A GLUING CONSTRUCTION FOR PERIODIC MONOPOLES

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ABSTRACT. In [4] and [6] Cherkis and Kapustin introduced the study of periodic monopoles (with singularities), *i.e.* monopoles on $\mathbb{R}^2 \times \mathbb{S}^1$ possibly singular at a finite collection of points. Four-dimensional moduli spaces of periodic monopoles with singularities are expected to provide examples of gravitational instantons, *i.e.* complete hyperkähler 4–manifolds with finite energy.

In a previous paper [9] we proved that the moduli space of charge k periodic monopoles with n singularities is either empty or generically a smooth hyperkähler manifold of dimension $4(k-1)$. In this paper we settle the existence question, constructing periodic monopoles (with singularities) by gluing methods.

1. INTRODUCTION

Let (X, g) be an oriented Riemannian 3–manifold and $P \to X$ a principal G–bundle, where G is a compact Lie group. (Magnetic) *monopoles* are solutions (A, Φ) to the *Bogomolny equation*

$$(1.1) \qquad *F_A = d_A \Phi.$$

Here $*$ is the Hodge star operator of (X, g); F_A is the curvature of a connection A on P; and Φ, the *Higgs field*, is a section of the adjoint bundle $\mathrm{ad}(P)$. The moduli space of monopoles on $P \to X$ is the space of equivalence classes of solutions to (1.1) with respect to the action of the gauge group $\mathrm{Aut}(P)$. The Bogomolny equation is the dimensional reduction of the Yang-Mills anti-self-duality (ASD) equation, *i.e.* monopoles on X are circle-invariant instantons on $X \times \mathbb{S}^1$.

An immediate consequence of equation (1.1) and the Bianchi identity is

$$(1.2) \qquad d_A^* d_A \Phi = 0.$$

In particular, when X is compact smooth monopoles coincide with reducible (assuming $\Phi \neq 0$) flat connections. In order to find irreducible solutions to (1.1) one has to consider a non-compact base manifold X, in the sense that either X is complete or we allow for singularities of the fields (A, Φ), or a combination of the two possibilities, as in this paper.

The classical case of smooth monopoles on $\mathbb{R}^3$ has been investigated from many different points of view, *cf.* Atiyah and Hitchin's book [2]. An important property of the moduli spaces of monopoles on $\mathbb{R}^3$ is that they are hyperkähler manifolds by virtue of an infinite dimensional hyperkähler quotient. In the 1980's Atiyah and Hitchin found an explicit formula for the metric on the moduli space of centred charge 2 $SU(2)$ monopoles on $\mathbb{R}^3$. From this formula it follows that the Atiyah–Hitchin manifold is a gravitational instanton (a complete hyperkähler 4–manifold with finite L^2–norm of the curvature) of type ALF: the volume of large geodesic balls of radius r grows like r^3, the complement of a large ball is a circle bundle over $\mathbb{R}^3/\mathbb{Z}_2$ and the metric is asymptotically adapted to this circle fibration.

Pursuing the idea that moduli spaces of solutions to dimensional reductions of the Yang–Mills ASD equations on $\mathbb{R}^4$ are "a natural place to look for gravitational instantons" [3], in [4], [5] and [6] Cherkis and Kapustin introduced the study of *periodic monopoles*, *i.e.* monopoles on $\mathbb{R}^2 \times \mathbb{S}^1$, possibly with isolated singularities at a finite collection of points. They argued that, when 4–dimensional, moduli spaces of periodic monopoles (with singularities) are gravitational instantons

of type ALG: the volume of large balls grows quadratically and the metric is asymptotically adapted to a fibration by 2–dimensional tori.

Let $\mathcal{M}_{n,k}$ be the moduli space of $SO(3)$ periodic monopoles of charge k with n fixed singularities. In [9] we proved that the open subset of $\mathcal{M}_{n,k}$ consisting of irreducible monopoles is either empty or a smooth hyperkähler manifold of dimension $4(k-1)$. Moreover, for generic choices of the parameters defining the boundary conditions (including the position of the n singularities), $\mathcal{M}_{n,k}$ cannot contain reducible monopoles. In this paper we address the existence question and construct periodic solutions to (1.1) by gluing methods. The main result of the paper is the following theorem. We refer to Corollary 6.9 for a more precise statement.

Theorem 1.3. *Under appropriate assumptions on the parameters defining the boundary conditions, there exist irreducible $SO(3)$ periodic monopoles (with singularities), i.e. the moduli space $\mathcal{M}_{n,k}$ contains smooth points.*

Monopoles on $\mathbb{R}^3$ (with structure group $SU(2)$ and without singularities) were themselves constructed via gluing methods in a seminal work by Taubes [10, Theorem 1.1 §IV.1]. Furthermore, Cherkis and Kapustin's physically-motivated computation of the asymptotics of the L^2–metric on the moduli space of periodic monopoles [5] is based on the idea that a charge k monopole is a non-linear superposition of particle-like charge 1 components. The theorem confirms this expectation. Moreover the gluing construction could be exploited to recover Cherkis and Kapustin's asymptotic formula for the L^2–metric on the moduli spaces as in recent work of Kottke–Singer [11] on smooth Euclidean monopoles.

The main steps and ingredients of Taubes's original gluing construction for Euclidean monopoles without singularities are:

(i) Charge 1 monopoles on $\mathbb{R}^3$ are completely explicit, as we will see in Section 2.3. Up to translations and scaling there exists a unique solution, localised around the origin in $\mathbb{R}^3$.

(ii) Given k points far apart in $\mathbb{R}^3$, Taubes constructs an approximate solution to (1.1) patching together k charge 1 monopoles each localised around one of these points; the choice of gluing maps accounts for a further $k-1$ parameters.

(iii) The approximate solution is deformed to a genuine monopole by an application of the Implicit Function Theorem.

The first difficulty to implement the construction in the periodic case is that not even charge 1 periodic monopoles are explicitly known. In fact, numerical experiments of Ward [15] show that a very different behaviour should be expected depending on the sign of the *mass*, the constant term v in the expansion of $|\Phi|$ at infinity, *cf.* Definition 3.13. When v is positive and large, charge 1 periodic monopoles are concentrated in an almost spherical region around their centre. When the mass is negative and large in absolute value, the monopoles are instead localised in a slab containing *two* maxima of the energy density.

As a consequence, the construction of a charge k periodic monopole as a superposition of k charge 1 monopoles can be carried out only when the charge 1 constituents have large positive mass. There are two ways of arranging this. On one side one can consider periodic monopoles with large mass v. By scaling, the large mass limit $v \to +\infty$ is equivalent to the large radius limit $\mathbb{R}^2 \times \mathbb{R}/2\pi v\mathbb{Z} \to \mathbb{R}^3$. Here nothing is special about the case $X = \mathbb{R}^2 \times \mathbb{S}^1$ and it is conceivable that large mass monopoles exist on any 3–manifold satisfying appropriate conditions. For example, Oliveira [13] has recently constructed large mass monopoles on asymptotically conical 3–manifolds with vanishing second Betti number. More interestingly, we will exploit the fact that the Green's function of $\mathbb{R}^2 \times \mathbb{S}^1$ grows logarithmically at infinity (*cf.* Lemma 2.8) to construct periodic monopoles with arbitrary mass v and $n < 2(k-1)$ singularities. These solutions are described qualitatively as the superposition of widely separated charge 1 components which get more and more concentrated around their respective centres as these recede from each other.

Plan of the paper. The proof of Theorem 1.3 consists of various steps. The basic building blocks used in the gluing construction—periodic Dirac monopoles and charge 1 Euclidean monopoles—are introduced in Section 2 together with further background material.

The starting point of the construction is a certain singular solution to the Bogomolny equation described in Section 3: given singularities $p_1, \ldots, p_n$ and k additional well-separated points $q_1, \ldots, q_k$, we construct a reducible solution to the Bogomolny equation on $(\mathbb{R}^2 \times \mathbb{S}^1) \setminus \{p_1, \ldots, p_n, q_1, \ldots, q_k\}$ by taking a sum of periodic Dirac monopoles. Consideration of the asymptotic behaviour of this reducible solution leads to the definition of boundary conditions as in Cherkis–Kapustin [4] and [6].

We want to think of $q_1, \ldots, q_k$ as the centres of highly concentrated charge 1 monopoles. To this end it is necessary to assume that either

(A) the mass v is sufficiently large, or
(B) when the number of singularities n is less than $2(k-1)$, $q_1, \ldots, q_k$ are sufficiently far away from each other and from the singularities $p_1, \ldots, p_n$.

