Annali di Matematica Pura ed Applicata (1923 -)
https://doi.org/10.1007/s10231-024-01506-3

Symplectic Grassmannians and cyclic quivers

Evgeny Feigin[1] · Martina Lanini[2] · Matteo Micheli[3] · Alexander Pütz[4]

Received: 16 July 2024 / Accepted: 9 September 2024

Abstract
The goal of this paper is to extend the quiver Grassmannian description of certain degenerations of Grassmann varieties to the symplectic case. We introduce a symplectic version of quiver Grassmannians studied in our previous papers and prove a number of results on these projective algebraic varieties. First, we construct a cellular decomposition of the symplectic quiver Grassmannians in question and develop combinatorics needed to compute Euler characteristics and Poincaré polynomials. Second, we show that the number of irreducible components of our varieties coincides with the Euler characteristic of the classical symplectic Grassmannians. Third, we describe the automorphism groups of the underlying symplectic quiver representations and show that the cells are the orbits of this group. Lastly, we provide an embedding into the affine flag varieties for the affine symplectic group.

Keywords Quiver representations · Quiver Grassmannians · Symplectic Grassmannians · Affine flag varieties

Mathematics Subject Classification 16G20 Representations of quivers and partially ordered sets · 14M15 Grassmannians, Schubert varieties, flag manifolds

✉ Matteo Micheli
matteo.micheli@uniroma1.it

Evgeny Feigin
evgfeig@gmail.com

Martina Lanini
lanini@mat.uniroma2.it

Alexander Pütz
alexander.puetz@math.uni-paderborn.de

[1] School of Mathematical Sciences, Tel Aviv University, 69978 Tel Aviv, Israel

[2] Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica 1, 00133 Rome, Italy

[3] Dipartimento di Matematica "Guido Castelnuovo", Sapienza Università di Roma, Piazzale Aldo Moro 5, 00185 Rome, Italy

[4] Institute of Mathematics, University Paderborn, Warburger Str. 100, D-33098 Paderborn, Germany

Published online: 14 October 2024

1 Introduction

The classical Grassmann varieties $\mathrm{Gr}(k,n)$ admit a flat degeneration into certain reducible algebraic varieties $X(k,n)$. The construction emerged from arithmetics as local models of Shimura varieties [1–3], but is also very natural from the point of view of complex algebraic geometry [4, 5]. In [6–8] the quiver Grassmannians approach describing these degenerations was developed. More precisely, the authors considered certain modules $U_{[n]}$ over the cyclic quivers Δ_n such that $X(k,n)$ is isomorphic to the quiver Grassmannian of subrepresentations of $U_{[n]}$ of dimension k at each vertex. This paper is the first step towards the extension of the quiver approach to the symplectic case (see [9–12] for the non cyclic type A case).

The varieties $X(k,n)$ admit many nice properties. Having a quiver Grassmannians realization at hand, one is able to use various techniques from the algebraic, combinatorial and geometric theories of quivers in order to study various structures related to $X(k,n)$. In particular, one can use the automorphism groups of the underlying representations in order to describe cellular decompositions and to establish a link with the positroid decomposition of totally nonnegative Grassmannians. Looking at the side of local models of Shimura varieties, one gets generalizations for other classical groups [13–15] (we note that only the Lagrangian case was considered). Hence it is natural to ask for the quiver realization with a bilinear form added as an extra piece of data. Our goal is to initiate the study of the corresponding quiver Grassmannians.

In this paper we deal only with the symplectic form on a $2n$ dimensional complex vector space. The symplectic Grassmannians $\mathrm{Gr}(k,2n)^{sp}$ have been extensively studied in the literature (see e.g. [16, 17]). For each $k=1,\dots,n$ we define a degeneration $X(k,2n)^{sp}$ of the corresponding symplectic Grassmannian as follows. The symplectic form on the $2n$ dimensional complex vector space as above induces a non degenerate skew-symmetric form on the representation space $U_{[2n]}$. Note that this is a symplectic representation in the sense of Derksen–Weyman [18]. Then $X(k,2n)^{sp}$ is defined as a subvariety of $X(k,2n)$ consisting of self-orthogonal subrepresentations. Our first theorem is as follows.

Theorem A (Theorem 4.14) *The intersection of a cell of $X(k,2n)$ with $X(k,2n)^{sp}$ is either empty or an affine cell. The top dimensional cells of $X(k,2n)^{sp}$ are obtained from top dimensional cells of $X(k,2n)$. The dimension of $X(k,2n)^{sp}$ is equal to the dimension of the classical symplectic Grassmannian $\mathrm{Gr}(k,2n)^{sp}$ and the number of irreducible components of $X(k,2n)^{sp}$ is equal to the Euler characteristic of $\mathrm{Gr}(k,2n)^{sp}$.*

Let G be the automorphism group of $U_{[2n]}$ and let G^{sp} be its subgroup preserving the symplectic form. It was shown in [6–8] that G is a degeneration of the general linear group GL_{2n} and the action of G can be used to describe various properties of the varieties $X(k,2n)$. We show that similar properties hold true in the symplectic case as well.

Theorem B (Theorem 3.12) *The group G^{sp} is a degeneration of the classical group Sp_{2n}; in particular, $\dim G^{sp}=2n^2+n$. The group G^{sp} acts on $X(k,2n)^{sp}$ with a finite number of orbits, the orbits are affine cells and are naturally labeled by certain combinatorial gadgets that we call symplectic juggling patterns.*

Recall that the varieties $X(k,2n)$ admit embeddings into the affine flag varieties of type A [5, 19, 20]. These embeddings have simple and transparent descriptions in terms of quiver representations via the lattice realization of affine Grassmannians and affine flag varieties ([7, 21, 22]). We show that one can extend the type A picture to the symplectic case. More precisely, we prove the following theorem (it is known for $k=n$ from the arithmetic side by [3]).

Theorem C (Theorem 5.4) *The varieties $X(k,2n)^{sp}$ admit an embedding into the flag variety for the affine symplectic group Sp_{2n}. The image of the embedding is a union of Schubert varieties.*

We put forward two conjectures:

- The degeneration from the symplectic Grassmannian $\mathrm{Gr}(k,2n)^{sp}$ to $X(k,2n)^{sp}$ is flat (see Sect. 2.4 for more detail);
- The poset structure on the set of cells of $X(k,2n)^{sp}$ is induced from the corresponding poset of cells of $X(k,2n)$ (see Conjecture 4.9).

Finally, let us mention several possible further directions. First, it would be interesting to study the projections $X_S(k,2n)^{sp}$ of the varieties $X(k,2n)^{sp}$ to the product of Grassmannians corresponding to a subset S of vertices of the quiver Δ_{2n}. Second, the varieties $X(k,2n)$ admit certain ω-generalizations $X(k,2n,\omega)$ [7], it would be interesting to develop the corresponding symplectic story. Third, one is interested in the projections $X_S(k,2n,\omega)^{sp}$. In all these cases one expects a link to a flat degeneration from Schubert varieties in affine Grassmannians to unions of Schubert varieties in affine flag varieties (see [2, 5, 8]). Fourth, the quiver Grassmannian realization allows one to use quiver techniques for the construction of resolutions of singularities [23–25]. It is natural to ask for a similar construction in the symplectic case (see [26] for a special case of such construction). Fifth, it is natural to expect a link between the topology of the varieties $X(k,2n)^{sp}$ and the symplectic version of the totally nonnegative Grassmannians (in the spirit of [6], see also [27–29]). Lastly, it is natural to study the orthogonal case (both even and odd) and the case of odd symplectic Grassmannians [30, 31].

Our paper is organized as follows. In Sect. 2 we provide the needed background; in particular, we introduce here the poset $JP(k,n)$ of (k,n)-juggling patterns and the juggling variety $X(k,n)$, as well as their symplectic analogues $JP(k,2n)^{sp}$ and $X(k,2n)^{sp}$. Section 3 is about reductive group actions: we recall the action of an appropriate degeneration of GL_{2n} on $X(k,2n)$, we define an involution on such a group and look at both its fixed point subgroup G^{sp} (the symplectic automorphism group) and at the corresponding Lie algebra. The symplectic automorphism group acts on $X(k,2n)^{sp}$ and we devote the rest of this section to the investigation of its orbits. More precisely, we show that they are parameterized by symplectic juggling patterns and that they are all affine spaces. In Sect. 4 we deepen the study of the G^{sp}-orbits on $X(k,2n)^{sp}$. More specifically, we connect the orbit closure inclusion relation to certain combinatorial moves on juggling patterns called mutations, and exploit them to determine the cell dimension. Moreover, we provide an explicit counting of the top dimensional cells, and hence determine the dimension of our variety of interest. Section 5 deals with the affine Grassmannians and flag variety: after recalling the embedding of $X(k,2n)$ into the GL_{2n} affine flag variety, we show that $X(k,2n)^{sp}$ can be embedded into the Sp_{2n}-affine flag variety. In Appendix A we exhibit Poincaré polynomials and Euler characteristics of $X(k,2n)$ and its symplectic analogue for $n = 1,2,3,4$ and $k \leq n$.

2 Background

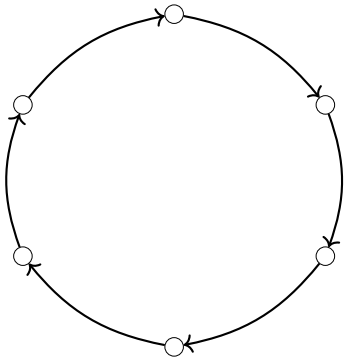
2.1 Juggling patterns and juggling variety

First we lay down some notation:

- for $n \leq m \in \mathbb{N}^+$, we will write $[n]$ for the set $\{1,2,\ldots,n\}$, $[n,m]$ for $\{n,n+1,\ldots,m\}$ and $\binom{[m]}{n}$ for the set of subsets of $[m]$ with cardinality n;

- $e_1, e_2, \ldots, e_n$ denote the elements of the standard basis for $\mathbb{C}^n$. Given a subset I of $[n]$, the coordinate subspace V_I is defined as $\mathrm{Span}\{e_i \mid i \in I\}$;
- if g is an invertible matrix, g^{-t} will be notation for $(g^{-1})^t$;
- the ring of integers modulo m will be denoted by $\mathbb{Z}_m$.

We denote with Δ_n the equioriented quiver of type $\widetilde{A}$ on n vertices.



The quiver Δ_6.

We label its vertices with the integers modulo n, so that the arrows are of the form $i \longrightarrow i+1$.

