MIXING TIME AND LOCAL EXPONENTIAL ERGODICITY OF THE EAST-LIKE PROCESS IN $\mathbb{Z}^d$

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ABSTRACT. The East process, a well known reversible linear chain of spins, represents the prototype of a general class of interacting particle systems with constraints modeling the dynamics of real glasses. In this paper we consider a generalization of the East process living in the d-dimensional lattice and we establish new progresses on the *out-of-equilibrium* behavior. In particular we prove a form of (local) exponential ergodicity when the initial distribution is far from the stationary one and we prove that the mixing time in a finite box grows linearly in the side of the box.

1. INTRODUCTION

Kinetically constrained spin models (KCMs) are interacting 0-1 particle systems, on general graphs, which evolve with a simple Glauber dynamics described as follows. At every site x the system tries to update the occupancy variable (or spin) at x to the value 1 or 0 with probability p and $q = 1 - p$[1], respectively. However the update at x is accepted only if the *current* local configuration satisfies a certain constraint, hence the models are "kinetically constrained". It is always assumed that the constraint at site x does not depend on the spin at x and therefore the product Bernoulli(p) probability measure π is the reversible measure. Constraints may require, for example, that a certain *number* of the neighbouring spins are in state 0, or more restrictively, that certain *preassigned* neighbouring spins are in state 0 (e.g. the children of x when the underlying graph is a rooted tree).

The main interest in the physical literature for KCMs (see e.g. [23, 31] for a review) stems from the fact that they display many key dynamical features of real glassy materials: ergodicity breaking transition at some critical value q_c, huge relaxation time for q close to q_c, dynamic heterogeneity (*i.e.* non-trivial spatio-temporal fluctuations of the local relaxation to equilibrium) and aging, just to mention a few. Mathematically, despite their simple definition, KCMs pose very challenging and interesting problems because of the hardness of the constraint, with ramifications towards bootstrap percolation problems [35], combinatorics [12, 36], coalescence processes [16, 17] and random walks on upper triangular matrices [30]. Some of the mathematical tools developed for the analysis of the relaxation process of KCMs [5] have proved to be quite powerful also in other contexts such as card shuffling problems [3] and random evolution of surfaces [7].

Among the KCMs, the most studied model is the *East process*, a one-dimensional spin system that was introduced in the physics literature by Jäckle and Eisinger [24] (cf.

[1]In the physical applications $q \approx e^{-c\beta}$ at low temperature, where β is the inverse-temperature and c is a constant.

also [33, 34] and [15] for a recent mathematical review). In this case the base graph is $\mathbb{Z}$ (or finite connected subsets of it) and the constraint at $x \in \mathbb{Z}$ requires that the vertex $x-1$ is empty (*i.e.* its associated spin is 0).

It is the properties of the East process before and towards reaching equilibrium which are of interest, with the standard gauges for the speed of convergence to stationarity given by the relaxation time $T_{\rm rel}$ ($\equiv$ inverse spectral-gap) and the total-variation mixing time $T_{\rm mix}$ on a finite interval $\Lambda = \{0, \dots, L\}$ (cf. e.g. [27]), where we fix $\eta(0) = 0$ for ergodicity.

That $T_{\rm rel} = O(1)$[2] in L for any q small enough was first proved in [2] and, later on, for all $q \in (0,1]$ in [5] by different methods. Subsequently the analysis of the relaxation time in the physical relevant setting $q \searrow 0$ (corresponding to the low temperature limit) was developed to a high level of precision in [8, 9] where the relevant questions of dynamical heterogeneities and time scale separation have been rigorously settled. The fact that the relaxation time is $O(1)$ in L implies, in particular, that $T_{\rm mix} \sim L$ (cf. Theorem 5.1). It is then natural to ask whether the finite volume East process exhibits the *cutoff phenomenon* (coined by Aldous and Diaconis [1]; see also [2, 13, 14] and the references therein): over a negligible period of time (the cutoff window) the distance from equilibrium drops from near 1 to near 0. As is easily seen, the cutoff problem is strongly linked to the following *front progression* problem for the East process. For a initial configuration η with $\sup\{x : \eta_x = 0\} < \infty$, call this rightmost 0 its *front* and denote by $X(\eta)$ its position. It is easy to verify that at any later time the process starting from η will also have a front $X(\eta_t)$ and the key step to prove cutoff is a detailed analysis of the asymptotic law of $X(\eta_t)$ as $t \to \infty$. In [4] a kind of shape theorem was proved: as $t \to \infty$ the law of the process behind the front converges to an invariant measure ν and $\frac{1}{t}X(\eta_t) \to v_\infty$ in probability for a suitable constant $v_\infty > 0$. As a consequence $T_{\rm mix} \sim L/v_\infty$. To prove cutoff one has to go beyond the law of large numbers and to control the fluctuations of the front around the mean value $v_\infty t$. In [19] it was proved that the latters obey a CLT, and cutoff with the optimal window $\sim \sqrt{L}$ follows.

In several interesting contributions (cf. [20–22] and references therein) a natural generalization of the East dynamics to higher dimensions $d > 1$, in the sequel referred to as the *East-like* process, appears to play a key role in *realistic* models of glass formers. In $d = 2$ the East-like process evolves similarly to the East process but now the kinetic constraint requires that the South *or* West neighbor of the updating vertex contains *at least* one vacancy (*i.e.* a zero spin). In general $\eta(x)$ can flip if $\eta(x - e) = 0$ for some e in the canonical basis of $\mathbb{Z}^d$.

In [10, 11] the authors thoroughly analyzed the East-like process with emphasis on its low temperature (small q) behavior. Among the main results it was proved that the process on $\mathbb{Z}^d$ is ergodic with a finite relaxation time $T_{\rm rel}(\mathbb{Z}^d)$ and that, as $q \searrow 0$, $T_{\rm rel}(\mathbb{Z}^d) \asymp T_{\rm rel}(\mathbb{Z})^{1/d}$, correcting in this way some heuristic conjectures which appeared in the physical literature. In finite volume with ergodic boundary conditions (see Section 2.2 below) the asymptotics of the relaxation time as $q \searrow 0$ was also computed quite precisely and it was shown to depend very strongly on the choice of the boundary

[2] We recall that $f = O(g)$ means that there exists a constant $C > 0$ such that $|f| \leqslant C|g|$ and that $f = \Omega(g) \Leftrightarrow g = O(f)$.

conditions. Finally the asymptotics as $q \searrow 0$ of the *persistence times* (see Section 4.3 below) was also established but only up to a spatial scale $O(1/q^{1/d})$.

Other natural questions concerning the genuine *out-of-equilibrium* behavior of the East-like process, for example the convergence as $t \to \infty$ of the particle density to the equilibrium value p for a general class of initial laws or the asymptotics as $L \to \infty$ of the mixing time $T_{\rm mix}$ in a box of side L remained open.

In this paper we fill this gap in the spirit of similar results for the East process [6]. In Section 4 (cf. Theorem 4.3 and Corollary 4.4), using key results from Section 3, we prove a form of *local exponential ergodicity* together with the above mentioned convergence of the particle density. In the same section we also prove an exponential tail for the law of the persistence time of a given vertex (cf. Theorem 4.6), a key step for our final main result, namely that the mixing time in a finite box grows linearly in the side of the box (cf. Theorem 5.1).

2. The Model

2.1. **Notation.** For any $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ we denote its ℓ^1-norm by $\|x\|_1 = \sum_{i=1}^d |x_i|$. For any $x \in \mathbb{Z}^d$ we define the positive quadrant $\mathbb{Z}^d_{x,+}$ at x as the set $\{y \in \mathbb{Z}^d : \ y_i \geq x_i, \ i = 1, \dots, d\}$. When x is the orgin we will simply write $\mathbb{Z}^d_+$. We will also let $\mathbb{Z}^d_{x,\uparrow} \triangleq \mathbb{Z}^d_{x,+} \setminus \{x\}$ and $\mathbb{Z}^d_{x,\downarrow} \triangleq \{y \in \mathbb{Z}^d : \, x \in \mathbb{Z}^d_{y,\uparrow}\}$. The interpretation of these two sets is as follows: $\mathbb{Z}^d_{x,\uparrow}$ is the set of vertices other than x which are *influenced* by the spin at x, while $\mathbb{Z}^d_{x,\downarrow}$ are those vertices other than x which *influence* the spin at x.

We denote by $\mathcal{B} = \{e_1, e_2, \dots, e_d\}$ the canonical basis of $\mathbb{Z}^d$ and, given a set $X \subset \mathbb{Z}^d$, we define its East-boundary by

$$\partial_E X \triangleq \{y \in \mathbb{Z}^d \setminus X \, : \, x + e_i \in X \text{ for some } e_i\}\,.$$

Given $\Lambda \subset \mathbb{Z}^d$, we will denote by Ω_Λ the product space $\{0,1\}^\Lambda$ endowed with the product topology. If $\Lambda = \mathbb{Z}^d$ we simply write Ω. In the sequel we will refer to the vertices of Λ where a given configuration $\eta \in \Omega_\Lambda$ is equal to one (zero) as the *particles* (*vacancies*) of η. If $V \subset \Lambda$ and $\eta \in \Omega_\Lambda$ we will write $\eta\restriction_V$ for the restriction of η to V. In particular we will simply write $\eta\restriction_V = 1$ to mean that $\eta(x) = 1$ for all $x \in V$. Finally, for any $\Lambda \subset \mathbb{Z}^d$, a configuration $\sigma \in \Omega_{\partial_E \Lambda}$ will be referred to as a *boundary condition*.