We refer to (A) and (B) as the *high mass* and *large distance* case respectively. Under either of these hypothesis, in Section 4 we construct initial approximate solutions to the Bogomolny equation by gluing scaled Euclidean charge 1 monopoles in a neighbourhood of $q_1, \ldots, q_k$ to resolve the singularities of the sum of periodic Dirac monopoles. By varying the centres and phases (thought of as fixing the choice of gluing maps) of the glued-in charge 1 monopoles, we obtain a $4(k-1)$–dimensional family of inequivalent approximate solutions.

The next step of the construction is to deform the initial approximate solutions into genuine monopoles by means of the Implicit Function Theorem. The crucial step, tackled in Section 5, is to study the linearised equation. A first difficulty arises from the fact that, if one fixes the boundary conditions (*i.e.* works with weighted Sobolev spaces forcing certain decay), there is a 3–dimensional space of obstructions to the solvability of the linearised equation. There are two ways to proceed:

(i) enlarge the Banach spaces in which to solve the Bogomolny equation by allowing the appropriate changes of asymptotics;
(ii) consider the centre of mass of the centres of the glued-in Euclidean charge 1 monopoles as a free parameter to be fixed only at the end of the construction to compensate for the obstructions.

We follow this second approach as it seems more appropriate to construct a whole $4(k-1)$–parameter family of monopoles in a fixed moduli space. In order to study the linearised problem, we adopt the strategy of studying the linearised equation separately on the building blocks, the charge 1 Euclidean monopoles and the sum of periodic Dirac monopoles. In the former case, there are no obstructions to the solvability of the linearised equation and the use of weighted Sobolev spaces allows to obtain uniform estimates for the norm of a right inverse. In the latter case, we can solve the linearised equation in the chosen weighted Sobolev spaces only modulo obstructions. Furthermore, for technical reasons we have to distinguish between the high mass and large distance case (A) and (B) above.

(A) Under the assumption that the points $q_1, \ldots, q_k$ are contained in a fixed compact set of $\mathbb{R}^2 \times \mathbb{S}^1$ and that the mass v is sufficiently large, we use the weighted Sobolev spaces and estimates of [9] without major difficulties.
(B) When the points $q_1, \ldots, q_k$ move off to infinity, instead, an additional technical difficulty arises from the following fact: it is well-known that for all $f \in C_0^\infty(\mathbb{R}^2)$ with mean value zero there exists a bounded solution u, unique up to the addition of a constant, to $\triangle u = f$ with $\|\nabla u\|_{L^2} < \infty$. However, if f is supported on the union of balls $B_1(z_1) \cup B_1(z_2)$, say, with non-zero mean value on each of them, then $\|\nabla u\|_{L^2}^2 \geq C \log |z_1 - z_2|$. As a consequence, in the large distance case the linearised operator is not well-behaved on a certain finite-dimensional space of sections which has to be dealt with separately.

In Section 5.4 we patch together the local inverses of the linearised operator on the different building blocks and by a simple iteration solve the linearised equation globally modulo obstructions. Finally, in Section 6 we conclude the construction of solutions to the Bogomolny equation satisfying the required boundary conditions by an application of the Implicit Function Theorem.

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2. Preliminaries

In this section we describe in some detail the simple components to be patched together in the gluing construction. We begin with periodic Dirac monopoles, *i.e.* abelian solutions to the Bogomolny equation on $\mathbb{R}^2 \times \mathbb{S}^1$ with a singularity at one point, recalling the asymptotic expansions proved in [9]. Secondly, we collect the main properties of the basic Euclidean monopole, the Prasad–Sommerfield monopole. As a preliminary and mainly to fix some notation we give a brief overview of the deformation theory of monopoles on an arbitrary 3–manifold.

2.1. **Deformation theory of monopoles.** Let (X, g) be a non-compact oriented Riemannian 3–manifold endowed with a principal G–bundle $P \to X$. Denote by $\mathcal{C}$ the infinite dimensional space of smooth pairs $c = (A, \Phi)$, where A is a connection on $P \to X$ and $\Phi \in \Omega^0(X; \mathrm{ad}P)$ a Higgs field. Since X is non compact, elements of $\mathcal{C}$ have to satisfy appropriate boundary conditions, which we suppose to be included in the definition of $\mathcal{C}$. The configuration space $\mathcal{C}$ is an affine space; the underlying vector space is the space of sections

$$\Omega(X; \mathrm{ad}P) := \Omega^1(X; \mathrm{ad}P) \oplus \Omega^0(X; \mathrm{ad}P)$$

satisfying appropriate decay conditions.

Let $\mathcal{G}$ be the group of bounded smooth sections of $\mathrm{Aut}(P)$ which preserve the chosen boundary conditions. Here $g \in Aut(P)$ acts on a pair $c = (A, \Phi) \in \mathcal{C}$ by $c \mapsto c + (d_1 g)g^{-1}$, where

$$(2.1) \qquad d_1 g = -(d_A g, [\Phi, g]) \in \Omega(X; \mathrm{ad}P).$$

Consider the gauge-equivariant map $\Psi \colon \mathcal{C} \to \Omega^1(X; \mathrm{ad}P)$ defined by

$$\Psi(A, \Phi) = *F_A - d_A \Phi.$$

By fixing a base point $c = (A, \Phi) \in \mathcal{C}$ we write $\Psi(A + a, \Phi + \psi) = \Psi(c) + d_2(a, \psi) + (a, \psi) \cdot (a, \psi)$ for all $(a, \psi) \in \Omega(X; \mathrm{ad}P)$. The linearisation d_2 of Ψ at c and the quadratic term are defined by:

$$(2.2) \qquad d_2(a, \psi) = *d_A a - d_A \psi + [\Phi, a]$$

$$(2.3) \qquad (a, \psi) \cdot (a, \psi) = *[a, a] - [a, \psi]$$

The linearisation at c of the action of $\mathcal{G}$ on $\mathcal{C}$ is the operator $d_1 \colon \Omega^0(X; \mathrm{ad}\, P) \to \Omega(X; \mathrm{ad}P)$ defined as in (2.1). We couple d_2 with d_1^* to obtain an elliptic operator

$$(2.4) \qquad D = d_2 \oplus d_1^* \colon \Omega(X; \mathrm{ad}P) \longrightarrow \Omega(X; \mathrm{ad}P).$$

The moduli space $\mathcal{M}$ of monopoles in $\mathcal{C}$ is $\mathcal{M} = \Psi^{-1}(0)/\mathcal{G}$. If $\mathcal{M}$ is smooth, the tangent space $T_{[c]}\mathcal{M}$ at a point $c = (A, \Phi)$ is identified with $\ker D$.

The operator D is a twisted Dirac operator on $\Omega(X; \mathrm{ad}P)$. To see this, recall that Clifford multiplication is defined by

$$(2.5) \qquad \gamma(\alpha)\beta = \alpha \wedge \beta - \alpha^{\sharp} \lrcorner \beta$$

for a 1–form α and a k–form β. Then $D = \tau D\!\!\!\!/_A + [\Phi, \cdot]$, where $D\!\!\!\!/_A$ is the twisted Dirac operator

$$\Omega^1 \oplus \Omega^0 \xrightarrow{(\text{id},*)} \Omega^1 \oplus \Omega^3 \xrightarrow{\gamma \circ \nabla_A} \Omega^2 \oplus \Omega^0 \xrightarrow{(*,\text{id})} \Omega^1 \oplus \Omega^0.$$

and τ is a sign operator with $\tau = 1$ on 1–forms and $\tau = -1$ on 0–forms.

2.2. **Periodic Dirac monopole.** When the structure group is $G = U(1)$, the Bogomolny equation (1.1) reduces to a linear equation. By (1.2) the Higgs field Φ is a harmonic function such that $\frac{*d\Phi}{2\pi i}$ represents the first Chern class of a line bundle. The assumption that Φ is bounded thus forces Φ to be constant and A to be flat. Given $v \in \mathbb{R}$ and $b \in \mathbb{R}/\mathbb{Z}$ let $L_{v,b}$ be the trivial line bundle on $\mathbb{R}^2 \times \mathbb{S}^1$ endowed with the trivial monopole given by a constant Higgs field $\Phi \equiv iv$ and the flat connection $d + ib\,dt$.

Non-trivial abelian solutions, so called *Dirac monopoles*, are obtained by allowing an isolated singularity. On $\mathbb{R}^3$ these are defined as follows.