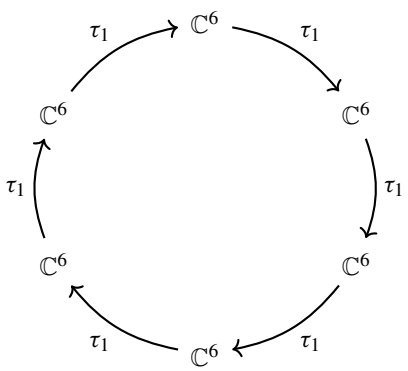
Definition 2.1 Let $k \leq n$ be natural numbers; we define the projective variety

$$X(k,n) := \left\{ V = (V_i)_{i \in \mathbb{Z}_n} \in \prod_{i \in \mathbb{Z}_n} Gr(k, \mathbb{C}^n) \mid \tau_1(V_i) \subseteq V_{i+1} \, \forall \, i \in \mathbb{Z}_n \right\}$$

where the endomorphism τ_1 of $\mathbb{C}^n$ is given by $\tau_1(e_i) = e_{i+1}$ for $i < n$ and $\tau_1(e_n) = 0$.

Notice that $X(k,n)$ is a quiver Grassmannian for Δ_n (see [6] for more detail): consider the nilpotent Δ_n-representation $U_{[n]}$ which has $\mathbb{C}^n$ over each vertex and τ_1 over each arrow. If we denote by $\underline{k} \in \mathbb{N}^n$ the dimension vector whose entries are all equal to $k \in \mathbb{N}$, we have that $X(k,n)$ is the locus of subrepresentations of $U_{[n]}$ of dimension vector $\underline{k}$, i.e.

$$X(k,n) = Gr(\underline{k}, U_{[n]}) .$$



The representation $U_{[6]}$.

Definition 2.2 Let $k \leq n$ be natural numbers. A (k, n)-juggling pattern is a collection $\mathcal{J} = (J_i)_{i \in \mathbb{Z}_n}$ of cardinality k subsets of $[n]$, such that $j \in J_i$ implies $j + 1 \in J_{i+1}$, for all $i \in \mathbb{Z}_n$ and all $j \in [n-1]$. The set of (k, n)-juggling patterns is denoted by $JP(k, n)$.

Remark 2.3 Observe that for each (k, n)-juggling pattern $\mathcal{J} = (J_i)_i$, the collection of coordinate vector spaces $p_{\mathcal{J}} := (V_{J_i})_i$ is a point in $X(k, n)$.

In [6] a different, but isomorphic, Δ_n-representation is chosen: the map on the arrow $i \longrightarrow i + 1$ is the projection along the i-th standard basis vector. The corresponding quiver Grassmannians are again isomorphic to ours, and the combinatorial gadgets resulting from this representation are called (k, n)-Grassmann necklaces. One can produce a juggling pattern from a Grassmann necklace and viceversa [6, 2.2], and all results we recall that use one family of objects still hold for the other. The (k, n)-juggling patterns admit certain ω-generalizations ([7]) which do not naturally exist for Grassmann necklaces. With this generalization of the symplectic setting in mind we decided to work with juggling patterns instead of Grassmann necklaces.

For all $k \in [0, n]$, the set $JP(k, n)$ can be equipped with the following partial order: $\mathcal{J} \leq \mathcal{J}'$ if and only if $J_i \geq J'_i$ for all $i \in \mathbb{Z}/n\mathbb{Z}$, where two sets $A, B \in \binom{[n]}{k}$ satisfy $A \leq B$ if, when written in increasing order $A = \{a_1 < a_2 < \cdots < a_k\}$, $B = \{b_1 < b_2 < \cdots < b_k\}$, one has $a_i \leq b_i$ for all $i \in [k]$.
Let $G := \mathrm{Aut}_{\Delta_n}(U_{[n]})$ be the automorphism group of the Δ_n-representation $U_{[n]}$. Then G acts on all quiver Grassmannians for $U_{[n]}$, in particular on $X(k, n)$. Each G-orbit in $X(k, n)$ contains exactly one point $p_{\mathcal{J}}$ labeled by a juggling pattern $\mathcal{J}$ as in Remark 2.3. Such an orbit is an affine cell by [6, Theorem 1] and we denote it by $C_{\mathcal{J}}$. Thus the closure inclusion order on the set of G-orbits induces a partial order on $JP(k, n)$, which coincides with the combinatorial order described above [7, Corollary 4.7].

2.2 Symplectic conditions on the juggling variety

To introduce symplectic conditions on such varieties we need the dimension of the vector spaces to be even, so we work in $X(k, 2n)$. We equip $\mathbb{C}^{2n}$ with the symplectic form

$$(v, w) = \sum_{i=1}^{n} (-1)^{i+1} \cdot v_i \cdot w_{2n-i+1} \ ,$$

where $v = \sum_{i=1}^{n} v_i e_i$ and $w = \sum_{i=1}^{n} w_i e_i$. In other words, this is the symplectic form whose Gram matrix Ω in the standard basis has zeros everywhere except on the antidiagonal, where it has alternating 1 s and -1s, with a 1 on the upper right corner.

$$\Omega = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

When n is fixed, we write $\tilde{i}$ for $2n - i + 1$ for brevity. This way we have $(e_i, e_j) = (-1)^{\tilde{i}} \delta_{\tilde{i}j}$.

Proposition 2.4 *Given k, n natural numbers with $0 \leq k \leq 2n$, the map*

$$\sigma : X(k, 2n) \longrightarrow X(2n-k, 2n)$$
$$(V_i)_i \longmapsto (V_{-i}^{\perp})_i$$

is well defined.

For $V \in X(k,n)$ and $W \in X(\ell,n)$ with $k \leq \ell$, we write $V \subseteq W$ whenever $V_i \subseteq W_i$ for all $i \in \mathbb{Z}_n$.

Proof of Proposition 2.4 To start with, observe that $-\Omega = \Omega^t = \Omega^{-1}$. Moreover, notice that, as matrices in the standard basis, $-\Omega \cdot \tau_1^t \cdot \Omega^{-1} = \Omega \tau_1^t \Omega = \tau_1$. From linear algebra we obtain: given $W \subseteq \mathbb{C}^{2n}$ a subspace and M a $2n \times 2n$ matrix, one has $\Omega M^t \Omega\big((M \cdot W)^{\perp}\big) \subseteq W^{\perp}$. Now if $V = (V_i)_i$ is a point in $X(k, 2n)$, its vector spaces satisfy $\tau_1 \cdot V_i \subseteq V_{i+1}$ for all i. Taking the orthogonal subspaces we get $V_{i+1}^{\perp} \subseteq (\tau_1 \cdot V_i)^{\perp}$, then applying τ_1 to both sides we find

$$\tau_1(V_{i+1}^{\perp}) \subseteq \tau_1\big((\tau_1 \cdot V_i)^{\perp}\big) = \Omega \tau_1^t \Omega\big((\tau_1 \cdot V_i)^{\perp}\big) \subseteq V_i^{\perp} \ .$$

That is, $\tau_1(\sigma V)_{-i-1} \subseteq (\sigma V)_{-i}$ for any $i \in \mathbb{Z}_{2n}$ and hence we obtain $\sigma V \in X(2n-k, 2n)$. □

Remark 2.5 The composition

$$X(k, 2n) \xrightarrow{\sigma} X(2n-k, 2n) \xrightarrow{\sigma} X(k, 2n)$$

is the identity map.

2.3 The main object

Now we introduce the main object of this paper:

Definition 2.6 Let $k \leq n$. Then

$$X(k, 2n)^{sp} := \{V = (V_i)_i \in X(k, 2n) \mid V \subseteq \sigma V\}.$$

This is the subvariety of isotropic, or symplectic, points in $X(k, 2n)$.

Remark 2.7 If instead $n \leq k \leq 2n$, we can define coisotropic points in $X(k, 2n)$ as those that satisfy $V \supseteq \sigma V$. Then the corresponding subvariety is isomorphic to $X(2n-k, 2n)^{sp}$ via σ.

For a subset $I \in \binom{[2n]}{k}$, let $RI := [2n] \backslash \{\tilde{i} \mid i \in I\} \in \binom{[2n]}{2n-k}$. Notice that V_{RI}, the coordinate subspace corresponding to RI, coincides with $V_I^{\perp}$. If $I \subseteq RI$ we say that I is isotropic, or coisotropic if $I \supseteq RI$; in either case we say that I is symplectic.

Next, we extend $R : \binom{[2n]}{k} \longrightarrow \binom{[2n]}{2n-k}$ to the poset of juggling patterns: for $\mathcal{J} = (J_i) \in JP(k, 2n)$, we define $R\mathcal{J}$ as the tuple of sets $(R\mathcal{J})_i = R(J_{-i})$; this is a $(2n-k, 2n)$-juggling pattern, since $\sigma(p_{\mathcal{J}}) \in X(2n-k, 2n)$ and it has a coordinate vector space $V_{R(J_{-i})} = V_{(R\mathcal{J})_i}$ on vertex i. Once again, R is a bijection and $R(R\mathcal{J}) = \mathcal{J}$. Given two juggling patterns $\mathcal{J} \in JP(k,n)$ and $\mathcal{J}' \in JP(\ell,n)$ with $k \leq \ell$, we write $\mathcal{J} \subseteq \mathcal{J}'$ if $p_{\mathcal{J}} \subseteq p_{\mathcal{J}'}$, that is, if $J_i \subseteq J_i'$ for all i.

Definition 2.8 Let $k \leq n$; a $(k, 2n)$-juggling pattern $\mathcal{J}$ is isotropic, or symplectic, if $p_{\mathcal{J}} \in X(k, 2n)^{sp}$ or, equivalently, if $\mathcal{J} \subseteq R\mathcal{J}$. By $JP(k, 2n)^{sp}$ we denote the set of $(k, 2n)$-symplectic juggling patterns.

Example 2.9 Let $k = n = 2$.
Then $\mathcal{J} = (J_0 = \{3,4\}, J_1 = \{2,4\}, J_2 = \{2,3\}, J_3 = \{3,4\})$ is a $(2,4)$-juggling pattern, but it is not symplectic, as $J_2 \nsubseteq RJ_2 = \{1,4\}$. Instead, $\mathcal{J}' = (J'_0 = \{3,4\}, J'_1 = \{2,4\}, J'_2 = \{3,4\}, J'_3 = \{2,4\}) \in JP(2,4)^{sp}$.

Remark 2.10 A $(k,2n)$-juggling pattern $\mathcal{J}$ is maximal in the partial order if for all $i, 2n \in J_i$ implies $1 \in J_{i+1}$ [6, Remark 4.12]. Then a maximal $\mathcal{J}$ is symplectic if and only if J_0 is symplectic, or equivalently if and only if J_n is symplectic.