2.2. **Finite volume process and boundary conditions.** Given a region $\Lambda \subset \mathbb{Z}^d$ and a configuration $\sigma \in \Omega_{\partial_E \Lambda}$, we define the *constraint* at site $x \in \Lambda$ with boundary condition σ, in the sequel denoted by $c_x^{\Lambda,\sigma}(\eta)$, as the indicator function of the event in Ω_Λ that there exists $e \in \mathcal{B}$ such that $(\sigma \cdot \eta)(x - e) = 0$, where $\sigma \cdot \eta \in \Omega_{\Lambda \cup \partial_E \Lambda}$ is the configuration equal to σ on $\partial_E \Lambda$ and η on Λ. Then the East-like process with parameter $q \in (0,1)$ and boundary configuration σ is the continuous time Markov chain with state space Ω_Λ and infinitesimal generator

$$\begin{aligned}\mathcal{L}^\sigma_\Lambda f(\eta) &= \sum_{x \in \Lambda} c_x^{\Lambda,\sigma}(\eta)\left[\eta_x q + (1-\eta_x)p\right] \cdot \left[f(\eta^x) - f(\eta)\right] \\ &= \sum_{x \in \Lambda} c_x^{\Lambda,\sigma}(\eta)\left[\pi_x(f) - f\right](\eta), \qquad (2.1)\end{aligned}$$

where $p = 1 - q$, η^x is the configuration in Ω_Λ obtained from η by flipping its value at x, and π_x is the Bernoulli(p) probability measure on the spin at x.

We call the boundary condition σ *ergodic* if the process given by (2.1) is ergodic. For example if $\Lambda = \prod_{i=1}^d [a_i, b_i]$ then a boundary condition σ is ergodic if and only if $\sigma(a-e) = 0$ for some $e \in \mathcal{B}$. If σ is such that by removing some single vacancy in σ one obtains a non-ergodic boundary condition then σ is said to be *minimal*. On the other hand if σ is identically equal to zero we call it a *maximal* boundary condition.

Since the local constraint $c_x^{\Lambda,\sigma}(\eta)$ does not depend on η_x, it is simple to check that the East-like process is reversible w.r.t. the product Bernoulli(p) probability measure $\pi_\Lambda = \prod_{x\in\Lambda} \pi_x$ on Ω_Λ. In what follows we will usually write π instead of π_Λ whenever no confusion can arise. Moreover, since our results are uniform in the choice of the ergodic boundary condition, we will usually drop the superscript σ in our notation and we will write $\mathbb{P}_\eta^\Lambda(\cdot)$ and $\mathbb{E}_\eta^\Lambda(\cdot)$ for the law and the associated expectation of the process with initial condition η. The only exception is when $\Lambda = \mathbb{Z}_+^d$ and the boundary condition σ is minimal, *i.e.* $\sigma(-e) = 0$ for some $e \in \mathcal{B}$ and $\sigma(x) = 1$ otherwise. In that case we will write $\mathbb{P}_\eta^{\min}(\cdot)$ and $\mathbb{E}_\eta^{\min}(\cdot)$.

2.3. **The infinite volume East-like process.** We also define the East process on the entire lattice $\mathbb{Z}^d$ as follows. Let $c_x(\eta) \triangleq \mathbb{1}(\exists\, e \in \mathcal{B}$ such that $\eta(x-e) = 0)$, be the constraint at x [3]. Then the East-like process on $\mathbb{Z}^d$ is the continuous time Markov process with state space Ω, with reversible measure given by the product Bernoulli(p) probability measure $\pi = \prod_{x\in\mathbb{Z}^d} \pi_x$ and infinitesimal generator $\mathcal{L}$ whose action on functions depending on finitely many spins is given by

$$\begin{aligned} \mathcal{L}f(\eta) &= \sum_{x\in\mathbb{Z}^d} c_x(\eta)\left[\eta_x q + (1-\eta_x)p\right] \cdot \left[f(\eta^x) - f(\eta)\right] \\ &= \sum_{x\in\mathbb{Z}^d} c_x(\eta)\big[\pi_x(f) - f\big](\eta)\,. \end{aligned} \tag{2.2}$$

We will denote by $\mathbb{P}_\eta(\cdot)$ and $\mathbb{E}_\eta(\cdot)$ the law and the associated expectation of the process started from η. In general, if the law of the initial configuration is ν, we will write $\mathbb{P}_\nu(\cdot)$, $\mathbb{E}_\nu(\cdot)$

2.4. **Spectral Gap.** In the sequel, the spectral gap of a reversible Markov generator $\mathcal{A}$ will be denoted by $\mathrm{gap}(\mathcal{A})$. It is defined as infimum over all non-constant functions f in the domain of $\mathcal{A}$ of the ratio between the Dirichlet form of f and its variance w.r.t. the reversible probability measure. In the finite dimensional case, if $\mathcal{A}$ is irreducible then $\mathrm{gap}(\mathcal{A}) > 0$. A positive spectral gap implies, in particular, that the reversible measure is mixing for the semigroup generated by $\mathcal{A}$ and that the variance contracts exponentially fast:

$$\mathrm{Var}(e^{t\mathcal{A}}f) \leqslant e^{-2t\,\mathrm{gap}(\mathcal{A})}\,\mathrm{Var}(f), \quad \forall\, f \in L^2\,. \tag{2.3}$$

We now recall some properties of the spectral gap for the East-like process which will be important in the following (for more details see [10, 11]). The infinite volume East-like process has a positive spectral gap for all $p \in (0,1)$. If $\Lambda \subset \mathbb{Z}^d$ then $\mathrm{gap}(\mathcal{L}_\Lambda^\sigma)$ is bounded from below by a positive constant uniformly in choice of the boundary condition σ

[3] We will adopt the notation $\mathbb{1}(A)$ for the indicator of the event A

among the ergodic ones. Moreover, if $\Lambda = \prod_{i=1}^d [1, \ell_i]$, then $\mathrm{gap}(\mathcal{L}_\Lambda^\sigma)$ is decreasing in $\{\ell_i\}_{i=1}^d$ *and* in the values of the boundary spins $\{\sigma(x)\}_{x\in\partial_E\Lambda}$.

2.5. **Graphical construction.** We recall a graphical construction of the Eastlike process on $\mathbb{Z}^d$, which will be very useful in the sequel. A similar construction, with slight modifications, holds also in the finite volume case. To each $x \in \mathbb{Z}^d$, we associate a rate one Poisson process and, independently, a family of independent Bernoulli(p) random variables $\{s_{x,\ell} : \ell \in \mathbb{N}\}$ ($\mathbb{N} := \{0, 1, \dots\}$). The occurrences of the Poisson process associated to x will be denoted by $\{t_{x,\ell} : \ell \in \mathbb{N}\}$, labelled in increasing order. We assume independence as x varies in $\mathbb{Z}^d$. Sometimes in the sequel we will refer to the collection $\{t_{x,\ell}, s_{x,\ell}\}_{\ell\in\mathbb{N}}$ as the *clock rings and coin tosses* associated to the vertex x. We write $(\Theta, \mathbb{P})$ for the probability space on which the above objects are defined. Notice that, $\mathbb{P}$-almost surely, all the occurrences $\{t_{x,\ell} : \ell \in \mathbb{N},\, x \in \mathbb{Z}^d\}$ are different (this property will be often used below without further mention).

Given a probability measure ν on Ω we consider the product probability measure $\mathbb{P}_\nu \triangleq \nu \otimes \mathbb{P}$ on the product space $\Omega \otimes \Theta$. Then on $(\Omega \otimes \Theta, \mathbb{P}_\nu)$ we can define a càdlàg Markov process $(\eta_t)_{t\geqslant 0}$ which is exactly the East-like process on $\mathbb{Z}^d$ given by (2.2) with initial distribution ν, as follows. The initial configuration η_0 associated to the element $(\eta, \vartheta) \in \Omega \otimes \Theta$ is given by η. At each time $t = t_{x,\ell}(\vartheta)$ the site x queries the state of its own constraint $c_x(\eta_{t-})$. If and only if the constraint is satisfied (*i.e.* $c_x(\eta_{t-}) = 1$), then t is called a *legal ring* and the configuration η_t is obtained from η_{t-} by resetting its value at site x to the value of the corresponding Bernoulli variable $s_{x,\ell}(\vartheta)$. Using the Harris's percolation argument [25] the above definition is well posed for $\mathbb{P}_\nu$ a.e. (η, ϑ). When $\nu = \delta_\eta$ we simply write $\mathbb{P}_\eta$ instead of $\mathbb{P}_\nu$[4].

Finally, for $\Lambda \subset \mathbb{Z}^d$ and $t > 0$, we let $\mathcal{F}_\Lambda$ ($\mathcal{F}_{\Lambda,t}$) be the σ-algebra generated by all clock rings and coin tosses associated to vertices in Λ (all the clock rings and coin tosses up to time t).

3. Local stationarity

In this section we prove two results showing a kind of local stationarity of the reversible measure π. In the first case the region where we want to prove stationarity of π is a non-random subset Λ of $\mathbb{Z}^d$. In the second case the set Λ will be random and determined by the dynamics itself in the graphical construction (the definition uses clock rings and coin tosses).

Proposition 3.1. *Let Λ be a finite subset of $\mathbb{Z}^d$ and assume that the initial distribution ν of the East-like process in $\mathbb{Z}^d$ is the product of its marginals on $\Omega_\Lambda, \Omega_{\Lambda^c}$ and that the marginal on Ω_Λ coincides with π_Λ. Then, for any $t > 0$ and any $\sigma \in \Omega_\Lambda$,*

$$\mathbb{P}_\nu(\eta_t\restriction_\Lambda = \sigma \mid \mathcal{F}_{\Lambda^c,t}) = \pi(\sigma).$$

[4] Although we have defined $\mathbb{P}_\eta, \mathbb{P}_\nu$ also for the law of the East–like process, it will be clear from the context when the notation is referred to the above graphical construction.