Definition 2.6. Fix a point $q \in \mathbb{R}^3$ and let H_q denote the radial extension of the Hopf line bundle to $\mathbb{R}^3 \setminus \{q\}$. Fix $k \in \mathbb{Z}$ and $v \in \mathbb{R}$. The *Euclidean Dirac monopole* of charge k and mass v with singularity at q is the abelian monopole (A^0, Φ^0) on H_q^k, where

$$\Phi^0 = i\left(v - \frac{k}{2|x - q|}\right),$$

$x \in \mathbb{R}^3$, and A^0 is the $SO(3)$–invariant connection on H_q^k with curvature $*d\Phi$.

Periodic Dirac monopoles are defined in a similar way. Fix coordinates $(z,t) \in \mathbb{C} \times \mathbb{R}/2\pi\mathbb{Z}$ and a point $q = (z_0, t_0) \in \mathbb{R}^2 \times \mathbb{S}^1$. Line bundles of a fixed degree on $(\mathbb{R}^2 \times \mathbb{S}^1) \setminus \{q\}$ differ by tensoring by flat line bundles. We can distinguish connections with the same curvature by comparing their holonomy around loops $\gamma_z := \{z\} \times \mathbb{S}^1_t$ for $z \neq z_0$. Set $\theta_q = \arg(z - z_0)$ and denote by L_q the degree 1 line bundle on $(\mathbb{R}^2 \times \mathbb{S}^1) \setminus \{q\}$ with connection A_q whose holonomy around γ_z is $e^{-i\theta_q}$. Any line bundle of degree 1 is obtained from L_q by tensoring with a flat line bundle.

Definition 2.7. Fix a point $q \in \mathbb{R}^2 \times \mathbb{S}^1$. The *periodic Dirac monopole* of charge $k \in \mathbb{Z}$, with singularity at q and twisted by the flat line bundle $L_{v,b}$ for some $v \in \mathbb{R}$ and $b \in \mathbb{R}/\mathbb{Z}$ is the pair (A, Φ) on $L_q^k \otimes L_{v,b}$, where

$$\Phi = i\left(v + kG_q\right)$$

and up to gauge transformations the connection is $A = kA_q + ib\,dt$. Here G_q is the Green's function of $\mathbb{R}^2 \times \mathbb{S}^1$ with singularity at q defined in Lemma 2.8 below.

In [9] we derived asymptotic expansions for the Green's function G_q and the connection A_q, both at infinity and close to the singularity. As these expansions will be essential for the gluing construction, we recall them here. By taking coordinates centred at $q \in \mathbb{R}^2 \times \mathbb{S}^1$, we can assume that the singularity is located at $q = 0$. We use polar coordinates $z = re^{i\theta} \in \mathbb{C}$.

Lemma 2.8 (Lemma 3.4 in [9]). *There exists a Green's function G of $\mathbb{R}^2 \times \mathbb{S}^1$ with singularity at* 0 *such that the following holds.*

(i) *There exists a constant $C_1 > 0$ such that*

$$\left|\nabla^k\left(G(z,t) - \frac{1}{2\pi}\log r\right)\right| \leq C_1 e^{-r}$$

for all $r \geq 2$ and $k = 0, 1, 2$.

(ii) *There exists a constant $C_2 > 0$ such that*

$$\left|\nabla^k\left(G(z,t) - \frac{a_0}{2} + \frac{1}{2\rho}\right)\right| \leq C_2\rho^{2-k}$$

for all (z,t) with $\rho = \sqrt{r^2 + t^2} < \frac{\pi}{2}$ and $k = 0, 1, 2$.

Fix a constant $v \in \mathbb{R}$ and consider the Higgs field $\Phi = iv + iG$. The 2–form $i * dG$ represents the curvature of a line bundle L over $(\mathbb{R}^2 \times \mathbb{S}^1) \setminus \{0\}$. In a neighbourhood of the singularity L is isomorphic to the Hopf line bundle extended radially from a small sphere $\mathbb{S}^2$ enclosing the origin. Any connection A on L with $F_A = *d\Phi$ is asymptotically gauge equivalent to A^0 as $\rho \to 0$.

At infinity L is isomorphic to the radial extension of a line bundle of degree 1 over the torus $\mathbb{T}^2_\infty$. We choose a representative for A in this asymptotic model as follows. Consider the connection $A^\infty = -i \frac{t}{2\pi} d\theta$ on the trivial line bundle $\underline{\mathbb{C}}$ over $\mathbb{S}^1_\theta \times \mathbb{R}_t$. Let τ be the map of $\underline{\mathbb{C}}$ into itself defined by $\tau(e^{i\theta}, t, \xi) = (e^{i\theta}, t + 2\pi, e^{i\theta}\xi)$ and define a line bundle with connection over $\mathbb{T}^2_{\theta,t}$ as the quotient $(\underline{\mathbb{C}}, A^\infty)/\tau$. As $r \to \infty$, up to gauge transformations A is asymptotic to $A^\infty + i\alpha \, d\theta + ib \, dt$ for some $\alpha, b \in \mathbb{R}/\mathbb{Z}$. The monodromy of this limiting connection is $e^{-i\theta - 2\pi ib}$ around the circle $\{\theta\} \times \mathbb{S}^1_t$ and $e^{it - 2\pi i\alpha}$ around the circle $\mathbb{S}^1_\theta \times \{t\}$. While b can be chosen arbitrarily, by [9, Lemma 3.6] α is fixed by the Bogomolny equation (1.1), $\alpha = \frac{1}{2}$ modulo $\mathbb{Z}$.

Lemma 2.9 (Lemma 3.6 in [9]). *Fix parameters $(v, b) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$. Let (A, Φ) be a solution to (1.1) with $\Phi = i(v + G)$ and such that the holonomy of A around circles $\{re^{i\theta}\} \times \mathbb{S}^1_t$, $r \neq 0$, is $e^{-i\theta - 2\pi ib}$.*

  (i) *In the region where $r \geq 2$ the connection A is gauge equivalent to*

$$A^\infty + \frac{i}{2} \, d\theta + ib \, dt + a$$

   for a 1–form a such that $d^*a = 0 = \partial_r \lrcorner \, a$ and $|a| + |\nabla a| = O(e^{-r})$.
  (ii) *In a ball of radius $\frac{\pi}{2}$ centred at the singular point $z = 0 = t$, A is gauge equivalent to $A^0 + a'$ where $|a'| + \rho|\nabla a'| = O(\rho^2)$ and $d^*a' = 0 = \partial_\rho \lrcorner \, a'$.*

Given an arbitrary point $q = (z_0, t_0)$ in $\mathbb{R}^2 \times \mathbb{S}^1$ the same formulas describe the asymptotic behaviour of the periodic Dirac monopole (A_q, Φ_q) with singularity at q in coordinates centred at q. It will be useful to express the behaviour of (A_q, Φ_q) at large distances from q in a fixed coordinate system.

Lemma 2.10 (Lemma 3.7 in [9]). *For $r \geq 2|z_0|$ we have*

$$\frac{1}{i}\Phi_q(z,t) = v + \frac{1}{2\pi}\log r - \frac{1}{2\pi} \, Re\left(\frac{z_0}{z}\right) + O(r^{-2})$$

$$A_q(z,t) = A^\infty + ib \, dt + i\frac{t_0 + \pi}{2\pi} \, d\theta - \frac{i}{2\pi} \, Im\left(\frac{z_0}{z}\right) dt + O(r^{-2}).$$

Finally, notice that the parameters $(v, b) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ are related to rotations and dilations. By a rotation in the z–plane, we can always assume that $b = 0$. On the other hand, given any $\lambda > 0$ consider the homothety

$$h_\lambda : \mathbb{R}^2 \times \mathbb{R}/2\pi\mathbb{Z} \longrightarrow \mathbb{R}^2 \times \mathbb{R}/2\pi\lambda\mathbb{Z}$$

of ratio λ. Since the Bogomolny equation is the dimensional reduction of the ASD equation, which is conformally invariant, $(h_\lambda^* A, \lambda \, h_\lambda^* \Phi)$ is a monopole on $\mathbb{R}^2 \times \mathbb{R}/2\pi\mathbb{Z}$ if and only if (A, Φ) solves the Bogomolny equation on $\mathbb{R}^2 \times \mathbb{R}/2\pi\lambda\mathbb{Z}$. A simple but crucial observation for the gluing construction is that given a periodic Dirac monopole (A_q, Φ_q) with mass v, then as $v \to \infty$

$$\lambda^{-1} h_{\lambda^{-1}}^* \Phi \longrightarrow i\left(1 - \frac{1}{2\sqrt{r^2 + t^2}}\right),$$

where we set $\lambda = v + \frac{a_0}{2}$. In other words, the limit $v \to \infty$ corresponds to the limit $\mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^3$ and in this limit a periodic Dirac monopole converges to an Euclidean Dirac monopole.

