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On the high accuracy to test dragging of inertial frames with the LARES 2 space experiment

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Abstract In this paper we treat some aspects of the LARES 2 space experiment to test the general relativistic phenomenon of dragging of inertial frames, or frame-dragging, in particular we discuss some aspects of its relative accuracy which can approach one part in a thousand. We then, once again respond to the criticisms of the author of a recent paper about the accuracy in the measurement of frame-dragging with LARES 2. The claims of such a paper are not reproducible in any independent analyses. Indeed, it claims that the accuracy in the test of frame-dragging, which can be reached by the LARES 2 space experiment, is several orders of magnitude larger than previously estimated in a number of papers. Here we show that such a paper is based on a number of significant misunderstandings and conceptual mistakes. Furthermore, it is puzzling to observe that previous papers by the same author contained completely opposite statements about the accuracy which can be reached using two satellites with supplementary inclinations, such as in the LARES 2 space experiment, and in general with laser-ranged satellites.

1 Introduction

The Laser-Ranged Satellite (SLR) LARES 2, of the Italian Space Agency (ASI), was successfully launched on July 13, 2022 [1–5] and together with the laser-ranged satellite LAGEOS (successfully launched in 1976 by NASA) [6] is dedicated to tests of the general theory of relativity and in particular to highly accurate tests of dragging of inertial frames, or frame-dragging. Here we discuss some aspects of the accuracy of the LARES 2 space experiment which, in a number of papers (see, e.g., [1–5] based on previous older studies [7–13]) was proven to be able to approach a relative accuracy of almost one part in a thousand to test frame-dragging, in particular we respond to a recent paper [14] by an author who, referring to the accuracy reachable in the measurement of frame-dragging, claims that: "…*both satellites do not presently allow the stated accuracy goal to be met, needing improvements of 3–4 orders of magnitude.*"

Here, we once again show that this paper [14] is based on a number of significant misunderstandings and conceptual mistakes.

In particular, in Sects. 2 and 3, we show that the claims by the paper [14] are wrong by a huge factor of about 10^7. Indeed, such a paper: (a) neglects the well-known and simple theory of propagation of errors (which, simply put, states that to calculate the uncertainty of a quantity, i.e. see below Eq. 6 of [14], due to the uncertainty of another quantity on which the first one depends, one needs to simply propagate the uncertainty of the second quantity in the first one [15–17]). (b) Paper [14] neglects the real accuracy in the SLR measurement of the orbital elements of LARES 2 and LAGEOS, since it confuses the limitations in the published, significant digits of their orbital elements, due to their

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time-variations [18], with their SLR measurement accuracy. Finally, (c) paper [14] neglects the accuracy in the determination of the Earth gravity field achieved by well-known space techniques, among which are the successful GRACE [19,20] and GRACE Follow-On [21,22] space missions. Today, the Earth quadrupole moment, measured by the even zonal harmonic coefficient J_2, is determined with a relative accuracy of about 10^{-7} (see, e.g., [23]).

In Sect. 4, we show that the relative uncertainty in the value of the Earth angular momentum $S_\oplus$ is significantly less than 10^{-3}, as clearly displayed by its significant digits published in the relevant literature (see, e.g., [24]). Thus, leading to an uncertainty in the test of frame-dragging of less than 10^{-3}.

Finally, in Sect. 5, we display a number of other miscellaneous misunderstandings and mistakes in [14], among which: the serious issues which would arise using the technique of solving for frame-dragging together with other Earth gravitational field parameters (Sect. 5.1); the wrong claim of the absence of alternative tests of frame-dragging with laser-ranged satellites (Sect. 5.2) and finally, in Sect. 5.3, the profound difference between the LAGEOS and LARES 2 passive satellite experiment, and a proposal using two active polar satellites and, in particular, the curious and puzzling fact that a number of previous papers by the same author (Iorio) claim results with profound differences with those claimed in [14] (an attentive reader is then puzzled about which statements of the same author should be then considered representing his true view of the experiment!)

2 The accuracy in the measurement of the Earth quadrupole moment, J_2, and the theory of propagation of errors

In Section 2 of [14] "2. *How Accurate Is the Cancellation of the Effect of the Earth's J_2 on the Sum of the Nodes?*", the author reports, in Eq. 6, the ratio between the sum of the nodal precession due to the Earth quadrupole moment parameter J_2 and the frame-dragging effect of LARES 2 and LAGEOS (incidentally, already precisely calculated in [7–13]), that is:

$$R^{J_2} = \frac{\dot{\Omega}^{L}_{J_2} + \dot{\Omega}^{LR2}_{J_2}}{\dot{\Omega}^{L}_{LT} + \dot{\Omega}^{LR2}_{LT}} = -\frac{3c^2 J_2 R^2 \sqrt{M}}{4\sqrt{G} S} \times \frac{\frac{\cos I_L}{a_L^{\frac{7}{2}}(1-e_L^2)^2} + \frac{\cos I_{LR2}}{a_{LR2}^{\frac{7}{2}}(1-e_{LR2}^2)^2}}{\frac{1}{a_L^3(1-e_L^2)^{\frac{3}{2}}} + \frac{1}{a_{LR2}^3(1-e_{LR2}^2)^{\frac{3}{2}}}} \quad (6)$$

The author then concludes: "*Eq.* (6), *calculated with the values of Equations (7)–(12)*", [i.e., the values of the parameters of the two satellites] *yields*

$$\left|R^{J_2}\right| = 4918,$$

that is, the sum of the nominal node precessions due to the first even zonal harmonic J_2 of the geopotential is still almost 5000 times larger than the sum of the theoretically predicted LT node precessions".

