Nonlinear Hamiltonian Systems Under Sampling

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***Abstract*—This article investigates the transformation of Hamiltonian structures under sampling. It is shown that the exact sampled equivalent model associated to a given port-Hamiltonian continuous-time dynamics exhibits a discrete-time representation in terms of the discrete gradient, with the same energy function but modified damping and interconnection matrices. By construction, the proposed sampled-data dynamics guarantees exact matching of both the state evolutions and the energy-balance at all sampling instants. Its generalization to port-controlled Hamiltonian dynamics leads to characterize a new power conjugate output so recovering the concept of average passivation. On these bases, energy-management control strategies can be proposed. An energetic interpretation of the approach is confirmed by its formulation in the Dirac formalism. Two classical examples are worked out to validate the proposed sampled-data modeling in a comparative way with the literature.**

***Index Terms*—Computational methods, energy systems, nonlinear systems, sampled-data control.**

I. Introduction

Gradient or Hamiltonian dynamics at large, endorsing straight relations with fundamental physical properties such as energy conservation or variation principles, have been widely investigated in the continuous-time setting (see [1] and [2] and references therein). Among the vast dedicated literature, two main conceptual approaches can be distinguished with their own features depending on the goal, modeling or control: the Dirac approach and the input-state-output approach. The first one, employing an abstract generalization of the bond-graph formalism, is efficient to model the interconnections and the energy flowing between the components of the system [3]–[5]. The second one, based on the differential representation of the Hamiltonian function, is well suited for energy-based design [6]–[12]. Nowadays, an efficient interplay between both approaches supports innovative research in various physical domains where energy and energy exchanges serve as lingua franca (e.g., [5], [8], and [13]).

All of this essentially concerns the continuous-time framework while a consensus on a unifying input-state-output representation of Hamiltonian structures in discrete time is still missing. As in continuous time, the aforementioned approaches coexist, describing port-Hamiltonian systems on discrete manifolds in the Dirac formalism [14]–[16] or Hamiltonian difference equations in a state-space-oriented approach [17]–[21]. In this latter case, the discrete gradient function that properly expresses the variation of a function between two points, is used in place of the usual gradient function. Such a choice, that opens toward discrete structures looking similar to the continuous-time ones, unfortunately brings to implicit difference state-space representations [18], [22]–[24]. The interest of the community recently extends to a digital environment including dynamics issued from sampling, with direct impact in real-time applications. In this context, several works find their roots in various sampling and time-integration devices so providing Hamiltonian forms through symplectic integration [25]–[27], geometric scattering [28], spatial discretization of the continuous-time energy-balance [29], [30], Galerkin discretization [31] up to numerical methods like Runge–Kutta [32], and Gauss–Legendre [33]. Despite these studies provide structures satisfying the required energy-balance properties, well identified links between all these forms are still missing.

The attempt of this work is to fill this gap starting from the understanding of the structure of Hamiltonian representations under sampling. The study is performed for nonlinear dynamics and set in a formal way, inspired from the Lie series framework involved in the characterization of sampled dynamics associated to nonlinear differential dynamics developed in [34]. Our analysis is motivated by a recent authors' contribution [24] where a discrete-time port-Hamiltonian representation embedding the required properties of energy conservation or dissipation has been introduced. These discrete-time forms are adopted in this study as target structures to recover under exact sampling.

Along these lines, this article addresses the preservation of Hamiltonian structures under sampling or time integration at large in a nonlinear context. In all cases (gradient, port-Hamiltonian, and port-controlled Hamiltonian), we show that similar discrete-time structures can be recovered under exact sampling with respect to the same Hamiltonian function with modified structural matrices (embedding the interconnection and the dissipation matrices). The relevant consequence of

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preserving the Hamiltonian while assuring exact matching of the state evolutions lies in the characterization of an energy-like representation that can be profitably employed in the design. The approach is constructive. The solutions are described by their series expansions in powers of the sampling period that can be computed, iteratively, in an approximation perspective. In addition, the continuous-time model is recovered in first approximation so preserving a physical interpretation of the results. To strengthen physical validity, the associated discrete Dirac structure is described and compared with the underlying continuous-time one.

The contributions of this work are specified in the sequel.

1) The formal expressions of discrete gradient, Jacobian, and Hessian along a given sampled-data dynamics proposed in Proposition 5 specify the basic formal instruments. These forms provide new computable series expansions of these objects around the usual gradient, Jacobian, and Hessian.
2) The sampled-data equivalent forms to continuous-time gradient and Hamiltonian structures are specified in Theorems 1 and 3. Gradient forms are transformed into Hamiltonian-like ones with modified dissipation matrix; conservative or dissipative Hamiltonian structures are transformed into equivalent structures that preserve the continuous-time energy-balance equality (EBE) at the sampling instants. All these forms are parameterized by the sampling period.
3) The characterization of port-controlled Hamiltonian dynamics under sampling is in Theorem 5 opening toward energy-based stabilizing control strategies. A power-conjugate output is defined and its relation with the average-passifying output introduced in [35] is clarified. Accordingly, it is briefly discussed how digital damping through negative output feedback may be worked out for stabilization.
4) The discrete Dirac structure sustaining the proposed sampled-data form is described in Theorem 7 enforcing its physical interpretation.

Besides their theoretical interests, the results are constructive and open wide perspectives to digital energy management and structure assignment designs as well as networked modeling of interconnected structures. A preliminary work making reference to linear time-invariant (LTI) models is in [36].

The rest of this article is organized as follows. Notations and recalls are in Section II to fix port-Hamiltonian input-state-output representations in both the continuous and discrete-time frameworks. The notions of discrete gradient, Jacobian, and Hessian are given. The section ends formalizing the general question addressed in thise article with the LTI case discussed as a motivating example. In Section III, instrumental results, developed in the formalism of the algebra of series, are proved as fundamental tools to reshape exact sampled-data models into discrete port-Hamiltonian structures. Section IV addresses the question for gradient dynamics as a preliminary attempt toward the case of port-Hamiltonian dynamics. The main results are in Sections V where sampled-data input-state-output representations of port-Hamiltonian dynamics are described and their structural and energetic properties highlighted. Section VI discusses the extension to port-controlled Hamiltonian dynamics. On these bases, energy-based stabilizing control strategies are briefly discussed. In Section VII, simulated case studies are worked out to illustrate computational aspects. Finally, Section VIII concludes this article.

II. Prolegomena

A. Some Notations

All functions and vector fields defining the dynamics are assumed smooth and complete over the respective definition spaces. $\mathbb{R}$ and $\mathbb{N}$ denote the set of real and natural numbers including 0, respectively. $\mathbb{R}^{n\times m}$ denotes the set of real valued $n\times m$ matrices. For any vector $v\in\mathbb{R}^n$, $|v|$ and $v^\top$ define the norm and transpose of v, respectively. I_d denotes the identity function on the definition space, while I denotes the identity operator and the identity matrix when related to a linear operator. For a matrix $A\in\mathbb{R}^{n\times n}$, we denote by A^{-1} its inverse, and for the sake of the notations, its pseudoinverse when singular. The symmetric and skew-symmetric parts of $A\in\mathbb{R}^{n\times n}$ are denoted by $\texttt{sym}(A)=\frac{1}{2}(A+A^\top)$ and $\texttt{skew}(A)=\frac{1}{2}(A-A^\top)$. Given a differentiable real-valued function $V(\cdot):\mathbb{R}^n\to\mathbb{R}$, ∇V represents the gradient column-vector with $\nabla=\mathrm{col}\{\frac{\partial}{\partial x_i}\}_{i=1,n}$ and $\nabla^2 V$ denotes its Hessian matrix with $\nabla^2=\{\frac{\partial^2}{\partial x_i,\partial x_j}\}_{i,j=1,n}$. Given a vector-valued function $F(x)=\mathrm{col}(F_1(x),\dots,F_n(x))$, $J[F(x)]=\{\frac{\partial}{\partial x_j}F_i(x)\}_{i,j=1,n}$ denotes the Jacobian of F.

The symbols $''>0''$ and $''<0''$ denote positive and negative definite functions (or matrices), respectively. Indicating by X a formal variable that can represent an operator (or a matrix in the linear case), one defines the formal exponential series $e^X=I+\sum_{p\geq 1}\frac{X^p}{p!}$ with X^p the power p composition of X with respect to a given product. Accordingly, formal manipulations are worked out as the inverse series denoted by $(I+X)^{-1}$ with $(I+X)^{-1}=I+\sum_{p\geq 1}(-1)^p X^p$, or formal quotient denoted by $\frac{e^X-1}{X}$, that stands for the formal cancellation of X in the numerator series e^X-1 so getting $\frac{e^X-I}{X}=I+\sum_{p\geq 1}\frac{X^p}{(p+1)!}$; similar rules apply to define formal algebraic relations along this article.

Given a smooth vector field f over $\mathbb{R}^n$, $\mathrm{L}_f=\sum_{i=1}^n f_i(x)\frac{\partial}{\partial x_i}$ denotes its associated Lie derivative operator. Setting $\mathrm{L}_f^0=I$, one iteratively defines the composition at power p as $\mathrm{L}_f^p=\mathrm{L}_f\mathrm{L}_f^{p-1}$; $e^f:=I+\sum_{p\geq 1}\frac{\mathrm{L}_f^p}{p!}$ represents the exponential Lie series operator recovering, for linear vector fields, the exponential of the matrix representing the operator. For any smooth function $h(\cdot):\mathbb{R}^n\to\mathbb{R}$, the equality of functions $e^f h(x)=h(e^f(x))=e^f h|_x$ holds true [37, p. 22], where $|_x$ denotes the evaluation of the function at x. Given a function of time $\gamma(t)$, $\gamma_k:=\gamma(k\delta)$ denotes its value at time $t=k\delta$, with $k\in\mathbb{N}$ for a fixed $\delta\in]0,T^\star[$, $T^\star>0$. The root-mean-square error (RMSE) between a continuous-time function $\gamma(t)$ and a discrete sequence $\gamma_d(k\delta)$ is given for $T>0$ by $\mathrm{RMSE}=\sqrt{\sum_{k=1}^T\frac{1}{T}(\gamma_d(k\delta)-\gamma(t)|_{t=k\delta})^2}$. A function $R(x,\delta):\mathcal{B}\times\mathbb{R}\to\mathbb{R}^n$ is said in $O(\delta^p)$ with $p\geq 1$, if it can be written as $R(x,\delta)=\delta^{p-1}\tilde{R}(x,\delta)$ for all $x\in\mathcal{B}$ and there exist a function $\theta\in\mathcal{K}_\infty$ and $\delta^\star>0$ s.t. $\forall\delta\leq\delta^\star$, $|\tilde{R}(x,\delta)|\leq\theta(\delta)$.

Given two matrices $A\in\mathbb{R}^{n_1\times n_2}$ and $B\in\mathbb{R}^{m_1\times m_2}$, the Kronecker product is denoted by $A\otimes B\in\mathbb{R}^{n_1m_1\times n_2m_2}$. Given a matrix-valued function $L(x):\mathbb{R}^n\to\mathbb{R}^{n\times n}$, the following representations are set under differentiation:

$$\frac{\partial L(x)}{\partial x}=\frac{\partial}{\partial x}\otimes L(x)=\begin{pmatrix}\frac{\partial L(x)}{\partial x_1} & \dots & \frac{\partial L(x)}{\partial x_n}\end{pmatrix}\in\mathbb{R}^{n\times n^2}$$

and consequently, for an n-dimensional vector, $N(x):\mathbb{R}^n\to\mathbb{R}^n$

$$\begin{aligned}\frac{\partial L(x)N(x)}{\partial x}&=\frac{\partial L(x)}{\partial x}(N(x)\otimes I)+L(x)\frac{\partial N(x)}{\partial x}\\&=L(x)\frac{\partial}{\partial x}N(x)+\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i}L(x)\right)N(x).\end{aligned}$$

B. Discrete Gradient Function

Let us first recall from the literature (see [18]–[20]) the definition of discrete gradient function.

Definition 1 (Discrete gradient): Given a smooth real-valued function $V(\cdot):\mathbb{R}^n\to\mathbb{R}$, its discrete gradient is a vector-valued function of two variables, $\bar\nabla V\Big|_{\cdot}^{\cdot}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ satisfying the variational equality

$$V(w)-V(v)=(w-v)^\top\bar\nabla V\Big|_v^w \tag{1}$$

with, by continuity, $\bar\nabla V\Big|_v^v=\nabla V(v)$.

Setting $\bar\nabla V\Big|_v^w=col\{\bar\nabla_i V\Big|_v^w\}_{i=1,n}$ with $v=\mathrm{col}\{v_1,\dots,v_n\}$ and $w=\mathrm{col}\{w_1,\dots,w_n\}$, the discrete gradient can be computed component-wise through the integral form

$$\bar\nabla_i V\Big|_v^w=\frac{1}{w_i-v_i}\int_{v_i}^{w_i}\frac{\partial V(w_1,\dots,w_{i-1},\xi,v_{i+1},\dots,v_n)}{\partial\xi}\mathrm{d}\xi. \tag{2}$$

From this definition, the extended notions of discrete Jacobian and discrete Hessian are now introduced.