2.3. **Charge 1 monopoles on $\mathbb{R}^3$.** As we will see in the next section, periodic Dirac monopoles will serve us to construct an initial *singular* solution to the Bogomolny equation. We now describe the model solution that we will use to desingularise this initial solution. In 1975 Prasad and Sommerfield [14] found an explicit smooth finite energy solution to the Bogomolny equation on $\mathbb{R}^3$ with structure group $SU(2)$. By translations and scaling, this explicit solution accounts for all $SU(2)$ Euclidean monopoles of charge 1. We collect the main properties of the Prasad–Sommerfield monopole following Atiyah–Hitchin [2] and Taubes [10].

First, we fix some notation. For $x \in \mathbb{R}^3$, set $\rho = |x|$ and $\hat{x} = \rho^{-1}x$. Denote by σ the vector $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3 \otimes \mathfrak{su}_2$, where $\{\sigma_1, \sigma_2, \sigma_3\}$ is the standard orthonormal basis of $\mathfrak{su}_2$ defined in terms of Pauli's matrices. In particular, $\sigma_3 = \operatorname{diag}\left(\frac{i}{2}, -\frac{i}{2}\right)$.

The Prasad–Sommerfield (PS) monopole $c_{\mathrm{PS}} = (A_{\mathrm{PS}}, \Phi_{\mathrm{PS}})$ is given by the explicit formula, *cf.* [10, IV.1, Equation 1.15]:

$$(2.11) \qquad \Phi_{\mathrm{PS}}(x) = \left(\frac{1}{\tanh(\rho)} - \frac{1}{\rho}\right)\hat{x} \cdot \sigma \qquad A_{\mathrm{PS}}(x) = \left(\frac{1}{\sinh(\rho)} - \frac{1}{\rho}\right)(\hat{x} \times \sigma) \cdot dx$$

Here $\cdot$ and $\times$ are the scalar and vector product in $\mathbb{R}^3$, respectively. To simplify the notation we will often drop the subscript PS throughout this section. In (2.11) we fixed the mass $v = 1$. A monopole with arbitrary mass $v > 0$ is obtained by scaling. The following properties of the PS monopole follow directly from (2.11), *cf.* [10, §IV.1].

Lemma 2.12. *The pair (A_{PS}, Φ_{PS}) is a solution of the Bogomolny equation (1.1) with finite energy.*
  (i) *Φ has exactly one zero, $\Phi(0) = 0$.*
  (ii) *$|\Phi(x)| < 1$ and $1 - |\Phi(x)| = \frac{1}{\rho} + O(e^{-2\rho})$.*
  (iii) *By (i), over $\mathbb{R}^3 \setminus \{0\}$ we can decompose each $\mathfrak{su}_2$–valued form u into diagonal and off-diagonal part $u = u_D + u_T$, where $u_D = |\Phi|^{-2}\langle u, \Phi\rangle\Phi$. Then $|(d_A\Phi)_D| = O(\rho^{-2})$ and $|(d_A\Phi)_T| = O(e^{-\rho})$.*

In particular, over $\mathbb{R}^3 \setminus \{0\}$ the trivial rank 2 complex vector bundle splits as a sum $H \oplus H^{-1}$ of eigenspaces of Φ. By Lemma 2.12.(i) H is the radial extension of the Hopf line bundle. The adjoint bundle $(\mathbb{R}^3 \setminus \{0\}) \times \mathfrak{su}_2$ splits as a sum $\mathbb{R} \oplus H^2$. We refer to such a gauge over $\mathbb{R}^3 \setminus \{0\}$ as the *asymptotically abelian gauge* because it yields an asymptotic isomorphism between the PS monopole and a charge 1 Euclidean Dirac monopole.

The isomorphism $\eta\colon \mathbb{C}^2 \to H \oplus H^{-1}$ can be made explicit, *cf.* [10, §IV.7, 7.1 and 7.2]. A direct computation then yields the following asymptotic expansions.

Lemma 2.13. *Let (A, Φ) be the PS monopole defined in (2.11).*
  (i) *There exists an isomorphism $\eta : \mathbb{C}^2 \to H \oplus H^{-1}$ over $\mathbb{R}^3 \setminus \{0\}$ such that $\eta(A, \Phi) = (A^0, \Phi^0)\sigma_3 + (a, \psi)$, with a and ψ a 1 and 0–form with values in the $SO(3)$–bundle $\mathbb{R} \oplus H^2$.*
  (ii) *(a, ψ) satisfies $d^*_{A^0\sigma_3}a = 0 = [\Phi^0\sigma_3, \psi]$ and $\partial_\rho \lrcorner a = 0$. Moreover, as $\rho \to \infty$:*
$$|a| + |\psi| + |d_{A^0\sigma_3}a| + |[\Phi^0\sigma_3, a]| + |d_{A^0\sigma_3}\psi| = O(e^{-\rho})$$

Therefore η puts (A, Φ) in "Coulomb gauge" with respect to $(A^0, \Phi^0)\sigma_3$. Without altering properties (i) and (ii), we have the freedom to change η by composing with an element in the stabiliser $U(1)$ of $(A^0, \Phi^0)\sigma_3$. By abuse of notation, we won't distinguish between η and the induced isomorphism of $SO(3)$–bundles $(\mathbb{R}^3 \setminus \{0\}) \times \mathfrak{su}_2 \simeq \mathbb{R} \oplus H^2$. As an isomorphism of $SO(3)$–bundles, we have the freedom to compose η with an element of $SO(2)$, where $U(1) \to SO(2)$ is the double cover induced by the adjoint representation $SU(2) \to SO(3)$.

The deformation theory of charge 1 monopoles on $\mathbb{R}^3$ can be understood explicitly. Let D be the Dirac operator (2.4) twisted by the PS monopole. The L^2–kernel of D is 4–dimensional, spanned

over $\mathbb{H}$ by the vector $(d_A\Phi, 0)$, *i.e.*

$$\ker D = \langle\, (d_A\Phi, 0), \gamma(dx_h)\,(d_A\Phi, 0), h = 1, 2, 3\,\rangle_\mathbb{R},$$

where $\gamma(dx_h)$ denotes the Clifford multiplication. We can explicitly integrate these infinitesimal deformations. Choose $x_0 \in \mathbb{R}^3$ and let T_{x_0} be the translation $x \mapsto x - x_0$. Then $T_{x_0}^* c_{\mathrm{PS}}$ is still a solution to the Bogomolny equation. The corresponding infinitesimal deformation is $-\gamma(x_0)\,(d_A\Phi, 0) = -x_0 \lrcorner\, (F_A, d_A\Phi)$. On the other hand, $(d_A\Phi, 0)$ is the infinitesimal action of the gauge transformation $\exp(-\Phi)$.

For future use, in the next lemma we put $T_{x_0}^* c_{\mathrm{PS}}$ in "Coulomb gauge" with respect to c_{PS} and derive some useful estimates.

Lemma 2.14. *There exists κ, C and $\rho_0 > 0$ such that the following holds. For any $x_0 \in \mathbb{R}^3$ with $|x_0| < \kappa$ there exists a solution (A_{x_0}, Φ_{x_0}) to the Bogomolny equation which can be written*

$$(A_{x_0}, \Phi_{x_0}) = (A, \Phi) - x_0 \lrcorner\, (F_A, d_A\Phi) + (a_{x_0}, \psi_{x_0}),$$

where $d_1^*(a_{x_0}, \psi_{x_0}) = 0$. Here d_1 is the linearisation at $c_{PS} = (A, \Phi)$ of the action of gauge transformations. Moreover, $|(a_{x_0}, \psi_{x_0})| \leq C \frac{|x_0|^2}{\rho^3}$ for all $\rho \geq \rho_0$.

Proof. We give a sketch of the proof as it seems that the statement should be well-known.

We will prove later, *cf.* Proposition 5.8, that the operator $DD^*\colon W_{w,\delta}^{2,2} \to L_{w,\delta-2}^2$ is an isomorphism for all $\delta \in (-1, 0)$. Here the spaces $W_{w,\delta}^{m,2}$ are defined in Definition 5.7. Using Lemma 2.12.(iii) to see that $d_A\Phi \in W_{w,\delta-1}^{1,2}$ one then shows that $DD^* - 2x_0 \lrcorner\, (F_A, d_A\Phi) \cdot D^*$ remains an isomorphism if κ is sufficiently small. The Implicit Function Theorem implies that there exists a unique solution $u \in W_{w,\delta}^{2,2}$ to the equation

$$DD^*u + \big(-x_0 \lrcorner\, (F_A, d_A\Phi) + D^*u\big) \cdot \big(-x_0 \lrcorner\, (F_A, d_A\Phi) + D^*u\big) = 0,$$

i.e. such that $(A, \Phi) - x_0 \lrcorner\, (F_A, d_A\Phi) + D^*u$ satisfies the Bogomolny equation and the gauge fixing condition $d_1^* D^*u = 0$. Moreover, $\|D^*u\|_{W_{w,\delta-1}^{1,2}} \leq C|x_0|^2$ and the map $x_0 \mapsto u \in W_{w,\delta}^{2,2}$ is smooth.

