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Rapidly convergent series and closed-form expressions for a class of integrals involving products of spherical Bessel functions

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ABSTRACT

Rapidly convergent series are derived to efficiently evaluate a class of integrals involving the product of spherical Bessel functions of the first kind occurring in acoustic and electromagnetic scattering from circular disks and apertures. Depending on the involved parameters, the series can be further reduced to closed-form expressions in terms of generalized hypergeometric functions or Meijer G-functions, which can immediately be evaluated through current mathematical toolboxes. Numerical results are provided showing that the accuracy of the series representation can easily be controlled and that the proposed solutions are at least 1000 times faster than specific quadrature schemes which, in general, have to deal with irregularly oscillating and slowly decaying functions.

1. Introduction

This paper deals with the analytical evaluation of the integrals

$$I(m,n,k,\alpha) = \int_0^\infty \frac{\mathrm{J}_{m+1/2}(\nu)\,\mathrm{J}_{n+1/2}(\nu)}{\nu^k\sqrt{\alpha^2-\nu^2}}\,\mathrm{d}\nu \tag{1}$$

and

$$J(m,n,k,\alpha) = \int_0^\infty \mathrm{J}_{m+1/2}(\nu)\,\mathrm{J}_{n+1/2}(\nu)\,\frac{\sqrt{\alpha^2-\nu^2}}{\nu^k}\,\mathrm{d}\nu\,, \tag{2}$$

where $\mathrm{J}_\mu(\cdot)$ are Bessel functions of the first kind and order μ while m, n, and k are non-negative integers satisfying suitable conditions of convergence (i.e., $m+n+2-k>0$ with $k>0$ for (1) and $k>1$ for (2)), α is a real parameter, and the square-root is defined according to $\mathrm{Im}\left[\sqrt{\alpha^2-\nu^2}\right] \le 0$. The Bessel functions of half-integer order $\mathrm{J}_{m+1/2}(\cdot)$ appearing in (1)–(2) can also be expressed in terms of spherical Bessel functions of the first kind $\mathrm{j}_m(\cdot)$, taking into account the identity

$$\mathrm{j}_m(\nu) = \sqrt{\frac{\pi}{2\nu}}\,\mathrm{J}_{m+1/2}(\nu) \tag{3}$$

so that (1) and (2) can alternatively be written as

$$I(m,n,k,\alpha) = \frac{2}{\pi}\int_0^\infty \frac{\mathrm{j}_m(\nu)\,\mathrm{j}_n(\nu)}{\nu^{k-1}\sqrt{\alpha^2-\nu^2}}\,\mathrm{d}\nu \tag{4}$$

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and

$$J(m,n,k,\alpha) = \frac{2}{\pi} \int_0^\infty \mathrm{j}_m(\nu)\,\mathrm{j}_n(\nu)\,\frac{\sqrt{\alpha^2-\nu^2}}{\nu^{k-1}}\,\mathrm{d}\nu\,. \tag{5}$$

The interest of the authors in such a class of integrals was born because their efficient evaluation is a crucial step in the solution of different problems involving electromagnetic (EM) scattering and shielding from circular disks or circular apertures in infinite screens [1–6], where the parameter α in (1) and (2) represents the normalized radius of the disk or of the aperture. More specifically, such integrals appear when the involved integral equation is transformed into a second-kind Fredholm infinite-matrix operator equation by means of a suitable choice of basis functions: the mentioned integrals represent the elements of the resulting matrix. In general such integrals are improper integrals of asymptotically oscillating and slowly decaying functions and their calculation is usually the bottleneck of an efficient formulation. This class of integrals have been encountered by other authors in similar problems [7–15] where the methods of analytical regularization and analytical preconditioners are used to convert first-kind integral equations and strongly singular second-kind integral equations to analytically regularized matrix equations and specific quadrature routines and techniques are proposed [16–19]. It is worth mentioning that in [20,21], for a subclass of integrals (1) and (2), alternative expressions involving fast converging proper integrals have been proposed, which however always require numerical quadrature routines. Also in other disciplines, like acoustics, the considered class of integrals is fundamental for the solution of the involved problems [22–24].

Because of the irregular oscillating behavior of the product of two Bessel functions, in addition to the above mentioned investigations, through the years much research has been devoted to finding efficient formulations or closed-form expressions for integrals containing this product as a multiplicative factor of the integrand function [25–35]. In fact, such integrals result to be of extreme importance in many other branches of theoretical and applied physics and engineering: in addition to electromagnetic and acoustic scattering, they find application in nuclear physics, calculation of Feynman diagrams [36], gravitational fields of astrophysical discs [37], hydrodynamics [38], and elasticity theory [39]. In this connection, the proposed approach can also be useful to address more general integrals containing a product of Bessel functions of arbitrary order and with different integrand functions.

1.1. Method of solution

Before presenting the proposed solution for the integrals (1)–(2), credit should be given to the work of George Fikioris, who introduced the powerful method of the Mellin transform technique for the evaluation of different finite and infinite one-dimensional integrals arising in EM theory (including integrals involving products of Bessel functions, as those considered here) [40–43]. The method and the analysis presented in this paper are strictly related to the Mellin-transform technique and have their roots in the fundamental work of Prudnikov et al. [44] although, in our case, we preferred to start directly from a Mellin–Barnes class of integrals derived from an integral representation of the product of Bessel functions originally pointed out in [45]. In fact, let us consider the general integral

$$\mathcal{I} = \int_0^\infty f(\nu)\,\mathrm{J}_\mu(\nu)\,\mathrm{J}_\xi(\nu)\,\mathrm{d}\nu \tag{6}$$

We first use the Mellin–Barnes integral representation of the product of Bessel functions of the first kind [45, Sec. 13.6]

$$\mathrm{J}_\mu(\nu)\,\mathrm{J}_\xi(\nu) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(-s)\,\Gamma(2s+\mu+\xi+1)\left(\frac{1}{2}\nu\right)^{2s+\mu+\xi}}{\Gamma(s+\mu+1)\,\Gamma(s+\xi+1)\,\Gamma(s+\mu+\xi+1)}\,\mathrm{d}s\,, \tag{7}$$

where $\Gamma(\cdot)$ is the Gamma function. By using the Legendre duplication formula for the Gamma function [46, 8.335], i.e.,

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\,\Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right), \tag{8}$$

the integral in (7) can also be written as

$$\mathrm{J}_\mu(\nu)\,\mathrm{J}_\xi(\nu) = \frac{1}{2\pi\sqrt{\pi}\,\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(-s)\,\Gamma\left(s+\frac{\mu+\xi+1}{2}\right)\Gamma\left(s+\frac{\mu+\xi}{2}+1\right)\nu^{2s+\mu+\xi}}{\Gamma(s+\mu+1)\,\Gamma(s+\xi+1)\,\Gamma(s+\mu+\xi+1)}\,\mathrm{d}s\,. \tag{9}$$

The singularities of the integrand in (9) are the poles of the Gamma functions in the numerator [46, 8.310.2], i.e., $s_p^{(1)} = p$ $(p = 0,1,2\ldots)$, $s_q^{(2)} = -q-(\mu+\xi+1)/2$ $(q = 0,1,2\ldots)$, and $s_r^{(3)} = -r-1-(\mu+\xi)/2$ $(r = 0,1,2\ldots)$: the parameter c is therefore a real number such that $-(\mu+\xi+1)/2 < c < 0$. Substituting (9) into (6) and interchanging the order of integrations we have

$$\mathcal{I} = \frac{1}{2\pi\sqrt{\pi}\,\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(-s)\,\Gamma\left(s+\frac{\mu+\xi+1}{2}\right)\Gamma\left(s+\frac{\mu+\xi}{2}+1\right)}{\Gamma(s+\mu+1)\,\Gamma(s+\xi+1)\,\Gamma(s+\mu+\xi+1)}\,\Psi(\mu,\xi,s)\,\mathrm{d}s \tag{10}$$

where

$$\Psi(\mu,\xi,s) = \int_0^\infty f(\nu)\,\nu^{2s+\mu+\xi}\,\mathrm{d}\nu\,. \tag{11}$$

In some cases the function $\Psi(\mu,\xi,s)$ of the complex variable s can be expressed in a closed form and it is possible to determine its singularities (moreover, in general, the existence of the function $\Psi(\mu,\xi,s)$ is limited to a domain of analyticity which can next be extended by analytic continuation). The integral $\mathcal{I}$ is then itself a Mellin–Barnes integral [47]. The original vertical integration path of the complex s plane in (10) can then be closed to its right or to the left (depending on the behavior of the integrand function) [44] enclosing all the poles of the integrand function in (10). If Jordan's Lemma can be applied, then the integral $\mathcal{I}$ is expressed as a sum of the residues at the enclosed poles. Very often the function $\Psi(\mu,\xi,s)$ is expressed as a product and ratio of Gamma functions and some exponentials and, in such a case, the integrals can then be expressed in terms of generalized hypergeometric [48], Meijer G-, or Fox H-functions [49].

We focus our work in the evaluation of integrals (1), where the Bessel functions have the same argument. This restriction is important since it allows for a representation of the product of Bessel functions as a Mellin–Barnes integral, which is fundamental for the successive derivations. Unfortunately, we are not aware of any Mellin–Barnes integral representation when the arguments of the involved Bessel functions are different. However, in this case one could think of different strategies, such as those proposed in [50].

For conciseness, the application of the method is detailed only for the integral I: in fact, although it can be applied for the evaluation of the integral J as well, the integral J can also be calculated as

$$J(m,n,k,\alpha) = \alpha^2 I(m,n,k,\alpha) - I(m,n,k-2,\alpha)\,. \tag{12}$$

1.2. Organization of the article

The paper is organized as follows. First, in Section 2 the most effective quadrature scheme for the evaluation of the considered class of integrals is presented, which is based on the work of Michalski and Mosig on numerical computation of generalized Sommerfeld-type integrals [51]. Section 3 is the core of the paper and presents a detailed analysis for the derivation of rapidly converging series based on a Mellin–Barnes integral representation of the product of Bessel functions and a careful application of the Cauchy residue theorem. Moreover, closed-form expressions for all the classes of considered integrals are provided in terms of generalized hypergeometric functions and/or Meijer G-functions. In Section 4, several numerical examples are presented to check the efficiency and accuracy of the proposed results. Finally, in Section 5 conclusion are drawn.

2. Evaluation of the integral through numerical quadrature

In this section, an efficient quadrature scheme for the calculation of the integral $I(m,n,k,\alpha)$ in (1) is illustrated using the method proposed by Michalski and Mosig [51]. The integral $J(m,n,k,\alpha)$ in (2) can be treated in a similar way. In fact, as is well known, the numerical evaluation of such integrals is a cumbersome task due to the potential irregularly oscillating nature of the integrand and the possible slow decay.

