



Singular limits of a coupled elasto-plastic damage system as viscosity and hardening vanish

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Abstract

The paper studies the asymptotic analysis of a model coupling elastoplasticity and damage depending on three parameters—governing viscosity, plastic hardening, and convergence rate of plastic strain and displacement to equilibrium—as they vanish in different orders. The notion of limit evolution obtained is proven to coincide in any case with a notion introduced by Crismale and Rossi (SIAM J Math Anal 53(3):3420–3492, 2021), moreover, such solutions are closely related to those obtained in the vanishing-viscosity limit by Crismale and Lazzaroni (Calc Var Part Differ Equ 55(1):17, 2016), for the analogous model where only the viscosity parameter was present.

Keywords Rate-independent systems · Variational models · Vanishing viscosity and hardening · Balanced Viscosity solutions · Damage · Elasto-plasticity

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1 Introduction

Rate-independent processes model evolutionary phenomena where the external loading is much slower than the internal oscillations of materials, while viscosities may be neglected. Despite a wide literature on the subject (see [24] and references therein), a further understanding is needed of the relations between the different notions of solution that have been proposed. In particular, a problem of interest in applications is to determine which kind of solution captures the limiting behavior of dynamical systems for small viscosity or inertia. For this reason, in this paper we compare different notions of solution, obtained with different approximation methods as viscosities tend to zero at different rates (but with no inertia).

We focus on a rate-independent system modeling damage in an elasto-plastic body occupying a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. The model was advanced and first studied in [5, 6], while the existence of *globally minimizing quasistatic evolutions* (or, equivalently, *Energetic solutions*) was first proved in [11]. In [8, 10], the *vanishing-viscosity* approach was instead exploited to find the so-called *Balanced Viscosity* solutions, obtaining the rate-independent system as the limit of a viscously perturbed system. Indeed, vanishing viscosity was advanced as a selection criterion for solutions with a mechanically feasible behavior at jumps, motivated by the observation that Energetic solutions jump ‘too long and too early’, cf. the characterization proved in [29] and the references therein. We refer to the pioneering [16], and the subsequent [26, 27], for the definition and properties of such solutions in the context of an ‘abstract’ rate-independent system. The vanishing-viscosity technique has also been adopted in various concrete applications, ranging from plasticity (cf., e.g., [7, 12, 14, 18, 32]), to damage, fracture, and fatigue (see for instance [1, 4, 9, 20, 21, 23]).

In this paper, we aim to gain further insight into the different ways of constructing *Balanced Viscosity* solutions to the model for damage and plasticity from [6], which were explored in [8] and [10]. We show that these notions of solutions essentially coincide if the hardening vanishes together with viscosities, while they retain different features if the hardening parameter is positive. In particular, it turns out that perfect plasticity coupled with damage may equivalently be approximated by means of processes where viscosity is confined to the flow rule for damage, or with viscosity also in the momentum equation and in the plastic flow rule.

1.1 The model

The rate-independent process we are going to address describes the evolution, in the time interval $(0, T)$, of the *displacement* $u : (0, T) \times \Omega \rightarrow \mathbb{R}^n$, of the plastic strain $p : (0, T) \times \Omega \rightarrow \mathbb{M}_D^{n \times n}$, and of the damage variable $z : (0, T) \times \Omega \rightarrow [0, 1]$ that describes the soundness of the material: for $z(t, x) = 1$ (respectively, $z(t, x) = 0$) the material is in the undamaged (fully damaged, resp.) state, at the time $t \in (0, T)$ and ‘locally’ around the point $x \in \Omega$. In fact, the related PDE system consists of

- The momentum balance

$$-\operatorname{div} \sigma = f \quad \text{in } \Omega \times (0, T), \quad \sigma n = g \text{ on } \Gamma_{\text{Neu}} \times (0, T), \quad (1.1a)$$

with f, g some external forces, n the outer unit normal vector to Ω , σ the *stress* tensor

$$\sigma = \mathbb{C}(z)e \quad \text{in } \Omega \times (0, T), \tag{1.1b}$$

\mathbb{C} the elastic stress tensor, and $e : (0, T) \times \Omega \rightarrow \mathbb{M}_{\text{sym}}^{n \times n}$ the elastic strain; together with the plastic strain p , the elastic strain e concurs to the kinematic admissibility condition for the strain $\frac{\sigma}{2}$, i.e.,

$$E(\cdot) = \cdot + \cdot \quad \Omega \times (\cdot) \tag{1.1c}$$

– The flow rule for the damage variable z

$$-\mathbb{C} \tag{1.1d}$$

where $\partial \mathbb{R} : \mathbb{R} \rightrightarrows \mathbb{R}$ denotes the convex analysis subdifferential of the density of dissipation potential

$$\mathbb{R} \rightarrow$$

encompassing the unidirectionality in the evolution of damage, A_m is the m -Laplacian operator, with $m > \frac{n}{2}$, and W is a suitable nonlinear, possibly nonsmooth, function;

– The flow rule for the plastic tensor

$$(1.1e)$$

with σ_D the deviatoric part of the stress tensor σ and $H(z, \cdot)$ the density of the plastic dissipation potential.

System (1.1a)–(1.1e) is complemented by the boundary conditions

$$(1.1f)$$

where Γ_{Dir} is the Dirichlet part of the boundary $\partial \Omega$ and w a time-dependent Dirichlet loading, while Γ_{Neu} is the Neumann part of $\partial \Omega$ and g an assigned traction.

Alternative models for damage and plasticity have been analyzed in, e.g., [15, 30, 31], albeit from a different perspective. In fact, those papers address the rate-independent evolution of the damage and plastic processes coupled with a *rate-dependent* momentum balance, featuring viscosity and even inertial terms. Therefore, the resulting system has a *mixed* rate-dependent/independent character and is formulated in terms of a weak, energetic-type notion of solution. Instead, both in [8] and [10], (two distinct) viscous regularization procedures, described below, were advanced to construct Balanced Viscosity solutions to the fully rate-independent system (1.1).

1.2 Balanced viscosity solutions

In [8], the vanishing-viscosity approximation of system (1.1) was carried out by perturbing the damage flow rule by a viscous term, which led to the *viscously regularized* system

$$\sigma \quad \sigma \quad \mathbb{C} \tag{1.2a}$$

$$-\mathbb{C} \tag{1.2b}$$

$$\tag{1.2c}$$

supplemented by the boundary conditions

$$\tag{1.2d}$$

Passing to the limit in a reparameterized version of (1.2) led to a first construction of solutions to system (1.1). We shall illustrate the notion of *parameterized* Balanced Viscosity solution thus obtained in the forthcoming Sect. 3.3. In what follows, for quicker reference we will call the solutions from [8] *solutions to system (1.1)*, where the subscript 0 indicates that the solutions are obtained in the limit as $\epsilon \rightarrow 0$.

In [10], a different construction of solutions to system (1.1) was proposed, based on a viscous regularization of the momentum balance and of the plastic flow rule, *in addition* to that of the damage flow rule. This alternative approach was first proposed in [28] in a finite-dimensional context and extended to infinite-dimensional systems in the recent [25]. Both papers address the vanishing-viscosity analysis of an abstract evolutionary system, that can be thought of as prototypical of rate-dependent systems in solid mechanics, governing the evolution of an elastic variable ϵ and of an internal variable η . In those papers, ϵ has relaxation time τ_ϵ , while η has a viscous damping with relaxation time τ_η . To emphasize the occurrence of these three time scales (the time scale τ_σ of the external loading, the relaxation time τ_ϵ of ϵ , and the - possibly different - relaxation time τ_η of η), the term ‘multi-rate system’ was used in [28]. Therein, as well as in [25], it was shown that, in the three cases $\tau_\epsilon/\tau_\sigma \rightarrow 0, 1$, and $\tau_\eta/\tau_\sigma \rightarrow 0$, the vanishing-viscosity analysis as $\epsilon \rightarrow 0$ leads to different notions of Balanced-Viscosity solutions, including in particular different descriptions of time discontinuities. Indeed, the transition corresponding to a jump in time is characterized by a PDE system. In contrast, in the case of global minimization, the only condition to be satisfied at jumps is energy conservation.

Thus, along the lines of [28], in [10] the authors addressed the following, alternative, viscous regularization of system (1.1):

$$\text{in } \Omega, \tag{1.3a}$$

$$-\mathbb{C} \tag{1.3b}$$

$$\tag{1.3c}$$

supplemented by the boundary conditions

$$\tag{1.3d}$$

where \mathbb{C} is a positive-definite fourth-order tensor. System (1.3) features a viscous regularization *both* in the damage flow rule *and* in the displacement equation and the plastic flow rule. Let us now illustrate the role of the various parameters appearing therein, namely the

- (Vanishing-)viscosity parameter τ_η ,
- (Vanishing-)hardening parameter τ_ϵ ,

- Additional parameter τ , which was required to fulfill (1.1) in order to get suitable a priori estimates. We have referred to τ as a *rate* parameter, since it sets the mutual rate at which, on the one hand, the displacement and the plastic strain converge to equilibrium and rate-independent evolution, and, on the other hand, the damage parameter converges to rate-independent evolution. More precisely, if τ stays fixed then u and p converge at the same rate as z , while their convergence occurs at a faster rate if $\tau \rightarrow 0$. This is clear if one chooses, e.g., $\tau = \epsilon^\alpha$, so that the viscous terms $\epsilon^\alpha \dot{u}$ and $\epsilon^\alpha \dot{p}$ in (1.3a) and (1.3c) are modulated by the coefficient $\epsilon^{2-\alpha}$, as opposed to the coefficient ϵ in the damage flow rule. Observe that, upon taking the vanishing-hardening limit $\epsilon \rightarrow 0$, the constraint $\tau \dot{z} \leq 0$ forces the joint vanishing-viscosity and vanishing-hardening limit to occur at a faster rate for u and p than for z .

We will refer to the vanishing-viscosity analyses in system (1.3) as *full*, as opposed to the *partial* vanishing-viscosity approximation provided by system (1.2), where only the damage flow rule is regularized.

Indeed, in [10] three *full* vanishing-viscosity analyses have been carried out for system (1.3), leading to three different notions of solution for system (1.1), possibly regularized by a hardening term. Let us briefly illustrate them.

