \quad (3.2)$$

with  $S(t)$  as in (2.1). We will give only sketchy proofs of standard results, which are virtually identical to the corresponding ones for semilinear wave equations on  $\mathbb{R}^n$  (for which we refer e.g. to Chapter 6 of [9]).

**Lemma 3.1** *Assume  $f : \mathbb{R} \rightarrow \mathbb{R}$  is (globally) Lipschitz with  $f(0) = 0$ . Then for any initial data  $(u_0, u_1) \in H_0^1 \times L^2(\Omega)$  Problem (3.1) has a unique solution  $u(t, x)$  in  $C H_0^1(\Omega) \cap C^1 L^2(\Omega)$ . The solution satisfies the energy bound for all  $t > 0$*

$$\|u(t, \cdot)\|_{H^1(\Omega)} + \|\partial_t u(t, \cdot)\|_{L^2(\Omega)} \leq C(f, t) \left[ \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} \right]. \quad (3.3)$$

**Proof** Apply a contraction argument in the space  $C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$  to (3.2), with  $T > 0$  sufficiently small, using the energy estimates (2.4), (2.5). The lifespan  $T$  depends only on the  $H^1 \times L^2$  norm of the data, thus we can iterate to a global solution. Estimate (3.3) is a byproduct of the proof.  $\square$

**Lemma 3.2** *Consider the solution  $u$  constructed in Lemma 3.1. Assume in addition that  $f \in C^2$ ,  $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ ,  $u_1 \in H_0^1(\Omega)$ . Then  $u$  belongs to  $C^2 L^2(\Omega) \cap C^1 H_0^1(\Omega) \cap C H^2(\Omega)$  and solves (3.1) in both distributional and a.e. sense.*

*If we further assume that  $0 \leq f(s)s \lesssim F(s)$  for  $s \in \mathbb{R}$ , where  $F(s) = \int_0^s f(\sigma)d\sigma$ , then the solution satisfies for all times the energy identity*

$$E(t) = E(0), \quad E(t) := \int_{\Omega} \left[ \frac{1}{2} |\nabla_x u|^2 + \frac{1}{2} |u_t|^2 + F(u) \right] dx. \quad (3.4)$$

**Proof** Differentiate the equation once w.r.to time and call formally  $v = u_t$ . Then  $v$  must solve

$$v_{tt} - \Delta v + f'(u)v = 0$$

where we may regard  $f'(u)$  as an  $L^\infty$  coefficient. As data we take  $v(0) = u_t \in H_0^1(\Omega)$  and  $v_t(0) = u_{tt}(0) = \Delta u(0) - f'(u(0))v(0) \in L^2(\Omega)$ . By the previous result, this problem has a global solution  $v(t, x)$ , satisfying an energy estimate like  $u$ . Since  $u(0) + \int_0^t v$  solves the original equation (3.1), by uniqueness for (3.1) we conclude that  $v = u_t$  and hence  $u$  has the claimed regularity. The energy conservation follows by direct computation.  $\square$

Note that the assumption  $0 \leq sf(s) \lesssim F(s)$  is sufficient to prove the existence of a global weak solution for data in  $H_0^1(\Omega) \times L^2(\Omega)$ , even if  $f$  is not Lipschitz (Segal's

Theorem). This can be proved like in the case of the whole space  $\mathbb{R}^n$  by approximating  $f$  with a sequence of truncated Lipschitz functions and using weak compactness. The weak solution thus constructed satisfies then a weaker energy inequality  $E(t) \leq E(0)$  (proved using Fatou’s Lemma). We shall not need this variant in the sequel.

For smoother data, one can prove a local existence theorem which does not require a global Lipschitz condition, similarly to the case  $\Omega = \mathbb{R}^n$ . However, one must assume suitable compatibility conditions, analogous to the linear ones from Definition 2.1. Define formally a sequence of functions  $\psi_j, j \geq 0$  as follows: differentiating the equation  $u_{tt} = \Delta u + f(u)$  with respect to time, set

$$\psi_0 = u(0, x) = u_0, \quad \psi_1 = u_t(0, x) = u_1, \quad \psi_j = \partial_t^j u(0, x) = \Delta \psi_{j-2} + \partial_t^{j-2}(f(u))|_{t=0},$$

where values of  $\partial_t^k u(0, x)$  for  $0 \leq k \leq j - 2$ , required to compute  $\psi_j$ , are set recursively equal to  $\psi_k$ . For instance,

$$\begin{aligned} \psi_2 &= \Delta u(0, x) + f(u(0, x)) = \Delta \psi_0 + f(\psi_0), \\ \psi_3 &= \Delta \psi_1 + f(u)u_t = \Delta \psi_1 + f'(\psi_0)\psi_1, \\ \psi_4 &= \Delta \psi_2 + f''(u)u_t^2 + f'(u)u_{tt} = \Delta \psi_2 + f''(\psi_0)\psi_1^2 + f'(\psi_0)\psi_2 \end{aligned}$$

and so on. Then we have:

**Definition 3.3** (Nonlinear compatibility conditions) We say that the data  $(u_0, u_1, f)$  satisfy the *nonlinear compatibility conditions of order  $N \geq 1$*  if  $(u_0, u_1) \in H^{N+1}(\Omega) \times H^N(\Omega), f \in C^{N-2}(\mathbb{R}; \mathbb{R})$  and we have

$$\psi_j \in H_0^1(\Omega) \quad \text{for } 0 \leq j \leq N. \tag{3.5}$$

**Remark 3.1** Conditions (3.5) are implied by a number of simpler assumptions on the data. For instance, if  $f \in C^{N-2}$  and one assumes

$$u_0 \in H_0^{\lfloor \frac{n}{2} \rfloor + N + 1}, \quad u_1 \in H_0^{\lfloor \frac{n}{2} \rfloor + N} \tag{3.6}$$

then one checks easily that  $(u_0, u_1, f)$  satisfy the nonlinear compatibility conditions of order  $N$ .

**Lemma 3.4** (Local existence) *Let  $N > \frac{n}{2}, (u_0, u_1) \in H^{N+1}(\Omega) \times H^N(\Omega), f \in C^N$  and assume  $(u_0, u_1, f)$  satisfy the nonlinear compatibility conditions of order  $N$ . Then there exists  $0 < T \leq +\infty$ , depending only on the  $H^{N+1} \times H^N$  norm of  $(u_0, u_1)$ , such that Problem (3.1) has a unique solution  $u \in C^k([0, T]; H^{N+1-k}(\Omega)), 0 \leq k \leq N+1$ . The solution belongs to  $C^N([0, T]; H_0^1(\Omega))$ .*

*Moreover, if  $T^*$  is the maximal time of existence of such a solution, then either  $T^* = +\infty$  or  $\|u(t, \cdot)\|_{L^\infty} \rightarrow \infty$  as  $t \uparrow T^*$ .*

**Proof** The existence part is completely standard; it is usually proved for more general quasilinear equations, which require higher smoothness of the data; see e.g. Theorem 3.5 in [10], where local existence is proved for a nonlinear term of the form

$f(t, x, \partial_t^j \partial_x^\alpha u)$  with  $j + |\alpha| \leq 2, j \leq 1$  (and a regularity of order  $\lfloor \frac{n}{2} \rfloor + 8$  is imposed on the data). The proof is based on a contraction mapping argument, combined with Moser type estimates of the nonlinear term. The final blow up alternative in the statement is a byproduct of the proof.  $\square$