Definition 2 (Discrete Jacobian and discrete Hessian): Given a vector function $F=\mathrm{col}\{F_1,\dots,F_n\}:\mathbb{R}^n\to\mathbb{R}^n$ with $F_i(\cdot):\mathbb{R}^n\to\mathbb{R}$, its discrete Jacobian is a matrix-valued function of two variables $\bar J[F]\Big|_{\cdot}^{\cdot}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{n\times n}$ satisfying

$$F(w)-F(v)=\bar J[F]\Big|_v^w(w-v)=col\left\{(w-v)^\top\bar\nabla F_i\Big|_v^w\right\}_{i=1,n} \tag{3}$$

and $\bar J[F]\Big|_v^v=J[F(v)]$ when defining $\bar J[F]\Big|_v^w=[\bar\nabla_j[F_i]\Big|_v^w]_{i,j=1,n}$ with $\bar\nabla F_i\Big|_v^w=col\{\bar\nabla_j F_i\Big|_v^w\}_{j=1,n}$.

When $F(\cdot)=\nabla V(\cdot)$, one gets analogously the discrete Hessian matrix of $V(\cdot)$ as $\bar\nabla^2 V\Big|_v^w=\bar J[\nabla V]\Big|_v^w$ with $\bar\nabla^2 V\Big|_v^v=\nabla^2 V(v)$.

According to the aforementioned definitions, the discrete gradient, Jacobian, or Hessian are not uniquely defined. In the following, the following general constructive forms are used.

Proposition 1: For a given smooth function $V(\cdot):\mathbb{R}^n\to\mathbb{R}$, by the mean value theorem, one gets

$$V(w)-V(v)=(w-v)^\top\int_0^1\nabla V\Big|_{v+s(w-v)}\mathrm{d}s$$

so providing the constructive form of the discrete gradient

$$\bar\nabla V\Big|_v^w=\int_0^1\nabla V\Big|_{v+s(w-v)}\mathrm{d}s \tag{4}$$

with $v+s(w-v)=\mathrm{col}\{v_1+s(w_1-v_1),\dots,v_n+s(w_n-v_n)\}$. For a given smooth vector function $F(\cdot):\mathbb{R}^n\to\mathbb{R}^n$, one gets

$$F(w)-F(v)=\int_0^1 J[F]\Big|_{v+s(w-v)}\mathrm{d}s(w-v)$$

so providing the constructive forms of the discrete Jacobian

$$\bar J[F]\Big|_v^w=\int_0^1 J[F]\Big|_{v+s(w-v)}\mathrm{d}s \tag{5}$$

and of the discrete Hessian when $F(\cdot)=\nabla V(\cdot)$

$$\bar\nabla^2 V\Big|_v^w=\bar J[\nabla V]\Big|_v^w=\int_0^1\nabla^2 V\Big|_{v+s(w-v)}\mathrm{d}s.$$

Remark 1: With reference to a quadratic function $V(v)=\frac{1}{2}v^\top Pv$ with $P=P^\top$, the discrete gradient takes the form

$$\bar\nabla V\Big|_v^w=\frac{1}{2}P(v+w)=\frac{1}{2}\nabla V(v+w). \tag{6}$$

Remark 2: With reference to a separable function $V(v)=\sum_{i=1}^n V_i(v_i)$, the integral form (2) simplifies for $i=1,\dots,n$ as

$$\bar\nabla_i V\Big|_v^w=\frac{1}{w_i-v_i}\int_{v_i}^{w_i}\frac{\mathrm{d}V_i(\xi)}{\mathrm{d}\xi}\mathrm{d}\xi=\int_0^1\nabla V_i\Big|_{v_i+s(w_i-v_i)}\mathrm{d}s.$$

C. Port-Hamiltonian Dynamics

Hamiltonian dynamics were historically expressed over $\mathbb{R}^{2n}$, in the canonical generalized coordinates (q,p) through a real-valued function $H(\cdot):\mathbb{R}^{2n}\to\mathbb{R}$. In the recent literature, port-Hamiltonian structures are commonly defined over $\mathbb{R}^n$, with the function $H(\cdot):\mathbb{R}^n\to\mathbb{R}$ catching the energy-like properties of the plant (e.g., [2] and [5]). Such dynamics are briefly recalled in the continuous- and discrete-time settings.

Definition 3: Given a smooth function $H(\cdot):\mathbb{R}^n\to\mathbb{R}$, a continuous-time port-Hamiltonian dynamics is given by

$$\dot x=f(x)=(J(x)-R(x))\nabla H(x) \tag{7}$$

with $J(x)=-J^\top(x)\in\mathbb{R}^{n\times n}$ and $R(x)=R^\top(x)\in\mathbb{R}^{n\times n}$ for $R(x)\geq 0$ the interconnection and damping matrices.

The following comments are in order:

1) all local extrema of $H(x)$ ($x_e\in\mathbb{R}^n$ such that $\nabla H(x_e)=0$) are equilibria of (7);
2) when $H(x)$ is bounded from below, stability of x_e follows from the variational inequality
$$\dot H(x)=\mathrm{L}_f H(x)=-\nabla^\top H(x)R(x)\nabla H(x)\leq 0$$
3) when $R(x)=0$, conservation of $H(x)$ follows;
4) when $J(x)$ and $R(x)$ are constant and $H(x)$ is a quadratic function, (7) specifies a linear Hamiltonian dynamics;
5) canonical Hamiltonian dynamics are referred to
$$R(x)=0\quad\text{and}\quad J(x)=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}$$
6) when $J(x)-R(x)=-I$, (7) is called *gradient dynamics*.

Port-Hamiltonian models employing the discrete gradient of $H(\cdot)$ have been proposed in discrete time as recalled here as follows (e.g., [19], [21]–[24], [38], and [39]).

Definition 4: Given a smooth function $H(\cdot):\mathbb{R}^n\to\mathbb{R}$, a discrete-time port-Hamiltonian dynamics can be described as

$$x^+=x+(J_d(x)-R_d(x))\bar\nabla H\Big|_x^{x^+} \tag{8}$$

where $J_d(x) = -J_d^\top(x) \in \mathbb{R}^{n\times n}$ and $R_d(x) = R_d^\top(x) \in \mathbb{R}^{n\times n}$ with $R_d(x) \geq 0$ are the interconnection and damping matrices; $x \in \mathbb{R}^n$ is the state at the discrete-time instant $k \geq 0$, while x^+ indicates its value one step ahead, that is at $k+1$.

Exploiting (8), one verifies the following expected energetic properties:

1) all local extrema $x_e \in \mathbb{R}^n$ of $H(\cdot)$ (i.e., $\bar\nabla H\Big|_{x_e}^{x_e} = \nabla H(x_e) = 0$) are equilibria of (8);
2) from (8) and Definition 1, one gets
$$H(x^+) - H(x) = -\bar\nabla^\top H\Big|_x^{x^+} R_d(x)\bar\nabla H\Big|_x^{x^+} \leq 0 \quad (9)$$
so that x_e is stable for (8) if $H(x)$ is bounded from below;
3) when $R_d(x) = 0$, the dynamics (8) is energy - conservative
$$H(x^+) - H(x) = \bar\nabla^\top H\Big|_x^{x^+} J_d(x)\bar\nabla H\Big|_x^{x^+} = 0.$$

The representation (8) specifies the state evolution through a set of implicit first-order difference equations. This represents the main source of difficulties: recovering an explicit state-space representation in the form of a map (say $x^+ = x + F(x)$) from (8) is not an easy task, except for the case of a quadratic Hamiltonian function.

Remark 3: Assuming $H(x) = \frac{1}{2}x^\top Px$, with $P = P^\top \geq 0$ and invoking Remark 1, the discrete dynamics (8) rewrites in explicit form as
$$x^+ = \left(I - \frac{1}{2}(J_d(x) - R_d(x))P\right)^{-1} \times \left(I + \frac{1}{2}(J_d(x) - R_d(x))P\right)x.$$

D. Problem Statement

Consider a generic nonlinear dynamics
$$\dot x = f(x) \quad (10)$$
with $f(\cdot): \mathbb{R}^n \to \mathbb{R}^n$. Denoting by $\delta > 0$ the sampling period, its equivalent sampled-data dynamics [34] specifying the one step-ahead evolution at time $t = (k+1)\delta$ starting from $t = k\delta$, gets the form of a map
$$x^+ = x + F^\delta(x) \quad (11)$$
with $x = x(k\delta)$, $x^+ = x((k+1)\delta)$. Whenever $f(x)$ is smooth, $F^\delta(x)$ admits for all $\delta \in]0, T^\star[$ the asymptotic expansion
$$F^\delta(x) = e^{\delta f}x - x = \delta f(x) + \sum_{i\geq 2}\frac{\delta^i}{i!}\mathrm{L}_f^{i-1}f(x). \quad (12)$$
with $e^{\delta f}x$ the Lie exponential applied to x, $T^\star > 0$ the convergence interval upper bound of the exponential expansion.

Assuming now a port-Hamiltonian continuous-time dynamics (7) with $f(x) = (J(x) - R(x))\nabla H(x)$, a natural question arises: Is the port-Hamiltonian structure preserved under sampling? More precisely, the problem we address is formally set as follows.

Problem 1: Consider a continuous-time port-Hamiltonian dynamics (7) in the sense of Definition 3 and let (11) be the sampled-data equivalent model. Compute, if any, matrices $J^\delta(f,x) = -J^{\delta^\top}(f,x)$ and $R^\delta(f,x) = R^{\delta^\top}(f,x) \geq 0$ verifying
$$F^\delta(x) = \delta\left(J^\delta(f,x) - R^\delta(f,x)\right)\bar\nabla H|_x^{x+F^\delta(x)}$$
i.e., its equivalent sampled-data model (12) admits a discrete-time port-Hamiltonian structure in the sense of Definition 4
$$x^+ = x + \delta\left(J^\delta(f,x) - R^\delta(f,x)\right)\bar\nabla H|_x^{x^+}$$
verifying for all $k \geq 0$, $x = x(k\delta)$ whenever $x_0 = x(0)$. $\triangleleft$

Solving this problem provides the following two immediate outcomes:

1) it provides a sampled-data Dirac structure;
2) it proves the preservation of port-controlled Hamiltonian structure under sampling [24].

We underline that for guaranteeing both exact sampling and preservation of the Hamiltonian (energetic) structure, the interconnection and damping matrices $J^\delta(f,x)$ and $R^\delta(f,x)$ will not be the same as in continuous time and will result to be explicitly parameterized by δ, the sampling period. This is motivated by the LTI case detailed as follows.

E. Case of LTI Dynamics

For LTI dynamics of the form
$$\dot x = (J - R)\nabla H(x) \quad (13)$$
with quadratic Hamiltonian $H(x) = \frac{1}{2}x^T Px$, a solution to Problem 1 has been given in [36]. Recalling in that case $\bar\nabla H|_x^{x^+} = \frac{1}{2}P(x + x^+)$, a discrete-time port-Hamiltonian representation of the sampled dynamics equivalent to (13) can be obtained by comparing the target implicit form (8) with the explicit sampled-data dynamics computed according to (12), i.e.,
$$x^+ = x + \frac{\delta}{2}(J^\delta - R^\delta)(x + x^+) \quad \equiv \quad x^+ = e^{\delta(J-R)P}x. \quad (14)$$

From the equivalence of the aforementioned forms, one deduces the matrix equalities
$$e^{\delta(J-R)P} = \left(I - \frac{\delta}{2}(J^\delta - R^\delta)\right)^{-1}\left(I + \frac{\delta}{2}(J^\delta - R^\delta)\right) \quad (15a)$$
$$\delta(J^\delta - R^\delta) = 2(e^{\delta(J-R)P} - I)(e^{\delta(J-R)P} + I)^{-1}P^{-1} \quad (15b)$$
with $J^\delta \to J$ and $R^\delta \to 0$ as $\delta \to 0$.

The modified structural matrices J^δ and R^δ are uniquely defined by (15b) and naturally depend on δ, the sampling period. As an example, for the simplest case of a canonical Hamiltonian dynamics over $\mathbb{R}^2$ ($R = 0, P = I$), one computes
$$e^{\delta J} = \begin{pmatrix}\cos\delta & \sin\delta\\ -\sin\delta & \cos\delta\end{pmatrix},\ \delta J^\delta = \begin{pmatrix}0 & \frac{2\sin\delta}{1+\cos\delta}\\ -\frac{2\sin\delta}{1+\cos\delta} & 0\end{pmatrix} \quad (16)$$
so characterizing the discrete-time Hamiltonian form as
$$x^+ = x + \delta J^\delta\bar\nabla H\Big|_x^{x+} = x + \begin{pmatrix}0 & \frac{\sin\delta}{1+\cos\delta}\\ -\frac{\sin\delta}{1+\cos\delta} & 0\end{pmatrix}(x + x^+) \quad (17)$$
matching the sampled evolutions of the continuous-time canonical port-Hamiltonian dynamics. The implicit representation (17) motivates the need of changing $J \to J^\delta$ to recover an equivalent sampled-data representation. Note that J^δ is not an arbitrary skew-symmetric matrix, but a proper δ-dependent matrix that infinitesimally recasts J as $\delta \to 0$.