- (1) τ_0 solutions The limit passage in a (reparameterized) version of (1.3) as $\epsilon \rightarrow 0$, while the positive parameters τ and α stayed fixed, has led to τ_0 solutions for a variant of the system (1.1), where the plastic flow rule was regularized by the hardening term $\tau_0 \dot{p}$, i.e., for the rate-independent system with hardening

$$\text{C} \quad \text{in} \quad 0, \quad (1.4a)$$

$$-\text{C} \quad (1.4b)$$

$$(1.4c)$$

coupled with the boundary conditions

$$(1.4d)$$

The τ_0 solutions to system (1.4) thus constructed reflect their origin from a system viscously regularized in all of the three variables u, z, p . In fact, in the jump regime the system may switch to viscous behavior in the three variables u, z , and p . This means that the states of the system before and after a time discontinuity are connected by a trajectory (reparameterized in a certain time scale), whose evolution may be governed by viscosity in u, z , and p , similarly to (1.3). Since the convergence of u, z , and p to elastic equilibrium and rate-independent evolution has occurred at the *same rate* (as τ stayed fixed), viscous behavior in u, z , and p may *equally* intervene in the jump regime. We shall refer to such solutions as τ_0 solutions to system (1.4). The subscript 0 suggests that they have been obtained in the vanishing-viscosity limit $\epsilon \rightarrow 0$, while the occurrence of the parameter τ_0 keeps track of the presence of hardening. Also, the parameter τ_0 appears in the notation, since it still features in the limiting evolution as a coefficient of the viscous terms in the displacement equation and in the plastic flow rule, which may be active in the jump regime.

- (2) $\overset{0}{0}$ solutions The limit passage in a (reparameterized) version of (1.3) as $\epsilon \rightarrow 0$ simultaneously with $\delta \rightarrow 0$, while η stayed fixed, has again led to $\overset{0}{0}$ solutions for the rate-independent elasto-plastic damage system with hardening (1.4). These solutions still have the feature that, in the jump regime, the system may switch to viscous behavior in u , z , and p . However, $\overset{0}{0}$ solutions thus obtained reflect the fact that the convergence of u and p to elastic equilibrium and rate-independent evolution has occurred at a *faster rate* (as $\delta \rightarrow 0$) than that for z . To emphasize this, such solutions were termed *solutions to the multi-rate system for damage with hardening*. We will refer to them as $\overset{0}{0}$ solutions to system (1.1). In this notation, the double occurrence of 0 relates to the fact that such solutions were obtained in the limit $\epsilon \rightarrow 0$, as opposed to the $\overset{0}{0}$ solutions from [8] (arising in the limit of system (1.3) as $\delta \rightarrow 0$ and $\eta \rightarrow 0$).
- (3) $\overset{0,0}{0}$ solutions The limit passage in a (reparameterized) version of (1.3) as $\epsilon \rightarrow 0$, $\delta \rightarrow 0$, $\eta \rightarrow 0$ jointly led to $\overset{0,0}{0}$ solutions to the multi-rate system for damage and perfect plasticity (1.1), again reflecting the fact that the convergence of u and p to elastic equilibrium and rate-independent, *perfectly plastic* evolution happened at a rate faster than that for z . The vanishing-viscosity solutions arising from this joint limit will receive specific attention in this paper. In what follows, we will refer to them as $\overset{0,0}{0}$ solutions to system (1.1). Here, the triple occurrence of 0 relates to the fact that such solutions were obtained in the limit $\epsilon \rightarrow 0$, $\delta \rightarrow 0$ and $\eta \rightarrow 0$ and thus immediately suggests the comparison with the $\overset{0}{0}$ solutions from [8].

1.3 Our results

The aim of this paper is twofold:

- (i) We propose to gain further insight into $\overset{0,0}{0}$ solutions to system (1.1) (cf. item $\overset{0}{0}$ in the above list). More precisely, first of all we shall provide a *differential characterization* of such solutions, cf. Proposition 3.11 ahead. This will be compared with a corresponding characterization of $\overset{0}{0}$ solutions proved in Proposition 3.5. Moreover, relying on Proposition 3.11, in Theorem 4.1 we will subsequently prove that, after an initial phase in which z is constant while u and p , evolving by viscosity, relax to elastic equilibrium and to rate-independent evolution, respectively, it turns out that u never leaves the equilibrium, and p the rate-independent regime. Afterward, the evolution of system (1.1) is captured by the notion of $\overset{0}{0}$ solution as obtained in [8] by taking the vanishing-viscosity limit as $\delta \rightarrow 0$ of system (1.2). In other words, viscosity in u and p (may) intervene only in an initial phase in the reparameterized time scale, corresponding to a time discontinuity in the time scale of the loading. After this initial phase, the $\overset{0,0}{0}$ solutions to the perfectly plastic system for damage (1.1) arising by the *full* vanishing-viscosity approach of [10] comply with the same notion of solution of [8], where viscosity for u and p was neglected.

We point out that an analogous characterization can be proved for the $\overset{0}{0}$ solutions to the multi-rate system with *fixed* hardening parameter η , obtained in the limit passage $\delta \rightarrow 0$ of the above list; see Remark 5.2 ahead.

- (ii) We aim to ‘close the circle’ in the analysis of the singular limits of system (1.3), by showing that, for two given sequences $\delta_n \rightarrow 0$, $\eta_n \rightarrow 0$, with $\delta_n / \eta_n \rightarrow \infty$ as $n \rightarrow \infty$,

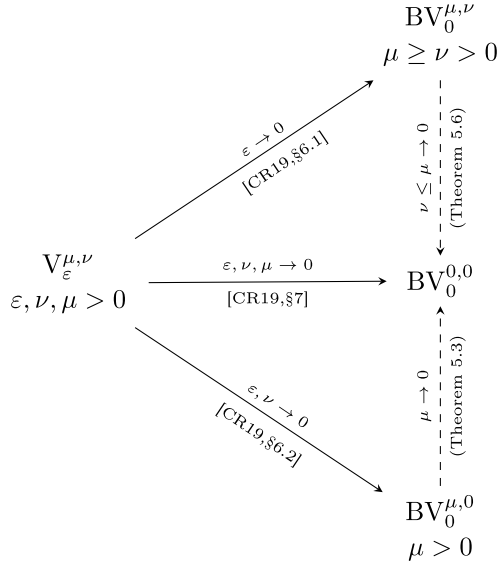


Fig. 1 The diagram displays the asymptotic relations between different notions of solution. The symbol $V_\varepsilon^{\mu, \nu}$ indicates solutions to the viscous system (1.3). Solid lines represent convergences (along sequences) to the limiting solutions of type $BV_0^{\mu, \nu}$, $BV_0^{0,0}$, $BV_0^{\mu,0}$, already proved in [10, Sections 6.1, 6.2, 7]. Dashed lines represent convergences proved in the present paper, the corresponding theorems being referred to in the diagram. Starting from $V_\varepsilon^{\mu, \nu}$ one may either pass to the limit as $\varepsilon \rightarrow 0$ and then as $\nu \leq \mu \rightarrow 0$; or pass to the limit as $\varepsilon, \nu, \mu \rightarrow 0$. Since there is no uniqueness, it is not guaranteed that one gets the very same solution found in the joint limit $\varepsilon, \nu, \mu \rightarrow 0$. However, we prove that through the three different procedures one finds evolutions satisfying the same *notion* of solution. In this sense, we may say that the diagram commutes

- (1) $BV_0^{\mu, \nu}$ solutions to the *single-rate* system with hardening converge as $\varepsilon \rightarrow 0$ to a $BV_0^{0,0}$ solution of the perfectly plastic damage system, which will be shown in Theorem 5.6 ahead;
- (2) $BV_0^{\mu,0}$ solutions to the *multi-rate* system with hardening converge as $\varepsilon \rightarrow 0$ to a $BV_0^{0,0}$ solution, cf. Theorem 5.3.

In particular, we will prove that the diagram in Fig. 1 commutes.

We emphasize that these results establish asymptotic relations between $BV_0^{\mu, \nu}$ solutions, already obtained as vanishing-viscosity limits. These convergence analyses show that $BV_0^{0,0}$ solutions are robust enough to capture the asymptotic behavior of a wide class of $BV_0^{\mu, \nu}$ solutions depending on different parameters. However, $BV_0^{\mu,0}$ solutions reduce to $BV_0^{0,0}$ solutions after an initial phase in which u and p converge to elastic equilibrium and stability, respectively; in particular, if the initial conditions are at equilibrium, then the two notions

of solution coincide. This feature may be traced back to the convex character of perfect plasticity and to the multi-rate character inherent to the system, since with μ we have forced faster convergence to equilibrium in u and stability in p .

1.4 Plan of the paper

In Sect. 2, we detail the setup of the problem, list our assumptions and provide some preliminary results. In Sect. 3, we illustrate the notion of μ solution to system (1.1) arising from the *partial* vanishing-viscosity approach of [8], and that of $\mu^{0,0}$ solution via the *full* vanishing-viscosity analysis in [10]. In Sect. 4, we establish Theorem 4.1, providing a complete characterization of $\mu^{0,0}$ solutions. Section 5 is devoted to the vanishing-hardening analysis of $\mu^{0,0}$ and μ solutions to the system with hardening. The proofs of Theorems 5.3 and 5.6 rely on some technical results collected in the “Appendix”.

2 Setup and preliminaries

Throughout the paper, we will use the following

Notation 2.1 (*General notation and preliminaries*) Let X be a Banach space. By $\langle \cdot, \cdot \rangle$, we denote the duality between X and X' or between X and X (whenever X is a Hilbert space, $\langle \cdot, \cdot \rangle$ will be the inner product), while $\|\cdot\|$ stands for the norm in X or in \mathbb{R}^n . The inner Euclidean product in \mathbb{R}^n , $n \geq 1$, is denoted by $\langle \cdot, \cdot \rangle$, and the Euclidean norm in \mathbb{R}^n by $|\cdot|$. The symbol B_r stands for the open ball in \mathbb{R}^n with radius r and center 0 .

We write $\|\cdot\|_m$ for the m -norm on the space C^m , with O a measurable subset of \mathbb{R}^n and $\|\cdot\|_{m,O}$, and similarly $\|\cdot\|_{m,m}$ for the norm of the Sobolev-Slobodskij space $W^{m,m}$, for $m \in \mathbb{N}$. The symbol \mathbb{R}^n stands for the space of \mathbb{R}^n -valued bounded Radon measures in O .

The space of symmetric $n \times n$ -matrices is denoted by \mathbb{M}^n , while the subspace of the deviatoric matrices with null trace is denoted by $\mathbb{M}^{n,n}$. One has $\mathbb{M}^n = \mathbb{M}^{n,n} \oplus \mathbb{R}I$, where I is the identity matrix, i.e., any $M \in \mathbb{M}^n$ can be decomposed as $M = P + \frac{\text{tr}(M)}{n}I$, where $P \in \mathbb{M}^{n,n}$ is the orthogonal projection of M onto $\mathbb{M}^{n,n}$. The latter is called the *deviatoric part* of M . The symbol $\mathbb{M}^{n,n} \times \mathbb{M}^{n,n}$ stands for the set of symmetric endomorphisms on $\mathbb{M}^{n,n}$.