The author [14] reports then this number in the abstract and conclusions, as a total bias in the measurement of the frame-dragging effect with LAGEOS and LARES 2. For example, in the "*Abstract*", the author writes: "*In fact, the actual orbital configurations of the two satellites do not allow one to attain the sought for mutual cancellation of their classical node precessions due to the Earth's quadrupole mass moment, as their sum is still* $\simeq$ 5000 *times larger than the added general relativistic rates.*" And in the "*Summary and Conclusions*", the author states: "*The orbital parameters of the newly launched laser-ranged geodetic satellite LARES 2 do not allow the canceling out of the sum of its classical oblateness-driven node precessions and those of its cousin LAGEOS; indeed, they still amount to about 5000 times the sum of the LT node rates"*.

Before explaining in detail how the data analysis of the orbits of LARES 2 and LAGEOS is *independently* performed with the orbital estimators GEODYN [25] (NASA and SIA-Sapienza, University of Rome), UTOPIA (CSR-University of Texas at Austin [10,11,26,27]) and EPOSOC (GFZ Munich-Helmholtz Center [28,29]), leading to an uncertainty of about 10^{-3} due to the uncertainty in the value of the Earth quadrupole moment (i.e., the even zonal harmonic of degree 2 and order 0), measured by the parameter J_2, we show here that the treatment reported in the paper [14] is wrong since it misses entirely the proper application of the propagation of errors, which can, for example, be found in many basic college textbooks [15–17].

The first significant mistake in the paper [14] and in its conclusions is indeed that the author fails to consider the error in J_2 and its propagation in his Eq. (6), reported here above (see, e.g., [30,31]). The value of J_2 is, since the XX century, extremely well measured by a number of space missions and Earth surface measurements and, among other measurements, by the dedicated, successful GRACE and GRACE Follow-On space missions. GRACE (Gravity Recovery and Climate Experiment), by NASA and DLR (the German Research Centre for Geosciences), was successfully launched in 2002 [19,20]. GRACE Follow-On, by NASA and DLR, was successfully launched on May 22, 2018 [21,22]. GRACE consisted of two identical spacecraft orbiting the Earth in a polar orbit in tandem, some 200–250 km apart. The satellites ranged each other via a K-band radar which accurately measured the inter-satellite distance variations. The non-gravitational forces were measured by very precise onboard accelerometers and the observed inter-satellite variations were used to improve the accuracy in the determination of the Earth gravity field. The GRACE and GRACE Follow-On space missions have successfully and dramati-

cally improved our knowledge of the Earth gravitational field, both of its static and time-dependent parts.
For example, by considering the value of J_2 and of its uncertainty as published in the older 2003-2013 model GGM05S [23]: $J_2 = 1.08264 \times 10^{-3}$ and $\delta J_2 = 2.62581 \times 10^{-10}$, we have a relative uncertainty in its value: $\frac{\delta J2}{J2} = \frac{2.62581\times10^{-10}}{1.08264\times10^{-3}} \cong 2.4 \times 10^{-7}$.

- By considering such a relative uncertain in the value of J_2, of about $\frac{\delta J2}{J2} \cong 2.4 \times 10^{-7}$ and by propagating this uncertainty in Eq. 6 of [14], it can be easily derived a relative error of $4918 \times 2.4 \times 10^{-7} \cong 0.001$, that is an uncertainty of about 10^{-3} in the measurement of frame-dragging due to the uncertainty in J_2, as reported in [1].

Let us now briefly explain in the next two paragraphs, how the orbital estimators GEODYN [25], UTOPIA [10,11,26, 27], EPOSOC [28,29] and other advanced orbital estimator can generate such an accurate data analysis of the orbits of LARES 2 and LAGEOS. A more detailed treatment of the orbital analysis performed with GEODYN and other orbital estimators and a list of all the physical perturbations included in the advanced orbital estimators can, e.g., be found, in [25]. The specific orbital models used in our analysis can, e.g., be found in [30].

The advanced orbital estimators contain the state-of-the-art determinations of the Earth gravity field, including the state-of-the-art, updated coefficients of the static part of the geopotential (e.g., degree 30 and order 30 of GGM05S [23]) and of its dynamical part, including state-of-the-art Earth tidal models. The orbital estimators also contain lunisolar and planetary perturbations (using JPL ephemerides) and GR (general relativity) post-Newtonian corrections [32–36] with the exception of frame-dragging (the weak-field and slow-motion Lense-Thirring effect) to be measured using the orbital analysis (or, alternatively including the standard GR Lense-Thirring effect for a measurement of its potential deviation from the GR prediction). Furthermore, they contain the state-of-the-art modelling of satellite perturbations due to direct solar radiation pressure, albedo radiation pressure and to anisotropic thermal re-radiation from a satellite (Yarkovsky–Rubincam effects [37,38]. The position of the laser-ranging stations is based on the International Terrestrial Reference Frame (current ITRF realization: ITRF2020 product, Greenbelt, MD, USA: NASA Crustal Dynamics Data Information System (CDDIS)), using updated ocean loading models, and polar motion and Earth rotation (determined with VLBI and GPS).