It remains to show that $(a_{x_0}, \psi_{x_0}) = D^*u = O(\rho^{-3})$. Consider the equation $D\xi + \xi \cdot \xi = 0$ for $\xi \in W_{w,\delta-1}^{1,2}$. By Lemma 2.13, in the asymptotically abelian gauge we write

$$D_0\xi + 2(a, \psi) \cdot \xi + \xi \cdot \xi = 0,$$

where (a, ψ) is exponentially decaying and D_0 is the Dirac operator twisted by the Dirac monopole (A^0, Φ^0). Decomposing $\xi = \xi_D + \xi_T$ into diagonal and transverse part, a Moser iteration argument as in [9, Lemma 7.10] yields $\xi_T = O(\rho^{-\mu}) = \triangle\xi_D$ for all μ.

Apply this result to $\xi = -x_0 \lrcorner\, (F_A, d_A\Phi) + D^*u$. It follows that we can write

$$D^*u = x_0' \lrcorner\, (F_A, d_A\Phi) + \tau\,(d_A\Phi, 0) + O(\rho^{-3})$$

for some $x_0' \in \mathbb{R}^3$ and $\tau \in \mathbb{R}$. However, since $u \in W_{w,\delta}^{2,2}$ and $\delta \in (-1, 0)$, an integration by parts shows that D^*u is L^2–orthogonal to $\ker D$ and therefore $x_0' = 0 = \tau$. $\qquad\square$

Of course, there exists a gauge transformation such that $g(A_{x_0}, \Phi_{x_0}) = T_{x_0}^*(A, \Phi)$. Indeed, there must exist $x_0' \in \mathbb{R}^3$ and g such that $g(A_{x_0}, \Phi_{x_0}) = T_{x_0'}^*(A, \Phi)$. On the other hand, comparing $|\Phi_{x_0}|$ with $|T_{x_0'}^*\Phi|$, one concludes that $x_0' = x_0$.

3. A SUM OF PERIODIC DIRAC MONOPOLES

In this section we construct a singular *reducible* solution c_{ext} to the Bogomolny equation (1.1) by summing periodic Dirac monopoles. Consideration of the asymptotic behaviour of this singular solution suggests the definition of boundary conditions for periodic monopoles (with singularities) as in Cherkis–Kapustin [4] and [6]. The aim of the rest of the paper will be to construct periodic monopoles by desingularising c_{ext} while preserving the boundary conditions.

The construction of the pair $c_{\text{ext}} = (A_{\text{ext}}, \Phi_{\text{ext}})$ depends on the choice of the following data:

(i) A *vacuum background*, *i.e.* constants $v \in \mathbb{R}$ and $b \in \mathbb{R}/\mathbb{Z}$ corresponding to the flat line bundle $L_{v,b}$ on $\mathbb{R}^2 \times \mathbb{S}^1$ with constant Higgs field iv and flat connection $d + ib\,dt$.

(ii) The *singularities*: a collection S of n distinct points $p_i = (m_i, a_i) \in \mathbb{R}^2 \times \mathbb{S}^1$ for $i = 1, \ldots, n$.

(iii) The *centres of non-abelian monopoles*: further k points, pair-wise distinct and distinct from the p_i's, which we denote by $q_j = (z_j, t_j)$ for $j = 1, \ldots, k$.

Fix an origin in $\mathbb{C} \times \mathbb{R}/2\pi\mathbb{Z}$ so that $z_1 + \ldots + z_k = 0$ and $t_1 + \ldots + t_k = 0$ modulo 2π. Set $\mu = m_1 + \ldots + m_n$ and $\alpha = a_1 + \ldots + a_n$ modulo 2π. Denote by d the *minimum distance*:

$$(3.1) \qquad d = \min\{|z_j - z_h|, |z_j - m_i| \text{ for all } j, h = 1, \ldots k, j \neq h, \text{ and all } i = 1, \ldots n\}$$

We assume that $d \geq 5$. Throughout the paper constants are allowed to depend on a lower bound for d and on the position of the points $p_1, \ldots, p_n$.

Define a harmonic function Φ_{ext} on $(\mathbb{R}^2 \times \mathbb{S}^1) \setminus (S \cup \{q_1, \ldots, q_k\})$ by

$$(3.2) \qquad \Phi_{\text{ext}} = v + 2\sum_{j=1}^{k} G_{q_j} - \sum_{i=1}^{n} G_{p_i}.$$

Here G_p is the Green's function of $\mathbb{R}^2 \times \mathbb{S}^1$ with singularity at p given in Lemma 2.8.

By Lemmas 2.8 and 2.10, for large $|z|$

$$(3.3) \qquad \Phi_{\text{ext}} = v + \frac{2k - n}{2\pi}\log|z| + \frac{1}{2\pi}\text{Re}\left(\frac{\mu}{z}\right) + O(|z|^{-2}),$$

while close to the singularity p_i

$$(3.4) \qquad \Phi_{\text{ext}} = \text{const} + \frac{1}{2\rho_i} + O(\rho_i).$$

Here $\rho_i = \text{dist}(p_i, \cdot)$ and the constant term is defined by

$$(3.5) \qquad \sum_{m \neq i} G_{p_m}(p_i) - \frac{a_0}{2} + v + \frac{1}{\pi}\sum_{j=1}^{k}\log|z_j - m_i| + O\left(e^{-d}\right).$$

Lemmas 2.8 and 2.10 yield similar expansions also for the derivatives of Φ_{ext}.

Finally, in the ball $B_{\frac{\pi}{2}}(q_j)$

$$(3.6) \qquad \Phi_{\text{ext}} = \lambda_j - \frac{1}{\rho_j} + O\left(\frac{\rho_j}{d} + \rho_j^2\right)$$

where the constant term λ_j, using once again Lemma 2.8, is defined by

$$(3.7) \qquad \lambda_j = v + a_0 + \frac{1}{\pi}\sum_{h=1, h \neq j}^{k}\log|z_h - z_j| - \frac{1}{2\pi}\sum_{i=1}^{n}\log|m_i - z_j| + O\left(e^{-d}\right).$$

We will refer to λ_j as the *local mass* attached to the point q_j.

Definition 3.8. Given $\lambda_0, K > 1$ and $d_0 \geq 5$ we say that $v \in \mathbb{R}$ and the points $p_1, \ldots, p_n, q_1, \ldots, q_k$ are (λ_0, d_0, K)–*admissible* if:

(i) The minimum distance d (3.1) satisfies $d \geq d_0$;
(ii) $\lambda := \min_j \lambda_j > \lambda_0$;
(iii) $\overline{\lambda} := \max_j \lambda_j \leq K\lambda$;
(iv) $v > 0$ if $n = 2k$.

The constant λ_0 has to be thought of as being very large. Therefore it is important to determine which set of data are admissible as $\lambda_0 \to \infty$.

(A) By (3.7), a first possibility is to fix the points $p_1, \ldots, p_n$, $q_1, \ldots, q_k$ so that (i) in Definition 3.8 is satisfied and then choose v sufficiently large. We refer to this as the *high mass* case.
(B) Consider instead the limit $d \to +\infty$ and assume that in addition to Definition 3.8 we have

$$\overline{d} = \max\{|z_j - z_h|, |z_j - m_i| \text{ for all } j, h = 1, \ldots k, j \neq h, i = 1, \ldots n\} \leq K'd$$

for some $K' > 1$. Then as $d \to \infty$

$$\lambda_j \sim \frac{1}{\pi}\left(k - 1 - \frac{n}{2}\right)\log d \tag{3.9}$$

for all $j = 1, \ldots, k$. Thus if $n < 2(k-1)$ we can fix v and $p_1, \ldots, p_n$ arbitrarily and then choose the additional k points $q_1, \ldots, q_k$ so that $d \geq d_0$ for d_0 sufficiently large. We call this the *large distance* case. It is the most interesting case because, in analogy with Taubes's result [10] for Euclidean monopoles, it is expected that it corresponds to (an open subset of) the end of the moduli space.