Given a function $v : (0, T) \rightarrow \mathbb{R}^n$ differentiable, w.r.t. time a.e. on $(0, T)$, its (almost everywhere defined) partial time derivative is indicated by $\partial_t v : (0, T) \rightarrow \mathbb{R}^n$. A different notation will be employed when considering v as a (Bochner) function, from $(0, T)$ with values in a Lebesgue or Sobolev space X (with the Radon–Nikodým property): if $v : (0, T) \rightarrow X$, then its (almost everywhere defined) time derivative is indicated by $\partial_t v : (0, T) \rightarrow X$.

The symbols c, c_1, c_2, \dots will denote positive constants whose precise value may vary from line to line (or within the same line). We will sometimes employ the symbols μ, ν, \dots , as place-holders for terms appearing in inequalities: also in this case, such symbols may appear in different proofs with different meaning.

2.1 Functions of bounded deformation

The state space for the displacement variable for the systems with hardening will be

(recall that Ω_D is the Dirichlet part of Ω , cf. (2.2) ahead).

For the perfectly plastic damage system, displacements will belong to the space of *functions of bounded deformations*, defined by

with $\mathcal{BD}(\Omega; \mathbb{R}^n)$ the space of \mathbb{R}^n -valued bounded Radon measures on Ω .

We recall that $\mathcal{BD}(\Omega; \mathbb{R}^n)$ can be identified with the dual of the space $\mathcal{C}_0(\Omega; \mathbb{R}^n)$ of continuous \mathbb{R}^n -valued functions vanishing at the boundary of Ω . The space $\mathcal{BD}(\Omega; \mathbb{R}^n)$ has a Banach structure if equipped with the norm

Indeed, $\mathcal{C}_0(\Omega; \mathbb{R}^n)$ is the dual of a normed space, cf. [33], and such duality provides a weak convergence on $\mathcal{BD}(\Omega; \mathbb{R}^n)$: a sequence μ_k converges to u weakly in $\mathcal{BD}(\Omega; \mathbb{R}^n)$ if $\int_{\Omega} \varphi \, d\mu_k \rightarrow \int_{\Omega} \varphi \, d\mu$ for every $\varphi \in \mathcal{C}_0(\Omega; \mathbb{R}^n)$. It holds $\mu_k \rightarrow u$ weakly* in $\mathcal{BD}(\Omega; \mathbb{R}^n)$. Up to subsequences, every bounded sequence in $\mathcal{BD}(\Omega; \mathbb{R}^n)$ converges weakly*, weakly in $\mathcal{BD}(\Omega; \mathbb{R}^n)$, and strongly in $\mathcal{BD}(\Omega; \mathbb{R}^n)$ for any $\varphi \in \mathcal{C}_0(\Omega; \mathbb{R}^n)$. Finally, we recall that the trace $\mu|_{\Omega_D}$ of a function $\mu \in \mathcal{BD}(\Omega; \mathbb{R}^n)$ is well defined and is an element in $\mathcal{BD}(\Omega_D; \mathbb{R}^n)$.

2.1.1 A divergence operator

First of all, we observe that any $\mu \in \mathcal{BD}(\Omega; \mathbb{R}^n)$ such that $\mu|_{\Omega_D} = 0$ induces the distribution $\text{div} \mu$ defined by

$$\text{div} \mu(\varphi) = - \int_{\Omega} \varphi \, d\mu \quad (2.1)$$

By [22, Theorem 1.2] and [13, (2.24)] we have that $\text{div} \mu \in \mathcal{D}'(\Omega; \mathbb{R}^n)$; moreover, if $\mu = \sigma \otimes n$, then the distribution $\text{div} \mu$ fulfills $\text{div} \mu = \text{div} \sigma$, where the right-hand side is the standard pointwise product of the matrix σ and the normal vector n in Ω .

For the treatment of the perfectly plastic system for damage, it will be crucial to work with the space

$$\mathcal{BD}(\Omega; \mathbb{R}^n) \quad (2.2)$$

Furthermore, our choice of external loadings (see (2.8a)) will ensure that the stress fields that we consider, at equilibrium, have the additional property that $\text{div} \sigma = 0$ and $\sigma \otimes n = 0$ (cf. Lemma 3.1). Therefore, any of such fields induces a functional via

$$\mu \mapsto \int_{\Omega} \varphi \, d\mu \quad (2.3)$$

With slight abuse of notation, we shall denote by $\mathcal{BD}(\Omega; \mathbb{R}^n)$ also the restriction of the above functional to $\mathcal{BD}(\Omega; \mathbb{R}^n)$.

2.1.2 The Δ -Laplacian

The damage flow rule features a gradient regularizing contribution in terms of the Δ -Laplacian operator, that is defined from the bilinear form $\langle \cdot, \cdot \rangle_{\Delta}$

$$\langle u, v \rangle_{\Delta} = \int_{\Omega} \nabla u \cdot \nabla v \, dx$$

Then,

$$\rightarrow$$

The inner product $\langle \cdot, \cdot \rangle_{\Delta}$ makes $H^1(\Omega)$ a Hilbert space.

Throughout the paper, we shall assume Ω - and rely on the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$.

2.2 Assumptions and preliminary results

This section and Sect. 2.3 collect all our assumptions on the constitutive functions of the model and on the problem data. We will omit to invoke them explicitly in the statement of the various results.

2.2.1 The reference configuration

In what follows, we will assume that $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$, is a bounded Lipschitz domain satisfying the so-called *Kohn–Temam condition*:

Notice that the additional C^1 regularity is only required close to the set Γ where the Dirichlet and the Neumann boundary meet. This is a technical assumption needed to interpret the stress-strain duality (2.9) as a measure.

2.2.2 The elasticity and viscosity tensors

We assume that the elastic tensor $\mathbb{C} : \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$ fulfills the following conditions

For the viscosity tensor η , we require that

—

Thus, η induces an equivalent (by a Korn–Poincaré-type inequality) Hilbert norm on V , i.e.,

$$(2.4)$$

The related ‘dual norm’ is

$$(2.5)$$

2.2.3 The potential energy for the damage variable

In addition to the regularizing, nonlocal gradient contribution featuring the bilinear form $a(\cdot, \cdot)$, the z -dependent part of the mechanical energy functional shall feature a further term with density W satisfying

$$(2.W1)$$

$$\rightarrow \rightarrow (2.W2)$$

where \rightarrow means that \rightarrow and \rightarrow if \rightarrow , in accordance with (2.1).

Indeed, the energy contribution involving W forces z to be strictly positive; consequently, the material never reaches the most damaged state at any point.

2.2.4 The plastic and the damage dissipation densities

The plastic dissipation potential shall reflect the requirement that the admissible stresses belong to given constraint sets which, in turn, depend on the damage variable z . More precisely, as in [8] we ask that the constraint sets \mathcal{K}_z fulfill

M

with d_H the Hausdorff distance between two subsets of \mathbb{R}^n , defined by

The associated support function $\chi_C : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, defined by

$$\chi_C(v) = \begin{cases} \sup_{x \in C} v \cdot x & \text{if } C \text{ is compact and convex} \\ +\infty & \text{otherwise} \end{cases} \quad (2.6)$$

will act as density function for the plastic dissipation potential, cf. (2.14) later on.

We choose as damage dissipation density the function $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ given by

$$\psi(\lambda) = \begin{cases} \frac{1}{2} \lambda^2 & \text{if } \lambda \leq \lambda_0 \\ \lambda_0 \lambda & \text{if } \lambda > \lambda_0 \end{cases} \quad (2.7)$$

with λ_0 a constant related to the toughness of the material.

2.2.5 Body and surface forces, Dirichlet loading, and initial data

We assume that the volume force f and the assigned traction g fulfill

$$f \in L^2(\Omega; \mathbb{R}^n), \quad g \in L^2(\Gamma_D; \mathbb{R}^n) \quad (2.8a)$$

The induced total load is the function $\ell : \mathbb{R}^n \rightarrow \mathbb{R}$,

Furthermore, as customary for perfect plasticity, we shall impose a *uniform safe load condition*, namely that there exists

$$\ell_0 \in \mathbb{R} \quad (2.8b)$$

and there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ (recall (2.1))

$$\ell_0 - \delta \leq \ell \leq \ell_0 + \delta \quad (2.8c)$$

$$\ell_0 - \delta \leq \ell \leq \ell_0 + \delta \quad (2.8d)$$

Observe that, combining (2.8a) with (2.8b)–(2.8d) yields $\ell_0 - \delta \leq \ell \leq \ell_0 + \delta$ for all $\delta \in (0, \delta_0)$.

As for the time-dependent Dirichlet boundary condition w , we assume that

$$w \in C([0, T]; H^1(\Gamma_D)) \quad (2.8e)$$

Finally, we shall consider initial data (u_0, v_0, θ_0) with

(2.8f)

2.2.6 The stress-strain duality

For the treatment of the perfectly plastic damage system, it is essential to resort to a suitable notion of stress-strain duality that we borrow from [13, 22], also relying on [17] for the extension to Lipschitz boundaries satisfying (2.Ω). Following [13], we introduce the class $A(w)$ of *admissible displacements and strains* associated with a function $\in (\mathbb{R} \mathbb{R})$, that is

$$\mathbb{M} \quad \mathbb{M}$$

where ν denotes the normal vector to $\partial\Omega$ and \otimes the symmetrized tensorial product. The *space of admissible plastic strains* is

Given σ (cf. (2.2)), ν , and \otimes such that $\nu \otimes \nu = \mathbb{I}$, we define

$$\mathbb{M} \quad (2.9)$$

for every $\sigma \in \mathbb{M}$; in fact, this definition is independent of u and e . Under these assumptions, $\sigma \in \mathbb{M}$ for every σ , and σ is a bounded Radon measure with $\sigma \in \mathbb{M}$ in \mathbb{R}^n . Restricting such measure to \mathbb{M} , we set

$$\sigma \quad \sigma \quad (2.10)$$

By (2.Ω) and (2.9), since $\mathbb{M} \subset \mathbb{R}$, we get the following integration by parts formula, valid if the distribution σ defined in (2.1) belongs to \mathbb{R} :

$$\sigma \quad \sigma \quad \sigma \quad \sigma$$

for every σ and ν . We refer to [17] for the properties mentioned above.