III. Formal Instrumental Results

The results proposed in this section exploit the exponential form representation (12) of sampled-data dynamics to describe the discrete gradient function computed over a fixed sampled-data dynamics. They are instrumental for the results and further computational aspects.

Given a continuous-time dynamics (10) described by the vector field $f(x)$, let (11) be its associated sampled-data model described in the form of a map as in (12) by the map $F^\delta(x)$. Let the formal differential operator $\mathcal{D}_f^\delta$ be defined as

$$\mathcal{D}_f^\delta = \frac{e^{\delta f} - I}{\delta f} = I + \sum_{i\geq 1} \frac{\delta^i}{(i+1)!} \mathrm{L}_f^i. \tag{18}$$

The first proposition provides a matrix form representation of $F^\delta(x)$ in terms of $f(x)$.

Proposition 2: Given a continuous-time dynamics (10) on $\mathbb{R}^n$, then its sampled-data equivalent dynamics satisfies for all $\delta \in]0, T^\star[$ the equality

$$x^+ - x = F^\delta(x) = \delta M^\delta(f, x) f(x) \tag{19}$$

with square matrix $M^\delta(f, x)$, locally nonsingular, given by

$$M^\delta(f, x) = \frac{1}{\delta} \int_0^\delta J[x(s)]\mathrm{d}s = J[\mathcal{D}_f^\delta(x)] \tag{20}$$

and $x(s) = e^{sf}x$.

Proof: The proof follows from the definition of $x^+ = e^{\delta f}x = x + F^\delta(x)$ and computing

$$\begin{aligned}
e^{\delta f}x - x &= \delta f(x) + \frac{\delta^2}{2!}\mathrm{L}_f f(x) + \frac{\delta^3}{3!}\mathrm{L}_f^2 f(x) + \dots \\
&= \left(\delta J[x] + \frac{\delta^2}{2!} J[f] + \frac{\delta^3}{3!} J[\mathrm{L}_f f] + \dots) f(x\right) \\
&= J\left[\delta x + \frac{\delta^2}{2!} f(x) + \frac{\delta^3}{3!} \mathrm{L}_f f(x) + \dots\right] f(x) \\
&= J[\delta \mathcal{D}_f^\delta(x)] f(x) \\
&= J\left[\int_0^\delta e^{sf} x \mathrm{d}s\right] f(x) = \int_0^\delta J[e^{sf}x]\mathrm{d}s f(x) \\
&= \delta M^\delta(f, x) f(x)
\end{aligned}$$

with $M^\delta(f, x)$ as in (20). The nonsingularity of $M^\delta(f, x)$ follows by construction for δ small enough. ■

It is worth to note that $M^\delta(f, x)$ coincides with the Jacobian, evaluated at x, of the result of the differential operator $\mathcal{D}_f^\delta$ applied to the identity function; it admits an asymptotic expansion in powers of δ of the form

$$\begin{aligned}
M^\delta(f, x) &= J[\mathcal{D}_f^\delta(x)] = J\left[\frac{e^{\delta f} - I}{\delta f}(x)\right] \\
&= I + \frac{\delta}{2} J[f(x)] + \frac{\delta^2}{3!} J[J[f(x)]f(x)] \\
&\quad + \sum_{i\geq 3} \frac{\delta^i}{(i+1)!} J[\mathrm{L}_f^{i-1} f(x)].
\end{aligned} \tag{21}$$

Remark 4: According to (19), truncating the expansion $M^\delta(f, x)$ in (21) at any finite order in δ, provides approximate sampled-data dynamics of increasing order. Setting $M^\delta(f, x) = I$, the Euler approximation is recovered.

Remark 5: With reference to an LTI dynamics, $f(x) = Ax$, one gets $\mathscr{D}_{Ax}^\delta = (e^{\delta A} - I)(\delta A)^{-1}$ and constant matrix $M^\delta = (e^{\delta A} - I)(\delta A)^{-1}$ parameterized by δ and satisfying (19), i.e.,

$$\begin{aligned}
x^+ - x &= \delta M^\delta A x \\
&= \delta\left(I + \frac{\delta}{2!}A + \frac{\delta^2}{3!}A^2 + \cdots\right) Ax = e^{\delta A}x - x.
\end{aligned}$$

The next proposition specifies the variation of a real-valued function $V(\cdot)$ along the sampled-data dynamics (11).

Proposition 3: Given a smooth vector field $f(\cdot)$ on $\mathbb{R}^n$ and a smooth real-valued function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$, then the variation of the function $V(\cdot)$ along the associated sampled-data dynamics (11) satisfies for all $\delta \in]0, T^\star[$ the equality

$$\frac{V(x^+) - V(x)}{\delta} = \nabla^\top V_{\mathrm{av}}^\delta(f, x) f(x) = \mathrm{L}_f V_{\mathrm{av}}^\delta(f, x) \tag{22}$$

with

$$V_{\mathrm{av}}^\delta(f, x) = \frac{1}{\delta}\int_0^\delta V(x(s))\mathrm{d}s = \mathcal{D}_f^\delta(V)(x) \tag{23}$$

and $x(s) = e^{sf}x$.

Proof: The proof follows from the definition of $V(x^+) = e^{\delta f}V(x)$ when $x^+ = e^{\delta f}x$, so getting

$$\begin{aligned}
&e^{\delta f}V(x) - V(x) = \delta \mathrm{L}_f V(x) + \frac{\delta^2}{2!} L_f^2 V(x) + \frac{\delta^3}{3!} L_f^3 V(x) + \cdots \\
&= \delta \mathrm{L}_f\left(V(x) + \frac{\delta}{2!}\mathrm{L}_f V(x) + \frac{\delta^3}{3!}\ \mathrm{L}_f V(x) + \cdots\right) \\
&= \nabla^\top\left(\delta V(x) + \frac{\delta^2}{2!}\mathrm{L}_f V(x) + \frac{\delta^3}{3!}\mathrm{L}_f V(x) + \cdots\right) f(x) \\
&= \nabla^\top\left(\int_0^\delta e^{sf} V(x)\mathrm{d}s\right) f(x) = \nabla^\top\left(\int_0^\delta V(x(s))\mathrm{d}s\right) f(x).
\end{aligned}$$

■

$V_{\mathrm{av}}^\delta(\cdot)$ in (23) can be computed through the application of $\mathcal{D}_f^\delta$ to $V(\cdot)$ and evaluating the result at x so getting the computational formula

$$V_{\mathrm{av}}^\delta(f, x) = \mathcal{D}_f^\delta(V)(x) = V(x) + \sum_{i\geq 1} \frac{\delta^i}{(i+1)!} \mathrm{L}_f^i V(x). \tag{24}$$

Remark 6: With reference to the LTI case, $f(x) = Ax, V(x) = \frac{1}{2}x^\top P x$, (24) specifies as $V_{\mathrm{av}}^\delta(f, x) = \frac{1}{2}x^\top P^\delta x$ with

$$\begin{aligned}
P^\delta &= \frac{1}{\delta}\int_0^\delta e^{sA^\top} P e^{As}\mathrm{d}s \\
&= (e^{\delta A^\top} + I)P(e^{\delta A} - I)(\delta A)^{-1} = P + \sum_{i\geq 1} \frac{\delta^i}{(i+1)!} P_i
\end{aligned}$$

and $P_i = P_{i-1}A + A^\top P_{i-1}$. From (22), the variation of $V(\cdot)$ along the linear sampled-data dynamics gives

$$V(x^+) - V(x) = \delta x^\top P^\delta A x = x^\top (e^{\delta A^\top} + I)P(e^{\delta A} - I)x.$$

To conclude, a constructive characterization of the discrete gradient of any function $V(\cdot)$ along the sampled dynamics $e^{\delta f}(\cdot)$ associated to any $f(\cdot)$ is given; such a form is clearly not uniquely defined.

Proposition 4: Given a smooth vector field $f(\cdot)$ on $\mathbb{R}^n$, then the discrete gradient of $V(\cdot)$ along the sampled dynamics (11) can be computed for all $\delta \in]0, T^\star[$ according to

$$\bar{\nabla}^\top V\Big|_x^{x^+} = \nabla^\top V^\delta_{\rm av}(f,x)(M^\delta(f,x))^{-1}$$
$$= \left(\mathcal{D}^\delta_f(\nabla V)(x)\right)^\top \left(J[\mathcal{D}^\delta_f(x)]\right)^{-1} \tag{25}$$

with $\mathcal{D}^\delta_f(\cdot)$ in (18) and $V^\delta_{\rm av}(f,\cdot)$ in 23.

Proof: From Definition 1, Propositions 2 and 3, one gets the equality

$$\bar{\nabla}^\top V\Big|_x^{x^+} M^\delta(f,x)f(x) = \nabla^\top V^\delta_{\rm av}(f,x)f(x)$$
$$= \left[\mathcal{D}^\delta_f(\nabla V)(x)\right]^\top f(x)$$

which proves that (25) is an admissible solution because the matrix $M^\delta(f,x)$ invertible for δ small enough. When $\delta \to 0$ and $x^+ \to x$, one recovers $\bar{\nabla}^\top V\Big|_x^x = \nabla V(x)$. ■

For the first terms in $O(\delta^3)$, one computes

$$(M^\delta(f,x))^{-1} = I - \frac{\delta}{2}J[f(x)]$$
$$-\frac{\delta^2}{3!}J[J[f(x)]f(x)] + \frac{\delta^2}{4}J[f(x)]J[f(x)] + O(\delta^3).$$

Remark 7: When $f(x) = Ax, V(x) = \frac{1}{2}x^\top Px$, one gets from (25)

$$\bar{\nabla} V\Big|_x^{x^+} = \delta A(e^{\delta A} - I)^{-1}P^\delta x. \tag{26}$$

Finally, a computational relation between the usual gradient of a given function $V(\cdot)$ and its discrete gradient along the sampled dynamics associated to a vector field $f(\cdot)$ can be set.

Proposition 5: Given a smooth vector field $f(\cdot)$ on $\mathbb{R}^n$, then the discrete gradient of $V(\cdot)$ along its sampled-data equivalent dynamics (11) satisfies for all $\delta \in]0, T^\star[$ the equality

$$\bar{\nabla} V\Big|_x^{x^+} = \nabla V(x) + \delta Q^\delta(V,f,x)f(x) \tag{27}$$

with square matrix $Q^\delta(V,f,x)$ given by

$$Q^\delta(V,f,x) = \left(\int_0^1\int_0^1 s_1 \nabla^2 V\Big|_{x+s_2s_1F^\delta(x)} \mathrm{d}s_2\mathrm{d}s_1\right) M^\delta(f,x). \tag{28}$$

Proof: The proof follows from Proposition 1 as

$$\bar{\nabla} V\Big|_x^{x^+} = \bar{\nabla} V\Big|_x^{x+F^\delta(x)} = \int_0^1 \nabla V(x + s_1F^\delta(x))\mathrm{d}s_1 =$$
$$= \nabla V(x) + \int_0^1 s_1 \bar{J}[\nabla V]\Big|_x^{x+s_1F^\delta(x)} F^\delta(x)\mathrm{d}s_1$$
$$= \nabla V(x) + \int_0^1 s_1 \bar{\nabla}^2 V\Big|_x^{x+s_1F^\delta(x)} F^\delta(x)\mathrm{d}s_1$$

so recovering (27) as $\bar{\nabla^2} V\Big|_x^{x+s_1F^\delta(x)} = \int_0^1 \nabla^2 V\Big|_{x+s_2s_1F^\delta(x)}\mathrm{d}s_2$ and $F^\delta(x) = \delta M^\delta(f,x)f(x)$ from Proposition 2. ■

For the first terms in $O(\delta^3)$, one computes

$$\delta Q^\delta(V,f,x) = \frac{\delta}{2}\nabla^2 V(x) + \frac{\delta^2}{4}\nabla^2 V(x)J[f(x)]$$
$$+ \frac{\delta^2}{3!}\frac{\partial \nabla^2 V(x)}{\partial x}(f(x)\otimes I) + O(\delta^3).$$

Remark 8: The discrete gradient (27) admits a computable power expansion in δ of the form

$$\bar{\nabla} H|_x^{x^+} = \nabla H(x) + \sum_{i>0}\frac{\delta^i}{(i+1)!}\bar{\nabla}_i H(x) \tag{29}$$

with, for the first terms

$$\bar{\nabla}_1 H(x) = \nabla^2 H(x) f(x)$$
$$\bar{\nabla}_2 H(x) = \frac{\partial \nabla^2 H(x)}{\partial x}(f(x)\otimes I)f(x)$$
$$+ \frac{3}{2}\nabla^2 H(x)J[f(x)]f(x).$$

Remark 9: When $f(x) = Ax, V = x^\top Px$, one gets $Q^\delta(\frac{1}{2}x^\top Px, Ax, x) = \frac{1}{2}P(e^{\delta A} - I)(\delta A)^{-1}$.

IV. Gradient Dynamics Under Sampling

Consider the continuous-time gradient dynamics

$$\dot{x} = f(x) = -\nabla H(x) \tag{30}$$

that satisfies the inequality

$$H(x^+) - H(x) = -\int_0^\delta \nabla^\top H(x(s))\nabla H(x(s))\mathrm{d}s \le 0. \tag{31}$$

We address the following question: Does the equivalent sampled-data dynamics admit a discrete gradient form? To this end, let us specify the Proposition 4 to the case $f = -\nabla H$.