Proposition 2.11 *For any $n \geq 1$, $X(1,2n)^{sp} = X(1,2n)$.*

Proof First we prove that any $(1,2n)$-juggling pattern is isotropic: assume there exists a non-symplectic $(1,2n)$-juggling pattern $\mathcal{J}$, with $J_i = \{x\}$ and $J_{-i} = \{\tilde{x}\} = \{2n - x + 1\}$ for some $x \in [2n]$ and $i \in \mathbb{Z}_{2n}$. Let m be the number of arrows on the minimal path from $-i$ to i, which is equivalent to $2i$ modulo $2n$ and therefore even. Since x and $\tilde{x}$ have different parity, we see that both between i and $-i-1$ and between $-i$ and $i-1$ there must be a vertex a such that $J_a = \{2n\}$; this means $x + m > 2n$ and $\tilde{x} + 2n - m > 2n$. These two inequalities are incompatible, thus we have found a contradiction.

Next we show that if $\mathcal{J}$ is a maximal $(1,2n)$-juggling pattern, then the whole orbit $C_{\mathcal{J}} = G \cdot p_{\mathcal{J}}$ consists of symplectic points: let i be the vertex with $J_i = \{1\}$, so we have $J_{i+m} = \{i+m\}$ for $m \in [0, 2n-1]$. Given a point V in the orbit $C_{\mathcal{J}}$, there exist coefficients $g_1, \ldots, g_{2n}$ such that

$$V_{i+m} = \mathrm{Span}_{\mathbb{C}}(\underbrace{0, \ldots, 0}_{m}, g_1, g_2, \ldots, g_{2n-m})$$

with $g_1 \neq 0$ (see [6, Theorem 3.10] and [32, Theorem 4.13]). Now let m' be the number in $[0, 2n-1]$ such that $i + m \equiv_{2n} -(i + m')$; observe that if $m + m' \geq 2n$, the symplectic form between the generators of V_{i+m} and $V_{i+m'}$ trivially vanishes, so we can assume $m' + 1 \leq 2n - m$. We compute the product and find

$$\sum_{s=m'+1}^{2n-m} (-1)^{s+1} g_{s-m'} \cdot g_{2n-m+1-s} \,.$$

There is an even number of summands since m and m' have the same parity, and hence $m'+1$ and $2n-m$ have different parity. Moreover, every summand appears twice with opposite signs, so that the sum is zero. We conclude the proof by observing that the union of all maximal cells is dense in $X(1,2n)$ and that $X(1,2n)^{sp}$ is a closed subvariety of $X(1,2n)$. □

2.4 Interpretation as degeneration of the isotropic Grassmannian

The construction of $X(k,2n)^{sp}$ mimics the definition of the classical isotropic Grassmannian $Gr(k,2n)^{sp}$, that is, the projective variety of isotropic subspaces of $\mathbb{C}^{2n}$. Recall from [6] that $X(k,n)$ is a degeneration of $Gr(k,n)$: for $z \in \mathbb{C}$ define $U_{[n]}(z)$ as the Δ_n-representation with $\mathbb{C}^n$ on every vertex and $\tau_{1,z}$ on every map, where the matrix of $\tau_{1,z}$ in the standard basis is

$$\tau_{1,z} = \begin{pmatrix} 0 & 0 & \cdots & 0 & z \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Then $Gr(k,n)$ is isomorphic to $Gr(\underline{k}, U_{[n]}(z))$ for all $z \neq 0$ (for example by taking the vector space on vertex 0). Since $\tau_{1,0} = \tau_1$, we get $U_{[n]}(0) = U_{[n]}$, so these varieties form a family Y equipped with a morphism $\pi : Y \longrightarrow \mathbb{C}$. The fibers over nonzero numbers are all isomorphic to the classical Grassmannian, and the fiber over 0 is $X(k,n)$. Now, we replace n by $2n$ and take $k \leq n$, in order to realize $X(k,2n)^{sp}$ as a degeneration of $Gr(k,2n)^{sp}$. Observe that $\Omega\tau_{1,z}^t\Omega = \tau_{1,z}$, so we can define a map

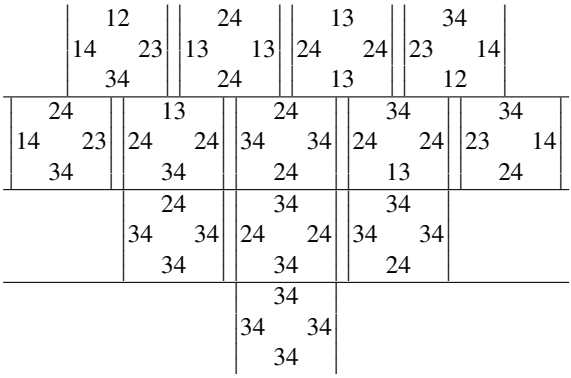
$$\sigma_z : Gr(\underline{k}, U_{[2n]}(z)) \longrightarrow Gr(\underline{2n-k}, U_{[2n]}(z))$$

as in Proposition 2.4. For every fiber $\pi^{-1}(z)$, we consider points that satisfy $V \subseteq \sigma_z(V)$. This condition is linear with $z \in \mathbb{C}$ and the fibers over nonzero numbers are isomorphic to the isotropic Grassmannian. Thus we have a subvariety Y^{sp} of Y such that $X(k,2n)^{sp}$ is the desired degeneration. Lastly we observe that $G = \mathrm{Aut}_{\Delta_n}(U_{[n]})$ is a degeneration of GL_n if we see it as the special fiber of the family over $\mathbb{C}$ whose fiber over $z \in \mathbb{C}$ is the algebraic group $\mathrm{Aut}_{\Delta_n}(U_{[n]}(z))$.

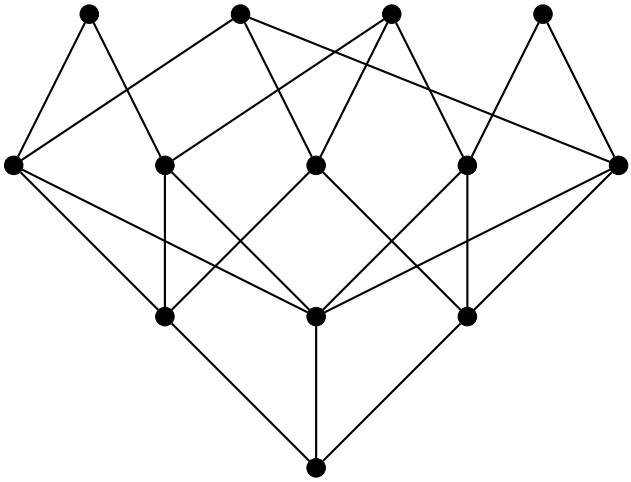
Example 2.12 We compute the poset of symplectic $(2,4)$-juggling patterns, with the combinatorial order inherited by $JP(2,4)$. We write ij for the set $\{i, j\}$, and we describe a juggling pattern $\mathcal{J}$ in the following way:

$$\begin{array}{ccc} & J_0 & \\ J_3 & & J_1 \\ & J_2 & \end{array}$$

Out of the 33 juggling patterns from $JP(2,4)$, only these 13 are symplectic:



The Hasse diagram of $JP(2,4)^{sp}$ is the following (the minimal vertex is on the bottom):



3 Symplectic automorphisms

Let us now discuss the largest subgroup of $G = \mathrm{Aut}_{\Delta_{2n}}(U_{[2n]})$ which preserves the symplectic form.

Definition 3.1 An automorphism $A = (A_i)_{i\in\mathbb{Z}_{2n}} \in G$ is symplectic if

$$(A_i(v), A_{-i}(w)) = (v, w) \tag{3.1}$$

for all $i \in \mathbb{Z}_{2n}$, $v \in U^{(i)}_{[2n]}$ and $w \in U^{(-i)}_{[2n]}$. The subgroup of such elements is denoted by G^{sp}.

Recall that the choice of Ω corresponds to the choice of an involutive (non trivial) automorphism of GL_{2n}, given by:

$$g \longmapsto -\Omega \cdot g^{-t} \cdot \Omega,$$

whose fixed-point subgroup is Sp_{2n}. We provide G with a similar automorphism $\sigma_G : G \to G$: for any $A = (A_i)_{i\in\mathbb{Z}_{2n}} \in G$ we define

$$\sigma_G(A) := \left(-\Omega \cdot (A^{-t}_{-i}) \cdot \Omega\right)_i . \tag{3.2}$$

Observe that two matrices A and B in GL_{2n} with $(A(v), B(w)) = (v, w)$ for all $v, w \in \mathbb{C}^{2n}$ must satisfy

$$B = -\Omega \cdot (A^{-t}) \cdot \Omega.$$

Hence the σ_G-fixed subgroup of G is precisely G^{sp}.

Recall the morphism $\sigma : X(k, 2n) \to X(2n - k, 2n)$ defined in Proposition 2.4.

Lemma 3.2 *The equality*

$$\sigma_G(A) \cdot V = \sigma\left(A \cdot (\sigma V)\right)$$

holds for all $V \in X(k, 2n)$ and $A \in G$.

Proof Since $\big(A \cdot (\sigma V)\big)_i = A_i \cdot (V_{-i}^{\perp})$ we have

$$\Big(\sigma\big(A \cdot (\sigma V)\big)\Big)_i = \Big(A_{-i} \cdot (V_i^{\perp})\Big)^{\perp}.$$

As in the proof of Proposition 2.4, we use the fact that given a matrix M and a subspace W, there is an inclusion $\Omega M^t \Omega\big((M \cdot W)^{\perp}\big) \subseteq W^{\perp}$ which is an equality whenever M is invertible. In this case the identity can be rewritten as $(M \cdot W)^{\perp} = \Omega M^{-t} \Omega(W^{\perp})$. We obtain the claim by plugging in $M = A_{-i}$ and $W = V_i^{\perp}$ since A consists of invertible matrices. □

Corollary 3.3 *The map $R\colon JP(k, 2n) \longrightarrow JP(2n-k, 2n)$ is order preserving.*

Proof Firstly, notice that R sends the minimal $(k, 2n)$-juggling pattern, whose sets are all equal to $\{2n-k+1, \ldots, 2n\}$ to the minimal $(2n-k, 2n)$-juggling pattern, which is constantly equal to $\{k+1, \ldots, 2n\}$. Now let $V = A \cdot p_{\mathcal{J}'}$ for some $A \in G$ and $\mathcal{J}'$ some $(k, 2n)$-juggling pattern. By Lemma 3.2 we obtain,

$$\sigma V = \sigma(A \cdot p_{\mathcal{J}'}) = \sigma(A \cdot \sigma(p_{R\mathcal{J}'})) = \sigma_G(A) \cdot p_{R\mathcal{J}'}.$$

Hence σ sends isomorphically the cell $C_{\mathcal{J}'}$ into the cell $C_{R\mathcal{J}'}$, since they are G-orbits. Recall that, for another juggling pattern $\mathcal{J}$, the condition $\mathcal{J} \leq \mathcal{J}'$, is equivalent to $p_{\mathcal{J}} \in \overline{C_{\mathcal{J}'}}$. Since σ is an isomorphism of varieties, we get $p_{R\mathcal{J}} \in \overline{C_{R\mathcal{J}'}}$ □

3.1 Explicit description of the symplectic automorphisms

Let $A = (A_i)_{i \in \mathbb{Z}_{2n}} \in G$. For any $i \in \mathbb{Z}_{2n}$, we denote by $\{e_1^{(i)}, \ldots, e_{2n}^{(i)}\}$ the standard basis of $U_{[2n]}^{(i)} = \mathbb{C}^{2n}$. With respect to such a basis, each A_i is a lower triangular matrix with nonzero diagonal entries and A is completely determined by the the entries of the first column of each of the A_i's [6, Proposition 4.5]. Let us denote the entries of the first column of A_i by $a_j^{(i)}$ for $i \in \mathbb{Z}_{2n}$ and $j \in [2n]$.