As suggested in [51], the first step is the adoption of the method of Lucas [17] based on the decomposition

$$\mathrm{J}_{m+1/2}(\nu)\,\mathrm{J}_{n+1/2}(\nu) = \frac{1}{2}\left[\mathcal{J}^{+}_{mn}(\nu) + \mathcal{J}^{-}_{mn}(\nu)\right], \tag{13}$$

where

$$\mathcal{J}^{\pm}_{mn}(\nu) = \mathrm{J}_{m+1/2}(\nu)\,\mathrm{J}_{n+1/2}(\nu) \mp \mathrm{Y}_{m+1/2}(\nu)\,\mathrm{Y}_{n+1/2}(\nu)\,. \tag{14}$$

It should be noted that, for large ν

$$\mathcal{J}^{\pm}_{mn}(\nu) \simeq \frac{1}{\pi\nu}\begin{cases} \sin\left[2\nu - (m+n+1)\frac{\pi}{2}\right] \\ \cos\left[(m-n)\frac{\pi}{2}\right] \end{cases} \tag{15}$$

so that $\mathcal{J}^{+}_{mn}(\nu)$ is asymptotically sinusoidal, while $\mathcal{J}^{-}_{mn}(\nu)$ tends to be a monotonically decreasing function. As noted by Lucas [17], the decomposition (13) should be used only for $\nu > a$ (where a is a suitable threshold value) to prevent the loss of accuracy due to the large values of the Bessel functions of the second kind for small arguments (and the consequent catastrophic cancellation). Lucas suggests to use

$$a = y_{p+1/2}, \qquad p = \max(m,n)\,, \tag{16}$$

where y_q denotes the first zero of $\mathrm{Y}_q(\nu)$ [17]. Therefore the original integral can be split as follows:

$$\begin{aligned} I(m,n,k,\alpha) &= \int_0^\infty \frac{\mathrm{J}_{m+1/2}(\nu)\,\mathrm{J}_{n+1/2}(\nu)}{\nu^k\sqrt{\alpha^2-\nu^2}}\,\mathrm{d}\nu \\ &= \underbrace{\int_0^\alpha \frac{\mathrm{J}_{m+1/2}(\nu)\,\mathrm{J}_{n+1/2}(\nu)}{\nu^k\sqrt{\alpha^2-\nu^2}}\,\mathrm{d}\nu}_{I_\alpha} + \underbrace{\int_\alpha^a \frac{\mathrm{J}_{m+1/2}(\nu)\,\mathrm{J}_{n+1/2}(\nu)}{\nu^k\sqrt{\alpha^2-\nu^2}}\,\mathrm{d}\nu}_{I_a} \\ &\quad + \underbrace{\int_a^\infty \frac{\mathcal{J}^{+}_{mn}(\nu)}{\nu^k\sqrt{\alpha^2-\nu^2}}\,\mathrm{d}\nu}_{I^+} + \underbrace{\int_a^\infty \frac{\mathcal{J}^{-}_{mn}(\nu)}{\nu^k\sqrt{\alpha^2-\nu^2}}\,\mathrm{d}\nu}_{I^-}\,. \end{aligned} \tag{17}$$

Now, both the integrals I_α and I_a can be computed through the tanh-sinh Double Exponential (DE) rule (taking into account the square-root singularity) [51–53]. On the other hand the integral I^- (which is not oscillatory) can be computed by the DE rule [51,52], while the integral I^+ can be computed by the Partition-Extrapolation (PE) method (with the modified Sidi partition) [51,54,55]. To do this, following [51] we must find b such that

$$\mathcal{J}^+_{mn}(b) = 0 \tag{18}$$

with $b > a$ and, consequently split the integral

$$I^+ = I_b + I_2^+ = \int_a^b \frac{\mathcal{J}^+_{mn}(\nu)}{\nu^k\sqrt{\alpha^2-\nu^2}}\,\mathrm{d}\nu + \int_b^\infty \frac{\mathcal{J}^+_{mn}(\nu)}{\nu^k\sqrt{\alpha^2-\nu^2}}\,\mathrm{d}\nu\,. \tag{19}$$

Again, the integral I_b can be computed through the tanh-sinh DE rule and I_2^+ by the PE method.

In summary:

$$\begin{aligned} I(m,n,k,\alpha) = {} & \underbrace{\int_0^\alpha \frac{\mathrm{J}_{m+1/2}(\nu)\mathrm{J}_{n+1/2}(\nu)}{\nu^k\sqrt{\alpha^2-\nu^2}}\,\mathrm{d}\nu}_{\text{tanh-sinh DE}} + \underbrace{\int_\alpha^a \frac{\mathrm{J}_{m+1/2}(\nu)\mathrm{J}_{n+1/2}(\nu)}{\nu^k\sqrt{\alpha^2-\nu^2}}\,\mathrm{d}\nu}_{\text{tanh-sinh DE}} \\ & + \underbrace{\int_a^b \frac{\mathcal{J}^+_{mn}(\nu)}{\nu^k\sqrt{\alpha^2-\nu^2}}\,\mathrm{d}\nu}_{\text{tanh-sinh DE}} + \underbrace{\int_b^\infty \frac{\mathcal{J}^+_{mn}(\nu)}{\nu^k\sqrt{\alpha^2-\nu^2}}\,\mathrm{d}\nu}_{\text{PE-Sidi}} + \underbrace{\int_a^\infty \frac{\mathcal{J}^-_{mn}(\nu)}{\nu^k\sqrt{\alpha^2-\nu^2}}\,\mathrm{d}\nu}_{\text{DE rule}}\,. \end{aligned} \tag{20}$$

3. Evaluation of the integral through rapidly convergent series

The integral $I(m,n,k,\alpha)$ in (1) is now evaluated as a rapidly convergent power series of α.

First of all the integral in (1) can be written as

$$I(m,n,k,\alpha) = I_\mathrm{R}(m,n,k,\alpha) + \mathrm{i}I_\mathrm{J}(m,n,k,\alpha)\,, \tag{21}$$

where

$$I_\mathrm{R}(m,n,k,\alpha) = \int_0^\alpha \frac{\mathrm{J}_{m+1/2}(\nu)\mathrm{J}_{n+1/2}(\nu)}{\nu^k\sqrt{\alpha^2-\nu^2}}\,\mathrm{d}\nu \tag{22}$$

and

$$I_\mathrm{J}(m,n,k,\alpha) = \int_\alpha^\infty \frac{\mathrm{J}_{m+1/2}(\nu)\mathrm{J}_{n+1/2}(\nu)}{\nu^k\sqrt{\nu^2-\alpha^2}}\,\mathrm{d}\nu\,. \tag{23}$$

As mentioned above, at the basis of the method there is the Mellin–Barnes integral representation of the product of Bessel functions of first kind (9).

3.1. Evaluation of $I_\mathrm{R}(m,n,k,\alpha)$

Let us start with the integral in (22). Taking into account that $\mu = m+1/2$ and $\xi = n+1/2$ in (9) (so that $\mu+\xi = m+n+1$), (22) can be written as

$$I_\mathrm{R}(m,n,k,\alpha) = \int_0^\alpha \frac{1}{\nu^k\sqrt{\alpha^2-\nu^2}}\,\frac{1}{2\pi\sqrt{\pi}\,\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(-s)\,\Gamma\left(s+\frac{m+n}{2}+1\right)\Gamma\left(s+\frac{m+n+3}{2}\right)\nu^{2s+m+n+1}}{\Gamma\left(s+m+\frac{3}{2}\right)\Gamma\left(s+n+\frac{3}{2}\right)\Gamma(s+m+n+2)}\,\mathrm{d}s\,\mathrm{d}\nu\,. \tag{24}$$

By interchanging the two integrals in (24), we have

$$I_\mathrm{R}(m,n,k,\alpha) = \frac{1}{2\pi\sqrt{\pi}\,\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(-s)\,\Gamma\left(s+\frac{m+n}{2}+1\right)\Gamma\left(s+\frac{m+n+3}{2}\right)}{\Gamma(s+m+n+2)\,\Gamma\left(s+m+\frac{3}{2}\right)\Gamma\left(s+n+\frac{3}{2}\right)}\left[\int_0^\alpha \frac{\nu^{2s+m+n+1-k}}{\sqrt{\alpha^2-\nu^2}}\,\mathrm{d}\nu\right]\mathrm{d}s\,. \tag{25}$$

The integral in [46, 3.251.1] furnishes

$$\int_0^1 x^{\mu-1}\left(1-x^\lambda\right)^{\nu-1}\mathrm{d}x = \frac{1}{\lambda}\mathrm{B}\left(\frac{\mu}{\lambda},\nu\right), \qquad \mu>0,\ \nu>0,\ \lambda>0\,, \tag{26}$$

where $\mathrm{B}(\cdot,\cdot)$ is the Beta function [46, 8.38]. From the relation [46, 8.384.1]

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \tag{27}$$

it thus follows that

$$\int_0^a \frac{y^\beta}{\sqrt{a^2-y^2}}\,\mathrm{d}y = \frac{a^\beta\sqrt{\pi}\,\Gamma\left(\frac{\beta+1}{2}\right)}{2\Gamma\left(1+\frac{\beta}{2}\right)}, \qquad \beta > -1 \tag{28}$$

and therefore, by using $\beta = 2s + m + n + 1 - k$, we have

$$I_{\mathrm{R}}(m,n,k,\alpha) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(-s)\,\Gamma\left(s+\frac{m+n}{2}+1\right)\Gamma\left(s+\frac{m+n+3}{2}\right)}{\Gamma(s+m+n+2)\,\Gamma\left(s+m+\frac{3}{2}\right)\Gamma\left(s+n+\frac{3}{2}\right)} \cdot \frac{\alpha^{2s+m+n+1-k}\sqrt{\pi}\,\Gamma\left(s+1+\frac{m+n-k}{2}\right)}{2\Gamma\left(s+\frac{m+n+3-k}{2}\right)}\,\mathrm{d}s\,, \quad \mathrm{Re}\,[s] > -\frac{m+n-k}{2} - 1\,. \tag{29}$$

By letting

$$\sigma = m+n, \qquad \delta = m-n \tag{30}$$

so that

$$m = \frac{\sigma+\delta}{2}\,, \qquad n = \frac{\sigma-\delta}{2}\,, \tag{31}$$

(29) can be written as

$$I_{\mathrm{R}}(m,n,k,\alpha) = \frac{1}{4\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(-s)\,\Gamma\left(s+\frac{\sigma}{2}+1\right)\Gamma\left(s+\frac{\sigma+3}{2}\right)\Gamma\left(s+1+\frac{\sigma-k}{2}\right)}{\Gamma(s+\sigma+2)\,\Gamma\left(s+\frac{\sigma+\delta+3}{2}\right)\Gamma\left(s+\frac{\sigma-\delta+3}{2}\right)\Gamma\left(s+\frac{\sigma+3-k}{2}\right)}\,\alpha^{2s+\sigma+1-k}\,\mathrm{d}s \tag{32}$$

provided that $\mathrm{Re}\,[s] > -(\sigma-k)/2 - 1$.