Proposition 6: Given a smooth gradient vector field $f(\cdot) = -\nabla H(\cdot)$ on $\mathbb{R}^n$, then the discrete gradient of $H(\cdot)$ along its sampled equivalent dynamics (11) is given for all $\delta \in]0, T^\star[$ by

$$\bar{\nabla}^\top H\Big|_x^{x^+} =$$
$$\nabla^\top H(x)\int_0^\delta (I - sJ[\nabla^s_{\rm av}H(x)])^\top$$
$$(I - sJ[\nabla^s_{\rm av}H(x)])\mathrm{d}s(\delta M^\delta(f,x))^{-1} \tag{32}$$

with from definition (23) and setting $x(\ell) = e^{\ell f}x$

$$s\nabla^s_{\rm av}H(x) = \int_0^s \nabla H(x(\ell))\mathrm{d}\ell = s\nabla H(x)$$
$$+ \sum_{i>1}\frac{s^i}{i!}\mathrm{L}_f^{i-1}\nabla H(x). \tag{33}$$

Proof: According to Proposition 4, when $f = -\nabla H$, the discrete gradient $H(\cdot)$ along its sampled dynamics satisfies the following equality:

$$-\delta\bar{\nabla}^\top H\Big|_x^{x^+} M^\delta(f,x)\nabla H(x) = -\int_0^\delta \nabla^\top H(x(s))\nabla H(x(s))\mathrm{d}s$$

with $x(s) = e^{sf}x$. Rewriting now $H(x^+) - H(x)$ along the continuous-time dynamics according to

$$\nabla H(x(s)) = (I - sJ[\nabla^s_{\rm av}H(x)])\,\nabla H(x)$$

one gets the equality

$$\delta\bar{\nabla}^\top H\Big|_x^{x^+} M^\delta(f,x)\nabla H(x) = \nabla^\top H(x)\int_0^\delta (I - sJ[\nabla^s_{\rm av}H(x)])^\top$$
$$\times (I - sJ[\nabla^s_{\rm av}H(x)])\mathrm{d}s\nabla H(x)$$

which is solved by the choice (32), so concluding the proof. ■

For the first terms in $O(\delta^3)$ in (32), one computes

$$\bar\nabla H\Big|_x^{x^+} = \left(I - \frac{\delta}{2}\nabla^2 H(x) + \frac{\delta^2}{4}\nabla^2 H(x)\nabla^2 H(x) + \frac{\delta^2}{3!}\frac{\partial \nabla^2 H(x)}{\partial x}(\nabla H(x)\otimes I)\right)\nabla H(x) + O(\delta^3).$$

On these bases, the following result can be proved.

Theorem 1: Given the continuous-time gradient dynamics (30), then for all $\delta \in]0, T^\star[$, its sampled-data equivalent dynamics (11) admits the discrete-time form as follows:

$$x^+ - x = -\delta I^\delta(H, -\nabla H, x)\bar\nabla H\Big|_x^{x^+} \tag{34}$$

with nonsingular symmetric positive-definite matrix

$$I^\delta(H,-\nabla H,x) = M^\delta(-\nabla H,x)(I-\delta Q^\delta(H,-\nabla H,x))^{-1} = \left(I^\delta(H,-\nabla H,x)\right)^\top > 0. \tag{35}$$

The function $H(\cdot)$ satisfies the variational inequality

$$H(x^+) - H(x) = -\delta\bar\nabla^\top H\Big|_x^{x^+} I^\delta(H,-\nabla H,x)\bar\nabla H\Big|_x^{x^+} < 0. \tag{36}$$

Proof: According to Proposition 5 as $(I - \delta Q^\delta(f,x))$ is nonsingular by construction, one rewrites $F^\delta(x)$ as

$$F^\delta(x) = -\delta M^\delta(-\nabla H,x)(I-\delta Q^\delta(H,-\nabla H,x))^{-1} (I-\delta Q^\delta(H,-\nabla H,x))\nabla H(x)$$

so getting the result when setting $I^\delta(H,-\nabla H,x)$ as in (35). Nonsingularity of $I^\delta(H,-\nabla(x),x)$ follows from nonsingularity of $M^\delta(f,x)$ and $(I-\delta Q^\delta(H,f,x))$. Moreover, exploiting Proposition 6 and the discrete-gradient (32), one rewrites (35) as

$$I^\delta(H,-\nabla H,x) = \delta M^\delta(f,x)\left(\int_0^\delta (I - sJ[\nabla^s_{\text{av}}H(x)])^\top \times (I - sJ[\nabla^s_{\text{av}}H(x)])\mathrm{d}s\right)^{-1}(M^\delta(f,x))^\top$$

which is symmetric and positive definite by construction. The equality (36) is obtained by expressing $H(x^+) - H(x)$ either as $H(x^+) - H(x) = \bar\nabla^\top H\Big|_x^{x^+}(x^+ - x)$, or equivalently as the integration of the Hamiltonian function along the continuous-time dynamics $H(x^+) - H(x) = \int_0^\delta \dot H(x(s))\mathrm{d}s$. Both are equal since the sampled-data dynamics matches at the sampling times the continuous-time one by definition. ∎

For the first terms in $O(\delta^3)$ in (35), one computes

$$I^\delta(H,-\nabla H,x) = \left(I - \frac{\delta}{2}\nabla^2 H(x) + \frac{\delta^2}{3!}J[\nabla^2 H(x)\nabla H(x)]\right) \times \left(\left(I - \frac{\delta}{2}\nabla^2 H(x) + \frac{\delta^2}{4}\nabla^2 H(x)\nabla^2 H(x) - \frac{\delta^2}{3!}\frac{\partial\nabla^2 H(x)}{\partial x}(\nabla H(x)\otimes I)\right)\nabla H(x)\right)^{-1} = I - \frac{\delta^2}{12}\nabla^2 H(x)\nabla^2 H(x) + O(\delta^3).$$

It is important to stress that the sampled equivalent dynamics (34) to the continuous-time gradient dynamics (30) does not exhibit a discrete gradient form in general.

When considering a quadratic Hamiltonian, Theorem 1 recovers the results in [36]. In detail, when $H(x) = \frac{1}{2}x^\top Px$ with $P \geq 0$, the gradient dynamics $\dot x = -\nabla H(x) = -Px$ admits the sampled-data representation (34) specified as

$$x^+ - x = -I^\delta\frac{P}{2}(x + x^+)$$

with $x^+ = e^{-\delta P}x$ and

$$I^\delta = 2\delta(I - e^{-\delta P})P^{-1}(I + e^{-\delta P})^{-1}. \tag{37}$$

In addition, the following result can be given.

Theorem 2: Given a LTI gradient-dynamics (30) (i.e., $f(x) = -Px$ and $H(x) = \frac{1}{2}x^\top Px$, there exists a new Hamiltonian function $H^\delta = \frac{1}{2}x^\top P^\delta x$ parameterized by $\delta > 0$ with

$$P^\delta = I^\delta P = (P^\delta)^\top > 0$$

such that the sampled-data equivalent dynamics preserves a discrete gradient form; namely, one gets

$$x^+ - x = -\bar\nabla H^\delta\Big|_x^{x^+} = -\frac{P^\delta}{2}(x + x^+). \tag{38}$$

Proof: From Theorem 1, $I^\delta = (I^\delta)^\top > 0$, so getting that the so-defined P^δ, coinciding with the one in Remark 6, is positive definite and symmetric. Also as $x^+ = x$ and $\delta \to 0$, one gets $\bar\nabla H^\delta|_x^x = Px$ so concluding the result. ∎

V. Port-Hamiltonian Dynamics Under Sampling

The results of the previous section are now generalized to port-Hamiltonian dynamics of the form (7). To this end, we specify to this context the results in Proposition 6, when setting for compactness $S(x) = J(x) - R(x)$.

Proposition 7: Given a smooth vector field $f(x) = S(x)\nabla H(x)$ on $\mathbb{R}^n$, then for all $\delta \in]0, T^\star[$, the discrete gradient of $H(\cdot)$ along its sampled equivalent can be rewritten as

$$\bar\nabla^\top H\Big|_x^{x^+} = \nabla^\top H(x) \times \int_0^\delta (I + sJ[\nabla^s_{\text{av}}H(x)]S(x))^\top S^\top(x(s)) (I + sJ[\nabla^s_{\text{av}}H(x)]S(x))\,\mathrm{d}s \times (\delta M^\delta(f,x)S(x))^{-1} \tag{39}$$

with $x^+ = e^{\delta f}x$, $x(s) = e^{sf}x$ and $s\nabla^s_{\text{av}}H(x)$ as in (33).

Proof: According to Proposition 4, when $f(x) = -S(x)\nabla H$, the discrete gradient of the function $H(\cdot)$ along its sampled dynamics satisfies the following equality:

$$\delta\bar\nabla^\top H\Big|_x^{x^+} M^\delta(f,x)S(x)\nabla H(x) = \int_0^\delta \nabla^\top H(x(s))S(x(s))\nabla H(x(s))\mathrm{d}s.$$

Rewriting $\nabla H(x(s)) = (I + sJ[\nabla^s_{\text{av}}H(x)]S(x))\nabla H(x)$, one gets

$$\delta\bar\nabla^\top H\Big|_x^{x^+} M^\delta(f,x)S(x)\nabla H(x) = \nabla^\top H(x) \times \int_0^\delta (I + sJ[\nabla^s_{\text{av}}H(x)]S(x))^\top S(x(s)) \times (I + sJ[\nabla^s_{\text{av}}H(x)]S(x))\,\mathrm{d}s\nabla H(x)$$

which is solved by the choice (39). ∎

Remark 10: From Propositions 7 and 5, it is a matter of computation to verify that

$$I+\delta Q^{\delta}(H,f,x)S(x)=(\delta M^{\delta}(f,x)S(x))^{-\top}\times\int_0^{\delta}\left(I+sJ[\nabla^s_{\text{av}}H(x)]S(x)\right)^{\top}S^{\top}(x(s))\times\left(I+sJ[\nabla^s_{\text{av}}H(x)]S(x)\right)\mathrm{d}s \quad (40)$$

with $x(s)=e^{sf}x$ and $s\nabla^s_{\text{av}}H(x)$ as in (33).

The main result in the following generalizes Theorem 1 to port-Hamiltonian dynamics.

Theorem 3: Given a continuous-time port-Hamiltonian dynamics as in (7), then for any $\delta\in]0,T^{\star}[$, its sampled equivalent model (11) admits the discrete-time port-Hamiltonian structure

$$x^+-x=\delta S^{\delta}_{J-R}(f,x)\bar{\nabla}H\Big|_x^{x^+} \quad (41)$$

with $f(x)=S(x)\nabla H(x)$, $x^+-e^{\delta f}x$ and

$$S^{\delta}_{J-R}(f,x)=M^{\delta}(f,x)S(x)\left(I+\delta Q^{\delta}(H,f,x)S(x)\right)^{-1}. \quad (42)$$

The following EBE is satisfied:

$$H(x^+)-H(x)=-\delta\bar{\nabla}^{\top}H\Big|_x^{x^+}S^{\delta}_{J-R}(f,x)\bar{\nabla}H\Big|_x^{x^+}=-\int_0^{\delta}\nabla^{\top}H(x(s))R(x(s))\nabla H(x(s))\mathrm{d}s\leq 0. \quad (43)$$

Proof: According to Proposition 2, one rewrites

$$x^+-x=F^{\delta}(x)=\delta M^{\delta}(f,x)S(x)\nabla H(x)$$

and the discrete gradient function satisfies

$$\bar{\nabla}H\Big|_x^{x^+}=\left(I+\delta\int_0^1 s\bar{\nabla}^2H\Big|_x^{x+sF^{\delta}(x)}\mathrm{d}sM^{\delta}(f,x)S(x)\right)\nabla H(x)$$

so getting (41). Moreover, by definition of the discrete gradient, the EBE rewrites as

$$H(x^+)-H(x)=\delta\bar{\nabla}^{\top}H\Big|_x^{x^+}S^{\delta}_{J-R}(f,x)\bar{\nabla}H\Big|_x^{x^+}=-\int_0^{\delta}\nabla^{\top}H(x(s))R(x(s))\nabla H(x(s))\mathrm{d}s\leq 0$$

so verifying energy dissipation (43). ■

From Remark 10, an alternate characterization of $S^{\delta}_{J-R}(f,x)$ in Theorem 3 is obtained by substituting (40) into (42).