Lemma 3.4 *We have that $A \in G^{sp}$ if and only if the following conditions hold true*

$$a_1^{(i)} a_1^{(j)} = 1, \quad i, j \in \mathbb{Z}_{2n},\ i + j = 1, \tag{3.3}$$

$$\sum_{\ell=0}^{r-1} (-1)^{\ell} a_{1+\ell}^{(i)} a_{r-\ell}^{(r-i)} = 0 \quad i \in \mathbb{Z}_{2n},\ r = 2, \ldots, 2n. \tag{3.4}$$

Remark 3.5 The relation (3.4) is trivial for $r = 2i$.

Proof of Lemma 3.4 By Definition 3.1, $A \in G^{sp}$ if and only if

$$(A_i e_{j_1}^{(i)}, A_{-i} e_{j_2}^{(-i)}) = (e_{j_1}^{(i)}, e_{j_2}^{(-i)})$$

holds for any $i \in \mathbb{Z}_{2n}$, any $j_1, j_2 \in [2n]$. Now the desired equations come from the explicit form of the matrices A_i as in [6, Proposition 4.5]. □

3.2 Lie algebra of the symplectic automorphism group

Now we compute the dimensions of G^{sp} and of $X(k, 2n)^{sp}$. Let $\mathfrak{g} := \mathrm{Lie}(G) = \mathrm{End}(U_{[2n]})$. These endomorphisms are explicitly described in [6, Proposition 4.5].

Let $\big(x(a,b) \mid a \in [2n], b \in \mathbb{Z}_{2n}\big)$ be the $\mathbb{C}$-basis of $\mathfrak{g}$ acting as

$$x(a,b)\big(e_{1+j}^{(b+j)}\big) = e_{a+j}^{(b+j)} \quad \text{for } j = 0, \ldots, 2n-a$$

and as zero when applied to any other basis vector. With respect to the basis $\left\{e_j^{(i)} \mid j \in [2n]\right\}_{i \in \mathbb{Z}_{2n}}$, the operator $x(a,b)$ is the matrix tuple $(x(a,b)_i)_{i \in \mathbb{Z}_{2n}}$ whose $b+j$-th block ($j \in [0, 2n-1]$) has (s,t)-entry equals to

$$x(a,b)_{b+j,(s,t)} = \begin{cases} 1 & \text{if } s - j = a \text{ and } t - j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

by [6, Proposition 4.5]. Notice that in particular the block $x(a,b)_{b+j}$ is the null matrix as soon as $a + j > 2n$.

Recall that the group G is equipped with the automorphism σ_G from (3.2). This induces a Lie algebra automorphism of $\mathfrak{g}$ of order 2:

$$\sigma_{\mathfrak{g}}(x_i) := (\Omega x_{-i}^t \Omega)_i \,.$$

Proposition 3.6 *The automorphism $\sigma_{\mathfrak{g}}$ acts on the basis of $\mathfrak{g}$ via:*

$$\sigma_{\mathfrak{g}}\big(x(a,b)\big) = (-1)^a \cdot x(a, a-b).$$

Proof Fix $a \in [2n]$, $b \in \mathbb{Z}_{2n}$ and let $-i = b + j$ for $j \in [0, 2n-1]$. If $j \leq 2n - a$ then all entries of $x(a,b)_{-i}$ are zero except for a 1 in position $(a+j, 1+j)$, hence its transpose has only a 1 in position $(1+j, a+j)$. Because $\Omega_{st} = (-1)^{s+1}\delta_{s+t,2n+1}$, for any matrix $A = (A_{st})_{st}$ we obtain

$$(\Omega A \Omega)_{st} = (-1)^{s+t+1} \cdot A_{2n-s+1,2n-t+1}.$$

Thus $\Omega x_{-i}^t \Omega$ has zeros everywhere except a $(-1)^{(1+j)+(a+j)+1} = (-1)^a$ in position $(2n - j, 2n - a - j + 1)$. If we let $\ell = 2n - a - j \in [0, 2n-a]$, then $i = a - b + \ell$ and $\sigma_{\mathfrak{g}}\big(x(a,b)\big)_i = (-1)^a \cdot x(a, a-b)_i$. If $j \geq 2n - a + 1$ then $x(a,b)_{-i} = 0$ and $\ell + 2n \geq 2n - a + 1$, so $\sigma_{\mathfrak{g}}\big(x(a,b)\big)_i = 0 = x(a, a-b)_i$. □

Definition 3.7 Let $y(a,b) := \frac{1}{2}\big[x(a,b) + (-1)^a \cdot x(a, a-b)\big]$.

Proposition 3.8 *The dimension of G^{sp} is $2n^2 + n$.*

Proof The elements from Definition 3.7 span $\mathfrak{g}^{\sigma_{\mathfrak{g}}}$, and since $y(a,b) = (-1)^a \cdot y(a, a-b)$, a basis of $\mathfrak{g}^{\sigma_{\mathfrak{g}}}$ made of such elements has cardinality $2n^2 + n$. We end the proof by remarking that $\mathfrak{g}^{\sigma_{\mathfrak{g}}}$ is the Lie algebra of G^{sp}, which therefore has the same dimension. □

3.3 Orbits of symplectic juggling patterns

Proposition 3.9 *If $p_{\mathcal{J}} \in X(k, 2n)^{sp}$, then*

$$C_{\mathcal{J}} \cap X(k, 2n)^{sp} = G^{sp}.p_{\mathcal{J}} \,.$$

Proof Let $V = (V_i)_i$ be a point in $C_{\mathcal{J}} \cap X(k, 2n)^{sp}$. We want to show that there exists $A = (A_i)_i \in G^{sp}$ such that $V = A.p_{\mathcal{J}}$. Observe that this A is not unique since $\dim G^{sp} > \dim X(k, 2n)^{sp}$. Let $e_j^{(i)}$ denote the j-th standard basis vector of the i-th copy of $\mathbb{C}^{2n}$ as vector space of the Δ_{2n}-representation $U_{[2n]}$. The point $p_{\mathcal{J}}$, seen as a Δ_{2n}-module,

is a direct sum of several indecomposables $p_{\mathcal{J},1} \oplus \cdots \oplus p_{\mathcal{J},s}$ (see [6, Example 3.3] and [7, Proposition 3.2]). For $c \in [s]$, let $e_{j_c}^{(i_c)}$ denote the basis vector corresponding to the starting point of the indecomposable $p_{\mathcal{J},c}$ viewed as a subrepresentation of $U_{[2n]}$. Then V is completely determined by vectors $v_{i_c} \in V_{i_c}$, for $c = 1, \ldots, s$ such that

$$v_{i_c} = \sum_{j=j_c}^{2n} b_{j,i_c} e_j^{(i_c)}, \quad \text{with } b_{j_c,i_c} \neq 0.$$

Since $V \in X(k, 2n)^{sp}$, we know that $(V_i, V_{-i}) = 0$. The equality $A.p_J = V$ means that s (out of $2n$) first columns start with the numbers $b_{\bullet,\bullet}$. More precisely, the following equalities hold for all $c \in [s]$:

$$a_1^{(i_c-j_c+1)} = b_{j_c,i_c}, \quad a_2^{(i_c-j_c+1)} = b_{j_c+1,i_c}, \quad \ldots \quad a_{2n-j_c+1}^{(i_c-j_c+1)} = b_{2n,i_c}. \tag{3.5}$$

We are left with the following problem: given several first entries of first columns of several matrices A_i satisfying the orthogonality conditions, we have to complete this data to an element $A \in G^{sp}$. We proceed by inductive application of Lemma 3.4 to prove this claim:

The equations from Lemma 3.4 satisfy the following properties:

- $a_1^\bullet$ is either fixed by (3.5) and (3.3) or free;
- for any pair $i_1 \neq i_2 \in \mathbb{Z}_{2n}$ there is a single relation involving $a_\bullet^{(i_1)}$ and $a_\bullet^{(i_2)}$;
- the relation

$$a_1^{(i)} a_r^{(r-i)} = a_2^{(i)} a_{r-1}^{(r-i)} - \cdots + (-1)^r a_r^{(i)} a_1^{(r-i)}$$

allows one to reconstruct $a_r^{(r-i)}$ starting from $a_{r'}^{(\bullet)}$ with $r' \leq r$.