2.3 Energetics

A key ingredient for the construction of solutions to the rate-independent systems (1.1) (damage with perfect plasticity) and (1.4) (damage and plasticity with hardening) is the observation that their rate-dependent regularizations (1.2) and (1.3) have a *gradient-system structure*. Namely, they can be reformulated in terms of the generalized gradient flow

$$\partial$$

for suitable choices of

- The state space for the triple $(\mathcal{U}, \mathcal{Z}, \mathcal{P})$;
- The driving energy functional $\mathcal{E} : \mathcal{U} \times \mathcal{Z} \rightarrow \mathbb{R}$;
- The dissipation potential $\mathcal{D} : \mathcal{Z} \rightarrow 0, \infty$, with convex analysis subdifferential $\partial \mathcal{D} \ni \dots$,

as rigorously proved in [10]. Observe that also the rate-independent systems (1.1) and (1.4) have a gradient structure that is, however, only formal due to the fact that the functions u , z , and p may have jumps as functions of time. Nonetheless, for our analysis it is crucial to detail the energetics underlying both the rate-dependent and the rate-independent systems.

2.3.1 The state spaces

The state space for the rate-dependent/independent damage systems with hardening is

For the rate-independent damage system with perfect plasticity, the displacements are just functions of bounded deformation and the plastic strains are only bounded Radon measures on Ω , so that the associated state space is

$$\mathcal{M} = \{ (u, z) \mid u \in \text{BD}(\Omega; \mathbb{R}^n), z \in \mathcal{M}(\Omega; \mathbb{R}^n) \} \quad (2.11)$$

Observe that in the definition of \mathcal{M} it is in fact required that $z \ll \mathcal{L}^n$: indeed, the condition $z \ll \mathcal{L}^n$ is a relaxation of the homogeneous Dirichlet condition on Γ_{Dir} .

2.3.2 The energy functionals

The energy functional governing the rate-dependent and rate-independent systems *with hardening* (1.3) and (1.4), respectively, consists of

- (1) a contribution featuring the elastic energy

$$-C$$

- (2) the potential energy for the damage variable and for the hardening term;
- (3) the time-dependent volume and surface forces.

Namely, for (u, z) given, $\mathcal{E} : \mathcal{U} \times \mathcal{Z} \rightarrow \mathbb{R}$ is defined by

$$-\quad -$$

In what follows, we will use the short-hand notation

for the elastic part of the strain tensor.

The energy functional for the rate-independent perfectly plastic damage system (1.1) is

$$\varphi_0 : \mathbb{M} \rightarrow \mathbb{R} \quad \text{given by} \quad (2.12)$$

Observe that

$$(2.13)$$

2.3.3 The dissipation potentials

The dissipation density from (2.7) clearly induces an integral functional $\psi : \mathbb{M} \rightarrow \mathbb{R}$. However, since the damage flow rule will be posed in (cf. (2.20) ahead), we will restrict ψ to the space \mathbb{M}_D , denoting the restriction by the same symbol. Hence, we will work with the functional

$$\psi : \mathbb{M}_D \rightarrow \mathbb{R}, \quad (2.14)$$

and with its convex analysis subdifferential $\partial \psi : \mathbb{M}_D \rightarrow \mathbb{M}$. We will often use the following characterization of $\partial \psi$, due to the 1-homogeneity of the potential ψ :

$$(2.15)$$

For the systems with hardening (i.e., (1.3) and (1.4)), the *plastic dissipation potential* $\psi : \mathbb{M}_D \rightarrow \mathbb{R}$ is defined by

$$\psi(\underline{D}) = \int_{\Omega} H(\underline{D}) \, dx, \quad (2.16)$$

where H is given by (2.6) and \underline{D} is a place-holder for the plastic rate \underline{D} . Its convex analysis subdifferential $\partial \psi : \mathbb{M}_D \rightarrow \mathbb{M}$, w.r.t. the second variable \underline{D} , given by

$$\mathbb{M}$$

fulfills

$$(2.17)$$

A characterization analogous to (2.17) holds for the subdifferential $\partial \psi : \mathbb{M}_D \rightarrow \mathbb{M}$.

In order to handle the perfectly plastic system for damage, the plastic dissipation potential $\bar{\psi}$, has to be extended to the space \mathbb{M}^n . We define $\bar{\psi} : \mathbb{M}^n \rightarrow \mathbb{R}$ by

$$\bar{\psi}(\mu) = \begin{cases} \psi(\mu) & \text{if } \mu \in \mathcal{M}^+, \\ +\infty & \text{otherwise,} \end{cases}$$

where \mathbb{M}^n is a positive measure such that $\mu \ll \bar{\mu}$ and $\bar{\mu}$ is the Radon–Nikodým derivative of μ with respect to $\bar{\mu}$; by one-homogeneity of ψ , the definition of $\bar{\psi}$ does not depend of $\bar{\mu}$. For the theory of convex functions of measures, we refer to [19]. By [2, Proposition 2.37], for every $z \in \mathbb{M}^n$ the functional $\bar{\psi}(\cdot + z)$ is convex and positively one-homogeneous. We recall that by Reshetnyak’s lower semicontinuity theorem, if $\mathcal{C} \subset \mathbb{M}^n$ and $\bar{\mu} \ll \mathbb{M}^n$ are such that $\mu \rightarrow \bar{\mu}$ in \mathbb{M}^n and $\mu \rightarrow \bar{\mu}$ weakly in \mathbb{M}^n , then

$$\bar{\psi}(\bar{\mu}) \leq \liminf \bar{\psi}(\mu)$$

Finally, from [17, Proposition 3.9] it follows that for every $\sigma \in \mathcal{M}^n$

$$\bar{\psi}(\sigma) \geq \sigma \tag{2.18}$$

In particular, we have

$$\bar{\psi}(\sigma) \geq \sigma \tag{2.19}$$

The (partially) viscously regularized system (1.2) also features the 2-homogeneous dissipation potential

$$\bar{\psi}_2(x) = |x|^2$$

while the (fully) viscously regularized system (1.3) additionally involves the quadratic potentials

$$\bar{\psi}_2(x) = |x|^2 + \dots$$

2.3.4 The gradient structure for system (1.3)

It was proved in [10, Lemma 3.3] that for every $t \in [0, T]$ the functional $\bar{\psi}_t : \mathcal{M}^n \rightarrow \mathbb{R}$ is Fréchet differentiable on its domain $\mathcal{D}(\bar{\psi}_t)$, with $\bar{\psi}_t'(\mu) > \bar{\mu}$, and that for all $\mu \in \mathcal{D}(\bar{\psi}_t)$ the function $\bar{\psi}_t(\cdot + \mu)$ belongs to \mathcal{M}^n . Relying on this, it was shown that system (1.3) reformulates as the generalized gradient system

$$\tag{2.20}$$

possible to reparameterize such curves, obtaining *parameterized curves* defined on an ‘artificial’ time interval $[0, S]$

and γ (suitable) arclength functions associated with the curves γ . Now, in terms of the parameterized curves γ , γ , γ , (3.1) translates into the *reparameterized* energy-dissipation balance

$$\int_0^S \left(\frac{1}{2} \rho \dot{\gamma}^2 + \rho \dot{\gamma} \dot{\gamma} \right) dt + \int_0^S \rho \dot{\gamma} \dot{\gamma} dt = \int_0^S \rho \dot{\gamma} \dot{\gamma} dt + \int_0^S \rho \dot{\gamma} \dot{\gamma} dt, \tag{3.2}$$

featuring the functional $\int_0^S \rho \dot{\gamma} \dot{\gamma} dt$, $\int_0^S \rho \dot{\gamma} \dot{\gamma} dt \rightarrow 0$,

$$\int_0^S \rho \dot{\gamma} \dot{\gamma} dt, \int_0^S \rho \dot{\gamma} \dot{\gamma} dt, \tag{3.3}$$

with the reduced functional

$$\int_0^S \rho \dot{\gamma} \dot{\gamma} dt$$

and

$$\int_0^S \rho \dot{\gamma} \dot{\gamma} dt + \int_0^S \rho \dot{\gamma} \dot{\gamma} dt = \int_0^S \rho \dot{\gamma} \dot{\gamma} dt + \int_0^S \rho \dot{\gamma} \dot{\gamma} dt \tag{3.4}$$

In (3.4), $\int_0^S \rho \dot{\gamma} \dot{\gamma} dt$ and $\int_0^S \rho \dot{\gamma} \dot{\gamma} dt$ are the norms introduced in (2.4) and (2.5), while the distance functional $\int_0^S \rho \dot{\gamma} \dot{\gamma} dt \rightarrow \int_0^S \rho \dot{\gamma} \dot{\gamma} dt$ is defined by

$$\int_0^S \rho \dot{\gamma} \dot{\gamma} dt \tag{3.5}$$

Clearly, the functional $\int_0^S \rho \dot{\gamma} \dot{\gamma} dt$ from (3.3) encompasses, in the energy-dissipation balance (3.2), the competition between viscous dissipation and tendency to relax toward equilibrium & rate-independent behavior. In fact, viscous dissipation is encoded in the term $\int_0^S \rho \dot{\gamma} \dot{\gamma} dt$, which is modulated by the viscosity parameter η . In turn, the relaxation to rate-independent behavior is encompassed in the term $\int_0^S \rho \dot{\gamma} \dot{\gamma} dt$, modulated by η .

Now, the concepts of $\eta \rightarrow 0$ and $\eta \rightarrow 0$ solutions to system (1.1) are defined in terms of parameterized energy-dissipation balances akin to (3.2). These energy identities involve a (positive) *vanishing-viscosity potential* ϕ that is defined on \mathcal{S} , \mathcal{S} , (recall that \mathcal{S} is the state space for the perfectly plastic damage system, cf. (2.11)). At least formally, ϕ arises as $\eta \rightarrow 0$ -limit

- Of the functionals $\int_0^S \rho \dot{\gamma} \dot{\gamma} dt$ as $\int_0^S \rho \dot{\gamma} \dot{\gamma} dt$, in the case of the vanishing-viscosity analysis in [8] (recall that system (1.2) is formally a particular case of (1.3), with $\eta \rightarrow 0$);

- Of \mathcal{E}_0 as \mathcal{E}_0 , in the case of the joint vanishing-viscosity and hardening analysis in [10].

However, in order to *rigorously* define the vanishing-viscosity contact potentials relevant for the two concepts of \mathcal{E}_0 solutions we will need to provide some technical preliminaries in the following section.