### 3.1 Proof of Proposition 1.2

Fix  $M > 0$  and define  $f_M(s) = \min\{|s|, M\}^{p-1}s$ . Then  $f_M$  is Lipschitz and the problem

$$\square u + f_M(u) = 0, \quad u(0, x) = u_0, \quad u_t(0, x) = u_1, \quad u(t, \cdot)|_{\partial\Omega} = 0 \quad (3.7)$$

has a global, unique, radially symmetric solution  $u \in C^2L^2(\Omega) \cap C^1H_0^1(\Omega) \cap CH^2(\Omega)$  satisfying the bound (3.4) with  $F = F_M = \int_0^s f_M$ . Combining (3.4) with (1.2) we get

$$|x|^{\frac{n}{2}-1}|u(t, x)| \leq C_0K, \quad K := \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2} + \|u_0\|_{L^{p+1}}^{\frac{p+1}{2}} \quad (3.8)$$

for some universal constant  $C_0$ . Since  $|x| \geq 1$  on  $\Omega$ , this gives

$$|u(t, x)| \leq C_0K$$

and if we choose  $M = C_0K + 1$  we see that  $f_M(u(t, x)) = f(u(t, x))$ , i.e.,  $u(t, x)$  is a global solution of the untruncated problem

$$\square u + |u|^{p-1}u = 0, \quad u(0, x) = u_0, \quad u_t(0, x) = u_1, \quad u(t, \cdot)|_{\partial\Omega} = 0. \quad (3.9)$$

The same argument guarantees also uniqueness of radially symmetric solutions. More generally, two solutions in  $H^2 \cap L^\infty$  with the same data must coincide, as an elementary argument based on classical energy estimates shows. Since energy estimates are valid also on domains of dependence, by finite speed of propagation, the same argument shows uniqueness of solutions in  $H^2 \cap L_{loc}^\infty$ . In particular, a locally bounded solution with radial data must coincide with the radial solution. The proof of Proposition 1.2 is concluded.

### 3.2 Proof of Theorem 1.3

We now prove the decay estimate (1.9), using the Penrose transform. We recall its definition. Describe the sphere  $\mathbb{S}^n$  using coordinates  $(\alpha, \theta)$  with  $\alpha \in (0, \pi)$  and  $\theta \in \mathbb{S}^{n-1}$ , as  $\mathbb{S}_{\alpha, \theta}^n = (0, \pi)_\alpha \times \mathbb{S}_\theta^{n-1}$ . Denoting with  $d\theta_{\mathbb{S}^{n-1}}^2$  the metric of  $\mathbb{S}_\theta^{n-1}$ , the metric on  $\mathbb{S}^n$  can then be written as

$$d\alpha^2 + (\sin \alpha)^2 d\theta_{\mathbb{S}^{n-1}}^2.$$

Similarly, on  $\mathbb{R}_x^n = \mathbb{R}_r^+ \times \mathbb{S}_\theta^{n-1}$ , use polar coordinates  $(r, \theta)$  with  $r \in (0, +\infty)$  and  $\theta \in \mathbb{S}^{n-1}$ , so that the euclidean metric can be written  $dr^2 + r^2 d\theta_{\mathbb{S}^{n-1}}^2$ . Then we can

define the Penrose map  $\Pi : \mathbb{R}_t \times \mathbb{R}_x \rightarrow \mathbb{R}_T \times \mathbb{S}^n$  as

$$\Pi : (t, r, \theta) \mapsto (T, \alpha, \theta)$$

where

$$T = \arctan(t + r) + \arctan(t - r), \quad \alpha = \arctan(t + r) - \arctan(t - r).$$

The map  $(t, r) \mapsto (T, \alpha)$  takes the quadrant

$$\{(t, r) : t \geq 0, r \geq 0\}$$

to the triangle

$$\{(T, \alpha) : T \geq 0, 0 \leq \alpha < \pi - T\}$$

so that  $\Pi$  maps  $\mathbb{R}^+ \times \mathbb{R}^n$  to the positive half of the Einstein diamond

$$\mathbb{E}^+ = \{(T, \alpha, \theta) : T \geq 0, 0 \leq \alpha < \pi - T, \theta \in \mathbb{S}^{n-1}\}$$

The boundary  $|x| = 1$  i.e.  $r = 1$  is mapped by  $\Pi$  to a curve described parametrically by

$$(T, \alpha) = (\arctan(t + 1) + \arctan(t - 1), \arctan(t + 1) - \arctan(t - 1)) \quad (3.10)$$

for  $t \geq 0$ . We denote this curve by

$$\alpha = \Gamma(T), \quad T \in [0, \pi) \quad \text{or} \quad T = \gamma(\alpha), \quad \alpha \in [0, \pi)$$

(i.e.,  $\gamma = \Gamma^{-1}$ ). One has the explicit (but not particularly useful) formulas

$$\Gamma(T) = \frac{\pi}{4} + \arcsin(2^{-\frac{1}{2}} \cos T), \quad \gamma(\alpha) = \arccos(\sin \alpha - \cos \alpha).$$

It is easy to check that

$$\gamma'(\alpha) < -1. \quad (3.11)$$

We denote by  $\omega$  the conformal factor

$$\omega = \cos T + \cos \alpha = \frac{2}{\langle t + r \rangle \langle t - r \rangle}, \quad \langle s \rangle = (1 + s^2)^{1/2}.$$

Note that  $\omega > 0$  on  $\mathbb{E}^+$ . The inverse of  $\Pi$  (defined on  $\mathbb{E}^+$ ) can be written

$$(t, r, \theta) = \Pi^{-1}(T, \alpha, \theta) = (\omega^{-1} \sin T, \omega^{-1} \sin \alpha, \theta).$$

We define a new function  $U(T, \alpha, \theta)$  via

$$u(t, r\theta) = \omega^{\frac{n-1}{2}} U \circ \Pi(t, r, \theta). \quad (3.12)$$

Since  $u, U$  are independent of  $\theta$ , we shall write simply  $U(T, \alpha)$ . Commuting  $\square$  with  $\Pi$  gives

$$\square u = \square_{\mathbb{R}_t \times \mathbb{R}^n} (\omega^{\frac{n-1}{2}} U \circ \Pi) = \omega^{\frac{n+3}{2}} (\square_{\mathbb{R}_T \times \mathbb{S}_{\alpha, \theta}^n} U + \frac{(n-1)^2}{4} U) \circ \Pi$$

so that  $U$  is a solution of the equation

$$\square_{\mathbb{R} \times \mathbb{S}^n} U + \frac{(n-1)^2}{4} U + \omega^v |U|^{p-1} U = 0, \quad v = \frac{n-1}{2} p - \frac{n+3}{2} \tag{3.13}$$

on the subset  $\mathbb{E}_\Omega$  of  $\mathbb{R} \times \mathbb{S}^n$  given by the conditions

$$\mathbb{E}_\Omega := \{(T, \alpha) : 0 \leq T < \pi, \Gamma(T) \leq \alpha < \pi - T\}$$

which is the image of  $\mathbb{R}_t^+ \times \Omega$  via  $\Pi$ . We introduce also the notation

$$D_T = \{(\alpha, \theta) \in \mathbb{S}^n : \Gamma(T) \leq \alpha < \pi - T\}, \quad 0 \leq T < \pi$$

for the slice of  $\Pi(\mathbb{R}^+ \times \Omega)$  at time  $T$ . Note that in coordinates, Eq. (3.13) reads

$$\partial_T^2 U - \partial_\alpha^2 U - (n-1) \frac{\cos \alpha}{\sin \alpha} \partial_\alpha U + \frac{(n-1)^2}{4} U + \omega^v |U|^{p-1} U = 0.$$