Proof. Let $x^* \in \Lambda$ be such that $\mathbb{Z}^d_{x^*,\uparrow} \cap \Lambda = \emptyset$, let $\Lambda^* \triangleq \Lambda \setminus \{x^*\}$ and let $\mathcal{F}^*_t \triangleq \mathcal{F}_{\mathbb{Z}^d \setminus \{x^*\},t}$. Clearly such a vertex always exists. Then

$$\begin{aligned}\mathbb{P}_\nu(\eta_t\restriction_\Lambda = \sigma \mid \mathcal{F}_{\Lambda^c,t}) &= \int d\nu(\eta)\mathbb{E}_\eta\left(\mathbb{1}(\eta_t\restriction_{\Lambda^*} = \sigma\restriction_{\Lambda^*})\mathbb{P}_\eta\big(\eta_t(x^*) = \sigma(x^*) \mid \mathcal{F}^*_t\big) \mid \mathcal{F}_{\Lambda^c,t}\right) \\ &= \int d\nu(\eta)\mathbb{E}_\eta\Big(\mathbb{1}(\eta_t\restriction_{\Lambda^*} = \sigma\restriction_{\Lambda^*}) \sum_{\xi\in\{0,1\}} \pi(\eta(x^*) = \xi)\mathbb{P}_{\eta,\xi}\big(\eta_t(x^*) = \sigma(x^*) \mid \mathcal{F}^*_t\big) \mid \mathcal{F}_{\Lambda^c,t}\Big),\end{aligned}$$

where the notation $\mathbb{P}_{\eta,\xi}(\cdot)$ indicates that the initial configuration is equal to η outside x^* and equal to ξ at x^*. Above we used the definition of x^* to guarantee that, once the initial condition η is given, the event $\{\eta_t\restriction_{\Lambda^*} = \sigma\restriction_{\Lambda^*}\}$ is measurable w.r.t. $\mathcal{F}^*_t$. Also, by the same reason, the event does not depend on the value $\eta(x^*)$. Finally, we have used the assumption on ν to perform a partial average over the initial value of $\eta(x^*)$.

We now claim that

$$\sum_{\xi\in\{0,1\}} \pi(\eta(x^*) = \xi)\mathbb{P}_{\eta,\xi}(\eta_t(x^*) = \sigma(x^*) \mid \mathcal{F}^*_t) = \pi(\sigma(x^*)).$$

To prove the claim we condition on the event $\mathcal{A}_t$ that there has been at least one legal ring at x^* before time t. Notice that also $\mathcal{A}_t$ does not depend on the initial value $\eta(x^*)$. Thus

$$\begin{aligned}&\sum_{\xi\in\{0,1\}} \pi(\eta(x^*) = \xi)\mathbb{P}_{\eta,\xi}(\eta_t(x^*) = \sigma(x^*) \mid \mathcal{F}^*_t) \\ &\quad= \sum_{\xi\in\{0,1\}} \pi(\eta(x^*) = \xi)\mathbb{P}_{\eta,\xi}(\eta_t(x^*) = \sigma(x^*) \mid \mathcal{A}_t, \mathcal{F}^*_t)\mathbb{P}_\eta(\mathcal{A}_t \mid \mathcal{F}^*_t) \\ &\quad+ \sum_{\xi\in\{0,1\}} \pi(\eta(x^*) = \xi)\mathbb{P}_{\eta,\xi}(\eta_t(x^*) = \sigma(x^*) \mid \mathcal{A}^c_t, \mathcal{F}^*_t)\mathbb{P}_\eta(\mathcal{A}^c_t \mid \mathcal{F}^*_t)\end{aligned}$$

We now observe that, for any η,

$$(i) \quad \mathbb{P}_{\eta,\xi}(\eta_t(x^*) = \sigma(x^*) \mid \mathcal{A}_t, \mathcal{F}^*_t) = \pi(\sigma(x^*)),$$

because in this case $\eta_t(x^*)$ takes the value of the last coin toss at x^* before t, and

$$(ii) \quad \sum_{\xi\in\{0,1\}} \pi(\eta(x^*) = \xi)\mathbb{P}_{\eta,\xi}(\eta_t(x^*) = \sigma(x^*) \mid \mathcal{A}^c_t, \mathcal{F}^*_t) = \pi(\sigma(x^*)),$$

because $\eta_t(x^*) = \xi$ on the event $\mathcal{A}^c_t$. Hence the claim.

In conclusion

$$\begin{aligned}\mathbb{P}_\nu(\eta_t\restriction_\Lambda = \sigma \mid \mathcal{F}_{\Lambda^c,t}) &= \mathbb{P}_\nu(\eta_t\restriction_{\Lambda^*} = \sigma\restriction_{\Lambda^*} \mid \mathcal{F}_{\Lambda^c,t})\ \pi(\sigma(x^*)) \\ &= \mathbb{P}_\nu(\eta_t\restriction_{\Lambda^*} = \sigma\restriction_{\Lambda^*} \mid \mathcal{F}_{(\Lambda^*)^c,t})\ \pi(\sigma(x^*)).\end{aligned}$$

Since π is a product measure, the term $\mathbb{P}_\nu(\eta_t\restriction_{\Lambda^*} = \sigma\restriction_{\Lambda^*} \mid \mathcal{F}_{(\Lambda^*)^c,t})$ has the required form on the reduced set Λ^* and the proof follows by iteration. □

Remark 3.2. *In the sequel we will use the above proposition but only in the "easy" case in which the set Λ has empty intersection with $\mathbb{Z}^d_{x,\downarrow}$ for any $x \in \partial_E\Lambda$ (e.g. Λ is given by a box, $\mathbb{Z}^d_+$ or some $\mathbb{Z}^d_{y,\uparrow}$). In this case, under the same assumptions of Proposition 3.1, the dynamics in $\partial_E\Lambda$ never queries the state of the dynamics in Λ. Moreover, calling*

$t_1 < t_2 < \cdots < t_n$ *the times at which a spin of* $\partial_E\Lambda$ *flips in the time window* $[0,t]$ *and setting* $t_0 \equiv 0$, $t_{n+1} \equiv t$, *for any* $i = 0,1,\dots,n$ *in the time interval* $[t_i, t_{i+1})$ *the projection on* Λ *of the East–like process equals a.s. the East–like process on* Λ *with fixed boundary condition* $\eta\restriction_{\partial_E\Lambda}(t_i)$.

We now extend Proposition 3.1 to a case in which the set Λ is itself random. Given a realization of the clock rings and coin tosses at all vertices and an initial configuration η such that $\eta(0) = 0$, let $\tau_0 = 0$, $z^{(0)} = 0$ and define

$$\tau_{k+1} = \inf\{s > \tau_k : \text{ at time } s \text{ there is a legal ring at } z^{(k)}\},$$
$$z^{(k+1)} = \min\{x \in \partial_E\{z^{(k)}\} : \ \eta_{\tau_{k+1}^-}(x) = 0\},$$

where the minimum is taken w.r.t. the lexicographic order. Notice that $\eta_t(z^{(k)}) = 0$ for any k, $t \in [\tau_k, \tau_{k+1})$ a.s. We will refer to this special vacancy as the *distinguished zero* (cf. [2] and [15] for an analogous definition for the one dimensional East process).

Definition 3.3. *We define the* trace *of the distinguished zero up to time* t *as the set*

$$\Gamma_t \triangleq \left\{z^{(0)}, z^{(1)}, \dots, z^{(\mathcal{N}_t - 1)}\right\},$$

where $\mathcal{N}_t \triangleq \max\{k \geqslant 0 \, : \, \tau_k \leqslant t\}$ *and we use the convention that* $\Gamma_t = \emptyset$ *if* $\mathcal{N}_t = 0$.

The above definition is well posed $\mathbb{P}_\eta$–a.s. since $\lim_{k\to\infty} \tau_k = +\infty$ $\mathbb{P}_\eta$–a.s. Let also $\mathcal{G}_0^{(1)}$ be the σ-algebra containing all information "below" the origin, up to the first legal ring τ_1. Formally $\mathcal{G}_0^{(1)}$ is generated by all events F such that $F \cap \{\tau_1 \leqslant s\} \in \mathcal{F}_{\mathbb{Z}^d_{0,\downarrow}, s}$ for any $s \geqslant 0$. Note that the event $\{z^{(1)} = z\}$ belongs to $\mathcal{G}_0^{(1)}$.

The above construction is of course valid for any initial vacancy of the initial configuration η. If the distinguished zero is initially at x then we will simply add a subscript "x" to the above notation.

We will now construct recursively the σ-algebras $\mathcal{G}_x^{(n)}$, $x \in \mathbb{Z}^d$, containing all the information on the first $n \geq 2$ steps of a distinguished zero initially at x and also on the clock rings and coin tosses "below" the successive positions of the distinguished zero and between consecutive jumps. In order to proceed more formally we first need the following definition.

Definition 3.4 (Time shift in Θ)**.** *Given* $s > 0$ *together with* $\omega \triangleq \{t_{y,j}, s_{y,j}\}_{y\in\mathbb{Z}^d, j\in\mathbb{N}} \in \Theta$, *let*

$$\theta_s\omega \triangleq \{t_{y,j+\nu_{y,s}}, s_{y,j+\nu_{y,s}}\}_{y\in\mathbb{Z}^d, j\in\mathbb{N}},$$

where $\nu_{y,s} \triangleq \min\{j : \ t_{y,j} \geq s\}$ *(recall that* $\mathbb{N} = \{0, 1, \dots\}$*). In other words the first ring and coin toss at* y *for* $\theta_s\omega$ *are the first ring and coin toss at* y after s *and so on.*

We then define recursively the family $\{\mathcal{G}_x^{(n)}\}_{n\geq 2}$ as follows. $\mathcal{G}_x^{(n)}$ is the σ-algebra generated by all events of the form

$$F^{(n)} = F^{(1)} \cap \{z^{(1)} = z\} \cap \{\theta_{\tau_1}\omega \in F^{(n-1)}\}, \quad z \in \partial_E\{x\}, \tag{3.1}$$

where $F^{(1)} \in \mathcal{G}_x^{(1)}$ and $F^{(n-1)} \in \mathcal{G}_z^{(n-1)}$.