3.2 Preliminary definitions

The vanishing-viscosity potentials at the core of the definitions of \mathcal{E}_0 solutions in [8] and $\mathcal{E}_0^{0,0}$ solutions in [10] have a structure akin to that of the functional \mathcal{E}_0 from (3.3), but they are tailored to the driving energy \mathcal{E}_0 (2.12) for the perfectly plastic system. Recalling the definition of \mathcal{E}_0 , it is thus clear that, in order to define such vanishing-viscosity potentials, one needs suitable surrogates of the \mathcal{E}_0 -norm of \mathcal{E}_0 , \mathcal{E}_0 , and of the \mathcal{E}_0 -distance of \mathcal{E}_0 , \mathcal{E}_0 from the stable set \mathcal{E}_0 , cf. (3.4). Indeed, such quantities are no longer well defined for the functional \mathcal{E}_0 at every \mathcal{E}_0 ; instead, note that the \mathcal{E}_0 -distance, in the sense of (3.5), of \mathcal{E}_0 , \mathcal{E}_0 from the stable set \mathcal{E}_0 still makes sense. Following [10, Sect. 7], we will set at every \mathcal{E}_0 (cf. (2.13))

$$(3.6a)$$

$$(3.6b)$$

with $\mathcal{E}_0 = \mathcal{E}_0(\mathcal{E}_0)$ and \mathcal{E}_0 . Observe that the expressions in (3.6a) and (3.6b) are well defined since $\mathcal{E}_0(t)$ and, a fortiori, \mathcal{E}_0 are elements in \mathcal{E}_0 whenever \mathcal{E}_0 . Formulae (3.6a) and (3.6b) have been inspired by the obvious fact that

and by the formula

which was proved in [10, Lemma 3.6]. In particular, we note that for every \mathcal{E}_0

$$(3.7a)$$

while

(3.7b)

For later use, we also record here the following result.

Lemma 3.1 *For every $\epsilon > 0$, there holds*

(3.8)

with $\chi_\epsilon(x) = \chi_\epsilon(x) \chi_\epsilon(x)$ and $\chi_\epsilon(x) = \chi_\epsilon(x)$.

Proof The implication $(i) \Rightarrow (ii)$ was proved in [10, Lemma 7.4]. A close perusal of the proof, also mimicking the convexity arguments from [13, Prop. 3.5], also yields the converse implication.

We then set

$$\chi_\epsilon(x) = \chi_\epsilon(x) \chi_\epsilon(x) \tag{3.9}$$

We are now in a position to define the vanishing-viscosity contact potentials involved in the definitions of ϵ -solutions in [8] and [10]. We will use the notation χ_ϵ for the indicator function of the singleton $\{x\}$, namely

3.3 ϵ -solutions via the partial vanishing-viscosity approach

The vanishing-viscosity contact potential for the ϵ -solutions from [8] is the functional

$$J_\epsilon(u) = \int_{\mathbb{R}^n} \chi_\epsilon(x) |u(x) - \min_{y \in \mathbb{R}^n} u(y)| \, dx \tag{3.10a}$$

where for $x \in \mathbb{R}^n$ and $\epsilon > 0$ we have

$$\chi_\epsilon(x) = \begin{cases} 1 & \text{if } |x| \leq \epsilon \\ 0 & \text{otherwise} \end{cases} \tag{3.10b}$$

and

(3.10c)

Observe that the definition of \mathcal{C}_0 , \mathcal{C}_0 , again reflects, in the jump regime, the tendencies of the system to evolve viscously and to relax toward equilibrium and rate-independent evolution. However, here viscous dissipation only affects the damage variable. Likewise, rate-independent behavior is solely encompassed by the relation

Let us now detail the properties of the parameterized curves providing solutions.

Definition 3.2 We call a parameterized curve *admissible* if it satisfies

$$(3.11a)$$

$$(3.11b)$$

$$(3.11c)$$

$$(3.11d)$$

We will denote by \mathcal{C}_0 the class of admissible parameterized curves from $[0, S]$ to 0 ,

We are now in a position to give the definition of Balanced Viscosity solution in the sense of [8].

Definition 3.3 We call an admissible parameterized curve a Balanced Viscosity solution for the perfectly plastic damage system (1.1) in the sense of [8] (a $\mathcal{C}_0^{0,0}$ solution, for short), if it satisfies the energy-dissipation balance

$$(3.12)$$

for every $\varphi \in \mathcal{C}_0$.

The existence of solutions was proved in [8, Thm. 5.4] under the condition that the initial data $\varphi_0 \in \mathcal{C}_0$, $\varphi_0 \geq 0$ for the perfectly plastic damage system (1.1) fulfill (2.8f) and the additional condition that

-

$$(3.13)$$

3.4 A differential characterization for solutions

We now aim to provide a *differential characterization* for the notion of solution from Definition 3.2, in terms of a suitable system of subdifferential inclusions for the displacement variable (which in fact shall satisfy the elastic equilibrium equation), the damage variable, and the plastic strain. In order to properly formulate the flow rule governing the latter, we need the following result; the proof of one implication can be found [8], in turn based on arguments from [13].

Lemma 3.4 *Let an admissible parameterized curve satisfy*

$$(3.14)$$

with $\sigma(\cdot) = \sigma(\cdot)$, and $\cdot \cdot$ the stress-strain duality from (2.10).

Then, γ satisfies Hill's maximum work principle

$$(3.15)$$

where the above equality holds in the sense of measures on \mathbb{M} , with the distribution defined in (2.9). Furthermore, defining μ for every $0, \dots$, there exists \mathbb{M} such that for almost all $\in (0, \dots)$ the following properties hold:

$$(3.16a)$$

$$(3.16b)$$

$$(3.16c)$$

Conversely, (3.16) imply (3.15) which, in turn, gives (3.14).

Proof We refer to [8, Prop. 6.5] for the proof of the fact that (3.14) implies (3.15) and (3.16). In turn, recalling (2.15), from (3.16c) we infer that

Combining this with (3.16a) and (3.16b), we conclude (3.15), which yields (3.14) in view of the definition (2.16) of σ , and of the definition (2.10) of the stress-strain duality product.

It is in the sense of (3.16) that we need to understand the (formally written) inclusion

We are now in a position to prove the following differential characterization for the concept of \mathbb{C} -solution from [8]. We mention in advance that, for notational simplicity, in (3.18b) we have simply written \mathbb{C} in place of \mathbb{C}^2 , with $\mathbb{C}^2 \rightarrow \mathbb{C}$ the Riesz operator.

Proposition 3.5 *An admissible parameterized curve $\gamma : [0, 1] \rightarrow \mathbb{C}$ is a solution to system (1.1) if and only if there exists a measurable function $\alpha : [0, 1] \rightarrow \mathbb{C}$ such that*

$$\gamma'(z) = \alpha(z) \gamma(z) \quad (3.17)$$

and α satisfies for a.a. $z \in (0, 1)$

$$(3.18a)$$

$$-\mathbb{C} \quad (3.18b)$$

$$(3.18c)$$

In fact, (3.18a) holds at every $z \in (0, 1)$.

Remark 3.6 System (3.18) illustrates in a clear way how viscous behavior, only w.r.t. the variable z , may arise in the jump regime, namely when the system still evolves while (i.e., the external time, recorded by the function α , is frozen). In that case, the parameter may be nonzero, thus activating the viscous contribution to the flow rule (3.18b).

Prior to carrying out the proof of Proposition 3.5, we record the following key chain-rule inequality, whose proof may be immediately inferred from that of [10, Lemma 7.6] (cf. also Lemma 3.13 ahead).

Lemma 3.7 *Along any admissible parameterized curve*

we have that $\gamma : [0, S] \rightarrow \mathbb{C}$ is absolutely continuous on $[0, S]$ and there holds

$$-\partial$$

$$(3.19)$$

As an immediate consequence of Lemma 3.7, we have the following characterization of the \mathbb{C} -solutions from Definition 3.2, which complements the other characterizations provided in [8, Prop. 5.3].

Corollary 3.8 *An admissible parameterized curve γ is a solution to system (1.1) if and only if the function δ_0 is absolutely continuous and there holds*

$$\begin{aligned}
 & \int_0^t \delta_0(s) \, ds = \delta_0(t) - \delta_0(0), \\
 & \delta_0(t) = \int_0^t \delta_0(s) \, ds + \delta_0(0),
 \end{aligned} \tag{3.20}$$

for almost all $t \in [0, \infty)$.

We are now in a position to carry out the

Proof of Proposition 3.5 By Corollary 3.8, solutions in the sense of Def. 3.2 can be characterized in terms of (3.20), whence we deduce that δ_0 is absolutely continuous for almost all $t \in [0, \infty)$. Therefore,

$$\delta_0(t) = \int_0^t \delta_0(s) \, ds + \delta_0(0) \tag{3.21}$$

In turn, (3.21) is equivalent to the validity of (3.18a). Observe that (3.18a) extends to every $t \in [0, \infty)$ by the continuity of the functions δ_0 and δ_0 and by the continuity w.r.t. the Hausdorff distance of γ thanks to (2.1).

In view of (3.21) and recalling the definition (3.10) of δ_0 , (3.20) can be then rewritten as

$$\int_0^t \delta_0(s) \, ds = \delta_0(t) - \delta_0(0),$$

for a.a. $t \in [0, \infty)$. Now, on the one hand the right-hand side of the above equality is positive thanks to (2.19). On the other hand, let δ_0 satisfy (3.21). Then, we may estimate the left-hand side of the above equality via

where we have used that δ_0 is absolutely continuous by the very definition of δ_0 (cf. (3.10c)), while thanks to (2.15). Therefore, (3.20) is ultimately equivalent to

$$\int_0^t \delta_0(s) \, ds = \delta_0(t) - \delta_0(0) \tag{3.22a}$$

$$\int_0^t \delta_0(s) \, ds = \delta_0(t) - \delta_0(0) \tag{3.22b}$$

Definition 3.9 We call a parameterized curve $\gamma : [0, S] \rightarrow \mathbb{R}^n$, $\gamma(0) = \gamma_0$, $\dot{\gamma}(0) = v_0$, $\gamma(S) = \gamma_S$, $\dot{\gamma}(S) = v_S$, γ *admissible in an enhanced sense* (‘enhanced admissible’ for short) if it satisfies (3.11a), (3.11b), (3.11c) and, in addition,

$$\int_0^S \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt > 0 \tag{3.24a}$$

$$\int_0^S \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt > 0 \tag{3.24b}$$

We will denote by $\mathcal{A}_{\text{enh}}(\gamma_0, v_0, \gamma_S, v_S)$ the class of enhanced admissible parameterized curves from $[0, S]$ to 0 ,

Hence, enhanced admissible curves enjoy better spatial regularity, with \mathbb{R}^n and \mathbb{M}^n , in the set in which either $\gamma_0 \neq \gamma_S$, $v_0 \neq v_S$ or $\gamma_0 = \gamma_S$, $v_0 \neq v_S$. With that definition at hand, we are now in a position to give the definition of γ solution in the sense of [10].