The singularities of the integrand function in (32) are among the poles of the Gamma functions in the numerator of (32), i.e.,

$$\begin{aligned} s_p^{(1)} &= p, & p &= 0,1,2\ldots\,, \\ s_q^{(2)} &= -\frac{\sigma}{2} - 1 - q, & q &= 0,1,2\ldots\,, \\ s_r^{(3)} &= -\frac{\sigma-k}{2} - 1 - r\,, & r &= 0,1,2,\ldots\,, \\ s_t^{(4)} &= -\frac{\sigma+3}{2} - t\,, & t &= 0,1,2,\ldots\,. \end{aligned} \tag{33}$$

while the zeros are among the poles of the Gamma functions in the denominator of (32), i.e.,

$$\begin{aligned} z_p^{(1)} &= -\sigma - 2 - p, & p &= 0,1,2\ldots\,, \\ z_q^{(2)} &= -\frac{\sigma+\delta+3}{2} - q, & q &= 0,1,2\ldots\,, \\ z_r^{(3)} &= -\frac{\sigma-\delta+3}{2} - r\,, & r &= 0,1,2,\ldots\,, \\ z_t^{(4)} &= -\frac{\sigma-k+3}{2} - t\,, & t &= 0,1,2,\ldots\,. \end{aligned} \tag{34}$$

It should be noted that for certain values of the parameters a pole-zero cancellation may occur, as it will be described later. The parameter c in (32) is thus chosen in the interval

$$-\frac{\sigma-k}{2} - 1 < c < 0 \tag{35}$$

so that all the poles $s_p^{(1)}$ are on the right of the integration path in (32) and all the poles $s_q^{(2)}$, $s_r^{(3)}$, and $s_t^{(4)}$ are on the left (it should be noted that it is certainly $-(\sigma-k)/2-1<0$). The original integration path in (32) (also known as Barnes integration path C_{B}) can then be closed to its right (or to the left) thus giving rise to a closed path C_{R} (or C_{L}) having clockwise (or anticlockwise) orientation enclosing all the poles $s_p^{(1)}$ (or all the poles $s_q^{(2)}$, $s_r^{(3)}$, and $s_t^{(4)}$). To determine where the integration path C_{B} has to be closed we need to examine the behavior of the integrand function $F(s)$ as $s \to \infty$. This can be done through the asymptotic expansion of the Gamma function [46, 8.327.1]:

$$\Gamma(z) \simeq \sqrt{2\pi}\, z^{z-1/2}\, \mathrm{e}^{-z} \tag{36}$$

for which

$$F(s) \simeq \left(\frac{\alpha \mathrm{e}}{s}\right)^{2s}\,. \tag{37}$$

The integration path can then be closed to its right, thus giving rise to a closed path C_{R} oriented clockwise and enclosing all the poles $s_p^{(1)}$. The situation is described in Fig. 1.

From the Residue theorem it then follows

$$I_{\mathrm{R}}(m,n,k,\alpha) = -\frac{\alpha^{\sigma-k}}{2} \sum_{p=0}^{\infty} \frac{\mathrm{Res}\,\left[\Gamma(-s)\right]_{s=p}\Gamma\left(p+\frac{\sigma}{2}+1\right)\Gamma\left(p+\frac{\sigma+3}{2}\right)\Gamma\left(p+1+\frac{\sigma-k}{2}\right)}{\Gamma(p+\sigma+2)\,\Gamma\left(p+\frac{\sigma+\delta+3}{2}\right)\Gamma\left(p+\frac{\sigma-\delta+3}{2}\right)\Gamma\left(p+\frac{\sigma+3-k}{2}\right)}\,\alpha^{2p+1}\,. \tag{38}$$

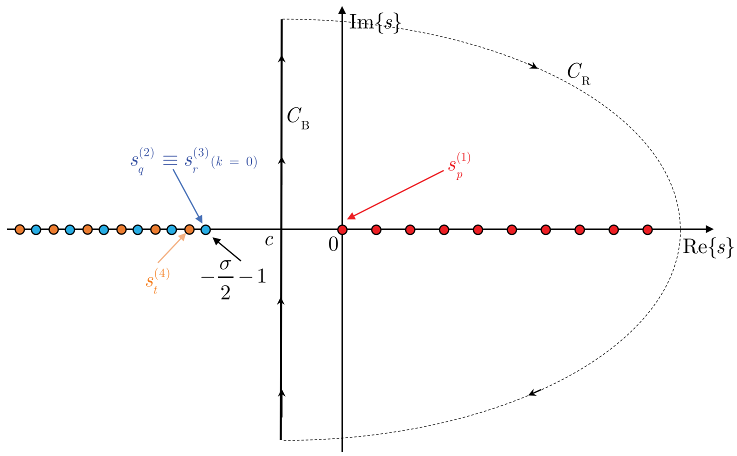


Fig. 1. Example of pole singularities of the Gamma functions in (32) in the s plane (in the case $k = 0$) together with the Barnes integration path C_B and the closed path C_R.

The residues of the function $\Gamma(-s)$ at the poles $s_p^{(1)} = p$ are

$$\text{Res }[\Gamma(-s)]_{s=p} = \lim_{s\to p}(s-p)\,\Gamma(-s) = \lim_{t\to 0} t\Gamma(-t-p) = -\lim_{t\to 0}\frac{\Gamma(-t+1)}{(-t-1)\cdots(-t-p)} = \frac{(-1)^{p+1}}{p!} = \frac{(-1)^{p+1}}{\Gamma(p+1)} \tag{39}$$

so that from (38) and (39) we obtain

$$I_R(m,n,k,\alpha) = \frac{\alpha^{\sigma-k}}{2}\sum_{p=0}^{\infty}\frac{(-1)^p\,\Gamma\left(p+\frac{\sigma}{2}+1\right)\Gamma\left(p+\frac{\sigma+3}{2}\right)\Gamma\left(p+\frac{\sigma-k}{2}+1\right)}{\Gamma(p+1)\,\Gamma(p+\sigma+2)\,\Gamma\left(p+\frac{\sigma+\delta+3}{2}\right)\Gamma\left(p+\frac{\sigma-\delta+3}{2}\right)\Gamma\left(p+\frac{\sigma-k+3}{2}\right)}\alpha^{2p+1}\,. \tag{40}$$

It is worth noting that the series representation in (40) can be used to express the result in terms of the generalized hypergeometric functions ${}_pF_q$ [46, 9.14]. In fact, by definition

$${}_pF_q\left(a_1,a_2,\dots,a_p;b_1,b_2,\dots,b_q;z\right) = \sum_{n=0}^{\infty}\frac{\left(a_1\right)_n\left(a_2\right)_n\cdots\left(a_p\right)_n}{\left(b_1\right)_n\left(b_2\right)_n\cdots\left(b_q\right)_n}\frac{z^n}{n!}\,, \tag{41}$$

where use is made of the Pochhammer symbol

$$(x)_y = \frac{\Gamma(x+y)}{\Gamma(x)}\,. \tag{42}$$

Noting that (40) can be written as

$$I_R(m,n,k,\alpha) = \frac{\alpha^{\sigma+1-k}\Gamma\left(1+\frac{\sigma}{2}\right)\Gamma\left(\frac{\sigma+3}{2}\right)\Gamma\left(1+\frac{\sigma-k}{2}\right)}{2\Gamma\left(\frac{\sigma+3+\delta}{2}\right)\Gamma\left(\frac{\sigma+3-\delta}{2}\right)\Gamma(\sigma+2)\,\Gamma\left(\frac{\sigma-k+3}{2}\right)}\sum_{p=0}^{\infty}\frac{\left(1+\frac{\sigma}{2}\right)_p\left(\frac{\sigma+3}{2}\right)_p\left(1+\frac{\sigma-k}{2}\right)_p}{\left(\frac{\sigma+3+\delta}{2}\right)_p\left(\frac{\sigma+3-\delta}{2}\right)_p(\sigma+2)_p\left(\frac{\sigma-k+3}{2}\right)_p}\frac{\left(-\alpha^2\right)^p}{p!} \tag{43}$$

using (41) and the properties of the Gamma function it also results

$$I_R(m,n,k,\alpha) = \frac{\alpha^{\sigma+1-k}\Gamma\left(1+\frac{\sigma}{2}\right)\Gamma\left(\frac{\sigma+3}{2}\right)\Gamma\left(1+\frac{\sigma-k}{2}\right)}{2\Gamma\left(\frac{\sigma+3+\delta}{2}\right)\Gamma\left(\frac{\sigma+3-\delta}{2}\right)\Gamma(\sigma+2)\,\Gamma\left(\frac{\sigma-k+3}{2}\right)}\cdot {}_3F_4\left(1+\frac{\sigma}{2},\frac{\sigma+3}{2},1+\frac{\sigma-k}{2};\sigma+2,\frac{\sigma-k+3}{2},\frac{\sigma+3+\delta}{2},\frac{\sigma+3-\delta}{2};-\alpha^2\right). \tag{44}$$

It should be taken into account that the closed-form expression (44) can further be simplified for particular values of the involved parameters because, for instance, the order of the generalized hypergeometric function is reduced when an upper and a lower Pochhammer symbol are equal. It is worth noting that, as already pointed out in [40,43], generalized hypergeometric functions can be automatically evaluated (also through definitions other than the series expansion (41)) and manipulated by modern routines in mathematical packages and their expression can be used for further steps in analytical studies.

3.2. Evaluation of $I_J(m,n,k,\alpha)$

Using similar arguments for the integral in (23) we have

$$I_J(m,n,k,\alpha) = \frac{1}{2\pi\sqrt{\pi}\,\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(-s)\,\Gamma\left(s+\frac{m+n}{2}+1\right)\Gamma\left(s+\frac{m+n+3}{2}\right)}{\Gamma(s+m+n+2)\,\Gamma\left(s+m+\frac{3}{2}\right)\Gamma\left(s+n+\frac{3}{2}\right)} \left[\int_{\alpha}^{+\infty} \frac{v^{2s+m+n+1-k}}{\sqrt{v^2-\alpha^2}}\,\mathrm{d}v\right]\mathrm{d}s\,. \tag{45}$$

The integral in [46, 3.251.3] furnishes

$$\int_1^{+\infty} x^{\mu-1}\left(x^p-1\right)^{\nu-1}\mathrm{d}x = \frac{1}{p}\,\mathrm{B}\left(1-\nu-\frac{\mu}{p},\nu\right), \qquad p>0,\ \nu>0,\ \mu<p-p\nu\,. \tag{46}$$

From (27) it thus follows that

$$\int_a^{\infty} \frac{y^{\beta}}{\sqrt{y^2-a^2}}\,\mathrm{d}y = \frac{a^{\beta}\sqrt{\pi}\,\Gamma\left(-\frac{\beta}{2}\right)}{2\Gamma\left(\frac{1-\beta}{2}\right)}\,, \qquad \beta<0 \tag{47}$$

and therefore, using $\beta = 2s+m+n+1-k$ and the same settings (30)–(31), we have

$$I_J(m,n,k,\alpha) = \frac{\alpha^{\sigma+1-k}}{4\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(-s)\,\Gamma\left(s+\frac{\sigma}{2}+1\right)\Gamma\left(-s-\frac{\sigma+1-k}{2}\right)\Gamma\left(s+\frac{\sigma+3}{2}\right)}{\Gamma(s+\sigma+2)\,\Gamma\left(s+\frac{\sigma+\delta+3}{2}\right)\Gamma\left(s+\frac{\sigma-\delta+3}{2}\right)\Gamma\left(-s-\frac{\sigma-k}{2}\right)}\,\alpha^{2s} \tag{48}$$

provided that $\mathrm{Re}[s] < -(\sigma+1-k)/2$.

The singularities of the integrand function in (48) are among the poles of the Gamma functions in the numerator of (48), i.e.,

$$\begin{aligned} s_p^{(1)} &= p, && p=0,1,2\ldots\,,\\ s_q^{(2)} &= -\frac{\sigma}{2}-1-q, && q=0,1,2\ldots\,,\\ s_r^{(3)} &= r-\frac{\sigma-k+1}{2}\,, && r=0,1,2,\ldots\,,\\ s_t^{(4)} &= -\frac{\sigma+3}{2}-t\,, && t=0,1,2,\ldots\,. \end{aligned} \tag{49}$$

while the zeros are among the poles of the Gamma functions in the denominator of (48), i.e.,

$$\begin{aligned} z_p^{(1)} &= -\sigma-2-p, && p=0,1,2\ldots\,,\\ z_q^{(2)} &= -\frac{\sigma+\delta+3}{2}-q, && q=0,1,2\ldots\,,\\ z_r^{(3)} &= -\frac{\sigma-\delta+3}{2}-r\,, && r=0,1,2,\ldots\,,\\ z_t^{(4)} &= t-\frac{\sigma-k}{2}\,, && t=0,1,2\ldots\,. \end{aligned} \tag{50}$$

The parameter c in (32) is thus chosen in the interval

$$-\frac{\sigma}{2}-1<c<-\frac{\sigma+1-k}{2} \tag{51}$$

so that all the poles $s_q^{(2)}$ and $s_t^{(4)}$ are on the left of the integration path C_B in (48) and all the poles $s_p^{(1)}$ and $s_r^{(3)}$ are on the right. The original integration path can then be closed to its right (or to its left) thus giving rise to a closed path having clockwise (or anticlockwise) orientation enclosing all the poles $s_p^{(1)}$ and $s_r^{(3)}$ (or all the poles $s_q^{(2)}$ and $s_t^{(4)}$). To determine where the integration path has to be closed we need to examine the behavior of the integrand function $F(s)$ as $s\to\infty$. This can be done again through the asymptotic expansion of the Gamma function so that

$$F(s) \simeq \left(\frac{\alpha\mathrm{e}}{s}\right)^{2s}. \tag{52}$$

The integration path can then be closed to its right thus giving rise to a closed path C_R run in a clockwise direction and enclosing all the poles $s_p^{(1)}$ and $s_r^{(3)}$.

