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RESEARCH

Decomposition and equivalence of general nonlinear dynamical control systems

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Abstract Projection and equivalence concepts have been widely studied in the literature. In the present paper two nonlinear systems with the same number of inputs, but not the same number of state variables, are considered. Under mild assumptions, necessary and sufficient conditions are given for the existence of a submersion such that the higher dimensional system projects locally onto the other one. The solution to this problem has relevant applications, for instance in robotics. It includes as a special case the equivalence of nonlinear systems with no particular structure.

Keywords Decomposition · Projection · Systems equivalence · Accessibility distribution · Invariant submanifolds

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1 Introduction

Transformations between two given systems to highlight properties such as linear equivalence [8], [28], observer canonical forms [6], [10], immersion [16], [21], bisimulation [13] and quotients [33], are largely investigated in the literature. The first systematic approach to classify linear dynamical systems was provided by Brunovsky in 1970 [9]. In [8] Brockett addressed the linear equivalence problem for nonlinear systems under change of coordinates. Jakubczyk and Respondek [15] extended these results to equivalence to a linear system through change of coordinates and regular feedback. A complete overview can be found in [18].

Herein, this problem is generalized. Given two nonlinear systems with the same number of inputs, conditions are given under which there exists a change of coordinates such that the largest dimensional system projects onto the smaller dimensional one. Robotics provides an obvious motivation to this theoretical problem. A walking robot which is a complex underactuated electro-mechanical system whose essential features are embodied by some smaller dimensional subsystem, as shown in [12] where a control for the 4-link walking robot with three actuators could be derived from the 2 DoF single input Acrobot. The dynamics of the Acrobot can be considered as a subsystem of the dynamics of a more complex 2-legged walking robot [24]. The same idea was exploited for state estimation in [2]. As shown below, the key tool to solve the problem is the geometrical concept of invariant manifold, already used in [11] where stability properties of a given system are analyzed through the properties of a lower dimensional submanifold by using the center manifold theory, or in [14] where the regulation problem for nonlinear systems was addressed and in [5] where adaptive control problems are considered, to cite a few.

The paper is organized as follows: in Sect. 2 some basic results for multi-input systems are recalled and the decomposition problem statement is given; Sect. 3 is devoted to some preliminary results while the solvability of the decomposition problem is fully characterized in Sect. 4. Finally, in Sect. 5 some examples are ruled out to show the results. Conclusions are given in Sect. 6.

The problem under interest is helpfully introduced by the following example.

Example 1 Consider the two nonlinear dynamics

$$\Sigma_1 : \begin{cases} \dot{x}_1 = u_1 \cos x_3 \\ \dot{x}_2 = u_1 \sin x_3 \\ \dot{x}_3 = u_2, \\ \dot{x}_4 = x_2 u_2, \end{cases} \quad \Sigma_2 : \begin{cases} \dot{z}_1 = u_2 \\ \dot{z}_2 = u_1 - z_3 u_2 \\ \dot{z}_3 = z_2 u_2, \end{cases} \tag{1}$$

where Σ_2 is the unicycle kinematics in appropriate coordinates. Consider the mapping

$$\varphi_1(x_1, x_2, x_3) = \begin{pmatrix} x_3 \\ x_2 \sin(x_3) + x_1 \cos(x_3) \\ x_1 \sin(x_3) - x_2 \cos(x_3) \end{pmatrix}. \tag{2}$$

It is easily seen that in the coordinates $\chi = \begin{pmatrix} \varphi_1(x_1, x_2, x_3) \\ x_4 \end{pmatrix}$, derived from (2), Σ_1 reads

$$\begin{cases} \dot{\chi}_1 = u_2 \\ \dot{\chi}_2 = u_1 - \chi_3 u_2 \\ \dot{\chi}_3 = \chi_2 u_2 \\ \dot{\chi}_4 = (\chi_2 \sin \chi_1 - \chi_3 \cos \chi_1)\, u_2 \end{cases} \tag{3}$$

that is, φ_1 projects the flow of Σ_1 onto Σ_2. In fact, Σ_1 can be decomposed into Σ_2 and a further scalar dynamics defined by the evolution of χ_4, thanks to the diffeomorphism $\chi = \begin{pmatrix} \varphi_1(x_1, x_2, x_3) \\ x_4 \end{pmatrix}$ and the canonical projection which maps χ onto (χ_1, χ_2, χ_3). ◁

The conditions that ensure the existence of the mapping (19) solving the problem are the topic of this paper.

2 Recalls and problem statement

In the present section, we first provide the essential notions upon which our results are derived. Then the problem under investigation is stated.

Consider the control affine nonlinear system

$$\Sigma : \dot{x} = f(x) + \sum_{j=1}^{m} g_j(x) u_j, \tag{4}$$

where $x \in R^n$. The input $u \in R^m$, and the vector fields $f, g_j, j = 1, ..., m$, are assumed to be smooth.

Given a set of functions $\Psi : R^n \to R^{\bar{n}}$, with $\bar{n} \leq n$, $S = \{x \in R^n \mid \Psi(x) = 0\}$ is a submanifold of R^n of dimension $q = n - \text{rank} \left[\frac{\partial \Psi(x)}{\partial x} \right]_{|\Psi(x)=0}$. If $\Psi = \emptyset$, then $S = R^n$.

Let us now recall that given a set of q vector fields $\tau_i(x)$, $i = 1, \cdots, q$, a distribution $\Delta = \text{span}\{\tau_1(x), \cdots, \tau_q(x)\}$, we denote by $ad_{\tau_i} \tau_j = [\tau_i, \tau_j] = \frac{\partial \tau_j}{\partial x} \tau_i - \frac{\partial \tau_i}{\partial x} \tau_j$ the standard Lie bracket of vector fields and accordingly by $ad^l_{\tau_i} \tau_j = [\tau_i, ad^{l-1}_{\tau_i} \tau_j]$. Moreover $\bar{\Delta}$ denotes the involutive closure of Δ that is the smallest distribution containing Δ and such that for any two vector fields τ_i,τ_j in $\bar{\Delta}$, also $[\tau_i, \tau_j] \in \bar{\Delta}$. Finally we denote by $\Delta^\perp$ the codistribution which is the left annihilator of Δ.