An important consequence of choosing data with large λ is that zeroes of Φ_{ext} are localised around the points $q_1, \ldots, q_k$.

Lemma 3.10. *Fix $d_0 = 5$ and suppose that $v > 1$ if $n = 2k$. There exists λ_0 such that the following holds. Suppose that the initial data are $(\lambda_0, 5, K)$–admissible. Then for all $j = 1, \ldots, k$ there exists $\frac{1}{\lambda_j} < \delta(\lambda_j) \leq \frac{2}{\lambda_j}$ such that $\Phi_{ext} \geq 1$ on $(\mathbb{R}^2 \times \mathbb{S}^1) \setminus \left(S \cup \bigcup_{j=1}^{k} B_{\delta(\lambda_j)}(q_j)\right)$.*

Proof. By Lemmas 2.8 and 2.10 there exists a constant C such that in the annulus $\delta \leq \rho_j \leq \frac{\pi}{2}$

$$\Phi_{\text{ext}} \geq \lambda_j - \frac{1}{\delta} - C.$$

Fix some λ_0 such that $\lambda_0 \geq 1 + \frac{2}{\pi} + C$ and set $\delta = \delta(\lambda_j) = (\lambda_j - 1 - C)^{-1}$. By choosing λ_0 larger if necessary, we can assume that $\lambda_0 \geq 2(1 + C)$, so that $\delta(\lambda_j) \leq \frac{2}{\lambda_j}$.

Now, if $\lambda_j > \lambda_0$ then $\delta < \frac{\pi}{2}$ and $\Phi_{\text{ext}} \geq 1$ in the annulus $\delta \leq \rho_j \leq \frac{\pi}{2}$. On the other hand, $\Phi_{\text{ext}} \geq 1$ in a neighbourhood of the singularities $p_1, \ldots, p_n$ and at infinity. Here we need to use the hypothesis $v > 1$ if $n = 2k$. The Lemma follows from the minimum principle. $\qquad\square$

The form $*d\Phi_{\text{ext}}$ is the curvature of the line bundle $M \to (\mathbb{R}^2 \times \mathbb{S}^1) \setminus (S \cup \{q_1, \ldots, q_k\})$

$$M = L_{v,b} \otimes \bigotimes_{j=1}^{k} L_{q_j}^2 \otimes \bigotimes_{i=1}^{n} L_{p_i}^{-1}, \tag{3.11}$$

where, in the notation of Section 2.2, $L_{v,b}$ is the trivial line bundle endowed with the flat abelian monopole with parameters $(v, b) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ and for $p \in \mathbb{R}^2 \times \mathbb{S}^1$ L_p is the line bundle on $(\mathbb{R}^2 \times \mathbb{S}^1) \setminus \{p\}$ of Definition 2.7.

Consider the reducible $SO(3)$–bundle $\mathbb{R} \oplus M$. By a result of Whitney [16, §III.7], isomorphism classes of $SO(3)$–bundles over a CW–complex of dimension at most 3 are completely classified by the second Stiefel–Whitney class w_2. In our case $w_2(\mathbb{R} \oplus M) = c_1(M)$ modulo 2, so that w_2 evaluated on the torus at infinity is $2k - n$ (mod 2), 1 on a small sphere enclosing any one of the n singularities and it vanishes on spheres enclosing each of the k points.

Denote by $\hat{\sigma}$ the trivialising section of the first factor in $\mathbb{R} \oplus M$. Multiplying by $\hat{\sigma}$, Φ_{ext} (3.2) defines a Higgs field on $\mathbb{R} \oplus M$. Fix the connection

$$(3.12) \qquad A_{\text{ext}} = \left(b\, dt + 2 \sum_{j=1}^{k} A_{q_j} - \sum_{i=1}^{n} A_{p_i} \right) \hat{\sigma}$$

on $\mathbb{R} \oplus M$, where A_p is the connection on L_p of Definition 2.7. Lemmas 2.9 and 2.10 yield asymptotic expansions of representatives of A_{ext} close to p_i, close to q_j and at infinity.

3.1. **Boundary conditions.** By Whitney's result mentioned above, given the collection S of n distinct points $p_1, \ldots, p_n \in \mathbb{R}^2 \times \mathbb{S}^1$, there exists a unique $SO(3)$–bundle V up to isomorphism on $(\mathbb{R}^2 \times \mathbb{S}^1) \setminus S$ with $w_2(V) \cdot [\mathbb{S}^2_{p_i}] = 1$ for all $i = 1, \ldots, n$. Notice that V does not lift to an $SU(2)$–bundle whenever $n > 0$ and this explains the choice of $SO(3)$ as structure group. Now, outside of $\{q_1, \ldots, q_k\}$ we have an isomorphism $V \simeq \mathbb{R} \oplus M$, showing that a desingularisation of the pair $(A_{\text{ext}}, \Phi_{\text{ext}})$ is topologically possible. Our aim is to find a desingularisation which solves the Bogomolny equation and satisfies the same asymptotic behaviour of $(A_{\text{ext}}, \Phi_{\text{ext}})$ at infinity and close to the points $p_1, \ldots, p_n$. This leads us to consider the boundary conditions for periodic monopoles (with singularities) introduced by Cherkis and Kapustin in [4, §1.4] and [6, §2].

Definition 3.13. Given a non-negative integer $k_\infty \in \mathbb{Z}_{\geq 0}$, parameters $(v, b) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ and a point $q = (\mu, \alpha) \in \mathbb{R}^2 \times \mathbb{S}^1$, let $\mathcal{C} = \mathcal{C}(p_1, \ldots, p_n, k_\infty, v, b, q)$ be the space of smooth pairs $c = (A, \Phi)$ of a connection A on V and a section Φ of V satisfying the following boundary conditions.

  (1) For each $p_i \in S$ there exists a ball $B_\sigma(p_i)$ and a gauge $V|_{B_\sigma(p_i) \setminus \{p_i\}} \simeq \mathbb{R} \oplus H_{p_i}$ such that (A, Φ) can be written

$$\Phi = -\frac{1}{2\rho_i} \hat{\sigma} + \psi \qquad A = A^0 \hat{\sigma} + a$$

  with $\xi = (a, \psi) = O(\rho_i^{-1+\tau})$ and $|\nabla_A \xi| + |[\Phi, \xi]| = O(\rho_i^{-2+\tau})$ for some rate $\tau > 0$. Here ρ_i is the distance from p_i and A^0 is the unique $SO(3)$–invariant connection on H_{p_i}.

  (2) There exists $R > 0$ and a gauge $V \simeq \mathbb{R} \oplus \left(L_q^{k_\infty} \otimes L_{v,b} \right)$ over $(\mathbb{R}^2 \setminus B_R) \times \mathbb{S}^1$ such that (A, Φ) can be written

$$\Phi = \left(v + \frac{k_\infty}{2\pi} \log r + \frac{1}{2\pi} \text{Re}\left(\frac{\mu}{z} \right) \right) \hat{\sigma} + \psi$$

$$A = \left(b\, dt + k_\infty A^\infty + \frac{1}{2\pi}(\alpha + k_\infty \pi)d\theta + \frac{1}{2\pi}\text{Im}\left(\frac{\mu}{z} \right) dt \right) \hat{\sigma} + a$$

  with $\xi = (a, \psi) = O(r^{-1-\tau})$ and $|\nabla_A \xi| + |[\Phi, \xi]| = O(r^{-2-\tau})$ for some $\tau > 0$. Here A^∞ is the connection on L_q of Lemma 2.9.

The reducible pair $(A_{\text{ext}}, \Phi_{\text{ext}})$ satisfies these boundary conditions with rate $\tau = 1$ and charge at infinity $k_\infty = 2k - n$. In fact, for topological reasons one always has $k_\infty \equiv n$ modulo 2 and then defines k by the formula above as the (non-abelian) *charge* of the $SO(3)$–monopole $(A, \Phi) \in \mathcal{C}$, *cf.* [9, Section 4.2]. We call the parameter q in Definition 3.13 the *centre* of the monopole. Observe that it is necessary to fix q in order to have L^2–integrable infinitesimal deformations.

4. The family of initial approximate solutions

Let c_{ext} be the reducible solution $(A_{\text{ext}}, \Phi_{\text{ext}})$ to the Bogomolny equation obtained in (3.2) and (3.12) from $(\lambda_0, 5, K)$–admissible data as in Definition 3.8. Here λ_0 is chosen large enough so that

Lemma 3.10 holds. In fact we will reserve the freedom to take λ_0 as large as needed until the end of the construction.