We plan to extend the solution beyond the line  $T + \alpha = \pi$ , i.e., in the region where  $\omega < 0$ . Thus we consider the following extended equation on  $(T, \alpha) \in [0, \pi]^2$ :

$$\partial_T^2 U - \partial_\alpha^2 U - (n-1) \frac{\cos \alpha}{\sin \alpha} \partial_\alpha U + \frac{(n-1)^2}{4} U + \tilde{\omega}^v |U|^{p-1} U = 0 \tag{3.14}$$

where we have replaced  $\omega$  by

$$\tilde{\omega} := \begin{cases} \omega & \text{if } T + \alpha \leq \pi, \\ 0 & \text{if } T + \alpha > \pi. \end{cases}$$

The solution  $U$  satisfies the identity

$$\begin{aligned} \partial_T \left\{ (\sin \alpha)^{n-1} \left( \frac{|\partial_T U|^2 + |\partial_\alpha U|^2}{2} + \frac{\tilde{\omega}^v}{p+1} |U|^{p+1} + \frac{(n-1)^2}{8} |U|^2 \right) \right\} = \\ = \partial_\alpha \left\{ (\sin \alpha)^{n-1} \partial_\alpha U \partial_T U \right\} - \frac{v \tilde{\omega}^{v-1}}{p+1} (\sin \alpha)^{n-1} (\sin T) |U|^{p+1}. \end{aligned} \tag{3.15}$$

We can now extend  $U$  to a larger domain in the cylinder  $\mathbb{R}_T \times \mathbb{S}^n$ . Recall that the data  $u_0, u_1$  satisfy  $C_M(u_0, u_1) < \infty$  with  $C_M(u_0, u_1)$  as in (1.7). Thus if we fix a smooth cutoff function  $\chi(x)$  equal to 0 for  $|x| \leq M+1$  and equal to 1 for  $|x| \geq M+2$ , we have

$$\|\chi u_0\|_{H^{N_0+1, N_0}} + \|\chi u_1\|_{H^{N_0, N_0+1}} < \infty.$$

Denote by  $\tilde{U}_0, \tilde{U}_1$  the functions obtained by applying the transformation (3.12) to  $\chi u_0, \chi u_1$  respectively (with  $t = 0$ ). By the first Lemma in Section 4 of [2] we have then

$$\|\tilde{U}_0\|_{H^{N_0+1}(\mathbb{S}^n)} + \|\tilde{U}_1\|_{H^{N_0}(\mathbb{S}^n)} < \infty.$$

In order to solve (3.13) locally via the energy method we require that the coefficient  $\tilde{\omega}^\nu$  be sufficiently smooth i.e.  $\tilde{\omega}^\nu \in C^{N_0}$ . This is true as soon as

$$\nu = \frac{n-1}{2}p - \frac{n+3}{2} > N_0 \quad \text{which is implied by } p > \frac{11}{2}.$$

Then a standard local existence result guarantees the existence of a local solution  $\tilde{U}$  to Eq. (3.13) with data  $\tilde{U}_0, \tilde{U}_1$  on some strip  $[0, \delta) \times \mathbb{S}^n$ . The lifespan  $\delta$ , which can be assumed  $\ll 1$ , depends only on

$$\delta = \delta(C_M(u_0, u_1), n, p) \tag{3.16}$$

where  $C_M(u_0, u_1)$  was defined in (1.7). Comparing  $\tilde{U}$  with the solution  $U$  constructed above, and noting that Eq. (3.13) has finite speed of propagation equal to 1, by local uniqueness we see that  $U, \tilde{U}$  must coincide on the forward dependence domain emanating from the set

$$T = 0, \quad 2 \arctan(M + 2) < \alpha < \pi.$$

Thus we can glue the two solutions, at least for  $0 < T < \delta/2$ , and we obtain an extended solution of (3.13), which we denote again  $U(T, \alpha)$ , defined on the larger domain

$$\mathbb{E}^+ \cup ([0, \frac{\delta}{2}) \times \mathbb{S}^n).$$

We next prove that the energy of  $U$ , defined as

$$E(T) = \int_{\Gamma(T)} (\sin \alpha)^{n-1} \left( \frac{|\partial_T U|^2 + |\partial_\alpha U|^2}{2} + \frac{\tilde{\omega}^\nu}{p+1} |U|^{p+1} + \frac{(n-1)^2}{8} |U|^2 \right) d\alpha$$

remains bounded for  $0 \leq T < \delta/2$ . To this end we integrate identity (3.15) on a slice

$$T_1 < T < T_2, \quad \Gamma(T) < \alpha < \pi,$$

for arbitrary times  $0 \leq T_1 \leq T_2 < \delta/2$ . Dropping negative terms at the RHS, we are left with the inequality

$$E(T_2) - E(T_1) \leq (\sin \alpha)^{n-1} \left[ \nu_\alpha \partial_\alpha U \partial_T U - \frac{1}{2} \nu_T (|\partial_T U|^2 + |\partial_\alpha U|^2) \right]_{T=\gamma(\alpha)} \tag{3.17}$$

where  $\nu_\alpha, \nu_T$  are the components of the exterior normal to the curve  $T = \gamma(\alpha)$ , i.e.,

$$\nu_T = -\frac{1}{\sqrt{1 + \gamma'(\alpha)^2}}, \quad \nu_\alpha = \frac{\gamma'(\alpha)}{\sqrt{1 + \gamma'(\alpha)^2}}.$$



The Dirichlet condition  $U(\gamma(\alpha), \alpha) = 0$  along the curve implies

$$(\partial_\alpha U + \gamma'(\alpha)\partial_T U)|_{T=\gamma(\alpha)} = 0.$$

Thus the RHS of (3.17) is equal to

$$= (\sin \alpha)^{n-1} \frac{1 - \gamma'(\alpha)^2}{2\sqrt{1 + \gamma'(\alpha)^2}} |\partial_T U|^2.$$

Recalling (3.11), we see that the RHS of (3.17) is negative, and we conclude that the energy is nonincreasing as claimed:

$$E(T) \leq E(0) \quad \text{for all } 0 \leq T < \delta/2. \tag{3.18}$$

Now, consider the set, which is a dependence domain for (3.13) (keep in mind that the speed of propagation for (3.13) is exactly 1):

$$\mathbb{D} = \{(T, \alpha) : \frac{\delta}{3} \leq T < \pi, \Gamma(T) < \alpha < \pi - T + \frac{\delta}{4}\}.$$

We have already extended  $U$  to the part of  $\mathbb{D}$  in the time strip  $\delta/3 < T < \delta/2$ , and our next goal is to prove that  $U$  can be extended to a bounded solution of (3.13) on the whole set  $\mathbb{D}$ . Clearly, it is sufficient to prove an a priori  $L^\infty$  bound of the solution on this domain in order to achieve the result via a continuation argument.