Definition 3.10 We call an enhanced admissible parameterized curve $\gamma : [0, S] \rightarrow \mathbb{R}^n$ a *Balanced Viscosity solution* for the perfectly plastic damage system (1.1) in the sense of [10] (a γ *0,0 solution*, for short), if it satisfies the energy-dissipation balance

$$\int_0^S \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt = \dots \tag{3.25}$$

∂

for every \dots .

The existence of γ $0,0$ solutions to system (1.1) was proved in [10, Thm. 7.9] for initial data $\gamma_0 \in \mathbb{R}^n$, $v_0 \in \mathbb{R}^n$, $\gamma_S \in \mathbb{R}^n$, $v_S \in \mathbb{R}^n$ complying with (2.8f) and (3.13).

3.6 A differential characterization for γ $0,0$ solutions

In this section, we provide a differential characterization for γ $0,0$ solutions. Preliminarily, we need to make precise in which sense we are going to understand the subdifferential inclusions governing the evolution of the reparameterized displacement and of the plastic variables. Indeed, by formally writing

with $\gamma : [0, S] \rightarrow \mathbb{R}^n$, $\gamma(0) = \gamma_0$, $\gamma(S) = \gamma_S$, $\dot{\gamma}(0) = v_0$, $\dot{\gamma}(S) = v_S$, a measurable function (below we will have $\gamma \in W^{1,1}$), we shall mean

$$\int_0^S \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt = \dots \tag{3.26a}$$

$$\int_0^S \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt = \dots \tag{3.26b}$$

where in (3.26) \mathcal{J}_0 denotes the restriction of the functional from (2.3) to \mathbb{R}^n . In particular, let us emphasize that, when \mathcal{J}_0 the displacement variable enjoys additional spatial regularity, and the quasistatic momentum balance (3.26b) allows for test functions in \mathbb{R}^n . Likewise, by writing

with $\chi: \mathbb{R}^n \rightarrow [0, 1]$ a measurable function, we shall mean

$$\mathcal{M}_\chi \tag{3.27a}$$

$$\mathcal{M} \tag{3.27b}$$

Namely, the plastic flow rule improves to a pointwise-in-space formulation in the set \mathcal{M}_χ , whereas in the set \mathcal{M} it only holds in the weak form (3.16).

We are now in a position to state our differential characterization of $\mathcal{M}_\chi^{0,0}$ solutions. Observe that the definition of enhanced admissible curve is tailored to the subdifferential inclusions (3.29).

Proposition 3.11 *An enhanced admissible parameterized curve γ is a $\mathcal{M}_\chi^{0,0}$ solution to system (1.1) if and only if there exist two measurable functions λ, μ such that*

$$\tag{3.28a}$$

$$\tag{3.28b}$$

and the curves γ satisfy for a.a. $t \in [0, 1]$, the system of subdifferential inclusions

$$\tag{3.29a}$$

$$-\mathcal{C} \tag{3.29b}$$

$$\tag{3.29c}$$

where (3.29a) and (3.29c) need to be interpreted as (3.26) and (3.27), respectively.

Remark 3.12 In comparison with the differential characterization for \mathcal{M}_χ solutions provided by system (3.5), system (3.29) features *two* parameters, instead of one. Both λ and μ have the role of activating the viscous contributions to the damage flow rule, and to the displacement equation/plastic flow rule, respectively, in the jump regime (i.e., when $\dot{\gamma} > 0$). In fact, the viscous terms in (3.29a) and (3.29c) are modulated by the same parameter, which reflects the fact that viscous behavior intervenes for the variables γ and $\dot{\gamma}$ equally (or, in other terms, that γ and $\dot{\gamma}$ relax to elastic equilibrium and rate-independent evolution at the same rate, faster than $\dot{\gamma}$).

As in the case of Prop. 3.5, the proof of Prop. 3.11 will rely on a suitable chain-rule inequality, which we recall below.

Lemma 3.13 [10, Lemma 7.6] *Along any enhanced admissible parameterized curve*

we have that

—

$$(3.30)$$

For later use, we also record the following consequence of the chain-rule inequality, cf. [10, Proposition 7.7].

Corollary 3.14 *An enhanced admissible parameterized curve is a ${}^{0,0}_0$ solution if and only if it satisfies one of the following equivalent conditions:*

- (1) *the energy-dissipation balance (3.25) holds as the inequality ;*
- (2) *fulfills (3.30) as a chain of equalities, i.e.,*

—

$$(3.31)$$

We are now in a position to carry out the

Proof of Proposition 3.11 We exploit the characterization of ${}^{0,0}_0$ solutions in terms of the chain of equalities (3.31). Now, we shall distinguish three cases:

. Then, by the definition (3.23) of ${}^{0,0}_0$, from ${}^{0,0}_0$, we infer and

$$((\cdot)) = ((\cdot)) =$$

By (3.8), the latter property is equivalent to (3.18a); hence, we find the validity of (3.29a) with $((\cdot)) =$. All in all, identity (3.31) reduces to

$$(3.32)$$

Now, since for a.a. $\in (0,)$, by (2.18) the above left-hand-side is negative. On the other hand, from we infer that

σ_0 , τ , for every γ . Therefore, the above right-hand side is positive. Hence, both sides are equal to zero. From

$$(3.33)$$

we infer (recall Lemma 3.4) that (3.29c) holds with $\gamma = \gamma$. Likewise, from

$$\sigma_0, \tau, \gamma, (3.34)$$

recalling (2.15) we deduce that σ_0, τ, γ , i.e., the validity of (3.29b) with $\gamma = \gamma$.

Conversely, from (3.29b) with $\gamma = \gamma$ and (3.29c) with $\gamma = \gamma$ we deduce (3.33) and (3.34), respectively, hence (3.32) which, in this case, is equivalent to (3.31).

(with the set from (3.24a)). Then, from σ_0, τ, γ we infer that

$$0, (3.35)$$

and (3.31) reduces to

$$\text{-----} (3.36)$$

$$\text{-----}$$

For (3.36), we have used that, on the left-hand side, the duality pairing involving σ_0 is between σ_0 and τ since the admissible curve γ enjoys the enhanced spatial regularity γ on the set Ω . In turn, the right-hand side of (3.36) has been rewritten in view of (3.7). Likewise, σ_0 and the stress-strain duality reduces to the scalar product in \mathbb{S}^n because $\sigma_0 \in \mathbb{S}^n$. Let us now consider a measurable selection γ such that

Then, (3.36) rewrites as

$$\text{-----}$$

While the left-hand side is negative by Cauchy–Schwarz inequality, the right-hand side is positive since $\sigma_0 \in \mathbb{S}^n$, cf. (2.15). All in all, we conclude that both sides are equal to zero. Now, combining the fact that $\sigma_0 \in \mathbb{S}^n$ with the identity

and again resorting to (2.15), we find that

(3.37)

From the equality

we infer that

(3.38a)

(3.38b)

(3.38c)

Combining (3.35), (3.37), and (3.38), we deduce the validity of system (3.29) with

Conversely, it can be easily checked that, if $\sigma \in \Sigma$ and $\epsilon \in \mathbb{R}^n$, the validity of system (3.29) yields (3.36), hence (3.31).

Hence, $\sigma \in \Sigma$, yielding

i.e., (3.29a) with $\sigma(\epsilon) = \sigma$. Then, (3.31) reduces to

(3.39)

and, arguing in the same way as in the proof of Proposition 3.5 we conclude the validity of (3.29b) and (3.29c) with $\sigma(\epsilon) = \sigma$ and some $\epsilon \in \mathbb{R}^n$.

Conversely, it can be proved that (3.29b) and (3.29c) with $\sigma(\epsilon) = \sigma$ and some $\epsilon \in \mathbb{R}^n$ yield (3.39).

With this, we conclude the proof.

4 A complete characterization of Σ solutions

The main result of this section is the following theorem, whose proof adapts that of [28, Prop. 5.5].

Theorem 4.1 *Let $\sigma \in \Sigma$ be a $\begin{smallmatrix} 0,0 \\ 0 \end{smallmatrix}$ solution to the perfectly plastic rate-independent system for damage (1.1). Suppose that σ is non-degenerate, namely that*

$$> \tag{4.1}$$

Set

Then, \mathcal{S} is either empty or it has the form $\mathcal{S} = \{(t, \theta) \mid t \in [t_0, \infty), \theta = \theta_0\}$ for some $t_0 \geq 0, \theta_0 \in \mathbb{R}$.

(a) Assume $\theta_0 > 0$, then for $t \geq t_0, \theta = \theta_0$, we have $\dot{\theta} = 0$ and $\dot{t} = 1$, whereas

$$0, \theta_0 > 0 \tag{4.2a}$$

and (t, θ_0) is a solution to the system

$$\tag{4.2b}$$

(b) Suppose that $\theta_0 = 0$. Then, $\dot{\theta} = 0, \dot{t} = 1$ for every $t \geq t_0$ and the curve $(t, 0)$ is a solution to system (1.1) in the sense of Definition 3.3.

Thus, Theorem 4.1 provides a complete characterization of (non-degenerate) $\begin{smallmatrix} 0,0 \\ 0 \end{smallmatrix}$ solutions. It asserts that, if a $\begin{smallmatrix} 0,0 \\ 0 \end{smallmatrix}$ solution starts from an unstable datum with $\theta_0 > 0$, then during an initial interval the damage variable θ is frozen and the pair (t, θ) evolves, possibly governed by viscosity in both variables. If it reaches, at some time t_0 , the state in which elastic equilibrium $(\theta_0, 0)$ and the plastic constraint $(\theta_0, 0)$ are fulfilled, then it never leaves that state afterwards, and subsequently (t, θ) behaves as a $\begin{smallmatrix} 0,0 \\ 0 \end{smallmatrix}$ solution.