We are finally ready to state our main result on the law of $\eta_t\restriction_{\Gamma_t}$.

Proposition 3.5. *For all $n \geq 1$, all η such that $\eta(0) = 0$ and all $t > 0$ the conditional distribution of $\eta_t\restriction_{\Gamma_t}$ given $\mathcal{G}_0^{(n)}$ and $\{\mathcal{N}_t = n\}$ coincides with the reversible measure π.*

Proof. Fix $n \geq 1$, $\sigma = (\sigma_0, \dots, \sigma_{n-1}) \in \{0,1\}^n$ and $z \in \partial_E\{0\}$. Let also $F^{(n)} \in \mathcal{G}_0^{(n)}$ be as in (3.1). It is enough to prove that[5]

$$\mathbb{P}_\eta(\mathcal{N}_t = n;\ F^{(n)};\ \eta_t\restriction_{\Gamma_t} = \sigma) = \pi(\sigma)\mathbb{P}_\eta(\mathcal{N}_t = n;\ F^{(n)}). \tag{3.2}$$

We will exploit (3.1) and use induction. Let $\hat\eta_{\tau_1}$ be the configuration in $\mathbb{Z}^d \setminus \{0\}$ equal to η_{τ_1} at all vertices different from the origin. Let also $\mathbb{P}_{\hat\eta_{\tau_1},\pi}(\cdot)$ be the law of the East-like process with initial condition equal to $\hat\eta_{\tau_1}$ outside the origin and sampled from π at the origin. The strong Markov property[6] w.r.t. the Markov time τ_1 together with (3.1) imply that

$$\begin{aligned}
&\mathbb{P}_\eta(\mathcal{N}_t = n;\, F^{(n)};\ \eta_t\restriction_{\Gamma_t} = \sigma) \\
&\quad = \mathbb{E}_\eta\Big[\mathbb{1}(F^{(1)} \cap \{z^{(1)} = z\})\mathbb{1}(\tau_1 < t)\ \times \qquad\qquad (3.3)\\
&\qquad\qquad \mathbb{P}_{\hat\eta_{\tau_1},\pi}\Big(\mathcal{N}_{z,t-\tau_1} = n-1;\ F^{(n-1)};\ \eta_{t-\tau_1}\restriction_{\Gamma_{z,t-\tau_1}} = \sigma';\ \eta_{t-\tau_1}(0) = \sigma_0\Big)\Big],
\end{aligned}$$

where $\sigma' = (\sigma_1, \dots, \sigma_{n-1})$ and we adopt the convention that the event $\{\eta_{t-\tau_1}\restriction_{\Gamma_{z,t-\tau_1}} = \sigma'\}$ is the sure event if $\Gamma_{z,t-\tau_1} = \emptyset$.

Observe now that, given $\hat\eta_{\tau_1}$, for all $s > 0$ the events $\{\mathcal{N}_{z,s} = n-1\}$, $F^{(n-1)}$ and $\{\eta_s\restriction_{\Gamma_{z,s}} = \sigma'\}$ are measurable w.r.t. $\mathcal{F}_{\mathbb{Z}^d_{0,\downarrow},s}$ while the event $\{\eta_s(0) = \sigma_0\}$ is measurable w.r.t. $\mathcal{F}_{\mathbb{Z}^d_{0,\downarrow}\cup\{0\},s}$. Therefore, using Proposition 3.1 applied to the set $\Lambda = \{0\}$,

$$\begin{aligned}
&\mathbb{P}_{\hat\eta_{\tau_1},\pi}\Big(\mathcal{N}_{z,s} = n-1;\ F^{(n-1)};\ \eta_s\restriction_{\Gamma_{z,s}} = \sigma';\ \eta_s(0) = \sigma_0\Big)\\
&= \mathbb{E}_{\hat\eta_\tau,\pi}\Big[\mathbb{1}(\mathcal{N}_{z,s} = 0)\mathbb{1}(F^{(n-1)})\mathbb{1}(\eta_s\restriction_{\Gamma_{z,s}} = \sigma')\mathbb{P}_{\hat\eta_\tau,\pi}(\eta_s(0) = \sigma_0 \mid \mathcal{F}_{\{0\}^c,s})\Big]\\
&\quad = \pi(\sigma_0)\mathbb{P}_{\hat\eta_\tau,\pi}(\mathcal{N}_{x,s} = 0;\ F^{(n-1)};\ \eta_s\restriction_{\Gamma_{z,s}} = \sigma').
\end{aligned} \tag{3.4}$$

If we now apply (3.4) to (3.3) we get

$$\begin{aligned}
&\mathbb{P}_\eta(\mathcal{N}_t = n;\ F^{(n)};\ \eta_t\restriction_{\Gamma_t} = \sigma)\\
&= \pi(\sigma_0)\mathbb{E}_\eta\Big[\mathbb{1}(F^{(1)} \cap \{z^{(1)} = z\})\mathbb{1}(\tau_1 < t)\mathbb{P}_{\hat\eta_\tau,\pi}(\mathcal{N}_{z,t-\tau_1} = n-1;\ F^{(n-1)};\ \eta_{t-\tau_1}\restriction_{\Gamma_{z,t-\tau_1}} = \sigma')\Big]
\end{aligned}$$

and (3.2) follows for $n = 1$. If $n \geq 2$ and we inductively assume (3.2) for $n-1$ to write $\mathbb{P}_{\hat\eta_\tau,\pi}(\mathcal{N}_{z,t-\tau_1} = n-1;\ F^{(n-1)};\ \eta_{t-\tau_1}\restriction_{\Gamma_{z,t-\tau_1}} = \sigma') = \pi(\sigma')\mathbb{P}_{\hat\eta_\tau,\pi}(\mathcal{N}_{z,t-\tau_1} = n-1;\ F^{(n-1)})$, we get

$$\begin{aligned}
&\mathbb{P}_\eta(\mathcal{N}_t = n;\ F^{(n)};\ \eta_t\restriction_{\Gamma_t} = \sigma)\\
&= \pi(\sigma)\mathbb{E}_\eta\Big[\,\mathbb{1}(F^{(1)} \cap \{z^{(1)} = z\})\mathbb{1}(\tau_1 < t)\mathbb{P}_{\hat\eta_\tau,\pi}(\mathcal{N}_{z,t-\tau_1} = n-1;\ F^{(n-1)})\,\Big]\\
&\quad = \pi(\sigma)\mathbb{P}_\eta\Big(\mathcal{N}_t = n;\ F^{(n)}\Big),
\end{aligned}$$

[5] We write $\mathbb{P}(A; B)$ instead of $\mathbb{P}(A \cap B)$ for shortness.

[6] Here we appeal to the strong Markov property of the underlying Poisson point process given by the clock rings.

i.e. (3.2) for n. □

4. Out-of-equilibrium results

In this section we begin by proving two results (cf. Lemma 4.1 and Corollary 4.2 below) showing that, given an initial vacancy at a site x, then at any given later time $t > 0$ it is very likely to find a vacancy in $\mathbb{Z}^d_{x,\downarrow} \cup \{x\}$ close to x. These results will be the keystone for the main outcome of this section (cf. Theorem 4.3), namely the fact that an initial vacancy is able to generate a wave of equilibrium (*i.e.* the reversible measure π) in front of itself. Finally, in Theorem 4.6 we will estimate the tail of the time needed to create a vacancy at a given site for the East–like process in the quadrant $\mathbb{Z}^d_+$.

4.1. Persistence of the vacancies.

Given $x \in \mathbb{Z}^d_+$ we say that x is of class $n \in \mathbb{N} = \{0, 1, \dots\}$ and write $x \in \mathcal{C}_n$ if $\min_i x_i \geq n$. Clearly $\mathcal{C}_n \subset \mathcal{C}_{n-1}$.

Lemma 4.1. *Let* $p_n \triangleq \sup_{x \in \mathcal{C}_n} \sup_{t \geq 0} \sup_{\eta:\, \eta(x)=0} \mathbb{P}_\eta(\eta_t\restriction_{\Lambda_x} = 1)$, *where* $\Lambda_x = \prod_{i=1}^d [0, x_i]$. *Then* $p_n \leqslant p^{n+1}$.

Proof. Let us fix $x \in \mathcal{C}_n$, $t > 0$ and an initial configuration η such that $\eta(x) = 0$ and recall Definition 3.3. If we make the initial vacancy at x "distinguished", then $\{\eta_t\restriction_{\Lambda_x} = 1\} \subset \{\mathcal{N}_t \geq n+1\} \cap \{\eta_t(z^{(k)}) = 1 \ \forall k = 0, 1, \dots, n\}$. Thus, using Proposition 3.5,

$$\mathbb{P}_\eta(\eta_t\restriction_{\Lambda_x} = 1) \leqslant p^{n+1} \mathbb{P}_\eta(\mathcal{N}_t \geq n+1) \leqslant p^{n+1}.$$

□

The next is a simple but useful consequence of the above result. Fix $\ell \geq 1$ and let $\mathcal{G}_t$ be the event that there exists a vertex $x \in V(\ell) \triangleq [-\ell + 1, 0]^d$ such that $\mathcal{T}_t(x) \geq t/\ell^d$, where

$$\mathcal{T}_t(x) \triangleq \int_0^t ds\, \mathbb{1}(\eta_s(x) = 0) \tag{4.1}$$

is the total time spent in the zero state by the spin at x up to time t.