To this end we prove an energy estimate similar to the previous one, but now we integrate identity (3.15) over the slice

$$T_1 < T < T_2, \quad \Gamma(T) < \alpha < \pi - T + \delta/4,$$

where  $\delta/3 \leq T_1 < T_2 < \pi$  are fixed, and we denote by  $F(T)$  the energy

$$F(T) := \int_{\Gamma(T)}^{\pi-T+\delta/4} (\sin \alpha)^{n-1} \left( \frac{|\partial_T U|^2 + |\partial_\alpha U|^2}{2} + \frac{\tilde{\omega}^v}{p+1} |U|^{p+1} + \frac{(n-1)^2}{8} |U|^2 \right) d\alpha.$$

After integration of (3.15), the terms on the side  $\alpha = \pi - T + \delta/4$  give a negative contribution at the RHS which can be dropped, as in the standard energy estimate, since the speed of propagation is 1. Proceeding as before, we are left with the inequality

$$F(T_2) - F(T_1) \leq (\sin \alpha)^{n-1} \left[ v_\alpha \partial_\alpha U \partial_T U - \frac{1}{2} v_T (|\partial_T U|^2 + |\partial_\alpha U|^2) \right]_{T=\gamma(\alpha)}$$

and again the RHS here is negative thanks to (3.11). We conclude that the energy  $F(T)$  is nonincreasing:

$$F(T) \leq F(\delta/3) \quad \text{for all } \delta/3 \leq T < \pi.$$

Since  $F(\delta/3) \leq E(\delta/3)$ , by (3.18) we conclude

$$F(T) \leq E(0) \quad \text{for all } \delta/3 \leq T < \pi. \tag{3.19}$$

To proceed, we need a Lemma:

**Lemma 3.5** *Let  $I \subseteq (0, +\infty)$  be a bounded interval and  $n \geq 3$ . Then for any  $V \in H^1(I)$  we have*

$$\sup_{s \in I} s^{\frac{n}{2}-1} |V(s)| \lesssim \left( \int_I s^{n-1} |V'(s)|^2 ds \right)^{\frac{1}{2}} + |I|^{-1} \left( \int_I s^{n-1} |V(s)|^2 ds \right)^{\frac{1}{2}} \quad (3.20)$$

with a constant independent of  $I$  and  $V$ .

**Proof of the Lemma** Pick any points  $\alpha, \beta \in I$  with  $\beta \geq \alpha/3$ . We first prove that

$$\alpha^{\frac{n}{2}-1} |V(\alpha)| \lesssim \left( \int_I s^{n-1} |V'|^2 \right)^{\frac{1}{2}} + \beta^{\frac{n}{2}-1} |V(\beta)|. \quad (3.21)$$

We have two cases: either  $\alpha \leq \beta$  or  $\alpha \geq \beta \geq \alpha/3$ . If  $\alpha \leq \beta$ , we write

$$|V(\alpha)| \leq \int_{\alpha}^{\beta} |V'| + |V(\beta)| \leq \left( \int_{\alpha}^{\beta} s^{n-1} |V'|^2 \right)^{\frac{1}{2}} \left( \int_{\alpha}^{\beta} s^{1-n} \right)^{\frac{1}{2}} + |V(\beta)|.$$

Using the inequality  $\int_{\alpha}^{\beta} s^{1-n} ds \leq \alpha^{2-n}$  and recalling that  $\alpha \leq \beta$  we obtain (3.21). If  $\alpha \geq \beta \geq \alpha/3$ , we have in a similar way

$$|V(\alpha)| \leq \int_{\beta}^{\alpha} |V'| + |V(\beta)| \leq \left( \int_{\beta}^{\alpha} s^{n-1} |V'|^2 \right)^{\frac{1}{2}} \left( \int_{\beta}^{\alpha} s^{1-n} \right)^{\frac{1}{2}} + |V(\beta)|.$$

Now  $\int_{\beta}^{\alpha} s^{1-n} ds \leq \beta^{2-n}$  and we get

$$\beta^{\frac{n}{2}-1} |V(\alpha)| \leq \left( \int_I s^{n-1} |V'|^2 \right)^{\frac{1}{2}} + \beta^{\frac{n}{2}-1} |V(\beta)|$$

and recalling that  $\beta \geq \alpha/3$  we obtain again (3.21).

Next, split  $I$  in thirds  $I = I_1 \cup I_2 \cup I_3$  (with  $I_1$  at the left and  $I_3$  at the right). If  $\alpha \in I_1 \cup I_2$ , pick  $\beta \in I_3$  arbitrary and apply (3.21) to get

$$\alpha^{\frac{n}{2}-1} |V(\alpha)| \lesssim \left( \int_I s^{n-1} |V'|^2 \right)^{\frac{1}{2}} + \inf_{\beta \in I_3} \beta^{\frac{n}{2}-1} |V(\beta)|.$$

Since

$$\left( \int_{I_3} s^{n-1} |V'|^2 \right)^{\frac{1}{2}} \geq \inf_{I_3} \beta^{\frac{n}{2}-1} |V(\beta)| \cdot \left( \int_{I_3} s ds \right)^{\frac{1}{2}} \gtrsim |I| \inf_{I_3} \beta^{\frac{n}{2}-1} |V(\beta)|$$

(indeed  $\int_a^b s ds = \frac{b^2-a^2}{2} \geq \frac{(b-a)^2}{2}$ ) our claim (3.20) is proved for the points in  $I_1 \cup I_2$ . On the other hand, if  $\alpha \in I_3$ , we pick  $\beta \in I_2$  arbitrary, we apply (3.21), and we get

$$\alpha^{\frac{n}{2}-1} |V(\alpha)| \lesssim \left( \int_I s^{n-1} |V'|^2 \right)^{\frac{1}{2}} + \inf_{\beta \in I_2} \beta^{\frac{n}{2}-1} |V(\beta)|$$

and the same argument gives again (3.20). □

We apply (3.20) to  $U(T, \alpha)$  at a fixed  $T \in (\delta/3, \pi)$  on the interval

$$I = [\Gamma(T), \pi - T + \delta/4] \subseteq (0, K) = (0, \pi - \frac{\delta}{12}];$$

note that  $|I| \geq \delta/4$ . We get

$$\sup_I \alpha^{\frac{n}{2}-1} |U(T, \alpha)| \lesssim \delta^{-1} \left[ \int_{\Gamma(T)}^{\pi-T+\delta/4} \alpha^{n-1} (|\partial_\alpha U|^2 + |U|^2) d\alpha \right]^{\frac{1}{2}}.$$

For  $\alpha \in [0, \pi - \frac{\delta}{12}]$  we have

$$\sin \alpha \geq \frac{\sin(\pi - \delta/12)}{\pi - \delta/12} \cdot \alpha \implies \alpha \lesssim \delta^{-1} \sin \alpha,$$

thus substituting in the previous inequality we get, for all  $\alpha \in [\Gamma(T), \pi - T + \delta/4]$ ,

$$(\sin \alpha)^{\frac{n}{2}-1} |U(T, \alpha)| \lesssim \delta^{-\frac{n+1}{2}} \left[ \int_{\Gamma(T)}^{\pi-T+\delta/4} (\sin \alpha)^{n-1} (|\partial_\alpha U|^2 + |U|^2) d\alpha \right]^{\frac{1}{2}}.$$

Recalling the definition of  $F(T)$  and (3.19) we conclude

$$(\sin \alpha)^{\frac{n}{2}-1} |U(T, \alpha)| \lesssim \delta^{-\frac{n+1}{2}} E(0)^{1/2}. \tag{3.22}$$

We now convert (3.22) into an estimate for  $u(t, x)$ . Since  $\sin \alpha = \omega r$ , we obtain

$$(\sin \alpha)^{\frac{n}{2}-1} |U(T, \alpha)| \gtrsim (\omega r)^{\frac{n}{2}-1} \omega^{\frac{1-n}{2}} |u(t, x)| = r^{\frac{n}{2}-1} \langle t+r \rangle^{\frac{1}{2}} \langle t-r \rangle^{\frac{1}{2}} |u(t, x)|.$$