Let us also recall that the notion of foliation of a manifold is common in differential geometry and dynamic systems' theory, see e.g [26, page 102]. More specifically let $\mathcal{M}$ be a smooth n-dimensional manifold, $T\mathcal{M}$ its tangent bundle and $V \subset T\mathcal{M}$ an involutive distribution. The set of all maximal integral submanifolds S_i of V is a foliation of $\mathcal{M}$ and the S_i are called leaves of the foliation. Two trajectories starting from some leaf S_i at some time t_i will go through the same leaf S_j at some time $t_j > t_i$.

The leaf S such that the trajectory starting on it remains on it, is called invariant. In the sequel, at some points, we will treat the case of non-constant dimensional distributions, where the Frobenius Theorem [1,29] is not valid any longer. The interested reader is referred to the pioneering works [17,30,31] for an attempt for the generalization of Frobenius Theorem to non constant dimensional distributions.

With the notation above, the strong accessibility property of a system is characterized as follows.

Definition 1 The strong accessibility distribution $\mathcal{L}$ of system Σ in (4) is defined as

$$\mathcal{L} = \overline{\text{span}\{ad_f^k g_j(x),\ \ j = 1, .., m, \forall k \geq 0\}}. \quad (5)$$

System Σ is said to be locally strongly accessible in a neighborhood V_{x_0} of x_0 if $\dim \mathcal{L} = n$ for any $x \in V_{x_0}$.

If the given system is strongly accessible, i.e. locally strongly accessible at almost all $x \in R^n$, then the previous distribution has dimension n for $0 \leq k \leq n-1$ for almost all $x \in R^n$, with the exclusion of some singular points. We will thus consider in the following

$$\mathcal{L}_n = \overline{\text{span}\{g_j(x), \cdots ad_f^{n-1} g_j(x),\ \ j = 1, .., m\}}. \quad (6)$$

and exclude initially the points where its dimension is less than n.

In the following, given two systems Σ_1 and Σ_2, those mappings are sought which allow to identify a special relation between them, as stated in the following problem statement.

2.1 The decomposition problem

Consider the two nonlinear systems driven by the same input $u \in R^m$, with state $x_i \in R^{n_i}$, $i = 1, 2$, $n_1 \geq n_2$:

$$\Sigma_i : \dot{x}_i = f_i(x_i) + \sum_{j=1}^{m} g_{ij}(x_i) u_j, \quad i = 1, 2, \quad (7)$$

where the vector fields f_i, g_{ij}, $i = 1, 2$, $j = 1, ..., m$, are smooth, and let the following assumption hold true:

Assumption 1 $\dim \mathcal{L}^i_{n_i} = n_i$, $\forall x_i \in V^i_{x_{i0}}$ a neighborhood of x_{i0}, for $i = 1, 2$.

Problem Statement: Find, if possible, a local diffeomorphism $\varphi = (\varphi_1^T, \varphi_2^T)^T : R^{n_1} \to R^{n_1}$, defined for any $x_1 \in V^1_{x_{10}} \subset R^{n_1}$, with $\dim(\varphi_1) = n_2$ such that, setting $z = \varphi(x)$ with $z_1 = \varphi_1(x_1)$ and $z_2 = \varphi_2(x_1)$, locally in the new coordinates system Σ_1 reads

$$\dot{z} = \tilde{f}(z) + \sum_{j=1}^{m} \tilde{g}_j(z) u$$

with $\tilde{f}(z) = \begin{pmatrix} f_2(z_1) \\ f_3(z_1, z_2) \end{pmatrix}$ and for $j = 1, \cdots, m$ $\tilde{g}_j(z) = \begin{pmatrix} g_{2j}(z_1) \\ g_{3j}(z_1, z_2) \end{pmatrix}$, for some f_3 and g_{3j}'s, so that

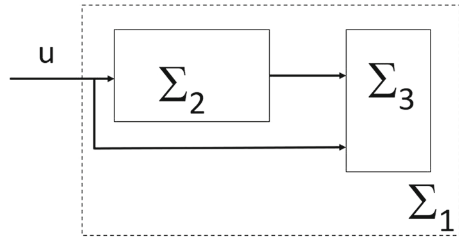

Fig. 1 Structure of system Σ_1

the transformed dynamics is decomposed in the triangular form

$$\dot{z}_1 = f_2(z_1) + \sum_{j=1}^{m} g_{2j}(z_1) u_j \quad (8)$$

$$\dot{z}_2 = f_3(z_1, z_2) + \sum_{j=1}^{m} g_{3j}(z_1, z_2) u_j. \quad (9)$$

Let $p(z)$ be the canonical projection which maps (z_1, z_2) onto z_1. The overall transformation $p \circ \varphi(x_1)$ maps locally Σ_1 onto Σ_2. If the decomposition problem is solvable, then after a possible change of coordinates, Σ_1 is decomposed as the cascade system in Fig. 1, including the upstream subsystem Σ_2.

The problem consists in characterizing the existence of φ_1. In this case, φ_2 will then be any basis completion which guarantees that φ is a local diffeomorphism. The change of coordinates to be applied to Σ_1 to display the dynamics of Σ_2 is derived later in this paper from the computation of non-controllable states in the so-called composite system, at least on some specific submanifold. The explicit procedure is however postponed for future work as outlined in the conclusions. Assumption 1 avoids to include non-controllable states which are specific to Σ_1 or Σ_2.