Corollary 4.2. *There exist positive constants* C, c *such that*

$$\sup_{\eta:\ \eta(0)=0} \mathbb{P}_\eta(\mathcal{G}_t) \geq 1 - Ct\,\ell^d e^{-c\ell}, \quad \forall \ell \geq 1.$$

Proof. Observe that, if the box $V(\ell)$ was never completely filled during the time-lag t, then necessarily the event $\mathcal{G}_t$ occurred. Thus it is sufficient to prove that

$$\sup_{\eta:\ \eta(0)=0} \mathbb{P}_\eta(\exists\, s \leqslant t:\ \eta_s\restriction_{V(\ell)} = 1) \leqslant Ct\ell^d e^{-c\ell},$$

for some constants $C, c > 0$. Furthermore, using a union bound over the possible rings in $V(\ell)$ within time t (cf. the discussion after equation (5.14) in [15]) it suffices to prove that

$$\sup_{\eta:\ \eta(0)=0} \sup_{s>0} \mathbb{P}_\eta(\eta_s\restriction_{V(\ell)} = 1) \leqslant e^{-c\ell}$$

for some $c > 0$. Such a bound follows from Lemma 4.1 with $c = -\log(p)$. □

4.2. Local exponential ergodicity in $\mathbb{Z}^d$.

Theorem 4.3. *There exist two positive constants C, c such that the following holds. Fix $t > 0$ and let f be a function depending only on the spins in $[1, t^{1/2d}]^d$ with $\|f\|_\infty = 1$. Then*

$$\sup_{\eta:\ \eta(0)=0} |\mathbb{E}_\eta(f(\eta_t)) - \pi(f)| \leqslant Ce^{-ct^{1/2d}}.$$

Corollary 4.4. *Let ν be a probability measure on $\Omega = \{0,1\}^{\mathbb{Z}^d}$ such that, for all $\ell \geq 1$ and all $x \in \mathbb{Z}^d$,*

$$\nu\left(\{\eta(y) = 1\ \forall y \in [x, x+\ell-1]^d\}\right) \leqslant e^{-m\ell},$$

for some positive m. Then there exist $\lambda = \lambda(m) > 0$ and $C > 0$ such that

$$\sup_x |\mathbb{E}_\nu\big(\eta_t(x)\big) - p| \leqslant Ce^{-\lambda t^{1/2d}}.$$

Proof of the Corollary. Without loss of generality consider only the case $x = 0$ and, for $\ell \geq 1$, write

$$|\mathbb{E}_\nu\left(\eta_t(0)\right) - p| \leqslant 2e^{-m\ell} + \int_{\{\eta:\ \exists x\in[-\ell,-1]^d:\ \eta(x)=0\}} d\nu(\eta)|\mathbb{E}_\eta\big(\eta_t(0)\big) - p|.$$

Using Theorem 4.3 the second term in the r.h.s. above is smaller than $Ce^{-ct^{1/2d}}$ for any $\ell \leqslant t^{1/2d}$. Hence the thesis. □

Remark 4.5. *A similar result holds if one replaces the spin at x with an arbitrary function f depending on finitely many spins. In that case the constant C will depend on f through $\|f\|_\infty$ and the size of the support of f while the constant λ will stay the same.*

Proof of Theorem 4.3. In what follows c will denote a generic constant depending on p which may vary from line to line. Fix f as in the theorem and assume for simplicity that $\pi(f) = 0$. Given η such that $\eta(0) = 0$, we use Corollary 4.2 with $\ell \equiv \lceil \delta t^{1/2d} \rceil$ for some small positive constant δ together with $\|f\|_\infty = 1$ to write

$$\mathbb{E}_\eta(f(\eta_t)) = \mathbb{E}_\eta(f(\eta_t)\mathbb{1}(\mathcal{G}_t)) + O(e^{-c\delta t^{1/2d}}),$$

where $\mathcal{G}_t$ is the event that there exists $x \in [-\ell+1, 0]^d$ with $\mathcal{T}_t(x) \geq t/\ell^d$. On the event $\mathcal{G}_t$ let $\xi = (\xi_1, \dots, \xi_d)$ be the position of the smallest (in the lexicographical order) vertex of $V(\ell) = [-\ell+1, 0]^d$ with the property that $\mathcal{T}_t(\xi) \geq t/\ell^d$ and let Λ_ξ be the box $[\xi_1 + 1, t^{1/2d}] \times \prod_{i=2}^d [\xi_i, t^{1/2d}]$ (see Figure 1). We then write

$$\mathbb{E}_\eta(f(\eta_t)\mathbb{1}(\mathcal{G}_t)) = \sum_{y\in V(\ell)} \mathbb{E}_\eta\left(\mathbb{1}(\hat{\mathcal{G}}_{t,y})\mathbb{E}_\eta(f(\eta_t) \,|\, \mathcal{F}_{\Lambda_y^c,t})\right),$$

where $\hat{\mathcal{G}}_{t,y} \triangleq \mathcal{G}_t \cap \{\xi = y\}$. Notice that, given η, the event $\hat{\mathcal{G}}_{t,y}$ is measurable w.r.t. $\mathcal{F}_{\Lambda_y^c,t}$. Clearly, for each $y \in V(\ell)$, one has

$$\begin{aligned} &|\mathbb{E}_\eta\left(\mathbb{1}(\hat{\mathcal{G}}_{t,y})\mathbb{E}_\eta(f(\eta_t) \,|\, \mathcal{F}_{\Lambda_y^c,t})\right)| \\ &\leqslant \frac{1}{\min_{\sigma\in\Omega_{\Lambda_y}} \pi(\sigma)} \mathbb{E}_\eta\Big(\mathbb{1}(\hat{\mathcal{G}}_{t,y}) \sum_{\sigma\in\Omega_{\Lambda_y}} \pi(\sigma)|\mathbb{E}_{\sigma\cdot\eta}(f(\sigma_t) \mid \mathcal{F}_{\Lambda_y^c,t})|\Big), \end{aligned}$$

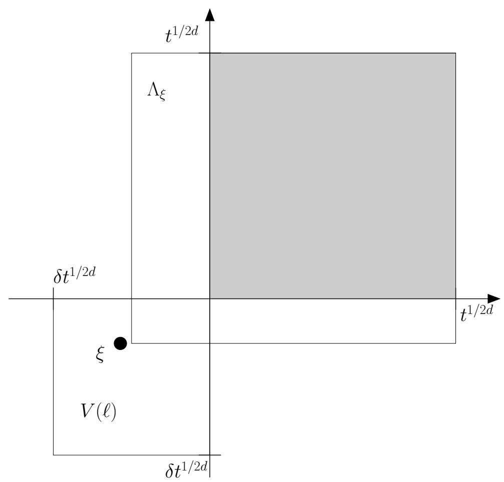

FIGURE 1. The box Λ_ξ

where $\sigma \cdot \eta$ means the configuration equal to σ in Λ_y and to η outside it. Above $\sigma_t \triangleq (\sigma \cdot \eta)_t \restriction_{\Lambda_y}$ and we used the fact that f depends only on the spins in Λ_y. We now observe that, given $\mathcal{F}_{\Lambda_y^c,t}$, the evolution in Λ_y up to time t is the standard East-like process with boundary conditions that vary at times say $t_1, t_2, \dots, t_n$. Moreover, once the initial η is given, the times $\{t_i\}_{i=1}^n$ and the actual value of the boundary conditions $\{\eta_{t_i}(z)\}_{z \in \partial_E \Lambda_y}$ become measurable w.r.t. $\mathcal{F}_{\Lambda_y^c}$. Call $\mathcal{L}^{(i)}$, $i = 0, \dots, n$, the generator of the East-like process in Λ with boundary conditions given by η_{t_i} (we set $t_0 \equiv 0$ and $t_{n+1} \equiv t$). If $\eta_{t_i}(y) = 0$, then η_{t_i} is an ergodic boundary condition and the spectral gap of $\mathcal{L}^{(i)}$ is not smaller than the spectral gap λ in the positive quadrant with minimal boundary conditions [10]. Thus, using the Cauchy-Schwarz inequality,

$$
\begin{aligned}
\sum_{\sigma \in \Omega_{\Lambda_y}} \pi(\sigma) |\mathbb{E}_{\sigma \cdot \eta}(f(\sigma_t) \mid \mathcal{F}_{\Lambda_y^c,t})| &= \sum_{\sigma \in \Omega_{\Lambda_y}} \pi(\sigma) \big| e^{t_1 \mathcal{L}^{(0)}} e^{(t_2 - t_1)\mathcal{L}^{(1)}} \dots e^{(t - t_n)\mathcal{L}^{(n)}} f(\sigma) \big| \\
&\leqslant \| e^{t_1 \mathcal{L}^{(0)}} e^{(t_2 - t_1)\mathcal{L}^{(1)}} \dots e^{(t - t_n)\mathcal{L}^{(n)}} f \|_\pi \\
&\leqslant \exp\big[-\lambda \sum_{i=0}^n (t_{i+1} - t_i) \mathbb{1}(\eta_{t_i}(y) = 0) \big]. \qquad (4.2)
\end{aligned}
$$

Above $\| \cdot \|_\pi$ denotes the norm in $L^2(\Lambda_y, \pi)$. Note that in the last step we applied the classical inequality $\mathrm{Var}(e^{t\mathcal{L}} f) \leqslant e^{-2\mathrm{gap}(\mathcal{L})} \mathrm{Var}(f)$, valid for any reversible continuous time Markov chain, to the chains with generators $\{\mathcal{L}^{(i)}\}_{i=0}^n$, together with the fact that $\pi\left(f^{(i)}\right) = 0$ for all functions

$$
f^{(i)} \triangleq e^{(t_i - t_{i-1})\mathcal{L}^{(i)}} \dots e^{(t - t_n)\mathcal{L}^{(n)}} f \, .
$$

This last property can be proved by induction on i from n to 0 using that $\pi(f) = 0$ and that $\mathcal{L}^{(i)}$ is reversible (hence stationary) in $L^2(\Lambda_y, \pi)$.