On the other hand, a change of variable shows that

$$\|U(0, \cdot)\|_{L^2(\tilde{\mathbb{S}}^n)} = \sqrt{2} \|\langle r \rangle^{-1} u_0\|_{L^2(\Omega)}$$

where  $\tilde{\mathbb{S}}^n$  denotes the image of  $\{t = 0\} \times \Omega$  via the Penrose transform,

$$\|U(0, \cdot)\|_{L^{p+1}(\tilde{\mathbb{S}}^n)} = \sqrt{2}^{1-n} \frac{p-1}{p+1} \|\langle r \rangle^{n \frac{p-1}{p+1}-1} u_0\|_{L^p(\Omega)},$$

$$\|\partial_T U(0, \cdot)\|_{L^2(\tilde{\mathbb{S}}^n)} = \sqrt{2}^{-1} \|\langle r \rangle u_1\|_{L^2(\Omega)}$$

and

$$\|\partial_\alpha U(0, \cdot)\|_{L^2(\tilde{\mathbb{S}}^n)} \lesssim \|\langle r \rangle^{-1} u_0\|_{L^2(\Omega)} + \|\langle r \rangle \nabla_x u_0\|_{L^2(\Omega)}.$$

This gives the estimate

$$E(0)^{1/2} \lesssim \|\langle x \rangle^{-1} u_0\|_{L^2} + \|\langle x \rangle^{n \frac{p-1}{p+1}-1} u_0\|_{L^{\frac{p+1}{2}}} + \|\langle x \rangle (|\nabla u_0| + |u_1|)\|_{L^2} \tag{3.23}$$

and recalling (3.22) (and the dependence of  $\delta$  in (3.16)), we conclude the proof of Theorem 1.3.

### 3.3 Proof of Theorem 1.4

Let  $u$  be the solution given by Proposition 1.2 and Theorem 1.3, and  $v$  be the local solution given by Lemma 3.4, maximally extended to a lifespan  $[0, T^*)$ . Since the data are radial,  $v$  is also radial, and by the uniqueness part of Proposition 1.2 we see that  $u \equiv v$ . In particular,  $v$  is bounded as  $t \uparrow T^*$ , hence  $T^* = +\infty$ . This proves the regularity of the radial solution.

It remains to prove the uniform bounds (1.11). For the  $L^2$  norm we have

$$\|u(t, \cdot)\|_{L^2} \leq \|u_0\|_{L^2} + \|\Lambda^{-1}u_1\|_{L^2} + \int_0^t \| |u|^p \|_{L^{\frac{2n}{n+2}}} ds.$$

Then we can write, if  $n \geq 3$ ,

$$\| |u|^p \|_{L^{\frac{2n}{n+2}}} \leq \|u\|_{L^\infty}^{p-\frac{n+2}{n-2}} \|u\|_{L^{\frac{2n}{n-2}}}^{\frac{n+2}{n-2}} \lesssim \|u\|_{L^\infty}^{p-\frac{n+2}{n-2}} \|\nabla_x u\|_{L^2(\Omega)}^{\frac{n+2}{n-2}}$$

by Hölder and Sobolev embedding, and we note the consequence of (1.9) (also valid only if  $n \geq 3$ )

$$\|u\|_{L^\infty} \lesssim C(\|(u_0, u_1)\|_M) \langle t \rangle^{-1} \tag{3.24}$$

and the conservation of energy (1.5). Summing up, we obtain

$$\|u(t, \cdot)\|_{L^2} \lesssim \|u_0\|_{L^2} + \|u_1\|_{L^{\frac{2n}{n+2}}} + \|(u_0, u_1)\|_M^{p-\frac{n+2}{n-2}} E(0)^{\frac{n+2}{2(n-2)}} \int_0^t \langle s \rangle^{-p+\frac{n+2}{n-2}} ds$$

where the integral converges since  $p > 1 + \frac{n+2}{n-2} = \frac{2n}{n-2}$ . Noting that

$$E(0) \leq C(\|(u_0, u_1)\|_M), \quad \|u_1\|_{L^{\frac{2n}{n+2}}} \lesssim \langle x \rangle u_1 \|_{L^2} \lesssim \|(u_0, u_1)\|_M,$$

we get the uniform bound

$$\|u(t, \cdot)\|_{L^2} \leq C(\|(u_0, u_1)\|_M). \tag{3.25}$$

Estimate (1.11) for  $k = 1$  is a consequence of the energy conservation  $E(t) = E(0)$  and of (3.25). For  $k > 1$  we proceed by induction on  $k$ . By the energy estimate (2.10) we have

$$\|\nabla_{t,x} u\|_{Y_T^{\infty,2;k}} \lesssim \|u_0\|_{H^{k+1}} + \|u_1\|_{H^k} + \| |u|^p \|_{Y_T^{1,2;k}}.$$

We apply Gagliardo–Nirenberg estimates to estimate the last term. Handling separately the highest order terms  $\sim \|u\|_{L^\infty}^{p-1} \sum_{j \leq k} \|\partial_t^j u\|_{H^{k-j}}$ , we get

$$\begin{aligned} \| |u|^p \|_{Y_T^{1,2;k}} &= \sum_{j+|\alpha| \leq k} \int_0^T \|\partial_t^j \partial_x^\alpha |u|^p\|_{L^2} dt \\ &\lesssim \int_0^T (\|u\|_{L^\infty}^{p-1} + \|u\|_{L^\infty}^{p-k}) \left( 1 + \sum_{\ell \leq k-1} \|\partial_t^\ell u\|_{L^\infty} \right)^k dt \cdot \|u\|_{Y_T^{\infty,2;k}} \end{aligned}$$

provided  $p > k + 1$ . Since  $u$  is radial in  $x$  we have

$$\sum_{\ell \leq k-1} \|\partial_t^\ell u\|_{L^\infty} \lesssim \|u\|_{Y_T^{\infty,2;k}}$$

which is bounded by the induction hypothesis; on the other hand  $\|u\|_{L^\infty}^{p-k}$  is integrable since  $p > k + 1$  by (3.24). Thus the right hand side is finite and this proves (1.11).

### 4 Global quasiradial solutions

This section is devoted to the proof of Theorem 1.5.

Let  $u$  be the global radial solution with initial data  $(u_0, u_1)$  given by Proposition 1.2 and Theorem 1.4, and let  $v$  be the local solution with data  $(v_0, v_1)$  given by Lemma 3.4. Denote by  $w = v - u$  the difference of the two solutions, which satisfies the equation

$$\square w + |u + w|^{p-1}(u + w) - |u|^{p-1}u = 0. \tag{4.1}$$

We can write

$$|u + w|^{p-1}(u + w) - |u|^{p-1}u = p|u|^{p-1}w - w^2 \cdot F[u, w]$$

where

$$F[u, w] = -p(p - 1) \int_0^1 |u + \sigma w|^{p-3}(u + \sigma w)(1 - \sigma) d\sigma \tag{4.2}$$

so that  $w$  satisfies

$$\square w + V(t, x)w = w^2 \cdot F[u, w], \quad V(t, x) = p|u|^{p-1}.$$

For  $j \leq p - 3$  we can write

$$\|\partial_t^j V\|_{W^{n,1}} \lesssim \sum_{\mu=0}^j \|u\|_{L^\infty}^{p-3-\mu} \|u\|_{L^2}^2 \|\partial_t^{j_1} u\|_{W^{n,\infty}} \dots \|\partial_t^{j_\mu} u\|_{W^{n,\infty}} \tag{4.3}$$

where  $j_1 + \dots + j_\mu = j$ . By Sobolev embedding and energy estimates (1.11) we have

$$\|\partial_t^{j_\mu} u\|_{W^{n,\infty}} \lesssim \|u\|_{Y^{\infty,2;N}} \leq C \left( \|(u_0, u_1)\|_M, \|u_0\|_{H^N}, \|u_1\|_{H^{N-1}} \right)$$

provided  $N > n + \frac{n}{2} + j$ , and this implies, recalling (3.24),

$$\|\partial_t^j V(t)\|_{W^{n,1}} \lesssim \langle t \rangle^{j+3-p} \tag{4.4}$$

provided

$$\|(u_0, u_1)\|_M + \|u_0\|_{H^N} + \|u_1\|_{H^{N-1}} < \infty, \quad N > \frac{3}{2}n + j. \tag{4.5}$$

Now let  $m \geq 1$  to be chosen and assume  $u_0, u_1$  satisfy (4.5) with

$$N > m + \frac{3}{2}n$$

while  $w_0, w_1$  satisfy the compatibility conditions of order  $m$ . We see that if  $p > m + 4$  the assumptions of Proposition 2.5 are satisfied and the estimates (2.22), (2.23) are valid.