Corollary 1: $S^{\delta}_{J-R}(f,x)$ in (42) can be equivalently rewritten in terms of $s\nabla^s_{\text{av}}H(x)$ in (33) as

$$S^{\delta}_{J-R}(f,x)=\delta M^{\delta}(f,x)S(x)\times\left(\int_0^{\delta}(I+sJ[\nabla^s_{\text{av}}H(x)]S(x))^{\top}S^{\top}(x(s))\times\left(I+sJ[\nabla^s_{\text{av}}H(x)]S(x)\right)\mathrm{d}s\right)^{-1}\times S^{\top}(x)(M^{\delta}(f,x))^{\top}. \quad (44)$$

Remark 11: The sampled-data structural matrix (42) can be described by its power expansion in δ according to

$$S^{\delta}_{J-R}(f,x)=S_0(x)+\sum_{i\geq 1}\frac{\delta^i}{(i+1)!}S_i(x) \quad (45)$$

with, for the first terms

$$\begin{aligned}
S_0(x)&=J(x)-R(x)\\
S_1(x)&=\left(\frac{\partial S_0(x)}{\partial x}(\nabla H(x)\otimes I)\right)S_0(x)\\
S_2(x)&=-\frac{1}{2}S_0(x)\nabla^2H(x)S_0(x)\nabla^2H(x)S_0(x)\\
&\quad-\frac{1}{2}S_0(x)\nabla^2H(x)S_1(x)-\frac{1}{2}S_1(x)\nabla^2H(x)S_0(x)\\
&\quad+\frac{\partial S_0(x)}{\partial x}(\nabla H(x)\otimes I)S_1(x)\\
&\quad-\nabla^2H(x)\frac{\partial S_0(x)}{\partial x}(f(x)\otimes I)S_0(x)\\
&\quad+\left(\frac{\partial}{\partial x}\left(\frac{\partial S_0(x)}{\partial x}(\nabla H(x)\otimes I)\right)\right)(f(x)\otimes I)S_0(x).
\end{aligned} \quad (46)$$

Remark 12: From (46), when R and J are constant, one gets

$$S_1=0,\quad S_2=-\frac{1}{2}S_0\nabla^2H(x)S_0\nabla^2H(x)S_0.$$

In the LTI case, the following theorem generalizes [36, Th. 4.2] and recovers the results.

Theorem 4: When $f(x)=(J-R)Px$, $H(x)=\frac{1}{2}x^{\top}Px$, its sampled port-Hamiltonian form (41) specifies as

$$x^+-x=\delta S^{\delta}_{J-R}P(x^++x)$$

with $x^+=e^{\delta(J-R)P}x$ and

$$S^{\delta}_{J-R}=2(e^{\delta(J-R)P}-I)(I+e^{\delta(J-R)P})^{-1}(\delta P)^{-1}. \quad (47)$$

The following corollaries characterize the structure matrix S^{δ}_{J-R} in (42) when the port-Hamiltonian system (7) is purely conservative ($R=0$) or in the degenerate case, dissipative ($J=0$).

Corollary 2: If $J(x)=0$ (i.e., $f(x)=-R(x)\nabla H(x)$), its sampled equivalent model takes the form

$$x^+-x=\delta S^{\delta}_{-R}(f,x)\bar{\nabla}H\Big|_x^{x^+}$$

with $x^+=e^{\delta f}x$ and symmetric, negative semidefinite matrix

$$S^{\delta}_{-R}(f,x)=-M^{\delta}(f,x)R(x)\times\left(\int_0^{\delta}(I-sJ[\nabla^s_{\text{av}}H(x)]R(x))^{\top}R(x(s))(I-sJ[\nabla^s_{\text{av}}H(x)]R(x))\mathrm{d}s\right)^{-1}\times R(x)(\delta M^{\delta}(f,x))^{\top}=(S^{\delta}_{-R}(f,x))^{\top}\leq 0.$$

Accordingly, one gets dissipation

$$H(x^+)-H(x)=\delta\bar{\nabla}^{\top}H\Big|_x^{x+}S^{\delta}_{-R}(f,x)\bar{\nabla}H\Big|_x^{x^+}=-\int_0^{\delta}\nabla^{\top}H(x(s))R(x(s))\nabla H(x(s))\mathrm{d}s\leq 0.$$

In conclusion, the purely dissipative structure of a port-Hamiltonian dynamics is preserved under sampling making reference to the new dissipation matrix $-S^{\delta}_{-R}(f,x)$.

Remark 13: In the LTI case (i.e., $f(x)=-RPx$), one recovers [36]

$$S^{\delta}_{-R}(-RPx,x)=2(e^{-\delta RP}-I)(I+e^{-\delta RP})^{-1}(\delta P)^{-1}$$

with symmetric and positive definite $-S^{\delta}_{-R}(-RPx, x)$ as proved in Corollary 2.

Corollary 3: If $R(x) = 0$ (i.e., $f(x) = J(x)\nabla H(x)$), its sampled equivalent model takes the form

$$x^+ - x = \delta S^{\delta}_J(f, x)\bar{\nabla} H\Big|_x^{x^+}$$

with $x^+ = e^{\delta J\nabla H}x$ and skew symmetric matrix

$$S^{\delta}_J(f, x) = -M^{\delta}(f, x)J(x)\times \left(\int_0^{\delta} (I + sJ[\nabla^s_{\text{av}} H(x)]J(x))^{\top} J(x(s)) \times (I + sJ[\nabla^s_{\text{av}} H(x)]J(x))\mathrm{d}s\right)^{-1} \times J(x)(\delta M^{\delta}(f, x))^{\top} = -(S^{\delta}_J(f, x))^{\top}.$$

Accordingly, the sampled equivalent dynamics is conservative

$$H(x^+) - H(x) = \delta\bar{\nabla}^{\top} H\Big|_x^{x^+} S^{\delta}_J(f, x)\bar{\nabla} H\Big|_x^{x^+} = 0. \quad (48)$$

One concludes that a purely conservative port-Hamiltonian structure is preserved under sampling making reference to the new interconnection skew symmetric matrix $S^{\delta}_J(f, x)$.

Remark 14: In the LTI case $f(x) = JPx$, one recovers [36]

$$S^{\delta}_J(JPx, x) = 2(e^{\delta JP} - I)(I + e^{\delta JP})^{-1}(\delta P)^{-1}$$

where, from Corollary 3, $S^{\delta}_J(JPx, x) = -(S^{\delta}_J(JPx, x))^{\top}$.

A. Sampled-Data Structural Matrix

Given the port-Hamiltonian representation (41), providing a unique characterization of the corresponding dissipating and conservative components of the structural matrix is not an easy task in general. Several options are possible for decomposing $S^{\delta}_{J-R}(f, x) = J^{\delta}(f, x) - R^{\delta}(f, x)$ into a suitably defined interconnection $(J^{\delta}(f, x))$ and damping $(R^{\delta}(f, x))$ matrices according to Definition 4. These matrices are uniquely defined in the LTI case discussed in Theorem 4, by setting

$$R^{\delta} = -\text{sym}(S^{\delta}_{J-R}), \quad J^{\delta} = \text{skew}(S^{\delta}_{J-R}).$$

Along these lines, in the nonlinear case, the easiest choice is to separate $S^{\delta}_{J-R}(f, x)$ into its symmetric and skew-symmetric components by setting

$$\begin{aligned} J^{\delta}(f, x) &= \text{skew}(S^{\delta}_{J-R}(f, x)) \\ R^{\delta}(f, x) &= -\,\text{sym}(S^{\delta}_{J-R}(f, x)) \succeq 0. \end{aligned} \quad (49)$$

However, the result is quite conservative as in general $J^{\delta}(f, x) \neq 0$ when $J(x) = 0$ or $R^{\delta}(f, x) \neq 0$ when $R(x) = 0$.

A more accurate choice is based on the computation of $R^{\delta}(f, x)$ as the solution to the dissipation-matching equality

$$\delta\bar{\nabla}^{\top} H\Big|_x^{x^+} R^{\delta}(f, x)\bar{\nabla} H\Big|_x^{x^+} = \int_0^{\delta} \nabla^{\top} H(x(s))R(x(s))\nabla H(x(s))\mathrm{d}s$$

with $x^+ = e^{\delta f}x$, $x(s) = e^{sf}x$ that gives

$$\delta R^{\delta}(f, x) = \left(I + \delta Q^{\delta}(H, f, x)S(x)\right)^{\top} \int_0^{\delta} (I + sJ[\nabla^s_{\text{av}} H(x)]S(x))^{\top} \times R(x(s))\,(I + sJ[\nabla^s_{\text{av}} H(x)]S(x))\,\mathrm{d}s \left(I + \delta Q^{\delta}(H, f, x)S(x)\right). \quad (50)$$

The so-defined matrix is by construction positive definite and symmetric with the EBE of the form

$$H(x^+) - H(x) = \delta\bar{\nabla}^{\top} H\Big|_x^{x^+} S^{\delta}_{J-R}(f, x)\bar{\nabla} H\Big|_x^{x^+} = -\int_0^{\delta} \nabla^{\top} H(x(s))R(x(s))\nabla H(x(s))\mathrm{d}s \leq 0$$

with

$$H(x^+) - H(x) = -\delta\bar{\nabla}^{\top} H\Big|_x^{x^+} R^{\delta}(f, x)\bar{\nabla} H\Big|_x^{x^+}.$$

Accordingly, energy dissipation (43) is verified by skew symmetry of the matrix $J(x)$.

The advantage of such a choice is to exactly match the continuous-time dissipation through the so-defined dissipation matrix $R^{\delta}(f, x)$. Accordingly, one sets $J^{\delta}(f, x) = R^{\delta}(f, x) + S^{\delta}_{J-R}(f, x)$ so preserving exact state sampling but not skew symmetry of $J^{\delta}(f, x)$ in general. However, it is guaranteed that $R^{\delta}(f, x) = 0$ when $R(x) = 0$, and analogously, $J^{\delta}(f, x) = 0$ when $J(x) = 0$.

B. Approximate Sampled-Data Port-Hamiltonian Models

Computing closed forms of the sampled port-Hamiltonian dynamics (41) is in general a difficult task that is, in most cases, not likely possible. The main issues basically rely on the following difficulties:

1) (i) the computation of the structural matrix $S^{\delta}(f, x)$ in (42);
2) the computation of the discrete gradient itself (1)
3) the exact inversion of the implicit model (41) involved in the explicit computation of x^+ that might be not possible, even for simple classes of Hamiltonian functions (e.g., separable).

Still exploiting the smooth δ-dependence of all mappings and matrices, one naturally defines approximations as truncations of the series expansions in powers of δ defining the exact solutions $S^{\delta}(f, x)$ in (45) and $\bar{\nabla} H|_x^{x^+}$ in (29), respectively.

More in detail, the pth-order approximation of the structural matrix (45) is defined as

$$S^{\delta,[p]}_{J-R}(f, x) := S(x) + \sum_{i=1}^{p} \frac{\delta^i}{(i+1)!} S_i(x) \quad (51)$$

with $p \geq 0$ resulting in the pth-order approximation of the implicit Hamiltonian model (41) as

$$x^+ - x = \delta S^{\delta,[p]}_{J-R}(f, x)\bar{\nabla} H\Big|_x^{x^+}. \quad (52)$$

When $p = 0$, one recovers the usual Euler-like approximate Hamiltonian model of the literature (e.g., [38]–[41]).

These approximate forms are instrumental when control strategies aimed at assigning a desired port-Hamiltonian structure (i.e., target structural matrix $R^{\delta}_d(\cdot) - J^{\delta}_d(\cdot)$ and Hamiltonian function $H_d(\cdot)$) are designed. However, the approximate form above is still implicitly defined in x^+.

Remark 15: When the Hamiltonian function is quadratic, the explicit approximate form associated to the implicit approximate form (52) given by

$$x^+ = \left(I - \frac{\delta}{2} S^{\delta,[p]}_{J-R}(f,x)P\right)^{-1} \left(I + \frac{\delta}{2} S^{\delta,[p]}_{J-R}(f,x)P\right) x$$

does not recover the truncation at the order $p \geq 0$ of the explicit map $e^{\delta f}x = x + F^\delta(x)$ in (12) with $f(x) = (J(x) - R(x))P$.

When the discrete-gradient function cannot be exactly computed as a function of x and x^+, one defines approximate implicit models of order (p, q) from the qth-order approximation of the discrete gradient (1) along the trajectories $x^+ = e^{\delta f}x$ so getting, for $q \geq 0$

$$\bar{\nabla}^{[q]} H\Big|_x^{x^+} := \nabla H(x) + \sum_{i=1}^{q} \frac{\delta^i}{(i+1)!} \bar{\nabla}_i H(x). \tag{53}$$

Accordingly, we define for $p, q > 0$ the (p, q)-order port-Hamiltonian form associated to (41)

$$x^+ - x = \delta S^{\delta,[p]}_{J-R}(f,x) \bar{\nabla}^{[q]} H\Big|_x^{x^+} \tag{54}$$

that is explicitly defined as a function of x but again does not coincide with the truncation at a certain order $r \leq q + p$ of the map $e^{\delta f}x = x + F^\delta(x)$ defined in (12). However, when $p = q = 0$, (54) recovers the usual Euler approximate model $x^+ = x + \delta f(x)$, associated to (7).

VI. Port-Controlled Hamiltonian Dynamics

We discuss now port-controlled Hamiltonian (pcH) dynamics and show the preservation under sampling of the energy-balance equalities as well as their use in damping-feedback design. For the sake of simplicity, we consider single-input single-output systems even if all results hold true, with simple modifications, in the general case.