3 Preliminary results

Under Assumption 1 some peculiarities of the decomposition property are pointed out. Note that if Σ_2 is a subsystem of Σ_1, then recalling the notation $\varphi = (\varphi_1^T, \varphi_2^T)^T$, necessarily,

- $n_1 \geq n_2$;
- the accessibility of Σ_2 yields that φ_1 is a submersion, i.e. rank $\left(\frac{\partial \varphi_1}{\partial x_1}\right) = n_2$.

The following result is in order.

Proposition 1 *Let Assumption 1 hold true. The decomposition property is reflexive and transitive.*

Proof Reflexivity and transitivity of the decomposition property are obvious. □

Using the F-relation property ([7], p.119), in our case $F = \varphi_1$, the solvability of the decomposition problem can be stated as follows:

Proposition 2 *Let Assumption 1 hold true. Given systems (7), the decomposition problem is locally solvable if and only if there exists a mapping $x_2 = \varphi_1(x_1)$ defined for any $x_1 \in V^1_{x10}$ such that the vector fields f_1 and g_{1j}, are φ_1-related respectively to f_2 and g_{2j} $j = 1, \cdots, m$, that is*

$$\begin{aligned} \frac{\partial\varphi_1}{\partial x_1} f_1(x_1) &= f_2(\varphi_1(x_1)) \\ \frac{\partial\varphi_1}{\partial x_1} g_{1j}(x_1) &= g_{2j}(\varphi_1(x_1)), \qquad j = 1, ..., m. \end{aligned} \tag{10}$$

The previous result, which is a special case of Proposition 2 in [4], relates the Lie algebra associated with the two systems as stated below.

Corollary 1 *Let Assumption 1 hold true. Given (7), if the decomposition problem is locally solvable through the mapping $x_2 = \varphi_1(x_1)$, defined for any $x_1 \in V^1_{x10}$, with $\text{rank}\,\frac{\partial\varphi_1}{\partial x_1} = n_2$ on V^1_{x10}, then for any $j = 1, ..., m$, $k \geq 1$, $ad^k_{f_1} g_{1j}(x_1)$ is φ_1-related to $ad^k_{f_2} g_{2j}(x_2)$.*

3.1 The composite system

Whether or not system Σ_2 is the projection of system Σ_1 through an appropriate mapping of the form $x_2 = \varphi_1(x_1)$ is now checked from the analysis of the composite system Σ obtained from Σ_1 and Σ_2 and given by (7). Setting $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $N = n_1 + n_2$, one gets:

$$\begin{aligned} \Sigma : \dot{x} = F(x) + \textstyle\sum_{j=1}^m G_j(x)u_j &= \begin{pmatrix} f_1(x_1) \\ f_2(x_2) \end{pmatrix} \\ &+ \textstyle\sum_{j=1}^m \begin{pmatrix} g_{1j}(x_1) \\ g_{2j}(x_2) \end{pmatrix} u_j. \end{aligned} \tag{11}$$

The accessibility distribution $\mathcal{L}_N$ of the composite system Σ has the following structure

$$\mathcal{L}_N = \text{span}\left\{ \begin{pmatrix} g_{1j}(x_1) \\ g_{2j}(x_2) \end{pmatrix}, \cdots, \begin{pmatrix} ad^{N-1}_{f_1} g_{1j}(x_1) \\ ad^{N-1}_{f_2} g_{2j}(x_2) \end{pmatrix}, j = 1, \cdots, m \right\}. \tag{12}$$

Let $\tau = \begin{pmatrix} \tau^1(x_1) \\ \tau^2(x_2) \end{pmatrix}$ denote a general vector field generating $\mathcal{L}_N$ as displayed in (12). The vector field $\tau^1(x_1)$ and the corresponding vector field $\tau^2(x_2)$ are obtained through the same sequence of brackets of vector fields, that is if

$$\begin{aligned} \tau^1(x_1) = [ad^{l_1}_{f_1} g_{1i_1} [ad^{l_2}_{f_1} g_{1i_2} \\ \cdots [ad^{l_q}_{f_1} g_{1i_q}, ad^{l_{q+1}}_{f_1} g_{1i_{q+1}}] \cdots]], \end{aligned} \tag{13}$$

for $l_\mu \geq 0$, $i_\mu \in [1, m]$, where $\mu = 1, \cdots q + 1$, then

$$\begin{aligned} \tau^2(x_2) = [ad^{l_1}_{f_2} g_{2i_1} [ad^{l_2}_{f_2} g_{2i_2} \\ \cdots [ad^{l_q}_{f_2} g_{2i_q}, ad^{l_{q+1}}_{f_2} g_{2i_{q+1}}] \cdots]] \end{aligned} \tag{14}$$

with the same sequence of indices. The following proposition follows from Corollary 1.

Proposition 3 *Suppose Assumption 1 holds true. Let*

$$\mathcal{L}^1_{n_1} = \text{span}\left\{ \tau^1_1(x_1), \tau^1_2(x_1), \cdots, \tau^1_{n_1}(x_1) \right\}. \tag{15}$$

If Σ_2 is the local projection of Σ_1 through $x_2 = \varphi_1(x_1)$, then any $\tau^1(x_1)$ of the form (13) which can be written as $\tau^1(x_1) = \sum_{i=1}^{n_1} \alpha_{i1}(x_1)\tau^1_i(x_1)$ and the corresponding $\tau^2(x_2)$ given by (14) are φ_1-related.

Furthermore,

Proposition 4 *Let Assumption 1 hold true. Assume that for the composite system $q = \dim \mathcal{L}_N$ is constant for any $x \in V_{x0}$. Then $n_i \leq q \leq N = n_1 + n_2$ for $i = 1, 2$.*

Proof The proof is straightforward if one considers that each vector field in the distribution has the form $X = (X^T_1(x_1), X^T_2(x_2))^T$. □

Proposition 4 leads to the following result.