In contrast with the case of the integral in (32), the zeros at $z_t^{(4)}$ may or may not cancel the poles $s_p^{(1)}$ in (49). In particular, if $\sigma-k$ is even, the zeros $z_t^{(4)}$ cancel the poles $s_p^{(1)}$ and only the poles $s_r^{(3)}$ contribute to the Residue theorem, while if $\sigma-k$ is odd no cancelation occurs and we end up with *two sets of poles* enclosed by C_R: a finite number of simple poles at $s_r^{(3)}$ for $r=0,\ldots,r_0$ (with $r_0=(\sigma-k-1)/2$) and an infinite number of *double* poles at $s_p^{(1)}$ (which are superimposed to the poles $s_r^{(3)}$ for $r>r_0$).

3.2.1. $\sigma-k$ *even*

If $\sigma-k$ is even, the poles enclosed by C_R are all the $s_r^{(3)}$ poles so that from the Residue theorem it follows

$$I_J(m,n,k,\alpha) = -\frac{1}{2}\sum_{r=0}^{\infty} \frac{\mathrm{Res}\left[\Gamma\left(-s-\frac{\sigma+1-k}{2}\right)\right]_{s=s_r^{(3)}}\Gamma\left(-r+\frac{\sigma-k+1}{2}\right)\Gamma\left(r+\frac{k+1}{2}\right)\Gamma\left(r+\frac{k}{2}+1\right)}{\Gamma\left(r+1+\frac{k+\delta}{2}\right)\Gamma\left(r+1+\frac{k-\delta}{2}\right)\Gamma\left(r+\frac{\sigma+k+3}{2}\right)\Gamma\left(-r+\frac{1}{2}\right)}\,\alpha^{2r}\,. \tag{53}$$

It results

$$\mathrm{Res}\left[\Gamma\left(-s-\frac{\sigma+1-k}{2}\right)\right]_{s=s_r^{(3)}}=\lim_{s\to s_r^{(3)}}\left(s-s_r^{(3)}\right)\Gamma\left(-s-\frac{\sigma+1-k}{2}\right)=\frac{(-1)^{r+1}}{r!}=\frac{(-1)^{r+1}}{\Gamma(r+1)} \tag{54}$$

so that from (53) and (54) we obtain

$$I_{\mathrm{J}}(m,n,k,\alpha)=\frac{1}{2}\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma\left(-r+\frac{\sigma-k+1}{2}\right)\Gamma\left(r+\frac{k+1}{2}\right)\Gamma\left(r+1+\frac{k}{2}\right)}{\Gamma\left(r+1+\frac{k+\delta}{2}\right)\Gamma\left(r+1+\frac{k-\delta}{2}\right)\Gamma(r+1)\Gamma\left(r+\frac{\sigma+k+3}{2}\right)\Gamma\left(-r+\frac{1}{2}\right)}\alpha^{2r}. \tag{55}$$

Using the reflection property of the Gamma function [46, 8.334.2]

$$\Gamma\left(-M+\frac{1}{2}\right)=\frac{\pi}{(-1)^M\Gamma\left(M+\frac{1}{2}\right)}, \tag{56}$$

where M is a positive integer, we obtain

$$I_{\mathrm{J}}(m,n,k,\alpha)=\frac{(-1)^{\frac{\sigma-k}{2}}}{2}\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(r+\frac{1}{2}+\frac{k}{2}\right)\Gamma\left(r+1+\frac{k}{2}\right)}{\Gamma\left(r+1+\frac{k+\delta}{2}\right)\Gamma\left(r+1+\frac{k-\delta}{2}\right)\Gamma(r+1)\Gamma\left(r+\frac{\sigma+k}{2}+\frac{3}{2}\right)\Gamma\left(r-\frac{\sigma-k}{2}+\frac{1}{2}\right)}\alpha^{2r}. \tag{57}$$

If $k\pm\delta<0$, some of the addends in the series (57) are identically zero. Therefore, in general, by indicating $r_0=\max\{0,-(k+\delta)/2,-(k-\delta)/2\}$, the sum in (57) starts with r_0, i.e.,

$$I_{\mathrm{J}}(m,n,k,\alpha)=\frac{(-1)^{\frac{\sigma-k}{2}}}{2}\sum_{r=r_0}^{\infty}\frac{(-1)^r\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(r+\frac{1}{2}+\frac{k}{2}\right)\Gamma\left(r+1+\frac{k}{2}\right)}{\Gamma\left(r+1+\frac{k+\delta}{2}\right)\Gamma\left(r+1+\frac{k-\delta}{2}\right)\Gamma(r+1)\Gamma\left(r+\frac{\sigma+k}{2}+\frac{3}{2}\right)\Gamma\left(r-\frac{\sigma-k}{2}+\frac{1}{2}\right)}\alpha^{2r}. \tag{58}$$

To represent (58) in terms of a general hypergeometric functions ${}_pF_q$, we thus redefine the index of summation as $q=r-r_0$. Recasting the series in a form equivalent to (41) and by letting for brevity $r_1=r_0+1$ and $r_2=r_0+1/2$ we arrive to

$$\begin{aligned}I_{\mathrm{J}}(m,n,k,\alpha)=&\frac{(-1)^{\frac{\sigma-k}{2}+r_0}\alpha^{2r_0}\Gamma(r_2)\Gamma\left(r_2+\frac{k}{2}\right)\Gamma\left(r_1+\frac{k}{2}\right)}{2\Gamma\left(r_1+\frac{k+\delta}{2}\right)\Gamma\left(r_1+\frac{k-\delta}{2}\right)\Gamma(r_1)\Gamma\left(r_1+\frac{\sigma+k+1}{2}\right)\Gamma\left(r_2-\frac{\sigma-k}{2}\right)}\cdot\\&{}_4F_5\left(r_2,r_2+\frac{k}{2},r_1+\frac{k}{2},1;r_1,r_1+\frac{k+\delta}{2},r_1+\frac{k-\delta}{2},r_1+\frac{\sigma+k+1}{2},r_2-\frac{\sigma-k}{2};-\alpha^2\right).\end{aligned} \tag{59}$$

Again, it should be noted that the closed-form expression (59) can further be simplified for particular values of the involved parameters because, for instance, the order is reduced when an upper and a lower Pochhammer symbol are equal.

3.2.2. $\sigma-k$ odd

As mentioned before, if $\sigma-k$ is an odd integer, when we apply the Cauchy Residue theorem to (48), both the $s_p^{(1)}$ and $s_r^{(3)}$ poles are enclosed: actually, for $r<(\sigma-k+1)/2$ the $s_r^{(3)}$ poles are simple poles of the integrand function, while for $r\geq(\sigma-k+1)/2$ the $s_r^{(3)}$ pole array coincide with the $s_p^{(1)}$ pole array and they represent a set of double poles for the integrand function in (48): this has to be carefully taken into account when applying the Residue theorem so that

$$I_{\mathrm{J}}(m,n,k,\alpha)=I_{\mathrm{J}}^{(1)}(m,n,k,\alpha)+I_{\mathrm{J}}^{(2)}(m,n,k,\alpha), \tag{60}$$

where

$$I_{\mathrm{J}}^{(1)}(m,n,k,\alpha)=-2\pi\mathrm{i}\sum_{r=0}^{L-1}\mathrm{Res}\,[F(s)]_{s=s_r} \tag{61}$$

and

$$I_{\mathrm{J}}^{(2)}(m,n,k,\alpha)=-2\pi\mathrm{i}\sum_{p=0}^{\infty}\mathrm{Res}\,[F(s)]_{s=s_p}, \tag{62}$$

where we have indicated

$$L=\frac{\sigma+1-k}{2} \tag{63}$$

and

$$F(s)=\frac{1}{4\pi\mathrm{i}}\frac{\Gamma(-s)\Gamma\left(s+\frac{\sigma}{2}+1\right)\Gamma\left(s+\frac{\sigma+3}{2}\right)\Gamma\left(-s-\frac{\sigma+1-k}{2}\right)}{\Gamma(s+\sigma+2)\Gamma\left(s+\frac{\sigma+\delta+3}{2}\right)\Gamma\left(s+\frac{\sigma-\delta+3}{2}\right)\Gamma\left(-s-\frac{\sigma-k}{2}\right)}\alpha^{2s+\sigma+1-k}. \tag{64}$$

From (61) and (64), calculating the residues at the first L simple poles $s_r^{(3)}$ and using the reflection property (56), we have

$$I_{\mathrm{J}}^{(1)}(m,n,k,\alpha)=\frac{1}{2\pi}\sum_{r=0}^{L-1}\frac{\Gamma\left(r+\frac{1}{2}\right)\Gamma(L-r)\Gamma\left(r+\frac{k+1}{2}\right)\Gamma\left(r+1+\frac{k}{2}\right)}{\Gamma\left(r+1+\frac{k+\delta}{2}\right)\Gamma\left(r+1+\frac{k-\delta}{2}\right)\Gamma\left(r+\frac{\sigma+k+3}{2}\right)\Gamma(r+1)}\alpha^{2r}. \tag{65}$$

When considering the evaluation of $I_{\mathrm{J}}^{(2)}(m,n,k,\alpha)$ we need to evaluate the residue of the $F(s)$ function in a double pole. The result is

$$
\begin{aligned}
I_{\mathrm{J}}^{(2)}(m,n,k,\alpha) = -\frac{\alpha^{\sigma-k+1}}{2\pi}\sum_{p=0}^{\infty} & \frac{(-1)^p\,\Gamma\left(p+1+\frac{\sigma}{2}\right)\Gamma\left(p+1+\frac{\sigma+1}{2}\right)\Gamma\left(p+L+\frac{1}{2}\right)}{\Gamma\left(p+\frac{\sigma+\delta+3}{2}\right)\Gamma\left(p+\frac{\sigma-\delta+3}{2}\right)\Gamma(p+1)\,\Gamma(p+L+1)\,\Gamma(p+\sigma+2)} \\
& \cdot\left[2\gamma - H_p - H_{p+L} + 2\ln\alpha + \psi\left(p+\frac{\sigma}{2}+1\right) + \psi\left(p+\frac{\sigma+3}{2}\right)\right. \\
& \left. -\psi\left(p+\frac{\sigma+\delta+3}{2}\right) - \psi\left(p+\frac{\sigma-\delta+3}{2}\right) - \psi(p+\sigma+2) + \psi\left(-p-L+\frac{1}{2}\right)\right]\alpha^{2p},
\end{aligned}
\tag{66}
$$

where H_p is the pth harmonic number (i.e., the sum of the reciprocals of the first p natural numbers) and the Digamma (or Psi) function $\psi(s)$ has been introduced [46, 8.36] which is defined as

$$
\psi(s) = \frac{\mathrm{d}}{\mathrm{d}s}\ln\Gamma(s)\,. \tag{67}
$$

Since the series in (66) includes a $\ln\alpha$ term (which in general appears when multiple poles are present), it is usually said to be logarithmic [44]. It is worth noting that expression (66) can be simplified when the evenness or oddness of σ and k are specified, as shown in the next Section. Since the derivation of (66) is not a trivial task, the relevant details are reported in Appendix.