Along with the literature (see [2] and the reference therein), consider a continuous-time port-controlled Hamiltonian system composed by the dynamics (7) with additional input-affine controlled part and a conjugate output map; i.e.,

$$\dot{x} = (J(x) - R(x))\nabla H(x) + ug(x) \tag{55a}$$

$$y = h(x) = g^\top(x)\nabla H(x) \tag{55b}$$

where $g(\cdot)$ is a smooth vector field over $\mathbb{R}^n$ and $u \in \mathbb{R}$. The following well-known facts are recalled:

1) the dynamics (55a) satisfies the EBE

$$H(x(t)) - H(x(0)) = -\int_0^t \nabla^\top H(x(s)R(x(s)) \times \nabla H(x(s))\mathrm{d}s + \int_0^t u(s)g^\top(x(s))\nabla H(x(s))\mathrm{d}s \tag{56}$$

2) the system (55) is passive with storage function $H(x)$ and dissipation rate $d(x) = \nabla H^\top(x)R(x)\nabla H(x)$; lossless when $R(x) \equiv 0$.

On these bases, stabilizing strategies under output feedback are developed in terms of passivity-based control (PBC) via damping [8].

A discrete-time port-controlled Hamiltonian structure has been recently introduced by the authors in [24]. Denoting by x^+ and $x^+(u)$ the one step ahead unforced and forced evolutions, respectively, the following definition is given.

Definition 5: A discrete-time port-controlled Hamiltonian system is given by

$$x^+(u) = x + (J_d(x) - R_d(x))\bar{\nabla} H\Big|_x^{x^+} + ug_d(x,u) \tag{57a}$$

$$y = h_d(x,u) = g_d^\top(x,u)\bar{\nabla} H\Big|_{x^+}^{x^+(u)} \tag{57b}$$

where $g_d(\cdot, u) : \mathbb{R}^n \to \mathbb{R}^n$ and $h_d(\cdot, u) : \mathbb{R}^n \to \mathbb{R}$ are smooth functions of the state and the control $u \in \mathbb{R}$.

By construction, the following holds true:

1) the dynamics (57a) satisfies the energy-balance equation, i.e., for any integer $k \geq 1$

$$H(x_k) - H(x_0) = -\sum_{i=0}^{k-1} \bar{\nabla}^\top H\Big|_{x_i}^{x_i^+} R_d(x_i)\bar{\nabla} H\Big|_{x_i}^{x_i^+} + \sum_{i=0}^{k-1} u_i g_d^\top(x_i, u_i)\bar{\nabla} H\Big|_{x_i^+}^{x_i^+(u_i)} \tag{58}$$

2) the system (57) is passive with storage function $H(x)$ and dissipation rate $d(x) = \bar{\nabla} H^\top\Big|_x^{x^+} R_d(x)\bar{\nabla} H\Big|_x^{x^+}$; it is lossless when $R_d(x) \equiv 0$.

Remark 16: When compared to the literature (e.g., [23] and [41]), two main differences hold. The dynamics (57) is defined in terms of the discrete gradient of the Hamiltonian along the free evolution only, the conjugate output is described in terms of the discrete gradient of the Hamiltonian along the control-dependent part of the evolution only. Accordingly, one gets passivity with respect to the so-defined conjugate output that recovers the average passivating output map introduced in [35].

A. Sampled-Data pcH Dynamics

Theorem 3 extends to controlled dynamics as follows.

Theorem 5: Consider a continuous-time port-controlled Hamiltonian system (55) and assume the control variable constant over time intervals of amplitude δ; i.e., $u(t) = u_k, \forall t \in [k\delta, (k+1)\delta[$. Then, for all $\delta \in]0, T^\star[$, its sampled equivalent model admits the discrete-time port-controlled Hamiltonian structure as follows:

$$x^+(u) - x = \delta S^\delta_{J-R}(f,x)\bar{\nabla} H\Big|_x^{x^+} + \delta g^\delta(x,u)u \tag{59a}$$

$$y^\delta = h^\delta(x,u) = (g^\delta(x,u))^\top \bar{\nabla} H\Big|_{x^+}^{x^+(u)} \tag{59b}$$

with $\delta u g^\delta(x,u) = e^{\delta(f+ug)}x - e^{\delta f}x$ and $S^\delta_{J-R}(f,x)$ given in (42). Moreover, the following properties hold:

1) the dynamics (59a) satisfies the EBE

$$H(x_k) - H(x_0) = -\delta \sum_{i=0}^{k-1} \bar{\nabla}^\top H\Big|_{x_i}^{x_i^+} S^\delta_{J-R}(f,x_i)\bar{\nabla} H\Big|_{x_i}^{x_i^+} + \delta \sum_{i=0}^{k-1} u_i (g^\delta(x_i,u_i))^\top \bar{\nabla} H\Big|_{x_i^+}^{x_i^+(u_i)} \tag{60}$$

2) the system (59) is passive with storage function $H(x)$ and dissipation rate $d^\delta(x) := \bar{\nabla} H^\top\Big|_x^{x^+} R^\delta(f,x)\bar{\nabla} H\Big|_x^{x^+}$; it is lossless when $R(x) \equiv 0$;

3) whenever the system is zero-state-detectable, any feedback $\bar{u} = \gamma^\delta(x)$ solving the algebraic equality

$$\bar{u} = -\kappa \left(g^\delta(x,\bar{u})\right)^\top \bar{\nabla} H\Big|_{x^+}^{x^+(\bar{u})},\ \kappa > 0 \tag{61}$$

is a sampled-data PBC making $x_\star$ asymptotically stable with increasing damping; i.e., one gets in closed loop the modified EBE, with $\bar{u}_i = \gamma^\delta(x_i)$

$$H(x_k) - H(x_0) = -\delta \sum_{i=0}^{k-1} \bar{\nabla}^\top H\Big|_{x_i}^{x_i^+} S^\delta_{J-R}(f, x_i) \bar{\nabla} H\Big|_{x_i}^{x_i^+}$$
$$- \delta\kappa \sum_{i=0}^{k-1} \bar{\nabla} H\Big|_{x_i^+}^{x_i^+(\bar{u}_i)} [g^\delta(x_i,\bar{u}_i)][g^\delta(x_i,\bar{u}_i)]^\top \bar{\nabla} H\Big|_{x_i^+}^{x_i^+(\bar{u}_i)}.$$

Proof: (1) and (2) are an immediate consequence of the fact that the sampled-data form (59) exhibits the same structure as (57). (3) specifies to the sampled-data form (59), PBC strategies developed in [35, Th. 4.1] for nonlinear discrete-time dynamics (negative output feedback design with respect to suitably defined passivating output map). ■

Remark 17: Under $\bar{u} = \gamma^\delta(x)$ solution to (61), one gets

$$x^+(\bar{u}) - x = \delta S^\delta_{J-R}(f, x) \bar{\nabla} H\Big|_x^{x^+}$$
$$- \kappa\delta g^\delta(x,\bar{u})(g^\delta(x,\bar{u}))^\top \bar{\nabla} H\Big|_{x^+}^{x^+(\bar{u})}$$

that does not properly exhibit a port-Hamiltonian structure of the form (57). Nevertheless, it properly adds damping to the natural one in free evolution through a well-structured term so getting in closed loop the EBE as follows:

$$H(x^+(\bar{u})) - H(x) = -\delta \bar{\nabla}^\top H\Big|_x^{x^+} S^\delta_{J-R}(f,x)) \bar{\nabla} H\Big|_x^{x^+}$$
$$- \delta\kappa \bar{\nabla}^\top H\Big|_{x^+}^{x^+(\bar{u})} g^\delta(x,\bar{u})(g^\delta(x,\bar{u}))^\top \bar{\nabla} H\Big|_{x^+}^{x^+(\bar{u})}.$$

Remark 18: The PBC $\bar{u} = \gamma^\delta(x)$ is defined as the solution to the nonlinear equality (61). Even though such an equality is solvable in virtue of the implicit function theorem [35], exact solutions are tough to compute in practice. However, as the solution can be expressed through its series expansion in powers of δ, computational approximations can be easily implemented (see [35] for details).

Let us specify Theorem 5 to the LTI case.

Theorem 6: Consider the LTI port-controlled Hamiltonian system

$$\dot{x}(t) = (J - R)Px + Bu$$
$$y = Cx = B^\top Px$$

with Hamiltonian $H(x) = \frac{1}{2}x^T Px$. Then, its sampled equivalent port-controlled Hamiltonian model is given by

$$x^+(u) - x = \delta S^\delta_{J-R} \bar{\nabla} H\Big|_x^{x^+} + \delta B^\delta u \tag{62a}$$

$$y^\delta = [B^\delta]^\top P x^+ + \frac{1}{2}[B^\delta]^\top P B^\delta u \tag{62b}$$

with $x^+ = e^{\delta(J-R)P}x$, $\delta B^\delta = \int_0^\delta e^{\tau(J-R)P} B d\tau$, and S^δ_{J-R} as in (47). In addition, the sampled-data PBC

$$\bar{u} = -\kappa\left(1 + \frac{1}{2}\kappa(B^\delta)^\top P B^\delta\right)^{-1} (B^\delta)^\top P x^+,\ \kappa > 0$$

solution to the damping equality (61) makes the closed-loop dynamics asymptotically stable.

B. Sampled-Data Dirac Structure

The energy-balance equation (58) satisfied by the sampled equivalent model to (59) can be recast through a discrete Dirac structure formulation [15]. Let us first recall the definition of the Dirac structure in the space of flows and efforts variables.

Definition 6 ([2]): Given a finite-dimensional linear space of flows $\mathbb{F}$ and efforts $\mathbb{E}$ (with $f \in \mathbb{F}$ and $e \in \mathbb{E}$), the space $\mathbb{E}$ being the dual of $\mathbb{F}$, then a subspace $\mathbb{D} \subset \mathbb{F} \times \mathbb{E}$ is a Dirac structure if it satisfies

1) $e^\top f = 0,\ \forall (f, e) \in \mathbb{D}$;
2) $\dim\mathbb{D} = \dim\mathbb{F}$.

Accordingly, the Dirac structure associated with the continuous-time port-controlled Hamiltonian dynamics (55) is described [2] through three pairs of port variables representing the energy storage (f_S, e_S), dissipation (f_R, e_R) and interconnection with the environment (f_I, e_I) and satisfying

$$\begin{pmatrix} f_S \\ f_R \\ f_I \end{pmatrix} = \begin{pmatrix} -J(x) & -g_R(x) & -g(x) \\ g_R^\top(x) & 0 & 0 \\ g^\top(x) & 0 & 0 \end{pmatrix} \begin{pmatrix} e_S \\ e_R \\ e_I \end{pmatrix}$$

with skew symmetric graph over $\dim\mathbb{D} = 2n + 1$

$$\left(\begin{pmatrix} -\dot{x} \\ f_R \\ y \end{pmatrix}, \begin{pmatrix} \nabla H(x) \\ -r(x) f_R \\ u \end{pmatrix} \right) \in \mathbb{D} \tag{63}$$

for $r(x) = r^\top(x) \succeq 0$, $g_R(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, and $R(x) = g_R(x) r(x) g_R^\top(x)$. Discrete-time Dirac structures can be associated to port-controlled Hamiltonian systems of the form (59) as described in [42, Th. 2.1]. Let us now show how the continuous-time Dirac structure is transformed under sampling by suitably defining storing, dissipating, and control ports, associated with the sampled-data structure (59).

Theorem 7: Given a continuous-time dynamics (7) with Dirac structure $\mathbb{D}$ of dimension $2n + 1$ as in (63), then for all $\delta \in]0, T^\star[$, its sampled equivalent model (59) admits a discrete Dirac structure $\mathbb{D}^\delta$ of dimension $3n + 1$ described in terms of then efforts and flow variables defined as follows:

$$\left(\begin{pmatrix} -(x^+ - x) \\ f_R^\delta \\ -(x^+(u) - x^+) \\ y^\delta \end{pmatrix}, \begin{pmatrix} \bar{\nabla} H|_x^{x^+} \\ -r^\delta(x) f_R^\delta \\ \bar{\nabla} H|_{x^+}^{x^+(u)} \\ u \end{pmatrix} \right) \in \mathbb{D}^\delta$$

where $J^\delta(f, x) = \mathtt{skew}(S^\delta_{J-R}(f, x))$, and $R^\delta(f, x) = -\mathtt{sym}(S^\delta_{J-R}(f, x)) = g_R^\delta(x) r^\delta(x) g_R^{\delta\top}(x)$ for $r^\delta(x) = r^{\delta\top}(x) \succeq 0$, $g_R^\delta(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, with a skew-symmetric graph

$$\begin{pmatrix} f^\delta_{S_f} \\ f^\delta_R \\ f^\delta_{S_u} \\ f^\delta_I \end{pmatrix} = \begin{pmatrix} -J^\delta(f,x) & -g_R^\delta(x) & 0 & 0 \\ g_R^{\delta\top}(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & -g^\delta(x, e_I^\delta) \\ 0 & 0 & g^{\delta\top}(x, e_I^\delta) & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} e^\delta_{S_f} \\ e^\delta_R \\ e^\delta_{S_u} \\ e^\delta_I \end{pmatrix}.$$