Corollary 2 *Let Assumption 1 hold true. If $\dim \mathcal{L}_N = q$ for any $x \in V_{x0}$, then there exist locally $N - q$ independent autonomous elements and $N - q \leq \min\{n_1, n_2\}$.*

Next result states that even if $\dim \mathcal{L}_N = q > n_1$ for almost any x, the dimension could drop on some specific invariant manifold. This is very specific to the class of nonlinear systems as it is argued below.

Proposition 5 *Let Assumption 1 hold true. If $\dim \mathcal{L}_N \geq n_1$ and there exists $\varphi_1(x_1)$ with $\text{rank}\,\frac{\partial\varphi_1}{\partial x_1} = n_2$ for any*

$x_1 \in V^1_{x_{10}}$ such that $d(x_2 - \varphi_1) \in \mathcal{L}_N|^{\perp}_{x_2=\varphi_1(x_1)}$ then $\mathcal{L}_N$ has dimension n_1 on $\Psi(x_1, x_2) = x_2 - \varphi_1(x_1) = 0$ and

$$S = \Big\{(x_1, x_2) \in R^N \mid \Psi(x_1, x_2) = x_2 - \varphi_1(x_1) = 0\Big\} \tag{16}$$

is an invariant manifold for the given composite system.

Proof: By assumption $\Psi(x_1, x_2) = x_2 - \varphi_1(x_1)$, is a set of n_2 independent functions. Furthermore $d\Psi \in \mathcal{L}_N|^{\perp}_{x_2=\varphi_1(x_1)}$, thus $\mathcal{L}_N$ has dimension n_1 on $\Psi(x_1, x_2) = x_2 - \varphi_1(x_1) = 0$ and since

$$\dot{\Psi}(x_1, x_2)|_{x_2=\varphi_1(x_1)} = 0$$

S defined by (16) is an invariant submanifold for the given composite system Σ.

The previous result is a key tool to solve the problem under interest, since the decomposition problem shows an important difference between the linear and the nonlinear case. In fact, under the strong accessibility assumption of two linear systems Σ_1 and Σ_2, Σ_2 is a subsystem of Σ_1 if and only if $\dim \mathcal{L}_N = n_1$ and $\mathcal{L}_N^{\perp}$ is the key to derive the required mapping $x_2 = Tx_1$ as $d(x_2 - Tx_1) \in \mathcal{L}_N^{\perp}$. This is no more the case when the Σ_i's are nonlinear as it is shown in Sect. 4.

4 Decomposition via change of coordinates

As mentioned above, the decomposition problem is solved in a straightforward way in the linear case, just from the left annihilator of $\mathcal{L}_N$ for the composite system which has to have dimension n_1.

Let us thus start with the case when $\dim \mathcal{L}_N = n_1$ for nonlinear systems, as a special case.

4.1 Special case $\dim \mathcal{L}_N = n_1$

The following introductory example shows that $\dim \mathcal{L}_N = n_1$ is not sufficient by itself, but some additional condition is required.

Example 2 Consider the two scalar systems Σ_1 : $\dot{x}_1 = x_1 u$ and Σ_2 : $\dot{x}_2 = x_2 + x_2 u$ that are strongly accessible and satisfy $\dim \mathcal{L}^i_{n_i} = n_i$ for $i = 1, 2$ respectively for $x_1 \neq 0$ and $x_2 \neq 0$. They are not equivalent under coordinates change since one is driftless and one is not. However for the composite system $\mathcal{L}_N = \text{span}\left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\}$ has dimension 1 for any $x \neq 0$. Its left-annihilator is spanned by the differential of any function of $\frac{x_2}{x_1}$, as $d(\frac{x_2}{x_1})$ or $d(Ln|\frac{x_2}{x_1}|)$. From the procedure in the linear case, $\frac{x_2}{x_1} = 0$ should yield a solution. This would imply $x_2 = 0$, so there is no way to get $x_2 = \varphi_1(x_1)$ as requested. ◁

As a matter of fact, $\dim \mathcal{L}_N = n_1$ is not sufficient anymore and some additional conditions are needed as shown in the next result.

Proposition 6 *Let Assumption 1 hold true. Consider the composite system Σ given by (11) and assume* $\dim \mathcal{L}_N = n_1$ *for any $x \in V_{x_0}$. The three following statements are equivalent.*

1. *Σ_1 projects onto Σ_2, $\forall x \in V_{x_0}$, through an appropriate mapping $x_2 = \varphi_1(x_1)$,*
2. *for the composite system Σ there exists an invariant submanifold S of constant dimension n_1 at x_0, defined as:*
$$S = \Big\{x \in R^N \mid x_2 - \varphi_1(x_1) = 0\Big\}$$
with rank $\left[\frac{\partial \varphi_1(x_1)}{\partial x_1}\right]\Big|_S = n_2$
3. *the n_2-dimensional foliation associated to $\mathcal{L}_N$ has an invariant leaf $\Psi(x_1, x_2) = 0$ with constant rank $\left[\frac{\partial \Psi}{\partial x_i}\right]\Big|_{\Psi(x_1,x_2)=0} = n_2$ for $i = 1, 2$.*

Proof (1.) implies (2.): Σ_1 and Σ_2 are given by (7). From (1.) there exists $\varphi_1(x_1)$ with rank$\left(\frac{\partial \varphi_1}{\partial x_1}\right) = n_2$ such that

$$\frac{d}{dt}[\varphi_1(x_1)] = f_2(\varphi_1(x_1)) + \sum_{j=1}^{m} g_{2j}(\varphi_1(x_1))u \tag{17}$$

Let $S = \{x \in R^N \mid x_2 = \varphi_1(x_1)\}$. Necessarily, from (17), $\frac{d}{dt}[x_2 - \varphi_1(x_1)]_{|x\in S} \equiv 0$ that is S is an invariant manifold.