Proof Step 1: As in the proof of [28, Prop. 5.5], we start by analyzing the behavior of a $\begin{smallmatrix} 0,0 \\ 0 \end{smallmatrix}$ solution (t, θ) on an interval $[t_0, \infty) \subset \mathbb{R}$. Since $\dot{\theta} = 0, \dot{t} = 1 > 0$ for all $t \in [t_0, \infty)$, we read from (3.29a) and (3.29c) that $\dot{\theta} = 0$ on $[t_0, \infty)$. Thus, (3.28a) yields $\dot{t} = 1$ on $[t_0, \infty)$, so that $t = t_0 + s$ for all $s \geq 0$. Furthermore, (3.28b) gives $\dot{\theta} = 0$ on $[t_0, \infty)$. Combining this with (3.29b), we gather that $\dot{\theta} = 0$ on $[t_0, \infty)$, so that $\theta = \theta_0$ for all $t \geq t_0$. From (4.1), we conclude that

$$\tag{4.3a}$$

and, therefore, from (3.29a) and (3.29c) we infer that $\dot{t} = 1$ for almost all $t \geq t_0$. Hence, the evolution of (t, θ) in $[t_0, \infty)$ is characterized by property (4.3a), joint with the previously found

$$\tag{4.3b}$$

$$\tag{4.3c}$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial x} = \theta \left(\frac{\partial \theta}{\partial x} + \frac{\partial \theta}{\partial t} \right) \tag{4.3d}$$

(observe that $\theta \geq 0$ with $\theta = 0$ from (3.38)). In turn, it can be easily checked that, for a *given* function $\theta \rightarrow \theta$, the Cauchy problem for system (4.3b)–(4.3d) does admit a unique solution.

Step 2: Since the function θ , $\theta \geq 0$, is lower semicontinuous by [10, Lemma 7.8], the set $\theta > 0$ is closed; hence, its complement $\theta = 0$ is relatively open, and thus, it is the finite or countable union of disjoint intervals. Its connected components are of the form (a, b) , $[a, b)$, or $[0, S]$ (if $\theta = 0$). By the lower semicontinuity of θ , it is immediate to check that $\theta = 0$ for all $x \in [0, S]$.

Now, we aim to show that connected components of the type (a, b) and $[a, b)$ cannot occur. To this end, let us study the properties of the $\theta^{0,0}$ solution θ on an interval of the type (a, b) or $[a, b)$, with $\theta = 0$. Since $\theta = 0$, $\theta = 0$, we have that $\theta = 0$, $\theta = 0$, $\theta = 0$ for $x \in (a, b)$. As shown in Step 1, the evolution on the intervals (a, b) and $[a, b)$ is characterized by (4.3). Recall that system (4.3b)–(4.3d) admits a unique solution. Now, since $\theta = 0$, $\theta = 0$, $\theta = 0$ for $x \in (a, b)$, it is immediate to check that the constant functions $\theta = 0$ and $\theta = 0$ provide the unique solution to (4.3b)–(4.3d). Thus, we conclude that $\theta^{0,0}$ solution θ on an interval of the type (a, b) or $[a, b)$ must be constant, which is a contradiction to (4.3a).

Therefore, θ does not possess connected components of the form (a, b) or $[a, b)$. Hence, either $\theta = 0$, or $\theta = 0$, for some $\delta > 0$. In the latter case, the calculations from Step 1 show that on $(0, \delta)$, $\theta = 0$, the evolution of θ is characterized by (4.2).

Step 3: Suppose that $\theta = 0$. Clearly, $\theta = 0$ for every $x \in [0, S]$. Hence, it satisfies system (3.29) with $\theta = 0$ for every $x \in [0, S]$, which coincides with system (3.18). This concludes the proof.

5 Vanishing-hardening limit of solutions

In this section, we carry out the asymptotic analysis as the hardening parameter ϵ tends to 0 for the solutions to system (1.4), both in the *single-rate* case (i.e., for $\theta^{0,0}$ solutions, with $\theta = 0$), and in the *multi-rate* case (i.e., for $\theta^{0,0}$ solutions). Indeed, we first address the latter case in Sect. 5.1 ahead, while the former will be sketched in Sect. 5.2. For both analyses, we will resort to some technical results collected in the ‘‘Appendix’’.

5.1 Vanishing-hardening analysis for multi-rate solutions

As recalled in the Introduction, solutions to the *multi-rate* system with hardening have been constructed in [10] (cf. Theorem 6.13 therein) by passing to the limit in the (reparameterized) version of (1.3) as the viscosity parameter ϵ tends to 0 simultaneously with

the rate parameter λ , while the hardening parameter μ stayed fixed. The solutions accordingly obtained, hereafter referred to as λ_0^0 solutions, thus account for multiple rates in the system with hardening. In particular, like for λ_0^0 solutions, the way in which viscous behavior in u , z , and p manifests itself in the jump regime reflects the fact that the convergence of u and p to elastic equilibrium and rate-independent evolution has occurred at a *faster rate* (as $\lambda \rightarrow 0$) than that for z , cf. Remark 5.2 ahead.

In order to recall the definition of λ_0^0 solutions for fixed μ , we need to introduce the related vanishing-viscosity contact potential

$$\begin{aligned} \lambda_0^0 &= 0, & \mu &= 0, & \lambda &\rightarrow 0, \\ \lambda_0^0 & \text{ , , , } & & & \lambda_0^0 & \text{ , , , , , } \end{aligned}$$

where

$$(5.1a)$$

$$(5.1b)$$

In (5.1), we have employed the notation

$$(5.1c)$$

We are now in a position to recall the notion of solution to system (1.4) from [10, Def. 6.10]. Observe that it involves (reparameterized) curves that are absolutely continuous on the *whole* interval $[0, S]$, with values in \mathbb{R}^n .

Definition 5.1 We call a parameterized curve γ a (parameterized) *Balanced Viscosity* solution to the *multi-rate* system with hardening (1.4) (a λ_0^0 solution, for short), if $\lambda \geq 0$, $\mu \geq 0$, λ is nondecreasing and γ fulfills for all $t \in [0, S]$ the energy-dissipation balance

$$(5.2)$$

We say that γ is *non-degenerate* if it fulfills (4.1).

Remark 5.2In [10, Prop. 6.11], a characterization was provided for λ_0^0 solutions in terms of a subdifferential system that features two positive parameters λ and μ activating

viscous terms in the displacement equation & plastic flow rule, and in the damage flow rule, respectively. We refrain from recalling that system because it is completely analogous to (3.29) (with the same *switching conditions* (3.28)).

Accordingly, repeating the very same arguments as in the proof of Theorem 4.1, it is possible to provide an additional characterization of \mathcal{S}_0^0 solutions completely analogous to that from the latter result. In particular, if a \mathcal{S}_0^0 solution (\mathbf{u}, \mathbf{d}) originates from an unstable datum $(\mathbf{u}_0, \mathbf{d}_0)$ with $\mathbf{u}_0 \cdot \mathbf{0}, \mathbf{d}_0 \cdot \mathbf{0}$, then, during an initial interval the damage variable \mathbf{d} is frozen and the pair (\mathbf{u}, \mathbf{d}) evolves, possibly in a viscous way. If (\mathbf{u}, \mathbf{d}) reaches, at some time t^* , the state in which elastic equilibrium $(\mathbf{u}^*, \mathbf{0})$ and stability $(\mathbf{u}^*, \mathbf{0})$ are fulfilled, then it never leaves that state afterwards and subsequently behaves as a Balanced Viscosity solution with viscous behavior in the variable \mathbf{u} , only (namely, the counterpart, for the system with hardening, of the \mathcal{S}_0^0 concept).

We mention in advance that the analogue of Theorem 4.1 does not hold, instead, for \mathcal{S}_0^0 solutions.

We now consider a vanishing sequence ϵ_k and set $\mathbf{u}_k, \mathbf{d}_k$. By [10, Theorem 6.13], under the assumptions of Sects. 2 and (3.13) (cf. also Remark 5.4 below), for any fixed k there exists a parameterized Balanced Viscosity solution $(\mathbf{u}_k, \mathbf{d}_k)$ in the sense of the previous definition. Moreover, for the sequence ϵ_k , we may assume the validity of the following a priori estimates:

$$\text{---} \tag{5.3}$$

Indeed, the existence of a sequence $(\mathbf{u}_k, \mathbf{d}_k)$ enjoying the bounds (5.3) follows by time-discretization, cf. [10, Prop. 4.4], and by a reparameterization argument. Up to a further time reparameterization, we may also assume that the solutions are non-degenerate, cf. (4.1) and [10, Remark 6.9].

Theorem 5.3 *Let ϵ_k be a vanishing sequence and $(\mathbf{u}_k, \mathbf{d}_k)$ be a sequence of \mathcal{S}_0^0 solutions to system (1.4), such that estimate (5.3) holds.*

Then, there exist a (not relabeled) subsequence and a curve such that

(1) *for all $\delta > 0$, the following convergences hold as $k \rightarrow \infty$*

$$\mathbb{M} \tag{5.4}$$

(2) *there exists $\bar{\epsilon} > 0$ such that for a.e. $\epsilon < \bar{\epsilon}$, there holds*

$$\mathbb{M} \tag{5.5}$$

(3) $\{u_k\}$ is a *Balanced Viscosity solution for the perfectly plastic damage system (1.1) in the sense of Definition 3.10.*

Remark 5.4 The validity of Theorem 5.3 extends to sequences $\{u_k\}$ originating from initial data (u_0, v_0, θ_0) fulfilling

with C independent of k .

In particular, the last condition yields $\{u_k\} \rightarrow u$ in \mathbb{M} , as needed for (5.10) below.

Proof The proof is divided in two steps.

Step 1: Compactness. For later use, we observe that, due to estimate (5.3),

$$(5.6)$$

By the assumptions on initial data and external loading and by (5.3), we have $\{u_k\}$ for a constant independent of k . In particular, this implies that, as $k \rightarrow \infty$,

$$\mathbb{M} \quad (5.7)$$

In view of (5.3), we find a (not relabeled) subsequence and a Lipschitz curve γ , such that the following convergences hold as $k \rightarrow \infty$

$$(5.8a)$$

$$(5.8b)$$

$$\mathbb{M} \quad \mathbb{M} \quad (5.8c)$$

where γ . Furthermore, an argument based on the Ascoli-Arzelà theorem (cf. [3, Prop. 3.3.1]) also yields

$$(5.9a)$$

$$\mathbb{M} \quad (5.9b)$$

$$(5.9c)$$

$$\mathbb{M} \quad (5.9d)$$

Indeed, convergences (5.9a) and (5.9d) are to be intended in the spaces $L^2(\Omega; \mathbb{M})$ and in $L^2(\Omega; \mathbb{M})$, where \mathbb{M} and \mathbb{M} metrize the weak topologies of \mathbb{M} and \mathbb{M} , respectively.

respectively, on the balls of radius R that contain φ and ψ , resp. (cf. (5.6); here, we use that \mathcal{H}^1 is the dual of a separable space). The second and third convergences have an analogous meaning. Hence, (5.4) follows.