Finally, it should be noted that the integral in (48), although not directly expressed as a suitable generalized hypergeometric function, it can be expressed in a closed form as a Meijer G-function [56–58] (which is, in some sense, a generalization of the generalized hypergeometric function). The latter is in fact defined through a Mellin–Barnes integral as

$$
G_{p,q}^{m,n}\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};z\right) = \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}}\frac{\prod_{j=1}^{m}\Gamma\left(b_j+s\right)\prod_{i=1}^{n}\Gamma\left(1-a_i-s\right)}{\prod_{i=n+1}^{p}\Gamma\left(a_i+s\right)\prod_{j=m+1}^{q}\Gamma\left(1-b_j-s\right)}z^{-s}\,\mathrm{d}s\,, \tag{68}
$$

where $\mathcal{L}$ is one of the contours which separate all poles $b_{jl} = b_j + l$ $(l = 0,1,\ldots)$ to the left and all poles $a_{ik} = 1 - a_i + k$ $(k = 0,1,\ldots)$ to the right of $\mathcal{L}$. By comparing (48) and (68) it is thus immediate to derive that

$$
I_{\mathrm{J}}(m,n,k,\alpha) = \frac{\alpha^{\sigma+1-k}}{2}G_{5,3}^{2,2}\left(\begin{matrix}1,\frac{\sigma+3-k}{2},\sigma+2,\frac{\sigma+3+\delta}{2},\frac{\sigma+3-\delta}{2}\\ 1+\frac{\sigma}{2},\frac{\sigma+3}{2},1+\frac{\sigma-k}{2}\end{matrix};\frac{1}{\alpha^2}\right). \tag{69}
$$

It should be noted that expressions (66) and (69) are consistent with the considerations in [59] concerning the series expansion of the Meijer G-function (Theorem 3.2 in [59]).

It is important to point out that reliable calculations of the Meijer G-functions (as of the generalized hypergeometric functions) are nowadays available in any commercial mathematical software like, e.g., Mathematica [60] or MATLAB [61].

3.3. Evaluation of $I(m,n,k,\alpha)$

The original integral $I(m,n,k,\alpha)$ is obtained as a combination of I_{R} and I_{J} through (21). From a computational point of view, the expressions for I_{R} and I_{J} obtained in the previous Section can be further simplified considering the parity of the parameters σ (and consequently δ) and k. For computational purposes we will also avoid the calculation of the Gamma and Digamma functions. In particular, by considering an integer M, we will make use of the properties of the Pochhammer symbol [46, 8.339.5-6]

$$
\frac{\Gamma(x+M)}{\Gamma(x)} = (x)_M = \begin{cases} x(x+1)\cdots(x+M-1)\,, & M>0\,,\\ 1\,, & M=0\,,\\ \dfrac{1}{(x+M)(x+M+1)\cdots(x-1)}\,, & M<0\,,\end{cases} \tag{70}
$$

the properties of the Gamma function [46, 8.339.1-2]

$$
\begin{aligned}
&\Gamma(M+1) = M!\,,\\
&\Gamma\left(M+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^M}(2M-1)!!
\end{aligned}
\tag{71}
$$

and the properties of the Digamma function [46, 8.365.4, 8.366.3]

$$
\begin{aligned}
&\psi(M+1) = -\gamma + H_M\\
&\psi\left(\pm M+\frac{1}{2}\right) = -\gamma - 2\ln 2 + 2H_M^{\mathrm{odd}}
\end{aligned}
\tag{72}
$$

where

$$
H_M^{\mathrm{odd}} = \sum_{h=1}^{M}\frac{1}{2h-1}\,. \tag{73}
$$

3.3.1. Case 1: $m + n$ even and k even

When both σ and k are even, from with $S = (m+n)/2$, $D = (m-n)/2$, and $K = k/2$, we have

$$I_{\mathrm{R}}(m,n,k,\alpha) = \frac{2^{2S+1}\alpha^{2S-2K}}{\pi} \sum_{p=0}^{\infty} \frac{(-1)^p\, 2^{2p}}{(2p+2S+2D+1)!!\,(2p+2S-2D+1)!!} \frac{(p+1)_{S-K}}{(p+S+1)_{S+1}\left(p+S+\frac{3}{2}\right)_{-K}} \alpha^{2p+1}, \tag{74}$$

while from (58), by indicating $r_0 = \min\{0, D-K, -D-K\}$, we have

$$I_{\mathrm{J}}(m,n,k,\alpha) = \frac{(-1)^{S-K}}{2} \sum_{r=r_0}^{\infty} \frac{(-1)^r}{(r+K+D)!\,(r+K-D)!} \frac{(r+1)_K}{\left(r+K+\frac{1}{2}\right)_{S+1}\left(r+\frac{1}{2}\right)_{K-S}} \alpha^{2r}. \tag{75}$$

It is worth noting that this class of integrals occurs in the solution of electromagnetic scattering from circular disks and apertures [20].

3.3.2. Case 2: $m + n$ odd and k odd

When both σ and k are odd, from (40) with $S = (m+n+1)/2$, $D = (m-n-1)/2$, and $K = (k+1)/2$, we have

$$I_{\mathrm{R}}(m,n,k,\alpha) = \frac{2^{2S}\alpha^{2S-2K}}{\pi} \sum_{p=0}^{\infty} \frac{(-1)^p\, 2^{2p}}{(2p+2S+2D+1)!!\,(2p+2S-2D-1)!!} \frac{(p+1)_{S-K}}{(p+S+1)_{S}\left(p+S+\frac{1}{2}\right)_{1-K}} \alpha^{2p+1} \tag{76}$$

while from (58) we have

$$I_{\mathrm{J}}(m,n,k,\alpha) = \frac{(-1)^{S-K}}{2} \sum_{r=r_0}^{\infty} \frac{(-1)^r}{(r+K+D)!\,(r+K-D-1)!} \frac{(r+1)_{K-1}}{\left(r+K+\frac{1}{2}\right)_{S}\left(r+K+\frac{1}{2}\right)_{K-S}} \alpha^{2r}, \tag{77}$$

where $r_0 = \max\{0, -K-D, -K+D+1\}$.

3.3.3. Case 3: $m + n$ even and k odd

If σ is even and k is odd, from (40) with $S = (m+n)/2$, $D = (m-n)/2$, and $K = (k+1)/2$ we have

$$I_{\mathrm{R}}(m,n,k,\alpha) = \frac{\alpha^{2S-2K+2}}{2} \sum_{p=0}^{\infty} \frac{(-1)^p}{(p+S+1-K)!\,p!} \frac{\left(p+S-D+\frac{3}{2}\right)_{D-K}}{\left(p+S+\frac{3}{2}\right)_{D}(p+S+1)_{S+1}} \alpha^{2p} \tag{78}$$

while we recall that

$$I_{\mathrm{J}}(m,n,k,\alpha) = I_{\mathrm{J}}^{(1)}(m,n,k,\alpha) + I_{\mathrm{J}}^{(2)}(m,n,k,\alpha). \tag{79}$$

In this case $L = S - K + 1$ and from (65) it also results

$$I_{\mathrm{J}}^{(1)}(m,n,k,\alpha) = \frac{1}{2\pi} \sum_{r=0}^{S-K} \frac{(r+1)_{K-1}}{\left(r+K+\frac{1}{2}\right)_{D}\left(r+\frac{1}{2}\right)_{K-D}(S+1-K-r)_{2r+2K}} \alpha^{2r} \tag{80}$$

while from (66), together with (70)–(72) we have

$$\begin{aligned} I_{\mathrm{J}}^{(2)}(m,n,k,\alpha) = &-\frac{\alpha^{2S-2K+2}}{2\pi} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\,(p+S-K+1)!} \frac{\left(p+S-D+\frac{3}{2}\right)_{D-K}}{\left(p+S+\frac{3}{2}\right)_{D}(p+S+1)_{S+1}} \\ &\cdot \left[2\gamma + 2\ln\alpha - H_p - H_{p+S-K+1} + H_{p+S} - H_{p+2S+1} + 2H^{\mathrm{odd}}_{p+S+1} - 2H^{\mathrm{odd}}_{p+S+D+1} - 2H^{\mathrm{odd}}_{p+S-D+1} + 2H^{\mathrm{odd}}_{p+S-K+1}\right]\alpha^{2p}. \end{aligned} \tag{81}$$

3.3.4. Case 4: $m + n$ odd and k even

If σ is odd and k is even, from (40) with $S = (m+n+1)/2$, $D = (m-n-1)/2$, and $K = k/2$ we have

$$I_{\mathrm{R}}(m,n,k,\alpha) = \frac{\alpha^{2S-2K}}{2} \sum_{p=0}^{\infty} \frac{(-1)^p}{(p+S-K)!\,p!} \frac{\left(p+S-D+\frac{1}{2}\right)_{D-K}}{\left(p+S+\frac{1}{2}\right)_{D+1}(p+S+1)_{S}} \alpha^{2p}. \tag{82}$$

In this case $L = S - K$ and from (65) it also results

$$I_{\mathrm{J}}^{(1)}(m,n,k,\alpha) = \frac{1}{2\pi} \sum_{r=0}^{S-K-1} \frac{(r+1)_{K}}{\left(r+K+\frac{1}{2}\right)_{D+1}\left(r+\frac{1}{2}\right)_{K-D}(S-K-r)_{2r+2K+1}} \alpha^{2r} \tag{83}$$

while from (66), together with (70)–(72) we have

$$I_{\mathrm{J}}^{(2)}(m,n,k,\alpha) = -\frac{\alpha^{2S-2K}}{2\pi}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!\,(p+S-K)!}\,\frac{\left(p+S-D+\frac{1}{2}\right)_{D-K}}{\left(p+S+\frac{1}{2}\right)_{D+1}(p+S+1)_S} \cdot\left[2\gamma+2\ln\alpha-H_p-H_{p+S-K}+H_{p+S}-H_{p+2S}+2H_{p+S}^{\mathrm{odd}}-2H_{p+S+D+1}^{\mathrm{odd}}-2H_{p+S-D}^{\mathrm{odd}}+2H_{p+S-K}^{\mathrm{odd}}\right]\alpha^{2p}. \tag{84}$$

It should be noted that although in the cases 3 and 4 the expressions for the general series term look more involved, H_n and H_n^{odd} are simple numbers that can be calculated once for all and stored.