Consider now the problem

$$\square w + V(t, x)w = w^2 \cdot F[u, w], \quad w(0, x) = w_0, \quad w_t(0, x) = w_1, \quad w(t, \cdot)|_{\partial\Omega} = 0 \tag{4.6}$$

where  $w_j = v_j - u_j, j = 0, 1$ , and  $V(t, x) = p|u|^{p-1}$ . Define the spacetime norm

$$M(T) = \|w\|_{Y_T^{\infty,2;m+1}} + \|w\|_{Y_T^{q,r;m}}$$

where the couple  $(q, r)$ , satisfying (2.14), will be made precise below. Our goal is to prove the a priori estimate

$$M(T) \lesssim \epsilon + M(T)^2 + M(T)^p \tag{4.7}$$

where  $\epsilon$  is a suitable norm of the data  $(w_0, w_1)$ .

Applying (2.22), (2.23) to (4.6) we get

$$M(T) \lesssim \|w_0\|_{H^{m+1}} + \|w_1\|_{H^m \cap L^{\frac{2n}{n+2}}} + \|w^2 F[u, w]\|_{Y_T^{1,2;m} \cap L^1_L \frac{2n}{n+2}} \tag{4.8}$$

We must estimate the last term. We begin by noticing that

$$\|w^2 F[u, w]\|_{L^{\frac{2n}{n+2}}} \lesssim \|w\|_{L^{\frac{4n}{n+2}}}^2 \left( \|u\|_{L^\infty}^{p-2} + \|w\|_{L^\infty}^{p-2} \right)$$

and if we assume  $m + 1 > \frac{n}{2}$  so that  $\|w\|_{L^\infty} \lesssim \|w\|_{H^{m+1}}$ , we get

$$\|w(t)\|_{L^{\frac{4n}{n+2}}}^2 \leq \|w\|_{L^2}^{\frac{n+2}{n}} \|w\|_{L^\infty}^{\frac{n-2}{n}} \lesssim \|w\|_{H^{m+1}}^2 \leq M(T)^2$$

for  $0 \leq t \leq T$ . If we assume in addition  $m > \frac{n}{r}$  and use the embedding  $\|w\|_{L^\infty} \lesssim \|w\|_{W^{m,r}}$ , we can write

$$\|w\|_{L^\infty}^{p-2} \leq \|w\|_{W^{m,r}}^q \|w\|_{H^{m+1}}^{p-2-q} \leq \|w\|_{W^{m,r}}^q M(T)^{p-q-2}$$

provided  $p > q + 2$ . Since  $|u(t, \cdot)| \lesssim \langle t \rangle^{-1}$  (recall (3.24)), we obtain

$$\|w^2 F[u, w]\|_{L^{\frac{2n}{n+2}}} \lesssim M(T)^2 \langle t \rangle^{2-p} + M(T)^{p-q} \|w\|_{W^{m,r}}^q$$

and integrating in time we conclude

$$\|w^2 F[u, w]\|_{L^1_T L^{\frac{2n}{n+2}}} ds \lesssim M(T)^2 + M(T)^p. \tag{4.9}$$

We next estimate

$$\|w^2 F[u, w]\|_{Y_T^{1,2;m}} = \sum_{j+|\alpha|\leq m} \|\partial_t^j \partial_x^\alpha (w^2 F[u, w])\|_{L^1_T L^2}.$$

Recalling the expression of  $F$  in (4.2), we may expand the derivative as a finite sum

$$\partial_t^j \partial_x^\alpha (w^2 F[u, w]) = \sum \int_0^1 G \cdot W_1 W_2 U_1 \cdots U_\nu d\sigma$$

where  $0 \leq \nu \leq m$  and, assuming  $p \geq m + 2$ ,

- $W_1 = \partial_t^{h_1} \partial_x^{\alpha_1} w$  and  $W_2 = \partial_t^{h_2} \partial_x^{\alpha_2} w$
- $U_k = \partial_t^{j_k} \partial_x^{\beta_k} (u + \sigma w)$
- $h_1 + h_2 + j_1 + \cdots + j_\nu = j$ ,  $\alpha_1 + \alpha_2 + \beta_1 + \cdots + \beta_\nu = \alpha$ , and  $j + |\alpha| \leq m$
- it is not restrictive to assume that  $j_\nu + |\beta_\nu| \geq j_k + |\beta_k|$  for all  $k$  and that  $h_2 + |\alpha_2| \geq h_1 + |\alpha_1|$
- $G$  satisfies  $|G| \lesssim |u + \sigma w|^{p-\nu-2}$  so that  $\|G\|_{L^\infty} \lesssim (\|u\|_{L^\infty} + \|w\|_{L^\infty})^{p-\nu-2}$ .

We take the  $L^2$  norm in  $x$  of each product, and we estimate it with the  $L^2$  norm of the derivative of highest order, times the  $L^\infty$  norm of the remaining factors; it may happen that the highest order derivative falls on  $w$  or on  $u + \sigma w$ . Thus we need to distinguish two cases:

(1) The highest order derivative is on  $u + \sigma w$ , that is to say  $j_\nu + |\beta_\nu| \geq h_2 + |\alpha_2|$ . Then we estimate

$$\|G W_1 W_2 U_1 \dots U_\nu\|_{L^2} \leq \|G\|_{L^\infty} \|W_1\|_{L^\infty} \|W_2\|_{L^\infty} \|U_1\|_{L^\infty} \dots \|U_{\nu-1}\|_{L^\infty} \|U_\nu\|_{L^2}.$$

We have for  $0 \leq t \leq T$

$$\|U_\nu\|_{L^2} \lesssim M(T).$$

Moreover for  $i < \nu$  we have  $j_i + |\alpha_i| \leq \frac{m}{2}$ , hence if we assume  $m > n - 2$  we have by Sobolev embedding

$$\|U_i\|_{L^\infty} \lesssim \|u\|_{Y_T^{\infty,2;m+1}} + \|w\|_{Y_T^{\infty,2;m+1}} \leq C_{m+1} + M(T),$$

where in the last step we used the energy estimates (1.11) for  $u$ , with

$$C_{m+1} = C(\|(u_0, u_1)\|_M, \|u_0\|_{H^{m+1}}, \|u_1\|_{H^m}).$$

In a similar way

$$\|W_i\|_{L^\infty} \lesssim M(T), \quad i = 1, 2$$

Finally, by (3.24) and Sobolev embedding

$$\|G\|_{L^\infty} \lesssim (\|u\|_{L^\infty} + \|w\|_{L^\infty})^{p-\nu-2} \lesssim (\langle t \rangle^{-1} + \|w\|_{W^{m,r}})^q (1 + M(T))^{p-\nu-q-2}$$

provided

$$m > \frac{n}{r} \quad \text{and} \quad p \geq m + q + 2.$$

Summing up we have, for  $m > \max\{n - 2, n/r\}$ ,  $p \geq m + q + 2$ ,  $0 \leq t \leq T$

$$\begin{aligned} & \|GW_1W_2U_1 \dots U_\nu\|_{L^2} \\ & \lesssim (\langle t \rangle^{-1} + \|w\|_{W^{m,r}})^q (1 + M(T))^{p-\nu-q-2} M(T)^2 (C_{m+1} + M(T))^\nu. \end{aligned}$$