Proof: The proof exploits the splitting of the Hamiltonian variation between two successive states into free and controlled parts as

$$H(x^+(u)) - H(x) = (H(x^+(u)) - H(x^+)) + (H(x^+) - H(x)).$$

Accordingly, one splits in the total energy storage the free elements $(f^\delta_{S_f}, e^\delta_{S_f}) \in \mathbb{F}^\delta_{S_f} \times \mathbb{E}^\delta_{S_f}$, from the controlled ones $(f^\delta_{S_u}, e^\delta_{S_u}) \in \mathbb{F}^\delta_{S_u} \times \mathbb{E}^\delta_{S_u}$ by, respectively, setting $-e^{\delta\top}_{S_f} f^\delta_{S_f} = \bar\nabla^\top H|_x^{x^+}(x^+ - x)$ and $-e^{\delta\top}_{S_u} f^\delta_{S_u} = \bar\nabla^\top H|_{x^+}^{x^+(u)}(x^+(u) - x^+)$ so verifying $e^{\delta\top} f^\delta = e^{\delta\top}_{S_f} f^\delta_{S_f} + e^{\delta\top}_{S_u} f^\delta_{S_u}$ with $-e^{\delta\top} f^\delta = \bar\nabla^\top H|_x^{x^+(u)}(x^+(u) - x)$. Regarding interconnection with the environment through $(f^\delta_I, e^\delta_I) \in \mathbb{F}^\delta_I \times \mathbb{E}^\delta_I$, one sets by definition of the conjugate output y^δ, $e^{\delta\top}_I f^\delta_I = \bar\nabla^\top H|_{x^+}^{x^+(u)}(x^+(u) - x^+) = uy^\delta$ so setting $f^\delta_I = g^{\delta\top}(x, e^\delta_I)e^\delta$ and $e_I = u$ that gives $e^{\delta\top}_{S_u} f^\delta_{S_u} + e^{\delta\top}_I f^\delta_I = 0$. Regarding the dissipative elements, $(f^\delta_R, e^\delta_R) \in \mathbb{F}^\delta_R \times \mathbb{E}^\delta_R$, setting the dissipating constraints $e^\delta_R = -r^\delta(x) f^\delta_R$ and $f_R = g^{\delta\top}_R(x) e^\delta_{S_f}$ for some $r^\delta(x) = r^{\delta\top}(x) \succeq 0$ and $g^\delta_R(\cdot): \mathbb{R}^n \to \mathbb{R}^n$ such that $R^\delta(f, x) = -\mathtt{sym}(S^\delta_{J-R}(f, x)) = g^\delta_R(x) r^\delta(x) g^{\delta\top}_R(x)$, one recovers the power balance equality

$$e^{\delta\top}_{S_f} f^\delta_{S_f} + e^{\delta\top}_R f^\delta_R + e^{\delta\top}_{S_u} f^\delta_{S_u} + e^{\delta\top}_I f^\delta_I = 0.$$

Finally, since $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$

$$\dim\mathbb{D}^\delta = \dim\mathbb{F}^\delta_{S_f} + \dim\mathbb{F}^\delta_R + \dim\mathbb{F}^\delta_{S_u} + \dim\mathbb{F}^\delta_I = 3n + 1.$$

■

VII. Illustrative Examples

In this section, two physical examples are worked out: the gravity pendulum system with a constant structure matrix and nonlinear Hamiltonian and the controlled rigid body with nonconstant structure matrix and quadratic Hamiltonian. Simulations are carried out to highlight advantages of the proposed model compared with the one of the literature (e.g., [23], [32], [38], [40], and [41]). In the lines of Section VI, digital stabilizing controllers are designed to highlight the advantages of the proposed structures in feedback design.

A. Gravity Pendulum

The gravity pendulum described by a separable Hamiltonian is an interesting case study to characterize the series expansion of $S^\delta_{J-R}(f, x)$ in (42). Setting $x = \mathrm{col}\{x_1, x_2\} = \mathrm{col}\{\vartheta, ml^2\dot\vartheta\}$, where θ is the angle between the vertical axis and the rod of the pendulum, the Hamiltonian dynamics is given by

$$\dot x = \begin{pmatrix} 0 & 1 \\ -1 & -r \end{pmatrix} \nabla H(x) \tag{64}$$

with Hamiltonian and gradient functions

$$H(x) = \frac{1}{2ml^2} x_2^2 + mgl(1 - \cos(x_1)),$$
$$\nabla H(x) = \left(mgl\sin(x_1) \; \tfrac{1}{ml^2} x_2\right)$$

and damping coefficient $r \geq 0$. For notational simplicity, we assume $ml^2 = 1$ and $mgl = 1$. According to Proposition 1, both the discrete gradient $\bar\nabla H|_x^{x^+}$ and the discrete Jacobian $\bar J[\nabla H]|_x^{x^+}$ in (4) and (5), respectively, can be exactly computed so getting

$$\bar\nabla H|_x^{x^+} = \left(-\frac{\cos(x_1^+) - \cos(x_1)}{x_1^+ - x_1} \; \frac{x_2^+ + x_2}{2}\right),$$
$$\bar J[\nabla H]|_x^{x^+} = \begin{pmatrix} \frac{\sin(x_1^+) - \sin(x_1)}{x_1^+ - x_1} & 0 \\ 0 & 1 \end{pmatrix}. \tag{65}$$

From Theorem 3 and (52), one computes for $p = 2$, the approximate system in $O(\delta^4)$ with matrix

$$S^{\delta,[2]}_{J-R}(f, x) = \left(0 \quad 1 - 1 - r\right) + \frac{\delta^2}{12}\begin{pmatrix} -r & (\cos(x_1) - r^2) \\ r^2 - \cos(x_1) & r^3 - r\cos(x_1) \end{pmatrix} \tag{66}$$

yielding the dissipation rate

$$H(x^+) - H(x) = -\frac{\delta}{4} r\left(1 + \frac{\delta^2}{6}(\cos(x_1) - \frac{1}{2}r^2)\right)$$
$$(x_2^+ + x_2)^2 \quad - r\frac{\delta^3}{12}\frac{(\cos(x_1^+) - \cos(x_1))^2}{(x_1^+ + x_1)^2} + O(\delta^4) \leq 0.$$

From the aforementioned expressions, whenever the pendulum is undamped ($r = 0$), the sampled equivalent dynamics is clearly conservative; namely, $H(x^+) - H(x) = 0$. For completeness, the Euler-like model proposed in [23] and [39] is recovered by setting $p = 0$ in (52), so yielding dissipation $H(x^+) - H(x) = -\frac{\delta}{4} r(x_2^+ + x_2)^2 + O(\delta^2)$.

1) Simulations: Fig. 1 shows the RMSE in the state and Hamiltonian evolutions at the sampling instants, between the approximate sampled-data model (52) with $p = 2$ and $p = 0$ and the continuous-time one (64). More in detail, the continuous-time (CT) dynamics (64) is compared with the approximate

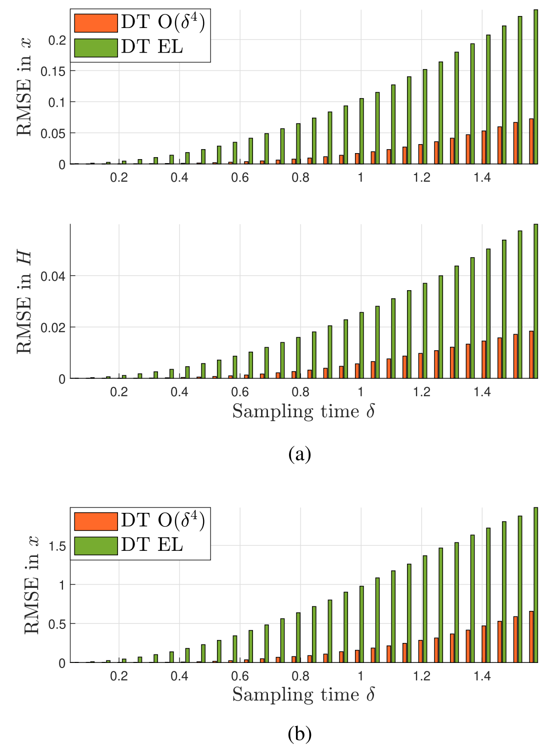


Fig. 1. Pendulum: RMSE in the state x and Hamiltonian H at the sampling instants $\delta \in [0, \frac{\pi}{2}]$ with $x_0 = \mathrm{col}(\frac{3}{2}\pi, 0)$.

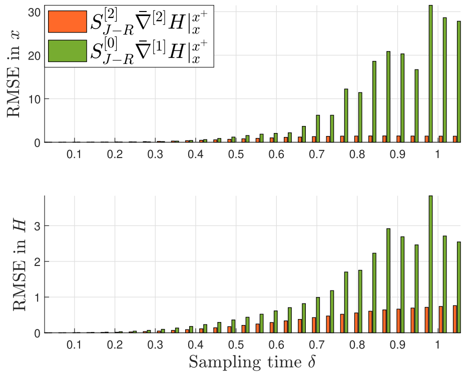


Fig. 2. Pendulum: RMSE in the state x and Hamiltonian H at the sampling instants $\delta \in [0, \frac{\pi}{3}]$ with $x_0 = \text{col}(\frac{3}{2}\pi, 0)$ and $r = 0$.

model (DT in $O(\delta^4)$) given in Theorem 3 with $S^{\delta,[2]}_{J-R}(f, x)$ and with the Euler-like (DT EL) of the literature $S^{\delta,[0]}_{J-R}(f, x)$. This is performed for both the dissipative and conservative cases, respectively. From Fig. 1, the performances improvement is clear. In the conservative case ($r = 0$), both (DT $O(\delta^4)$) and (DT EL) preserve the energy conservation property over the function $H(x) = 0$ although the advantage of the (DT $O(\delta^4)$) model is notable for state matching achievement as depicted in Fig. 1(b).

2) Gradient Approximation: In what follows, we illustrate the role of approximating $S^{\delta}_{J}(f, x)$ and $\bar{\nabla}H|^{x^+}_x$ when obtaining exact solutions is not possible. When approximating the discrete gradient in (53) up to $q = 2$, one gets

$$\bar{\nabla}^{[1]}H|^{x^+}_x = \big(\sin(x_1)x_2\big) + \frac{\delta}{2}\begin{pmatrix}\cos(x_1)x_2\\ -(\sin(x_1)+rx_2)\end{pmatrix}$$

$$\bar{\nabla}^{[2]}H|^{x^+}_x = \bar{\nabla}^{[1]}H|^{x^+}_x - \times\frac{\delta^2}{4}\begin{pmatrix}\frac{2}{3}\sin(x_1)x_2^2 + \cos(x_1)(\sin(x_1)+rx_2)\\ \cos(x_1)x_2 - r\sin(x_1) - r^2x_2\end{pmatrix}.$$

In Fig. 2, the improvement is clear when increasing the orders of approximation; $p = 2, q = 2$ versus $p = 0, q = 1$.

3) Digital Feedback Design: To highlight the improvement under digital damping (performed over the proposed conjugate output), we consider the controlled gravity pendulum dynamics

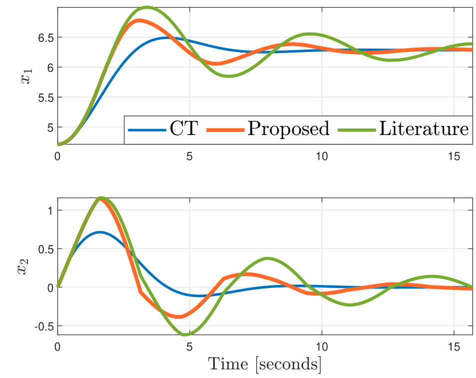
$$\dot{x} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\nabla H(x) + \begin{pmatrix}0\\ 1\end{pmatrix}u, \quad y = \begin{pmatrix}0 & 1\end{pmatrix}x \tag{67}$$

with $g(x) = B = (0\ 1)^\top$ deduced from (64) with $r = 0$. According to Theorem 5, the sampled-data equivalent model is of the form (59) with $S^{\delta,[2]}_{J-R}(f, x)$ in (66)

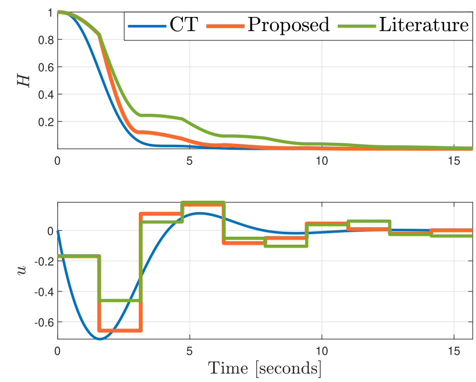
$$g^{\delta}(x, u) = \begin{pmatrix}g^{\delta}_1(x, u)\\ g^{\delta}_2(x, u)\end{pmatrix} = \delta\begin{pmatrix}0\\ 1\end{pmatrix} + \frac{\delta^2}{2}\begin{pmatrix}1\\ 0\end{pmatrix} - \frac{\delta^3}{6}\begin{pmatrix}0\\ \cos x_1\end{pmatrix}$$

and discrete gradient evaluated from x^+ to $x^+(u)$ as

$$\bar{\nabla}H|^{x^+(u)}_{x^+} = \begin{pmatrix}\sin x_1\\ x_2\end{pmatrix} + \delta\begin{pmatrix}x_2\cos x_1\\ \frac{1}{2}u - \sin x_1\end{pmatrix} + \frac{\delta^2}{2}\begin{pmatrix}-\sin x_1 x_2^2 - \cos x_1(\sin x_1 + \frac{1}{2}u)\\ -x_2\cos x_1\end{pmatrix} + O(\delta^3).$$



(a)



(b)

Fig. 3. Pendulum: Stabilization with the sampled period $\delta = \frac{\pi}{2}$, initial condition $x(0) = \text{col}(\frac{3}{2}\pi, 0)$, and $\kappa = 1$.