The laser-ranging observations (normal points) from the SLR stations around the world, are then fitted over an arc (usually of 15-day or 7-day length), using the orbital estimators and the state-of-the-art very accurate gravitational and non-gravitational perturbations described above. The orbital residuals are then determined according to the following simplified description (for a precise description see, e.g., [30,39,40]). First, using one of the advanced orbital estimators, we fit the SLR observations over a short arc, to determine the satellite initial conditions, i.e., its six Keplerian orbital elements (and its initial position and velocity). Then, using these orbital elements as initial conditions for the next arc, e.g., a 15-day arc, and using the orbital estimator, the evolution of the Keplerian orbital elements over this next arc is both determined, by fitting the SLR observations, or simply propagated over the arc (from the initial conditions). Propagating the initial orbital elements means to simulate the orbit of the satellite from its initial conditions by using all the a priori considered orbital perturbations contained in the orbital estimator (described above), without any fit of the SLR observations, i.e., without any consideration of the information in the actual SLR observations of the satellite. Finally, the post-fit orbital residuals are generated by taking the difference between the fitted orbital elements, i.e., "observed", using the real SLR satellite observational data, minus the "calculated" ones, i.e., those obtained by simply propagating the initial orbital conditions of the satellite using the orbital estimator with its full set of a priori orbital perturbations. That is, the post-fit orbital residuals are obtained by the differences between the observed orbital elements of a satellite minus its calculated ones. Therefore, the post-fit orbital residuals contain the errors in the modelling of the gravitational and non-gravitational orbital perturbations a priori considered in the orbital estimators, such as frame-dragging, which is not included in the set of orbital perturbations. Alternatively, frame-dragging can be included in the set of orbital perturbations with some *a priori* value, e.g., its GR theoretical value, and any deviation from this assumed frame-dragging value can then be measured in the post-fit orbital residuals. The technique of analyzing the post-fit residuals, together with precise descriptions of other profound misunderstandings and contradictions in the papers of Iorio, are reported in [40].

3 The real accuracy in the measurement of the orbital parameters of LARES 2 and LAGEOS and the correct propagation of the associated error

An additional fundamental error and misunderstanding of [14] is contained in its second part of Sect. 2, where it is written: "*From the number of significant digits quoted in* ([50], *Table* 1) [i.e., Ciufolini et al., 2023] *and reported in Equations (7)–(12), it can be argued that the errors in the orbital elements of L and LR 2 should be as follows...*" and where the author reports the figures of what he erroneously claims to be the uncertainty in the LARES 2 and LAGEOS orbital elements, inferred from the displayed significant digits as reported in [41].

Here the basic error of [14] is that the values of the orbital elements of LARES 2 and LAGEOS reported in [41] are some average values of the orbital elements which indeed, as in the case of the semimajor axis, eccentricity and orbital inclination, have small variations, which are well-determined by the technique of laser-ranging with exceptionally small measurement uncertainty. Only the error in the measurement of the orbital parameters of LARES 2 and LAGEOS, and not their well-measured temporal variations, can imply an error in the post-fit residuals.

In particular, the very large and wrong uncertainty reported in the second part of Section 2 of [14] is by far due mainly to his assumed over-estimated error in the orbital inclinations of LARES 2 and LAGEOS.

For example, the uncertainty in the nodal rate of LARES 2 due to J_2 inferred from the uncertainty in the semimajor axis of LARES 2, δa, that [14] reports to be 0.1 mm, can be simply estimated as:

$$\frac{7}{2}\delta a/a \times \text{ (Nodal rate of LARES 2 due to } J_2) \\ \cong \frac{7}{2} \times (0.1 \text{ mm})/(1.2 \times 10^6 \text{ mm}) \times 126^\circ/\text{yr} \\ \times (3.6 \times 10^6 \text{ milliarcsec/degree}) \cong 0.01 \text{ milliarcsec/yr}$$

That is, relative to the frame-dragging nodal effect on LARES 2 of about 31 milliarcsec/year, (see, e.g., [7,8]) corresponds to a 4×10^{-4} relative uncertainty only. Similarly, for the error in the eccentricities of LARES 2 and LAGEOS.

Therefore, the large uncertainty reported in [14] is due to his erroneously calculated large uncertainty in the inclinations of both LARES 2 and LAGEOS.

As stressed above, the fundamental error of [14] is the assumption that the uncertainty in the inclinations is represented by the number of significant digits reported in [41] which instead corresponds to an arbitrary average value of the orbital inclinations of LARES 2 and LAGEOS, which of course have variations in time (see, e.g., [18]). In summary, the significant digits which are reported in [41] are not indicative of the measurement uncertainties of the orbital inclinations of LARES 2 and LAGEOS but they simply represent an arbitrary average of their well-measured temporal variations, while only the measurement uncertainties can imply an error in the post-fit residuals.