4. Numerical examples

First of all, we want to point out that the proposed expressions can be used to evaluate the well-known integral

$$I_0 = \int_0^{\infty}\frac{\mathrm{J}_{m+1/2}(\nu)\,\mathrm{J}_{n+1/2}(\nu)}{\nu^{\lambda}}\,\mathrm{d}\nu \tag{85}$$

(where $m+n+2>\lambda>0$ to ensure the convergence of the integral), which is a special case of the Weber–Schafheitlin integral [45]. In fact, by letting $\alpha=0$ in (1) we have

$$I(m,n,k,0) = \mathrm{i}\int_0^{\infty}\frac{\mathrm{J}_{m+1/2}(\nu)\,\mathrm{J}_{n+1/2}(\nu)}{\nu^{k+1}}\,\mathrm{d}\nu \tag{86}$$

so that $I_0=-\mathrm{i}I(m,n,\lambda-1,0)$. On the other hand, it is immediate to check that it always results

$$\begin{aligned} &I_{\mathrm{R}}(m,n,\lambda-1,0)=0\,,\\ &I_{\mathrm{J}}^{(2)}(m,n,\lambda-1,0)=0 \end{aligned} \tag{87}$$

and, both for $m+n+1-\lambda$ even and odd from (57) and (65), after applying (8) and the property [46, 8.334.3]

$$\Gamma(1-x) = \frac{\pi}{\sin(\pi x)\,\Gamma(x)}\,, \tag{88}$$

it results

$$I_{\mathrm{J}}(m,n,\lambda-1,0) = \mathrm{i}\frac{\Gamma(\lambda)\,\Gamma\left(\frac{m+n-\lambda+2}{2}\right)}{2^{\lambda}\Gamma\left(\frac{-m+n+\lambda+1}{2}\right)\Gamma\left(\frac{m+n+\lambda+2}{2}\right)\Gamma\left(\frac{m-n+\lambda+1}{2}\right)} \tag{89}$$

so that

$$I_0 = -\mathrm{i}I(m,n,\lambda-1,0) = \frac{\Gamma(\lambda)\,\Gamma\left(\frac{m+n-\lambda+2}{2}\right)}{2^{\lambda}\Gamma\left(\frac{-m+n+\lambda+1}{2}\right)\Gamma\left(\frac{m+n+\lambda+2}{2}\right)\Gamma\left(\frac{m-n+\lambda+1}{2}\right)}\,, \tag{90}$$

consistent with the closed-form result in [46, 6.574.2]. It should be noted that the expressions derived in the present paper are strictly valid for $\alpha>0$, but the results continue to hold for $\alpha=0$ by analytic continuation.

We now evaluate the performance of the proposed formulas by comparing them with the results obtained with the quadrature procedure illustrated in Section 2. For all the presented results the precision $\varepsilon_{\mathrm{rel}}$ of the quadrature algorithm has to be set and the relevant results are taken as reference values.

In a numerical code it is simple to fix the relative error $\varepsilon_{\mathrm{rel}}$ for each presented series. In fact, we assume for both I_{R} and I_{J} a representation

$$I_{\mathrm{R/J}} = \sum_{p=0}^{\infty}a_p^{\mathrm{R/J}} \simeq \sum_{p=0}^{N_{\mathrm{R/J}}}a_p^{\mathrm{R/J}} = I_{\mathrm{R/J}}^{N_{\mathrm{R/J}}}\,,$$

where $N_{\mathrm{R/J}}$ is such that

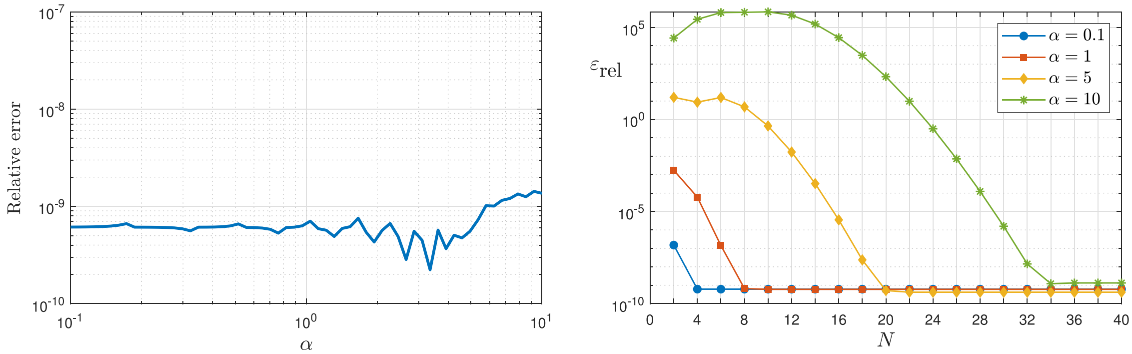
$$\left|a_{N_{\mathrm{R/J}}}\right| < \varepsilon_{\mathrm{rel}}\left|I_{\mathrm{R/J}}^{N_{\mathrm{R/J}}-1}\right|\,. \tag{91}$$

We first consider the integral $I(m,n,k,\alpha)$ in (1) with parameters $m=n=3$, $k=0$, (*Case 1*, σ even and k even) and different values of α. In Table 1, for each value of α and $\varepsilon_{\mathrm{rel}}$, we report the results obtained through the quadrature scheme together with the number N of function evaluations to reach the desired precision $\varepsilon_{\mathrm{rel}}$ and the results obtained through the proposed rapidly converging series together with the total number of addends $N_{\mathrm{TOT}}=N_{\mathrm{R}}+N_{\mathrm{J}}$ needed to reach the relative error $\varepsilon_{\mathrm{rel}}$ both in the real and imaginary parts of the integral. In the results, the correct digits are underlined. As it can be seen, on one hand the quadrature scheme may reach a certain accuracy which can be smaller than the prescribed tolerance [51], on the other hand the number of terms in the series can be very small, especially for smaller values of α.

In Fig. 2(a) we report the relative error between the proposed series formulation (74)–(75) and the quadrature scheme when the relative error and the precision has been set to $\varepsilon_{\mathrm{rel}}=10^{-8}$ for a wide range of α, from $\alpha=0.1$ to $\alpha=10$. As expected, the relative error

Table 1
Comparison between quadrature and series calculation for the integral $I(m,n,k,\alpha)$ with $m=n=3$ and $k=0$, where the correct digits are underlined.

α	ε_{rel}	Quadrature (20)	N	Series (74)–(75)	N_{TOT}
0.1	1.E−4	i0.142888990	332	i0.142888911	5
0.1	1.E−6	i0.142888910	1624	i0.142888911	5
0.1	1.E−8	i0.142888911	1816	i0.142888911	7
1	1.E−4	0.00002391 + i0.14628566	332	0.00002391 + i0.14628500	9
1	1.E−6	0.00002391 + i0.14628558	1624	0.00002391 + i0.14628558	12
1	1.E−8	0.00002391 + i0.14628558	1816	0.00002391 + i0.14628558	14
10	1.E−4	0.079307052 + i0.041829593	412	0.079306618 + i0.041829731	55
10	1.E−6	0.079307052 + i0.041829589	544	0.079307048 + i0.041829590	59
10	1.E−8	0.079307052 + i0.041829589	728	0.079307052 + i0.041829589	63



(a) (b)

Fig. 2. (a) Relative error of the proposed series formulation (74)–(75) as a function of α for the integral $I(3,3,0,\alpha)$ when the relative error in the sum of the series and in the calculation of the integral has been set to 10^{-8}; (b) relative error as a function of the total number of addends N for different values of α.

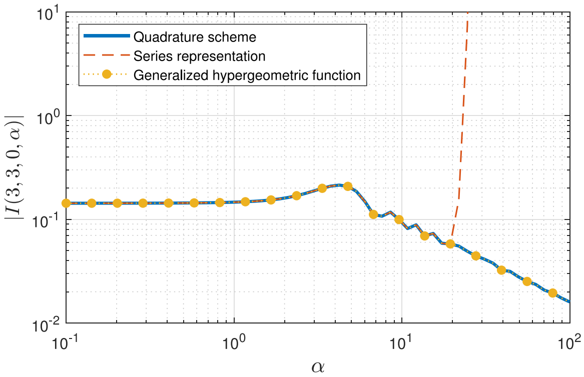
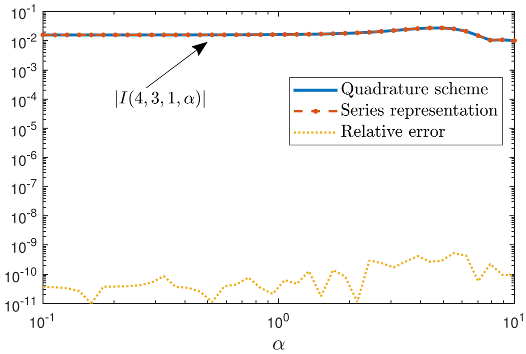


Fig. 3. Absolute value of $I(3,3,0,\alpha)$ as a function of α.

is always below ε_{rel}, but this allow us to compare the computation time required by the two formulations. In particular, to compare the efficiency of the algorithms we computed $I(3,3,0,\alpha)$ in 10^6 points: the quadrature scheme required more than 15 000 s, while the series representation requires about 4.3 s, thus showing the enormous computational advantage since the proposed formulation is more than 3000 faster (the simulations have been performed through the commercial software MATLAB with a processor 12th Gen Intel(R) Core(TM) i7-1255U 1.70 GHz). We have also compared our series formulation to the very specific fast converging finite proper integrals proposed in [20] (formula (11) with $n=0$, $k=1$, and $h=1$) obtaining a speed-up factor larger than 50.

In order to show the number of addends in the series necessary to achieve a given accuracy, in Fig. 2(b) we report the relative error ε_{rel} as a function of the number of addends N. The relative error is calculated with respect to the results obtained with a quadrature scheme computed with an accuracy 10^{-8}. At each step, the reported number of addends N takes into account an addend for the calculation of I_R and an addend for the calculation of I_J, so that N results to be an even number. In Fig. 2(b), different values of α are considered, i.e., $\alpha=0.1$, 1, 5, and 10. As it can be expected, by increasing α the number of addends N necessary for a given accuracy increases as well and, for small α, an acceptable error is obtained with only one addend for each



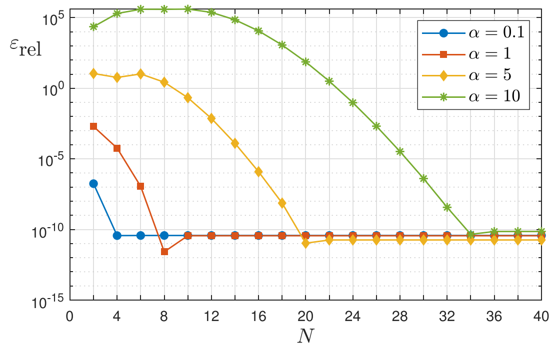


Fig. 4. (a) Absolute value of $I(4,3,1,\alpha)$ as a function of α and relevant relative error; (b) relative error for $I(4,3,1,\alpha)$ as a function of the total number of addends N for different values of α.

series. It should be noted that in electromagnetic scattering the parameter α corresponds to the normalized radius of the circular disk (or aperture) and usually values of interest range between 0.1 and 10 since for $\alpha < 0.1$ and $\alpha > 10$ low- and high-frequency approximations are often used.