We now observe that $\sum_{i=0}^n (t_{i+1} - t_i) \mathbb{1}(\eta_{t_i}(y) = 0) = \mathcal{T}_t(y)$, the total time spent in state zero by the spin at y. By construction the latter is at least $t/\ell^d \geq \delta^{-d}\sqrt{t}/2$ for t

large enough. In conclusion

$$\frac{1}{\min_{\sigma\in\Omega_{\Lambda_y}}\pi(\sigma)}\mathbb{E}_\eta\Big(\mathbb{1}(\hat{\mathcal{G}}_{t,y})\sum_{\sigma\in\Omega_{\Lambda_y}}\pi(\sigma)|\mathbb{E}_{\sigma\cdot\eta}(f(\sigma_t)\mid\mathcal{F}_{\Lambda_y^c,t})|\Big)$$
$$\leqslant\Big(\frac{1}{p\wedge q}\Big)^{|\Lambda_y|}e^{-\lambda\delta^{-d}\sqrt{t}/2}=O(e^{-c\sqrt{t}})$$

for δ small enough and t large enough. Thus

$$\big|\mathbb{E}_\eta(f(\eta_t)\mathbb{1}(\mathcal{G}_t))\big|\leqslant C\ell^d\,e^{-c\sqrt{t}}$$

for some constant C and the result follows. □

4.3. **Exponential tail of the persistence times in $\mathbb{Z}_+^d$.** Consider the East-like process in $\mathbb{Z}_+^d$ with minimal boundary condition, and let $\mathbb{P}_\eta^{\min}(\cdot)$ denote its law when the initial configuration is η. Recall also definition (4.1) of the random variable $\mathcal{T}_t(x)$.

Theorem 4.6. *There exist $\kappa,\lambda>0$ and $\delta\in(0,1)$ such that the following holds. For all $x\in\mathbb{Z}_+^d$ and $t\geq\kappa\|x\|_1$,*

$$\sup_\eta\mathbb{P}_\eta^{\min}(\mathcal{T}_t(x)\leqslant\delta^d t)\leqslant de^{-\lambda\delta t}. \tag{4.3}$$

In particular

$$\sup_\eta\mathbb{P}_\eta^{\min}(\tau_x\geq t)\leqslant de^{-\lambda\delta t}, \tag{4.4}$$

where $\tau_x\triangleq\inf\{t:\ \eta_t(x)=0\}$.

Remark 4.7. *For simplicity we have stated the result with minimal boundary conditions. Actually the same proof, with minor modifications, holds for any ergodic boundary conditions with uniform constants κ,λ,δ.*

Proof. It is obvious that (4.4) follows from (4.3). To prove the latter we first need a technical lemma. Let $x\in\mathbb{Z}_+^d$ and write $x=(x^*,x_d)$ where $x^*=(x_1,\dots,x_{d-1})\in\mathbb{Z}_+^{d-1}$. Let $I=\{y=(x^*,j),\ j=1,\dots x_d\}$ if $x_d>0$ and $I=\emptyset$ otherwise (see Figure 2). For

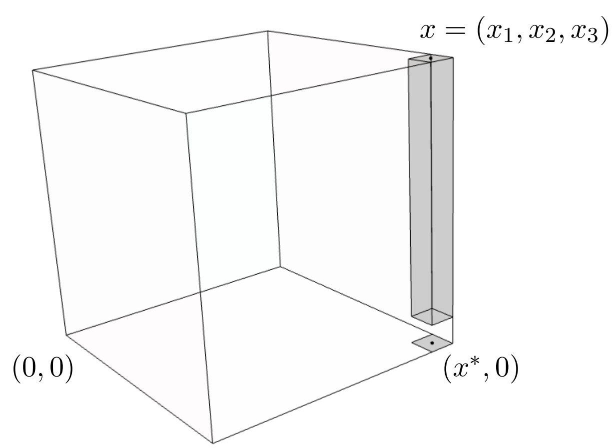

FIGURE 2. The interval I and the point x^* in three dimensions.

the East-like process in $\mathbb{Z}_+^d$ with ergodic boundary conditions and initial configuration η, the variable $\mathcal{T}_t((x^*,0))$ is measurable w.r.t. $\mathcal{F}_{I^c,t}$. Moreover, since the projection on the slab $\mathbb{Z}_+^{d-1}\times\{0\}$ of the East-like process in $\mathbb{Z}_+^d$ with minimal boundary conditions coincides with the East-like process on $\mathbb{Z}_+^{d-1}$ with minimal boundary conditions, the law of $\mathcal{T}_t((x^*,0))$ under $\mathbb{P}_\eta^{\min}$ coincides with the law of $\mathcal{T}_t(x^*)$ in $(d-1)$ dimensions. In the sequel, for notation convenience, we will simply write $\mathcal{T}_t(x^*)$ instead of the more precise $\mathcal{T}_t((x^*,0))$.

Lemma 4.8. *There exist constants $\delta\in(0,1)$ and $c>0$ such that*

$$\sup_\eta \mathbb{P}_\eta^{\min}(\mathcal{T}_t(x)\leqslant\delta\mathcal{T}_t(x^*)\,|\,\mathcal{F}_{I^c,t})\leqslant(p\wedge q)^{-x_d}\;e^{-c\mathcal{T}_t(x^*)}.$$

We postpone the proof of the lemma and write

$$\begin{aligned}\sup_\eta \mathbb{P}_\eta^{\min}\Big(\mathcal{T}_t(x)\leqslant\delta^d t\Big)\;&\leqslant\;\sup_\eta \mathbb{P}_\eta^{\min}\Big(\mathcal{T}_t(x^*)\leqslant\delta^{d-1}t\Big)\\&\quad+\sup_\eta \mathbb{P}_\eta^{\min}\Big(\mathcal{T}_t(x)\leqslant\delta^d t\,;\,\mathcal{T}_t(x^*)\geq\delta^{d-1}t\Big)\\&\leqslant\;\sup_\eta \mathbb{P}_\eta^{\min}\Big(\mathcal{T}_t(x^*)\leqslant\delta^{d-1}t\Big)+(p\wedge q)^{-x_d}\;\;e^{-c\delta^{d-1}t},\end{aligned}$$

where we used Lemma 4.8 to bound the second term in the r.h.s. of the first inequality. Notice that the first term in the r.h.s above has the same form of the starting quantity but now in $(d-1)$ dimensions. We can then iterate on the dimension d to get

$$\sup_\eta \mathbb{P}_\eta^{\min}\Big(\mathcal{T}_t(x)\leqslant\delta^d t\Big)\;\leqslant\;\sup_\eta \mathbb{P}_\eta^{\mathrm{East}}\left(\mathcal{T}_t(x_1)\leqslant\delta t\right)+\sum_{i=2}^{d}(p\wedge q)^{-x_i}\;e^{-c\delta^{i-1}t},$$

where $\mathbb{P}_\eta^{\mathrm{East}}(\cdot)$ is the law of the one dimensional East process on the interval $[0,x_1]$ with ergodic boundary conditions. As in [5, Theorem 3.6], the first term in the r.h.s. above can be bounded by

$$\sup_\eta \mathbb{P}_\eta^{\mathrm{East}}(\mathcal{T}_t(x_1)\leqslant\delta t)\leqslant(p\wedge q)^{-x_1}\,\mathbb{P}_\pi^{\mathrm{East}}(\mathcal{T}_t(x_1)\leqslant\delta t)\leqslant(p\wedge q)^{-x_1}\;e^{-c'\delta t},$$

for some constant $c'>0$ provided that δ is small enough (independent of t). The proof of (4.3) is finished by choosing e.g. $\kappa=2\log\left(1/(p\wedge q)\right)/(c\wedge c')\delta$ and $\lambda=(c\wedge c')/2$. ☐

Proof of Lemma 4.8. Using the exponential Chebyshev inequality we get

$$\begin{aligned}&\sup_\eta \mathbb{P}_\eta^{\min}\Big(\mathcal{T}_t(x)\leqslant\delta\mathcal{T}_t(x^*)\,|\,\mathcal{F}_{I^c,t}\Big)\\&\qquad\leqslant\;\sup_\eta\inf_{\gamma>0}e^{-\gamma(t-\delta\mathcal{T}_t(x^*))}\;\mathbb{E}_\eta^{\min}\left(e^{\gamma(t-\mathcal{T}_t(x))}\,|\,\mathcal{F}_{I^c,t}\right)\\&\qquad\leqslant\;(1/p\wedge q)^{x_d}\sup_\eta\inf_{\gamma>0}e^{-\gamma(t-\delta\mathcal{T}_t(x^*))}\;\mathbb{E}_{\eta,\pi}^{\min}\left(e^{\gamma(t-\mathcal{T}_t(x))}\,|\,\mathcal{F}_{I^c,t}\right),\end{aligned}$$

where $\mathbb{E}_{\eta,\pi}^{\min}(\cdot)$ denotes the expectation w.r.t. the East-like process with initial law $\nu(\eta')=\pi(\eta'\restriction_I)\mathbb{1}(\eta'\restriction_{\mathbb{Z}_+^d\setminus I}=\eta\restriction_{\mathbb{Z}_+^d\setminus I})$.