A precise analysis of the variations of the inclinations of LAGEOS and LAGEOS 2 was studied in [18]. Since the uncertainty in the orbital inclinations must be propagated in the equations of the nodal rate of LARES 2 and LAGEOS (but not the average value of each inclination), in [30] the measurement uncertainty in the inclination, which is due to atmospheric refraction, is analyzed and estimated according to the work of [42]. It is shown that the average uncertainty in the inclinations of LAGEOS 2 and LAGEOS is, over a long period of observations, at the level of about 0.01milliarcsec. Such an uncertainty, when propagated in the equation for the nodal rate of LAGEOS due to the even zonal harmonics, corresponds to an uncertainty in the nodal rate of about 1.6×10^{-3} of the frame-dragging effect of LAGEOS [30]. Furthermore, for two satellites with supplementary inclinations, such as LARES 2 and LAGEOS, over a long period of observations, there is an additional reduction of the uncertainty in the test of frame-dragging due to the measurement error in their orbital inclinations induced by atmospheric refraction mismodelling.

In conclusion, the uncertainty in the nodal rate of LARES 2 due to the uncertainty in the measurement of the inclination and due to J_2 is equal to:

$$\text{(nodal rate of LARES 2 due to } J_2) \times Tan[70^\circ] \times \delta I/I \\ \cong 126^\circ/\text{yr} \times (3.6 \times 10^6 \text{ milliarcsec}/^\circ) \times 2.75 \\ \times (0.01)/(70^\circ \times 3.6 \times 10^6 \text{ milliarcsec}/^\circ) \\ \cong 0.05 \text{ milliarcsec/year},$$

which implies a relative uncertainty in the measurement of the frame-dragging nodal effect of LARES 2 (of about 31 milliarcsec/year), of at most 1.6×10^{-3} only.

Incidentally, a simple intuitive argument to prove that paper [14] reports extremely wrong numbers and conclusions is that using one satellite only, e.g. LAGEOS (thus with no cancellation at all of the errors due to the Earth even zonal harmonics, J_{2n}, with the LARES 2-LAGEOS configuration which allows to eliminate the majority of the uncertainty due to all of the Earth even zonal harmonics J_{2n}), it is possible to obtain a test of frame-dragging with an uncertainty of approximately the same order of magnitude of the GR prediction of the Lense–Thirring effect, i.e., the uncertainty due to the leading error δJ_2 in the even zonal harmonic J_2. Nevertheless, [14] claims an error, using both LARES 2 and LAGEOS, several orders of magnitude larger than the GR prediction of the Lense–Thirring effect.

4 The real uncertainty in the Earth angular momentum

In [14] is claimed that "*Another major issue arising because of the too large value of Equation (6) is the uncertainty δS in the Earth's angular momentum S*". Here below, we show that this statement is wrong by several orders of magnitude.

Frame-dragging is the dragging of the local inertial frames of reference with respect to asymptotic inertial space, i.e., to distant stars, due to mass-energy currents, e.g., by the angular momentum of a body [36,43–46]. The Lense–Thirring effect is the weak-field and slow-motion frame-dragging of the orbit of a satellite due to the angular momentum of a central body with respect to distant stars [46], that is:

$$\Omega_{\mathrm{LT}} = \frac{2\,S}{a^3(1-e^2)^{3/2}}$$

where Ω_{LT} is the nodal rate of the satellite (the nodal line is the intersection of the satellite orbital plane with the equatorial plane of the central body), a and e are respectively its semimajor axis and eccentricity, and S is the angular momentum of the central body.

The angular momentum of a rigid body (the higher order term representing the tiny non-rigidity of the Earth can be neglected in the present calculation) along its rotation axis can be written: $S = I\omega$, where I is the body moment of inertia along its rotation axis and ω its angular velocity. Thus, the angular momentum of the Earth, $S_\oplus$, can be written as the product of the Earth axial moment of inertia $I_\oplus$ times its angular velocity $\omega_\oplus$.

Updated values of the Earth axial moment of inertia $I_\oplus$ have been determined by [47], using the Earth gravity models, EIGEN-GL05C and EGM2008, and by [48] for the much smaller linear trend correction of the Earth moment of inertia due to the mass redistribution (see below here). Table 1 provides the result of [47] and the previous, older result of [49], including the accuracy of each value.

Furthermore, the mean Earth angular velocity $\omega_\oplus$ with respect to distant stars is exceptionally well measured using the technique of VLBI (Very Long Baseline Interferometry) [50]. Its value is: $\omega_\oplus = 7.2921150 \times 10^{-5} \pm 1 \times 10^{-7}$ rad/s. Therefore, by using the values provided in Table 1 and the IERS mean Earth angular velocity, $\omega_\oplus$, we get the values of the Earth angular momentum, $S_\oplus$, with its relative uncertainty, $\delta S_\oplus/S_\oplus \cong (\delta I_\oplus \omega_\oplus + I_\oplus \delta\omega_\oplus + \delta I_\oplus \delta\omega_\oplus)/S_\oplus \approx (\delta I_\oplus \omega_\oplus)/S_\oplus = (\delta I_\oplus \omega_\oplus)/(I_\oplus \omega_\oplus) = \delta I_\oplus/I_\oplus$, shown in Table 2.