The aim of this section is to construct a family of approximate solutions to the Bogomolny equation (1.1). We desingularise the singular solution c_{ext} by gluing rescaled PS monopoles in small balls centred at the k points $q_1, \ldots, q_k$. By varying the centres of the PS monopoles and the gluing maps, we obtain a $(4k-1)$–parameter family of inequivalent smooth pairs $c(x_0, \tau)$ on $(\mathbb{R}^2 \times \mathbb{S}^1) \setminus S$.

Definition 4.1. We define *gluing regions*, adapted cut-off functions and *gluing parameters*.

(i) Let $N > 2$ be a number to be fixed later and set $\delta_j = \lambda_j^{-\frac{1}{2}}$. Taking λ_0 large enough depending on N, we assume that $2N\delta_j < \frac{1}{2}$ and $\frac{\delta_j}{2N} > \frac{2}{\lambda_j}$ for all j. Write $(\mathbb{R}^2 \times \mathbb{S}^1) \setminus S$ as the union of open sets

$$U_j = B_{N\delta_j}(q_j) \text{ for } j = 1, \ldots, k, \qquad U_{\text{ext}} = (\mathbb{R}^2 \times \mathbb{S}^1) \setminus \left(S \cup \bigcup_{j=1}^k B_{N^{-1}\delta_j}(q_j) \right).$$

Let Ann_j, $\text{Ann}_{j,\text{ext}}$, $\text{Ann}_{j,\text{int}}$ be the annuli, respectively,

$$U_j \cap U_{\text{ext}}, \qquad B_{2N\delta_j}(q_j) \setminus B_{N\delta_j}(q_j), \qquad B_{N^{-1}\delta_j}(q_j) \setminus B_{(2N)^{-1}\delta_j}(q_j).$$

(ii) Let χ be a smooth non-increasing function of one variable such that $\chi(s) \equiv 1$ if $s \leq 1$ and $\chi(s) \equiv 0$ when $s \geq 2$. For each j, let ρ_j be the distance function from q_j and define cut-off functions $\chi_{\text{int}}^j = \chi\left(\frac{\delta_j \rho_j}{2N}\right)$ and $\chi_{\text{ext}}^j = 1 - \chi\left(N\delta_j\rho_j\right)$ on $B_1(q_j)$.

(iii) Fix $\kappa \in (0,1)$ so that Lemma 2.14 holds. Let $\mathcal{P} = \mathcal{P}_\kappa$ be the trivial $\mathbb{T}^k$–bundle over the product of k balls $B_\kappa(0) \times \cdots \times B_\kappa(0) \subset (\mathbb{R}^3)^k$. We denote points in $\mathcal{P}$ by k–tuples (x_0, τ) of points $\left(x_0^j, \exp\left(\tau^j \hat{\sigma}\right)\right) \in B_\kappa(0) \times SO(2)$.

We think of $x_0^j \in B_\kappa(0)$ as parametrising the charge 1 monopole $(A_{x_0^j}, \Phi_{x_0^j})$ on $\mathbb{R}^3$ of Lemma 2.14. Identifying the ball $B_\pi(q_j) \subset \mathbb{R}^2 \times \mathbb{S}^1$ with $B_{\lambda_j \pi}(0) \subset \mathbb{R}^3$ via the homothety

$$(4.2) \qquad\qquad\qquad h_j \colon q_j + x \longmapsto \lambda_j x,$$

the jth copy of $B_\kappa(0)$ in the definition of $\mathcal{P}$ can also be thought of as $B_{\lambda_j^{-1}\kappa}(q_j) \subset \mathbb{R}^2 \times \mathbb{S}^1$.

As for the interpretation of the phase factors $(\tau_1, \ldots, \tau_k)$, let

$$(4.3) \qquad\qquad\qquad \eta_j \colon \text{Ann}_j \times \mathfrak{su}(2) \to \mathbb{R} \oplus H^2 \simeq \mathbb{R} \oplus M$$

be the isomorphism obtained by composing the gauge transformation η of Lemma 2.13 with a fixed isomorphism $H^2 \simeq M$ over Ann_j. The choice of τ_j fixes the freedom to compose η_j with a constant diagonal gauge transformation $\exp\left(\tau_j \hat{\sigma}\right)$.

It is clear how to define a family of $SO(3)$–bundles $V(\tau)$ over $(\mathbb{R}^2 \times \mathbb{S}^1) \setminus S$ with an isomorphism $V(\tau) \simeq \mathbb{R} \oplus M$ outside of $\{q_1, \ldots, q_k\}$: given a k–tuple $\tau = (e^{\tau_1 \hat{\sigma}}, \ldots, e^{\tau_k \hat{\sigma}}) \in SO(2) \times \ldots \times SO(2)$, define $V(\tau)$ identifying $(U_j, U_j \times \mathfrak{su}(2))$ and $(U_{\text{ext}}, \mathbb{R} \oplus M)$ over Ann_j using $\exp\left(\tau_j \hat{\sigma}\right) \circ \eta_j$. Since $w_2\left(V(\tau)\right) \cdot [\mathbb{S}_{p_i}^2] \equiv 1$ the isomorphism class of $V(\tau)$ does not depend on the choice of τ.

In order to define a smooth pair $c(x_0, \tau)$ on $V(\tau)$ for all $(x_0, \tau) \in \mathcal{P}$ we are now going to patch together c_{ext} and a rescaled Prasad–Sommerfield monopole in a neighbourhood of each of the points $q_1, \ldots, q_k$. Some care is needed to implement the construction in such a way that the resulting family of approximate solutions to the Bogomolny equation satisfies a number of properties. In particular, the most naïve approach to the construction would yield an error term in the Bogomolny equation not sufficiently small to apply the Implicit Function Theorem. Obstructions to match c_{ext} with scaled PS monopoles at a higher order appear if we also require a fixed behaviour at infinity.

Pull-back a rescaled Prasad–Sommerfield monopole with centre x_0^j to U_j via the homothety h_j (4.2) obtaining a pair $c_j(x_0) = h_j^*(A_{x_0^j}, \lambda_j \Phi_{x_0^j})$. By the abelian gauge of Lemma 2.13 and Lemma 2.14, over $U_j \setminus \{q_j\}$ we write $e^{\tau_j \hat\sigma} \eta_j \left(c_j(x_0)\right) = c_j^0(x_0) + (a, \psi)$, where $|(a, \psi)| = O(\lambda_j^{-2} \rho_j^{-3})$ and the leading order term $c_j^0(x_0)$ is

$$(4.4) \qquad c_j^0(x_0) = c_j^0 - \left(\frac{\langle x \times x_0^j, dx \rangle}{\lambda_j \rho_j^3}, \frac{\langle x, x_0^j \rangle}{\lambda_j \rho_j^3} \right) \hat\sigma.$$

Here c_j^0 is the Euclidean Dirac monopole of charge 2, mass λ_j and singularity at the origin pulled-back to a neighbourhood of q_j in $\mathbb{R}^2 \times \mathbb{S}^1$. Define a modified pair $c_j'(x_0)$ by

$$(4.5) \qquad e^{\tau_j \hat\sigma} \eta_j \left(c_j'(x_0) \right) = c_j^0(x_0) + \chi_{\mathrm{int}}^j(a, \psi).$$

Next, we modify c_{ext} so that it coincides with $c_j^0(x_0)$ over the annulus Ann_j. Set

$$(4.6) \qquad c_{\mathrm{ext}}(x_0) = c_{\mathrm{ext}} - 2 \sum_{j=1}^k \frac{x_0^j}{\lambda_j} \lrcorner \left(*dG_{q_j}, dG_{q_j} \right) \otimes \hat\sigma.$$

By Lemmas 2.8 and 2.10, over the punctured ball $B_{\frac{\pi}{2}}(q_j) \setminus \{q_j\}$ we can write $c_{\mathrm{ext}}(x_0) = c_j^0(x_0) + (a, \psi)$ with $|(a, \psi)| = O\left(\frac{\rho_j}{d} + \rho_j^2 + \frac{1}{\lambda_j} \right)$. Define $c_{\mathrm{ext}}'(x_0)$ by

$$(4.7) \qquad c_{\mathrm{ext}}'(x_0) = c_j^0(x_0) + \chi_{\mathrm{ext}}^j(a, \psi).$$

The collection $(U_{\mathrm{ext}}, c_{\mathrm{ext}}'(x_0))$ and $\left(U_j, c_j'(x_0) \right)$ for $j = 1, \ldots, k$ defines a pair $c(x_0, \tau)$ on $V(\tau)$.