(2.) implies (3.): From (2.) $S = \{x \in R^N \mid x_2 - \varphi_1(x_1) = 0\}$ is an invariant submanifold for system Σ. Let $\Psi = x_2 - \varphi_1(x_1) = 0$. Invariance of S yields by definition $\dot{\Psi} = 0$; thus $\Psi(x_1, x_2)$ defines a set of n_2 independent autonomous elements for Σ, which implies that $d\Psi \in \mathcal{L}_N^{\perp}$, and $\mathcal{L}_N$ is an involutive distribution of dimension n_1, which generates a foliation of dimension n_2 and $\Psi(x_1, x_2) = 0$ is an invariant leaf with rank$\left(\frac{\partial \Psi(x_1,x_2))}{\partial x_i}\right)\Big|_{\Psi(x_1,x_2)=0} = n_2$ for $i = 1, 2$.

(3.) implies (1.): By assumption $\dim \mathcal{L}_N = n_1$ locally, is involutive, $\Psi(x_1, x_2) = 0$ is an invariant leaf so that $\dot{\Psi} = 0$ and $d\Psi \in \mathcal{L}_N^{\perp}$ with

$\text{rank}\left[\frac{\partial\Psi}{\partial x_i}\right]_{|\Psi(x_1,x_2)=0} = n_2$ for $i = 1, 2$. Consequently one can locally compute the solution $x_2 = \varphi_1(x_1)$ satisfying $\Psi(x_1, \varphi_1(x_1)) = 0$. Set now $z = x_2 - \varphi_1(x_1)$ By construction $z \equiv 0$ on the invariant leaf, and thus

$$\dot{x}_2|_{x_2=\varphi_1(x_1)} - \frac{\partial\varphi_1}{\partial x_1}\dot{x}_1 = 0,$$

that is

$$f_2(\varphi_1(x_1)) = \frac{\partial\varphi_1(x_1)}{\partial x_1} f_1(x_1)$$
$$g_{2j}(\varphi_1(x_1)) = \frac{\partial\varphi_1(x_1)}{\partial x_1} g_{1j}(x_1), \qquad j = 1, \cdots, m,$$

which shows that Σ_1 projects onto Σ_2, which ends the proof. □

4.2 The general case

Before giving the main result which fully characterizes the existence of a solution to the decomposition problem, let's work out the following illustrative example.

Example 3 Consider systems Σ_1 and Σ_2

$$\Sigma_1 : \begin{cases} \dot{x}_1 = x_1^2 + x_1 u \\ \dot{x}_2 = x_1^2 \end{cases} \qquad \Sigma_2 : \{ \dot{z}_1 = z_1^2 + z_1 u.$$

which are strongly accessible and satisfy $\dim \mathcal{L}^i_{n_i} = n_i$ for $i = 1, 2$ respectively for $x_1 \neq 0$ and $z_1 \neq 0$. Clearly Σ_2 is a subsystem of Σ_1. Nevertheless, for the composite system

$$\mathcal{L}_N = \text{span}\left\{ \begin{pmatrix} x_1 \\ 0 \\ z_1 \end{pmatrix}, \begin{pmatrix} -x_1^2 \\ -2x_1^2 \\ -z_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2x_1^3 \\ 0 \end{pmatrix} \right\}.$$

has and $\dim \mathcal{L}_N = 3 > n_1$ for $2x_1^4 z_1(x_1 - z_1) \neq 0$. ◁

Example 2 and 3 stress the fact that $\dim \mathcal{L}_N = n_1$ is neither necessary nor sufficient. In fact, in general when a solution exists, for the composite system $\dim \mathcal{L}_N = n_1 + \ell$ for some $0 \leq \ell \leq n_2$.

In this section necessary and sufficient conditions are derived for the solvability of the decomposition problem. Theorem 1 below represents the main result.

Theorem 1 *Let Assumption 1 hold true. Σ_2 is the projection of Σ_1 through the mapping $x_2 = \varphi_1(x_1)$ with* $\text{rank}\left(\frac{\partial\varphi_1}{\partial x_1}\right) = n_2$*, locally for any $x_1 \in V^1_{x_{10}}$, if and only if, setting $x = (x_1^T, x_2^T)^T$ and $\Psi(x) = x_2 - \varphi_1(x_1)$,*

$$S = \left\{ x \in R^N \mid \Psi(x) = 0, \text{rank}\left(\frac{\partial\varphi_1}{\partial x_1}\right) = n_2 \right\} \tag{18}$$

is an invariant manifold of dimension n_1 for the composite system Σ and the distribution $\mathcal{L}_N$ associated to the composite system Σ has dimension n_1 on S.

Accordingly, there exists $\varphi_2(x_1)$ such that $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \varphi_1(x_1) \\ \varphi_2(x_1) \end{pmatrix}$ *defines a change of coordinates in which the system admits the triangular form (8), (9).*

Proof **Necessity.** If Σ_1 projects onto Σ_2 through the mapping $x_2 = \varphi_1(x_1)$, then equations (10) hold true. Set $\Psi(x) = x_2 - \varphi_1(x_1) = 0$ so that

$$\begin{aligned} \frac{d}{dt}\Psi(x)\bigg|_{\Psi(x)=0} &= \dot{x}_2|_{\Psi(x)=0} - \frac{\partial\varphi_1(x_1)}{\partial x_1}\dot{x}_1 \\ &= f_2(\varphi_1) + \sum_{j=1}^m g_{2j}(\varphi_1)u_j \\ &\quad - \frac{\partial\varphi_1(x_1)}{\partial x_1}\left(f_1 + \sum_{j=1}^m g_{1j}u_j\right) = 0 \end{aligned}$$