We now integrate in  $t$  on  $[0, T]$  to get

$$\begin{aligned} \|GW_1W_2U_1 \dots U_\nu\|_{L^1_T L^2} & \lesssim M(T)^2 (1 + M(T))^{p-q-2} \int_0^T (\langle t \rangle^{-q} + \|w\|_{W^{m,r}}^q) dt \\ & \lesssim M(T)^2 (1 + M(T))^{p-2} \end{aligned} \tag{4.10}$$

with an implicit constant depending on  $\|(u_0, u_1)\|_M, \|u_0\|_{H^{m+1}}, \|u_1\|_{H^m}$ .

(2) The highest order derivative is on  $w$ , that is to say  $j_\nu + |\beta_\nu| \leq h_2 + |\alpha_2|$ . In this case we estimate

$$\|GW_1W_2U_1 \dots U_\nu\|_{L^2} \leq \|G\|_{L^\infty} \|W_1\|_{L^\infty} \|W_2\|_{L^2} \|U_1\|_{L^\infty} \dots \|U_{\nu-1}\|_{L^\infty} \|U_\nu\|_{L^\infty}.$$

Proceeding in a similar way, we obtain again (4.10).

Summing up, recalling (4.8), we have proved

$$M(T) \lesssim \|w_0\|_{H^{m+1}} + \|w_1\|_{H^m \cap L^{\frac{2n}{n+2}}} + M(T)^2 + M(T)^p. \tag{4.11}$$

The implicit constant depends on  $\|(u_0, u_1)\|_M + \|u_0\|_{H^{m+1}} + \|u_1\|_{H^m}$ . The conditions on the parameters are

$$p \geq m + q + 2, \quad m > \frac{n}{r}, \quad m > n - 2$$

and  $(q, r)$  satisfy (2.14), that is to say

$$\delta \in (0, 1), \quad p \geq m + \frac{2}{\delta} + 2, \quad m > n - 2, \quad m > \frac{n}{2} - 1 - \frac{\delta}{2}.$$

All the constraints are satisfied if

$$m = n - 1, \quad p > n + 3 \tag{4.12}$$

with  $\delta < 1$  chosen accordingly. Recall also that in order to apply Proposition 2.5 we assumed  $p > m + 4 = n + 3$  and that  $u_0, u_1$  are in  $H^N \times H^{N-1}$  with compatibility conditions of order  $N$ , where  $N > m + \frac{3}{2}n = \frac{5}{2}n - 1$ .

To conclude the proof it is then sufficient to apply a standard continuation argument; if the norm

$$\|w_0\|_{H^{m+1}} + \|w_1\|_{H^m \cap L^{\frac{2n}{n+2}}}$$



of the initial data is sufficiently small with respect to the hidden constant

$$\|(u_0, u_1)\|_M + \|u_0\|_{H^{m+1}} + \|u_1\|_{H^m}$$

then the quantity  $M(T)$  must remain finite as  $T \rightarrow +\infty$  and global existence is proved.

### 5 Weak–strong uniqueness

Following [12], we prove a more general stability result which implies Theorem 1.6 as a special case. We use the notation

$$E(u) = E(u(t)) = \int_{\Omega} \left( \frac{|\partial_t u|^2 + |\nabla_x u|^2}{2} + \frac{|u|^{p+1}}{p+1} \right) dx, \quad \Omega = \{|x| > 1\}$$

for the energy of a solution  $u(t, x)$  at time  $t$  of the Cauchy problem

$$\square u + |u|^{p-1}u = 0, \quad u(t, \cdot)|_{\partial\Omega} = 0. \tag{5.1}$$

**Theorem 5.1** *Let  $I$  be an open interval containing  $[0, T]$ ,  $T > 0$ . Let  $u, v$  be two distributional solutions to (5.1) on  $I \times \Omega$  such that*

$$\begin{aligned} u &\in C(I; H^2(\Omega)) \cap C^1(I; H_0^1(\Omega)) \cap C^2(I; L^2(\Omega)), & u, u_t &\in L^\infty(I \times \Omega), \\ v &\in C(I; H_0^1(\Omega)) \cap C^1(I; L^2(\Omega)) \cap L^\infty(I; L^{p+1}(\Omega)). \end{aligned}$$

*Assume in addition that  $v$  satisfies an energy inequality*

$$E(v(t)) \leq E(v(0)).$$

*Then the difference  $w = v - u$  satisfies the energy estimate*

$$E(w(t)) \leq C e^{Ct} (E(w(0)) + \|w(0)\|_{L^2(\Omega)}^2), \quad t \in [0, T]$$

*where  $C$  is a constant depending on*

$$C = C(p, T, \| |u| + |u_t| \|_{L^\infty([0, T] \times \Omega)}). \tag{5.2}$$

**Proof** We shall need the following easy estimate of the  $L^2$  norm of  $w = v - u$ :

$$\|w(t)\|_{L^2}^2 \leq 2\| \int_0^t w_t ds \|_{L^2}^2 + 2\|w(0)\|_{L^2}^2 \leq 2T \int_0^t E(w(s)) ds + 2\|w(0)\|_{L^2}^2. \tag{5.3}$$

The difference  $w = v - u$  satisfies in the sense of distributions

$$\square w + |u + v|^{p-1}(u + w) - |u|^{p-1}u = 0.$$

We expand

$$E(v) = E(u) + A(t) + B(t)$$

where

$$A(t) = \int \frac{|\partial_t w|^2 + |\nabla_x w|^2}{2} dx + \int \left( \frac{|u + w|^{p+1} - |u|^{p+1}}{p + 1} - |u|^{p-1}uw \right) dx$$

$$B(t) = \int (u_t w_t + \nabla u \cdot \nabla w + |u|^{p-1}uw) dx.$$

From the energy inequality for  $v$ , and the conservation of energy for the smooth solution  $u$ , we have

$$0 \leq E(v(0)) - E(v(t)) = A(0) - A(t) + B(0) - B(t). \tag{5.4}$$

We get easily

$$\begin{aligned} \frac{|u+w|^{p+1}-|u|^{p+1}}{p+1} - |u|^{p-1}uw &= p \int_0^1 \int_0^\sigma |u + \tau w|^{p-1} d\tau d\sigma |w|^2 \\ &\geq \frac{2^{2-p}}{p+1} |w|^{p+1} - \frac{p}{2} |u|^{p-1} |w|^2 \end{aligned}$$

which implies

$$A(t) \geq 2^{-p} E(w(t)) - C \|w\|_{L^2}^2, \quad C = \frac{p}{2} \|u\|_{L^\infty((0,T] \times \Omega)}^{p-1}.$$

Recalling (5.3), this gives

$$A(t) \geq 2^{-p} E(w(t)) - C \int_0^t E(w(s)) ds - C \|w(0)\|_{L^2}^2$$

for some  $C = C(T, \|u\|_{L^\infty((0,T] \times \Omega)})$ . On the other hand

$$\frac{|u+w|^{p+1}-|u|^{p+1}}{p+1} - |u|^{p-1}uw \leq C(p, \|u\|_{L^\infty}) (|w|^{p+1} + |w|^2)$$

which implies

$$A(0) \leq CE(w(0)) + C \|w(0)\|_{L^2}^2$$

and in conclusion

$$A(0) - A(t) \leq -2^{-p} E(w(t)) + C \int_0^t E(w(s)) ds + C \|w(0)\|_{L^2}^2, \tag{5.5}$$

with  $C = C(p, T, \|u\|_{L^\infty})$ .