Observe that the number of free parameters depends on $\dim \mathrm{Stab}_{G^{sp}} p_{\mathcal{J}}$. The procedure of recovering $A \in G^{sp}$ is as follows. Start with $a_1^\bullet$: some of them are known from V and all the others are free or recovered via relation (3.3). We proceed with $a_2^\bullet$ and so on. At each step some coefficients are known from the data given by V as in (3.5). All the others are either free or recovered by using the relations from Lemma 3.4. □

Example 3.10 Let $k = n = 2$ and let $\mathcal{J} = (J_0, J_1, J_2, J_3)$ be given by $(\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\})$. Then $p_{\mathcal{J}} = p_{\mathcal{J},1} \oplus p_{\mathcal{J},2}$, where both indecomposables $p_{\mathcal{J},1}$ and $p_{\mathcal{J},2}$ are four-dimensional, $p_{\mathcal{J},1}$ starts at vertex 0 and $p_{\mathcal{J},1}$ starts at vertex 3 (hence $s = 2, i_1 = 0, i_2 = 3$). A point $V = (V_i)_i \in C_{\mathcal{J}} \cap X(k, 2n)^{sp}$ is completely determined by two vectors:

$$v_0 = b_{1,0} e_1^{(0)} + b_{2,0} e_2^{(0)} + b_{3,0} e_3^{(0)} + b_{4,0} e_4^{(0)} \in V_0, \ b_{1,0} \neq 0,$$
$$v_3 = b_{1,3} e_1^{(3)} + b_{2,3} e_2^{(3)} + b_{3,3} e_3^{(3)} + b_{4,3} e_4^{(3)} \in V_3, \ b_{1,3} \neq 0.$$

Hence one gets $a_j^{(0)} = b_{j,0}$, $a_j^{(3)} = b_{j,3}$ for $1 \leq j \leq 4$. The elements $a_j^{(1)}$ and $a_j^{(2)}$ are subject to the following relations:

$$a_1^{(2)} = b_{1,3}^{-1}, \ a_1^{(1)} = b_{1,0}^{-1}, \ a_2^{(2)} b_{1,0} = b_{2,0} b_{1,3}^{-1},$$
$$a_1^{(2)} a_3^{(1)} - a_2^{(2)} a_2^{(1)} + a_3^{(2)} a_1^{(1)} = 0, \ a_1^{(1)} b_{4,3} - a_2^{(1)} b_{3,3} + a_3^{(1)} b_{2,3} - a_4^{(1)} b_{1,3} = 0.$$

One easily sees that this system does have solutions.

Remark 3.11 Lemma 3.4 together with Proposition 3.9 allows to compute the dimension of the cells $C_{\mathcal{J}}^{sp} := C_{\mathcal{J}} \cap X(k, 2n)^{sp}$. In Lemma 4.10 and Proposition 4.12 we provide combinatorial formulas.

3.4 Orbit structure of the main object

Theorem 3.12 *The group G^{sp} is a degeneration of the classical group Sp_{2n}; in particular,* $\dim G^{sp} = 2n^2 + n$. *The group G^{sp} acts on $X(k, 2n)^{sp}$ with a finite number of orbits, the orbits are affine cells and are naturally labeled by the symplectic juggling patterns.*

Proof The degeneration procedure is analogous to $X(k, 2n)^{sp}$ as degeneration of $Gr(k, 2n)^{sp}$, as described in Sect. 2.4. The dimension of G^{sp} is computed in Proposition 3.8. In Proposition 3.9, the symplectic subsets of cells are obtained as G^{sp}-orbits of the points corresponding to juggling patterns as defined in Remark 2.3. This explicit description of the symplectic subsets of the G-orbits, together with the equations from Lemma 3.4, imply that they are affine cells, since this holds for the G-orbits by [6, Theorem 1]. □

4 Properties of the symplectic orbits

In this section we examine the poset structures on $JP(k, 2n)^{sp}$ and the dimension of the cells $C^{sp}_{\mathcal{J}} \subset X(k, 2n)^{sp}$ for $\mathcal{J} \in JP(k, 2n)^{sp}$. The computations are based on the combinatorics of the so called coefficient quivers associated to the juggling patterns.

4.1 Mutations of coefficient quivers

Let $B = \{e^{(i)}_j : j \in [2n]\}_{i \in \mathbb{Z}_{2n}}$, where $e^{(i)}_j$ is as usual the j-th standard basis vector in the i-th copy of $\mathbb{C}^{2n}$ as vector space of the Δ_{2n}-representation $U_{[2n]}$.

Definition 4.1 Let $Q(U_{[2n]}, B)$ denote the coefficient quiver of $U_{[2n]}$ with respect to the basis B. Thus Q has B as vertex set and arrows $e^{(i)}_j \to e^{(i+1)}_{j+1}$ for any $i \in \mathbb{Z}_{2n}$ and any $j \in [2n-1]$ (see [6, Definition 2.2, Definition 3.3] for more detail).

Note that every $(k, 2n)$-juggling pattern $\mathcal{J} \in JP(k, 2n)$ can be identified with an appropriate successor-closed subquiver $S_{\mathcal{J}}$ in $Q(U_{[2n]}, B)$ as follows: the subquiver $S_{\mathcal{J}}$ contains a vertex $e^{(i)}_j$ if $j \in J_i$; moreover, it contains an arrow of $Q(U_{[2n]}, B)$ if its source and target are contained in the vertex set of $S_{\mathcal{J}}$. Then $S_{\mathcal{J}}$ is successor-closed in $Q(U_{[2n]}, B)$ (we write $S_{\mathcal{J}} \subseteq^{sc} Q(U_{[2n]}, B)$ for short), i.e. if a vertex $v \in S_{\mathcal{J}}$ is a source of an arrow $\alpha : v \to w$ of $Q(U_{[2n]}, B)$ then $S_{\mathcal{J}}$ also contains w (and hence α). The above identification defines an isomorphism between $JP(k, 2n)$ and the set

$$SC(k, 2n) := \left\{ S \subseteq^{sc} Q(U_{[2n]}, B) \ \#S \cap \{e^{(i)}_j \ \ j \in [2n]\} = k \text{ for all } i \in \mathbb{Z}_n \right\}.$$

Definition 4.2 Two elements $S, S' \in SC(k, 2n)$ are connected by a mutation $\mu : S' \to S$ if they differ by the position of a (predecessor closed) segment, i.e.:

$$S' \setminus S = e^{(i)}_j \to e^{(i+1)}_{j+1} \to \cdots \to e^{(i+\ell)}_{j+\ell},$$
$$S \setminus S' = e^{(i)}_{j+s} \to e^{(i+1)}_{j+1+s} \to \cdots \to e^{(i+\ell)}_{j+\ell+s}.$$

We write $S' \geq^{\mu} S$ if there is a sequence of mutations $\mu_1, \ldots, \mu_r$ such that

$$S = \mu_r \circ \cdots \circ \mu_1(S').$$

By [33, Corollary 2.22] this defines a partial order on $SC(k, 2n)$. In [7, Theorem 4.6] we prove the following statement.

Proposition 4.3 *There is an order preserving poset isomorphism between $JP(k, 2n)$ with the order induced by cell closures in $X(k, 2n)$ and $SC(k, 2n)$ with the order induced by mutation sequences.*

4.2 Maximal symplectic juggling patterns

The above poset isomorphism allows us to apply the combinatorics of mutations to examine the structure of cell closures in $X(k, 2n)$ and its subvariety $X(k, 2n)^{sp}$. We say $S' \geq^{\mu} S$ are *adjacent* if there is no sequence of mutations from S' to S with $r > 1$. Analogously we define adjacent juggling patterns.

Lemma 4.4 *Let $\mathcal{J}$ be a non-maximal $(k, 2n)$-juggling pattern such that $\mathcal{J} \subseteq R\mathcal{J}$. Now let $\mathcal{J}' \geq \mathcal{J}$ be another $(k, 2n)$-juggling pattern, adjacent to $\mathcal{J}$. Suppose $\mathcal{J}' \nsubseteq R\mathcal{J}'$. Then there exists $\mathcal{J}'' \geq \mathcal{J}'$, adjacent to $\mathcal{J}'$, such that $\mathcal{J}'' \subseteq R\mathcal{J}''$.*

Proof First we set some notation: given $\mathcal{J}$ a juggling pattern, $x \in [2n]$ and $a \in \mathbb{Z}_{2n}$, we write $x^{(a)} \in \mathcal{J}$ to indicate $x \in J_a$.

We consider the inverse simple mutation in the coefficient quiver that links the successor-closed subquivers associated to $\mathcal{J}$ and $\mathcal{J}'$: let

$$x^{(a)} \to (x+1)^{(a+1)} \to \cdots \to (x+\ell)^{(a+\ell)}$$

be the elements of the sets of $\mathcal{J}$ that mutate into different ones. There exists an integer $s \geq 1$ such that these numbers are substituted with

$$(x-s)^{(a)} \to (x+1-s)^{(a+1)} \to \cdots \to (x+\ell-s)^{(a+\ell)} \tag{4.1}$$

and $\mathcal{J}'$ differs from $\mathcal{J}$ only by this change, since they are adjacent. Since this is a mutation, we have the following:

- $x \geq 1+s \geq 2$;
- $(x-1)^{(a-1)} \notin \mathcal{J}$;
- $(x-s-1)^{(a-1)} \notin \mathcal{J}'$;
- $x+\ell \leq 2n$, and therefore $\ell \leq 2n-2$;
- $x+\ell-s+1 \in J_{a+\ell+1} \cap J'_{a+\ell+1}$.

Recall now that $\mathcal{J}$ is symplectic, so we know that for all $a \in \mathbb{Z}_{2n}$ and all $y \in J_a$, the element $2n+1-y$ is not in J_{-a}. By our assumption, $\mathcal{J}'$ does not have this property, that is, there exist one vertex b and one element $y \in J'_b$ such that $2n+1-b \in J'_{-b}$, and it must be one that appears in segment (4.1). We consider the elements paired to the elements of (4.1) by the symplectic form. They again form a segment (not necessarily contained completely in $\mathcal{J}'$):

$$(2n-x-\ell+s+1)^{(-a-\ell)} \to \cdots \to (2n-x+s)^{(-a-1)} \to (2n-x+s+1)^{(-a)}. \tag{4.2}$$

We will call the elements of this segment which are in $\mathcal{J}'$ *problems*, and we know there is at least one, say $(2n-x-j+s+1)^{(-a-j)}$ for some $j \in [0, \ell]$. Then for all $0 \leq i \leq j$, $2n-x-i+s+1$ is in J'_{-a-i}, because $\mathcal{J}'$ is a juggling pattern. So the problems make up a successor closed subsegment of (4.2). We assume now that $(2n-x-j+s+1)^{(-a-j)}$ is the leftmost problem, so that it starts a segment in the successor-closed subquiver representing $\mathcal{J}'$. To obtain a new symplectic juggling pattern from $\mathcal{J}'$, we want to apply a mutation to remove this exact segment, again by subtracting the integer s. We can do so since $2n-x-j+s+1 \geq$

$s+1$ (recall that $s \geq 1$, $j \leq \ell$ and $x+\ell \leq 2n$), and it creates a juggling pattern $\mathcal{J}''$ which is symplectic, because any problem with it would be of the form $(2n-x+1+i)^{(-a+i)}$, with $i \geq 2n-j$, and it is not possible since the element paired to it is neither in $\mathcal{J}'$ nor $\mathcal{J}''$, since they were removed with the first mutation (4.1). □

4.3 Symplectic mutations and closures of symplectic cells

Corollary 4.5 *If $\mathcal{J}'' > \mathcal{J}$ are symplectic and adjacent in $JP(k,2n)$, or they are as in the previous lemma, then $p_{\mathcal{J}}$ is in the closure of the G^{sp}-orbit of $p_{\mathcal{J}''}$ in $X(k,2n)^{sp}$.*

Proof We define the path $V(t)$, $t \in \mathbb{C}$, as follows: for $a \in \mathbb{Z}_{2n}$ we set

$$V(t)_a := \mathrm{Span}\Big(\{e_j^{(a)} \mid j \in J_a \cap J_a''\} \cup \{e_{j-s}^{(a)} + t \cdot e_j^{(a)} \mid j \in J_a \backslash J_a''\}\Big),$$

where s is as in Definition 4.2 if $\mathcal{J}$ and $\mathcal{J}''$ are adjacent, or as in the proof of the previous Lemma if they are not. It follows from the explicit description of the mutations as in Lemma 4.4 that $V(t)$ is a point in the cell $C_{\mathcal{J}''}$ of $X(k,2n)$ for all $t \in \mathbb{C}$, since it satisfies the equations describing the cells as computed in the proof of [32, Theorem 4.13].