which shows that $S = \{x \in R^N \mid \Psi(x) = 0\}$ is an invariant manifold. Since $\text{rank}\frac{\partial\Psi}{\partial x_i} = n_2$, for $i = 1, 2$, and $x_1 \in V^1_{x_{10}}$ by assumption, $\dim S(\cdot) = n_1$. To show that $\dim \mathcal{L}_N(S) = n_1$ recall that according to Corollary 1, $ad^k_{f_2} g_{2j}$ and $ad^k_{f_1} g_{1j}$ are φ_1-related so that it is easily verified that $d\Psi = (-\frac{\partial\varphi_1}{\partial x_1}, 1) = d\Psi|_{\Psi=0} \in \mathcal{L}_N|^{\perp}_{\Psi=0}$. Consequently $\dim \mathcal{L}_N(S) = n_1$. Since $\text{rank}\frac{\partial\varphi_1(x_1)}{\partial x_1} = n_2$ there exists a function $\varphi_2(x_1)$ such that $\varphi = (\varphi_1^T, \varphi_2^T)$ defines a local change of coordinates. Set $z_1 = \varphi_1$ and $z_2 = \varphi_2$. One gets from

$$\begin{aligned} \dot{z}_1 &= \left[\frac{\partial\varphi_1}{\partial x_1}(f_1(x_1) + \sum_{j=1}^m g_{1j}(x_1)u_j)\right]\bigg|_{\varphi^{-1}(z)} \\ &= \left[f_2(\varphi_1(x_1)) + \sum_{j=1}^m g_{2j}(\varphi_1(x_1))u_j\right]\bigg|_{\varphi^{-1}(z)} \\ &= f_2(z_1) + \sum_{j=1}^m g_{2j}(z_1)u_j \end{aligned}$$

$$\begin{aligned} \dot{z}_2 &= \left[\frac{\partial\varphi_2}{\partial x_1}(f_1(x_1) + \sum_{j=1}^m g_{1j}(x_1)u_j)\right]\bigg|_{\varphi^{-1}} \\ &= f_3(z_1, z_2) + \sum_{j=1}^m g_{3j}(z_1, z_2)u_j \end{aligned}$$

which show that the system is in the form (8), (9).

Sufficiency. Assume that there exists an invariant manifold S given by (18) and that the distribution $\mathcal{L}_N$ has dimension n_1 on S. Since S is invariant, $\frac{d}{dt}\Psi(x)|_{\Psi=0} = 0$, which means that

$$f_2(\varphi_1(x_1)) + \sum_{j=1}^{m} g_{2j}(\varphi_1(x_1))u_j - \frac{\partial \varphi_1(x_1)}{\partial x_1}(f_1(x_1) + \sum_{j=1}^{m} g_{1j}(x_1)u_j)) = 0.$$

For $u = 0$, we get that the first relation in (10) is satisfied and consequently for $j = 1, \cdots, m$ the second relation in (10) is also satisfied. Since rank $\frac{\partial \varphi_1(x_1)}{\partial x_1} = n_2$, there exists $\varphi_2(x_1)$ such that in the coordinates $z_1 = \varphi_1$ and $z_2 = \varphi_2$ the system reads (8), (9). Finally Σ_1 projects onto Σ_2 through the mapping $x_2 = \varphi_1(x_1)$. □

Remark 1 It should be noted that while the decomposition is computed starting from $\mathcal{L}_N$, its validity could hold also in the points where $\mathcal{L}_N$ has not dimension n_1. This is the case in Example (3) where the decomposition holds true also for $x_1 = z_1 = 0$

Remark 2 As already noted while in the linear case, the decomposition problem has a solution if and only if $\mathcal{L}_N$ has dimension n_1, this condition is neither necessary nor sufficient in the nonlinear case, where instead the solution is linked to finding an invariant submanifold S such that $\dim \mathcal{L}_N(S) = n_1$. This is a major difference that shows how the conditions in the nonlinear case are more involved. Example 3 and Example 4 in the next section aim to highlight this peculiarity.

Remark 3 The computation of the desired change of coordinates comes from Proposition 6 and Theorem 7. In general $\dim(\mathcal{L}_N) \geq n_1$. If the dimension is greater than n_1, then one has to compute constraints of the form $\ell(x_1, x_2) = 0$ which ensure the necessary condition $\dim(\mathcal{L}_N|_{\ell=0}) = n_1$ and which partially define S. One possible procedure consists in considering the matrix $M(x_1, x_2)$ whose columns generate $\mathcal{L}_N(x_1, x_2)$. The practical computation of such constraints $\ell(x_1, x_2) = 0$ comes from zeroing the minors of $M(x_1, x_2)$, as done in Example 3.

However, the obtained constraints $\ell(x_1, x_2) = 0$ may not yield a solution as in the case of example 2. In this case no solution exists at all.

Otherwise, it is also necessary that $d\ell(x) \perp \mathcal{L}_N|_{\ell=0}$ and the required change of coordinates is then obtained by integrating the left-annihilator of $\mathcal{L}_N|_{\ell=0}$, as in Example 4.

In the next section some examples are ruled out to show how to apply the hints in Remark 3 to compute the solution whenever it exists.

We end this Section by showing that the equivalence of a nonlinear system to a linear one is a specific case of the more general result given here.