In order to estimate  $B(t)$ , we need the following remark. Assume the function

$$W(t, x) \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \tag{5.6}$$

satisfies in  $\mathcal{D}'((0, T) \times \Omega)$ , for some  $q \in (1, \infty)$ ,

$$\square W = F(t, x), \quad F \in L^q((0, T) \times \Omega). \tag{5.7}$$

Then for all  $\chi(t) \in C_c^\infty((0, T))$  and  $U(t, x) \in C_c^\infty(\mathbb{R} \times \Omega)$  we have the identity

$$\iint_{\Omega} \chi'(t)(U_t W_t + \nabla U \cdot \nabla W) dx dt = - \iint_{\Omega} \chi(t)(W_t \square U + U_t F) dx dt. \quad (5.8)$$

Now assume

$$U \in C(I; H^2(\Omega)) \cap C^1(I; H_0^1(\Omega)) \cap C^2(I; L^2(\Omega)), \quad U, U_t \in L^\infty(I \times \Omega) \quad (5.9)$$

on an open interval  $I \supset [0, T]$ , so that  $\square U \in C(I; L^2(\Omega))$  and  $U_t \in L^q(I \times \Omega)$ . We can approximate  $U$  with a sequence  $U_\epsilon \in C_c^\infty(I \times \Omega)$  in such a way that

$$\partial_t U_\epsilon \rightarrow U_t, \quad \nabla U_\epsilon \rightarrow \nabla U, \quad \square U_\epsilon \rightarrow \square U \quad \text{in } L^2((0, T) \times \Omega)$$

and

$$\partial_t U_\epsilon \rightarrow U_t \quad \text{in } L^q((0, T) \times \Omega)$$

as  $\epsilon \rightarrow 0$  (e.g., extend  $U$  as 0 on  $\mathbb{R}^n \setminus \Omega$ , apply a radial change of variables  $\tilde{u}(t, x) = u(t, (1-\epsilon)x)$ , truncate with a smooth cutoff  $\psi(|x|-\epsilon^{-1})$  where  $\psi = 1$  for  $|x| \leq 1$  and  $\psi = 0$  for  $|x| \geq 2$ , and finally convolve with a delta sequence of the form  $\rho_\epsilon(x)\sigma_\epsilon(t)$ ). Applying (5.8) to  $U_\epsilon$  and letting  $\epsilon \rightarrow 0$  we obtain that (5.8) holds for any functions  $U, W$  satisfying (5.6), (5.7), (5.9) and any  $\chi \in C_c^\infty((0, T))$ .

Now, consider a sequence of test functions  $\chi_k(t) \in C_c^\infty((0, T))$ , non negative, such that  $\chi_k \uparrow \mathbf{1}_{[0,t]}$  pointwise,  $t \in (0, T]$ . We can write

$$B(t) = B(0) + \lim \iint \chi_k'(t)(u_t w_t + \nabla u \cdot \nabla w + |u|^{p-1} u w) dx dt.$$

Applying formula (5.8) with  $U = u, W = w, F = |u|^{p-1} u - |u + v|^{p-1}(u + w)$  (with  $q = (p + 1)/p$ ) and using the equations for  $u, w$  we obtain the identity

$$\begin{aligned} B(t) - B(0) &= \lim \iint \chi_k(s)(|u|^{p-1} u w_t - F u_t) - \lim \iint \chi_k(s) \partial_t (|u|^{p-1} u w) \\ &= \lim \iint \chi_k(s)(|u + v|^{p-1}(u + w) - |u|^{p-1} u - p|u|^{p-1} w) u_t dx ds. \end{aligned}$$

We have

$$\begin{aligned} &|u + v|^{p-1}(u + w) - |u|^{p-1} u - p|u|^{p-1} w \\ &= p(p - 1) \int_0^1 \int_0^\sigma |u + \tau w|^{p-2}(u + \tau w) d\tau d\sigma |w|^2 \end{aligned}$$

which implies

$$\left| |u + v|^{p-1}(u + w) - |u|^{p-1} u - p|u|^{p-1} w \right| \leq C(p, \|u\|_{L^\infty})(|w|^{p+1} + |w|^2).$$

Thus we get

$$B(0) - B(t) \leq C \int_0^t [E(w(s)) + \|w(s)\|_{L^2(\Omega)}^2] ds, \quad C = C(p, \| |u| + |u_t| \|_{L^\infty([0, T] \times \Omega)}).$$

Using (5.3) we conclude

$$B(0) - B(t) \leq C \int_0^t E(w(s)) ds + C \|w(0)\|_{L^2}^2$$

for some  $C$  as in (5.2). Plugging (5.5), (5.2) in (5.4) we arrive at

$$E(w(t)) \leq C \int_0^t E(w(s)) ds + CE(w(0)) + C \|w(0)\|_{L^2}^2$$

with  $C$  as in (5.2), and by Gronwall's Lemma we conclude the proof.  $\square$

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## References

1. Burq, N.: Global Strichartz estimates for nontrapping geometries: about an article by H. F. Smith and C. D. Sogge. *Commun. Partial Differ. Equ.* **9–10**, 1675–1683 (2003)
2. Christodoulou, D.: Global solutions of nonlinear hyperbolic equations for small initial data. *Commun. Pure Appl. Math.* **39**(2), 267–282 (1986)
3. D'Ancona, P., Lucà, R.: Stein–Weiss and Caffarelli–Kohn–Nirenberg inequalities with angular integrability. *J. Math. Anal. Appl.* **388**(2), 1061–1079 (2012)
4. Ginibre, J., Velo, G.: Regularity of solutions of critical and subcritical nonlinear wave equations. *Nonlinear Anal.* **22**(1), 1–19 (1994)
5. Kapitanski, L.: Global and unique weak solutions of nonlinear wave equations. *Math. Res. Lett.* **1**(2), 211–223 (1994)
6. Metcalfe, J.L.: Global Strichartz estimates for solutions to the wave equation exterior to a convex obstacle. *Trans. Am. Math. Soc.* **12**, 4839–4855 (2004). (electronic)
7. Shatah, J., Struwe, M.: Regularity results for nonlinear wave equations. *Ann. Math. (2)* **138**(3), 503–518 (1993)
8. Shatah, J., Struwe, M.: Well-posedness in the energy space for semilinear wave equations with critical growth. *Int. Math. Res. Not.*, (7):303ff., approx. 7 pp. (electronic), (1994)
9. Shatah, J., Struwe, M.: *Geometric Wave Equations*. Courant Lecture Notes in Mathematics, vol. 2. New York University Courant Institute of Mathematical Sciences, New York (1998)
10. Shibata, Y., Tsutsumi, Y.: On a global existence theorem of small amplitude solutions for nonlinear wave equations in an exterior domain. *Math. Z.* **191**(2), 165–199 (1986)
11. Smith, H.F., Sogge, C.D.: Global Strichartz estimates for nontrapping perturbations of the Laplacian. *Commun. Partial Differ. Equ.* **11–12**, 2171–2183 (2000)
12. Struwe, M.: On uniqueness and stability for supercritical nonlinear wave and Schrödinger equations. *Int. Math. Res. Not. Art. ID 76737*, 14 (2006)

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