We will now bound the term $\mathbb{E}_{\eta,\pi}^{\min}\left(e^{\gamma(t-\mathcal{T}_t(x))}\,|\,\mathcal{F}_{I^c,t}\right)=\mathbb{E}_{\eta,\pi}^{\min}\left(e^{\gamma\int_0^t ds\,\mathbb{1}(\eta_s(x)=1)}\,|\,\mathcal{F}_{I^c,t}\right)$ using the Feynman-Kac formula. Firstly notice that it is enough to compute the above expectation w.r.t. the process in the box $\Lambda_x = \prod_{i=1}^d [0, x_i]$. Secondly, since the East boundary of any vertex in $\Lambda_x \setminus I$ does not intersect I and since the initial configuration in $\Lambda_x \setminus I$ is deterministic, the projection of the process in $\Lambda_x \setminus I$ is measurable w.r.t. $\mathcal{F}_{I^c,t}$. Denote by $0 < t_1 < t_2, \dots, < t_n < t$ the successive times in $[0, t]$ at which one of the spins at the East boundary of the interval I changes. During any time interval of the form $[t_i, t_{i+1})$ (define $t_0 \equiv 0$ and $t_{n+1} \equiv t$) the process in I is nothing but the usual one dimensional East process with possibly certain vertices which are unconstrained, namely those with a zero spin at their East boundary (in particular, the initial law π in I is preserved at any later time). Let $\mathcal{L}^{(i)}$ be the corresponding Markov generator. Because of what we just said $\mathcal{L}^{(i)}$ is self-adjoint in $L^2(I, \pi)$ and, if the boundary spin at $(x^*, 0)$ is zero, it has also a spectral gap which is not smaller than the spectral gap $\text{gap}(\mathcal{L}_{\text{East}})$ on the East process on $\mathbb{Z}$ [5]. Moreover, for any function $F : \Omega_I \mapsto \mathbb{R}$ and any $s > 0$, the Feynman-Kac formula

$$\mathbb{E}^{(i)}_\sigma \left(F(\sigma_{t_{i+1}-t_i}) e^{\gamma \int_0^{t_{i+1}-t_i} ds\, \mathbb{1}(\eta_s(x)=1)} \right) = \left(e^{(t_{i+1}-t_i)(\mathcal{L}^{(i)}+\gamma \mathcal{A}_x)} F \right)(\sigma)$$

holds, with $\mathcal{A}_x F(\sigma) = \mathbb{1}(\sigma(x) = 1) F(\sigma)$ and $\mathbb{E}^{(i)}_\sigma(\cdot)$ being the expectation over the process with generator $\mathcal{L}^{(i)}$ and initial condition σ.

Thus

$$\begin{aligned} &\mathbb{E}^{\min}_{\eta,\pi} \left(e^{\gamma \int_0^t ds\, \mathbb{1}(\eta_s(x)=1)} \mid \mathcal{F}_{I^c,t} \right) \\ &\quad = \langle \mathbf{1}, e^{t_1(\mathcal{L}^{(0)}+\gamma\mathcal{A}_x)} \cdot e^{(t_2-t_1)(\mathcal{L}^{(1)}+\gamma\mathcal{A}_x)} \cdot \dots \cdot e^{(t-t_n)(\mathcal{L}^{(n)}+\gamma\mathcal{A}_x)} \mathbf{1} \rangle_\pi \\ &\quad \leqslant \prod_{i=0}^n \| e^{(t_{i+1}-t_i)(\mathcal{L}^{(i)}+\gamma\mathcal{A}_x)} \|_\pi \end{aligned} \tag{4.5}$$

where $\langle \cdot \rangle_\pi$ denotes the scalar product in $L^2(I, \pi)$ and $\| \cdot \|_\pi$ denotes the operator norm on $L^2(I, \pi)$.

For i such that $\eta_s((x^*, 0)) = 1,\ s \in [t_i, t_{i+1})$, we simply bound $\| e^{(t_{i+1}-t_i)(\mathcal{L}^{(1)}+\gamma\mathcal{A}_x)} \|_\pi$ by $e^{\gamma(t_{i+1}-t_i)}$. Indeed, $\mathcal{L}^{(1)} + \gamma \mathcal{A}_x \leqslant \gamma \mathbb{1}$ as self-adjoint operators. In the opposite case and provided that γ is small enough (e.g. $\gamma = \text{gap}(\mathcal{L}_{\text{East}})/2$) we can use a result from [5, Proof of Theorem 3.6] to get that

$$\| e^{(t_{i+1}-t_i)(\mathcal{L}^{(1)}+\gamma\mathcal{A}_x)} \|_\pi \leqslant e^{\beta\gamma(t_{i+1}-t_i)}$$

where $\beta = pq/(1 + p) + p < 1$.

In conclusion

$$\prod_{i=0}^n \| e^{(t_{i+1}-t_i)(\mathcal{L}^{(1)}+\gamma\mathcal{A}_x)} \|_\pi \leqslant e^{\beta\gamma \mathcal{T}_t(x^*) + \gamma(t - \mathcal{T}_t(x^*))}$$

and

$$\sup_\eta \mathbb{P}^{\min}_\eta \Big(\mathcal{T}_t(x) \leqslant \delta \mathcal{T}_t(x^*) \mid \mathcal{F}_{I^c,t} \Big) \leqslant (1/p \wedge q)^{x_d} e^{-\gamma(1-\beta-\delta)\mathcal{T}_t(x^*)}.$$

The proof is complete if we take e.g. $\delta = (1 - \beta)/2$. □

5. Mixing time of the East-like process

Consider the East-like process in $\Lambda = [1, L]^d$ with ergodic boundary conditions and let

$$T_{\rm mix} \triangleq \inf\{t > 0 : \max_{\eta} \|\mathbb{P}^{\Lambda}_{\eta}(\eta_t \in \cdot) - \pi\|_{\rm TV} \leqslant 1/4\}$$

be its mixing time.

Theorem 5.1. *There exists $C > 0$ such that $C^{-1}L \leqslant T_{\rm mix} \leqslant CL$ for all $L \geqslant 1$.*

Remark 5.2. *Notice that the standard inequality $T_{\rm mix} \leqslant$ const. $\log(1/\pi^*) \times T_{\rm rel}$ with $\pi* = \min_{\eta\in\Omega_\Lambda} \pi(\eta)$ (see e.g. [32]) only gives $T_{\rm mix} = O(L^d)$. Also, if the boundary conditions are minimal, it is easy to see that the logarithmic Sobolev constant α_Λ (cf. [32]) satisfies $\alpha_\Lambda \sim L^{-d}$. The upper bound is proved by plugging into the variational characterization of α_Λ the test function given by the indicator of the configuration without vacancies. The lower bound follows at once from the general bound $\alpha_\Lambda \geq \mathrm{gap}(\mathcal{L}^{\rm min}_\Lambda)/(2 + \log(1/\pi^*))$. Thus the East-like process with minimal boundary conditions has $T_{\rm rel} = O(1)$, $T_{\rm mix} \sim L$ and a logarithmic Sobolev constant $\sim L^{-d}$.*

We first establish three key preliminary results before proving Theorem 5.1. We denote by $\hat{\Omega}_\Lambda$ the set of configurations such that in *any* interval $I \subset \Lambda$ parallel to one of the coordinate axes and of length $\lfloor(\log L)^2\rfloor$ there exists at least one vacancy. The first result says that any initial configuration will, with high probability, evolve into $\hat{\Omega}_\Lambda$ in a time $t = O(L)$.

Lemma 5.3. *For any $\varepsilon \in (0, 1)$ there exists M such that, for all initial configurations $\eta \in \Omega_\Lambda$ and all $L \geqslant 1$,*

$$\sup_{t\geq ML} \mathbb{P}^{\Lambda}_{\eta}\left(\eta_t \notin \hat{\Omega}_\Lambda\right) \leqslant \varepsilon.$$

Proof. Fix an interval $I \subset [1, L]^d$ of the form $I = [x, x + \lfloor(\log L)^2\rfloor e]$, $e \in \mathcal{B}$. Using Theorem 4.6 and its notation, there exist $M > 0$ and $c > 0$ independent of L and of the initial configuration such that

$$\mathbb{P}^{\Lambda}_{\eta}(\tau_x \geq ML/2) \leqslant e^{-cML/2}.$$

The strong Markov property at the hitting time τ_x together with Theorem 4.3 gives that, for $t \geq ML$,

$$\begin{aligned}
&\mathbb{P}^{\Lambda}_{\eta}(\tau_x \leqslant ML/2\,;\, \eta_t(z) = 1\ \forall\, z \in I)\\
&= \mathbb{E}^{\Lambda}_{\eta}\left(\mathbb{1}(\tau_x \leqslant ML/2)\mathbb{P}^{\Lambda}_{\eta_{\tau_x}}(\eta_{t-\tau_x}(z) = 1\ \forall\, z \in I)\right)\\
&\leqslant \pi(\eta(z) = 1\ \forall\, z \in I) + Ce^{-c(t-ML/2)^{1/2d}} = O(e^{-c'(\log L)^2}).
\end{aligned}$$

Since the number of such intervals I is $O(e^{c'' \log L})$, a union bound over the choice of I finishes the proof. □

The next result is a small refinement of the arguments used in the proof of Theorem 4.3. Consider the East-like process in $\mathbb{Z}^d_+$ with ergodic boundary conditions and let $\mathcal{G}_t(x; \Delta)$ be the event that the spin at x was unconstrained (*i.e.* with at least a vacancy in its East-boundary) for a total time $0 \leqslant \Delta \leqslant t$ during the time interval $[0, t]$. By construction, given the initial η, $\mathcal{G}_t(x; \Delta)$ is measurable w.r.t. $\mathcal{F}_{\mathbb{Z}^d_{x,\downarrow},t}$.