From Table 2, the relative uncertainty in the value of $S_\oplus$ is, at most, 2.5×10^{-5}, using the older value of $I_\oplus$ of [49], and 4×10^{-6} using the 2010 value of $I_\oplus$ from [47]. Furthermore, from Table 2, the largest difference between the various values of $S_\oplus$ is:
$(5.86031 - 5.86024)$ g cm^2 s^{-1} = 0.00007 g cm^2 s^{-1}, therefore with a relative uncertainty of, at most, 1×10^{-5}.

In conclusion, the relative uncertainty in the value of the Earth angular momentum $S_\oplus$ is significantly less than 10^{-3} and since the Lense–Thirring effect, Eq. 1, is linear in $S_\oplus$, its relative uncertainty due to the uncertainty $\delta S_\oplus$ in $S_\oplus$ is significantly less than 10^{-3}.

Such an uncertainty in the Earth angular momentum, $S_\oplus$, agrees with the significant digits provided in the reference text of Allen's astrophysical quantities [24] where $S_\oplus = 5.861 \times 10^{40}$ g cm^2 s^{-1}, thus with a relative uncertainty in $\delta S_\oplus/S_\oplus \cong 0.001/5.861 = 1.7 \times 10^{-4}$, i.e., significantly less than 10^{-3}.

Incidentally, [14] cites Ren, Leslie, Huang and Hu, 2022 [48]. However, in this paper, by using the 15-year GRACE data sets, it is shown that the linear trend correction due to the mass redistribution increases the Earth moment of inertia, reaching about 10.1×10^{27} kg m^2/year. Nevertheless, such a linear trend, even when integrated over 20 years, is less than 10^{-8} times the value of the axial moment of inertia which is about 8×10^{37} kg m^2. Therefore, such a correction, $\delta I_\oplus/I_\oplus$, is at the level of less than 10^{-8} of $I_\oplus$ and thus might only affect the value of the Earth angular momentum, $S_\oplus$, at a level of less than 10^{-8} of $S_\oplus$, i.e., at a level extremely smaller than 10^{-3} in $\delta S_\oplus/S_\oplus$, which is needed for a test of frame-dragging with a relative uncertainty, $\delta\Omega_{\mathrm{LT}}/\Omega_{\mathrm{LT}}$, of approximately 10^{-3}.

Furthermore, in Sect. 4 *"The Impact of the Uncertainty in the Earth's Angular Momentum"* of [14], another serious error is to use a severely rounded value of the Earth angular velocity, $\omega_\oplus$, with a relative uncertainty of about 10^{-3} and thus inferring from it a relative uncertainty in the value of $S_\oplus$ of about 10^{-3}. However, as reported above, the mean Earth angular velocity, $\omega_\oplus$, published by IERS, is extremely well measured by VLBI with a relative uncertainty of about 10^{-8} [50]. As shown above, a similar error is made in [14] in the case of the relative uncertainty of the Earth moment of inertia.

In particular, at the end of Sect. 4, the author of [14] concludes: *"The bias corresponding to Equation (32) is even worse, amounting to*

$$\delta R_S \simeq 49.2,$$

implying a staggering 4918 percent error in the added LT precessions."

Once again, as explained in Sect. 2, a major error of [14] is to multiply the relative uncertainty, $\delta S_\oplus/S_\oplus$, in the Earth angular momentum or the difference between the two nodal rates of LARES 2 and LAGEOS. If done correctly, one would then get an uncertainty in the test of frame-dragging considerably smaller than 10^{-3}. Indeed, the uncertainty in the test of frame-dragging, using the proper theory of the propagation of the errors, is simply, at the relevant order of approximation, the sum of all the uncertainties in the observable quantity. Such an uncertainty, in the case of the static gravitational effects, is mainly the sum of the uncertainties due to the even zonal harmonics J_{2n}, as explained here in Sect. 2 (see also, e.g., [30]). To such an uncertainty, when compared with the Lense–Thirring effect predicted by GR, Eq. 2, we must add the relative uncertainty in the value of the Lense–Thirring effect due to the relative uncertainty, $\delta S_\oplus/S_\oplus$, in the Earth angular momentum, which, as shown above, is significantly less than 10^{-3}.

In other words, we can also perform a null experiment, i.e., we can perform the orbital analysis with the generation of the orbital residuals by assuming a complete set of gen-

Table 1 The Earth axial moment of inertia in units of 10^{37} kg m^2 (Chen and Shen 2010 [47])

Earth moment of inertia $I_\oplus$	Groten 2004 [49]	Chen and Shen 2010 [47], with EIGEN-GL05C	Chen and Shen 2010 [47], with EGM2008
Zero-tide	8.0365 ± 0.0002	8.036483 ± 0.000030	8.0364807 ± 0.0000084
Tide-free	Not provided	8.036414 ± 0.000030	8.0364114 ± 0.0000084