One possible drawback of the series representation consists in the loss of accuracy due to catastrophic cancellation when α is large. In fact, α large implies a larger and larger number of addends to sum in the series and such terms can significantly grow before decreasing (as observed in Fig. 2(b)), thus generating overflow or numerical cancellation. For the case considered above, this happens for $\alpha > 20$, as it can be seen in Fig. 3 where the absolute value of $I(3,3,0,\alpha)$ is reported as a function of α for both the quadrature scheme and the series representation. However, the hypergeometric-function representation (44)–(59) does not suffer of these problems since, for the computation of ${}_pF_q$ functions, packaged routines do not rely exclusively on the series definition (41) [40]. The result of the hypergeometric-function representation (obtained with the MATLAB routine `hypergeom`) is also reported in Fig. 3, which is completely superimposed to the quadrature scheme results (although not shown the relative error is always smaller than 10^{-8}). Moreover, it should be taken into account that the closed-form expressions (44)–(59) can be further simplified for particular values of the involved parameters because, for instance, the order is reduced when an upper and a lower Pochhammer symbol are equal. For example, when $m = n$ and $k = 0$, from (44)–(59) we have

$$I_{\rm R}(m,m,0,\alpha) = \frac{(2\alpha)^{2m+1}(m!)^2}{\pi(2m+1)!\,[(2m+1)!!]^2}\,{}_2F_3\left(m+1,m+1;2m+2,m+\frac{3}{2},m+\frac{3}{2};-\alpha^2\right), \tag{92}$$

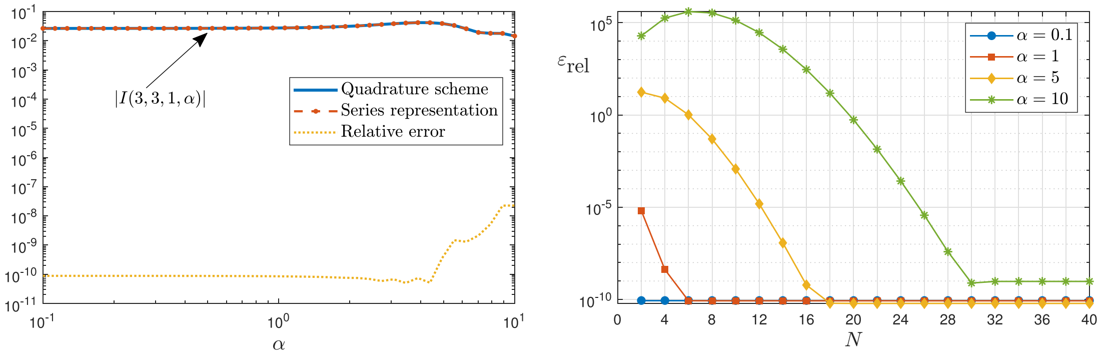
$$I_{\rm J}(m,m,0,\alpha) = \frac{(2m-1)!!}{(2m+1)!!}\,{}_2F_3\left(\frac{1}{2},\frac{1}{2};1,m+\frac{3}{2},-m+\frac{1}{2};-\alpha^2\right). \tag{93}$$

Although the closed-form formulas (44)–(59) do not present a significant computational saving with respect to the quadrature scheme, they do not require any need of develop an ad-hoc code (with all the relevant details illustrated in Section 2) and the packaged routines can be used without any specific knowledge from the user.

We next consider the integral $I(m,n,k,\alpha)$ in (1) with parameters $m = 4$, $n = 3$, $k = 1$, (*Case 2*, σ odd and k odd) and different values of α. In Fig. 4(a) we report the results for the integral obtained through the proposed series formulation (76)–(77) and the quadrature scheme when the relative error and the precision have been set to $\varepsilon_{\rm rel} = 10^{-8}$. Again, we have considered a range of α from $\alpha = 0.1$ to $\alpha = 10$ using 10^6 points. The relative error between the formulation is also reported for a fair comparison: as expected, the relative error is always below $\varepsilon_{\rm rel}$. The most important aspect, as in the previous case, is that the computation time of the quadrature scheme more than 11 300 s, while the series representation requires about 4.7 s, thus allowing for an enormous computational saving with a speed up larger than 2000. In Fig. 4(b) we report the relative error $\varepsilon_{\rm rel}$ as a function of the number of addends N used in the series representation as in Fig. 2(b). Different values of α are considered, i.e., $\alpha = 0.1$, 1, 5, and 10. Also in this case, by increasing α the number of addends N necessary for a given accuracy increases as well and, for small α, an acceptable error is obtained with only one addend for each series. Also for these integrals loss of accuracy due to catastrophic cancellation occurs for $\alpha > 20$, so that the hypergeometric-function representation (44)–(59), which does not suffer from this problem, is convenient.

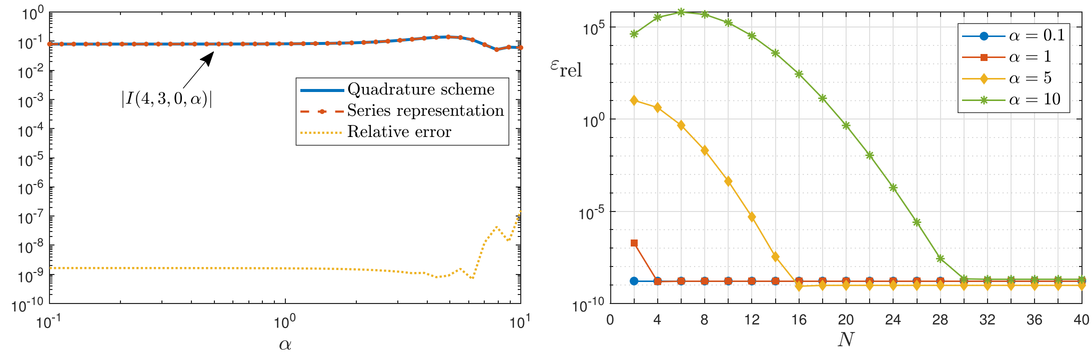
We next consider the integral $I(m,n,k,\alpha)$ in (1) with parameters $m = 3$, $n = 3$, $k = 1$, (*Case 3*, σ even and k odd). In Fig. 5(a) we report the results for the integral obtained through the proposed series formulation (78)–(81) and the quadrature scheme when the relative error and the precision have been set to $\varepsilon_{\rm rel} = 10^{-8}$: the relative error is also reported which is always below $\varepsilon_{\rm rel}$. To compute 10^6 points between $\alpha = 0.1$ and $\alpha = 10$ the quadrature scheme requires more than 13 000 s while the series representation requires about 6.7 s with a speed up larger than 2000. In Fig. 5(b) we also report the relative error $\varepsilon_{\rm rel}$ as a function of the number of addends N used in the series representation as in Fig. 2(b) for different values of α.

Again, loss of accuracy due to catastrophic cancellation occurs for $\alpha > 20$: however, in this case the closed form for $I_{\rm R}$ (44) in terms of a suitable generalized hypergeometric function and the closed form (69) for $I_{\rm J}$ in terms of a suitable Meijer G-function can be used. It should also be noted that, for sufficiently low precisions, asymptotic expressions can be used [56,62,63].



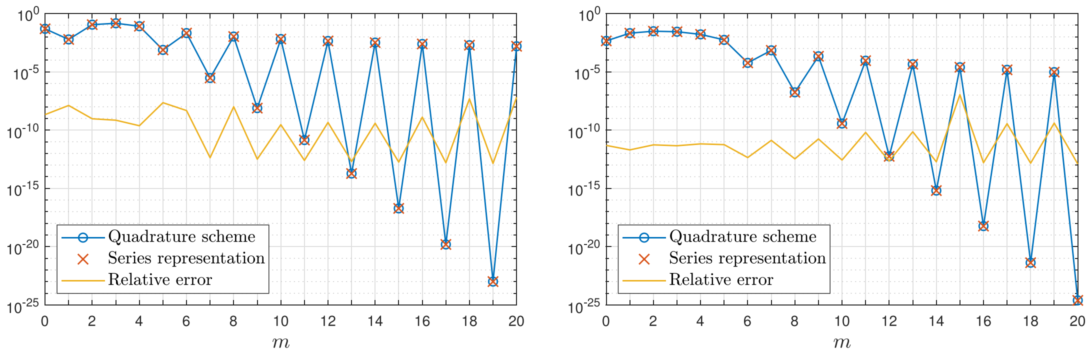
(a) (b)

Fig. 5. (a) Absolute value of $I(3,3,1,\alpha)$ as a function of α and relevant relative error; (b) relative error for $I(3,3,1,\alpha)$ as a function of the total number of addends N for different values of α.



(a) (b)

Fig. 6. (a) Absolute value of $I(4,3,0,\alpha)$ as a function of α and relevant relative error; (b) relative error for $I(4,3,0,\alpha)$ as a function of the total number of addends N for different values of α.



(a) (b)

Fig. 7. (a) Absolute value of $I(m,3,0,1)$ as a function of m and relevant relative error; (b) Absolute value of $I(m,3,1,1)$ as a function of m and relevant relative error.

This example also offers us the possibility to briefly clarify the role of the poles that lie on the left of the original Barnes integration path C_B [43]. In fact, in order to obtain an asymptotic representation, it is convenient to consider the integral along a closed rectangular path C_L in the counterclockwise direction, with the downward vertical line going from $c_1 + \mathrm{i}\infty$ to $c_1 - \mathrm{i}\infty$ (with $c_1 < c$). Such a contour C_L encloses the poles that lie between c_1 and c: it can be shown that the contributions along the horizontal line portions of the contour are negligible and the integral along the part of the path from $c_1 + \mathrm{i}\infty$ to $c_1 - \mathrm{i}\infty$ can be of higher order with respect to the poles contribution [63] so that the sum of the contribution of the enclosed poles (which are constituted by a sum of

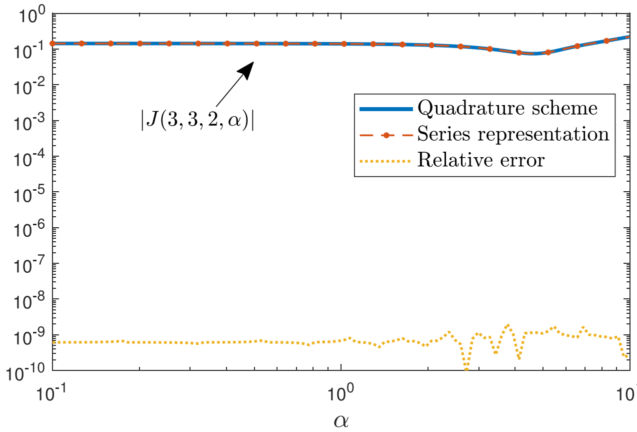


Fig. 8. Absolute value of $J(3,3,2,\alpha)$ as a function of α and relevant relative error.

terms of inverse powers of α) may represent an asymptotic expansion of the integral valid for large α. Just as an example, when considering the asymptotic evaluation of I_J (which allows a simpler evaluation with respect to I_R), by observing the poles and zeros in (49) and (50), it can be easily inferred that the poles $s_p^{(1)}$ and $s_r^{(3)}$ are always on the right of the original Barnes integration path C_B, the poles $s_t^{(4)}$ are canceled either by $z_q^{(2)}$ or $z_r^{(3)}$ and the only poles on the left of C_B are $s_q^{(2)} = -S - 1 - q$ for $q = 0, \ldots, S$, which are finite. Therefore, by choosing $c_1 < -2S - 1$ and applying the Residue theorem to the path C_L, from (48) we have

$$I_J(m,n,k,\alpha) \simeq \frac{1}{2\pi\alpha^{2K}} \sum_{q=0}^{S} \frac{\left(q+\frac{1}{2}\right)_D (-q+S+1)_{2q}}{(q+K)_{1-K}\left(q-D+\frac{1}{2}\right)_{K+D}} \frac{1}{\alpha^{2q}} \tag{94}$$

with $S = (m+n)/2$, $D = (m-n)/2$, and $K = (k+1)/2$, which usually gives an error less than 10^{-2} for $\alpha > 15$.

We then consider the fourth case for the integral $I(m,n,k,\alpha)$ in (1) with parameters $m = 4$, $n = 3$, $k = 0$, (*Case 4*, σ even and k odd) and different values of α. In Fig. 6(a) we report the results for the integral obtained through the proposed series formulation (82)–(84) and the quadrature scheme when the relative error and the precision have been set to $\varepsilon_{\text{rel}} = 10^{-8}$: the relative error is also reported which is almost always below ε_{rel}. To compute 10^6 points between $\alpha = 0.1$ and $\alpha = 10$ the quadrature scheme requires more than 12 500 s while the series representation requires about 6.7 s with a speed up about 2000. In Fig. 6(b) we also report the relative error ε_{rel} as a function of the number of addends N used in the series representation as in Fig. 2(b) for different values of α.