Proposition 7 *Consider the continuous time systems Σ_1 and Σ_2 defined by (7) with $f_2(x_2) = A_2x_2$ and $g_{2i} = B_{2i}$ for $i = 1, \cdots, m$. Let $n_1 = n_2$ and let Assumption 1 hold true. Then Σ_1 is equivalent to the linear system Σ_2 if and only if for the composite system Σ given by (11) the accessibility distribution $\mathcal{L}_N$ has dimension n_1.*

Proof **Necessity.** Assume that they are equivalent. Then there exists $z = \phi(x_1)$ such that

$$\left(\frac{\partial \phi(x_1)}{\partial x_1} ad^s_{f_1(x_1)} g_{1j}(x_1)\right)\Bigg|_{x_1=\phi^{-1}(z)} = A_2^r B_{2j},\ for\ r \geq 0,\ \ j \in [1, m].$$

Since ϕ is a diffeomorphism, one also has

$$\left[\ \frac{\partial \phi(x_1)}{\partial x_1}\ -I\ \right]\left[\begin{pmatrix} g_{1j}(x_1) \\ B_{2j} \end{pmatrix}, \begin{pmatrix} ad_{f_1(x_1)} g_{1j}(x_1) \\ A_2 B_{2j} \end{pmatrix}, \cdots, \begin{pmatrix} ad^{N-1}_{f_1(x_1)} g_{1j}(x_1) \\ A_2^{N-1} B_{2j} \end{pmatrix},\ j = 1, \cdots, m\right] = 0.$$

so that necessarily $\mathcal{L}_N$ must have dimension n_1.

Sufficiency. Assume that $\mathcal{L}_N$ has dimension n_1. Due to the linearity of Σ_2, by construction, the last n_1 rows have constant elements. Due to the controllability assumption on Σ_2, let $k_1, \cdots, k_m$ be the associated controllability indices. Then $\mathcal{L}_N$ can be chosen as

$$\mathcal{L}_N = \text{span}\left\{\begin{pmatrix} g_{1j}(x_1) \\ B_{2j} \end{pmatrix}, \cdots \begin{pmatrix} ad^{k_j-1}_{f_1(x_1)} g_{1j}(x_1) \\ A_2^{k_j-1} B_{2j} \end{pmatrix}, j = 1, \cdots, m\right\},$$

Consequently $\mathcal{L}^1_{n_1} = \text{span}\{g_{1j}(x_1), \cdots, ad^{k_j-1}_{f_1(x_1)} g_{1j}(x_1), j = 1, \cdots, m\}$ and any vector $ad^s_{f_1(x_1)} g_{1i}(x_1) \in \text{span}_R \mathcal{L}_{n_1}$.

Thus any two vector fields τ_i and τ_j in $\mathcal{L}_N$ satisfy the condition $[\tau_i(x_1, x_2), \tau_j(x_1, x_2)] = 0$ that is $\mathcal{L}_N$ is not only involutive but also nilpotent. As a consequence $[ad^l_{f_1} g_{1i}(x_1), ad^s_{f_1} g_{1j}(x_1)] = 0$ for all $l, s \geq 0$ and $i, j \in [1, m]$, which proves that Σ_1 is necessarily equivalent to a linear system [15,27].

System Σ_1 can then be written as a linear system in some suitable coordinates $z = \varphi(x_1)$. The dimension of $\mathcal{L}_N$ remains unchanged in the coordinates (z, x_2) and equals n_1. Thanks to Remark 2, the linear system Σ_2 is equivalent to the linear system Σ_1 and the result follows. □

5 Examples

In this section, the main results are now tested on the examples.

Example 1 cont'd.

The distribution $\mathcal{L}^1_{n_1}$ of Σ_1 is

$$\mathcal{L}^1_{n_1} = \text{span}\left\{\begin{pmatrix}\cos x_3\\ \sin x_3\\ 0\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 0\\ 1\\ x_2\end{pmatrix}, \begin{pmatrix}\sin x_3\\ -\cos x_3\\ 0\\ \sin x_3\end{pmatrix}, \begin{pmatrix}\cos x_3\\ \sin x_3\\ 0\\ 2\cos x_3\end{pmatrix}\right\}$$

For the composite system, the distribution $\mathcal{L}_N$ is computed as

$$\mathcal{L}_N = \text{span}\left\{\begin{pmatrix}\cos x_3\\ \sin x_3\\ 0\\ 0\\ 0\\ 1\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 0\\ 1\\ x_2\\ 1\\ -z_3\\ z_2\end{pmatrix}, \begin{pmatrix}\sin x_3\\ -\cos x_3\\ 0\\ \sin x_3\\ 0\\ 0\\ 1\end{pmatrix}, \begin{pmatrix}\cos x_3\\ \sin x_3\\ 0\\ 2\cos x_3\\ 0\\ 1\\ 0\end{pmatrix}\right\}$$

which has dimension 4. Its left annihilator has dimension 3 and standard computations lead to the generators

$$d\phi = \begin{pmatrix} d(x_3 - z_1)\\ d(x_2 - z_2 \sin x_3 + z_3 \cos x_3)\\ d(x_1 - z_3 \sin x_3 - z_2 \cos x_3)\end{pmatrix}.$$

Since Assumption 1 is verified, according to Proposition 6 and its notations one gets

$$z = \varphi_1(x) = \begin{pmatrix} x_3\\ x_2 \sin(x_3) + x_1 \cos(x_3)\\ x_1 \sin(x_3) - x_2 \cos(x_3)\end{pmatrix}.$$

which corresponds to the mapping (19) as expected. The invariant leaf is defined by $\Psi = 0$ with $\Psi(x, z) = \begin{pmatrix} z_1 - x_3\\ z_2 - x_2 \sin(x_3) - x_1 \cos(x_3)\\ z_3 - x_1 \sin(x_3) + x_2 \cos(x_3)\end{pmatrix}$.

Compute $d\Psi(x, z)/dt$. From (1), $\dot{z}_1 - \dot{x}_3$ is zero.