Now, we show that $V(t)$ is contained in $X(k,2n)^{sp}$. For any $j \in J_a \cap J_a''$, the vector $e_j^{(a)}$ pairs trivially with any other element in $V(t)_{-a}$, so we can take $j \in J_a \backslash J_a''$ and $y \in J_{-a} \backslash J_{-a}''$. Then

$$(t \cdot e_j^{(a)} + e_{j-s}^{(a)}, t \cdot e_y^{(-a)} + e_{y-s}^{(-a)}) = (t \cdot e_j^{(a)}, e_{y-s}^{(-a)}) + (e_{j-s}^{(a)}, t \cdot e_y^{(-a)}) = 0\,.$$

The expression is zero because the two summands are opposites of each other, since $j-s$ and $y-s$ are respectively in $J_a'' \backslash J_a$ and in $J_{-a}'' \backslash J_{-a}$.

Finally, we observe that $V(0) = p_{\mathcal{J}''}$ and the boundary point is $p_{\mathcal{J}}$. □

This motivates the following definition:

Definition 4.6 A symplectic mutation is either a mutation $\mu : S' \to S$ such that both S and S' are symplectic or a pair of two mutations $\mu_2 : S' \to S$ and $\mu_1 : S'' \to S'$ where S and S'' are symplectic and μ_1, μ_2 are as described in Lemma 4.4.

Remark 4.7 Sequences of symplectic mutations define a partial order on $SC(k,2n)^{sp}$ (i.e. the subset of $SC(k,2n)$ corresponding to the juggling patterns in $JP(k,2n)^{sp}$). Hence it is a natural question to ask if this partial order is induced by the mutation order on $SC(k,2n)$.

Example 4.8 In Example 2.12, we represented the Hasse diagram of $JP(2,4)^{sp}$ equipped with the combinatorial order. We grouped the elements horizontally in tiers by the dimension of their G^{sp}-orbit, from 3 to 0, top to bottom. In this case, the closure of a cell $C_{\mathcal{J}}^{sp}$ coincides with the union of symplectic cells for lower (symplectic) juggling patterns That is, the combinatorial order of symplectic $(2,4)$-juggling patterns coincides with the closure inclusion order on the set of G^{sp}-orbits in $X(2,4)^{sp}$. Otherwise stated, closure inclusion order on the set of G^{sp}-orbits in $X(2,4)^{sp}$ is induced by the closure inclusion order on the set of G-orbits in $X(2,4)$. We conjecture that this is the case in general.

Conjecture 4.9 *Let $\mathcal{J}, \mathcal{J}' \in JP(k,2n)^{sp}$ such that $p_{\mathcal{J}} \in \overline{C_{\mathcal{J}'}} \subset X(k,2n)$. Then $p_{\mathcal{J}} \in \overline{C_{\mathcal{J}'}^{sp}} \subset X^{sp}(k,2n)$.*

In other words: If $S, S' \in SC(k,2n)^{sp}$ are connected by a sequence of mutations, does a sequence of symplectic mutations connecting S to S' exist?

Lemma 4.10 *For $\mathcal{J} \in JP(k,2n)^{sp}$ the dimension of $C^{sp}_{\mathcal{J}}$ equals the number of symplectic mutations starting at $S_{\mathcal{J}} \in SC(k,2n)$.*

Proof In [34, Corollary 6.5, Theorem 6.15], we prove that $\dim_{\mathbb{C}} C_{\mathcal{J}}$ equals the number of mutations starting at $S_{\mathcal{J}}$, that is, mutations of the form $\mu\colon S \to S_{\mathcal{J}}$. Every mutation is linked to one so-called terminal parameter in the equations describing the cells (see [34, (5.9), (5.10)]). Now, Lemma 4.4 implies that these parameters are only independent if they belong to different symplectic mutations. Clearly, every symplectic mutation allows to choose one independent parameter. This implies the desired dimension formula. □

Remark 4.11 In particular, every mutation starting at $S_{\mathcal{J}}$ is either symplectic or part of a symplectic pair as described in Lemma 4.4.

4.4 Top dimensional cells

Proposition 4.12 *The dimension of any top dimensional cell in $X(k,2n)^{sp}$ is $k(2n-k) - \frac{k(k-1)}{2}$.*

Proof By [6, Theorem 3] every top dimensional cell $C_{\mathcal{J}}$ of $X(k,2n)$ is of dimension $k(2n-k)$. This equals the number of mutations starting at $S_{\mathcal{J}}$ by [34, Corollary 6.5, Theorem 6.15]. To apply Lemma 4.10, we count the symplectic mutations starting at $S_{\mathcal{J}}$. At each of the k-many segments in $S_{\mathcal{J}}$ of length $2n$, there start $2n-k$-many mutations. But $k-1$ of them make $\mathcal{J}$ non-symplectic. They have to be paired with their correction move as described in Lemma 4.4. In total there start $k(2n-k) - k(k-1)$ single mutations at $S_{\mathcal{J}}$ which are symplectic and $\frac{k(k-1)}{2}$ symplectic pairs as described in Lemma 4.4. This sums up to the desired formula. □

Proposition 4.13 *The number of top dimensional cells in $X(k,2n)^{sp}$ is*

$$\frac{(2n)!!}{k!(2n-2k)!!}.$$

Proof By Lemma 4.4, the top dimensional cells $C^{sp}_{\mathcal{J}}$ in $X(k,2n)^{2p}$ are the intersection of the top dimensional cells $C_{\mathcal{J}} \subset X(k,2n)$ with $X(k,2n)^{sp}$, where $\mathcal{J}$ is symplectic. Their number is given by the number of cardinality k subsets of $[2n]$ that contain no pair $(i,\tilde{i})$. There are $2^k \cdot \binom{n}{k} = \frac{(2n)!!}{k!(2n-2k)!!}$ such sets. □

Theorem 4.14 *The intersection of a cell of $X(k,2n)$ with $X(k,2n)^{sp}$ is either empty or an affine cell. The top dimensional cells of $X(k,2n)^{sp}$ are obtained from top dimensional cells of $X(k,2n)$. The dimension of $X(k,2n)^{sp}$ is equal to the dimension of the classical symplectic Grassmannian $\mathrm{Gr}(k,2n)^{sp}$ and the number of irreducible components of $X(k,2n)^{sp}$ is equal to the Euler characteristic of $\mathrm{Gr}(k,2n)^{sp}$.*

Proof In Theorem 3.12 the intersection of the cell $C_{\mathcal{J}}$ with the symplectic subvariety is described as the G^{sp}-orbit of the point $p_{\mathcal{J}}$ corresponding to a symplectic juggling pattern $\mathcal{J}$. In particular, it is an affine cell. If $\mathcal{J}$ is not symplectic, then $\mathcal{J} \nsubseteq R\mathcal{J}$ and hence there is at least one $i \in \mathbb{Z}_{2n}$ and an $a \in [2n]$ such that $a \in J_i$ and $2n-a+1 \in J_{-i}$. By the description of the G-orbits in $X(k,2n)$ from [6], we know that every point $(V_s)_s \in C_{\mathcal{J}}$ is such that the space V_i contains a vector of the form $v = e^{(i)}_a + \sum_{s=a+1}^{2n} c_s e^{(i)}_s$, while V_{-i} contains a vector of the form $w = e^{(-i)}_{2n+1-a} + \sum_{s=2n+2-a}^{2n} d_s e^{(-i)}_s$ for some complex numbers $c_s, d_s \in \mathbb{C}$. It is now clear that $(v,w) \neq 0$ and hence $(V_s)_s \notin X(k,2n)^{sp}$. For the maximal symplectic juggling

patterns, the dimension of the corresponding cell is computed in Proposition 4.12. It equals the dimension of $\mathrm{Gr}(k, 2n)^{sp}$. The irreducible components are obtained as closure of the top dimensional cells by Lemma 4.4. Hence, their number matches the Euler characteristic of $\mathrm{Gr}(k, 2n)^{sp}$ by Proposition 4.13 (see [35] where a similar property holds in a more general setup). □

5 Affine Grassmannians and flag varieties

5.1 The type A case

Here we briefly recall some background on type A affine Grassmannians and flag varieties (more details can be found, for example, in [32] or in [36] in much greater generality). Let us fix a positive integer number N. Let V be an N-dimensional complex vector space with a basis $v_1, \dots, v_N$. We consider the set of lattices inside the space $V[t, t^{-1}] = V \otimes \mathbb{C}[t, t^{-1}]$, i.e t-invariant subspaces L such that for some $m \in \mathbb{Z}_{>0}$ and $c \in \mathbb{Z}$ one has

$$t^m V[t] \subset L \subset t^{-m} V[t], \qquad \dim L/t^m V[t] = mN + c. \tag{5.1}$$

We denote by $\mathcal{AG}_{N,c}$ the set of lattices satisfying (5.1) for a fixed $c \in \mathbb{Z}$. For example, one has a distinguished lattice $\mathring{L}_c \in \mathcal{AG}_{N,c}$ defined as follows. Let us write $c = Nd + r$, where $d \in \mathbb{Z}$ and $r = 0, \dots, N-1$. Then

$$\mathring{L}_c = t^{-d} \left(V[t] \oplus \mathrm{Span}\{v_1 t^{-1}, \dots, v_r t^{-1}\} \right). \tag{5.2}$$

The affine Grassmannians $\mathcal{AG}_{N,c}$ are endowed with a structure of infinite-dimensional ind-varieties and the multiplication by t^{-1} induces a natural isomorphism $\mathcal{AG}_{N,c} \simeq \mathcal{AG}_{N,c+N}$. These ind-varieties are equipped with a transitive action of the affine Kac-Moody Lie group $\widehat{SL}_N$. One has $\mathcal{AG}_{N,c} \simeq \widehat{SL}_N / P_c$ for $c = 0, \dots, N-1$, where P_c is the parahoric subgroup corresponding to the fundamental weight Λ_c. The disjoint union $\sqcup_{c=0}^{N-1} \mathcal{AG}_{N,c}$ is identified with the affine Grassmannian for the affine GL_N group.