Table 2 Earth angular momentum, $S_\oplus$, in units of 10^{40} g cm^2 s^{-1}

	Using $I_\oplus$ from Groten 2004	Using $I_\oplus$ from Chen and Shen 2010 with EIGEN-GL05C (zero-tide)	Using $I_\oplus$ from Chen and Shen 2010 with EIGEN-GL05C (tide-free)	Using $I_\oplus$ from Chen and Shen 2010 with EGM2008 (zero-tide)	Using $I_\oplus$ from Chen and Shen 2010 with EGM2008 (tide-free)
Earth angular momentum $S_\oplus$	5.86031 ± 0.0001	5.8603 ± 0.00002	5.86025 ± 0.00002	5.86029 ± 0.000006	5.86024 ± 0.000006
Relative uncertainty, $\delta S_\oplus / S_\oplus$	0.000025	4×10^{-6}	4×10^{-6}	1×10^{-6}	1×10^{-6}

eral relativistic perturbations, including frame-dragging. The orbital residuals will then be affected by: (a) the errors in the non-relativistic Newtonian gravitational perturbations, i.e., mainly the errors in the values of the assumed Earth spherical harmonics and of the Earth tides; (b) the error in the assumed non-gravitational perturbations and (c) the errors in the assumed GR perturbations and in particular in the assumed value of the Lense–Thirring effect, Eq. 2. Since the post-Newtonian corrections are measured with high accuracy [32–36] (with the exception of frame-dragging which is measured with a relative uncertainty of about a few parts in one hundred only [51,52] see also: [30,45]), the errors in the assumed GR perturbations would then be due to: (c1) the error, $\delta S_\oplus$, in the assumed value of the Earth angular momentum, $S_\oplus$, in Eq. 2; (c2) the errors δa and δe in the measured values of semimajor axis, a, and eccentricity, e, in Eq. 2 and to (c3) any potential difference between the prediction of frame-dragging by some alternative gravitational theory (see, e.g., [53,54]) and GR. Since, as shown above, all the errors: a, b, c1 (including the error in the assumed value of the Earth angular momentum $S_\oplus$) and c2 (as discussed in Sect. 3) are at the level of about 10^{-3} or less of the GR theoretical value of frame-dragging, any potential deviation of the orbital residuals from zero would represent a deviation from the GR prediction of frame-dragging with an uncertainty of approximately 10^{-3}.

5 Additional remarks

5.1 About the technique of solving for frame-dragging together with other parameters

In [14] it is claimed that *"From a practical point of view, the LT effect would be nothing more than one of the many other dynamic features, of various origins, entering the equations of motion of the satellites, and whose characteristic parameter(s) are to be estimated in the data reductions along with those describing the behaviour of measuring devices, the propagation of electromagnetic waves, the spacecraft's state vector, etc. Indeed, the common practice in satellite geodesy, astrodynamics, and astronomy is that, to test a certain dynamical feature, X, it should be explicitly modelled along with the rest of the known dynamics and other pieces of the measurement process, and simultaneously one or more parameters characterising it should be estimated, along with many other ones, taking into account other accelerations, etc., and inspecting the resulting covariance matrix to look at their mutual correlations."* And also, in [14]: *"In principle, there should be nothing easier for so many competent and expert people worldwide than adding one more acceleration into the data reduction software and estimating one more parameter."*

Here, the paper [14] shows that its author is not very familiar with *"the common practice in satellite geodesy"*, but it seems to be confusing the techniques of space geodesy with those of other space experiments, such as Gravity Probe B. Indeed, in [14] is for example claimed that *"the propagation of electromagnetic waves"* and *"the behaviour of measuring devices"* are, in space geodesy, solved for together with the geophysical parameters. In [14] is once again claimed: *"In other words, the gravitomagnetic field of the Earth should be simultaneously estimated..."*. Of course, in standard space geodesy there is not such a thing as *"simultaneously"* estimating the coefficients of the Earth gravitational field with *"the propagation of electromagnetic waves"* and *"the behaviour of measuring devices"*. On the contrary, a number of physical effects are analyzed and estimated using the orbital residuals such as, for example, those due to anisotropic thermal radiation from laser-ranged satellites (the Yarkovsky and Yarkovsky-Rubincam effects [37,38]). As a further example, in [18] the orbital residuals are used to analyze and estimate the surface mean radiation pressure coefficient of LAGEOS and other physical effects.

Furthermore, about the possibility of estimating contemporarily the Lense–Thirring effect and the coefficients of the Earth gravitational field using, e.g., LARES 2 and LAGEOS, we stress that such an analysis must be performed by maintaining the basic idea of using two laser-ranged satellites with supplementary inclinations (to eliminate the nodal perturbations of the Earth even zonal harmonics and by contemporarily measure the Lense–Thirring effect). Therefore, such analysis should not be performed by contemporarily using the data of all the geodetic satellites, including LARES 2 and LAGEOS with the same weight. However, to accurately measure the Lense–Thirring effect, one needs to use some highly accurate Earth gravitational field model generated using a large amount of data from many geodetic satellites collected over an extended time period and then use it in a subsequent step to analyze only the orbits of LARES 2 and LAGEOS. In summary, for an accurate test of the Lense–Thirring effect, a feasible technique is that of using the post-fit residuals of two satellites with supplementary inclinations, and these post-fit orbital residuals must be obtained with the orbital estimators by using accurate Earth gravitational field models generated on the basis of multiple types of data collected from many space geodetic satellites, including GRACE and GRACE Follow-On. Such an approach was indeed validated by the 1989 NASA-ASI study [11,12].