With the very same arguments as in the proof of [10, Prop. 7.9], based on the estimate

we also prove the enhanced regularity

Step 2: energy-dissipation upper estimate. By Corollary 3.14, it is sufficient to show that the pair (φ, ψ) complies with the energy-dissipation inequality

$$\int_0^t \int_{\mathbb{R}^d} \varphi \psi \, dx \, ds \leq \int_0^t \int_{\mathbb{R}^d} \varphi \psi \, dx \, ds + \int_0^t \int_{\mathbb{R}^d} \varphi \psi \, dx \, ds,$$

for every $t \geq 0$. We start from (5.2) for the solution (φ, ψ) with $(\varphi, \psi) \in \mathcal{H}^1$. It is straightforward to see that

$$(5.10)$$

and

It remains to show that

$$(5.11)$$

In fact, it will be sufficient to obtain the above estimate only for the reduced functionals $\varphi_{0,0}$ and $\psi_{0,0}$. In view of (3.23), we distinguish two cases. Let $\varphi_{0,0} \in \mathcal{H}^1$ and $\psi_{0,0} \in \mathcal{H}^1$.

Case 1. We prove that the function $\varphi_{0,0} \psi_{0,0}$ is finite for a.a. $x \in \mathbb{R}^d$. We apply Lemma A.1 ahead with the choices $\varphi = \varphi_{0,0}$ and $\psi = \psi_{0,0}$. Thus, we conclude that for a.a. $x \in \mathbb{R}^d$ there is a subsequence $(j_k)_{k \in \mathbb{N}}$ and, for every j , there is $\epsilon_j \in (0, 1)$ such that $\varphi_{0,0} \psi_{0,0} \leq \epsilon_j$ - and $\varphi_{0,0} \psi_{0,0} \geq -\epsilon_j$. In particular, $\varphi_{0,0} \psi_{0,0} \in L^1_{loc}$. By (3.23), (5.7), convergences (5.9), and Lemma A.2 ahead, we obtain

which is equivalent to state that $\varphi_{0,0} \psi_{0,0} \in L^1_{loc}$. Hence, we obviously have the *pointwise* estimate

$$(5.12)$$

Case 2. By virtue of Lemma A.2, we are in a position to apply Lemma A.3 below in the context of the space \mathcal{H}^1 , with φ the ball of radius R from (5.6), to the functionals $\varphi_{0,0}$ and $\psi_{0,0}$, with $\varphi_{0,0} \psi_{0,0}$ extended

to \dots by setting \dots when q or \dots . We may then observe that the \dots estimate (A.2) in Lemma A.3 follows from Lemma A.2. Thus, we conclude

All in all, (5.11) follows.
This finishes the proof.

5.2 Vanishing-hardening analysis for single-rate solutions

solutions to the *single-rate* system with hardening have been obtained in [10, Section 6.1] by performing the asymptotic analysis of the reparameterized energy-dissipation balance (3.2) as the viscosity parameter ν tends to 0, while keeping the hardening and the rate parameters μ and β fixed. That is why we refer to them as ν -solutions to the system with hardening. Their definition involves the corresponding vanishing-viscosity contact potential $\Phi_\nu \rightarrow \Phi$ given by

$$(5.13a)$$

$$(5.13b)$$

For better readability, we also recall that

It is worthwhile to remark that the *reduced* functional \mathcal{F}_ν , at $\nu = 0$, *simultaneously* encompasses viscosity for the three variables u , z , and p . Instead, its counterpart \mathcal{F} for $\nu = 0$ solutions features, in the jump regime \dots , a viscous contribution in the variables (u, p) when \dots , and viscosity in z when \dots , i.e., when u is at elastic equilibrium and p is locally stable.

We are now in a position to recall the notion of ν -solution, cf. [10, Definition 6.2].

Definition 5.5 We call a parameterized curve γ_ν a *(parameterized) Balanced Viscosity* solution to the *single-rate* system with hardening (1.4) (a ν -solution, for short), if $\gamma_\nu(0) = 0$, $\gamma_\nu(1) = 0$, γ_ν is nondecreasing and γ_ν fulfills for all $\nu > 0$ the energy-dissipation balance

$$(5.14)$$

We say that \mathcal{S}_ϵ is *non-degenerate* if it fulfills (4.1).

Let us now address the asymptotic analysis of the above solutions for a vanishing sequence $\epsilon \rightarrow 0$. As mentioned in the Introduction, in the construction of \mathcal{S}_ϵ solutions the rate parameter is always supposed smaller than the hardening parameter, which forces us to also consider a sequence $\delta \rightarrow 0$ such that $\delta \ll \epsilon$ for all ϵ , so that $\delta/\epsilon \rightarrow 0$ as well. In fact, the technical condition $\delta \ll \epsilon$ comes into play in the proof of [10, Prop. 4.4]. The latter result and [10, Theorem 6.8] ensure the existence of \mathcal{S}_ϵ solutions enjoying the following a priori estimates

$$\|u_\epsilon\|_{C^0(\bar{\Omega})} \leq C \quad (5.15)$$

(and, up to a reparametrization, the non-degeneracy condition).

In Theorem 5.6 below, we are going to show that, as the hardening and rate parameters δ and ϵ vanish, (up to a subsequence) \mathcal{S}_ϵ solutions converge to a $\begin{smallmatrix} 0,0 \\ 0 \end{smallmatrix}$ solution of system (1.1).

Theorem 5.6 *Let δ_ϵ and ϵ_ϵ be two vanishing sequences, and let $\mathcal{S}_{\delta_\epsilon, \epsilon_\epsilon}$ be a sequence of solutions to system (1.4) such that estimate (5.15) holds.*

Then, there exist a (not relabeled) subsequence and a curve γ such that items (1), (2), (3) of the statement of

Theorem 5.3 hold.

Proof The argument is split in the same steps as the proof of Theorem 5.3.

Step 1: Compactness. With minor changes, from estimate (5.15) we derive estimate (5.6) and convergences (5.8) and (5.9), whence convergences (5.4), for the sequence of solutions. Analogously, for the limiting curve estimate (5.5) holds.

Step 2a: energy-dissipation upper estimate when $\delta \ll \epsilon$. The analogue of (5.12) at all $\delta \ll \epsilon$, $\delta \rightarrow 0$ can be obtained in the same way as in the proof of Theorem 5.3, taking into account that

Step 2b: energy-dissipation upper estimate when $\delta \ll \epsilon$. We will now show that

$$(5.16)$$

As in the proof of Thm. 5.3, we will apply Lemma A.3 below in the context of the space $\mathcal{H}^1(\Omega; \mathbb{R}^n)$, with B_R the ball of radius R from (5.6), to the functionals

(extended to $\mathcal{H}^1(\Omega; \mathbb{R}^n)$ as described for $\begin{smallmatrix} 0,0 \\ 0 \end{smallmatrix}$ in the proof of

Thm 5.3), and $\lim_{\delta \rightarrow 0} \frac{0,0}{0,}$. With this aim, we only need to check that the δ -liminf estimate in (A.2) holds in our context. Clearly, it is sufficient to check that for any sequence $\{z_n\} \subset \mathbb{R}$ with $z_n \rightarrow \infty$ there holds

$$\liminf_{n \rightarrow \infty} \dots \geq \dots \quad (5.17)$$

The above estimate easily follows from Lemma A.2 in the case in which \dots admits a strictly positive subsequence. Instead, if there exists \dots such that $\dots \geq \dots$, then \dots , for all $\dots \geq \dots$ and we may argue in the following way. When $z_n \rightarrow \infty$ and $\dots > 0$, we use that

$$\geq \dots$$

which follows by neglecting some terms in the expression for \dots . Then, as in Thm. 5.3 we use Lemma A.2 to pass to the limit in the two terms on the right-hand side of the above inequality in the cases $z_n \rightarrow \infty$ and $\dots \rightarrow 0$, respectively. In the remaining case $\dots > \dots$, it holds \dots : indeed, by Lemma A.2 and since \dots in \dots^2 , we have that \dots for a suitable $\dots > \dots$. Since

$$\geq \dots$$

we again conclude estimate (5.17) as \dots and \dots vanish.

Thus, by Lemma A.3, we have proven (5.16). This finishes the proof.

Appendix A: Some technical results

We collect the results employed in the proofs of Theorems 5.3 and 5.6.

Lemma A.1 *Let \dots, \dots, \dots be nondecreasing functions such that \dots uniformly. Let $\dots, \dots > 0$. Then for a.a. \dots , there is a subsequence \dots and, for every j , there is \dots, \dots such that \dots - and $\dots > \dots$.*

Proof Let

We shall prove that \dots is at most countable, which implies the statement of the lemma. Let \dots . Then, one has \dots ; indeed, since \dots is nondecreasing, $(\dots) = (\dots)$ would imply that \dots is constant in \dots , which is in contrast with the assumption (and thus $\dots > \dots$). Let now \dots for \dots . Since \dots for \dots , for k sufficiently large one has $\dots > \dots > \dots$. Let \dots, \dots denote the inverse function of \dots . It turns out that \dots are both jump points of \dots for every $\dots > \dots$. Since the jump points of a \dots function are countable, it follows that \dots is countable, too.

Lemma A.2 [10, Lemma 7.8] Let $\{t_n\} \rightarrow \infty$ in $[0, T]$, $\{s_n\} \rightarrow s$, $\{x_n\} \rightarrow x$, $\{y_n\} \rightarrow y$, $C \subset \mathbb{M}$ such that the following convergences hold as $n \rightarrow \infty$: $\|x_n - x\| \rightarrow 0$, $\|y_n - y\| \rightarrow 0$ in \mathbb{M} , and $\|C - C\| \rightarrow 0$ in \mathbb{M} . Then,

$$\|x_n - x\| \rightarrow 0 \tag{A.1a}$$

$$\|y_n - y\| \rightarrow 0 \tag{A.1b}$$

$$\|C - C\| \rightarrow 0 \tag{A.1c}$$

We borrow our final auxiliary result from [25]. The proof, therein developed in the case of a sequence $\{x_n\}$ with values in \mathbb{R} with a reflexive space, can be straightforwardly adapted to the case of the dual of a separable space.

Lemma A.3 [25, Prop. 5.2] Let X be the dual of a separable Banach space, let S be a weakly closed subset of X , and let $\{f_n\} \subset \mathbb{R} \rightarrow \mathbb{R} \rightarrow 0$, $\{g_n\}$ be measurable and weakly lower semicontinuous functionals fulfilling the ϵ -estimate

$$\int_{\mathbb{R}} f_n(x) g_n(x) dx \rightarrow 0 \tag{A.2}$$

Suppose that, the functionals f_0, \dots, f_n and g_0, \dots, g_n are convex for every $n \in \mathbb{N}$ and $\{f_n\} \subset \mathbb{R}$. Let $C \subset \mathbb{R}$ fulfill

$$\int_{\mathbb{R}} f_n(x) dx \rightarrow 0$$

Then,

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