The affine flag variety $\mathcal{AF}_N$ sits inside the product $\prod_{c \in \mathbb{Z}} \mathcal{AG}_{N,c}$ and consists of collections $(L_c)_{c \in \mathbb{Z}}$ such that $L_c \subset L_{c+1}$ and $L_{c+N} = t^{-1} L_c$. The ind-variety $\mathcal{AF}_N$ is isomorphic to the quotient $\widehat{SL}_N / \mathbb{I}$, where $\mathbb{I}$ is the Iwahori subgroup defined as the preimage of the Borel subgroup $B \subset SL_N$ under the $t = 0$ evaluation map $SL_N[t] \to SL_N$. We note that $\mathcal{AF}_N$ contains the distinguished point $\mathring{L} = (\mathring{L}_c)_{c \in \mathbb{Z}}$.

5.2 Symplectic version

Assume that $N = 2n$ is even. We endow the space V with a skew-symmetric nondegenerate bilinear form defined by $(v_i, v_j) = (-1)^{i+1} \delta_{i+j, N+1}$. The form induces a skew-symmetric nondegenerate form on $V[t, t^{-1}]$ given by $(vt^r, ut^s) = (v, u)\delta_{r+s,-1}$. For example, the lattice $\mathring{L}_0$ is Lagrangian with respect to this form.

The symplectic affine flag variety $\mathcal{AF}_{2n}^{\mathfrak{sp}}$ is the subvariety of the type A affine flag variety $\mathcal{AF}_{2n}$ defined by

$$(L_c)_{c \in \mathbb{Z}} \in \mathcal{AF}_{2n}^{\mathfrak{sp}} \ \text{ if } \ (L_c)_{c \in \mathbb{Z}} \in \mathcal{AF}_{2n} \text{ and } L_{-c} = L_c^{\perp}. \tag{5.3}$$

In other words, if we consider a degree two automorphism σ of $\mathcal{AF}_{2n}$ defined by $\sigma.(L_c)_{c \in \mathbb{Z}} = (L_{-c}^{\perp})_{c \in \mathbb{Z}}$, then $\mathcal{AF}_{2n}^{sp}$ is the set of σ-fixed points. Note that the symplectic affine flag variety contains the distinguished point $\mathring{L} = (\mathring{L}_c)_{c \in \mathbb{Z}}$.

Let $Sp_{2n} \subset SL_{2n}$ be the Lie group preserving the above defined skew-symmetric form on V. Let $B^{sp} := B \cap Sp_{2n}$ be the Borel subgroup of Sp_{2n}, consisting of symplectic upper triangular matrices (or rather, one needs the intersection with a Borel subgroup closed under the involution on SL_{2n} given by the chosen symplectic form). We denote by $\widehat{Sp}_{2n} \subset \widehat{SL}_{2n}$ the symplectic affine group. As in type A, the Iwahori subgroup $\mathbb{I}^{sp}$ is the preimage of the Borel subgroup $B^{sp} \subset Sp_{2n}$ under the evaluation map $Sp_{2n}[t] \to Sp_{2n}$, $t \mapsto 0$. Then $\mathcal{AF}^{sp}_{2n} \simeq \widehat{Sp}_{2n}/\mathbb{I}^{sp}$.

5.3 Embeddings of quiver Grassmannians

As usual, for $j \in [2n]$, we denote by e_j the j-th standard basis vector of $\mathbb{C}^{2n}$; moreover, whenever we identify $\mathbb{C}^{2n}$ with with $U^{(i)}_{[2n]}$, that is the i-th space of our Δ_{2n}-representation $U_{[2n]}$, we will use the notation $e^{(i)}_1, \dots, e^{(i)}_{2n}$ for the standard basis vectors.
For $k = 1, \dots, n$ consider the juggling variety $X(k, 2n)$. We recall the embedding $\varphi : X(k, 2n) \to \mathcal{AF}_{2n}$ from [7, Section 6.1]. Given a point $U = (U_i)_{i=0}^{2n-1}$ we describe the components $(\varphi U)_c$ of the point $\varphi U \in \mathcal{AF}_{2n}$ below.

Let us define a family $\eta_{j,d}$ of embeddings $\mathbb{C}^{2n} \to V[t, t^{-1}]$. The embeddings are labeled by $d \in \mathbb{Z}$, $j \in [2n]$; $\eta_{j,d}$ is defined by

$$e_{2n} \mapsto v_j t^d, \quad e_{2n-1} \mapsto v_{j+1} t^d, \dots, \ e_j \mapsto v_{2n} t^d,$$
$$e_{j-1} \mapsto v_1 t^{d-1}, \ e_{j-2} \mapsto v_2 t^{d-1}, \dots, \ e_1 \mapsto v_{j-1} t^{d-1}.$$

Now we define $\varphi : X(k, 2n) \to \mathcal{AF}_{2n}$ in the following way: (recall that $\mathring{L}_{-n} = tV[t] \oplus \mathrm{Span}\{v_1 t^0, \dots, v_n t^0\}$):

$$\begin{gathered}
(\varphi U)_0 = \mathring{L}_{-n} \oplus \eta_{n+1,0} U_0, \\
(\varphi U)_1 = \mathring{L}_{-n+1} \oplus \eta_{n+2,0} U_1, \\
\dots\dots\dots\dots\dots \\
(\varphi U)_{n-1} = \mathring{L}_{-1} \oplus \eta_{2n,0} U_{n-1}, \\
(\varphi U)_n = \mathring{L}_0 \oplus \eta_{1,-1} U_n, \\
\dots\dots\dots\dots\dots \\
(\varphi U)_{2n-1} = \mathring{L}_{n-1} \oplus \eta_{n,-1} U_{2n-1}.
\end{gathered}$$

We also set $(\varphi U)_{c+2n} = t^{-1}(\varphi U)_c$.

Remark 5.1 One easily sees (see [7, Lemmas 6.2 and 6.4]) that $(\varphi U)_c \in \mathcal{AG}_{2n,c}$ and that $(U_i)_i \in X(k, N)$ implies $\varphi(U) \in \mathcal{AF}_{2n}$.

Example 5.2 Let us take the point $(U_i)_{i=0}^{2n-1} \in X(k, 2n)$ given by $U_i = \mathrm{Span}\{e^{(i)}_{2n-k+1}, \dots, e^{(i)}_{2n}\}$ for all $i \in \mathbb{Z}_{2n}$. Then $\varphi U = \mathring{L}$.

Now we are ready to formulate the main result of this section.

Proposition 5.3 *For any $k = 1, \dots, n$ the image $\varphi(X(k, 2n)^{sp})$ belongs to $\mathcal{AF}^{sp}_{2n}$ seen inside $\mathcal{AF}_{2n}$.*

Proof Let us take a point $(U_i)_i \in X(k, 2n)^{sp}$. We need to show that the condition $(U_i, U_{-i}) = 0$ implies $(\varphi U)_j \subset (\varphi U)^{\perp}_{-j}$. Let us start with $j = 0$. By definition

$$(\varphi U)_0 = \mathring{L}_{-n} \oplus \eta_{n+1,0} U_0.$$

Since $\mathring{L}_{-n} = tV[t] \oplus \mathrm{Span}\{v_1 t^0, \ldots, v_n t^0\}$ and the image of $\eta_{n+1,0}$ is the subspace $\mathrm{Span}\{v_{n+1}t^0, \ldots, v_{2n}t^0, v_1 t^{-1}, \ldots, v_n t^{-1}\}$, we conclude that

$$(\mathring{L}_{-n}, \mathring{L}_{-n} \oplus \eta_{n+1,0}U_0) = 0$$

(recall that the skew-symmetric residue pairing is given by $(v_i t^a, v_j t^b) = \delta_{a+b,-1}\delta_{i+j,2n+1}(-1)^{i+1}$). We are left to show that $(\eta_{n+1,0}U_0, \eta_{n+1,0}U_0) = 0$. By definition the map

$$\eta_{n+1,0} : W \to \mathrm{Span}\{v_{n+1}t^0, \ldots, v_{2n}t^0, v_1 t^{-1}, \ldots, v_n t^{-1}\}$$

sends the form on W to the restriction of the residue form to the image). Hence the condition $(U_0, U_0) = 0$ implies $((\varphi U)_0, (\varphi U)_0) = 0$.

Now let us take $i = 1, \ldots, n$ and let us prove that $((\varphi U)_i, (\varphi U)_{-i}) = 0$. It suffices to show that

$$\left(\mathring{L}_{-n+i} \oplus \eta_{n+i+1,0}U_i, t(\mathring{L}_{n-i} \oplus \eta_{n-i+1,-1}(U_{2n-i}))\right) = 0$$

(recall $(\varphi U)_{-i} = t(\varphi U)_{2n-i}$). All the pairings except for the single term $(\eta_{n+i+1,0}U_i, t\eta_{n-i+1,-1}U_{-i})$ vanish by the definition of the residue pairing. The remaining term is zero due to the condition $(U_i, U_{-i}) = 0$. □

We conclude with the following theorem, which is proved exactly as in [6, 7].

Theorem 5.4 *The image of $X(k, 2n)^{sp}$ inside $\mathcal{AF}^{sp}_{2n}$ is equal to a union of Schubert cells. The embedding $\varphi : X(k, 2n)^{sp} \to \mathcal{AF}^{sp}_{2n}$ translates the action of the symplectic automorphism group on $X(k, 2n)^{sp}$ into the action of the Iwahori sugroup $\mathbb{I}^{sp}$ on $\mathcal{AF}^{sp}_{2n}$.*

Appendix A Numerical data

In this section we provide the Euler characteristics $\chi_{k,2n}$ and $\chi^{sp}_{k,2n}$ of the varieties $X(k, 2n)$ and $X(k, 2n)^{sp}$ (the numbers of juggling patterns and of the symplectic juggling patterns) as well as the corresponding Poincaré polynomials $P_{k,2n}(t)$ and $P^{sp}_{k,2n}(t)$. For the symplectic setting we use Lemma 4.10.