5.2 The wrong claim of the absence of alternative tests of frame-dragging using laser-ranged satellites

In [14] it is claimed that "*Another puzzling issue, is that there are several SLR stations scattered around the globe [87] where skilled teams of space geodesists routinely process laser ranging data from many geodetic satellites with a range of dedicated software [88]; yet, despite this, no one has ever tried to (correctly) perform LT tests independently of Ciufolini, or, if anyone has done so, they have not made their results public in the peer-reviewed literature. There are just some conference proceedings [89–92] in which the authors did not model and estimate the LT acceleration either. The same holds also for a few independent studies recently published in peer-reviewed journals by former coworkers of Ciufolini [93,94]*".

Such a claim of the absence of alternative tests of frame-dragging using laser-ranged satellites is profoundly misleading since in the past three different and independent teams, one of Sapienza and Salento universities, and Maryland University-NASA (see, e.g., [45]), a second one of the University of Texas at Austin and JPL (see: [26,27]), and the third one of GFZ-Helmholtz Institute of Potsdam-Munich (see: [28,29]), using three different orbital estimator, have reported and published *independent* confirmations and measurements of frame-dragging using laser-ranged satellites. It is of course not relevant the fact that these confirmations were published in proceedings of international conferences which however, had reviewers. Incidentally, some of these international conferences were quite prestigious such as the International Astronomical Union Symposium 261, Relativity in Fundamental Astronomy and the 16th International Workshop on Laser Ranging [26,27]. Explicit independent measurements of the Lense–Thirring effect using the independent orbital estimator EPOSOC were presented [28] and published [29]. It is of course not relevant that later on, these three independent teams merged to jointly publish some papers.

Furthermore, there are several papers (see, e.g., [52]) where completely independent results of the measurement of the Lense–Thirring effect using laser-ranged satellites were published by a "competing" team. It is of course not relevant that some authors, but not all, of such papers were "*former coworkers of Ciufolini*".

5.3 Other wrong claims of the 2023 paper of Iorio and previous papers of Iorio claiming results at variance with such a paper

In [14] is claimed "*Ciufolini et al.* [50], *referring to an earlier proposal* [51] *equivalent to the strategy proposed by van Patten and Everitt* [48,49]" [here ref. [51] of [14] is: Ciufolini, *Phys. Rev. Lett.* 1986, i.e., ref. [8] of the present paper, here referred to as "UT/ASI", and ref. [49] of [14] is: Van Patten and Everitt, *Cel. Mech.* 1976, i.e., ref. [55] of the present paper, here referred to as "VP&E"].

Since a discussion of the profound differences between two proposals was already presented in [40], here we simply, briefly list the main factual differences between the two proposals claimed to be "*equivalent*" in [14].
• VP&E proposed two polar satellites with 90° of inclination, whereas UT/ASI proposed two satellites with 70° and 110° of inclination (really, UT/ASI proposed the launch of just one satellite, called LAGEOS 3, with about 70° of inclination, to couple to the already orbiting satellite LAGEOS with about 110° of inclination). • VP&E proposed two satellites with extremely accurate drag-free systems, whereas UT/ASI proposed two (really one) completely passive satellites (endowed with retro-reflectors). • VP&E proposed two satellites very accurately ranging at each other, whereas UT/ASI proposed two (really one) completely passive satellites ranged by the SLR stations on the Earth. • VP&E proposed two satellites whose orbits must be maneuvered from Earth, whereas UT/ASI proposed two (really one) completely passive laser-ranged satellites. • Therefore, the duration of the mission proposed by VP&E would have been of a few months, whereas the duration of the mission proposed by UT/ASI is at least of several tens of years (since the

LAGEOS and LARES satellites are completely passive). • Consequently, the final estimated accuracy of the mission proposed by VP&E would have been about 2.5% [55], whereas the final estimated accuracy of the UT/ASI proposal, today using LARES 2, is about 0.2% [1–5]. • Incidentally, the cost of the mission proposed by VP&E may be estimated to be several hundred million dollars, whereas the cost of the mission proposed by UT/ASI is just a few millions of dollars. The only relevant, similar feature of the two proposals is that they both use satellites with supplementary inclinations (i.e., with sum equal to 180°) to eliminate the biasing nodal rate due to the uncertainty in J_2. Polar satellites, if exactly polar, have null nodal rates due to J_2 and the supplementary inclination of the VP&E proposal would have allowed to eliminate the J_2 biasing nodal rate due to the non-exactly polar orbits which could be achieved by the two VP&E satellites.