Finally, thanks to approximations in $O(\delta^3)$, the digital PBC feedback solution to (61) is given by

$$\bar{u} = -\kappa x_2 + \frac{\delta}{2}\kappa(\kappa x_2 + \sin x_1) + \frac{\delta^2}{6}\kappa\left(x_2\cos x_1 - \frac{3}{2}\kappa(\kappa x_2 + \sin x_1)\right) + O(\delta^3). \tag{68}$$

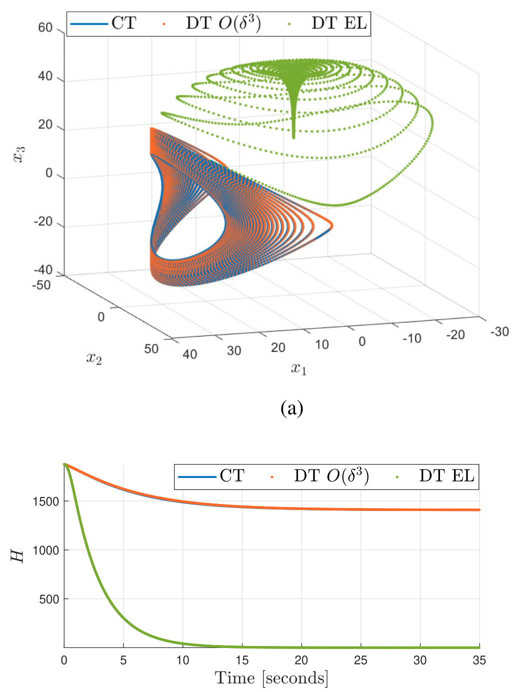
Accordingly, the origin is made asymptotically stable in a closed loop. The improvements achieved under digital damping performed over the proposed sampled port-Hamiltonian representation are illustrated in Fig. 3 with respect to more usual feedback strategies described in [23], [41], and [43] given by

$$u_l = -\kappa B^\top\bar{\nabla}H|^{x^+(u_l)}_x = -\kappa x_2 + \frac{\delta}{2}\kappa(\kappa x_2 + \sin x_1) + \frac{\delta^2}{4}\kappa\left(x_2\cos x_1 - \kappa(\kappa x_2 + \sin(x_1))\right) + O(\delta^3). \tag{69}$$

B. Controlled Rigid Body

Consider the dynamics of the angular velocities of a rigid body in the absence of gravity [2] given by

$$\dot{x} = \begin{pmatrix}-r_1 & -x_3 & x_2\\ x_3 & -r_2 & -x_1\\ -x_2 & x_1 & -r_3\end{pmatrix}\nabla H(x) \tag{70}$$



(a)

(b)

Fig. 4. Rigid Body with $\delta = 10^{-2}$, $r_1 = r_2 = 0$, and $r_3 = 0.2$.

where $x = (x_1, x_2, x_3)$ denote the components of the angular momentum along the three principal axes, $r_1, r_2, r_3 \geq 0$ denote the decay of the angular momentum, and $H(x)$ describes the kinetic energy with inertia $I_{x_1}, I_{x_2}, I_{x_3} > 0$, i.e.,

$$H(x) = \frac{x_1^2}{2I_{x_1}} + \frac{x_2^2}{2I_{x_2}} + \frac{x_3^2}{2I_{x_3}}. \tag{71}$$

Note that the unforced dynamics (70) with (71) is dissipative for some $r_i > 0$, and conversely, conservative for all $r_i = 0$.

In the uncontrolled case, invoking Theorem 5.1 and Remark 5.3, one computes $S^{\delta,[1]}_{J-R}(f, x) = S(x) + \frac{\delta}{2} S_1(x)$ in (52) with

$$S_1(x) = \begin{pmatrix} \frac{I_{x_3}x_2^2+I_{x_2}x_3^2}{I_{x_2}I_{x_3}} & -\frac{I_{x_2}r_2x_3+I_{x_3}x_1x_2}{I_{x_2}I_{x_3}} & \frac{I_{x_3}r_3x_2-I_{x_2}x_1x_3}{I_{x_2}I_{x_3}} \\ \frac{I_{x_1}r_1x_3-I_{x_3}x_1x_2}{I_{x_1}I_{x_3}} & \frac{I_{x_3}x_1^2+I_{x_1}x_3^2}{I_{x_1}I_{x_3}} & -\frac{I_{x_3}r_3x_1+I_{x_1}x_2x_3}{I_{x_1}I_{x_3}} \\ -\frac{I_{x_1}r_1x_2+I_{x_2}x_1x_3}{I_{x_1}I_{x_2}} & \frac{I_{x_2}r_2x_1-I_{x_1}x_2x_3}{I_{x_1}I_{x_2}} & \frac{I_{x_2}x_1^2+I_{x_1}x_2^2}{I_{x_1}I_{x_2}} \end{pmatrix}.$$

The Euler-like model [23], [39] is recovered with $p = 0$ in (52).

1) Simulations: Simulations have been performed fixing the parameters of the rigid body as in [23] ($I_{x_1} = \frac{1}{3}, I_{x_2} = \frac{1}{2}, I_{x_3} = 1$) and initial conditions $x_0 = (25, 25, 25)$. The improvement due to computing a sampled dynamics in $O(\delta^3)$ rather than in $O(\delta^2)$ is depicted in Fig. 4 that shows the phase portrait and the Hamiltonian evolution. The benefit of our model, with respect to the one with $p = 0$, stands in a better preservation of the state trajectory and the decay of the dissipation rate as clearly shown by simulations.

2) Digital Feedback Design: Consider the port-controlled dynamics of the form [13]

$$\dot{x} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \nabla H(x) + \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} u \tag{72a}$$

$$y = \frac{g_1}{I_{x_1}}x_1 + \frac{g_2}{I_{x_2}}x_2 + \frac{g_3}{I_{x_3}}x_3 \tag{72b}$$

where y is the passive output. According to Theorem 6.1, the digital passivity-based feedback stabilizing the angular velocity at zero is the solution to the equality (61) detailed in $O(\delta^2)$ as

$$\begin{aligned} \bar{u} = & -\kappa\Bigg(\sum_{i=1}^{3} \frac{g_i(x_i^+(\bar{u}) + x_i^+)}{2I_{x_i}} \\ & - \frac{\delta(g_2x_3+g_3x_2)(I_{x_2}-I_{x_3})(x_1^+(\bar{u})+x_1^+)}{4I_{x_2}I_{x_3}} \\ & - \frac{\delta((g_1I_{x_3}-g_3I_{x_1})x_3+g_3(I_{x_3}-I_{x_1})x_1)\,(x_2^+(\bar{u})+x_2^+)}{4I_{x_1}I_{x_3}} \\ & - \frac{\delta(g_1x_2+g_2x_1)(I_{x_1}-I_{x_2})(x_3^+(\bar{u})+x_3^+)}{4I_{x_1}I_{x_2}}\Bigg) \end{aligned} \tag{73}$$

when setting

$$g^\delta(x,u) = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} + \frac{\delta}{2}\begin{pmatrix} \frac{g_2I_{x_2}x_3+g_3I_{x_2}x_2-g_2I_{x_3}x_3-g_3I_{x_3}x_2}{I_{x_2}I_{x_3}} \\ \frac{g_1I_{x_3}x_3-g_3I_{x_1}x_3-g_3I_{x_1}x_1+g_3I_{x_3}x_1}{I_{x_1}I_{x_3}} \\ \frac{g_1I_{x_1}x_2+g_2I_{x_1}x_1-g_1I_{x_2}x_2-g_2I_{x_2}x_1}{I_{x_1}I_{x_2}} \end{pmatrix} + O(\delta^2).$$

3) Simulations: The solution to (73) truncated in $O(\delta^3)$ is injected into the continuous-time dynamics (72) to digitally asymptotically stabilize the equilibria. As in [23], we assume $g_1 = I_{x_1} = \frac{1}{3}$, $g_2 = I_{x_2} = \frac{1}{2}$, $g_3 = I_{x_3} = 1$, $\kappa = 1$, and initial conditions $x_0 = (25, 25, 25)$ so that (73) becomes

$$\begin{aligned} \bar{u} = & -\rho_x + \frac{\delta}{2}\Bigg(s_I\rho_x + \frac{x_1}{I_{x_1}}(x_2-x_3) + \frac{x_2}{I_{x_2}}(x_3-x_1) \\ & + \frac{x_3}{I_{x_3}}(x_1-x_2)\Bigg) \\ & - \frac{\delta^2}{6}\Bigg(\frac{s_I}{2} + \Bigg(I_{x_1}\rho_x + \frac{x_2x_3}{I_{x_2}} - \frac{x_2x_3}{I_{x_3}}\Bigg) \\ & \Bigg(s_I + \frac{x_2-x_3}{I_{x_1}} - \frac{x_2}{I_{x_2}} + \frac{x_3}{I_{x_3}}\Bigg) \\ & + \Bigg(I_{x_2}\rho_x - \frac{x_1x_3}{I_{x_1}} + \frac{x_1x_3}{I_{x_3}}\Bigg)\Bigg(s_I + \frac{x_1}{I_{x_1}} + \frac{x_3-x_1}{I_{x_2}} - \frac{x_3}{I_{x_3}}\Bigg) \\ & + \Bigg(I_{x_3}\rho_x + \frac{x_1x_2}{I_{x_1}} - \frac{x_1x_2}{I_{x_2}}\Bigg)\Bigg(s_I - \frac{x_1}{I_{x_1}} + \frac{x_2}{I_{x_2}} + \frac{x_1-x_2}{I_{x_3}}\Bigg)\Bigg) \end{aligned} \tag{74}$$

with $\rho_x = x_1 + x_2 + x_3$ and $s_I = I_{x_1} + I_{x_2} + I_{x_3}$. We have considered the same simulation as in [23] with an increased step-size up to $\delta = 0.675$. In Fig. 5, the comparison is made with respect to the digital control, respectively, described in [22] and [23]. Differently from the control considered in the literature, the proposed control (74) shows the preservation of the phase portrait under digital feedback and a decay of the Hamiltonian function at the sampling instants.

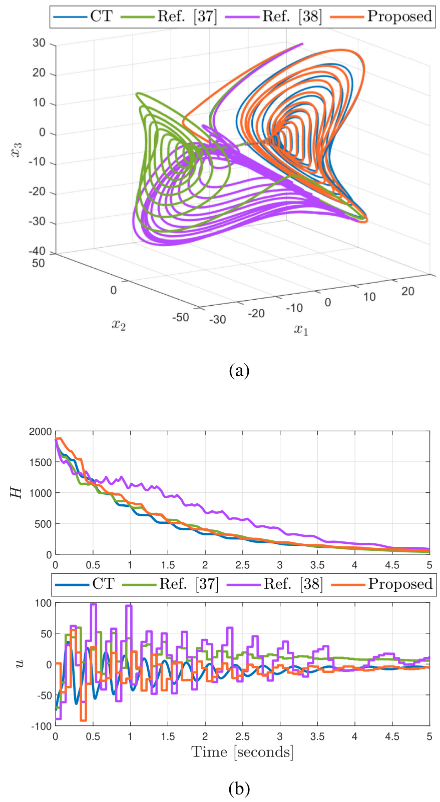


Fig. 5. Rigid body under digital control for $\delta = 0.0675$.

VIII. Conclusion

New forms describing sampled equivalent models of continuous-time gradient and port-Hamiltonian dynamics have been provided. It has been shown that it is always possible to recover under sampling a discrete-time equivalent model exhibiting a discrete-time port-Hamiltonian structure with respect to the same Hamiltonian function as in continuous time with modified structure matrices that depend on the sampling period. The deduced model preserves, beyond the structure, the same energetic properties as the continuous-time one at all sampling instants, conservativeness, or dissipation. The approach is constructive through an algorithmic procedure and allows the definition and computation of approximate models. The case of port-controlled Hamiltonian systems has been briefly discussed to show the benefits of the proposed approach in stabilization through digital damping. As an interesting outcome, we stress that the sampled-data equivalent model we propose evolves over a Dirac structure when properly defining effort and flow variables into the storing and dissipating ports and the environmental interaction through the input. Perspectives concern the use of these models to further investigate energy management-based control strategies of port-controlled-Hamiltonian systems along the lines of preliminary results set for purely discrete-time systems [44]. Further perspectives concern the time discretization of distributed port-Hamiltonian systems [45], [46].

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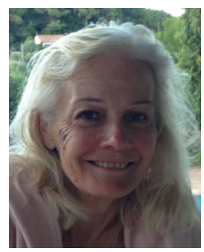

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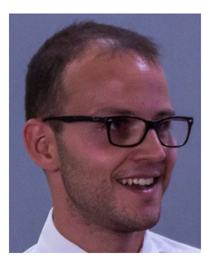

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