Let $n = 1$. Then

$$\chi_{1,2} = \chi^{sp}_{1,2} = 3, \;\; P_{1,2}(t) = P^{sp}_{1,2}(t) = 2t + 1.$$

Let $n = 2$. Then

$$\begin{aligned}
&\chi_{1,4} = \chi^{sp}_{1,4} = 15,\\
&P_{1,4}(t) = P^{sp}_{1,4}(t) = 4t^3 + 6t^2 + 4t + 1,\\
&\chi_{2,4} = 33, \quad \chi^{sp}_{2,4} = 13,\\
&P_{2,4}(t) = 6t^4 + 12t^3 + 10t^2 + 4t^1 + 1,\\
&P^{sp}_{2,4}(t) = 4t^3 + 5t^2 + 3t^1 + 1.
\end{aligned}$$

Let $n = 3$. Then

$$\begin{aligned}
&\chi_{1,6} = \chi^{sp}_{1,6} = 63,\\
&P_{1,6}(t) = P^{sp}_{1,6}(t) = 6t^5 + 15t^4 + 20t^3 + 15t^2 + 6t^1 + 1,\\
&\chi_{2,6} = 473, \quad \chi^{sp}_{2,6} = 293,
\end{aligned}$$

$$P_{2,6}(t) = 15t^8 + 60t^7 + 110t^6 + 120t^5 + 90t^4 + 50t^3 + 21t^2 + 6t^1 + 1,$$
$$P^{sp}_{2,6}(t) = 12t^7 + 47t^6 + 81t^5 + 77t^4 + 48t^3 + 21t^2 + 6t^1 + 1,$$
$$\chi_{3,6} = 883, \quad \chi^{sp}_{3,6} = 79,$$
$$P_{3,6}(t) = 20t^9 + 90t^8 + 180t^7 + 215t^6 + 180t^5 + 114t^4 + 56t^3 + 21t^2 + 6t^1 + 1,$$
$$P^{sp}_{3,6}(t) = 8t^6 + 18t^5 + 22t^4 + 17t^3 + 9t^2 + 4t^1 + 1.$$

Let $n = 4$. Then

$$\chi_{1,8} = \chi^{sp}_{1,8} = 255,$$
$$P_{1,8}(t) = P^{sp}_{1,8}(t) = 8t^7 + 28t^6 + 56t^5 + 70t^4 + 56t^3 + 28t^2 + 8t^1 + 1,$$
$$\chi_{2,8} = 5281, \quad \chi^{sp}_{2,8} = 4053,$$
$$P_{2,8}(t) = 28t^{12} + 168t^{11} + 476t^{10} + 840t^9 + 1050t^8 + 1008t^7 + 784t^6 + 504t^5 + 266t^4 + 112t^3 + 36t^2 + 8t^1 + 1,$$
$$P^{sp}_{2,8}(t) = 24t^{11} + 166t^{10} + 478t^9 + 798t^8 + 904t^7 + 759t^6 + 501t^5 + 266t^4 + 112t^3 + 36t^2 + 8t^1 + 1,$$
$$\chi_{3,8} = 26799, \quad \chi^{sp}_{3,8} = 7507,$$
$$P_{3,8}(t) = 56t^{15} + 420t^{14} + 1400t^{13} + 2870t^{12} + 4200t^{11} + 4788t^{10} + 4480t^9 + 3542t^8 + 2408t^7 + 1420t^6 + 728t^5 + 322t^4 + 120t^3 + 36t^2 + 8t^1 + 1,$$
$$P^{sp}_{3,8}(t) = 33t^{12} + 251t^{11} + 757t^{10} + 1319t^9 + 1588t^8 + 1445t^7 + 1042t^6 + 613t^5 + 297t^4 + 117t^3 + 36t^2 + 8t^1 + 1.$$
$$\chi_{4,8} = 44929, \quad \chi^{sp}_{4,8} = 633,$$
$$P_{4,8}(t) = 70t^{16} + 560t^{15} + 1960t^{14} + 4200t^{13} + 6426t^{12} + 7672t^{11} + 7532t^{10} + 6272t^9 + 4522t^8 + 2856t^7 + 1588t^6 + 776t^5 + 330t^4 + 120t^3 + 36t^2 + 8t^1 + 1,$$
$$P^{sp}_{4,8}(t) = 16t^{10} + 56t^9 + 106t^8 + 131t^7 + 121t^6 + 93t^5 + 59t^4 + 31t^3 + 14t^2 + 5t^1 + 1.$$

Acknowledgements E.F. was partially supported by the ISF grant 493/24. M.L. acknowledges the MUR Excellence Department Project 2023–2027 awarded to the Department of Mathematics, University of Rome Tor Vergata CUP E83C18000100006, and the PRIN2022 CUP E53D23005550006. A.P. was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - SFB-TRR 358/1 2023 - 491392403. M.L and M.M. are members of the network INdAM-G.N.S.A.G.A.

Funding Open access funding provided by Università degli Studi di Roma La Sapienza within the CRUI-CARE Agreement.

References

1. Görtz, U.: On the flatness of models of certain Shimura varieties of PEL-type. Math. Ann. **321**, 689–727 (2001)
2. Gaitsgory, D.: Construction of central elements in the affine Hecke algebra via nearby cycles. Invent. Math. **144**(2), 253–280 (2001)
3. Pappas, G., Rapoport, M., Smithling, B.: Local models of Shimura varieties, I. Geometry and combinatorics, Handbook of moduli. vol. III, Adv. Lect. Math. (ALM), vol. 26, Int. Press, Somerville, MA, 2013, pp. 135–217
4. Knutson, A.: The cyclic Bruhat decomposition of $\mathrm{Gr}_k(\mathbb{C}^n)$ from the affine Bruhat decomposition of $AFlag_k^\circ$, talk at Bert Kostant's 80th birthday conference (2008). http://pi.math.cornell.edu/~allenk/positroid.pdf
5. Zhou, Q.: Convex polytopes for the central degeneration of the affine Grassmannian. Adv. Math. **348**, 541–582 (2019)
6. Feigin, E., Lanini, M., Pütz, A.: Totally nonnegative Grassmannians, Grassmann necklaces and quiver Grassmannians. Can. J. Math. **75**(4), 1076–1109 (2023)
7. Feigin, E., Lanini, M., Pütz, A.: Generalized juggling patterns, quiver Grassmannians and affine flag varieties. Preprint at arxiv:2302.00304
8. Feigin, E., Lanini, M., Pütz, A.: Laumon parahoric local models via quiver Grassmannians. Preprint at arXiv:2307.00776
9. Bigeni, A., Feigin, E.: Symmetric Dellac configurations and symplectic/orthogonal flag varieties. Linear Algebra Appl. **573**, 54–79 (2019)
10. Bigeni, A., Feigin, E.: Symmetric Dellac configurations. J. Integer Seq. **23**(4), 32 (2020)
11. Boos, M., Irelli, G.C.: On degenerations and extensions of symplectic and orthogonal quiver representations. Preprint at arXiv:2106.08666
12. Boos, M., Irelli, G.C., Fang, X., Fourier, G.: Linear degenerate symplectic flag varieties: symmetric degenerations and PBW locus. Preprint at arXiv:2405.02739
13. Görtz, U.: On the flatness of local models for the symplectic group. Adv. Math. **176**(1), 89–115 (2003)
14. Pappas, G.: Arithmetic Models for Shimura Varieties. In: Proceedings of the ICM – Rio 2018, vol. II. Invited lectures, pp. 377–398, World Scientific Publishing, Hackensack, NJ, (2018)
15. Pappas, G., Zachos, I.: Regular integral models for Shimura varieties of orthogonal type. Compos. Math. **158**(4), 831–867 (2022)
16. De Concini, C.: Symplectic standard tableaux. Adv. Math. **34**(1), 1–27 (1979)
17. Fulton, W., Harris, J.: Representation Theory: A First Course, vol. 129. Springer Science & Business Media, Cham (2013)
18. Derksen, H., Weyman, J.: Generalized quivers associated to reductive groups. Colloq. Math. **94**(2), 151–173 (2002)
19. Haines, T., Richarz, T.: Smoothness of Schubert varieties in twisted affine Grassmannians. Duke Math. J. **169**(17), 3223–3260 (2020)
20. Zhu, X.: On the coherence conjecture of Pappas and Rapoport. Ann. Math. (2) **180**(1), 1–85 (2014)
21. He, X., Zhang, N.: Degenerations of Grassmannians via lattice configurations. Int. Math. Res. Not. **2023**(1), 298–349 (2023)
22. He, X., Zhang, N.: Degenerations of Grassmannians via lattice configurations II. Preprint at arXiv:2305.00158
23. Feigin, E., Finkelberg, M.: Degenerate flag varieties of type A: Frobenius splitting and BW theorem. Math. Z. **275**(1–2), 55–77 (2013)
24. Irelli, G.C., Feigin, E., Reineke, M.: Desingularisation of quiver Grassmannians for Dynkin quivers. Adv. Math. **245**, 182–207 (2013)
25. Pütz, A., Reineke, M.: Desingularizations of quiver Grassmannians for the equioriented cycle quiver. Pac. J. Math. **326**(1), 109–133 (2023)
26. Feigin, E., Finkelberg, M., Littelmann, P.: Symplectic degenerate flag varieties. Can. J. Math. **66**(3), 1250–1286 (2014)
27. Karpman, R.: Total positivity for the Lagrangian Grassmannian. Adv. Appl. Math. **98**, 25–76 (2018)
28. Williams, L.: Enumeration of totally positive Grassmann cells. Adv. Math. **190**(2), 319–342 (2005)
29. Postnikov, A.: Total positivity, Grassmannians, and networks. Preprint at http://math.mit.edu/~apost/papers/tpgrass.pdf (2006)
30. Mihai, I.A.: Odd symplectic flag manifolds. Transform. Gr. **12**(3), 573–599 (2007)
31. Proctor, R.A.: Odd symplectic groups. Invent. Math. **92**(2), 307–332 (1988)
32. Pütz, A.: Degenerate affine flag varieties and quiver Grassmannians. Algebr. Represent. Theory **25**, 91–119 (2022)

33. Lanini, M., Pütz, A.: Permutation actions on quiver Grassmannians for the equioriented cycle via GKM-theory. J. Algebr. Comb. **57**, 915–956 (2023)
34. Lanini, M., Pütz, A.: GKM-theory for torus actions on cyclic quiver Grassmannians. Algebr. Number Theory **17**(12), 2055–2096 (2023). arxiv:2008.13138
35. Brion, M.: Group completions via Hilbert schemes. J. Algebraic Geom. **12**, 605–626 (2003)
36. Kumar, S.: Kac-Moody Groups, Their Flag Varieties and Representation Theory. Progress in Mathematics, vol. 204 (Birkhäuser, Boston, 2002)