Also in this case, the loss of accuracy occurring for $\alpha > 20$ can be simply overcome with the use of the closed-form representations in terms of generalized hypergeometric and Meijer G-functions, i.e., (44) and (69), respectively (in MATLAB they are computed through the routine `meijerG`).

In Fig. 7 we also report the results for the integrals $I(m,3,0,1)$ and $I(m,3,1,1)$ by varying m. In particular, in Fig. 7(a) the results obtained through the proposed series formulation and the quadrature scheme when the relative error and the precision have been set to $\varepsilon_{\text{rel}} = 10^{-8}$ are reported for $I(m,3,0,1)$ together with the resulting actual relative error which is almost always below ε_{rel}, while in Fig. 7(b) the same is reported for $I(m,3,1,1)$. By varying m with $k = 0$ and $k = 1$ different combinations of evenness and oddness for σ and k are considered. While the series representation always converges very rapidly, increasing m (in particular, the difference $|m-n|$) requires suitable modifications of the quadrature scheme to obtain accurate and reliable results (if compared to the closed-form expressions in terms of generalized hypergeometric and Meijer G-functions).

Finally, we also address an example for the integral $J(m,n,k,\alpha)$ in (12) with parameters $m = 3$, $n = 3$, $k = 2$ and different values of α. In Fig. 8 we report the results for the integral obtained through the proposed series formulation according to (12) and the quadrature scheme when the relative error and the precision have been set to $\varepsilon_{\text{rel}} = 10^{-8}$: the relative error is also reported which is almost always below ε_{rel}. As in all the other cases, to compute 10^6 points between $\alpha = 0.1$ and $\alpha = 10$ the quadrature scheme requires about 14 900 s while the series representation takes about 13.4 s, thus being more than 1000 times faster.

5. Conclusion

Rapidly convergent series representations for a class of integrals involving the product of spherical Bessel functions of the first kind have been presented. The efficient evaluation of such integrals is required, e.g., for the solution of different acoustic and electromagnetic scattering problems from circular disks and apertures. However, the considered class of integrals is more general and it is shown that the series representations can be used to obtain closed-form expressions in terms of generalized hypergeometric functions and/or Meijer G-functions. A detailed numerical analysis is also provided to show the accuracy of the proposed solutions, which turns out to be more than 1000 times faster than the most efficient quadrature schemes, thus furnishing an enormous computational saving. The formulation can easily be extended to treat similar integrals involving Bessel functions of integer (or, more generally, real) order. The case where the Bessel functions have different arguments is left for future investigations.

Data availability

Data will be made available on request.

Acknowledgment

G. L. wishes to acknowledge helpful communications regarding this research with K. A. Michalski concerning the adopted numerical quadrature scheme.

Appendix. Derivation of $I_{\mathbf{J}}^{(2)}$

In this Appendix we give the details for the derivation of (66). The starting point is (62) together with (64). To evaluate the residue in the double poles we consider the following decomposition:

$$F(s) = \frac{1}{2\pi \mathrm{i}} \Gamma(-s) \Gamma(-s-L) G(s) \,, \tag{A.1}$$

where

$$G(s) = \frac{\Gamma\left(s+\frac{\sigma}{2}+1\right) \Gamma\left(s+\frac{\sigma+3}{2}\right)}{2\Gamma\left(s+\frac{\sigma+\delta+3}{2}\right) \Gamma\left(s+\frac{\sigma-\delta+3}{2}\right) \Gamma(s+\sigma+2) \Gamma\left(-s-L+\frac{1}{2}\right)} \alpha^{2s+2L} . \tag{A.2}$$

Since $s = s_p^{(1)} = p$ is a double pole, we have

$$\operatorname*{Res}_{s\to p} F(s) = \lim_{s\to p} \frac{\mathrm{d}}{\mathrm{d}s} \left[(s-p)^2 F(s)\right] = A_p B_p \,, \tag{A.3}$$

where

$$\begin{aligned} A_p &= \lim_{s\to p} (s-p)^2 F(s) \,, \\ B_p &= \lim_{s\to p} \left[\frac{2}{s-p} + \frac{F'(s)}{F(s)} \right] . \end{aligned} \tag{A.4}$$

Now

$$A_p = G(p) A_{p1} A_{p2} \,, \tag{A.5}$$

where

$$\begin{aligned} A_{p1} &= \lim_{s\to p} (s-p) \Gamma(-s) = \lim_{t\to 0} t\Gamma(-t-p) \,, \\ A_{p2} &= \lim_{s\to p} (s-p) \Gamma(-s-L) = \lim_{t\to 0} t\Gamma(-t-p-L) \,. \end{aligned} \tag{A.6}$$

Since

$$\Gamma(x-M) = \frac{\Gamma(x+1)}{x(x-1)\cdots(x-M)} \,, \tag{A.7}$$

it thus results

$$\begin{aligned} A_{p1} &= \lim_{t\to 0} t\Gamma(-t-p) = -\lim_{t\to 0} \frac{\Gamma(-t+1)}{(-t-1)\cdots(-t-p)} = \frac{(-1)^{p+1}}{p!} \,, \\ A_{p2} &= \lim_{s\to p} (s-p) \Gamma(-s-L) = -\lim_{t\to 0} \frac{\Gamma(-t+1)}{(t-1)\cdots(t-p-L)} = \frac{(-1)^{p+L+1}}{(p+L)!} \,, \end{aligned} \tag{A.8}$$

so that

$$A_p = G(p) A_{p1} A_{p2} = \frac{(-1)^L}{2\Gamma\left(p+\frac{\sigma+\delta+3}{2}\right) \Gamma\left(p+\frac{\sigma-\delta+3}{2}\right)} \frac{\Gamma\left(p+1+\frac{\sigma}{2}\right) \Gamma\left(p+1+\frac{\sigma+1}{2}\right)}{\Gamma(p+1) \Gamma(p+L+1) \Gamma(p+\sigma+2) \Gamma\left(-p-L+\frac{1}{2}\right)} \alpha^{2p+2L} , \tag{A.9}$$

i.e.,

$$A_p = \frac{(-1)^p \Gamma\left(p+1+\frac{\sigma}{2}\right) \Gamma\left(p+1+\frac{\sigma+1}{2}\right) \Gamma\left(p+L+\frac{1}{2}\right)}{2\pi \Gamma\left(p+\frac{\sigma+\delta+3}{2}\right) \Gamma\left(p+\frac{\sigma-\delta+3}{2}\right) \Gamma(p+1) \Gamma(p+L+1) \Gamma(p+\sigma+2)} \alpha^{2p+\sigma-k+1} . \tag{A.10}$$

On the other hand it also results

$$B_p = \lim_{s\to p} \left[\frac{2}{s-p} + \frac{F'(s)}{F(s)} \right] = \lim_{s\to p} \left[\frac{2}{s-p} + \frac{\Gamma'(-s)}{\Gamma(-s)} + \frac{\Gamma'(-s-L)}{\Gamma(-s-L)} + \frac{G'(s)}{G(s)} \right] , \tag{A.11}$$

so that we can express

$$B_p = B_{p1} + B_{p2} + \frac{\mathrm{d}}{\mathrm{d}s} \ln [G(s)]\Big|_{s=p} , \tag{A.12}$$

where

$$\begin{aligned} B_{p1} &= \lim_{s\to p} \left[\frac{1}{(s-p)} - \psi(-s) \right] , \\ B_{p2} &= \lim_{s\to p} \left[\frac{1}{(s-p)} - \psi(-s-L) \right] \end{aligned} \tag{A.13}$$

and the Digamma (or Psi) function $\psi(s)$ has been introduced in (67).

For the evaluation of B_{p1} let us consider the functional relation

$$\psi(x) = \psi(x+1) - \frac{1}{x} \tag{A.14}$$

from which we have

$$\begin{aligned} \psi(-s) &= \psi(-s+1) + \frac{1}{s} = -\frac{1}{-s} + \psi(-s+2) - \frac{1}{-s+1} = \cdots = \psi(-s+p+1) - \sum_{q=0}^{p} \frac{1}{-s+q} \\ &= \psi(-s+p+1) + \sum_{q=0}^{p} \frac{1}{s-q} \end{aligned} \tag{A.15}$$

so that

$$\begin{aligned} B_{p1} &= \lim_{s\to p} \left[\frac{1}{(s-p)} - \psi(-s) \right] = \lim_{s\to p} \left[\frac{1}{(s-p)} - \psi(-s+p+1) - \sum_{q=0}^{p} \frac{1}{s-q} \right] \\ &= -\psi(1) - \sum_{q=0}^{p-1} \frac{1}{p-q} = -\psi(1) - \sum_{h=1}^{p} \frac{1}{h} = \gamma - H_p , \end{aligned} \tag{A.16}$$

where H_p is the pth harmonic number (i.e., the sum of the reciprocals of the first p natural numbers).

In the same way

$$\begin{aligned} B_{p2} &= \lim_{s\to p} \left[\frac{1}{(s-p)} - \psi(-s-L) \right] = \lim_{s\to p} \left[\frac{1}{(s-p)} - \psi(-s-L+p+L+1) - \sum_{q=0}^{p+L} \frac{1}{s+L-q} \right] \\ &= -\psi(1) - \sum_{q=0}^{p+L-1} \frac{1}{p-q} = -\psi(1) - \sum_{h=1}^{p+L} \frac{1}{h} = \gamma - H_{p+L} . \end{aligned} \tag{A.17}$$

Now, the final step is the evaluation of the term

$$\frac{\mathrm{d}}{\mathrm{d}s} \ln [G(s)]\Big|_{s=p} . \tag{A.18}$$

First, from (A.2) we have

$$\begin{aligned} \ln [G(s)] &= \ln \frac{1}{2} + \ln \Gamma \left(s + \frac{\sigma}{2} + 1 \right) + \ln \Gamma \left(s + \frac{\sigma+3}{2} \right) - \ln \Gamma \left(s + \frac{\sigma+\delta+3}{2} \right) - \ln \Gamma \left(s + \frac{\sigma-\delta+3}{2} \right) \\ &\quad - \ln \Gamma (s+\sigma+2) - \ln \Gamma \left(-s - L + \frac{1}{2} \right) + (2s+2L) \ln \alpha . \end{aligned} \tag{A.19}$$

Therefore

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} \ln [G(s)]\Big|_{s=p} &= 2 \ln \alpha + \psi \left(p + \frac{\sigma}{2} + 1 \right) + \psi \left(p + \frac{\sigma+3}{2} \right) - \psi \left(p + \frac{\sigma+\delta+3}{2} \right) - \psi \left(p + \frac{\sigma-\delta+3}{2} \right) \\ &\quad - \psi (p+\sigma+2) + \psi \left(-p - L + \frac{1}{2} \right) \end{aligned} \tag{A.20}$$

so that from (A.12) we have

$$\begin{aligned} B_p = &2\gamma - H_p - H_{p+L} + 2 \ln \alpha + \psi \left(p + \frac{\sigma}{2} + 1 \right) + \psi \left(p + \frac{\sigma+3}{2} \right) - \psi \left(p + \frac{\sigma+\delta+3}{2} \right) - \psi \left(p + \frac{\sigma-\delta+3}{2} \right) \\ &- \psi (p+\sigma+2) + \psi \left(-p - L + \frac{1}{2} \right) . \end{aligned} \tag{A.21}$$

From (62), (A.3), (A.10), and (A.21), (66) is derived.

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