The second component $d(z_2 - x_2 \sin x_3 + x_1 \cos x_3)/dt$ equals

$$-z_3 u_2 - x_2 u_2 \cos x_3 + x_1 u_2 \sin x_3 =$$
$$-u_2(z_3 - x_1 \sin x_3 + x_2 \cos x_3).$$

The third component $d(z_3 - x_1 \sin x_3 + x_2 \cos x_3)/dt$ is computed as

$$u_2(z_2 - x_1 \cos x_3 - x_2 \sin x_3).$$

Any trajectory starting on the leaf $\Psi = 0$ remains on the leaf $\Psi = 0$ which is thus an invariant leaf. □

Example 2 cont'd. As already noted the left annihilator of the distribution $\mathcal{L}_N$ associated to the composite system is spanned by $d\lambda = d(\frac{x_2}{x_1})$. The associated autonomous dynamics $\dot{\lambda} = \lambda$, has an equilibrium at the origin, which corresponds to $x_2 = 0$ where Assumption 1 does not hold anymore, so Proposition 6 and Theorem 1 do not apply. Furthermore $\text{rank}\left[\frac{\partial\lambda}{\partial x_2}\right]_{|\lambda(x_1,x_2)=0} = 0$, and the implicit function theorem does not apply to derive $x_2 = \varphi_1(x_1)$. □

Example 3 cont'd As already noted $\dim \mathcal{L}_{n_i} = n_i$ for $i = 1, 2$ and $\dim \mathcal{L}_N = 3 > n_1$ for $2x_1^4 z_1(x_1 - z_1) \neq 0$. It is easy to verify that for $x_1 = z_1$ $\dim \mathcal{L}_N = 2 = n_1$ and $d\phi = d(z_1 - x_1) \in \mathcal{L}_N^{\perp}|_{\phi=0}$.

As a matter of fact Σ_1 projects onto Σ_2 through the mapping obtained by $z_1 = x_1$, that is $\phi = 0$. The invariant submanifold previewed in Theorem 1 is thus S defined in (18) with $\Psi = \phi = z_1 - x_1 = 0$. ◁

Example 4 Let us now consider the two systems

$$\Sigma_1 : \begin{cases} \dot{x}_{11} = x_{12}\\ \dot{x}_{12} = x_{12}^2 - 2x_{12}x_{13}^2 + x_{13}^4 + x_{12}x_{13} - x_{13}^3 + 2x_{13}u\\ \dot{x}_{13} = u. \end{cases}$$

$$\Sigma_2 : \begin{cases} \dot{x}_{21} = x_{21}^2 + x_{21}x_{22}\\ \dot{x}_{22} = u \end{cases}$$

and let us check if Σ_2 projects onto Σ_1. $\dim \mathcal{L}^1_{n_1} = n_1$ for $x_{12} - x_{13}^2 \neq 0$ and $x_{13} \neq 1/2$ and $\dim \mathcal{L}^2_{n_2} = n_2$ for $x_{21} \neq 0$.

For the composite system denoting by $\theta(x) = x_{12} - x_{13}^2$, one has

$$\mathcal{L}_N = \text{span}\left\{\begin{pmatrix}0\\ 2x_{13}\\ 1\\ 0\\ 1\end{pmatrix}, \begin{pmatrix}-2x_{13}\\ -\theta(x)\\ 0\\ -x_{21}\\ 0\end{pmatrix}, \begin{pmatrix}\theta(x)\\ \theta^2(x)\\ 0\\ x_{21}^2\\ 0\end{pmatrix}, \begin{pmatrix}x_{13}\theta(x)\\ x_{13}\theta^2(x)\\ 0\\ x_{21}^2 x_{22}\\ 0\end{pmatrix}\right\}$$

Note that $\dim \mathcal{L}_N = 4$ almost everywhere. The problem under consideration can be solved for example by following the procedure sketched in Remark 3.

Compute the minor consisting of the first 4 rows of $\mathcal{L}_N$. This minor is zero for $x_{21} = \theta(x)$ and $x_{22} = x_{13}$. In fact setting $\ell(x) = \begin{pmatrix} x_{21} - \theta(x)\\ x_{22} - x_{13}\end{pmatrix}$, one gets that

$\dim\,(\mathcal{L}_n|_{\ell(x)=0}) = n_1$ or equivalently, looking at vector fields, that

$$\left.\begin{pmatrix} x_{13}\theta(x) \\ x_{13}\theta^2(x) \\ 0 \\ x_{21}^2 x_{22} \\ 0 \end{pmatrix}\right|_{\substack{x_{21}=x_{12}-x_{13}^2 \\ x_{22}=x_{13}}} \in \operatorname{span}\left.\left\{\begin{pmatrix} 0 \\ 2x_{13} \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2x_{13} \\ -\theta(x) \\ 0 \\ -x_{21} \\ 0 \end{pmatrix}, \begin{pmatrix} \theta(x) \\ \theta^2(x) \\ 0 \\ x_{21}^2 \\ 0 \end{pmatrix}\right\}\right|_{\substack{x_{21}=x_{12}-x_{13}^2 \\ x_{22}=x_{13}}}.$$

Actually $\ell(x) = 0$ defines the submanifold S which is sought. In fact, the computation and integration of the left-annihilator of $\mathcal{L}_N|_{\ell(x)=0}$ shows that $d\ell(x) \in \mathcal{L}_N|_{\ell(x)=0}^{\perp}$ and allows to define the change of coordinates which solves the problem.

Summarizing, the solution $x_2 = \varphi_1(x_1)$ which is sought is given by

$$x_{21} = x_{12} - x_{13}^2$$
$$x_{22} = x_{13}$$

and defines the projection of Σ_1 onto Σ_2. □

6 Conclusions

Motivated by the study of robotic systems, the conditions under which two given nonlinear dynamics are related by a projection operation are fully characterized. The results are derived under the only assumption that the systems are smooth and accessible over their domain of definition. The main result is stated in Theorem 1 and gives necessary and sufficient conditions for a system to project onto another one. This result also gives a way to how the coordinates transformation φ can be identified and the definition of a procedure for computing it is the topic of future works in this area.

The avenues for further research are numerous: normal forms for classes of nonlinear systems, explicit equivalence for some classes of physical systems, the definition of a procedure to compute the mapping, the definition of the conditions when also feedback laws are considered as well as considering parameter uncertainties in the modeling of the dynamics.

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