Lemma 5.4. *Given $x \in \mathbb{Z}_+^d$, let V be a box of the form $V = \prod_{i=1}^d [x_i, x_i + \ell - 1]$, $\ell \geqslant 1$, and let $f : \mathbb{Z}_+^d \mapsto \mathbb{R}$ be a bounded function which does not depend on the spins in $\mathbb{Z}_{x,\uparrow}^d \setminus V$. There exist positive constants c, λ so that, on the event $\mathcal{G}_t(x;\Delta)$,*

$$\max_\eta |\mathbb{E}_\eta^{\mathbb{Z}_+^d}\big(f(\eta_t) - f^V(\eta_t) \mid \mathcal{F}_{V^c,t}\big)| \leqslant C\|f\|_\infty \, e^{c\ell^d - \lambda\Delta},$$

where $f^V(\eta) \triangleq \pi_V(f)(\eta)$ is the equilibrium average in V of f. The constant λ can be chosen as the spectral gap of the process in $\mathbb{Z}_+^d$ with minimal boundary conditions.

Proof. Let $W = V \cup (\mathbb{Z}_+^d \setminus \mathbb{Z}_{x,\uparrow}^d)$ (see Figure 3).

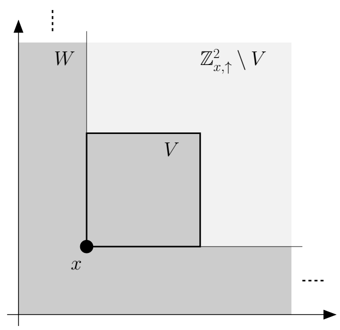

FIGURE 3. The set W when $d = 2$.

Using the assumption that f does not depend on the spins in $\mathbb{Z}_{x,\uparrow}^d \setminus V$ and the oriented nature of the constraints, we can safely replace the East-like process in $\mathbb{Z}_+^d$ with the East-like process in W, i.e.

$$\mathbb{E}_\eta^{\mathbb{Z}_+^d}\big(f(\eta_t) - f^V(\eta_t) \,|\, \mathcal{F}_{V^c,t}\big) = \mathbb{E}_\eta^W\big(f(\eta_t) - f^V(\eta_t) \,|\, \mathcal{F}_{V^c,t}\big).$$

Clearly, given the initial η, the dynamics of the spins in $W \setminus V$ is measurable w.r.t. $\mathcal{F}_{V^c,t}$. Moreover, using Proposition 3.1,

$$\pi_V\left(\mathbb{E}_\eta^W\left(f(\eta_s) \,|\, \mathcal{F}_{V^c,s}\right)\right) = f^V(\eta_s) \quad a.s. \quad \forall\, s \in [0,t],$$

where π_V averages only over the spins in V. The proof now follows the pattern of the proof of Theorem 4.3 (cf. (4.2)). We write

$$\begin{aligned}
&\mathbb{1}(\mathcal{G}_t(x;\Delta))|\mathbb{E}_\eta^W\left(f(\eta_t) - f^V(\eta_t) \,|\, \mathcal{F}_{V^c,t}\right)| \\
&\leqslant 1/(p\wedge q)^{\ell^d}\mathbb{1}(\mathcal{G}_t(x;\Delta))\pi_V\left(|\mathbb{E}_\eta^W\left(f(\eta_t) - f^V(\eta_t) \mid \mathcal{F}_{V^c,t}\right)|\right) \\
&\leqslant 1/(p\wedge q)^{\ell^d}\mathbb{1}(\mathcal{G}_t(x;\Delta))\left[\mathrm{Var}_{\pi_V}\left(\mathbb{E}_\eta^W\left(f(\eta_t) \,|\, \mathcal{F}_{V^c,t}\right)\right)\right]^{1/2} \leqslant \|f\|_\infty \, e^{c\ell^d - \lambda\Delta},
\end{aligned}$$

where Var_{π_V} denotes the variance w.r.t to π_V and $c = -\log(p\wedge q)$. The fact that λ can be taken equal to the spectral gap of the first quadrant with minimal boundary conditions follows immediately from the monotonicity of the spectral gap in the boundary conditions and in the volume. □

The third result says that the East-like process in the box $\Lambda = [1,L]^d$ with ergodic boundary conditions reaches equilibrium (in total variation distance) in a time lag

$O(\log(L)^{4d})$ if the initial configuration belongs to $\hat\Omega_\Lambda$. More precisely, let μ_t^η denote the law of η_t under $\mathbb{P}^\Lambda_\eta(\cdot)$.

Lemma 5.5. *For any $\varepsilon \in (0,1)$ there exists L_0 such that the following holds. Let $T = (\log L)^{5d}$ and let $d(T) = \max_{\eta\in\hat\Omega_\Lambda} \|\mu_T^\eta - \pi_\Lambda\|_{\rm TV}$. Then $\sup_{L\geq L_0} d(T) \leqslant \varepsilon$.*

Proof. Fix $\eta \in \hat\Omega_\Lambda$ and let us order the vertices of $\Lambda = [1,L]^d$ as follows. We first choose some order of the vertices on each hyperplane $\mathcal{P} = \{x \in \mathbb{Z}^d_+ :\ \|x\|_1 = \text{const.}\}$ in such a way that two consecutive vertices have distance $O(1)$ for large L (e.g. they have exactly two coordinates where they differ by one). Then, for any pair $x,y\in\Lambda$, we say that $x \prec y$ if either $\|x\|_1 < \|y\|_1$ or if x comes before y when they have the same ℓ^1-norm. The i^{th} vertex in the above order will be denoted by $x^{(i)}$.

Next, for any $f : \Omega_\Lambda \mapsto \mathbb{R}$ with $\|f\|_\infty = 1$ and any $j = 1,\dots,n$, where $n = L^d$, we denote by $f^{(j)}$ the new function obtained by averaging f w.r.t. the equilibrium measure π over the last j spins in the above order. For notation convenience we set $f^{(0)} \equiv f$. Notice that $f^{(j)} = \pi_j(f^{(j-1)})$, where $\pi_j(\cdot)$ denotes the marginal of π over the $(L^d - j + 1)^{th}$-spin, and that $f^{(n)} = \pi(f)$. Then

$$|\mu_t^\eta(f) - \pi(f)| \leqslant \sum_{j=1}^n |\mu_t^\eta(f^{(j-1)}) - \mu_t^\eta(f^{(j)})|, \quad \forall t \geq 0.$$

We now choose $t = (\log L)^{5d}$ and $\eta \in \hat\Omega_\Lambda$ and prove that each term in the above sum is smaller than $O(e^{-(\log L)^2})$ for large enough L.

Let Λ_j be the set $\{x^{(1)}, x^{(2)}, \dots, x^{(n-j)}\}$. Using the assumption that the initial con-

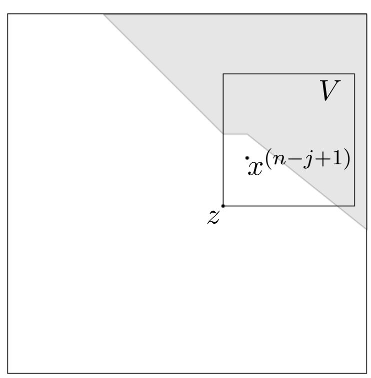

FIGURE 4. The shaded quasi-triangle corresponds to vertices larger than $x^{(n-j+1)}$

figuration η belongs to $\hat\Omega_\Lambda$, there exists a vertex $y \in (\Lambda_j \cup \partial_E\Lambda_j) \cap \mathbb{Z}^d_{x^{(n-j+1)},\downarrow}$ such that $|y - x^{(n-j+1)}| \leqslant (\log L)^2$ and $\eta(y) = 0$. Corollary 4.2 implies that, with probability greater than $1 - e^{-(\log L)^2}$, there exists $C > 0$ independent of j and a vertex $z \in (\Lambda_j \cup \partial_E\Lambda_j) \cap \mathbb{Z}^d_{x^{(n-j+1)},\downarrow}$ such that:

(i) $|z - x^{(n-j+1)}| \leqslant C(\log L)^2$,

(ii) one spin in the East boundary of z spends a time greater than $(\log L)^{3d}$ in state 0 up to time t.

Notice that, if $V \equiv \prod_{i=1}^{d}[z_i, z_i + 3C(\log L)^2]$, then neither $f^{(j)}$ nor $f^{(j-1)}$ depend on the spins in $\mathbb{Z}^d_{z,\uparrow} \setminus V$ (see Figure 4) and we can apply Lemma 5.4 to both $(V, f^{(j-1)})$ and $(V, f^{(j)})$. Using the fact that $\pi_V(f^{(j-1)}) = \pi_V(f^{(j)})$ we finally get that

$$|\mu^\eta_t(f^{(j-1)}) - \mu^\eta_t(f^{(j)})| \leqslant e^{-(\log L)^2} + C' e^{c(\log L)^{2d} - \lambda(\log L)^{3d}}.$$

In conclusion, for any L large enough

$$|\mu_t(f) - \pi(f)| \leqslant 2L^d e^{-(\log L)^2}$$

and the lemma follows. □

We are finally in a position to prove our main result.

Proof of Theorem 5.1. The lower bound is straightforward by choosing as initial condition the configuration without vacancies and using the finite speed of propagation of information (see e.g. [19, Section 2.4]) to prove that, with high probability, the process is not able to create vacancies near the vertex $\hat{x} = (L, \dots, L)$ in a time $t = \delta L$, if δ is small enough.

To prove the upper bound we proceed as follows. Using Lemma 5.3, uniformly in the initial condition, in a time $t = O(L)$ the East-like chain will enter the good set $\hat{\Omega}_\Lambda$ with probability e.g. greater than $7/8$. Using the Markov property and Lemma 5.5, in an additional time lag $O((\log L)^{5d})$ the chain will reduce its variation distance from the target distribution π_Λ to less than $1/8$. □

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