Nevertheless, it is quite puzzling and curious that several previous papers by Iorio (see, e.g., [56–58]) contain statements completely opposite to what is published in [14]. For instance, in [56], it is stated: "*The idea of using a pair of twin satellites, denoted as S1 and S2, in identical orbits with the same semimajor axes a and eccentricities e, except for the inclinations i of their orbital planes, which should be supplementary, in order to measure the general relativistic Lense–Thirring effect (Lense and Thirring 1918) in the gravitational field of the Earth*[1] *was put forth for the first time by Ciufolini with the proposed LAGEOS–LAGEOS III mission (Ciufolini 1986).*" Furthermore, in [57], it is stated: "*In order to achieve a few percent accuracy, in 9* [i.e., Ciufolini, I., Measurement of Lense–Thirring drag on high-altitude, laser ranged artificial satellites, Phys. Rev. Lett., 56, 278-281, 1986.] *it was proposed to launch a passive geodetic laser-ranged satellite-the former LAGEOS III - with the same orbital parameters of LAGEOS apart from its inclination which should be supplementary to that of LAGEOS. This orbital configuration would be able to cancel out exactly the classical nodal precessions, which are proportional to cos i, provided that the observable to be adopted is the sum of the residuals of the nodal precessions of LAGEOS III and LAGEOS*". In [58], it is also stated that: "*The use of the proposed LAGEOS III/LARES satellite would greatly increase the accuracy of such space–based measurement (Ciufolini, 1986). LARES would be a LAGEOS-type satellite to be placed in the same orbit as of LAGEOS except for the eccentricity, which should be one order of magnitude larger, and, especially, the inclination which should be supplementary to that of LAGEOS.*"

An attentive and careful reader may wonder which of these contradictory statements of Iorio should then be seriously taken as his true opinion on the subject!

6 Conclusions

We have discussed some aspects of the high accuracy which can be achieved by the LARES 2 space experiment to test the general relativistic phenomenon of frame-dragging. We have then pointed out the seriously erroneous statements contained in a recent paper [14] which are either manifestly wrong or significantly misleading.

In Sects. 2 and 3, we have shown that the misleading claim in [14] of an error of three to four orders of magnitude the Lense–Thirring effect is, using a proper combination of the LARES 2 and LAGEOS orbital elements, seriously wrong by a factor of about seven orders of magnitude. Indeed, [14] neglects: a) to apply the correct theory of the propagation of the errors; b) the accuracy achieved today in modelling of the Earth gravitational field by GRACE, GRACE Follow-On (in addition to other geodetic and space techniques), and c) the *real* measurement uncertainty of the LARES 2 and LAGEOS orbital parameters. In fact, among other errors, the paper [14] infers the uncertainty in the satellites' orbital parameters using the number of the published significant digits of their orbital elements, however the number of the significant digits are not indicative of their measurement accuracy but are simply showing the mean value of these elements which have some larger (than their accuracy) measured temporal variations.

In Sect. 4, we have shown that the Earth moment of inertia and the Earth angular velocity are both extremely well measured. For example, using VLBI, the mean Earth angular velocity is measured with an accuracy of better than 1 part in ten million (10^{-7}) but [14] uses a profoundly wrong and misleading uncertainty of about 10^{-3} in the mean Earth angular velocity, i.e., an uncertainty enormously wrong by a factor of about ten thousand. Such a serious sizable error disregards the extremely well-measured Earth physical quantities used in [14] and is not directly related to the LARES 2 parameters. We thus show that the uncertainty in the mean Earth angular momentum is less than one part in a thousand, in agreement with its significant digits published in some relevant Earth science literature (see, e.g., Allen, Astrophysical quantities). In summary, the real accuracy of the Earth angular momentum implies an error of less than 10^{-3} in the test of frame-dragging using LARES 2 and LAGEOS.

In Sect. 5, we have shown that the additional various claims of [14] are all unfounded and misleading.

In Sect. 5.1, we explain that the technique of solving for frame-dragging together with other Earth parameters, using LARES 2 and LAGEOS, is not *"practical"* and easily feasible as claimed in [14]. Indeed, [14] shows no experience in the generation of Earth gravitational field models and in the data analysis of laser-ranged satellites, quite different from the data analysis of, e.g., Gravity Probe-B.

In Sect. 5.2, we show that the claim of [14] of the absence of alternative tests of frame-dragging with laser-ranged satellites is not true and profoundly misleading. In this Section we indeed list the results of three different and independent teams: one using the orbital estimator GEODYN (from the universities of Sapienza, Salento and Maryland, and NASA, see, e.g., [45]; a second one using the orbital estimator UTOPIA (from the University of Texas at Austin and JPL, see. e.g., [26,27]) and the third one using the orbital estimator EPOSOC (from the GFZ-Helmholtz Institute of Germany, see, e.g., [28,29]). These international teams reported and published independent confirmations and measurements of frame-dragging, with laser-ranged satellites, using their *independent* orbital estimator. We also refer to the independent publications of a competing team by the Italian Institute of Astrophysics (INAF) and other institutions.

Incidentally, in Sect. 5.3, we have listed the profound factual differences between the passive satellite experiment LARES 2/LAGEOS 3 proposed by UT-ASI and that with two complex and expensive active polar satellites. It is puzzling that the wrong claim that these two space proposals are "*equivalent*" is repeated several times in [14].

In particular, in Sect. 5.3, we present the puzzling and curious contradictions and opposite statements contained in some papers by Iorio. Indeed, previous publications by Iorio report completely opposite conclusions than [14] in regard to the LAGEOS 3/LARES 2 space experiment and other similar experiments with laser-ranged satellites (see, e.g., [56–58]).

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Code Availability Statement This manuscript has associated code/software in a data repository. [Author's comment: The GEODYN orbit determination and geodetic parameter estimation program is available at: https://space-geodesy.nasa.gov/techniques/tools/GEODYN/GEODYN.html]

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