Remark 4.8. Notice that the choice $\delta_j = \lambda_j^{-\frac{1}{2}}$ minimises the size of both $e^{\tau_j \hat\sigma} \eta_j \left(c_j'(x_0) \right) - c_0^j(x_0)$ and $c_{\mathrm{ext}}(x_0) - c_0^j(x_0)$.

In fact, as in [7, Lemma 7.2.46], it is more convenient to fix a base point $\tau_0 = (\mathrm{id}, \ldots, \mathrm{id})$ and regard the pairs $c(x_0, \tau)$ as a family of configurations on the fixed $SO(3)$–bundle $V = V(\tau_0)$. Let $\gamma_1, \ldots, \gamma_k, \gamma_{\mathrm{ext}}$ be a partition of unity subordinate to the cover $U_1, \ldots, U_k, U_{\mathrm{ext}}$. Define a gauge transformation g_j on U_j by $\eta_j \circ g_j \circ \eta_j^{-1} = \exp\left(\tau_j \gamma_{\mathrm{ext}} \hat\sigma\right)$. Similarly, let g_{ext} be the gauge transformation on U_{ext} with the properties that $g_{\mathrm{ext}} = \exp\left(-\tau_j \gamma_j \hat\sigma\right)$ on Ann_j and $g_{\mathrm{ext}} \equiv 1$ on the complement of $\mathrm{Ann}_1 \cup \ldots \cup \mathrm{Ann}_k$. Then $\eta_j g_j \eta_j^{-1} g_{\mathrm{ext}}^{-1} = e^{\tau_j \hat\sigma}$ over Ann_j and therefore $\eta_j g_j \left(c_j'(x_0) \right) = g_{\mathrm{ext}} \left(c_{\mathrm{ext}}'(x_0) \right)$. Define a pair on V by

$$(4.9) \qquad \begin{cases} g_j \left(c_j'(x_0) \right) & \text{on } U_j, \\ g_{\mathrm{ext}} \left(c_{\mathrm{ext}}'(x_0) \right) & \text{on } U_{\mathrm{ext}}. \end{cases}$$

Then $(g_1, \ldots, g_k, g_{\mathrm{ext}})$ defines an isomorphism $g \colon V(\tau) \xrightarrow{\sim} V$ such that $g\left(c(x_0, \tau)\right)$ is precisely (4.9).

Let Γ be the stabiliser of c_{ext}, *i.e.* Γ is the group of constant diagonal gauge transformations of $\mathbb{R} \oplus M$. There is natural diagonal action of Γ on τ by composition on the left. Since the Prasad–Sommerfield monopole is irreducible $c(x_0, \tau)$ and $c(x_0', \tau')$ are gauge equivalent if and only if $x_0' = x_0$ and τ' belongs to the Γ–orbit of τ.

4.1. **The centre of mass.** By (4.6), the family $c(x_0, \tau)$ does not satisfy fixed boundary conditions as (x_0, τ) varies in $\mathcal{P}$. Indeed, the centre of the pair $c(x_0, \tau)$ in the sense of Definition 3.13 depends on the *centre of mass*

$$(4.10) \qquad \zeta = - \sum_{j=1}^k \frac{x_0^j}{\lambda_j}$$

of the points $\frac{x_0^1}{\lambda_1}, \ldots, \frac{x_0^k}{\lambda_k}$. Thus the family $\{c(x_0, \tau) \,|\, (x_0, \tau) \in \mathcal{P}\}$ belongs to the fixed configuration space $\mathcal{C}$ of Definition 3.13 if and only if (x_0, τ) satisfies the "balancing" condition $\zeta = 0$. Notice that the necessity of this constraint and the action of the stabiliser of c_{ext} agree with the fact that the dimension of the moduli space of periodic monopoles of charge k is $4(k-1)$ [9, Theorem 1.5].

Nonetheless, we will not require $\zeta = 0$ at this stage. Since $\mathbb{R}^2 \times \mathbb{S}^1$ is a *parabolic* manifold, *i.e.* it has no strictly positive Green's function, if $\triangle u = f \in C_0^\infty(\mathbb{R}^2 \times \mathbb{S}^1)$, then u grows logarithmically at infinity unless f has mean value zero. As a consequence, when deforming the approximate solution $c(x_0, \tau)$ into a genuine monopole by the Implicit Function Theorem it is necessary to allow appropriate changes of the asymptotics at infinity by varying the centre q in Definition 3.13. Since our aim is to construct a whole family of solutions to the Bogomolny equation in a fixed moduli space, however, we regard the gluing problem as obstructed. In order to compensate for the obstructions we have to introduce a family of initial approximate solutions depending on parameters. These are precisely the coordinates of the centre of mass ζ. Thus we don't require the "balancing" condition $\zeta = 0$ at the beginning but will rather fix ζ only at the end of the construction. This brief discussion motivates the following definition.

Definition 4.11. Define sections $o_1, \ldots, o_4$ of $(\Lambda^1 \oplus \Lambda^0) \otimes V$ over $(\mathbb{R}^2 \times \mathbb{S}^1) \setminus S$ by

$$o_h = -\frac{1}{2\pi k} \sum_{j=1}^k \gamma(dx_h)\left(\chi_{\text{ext}}^j \, dG_{q_j}, 0\right) \otimes \hat{\sigma} \qquad o_4 = -\frac{1}{2\pi k} \sum_{j=1}^k \left(\chi_{\text{ext}}^j \, dG_{q_j}, 0\right) \otimes \hat{\sigma},$$

Here $h = 1, 2, 3$, $\left(dG_{q_j}, 0\right)$ is an element of $\Omega(\mathbb{R} \oplus M)$ and, by abuse of notation, $dx_1 = dx$, $dx_2 = dy$ and $dx_3 = dt$. Moreover, $\gamma(dx_h)(a, 0) = \partial_{x_h} \lrcorner (*a, a)$ is the Clifford multiplication (2.5) of dx_h with the 1–form a.

As we will see later, the span of o_1, o_2, o_3 is the space of obstructions to solve the Bogomolny equation with fixed asymptotics. By Lemmas 2.8 and 2.10 and Definition 4.1.(ii) there exists a constant $C > 0$ such that:

$$(4.12) \qquad \begin{cases} |o_h| \le C\rho_j^{-2} & \text{in } B_1(q_j) \setminus B_{N\delta_j}(q_j) \\ |o_h| \le C & \text{outside of } \bigcup_{j=1}^k B_{\frac{1}{2}}(q_j) \end{cases} \qquad \begin{cases} |\nabla o_h| \le C\rho_j^{-3} & \text{in } B_1(q_j) \setminus B_{N\delta_j}(q_j) \\ |\nabla o_h| \le C & \text{outside of } \bigcup_{j=1}^k B_{\frac{1}{2}}(q_j) \end{cases}$$

We complete the definition of the pair $c(x_0, \tau)$ replacing (4.9) by

$$(4.13) \qquad\qquad c(x_0, \tau) + 4\pi \sum_{h=1}^3 \zeta_h \, o_h$$

where ζ is given by (4.10). By Lemmas 2.8, 2.9 and 2.10 this modification guarantees that the pairs $c(x_0, \tau)$ satisfy fixed asymptotics for all $(x_0, \tau) \in \mathcal{P}$, *i.e.* $c(x_0, \tau)$ lies in the fixed configuration space $\mathcal{C}$ as (x_0, τ) varies in $\mathcal{P}$. By abuse of notation, in the rest of the paper we will take (4.13) as the definition of the pair $c(x_0, \tau)$.

4.2. **Estimate of the error and the geometry of the approximate solutions.** Fix a point $(x_0, \tau) \in \mathcal{P}$ and set $(A, \Phi) = c(x_0, \tau)$. We want to estimate how far (A, Φ) is from being a solution to the Bogomolny equation, *i.e.* we want to control $\Psi(x_0, \tau) = *F_A - d_A\Phi$. For later use, we also estimate the size of the Higgs field Φ and of the "curvature" term $d_A\Phi = *F_A - \Psi(x_0, \tau)$.

Proposition 4.14. *There exists λ_0 and κ such that the following holds. Suppose that $v, S, q_1, \ldots, q_k$ are $(\lambda_0, 5, K)$–admissible and let $\mathcal{P} = \mathcal{P}_\kappa$ be as in Definition 4.1.(iii). Then the construction of $c(x_0, \tau)$ in (4.13) defines a map $c \colon \mathcal{P} \to \mathcal{C}$, the pre-gluing map, that factors through $c \colon \mathcal{P}/\Gamma \to \mathcal{C}/\mathcal{G}$, where $\Gamma \simeq SO(2)$ acts on $\mathcal{P}$ diagonally and $\mathcal{G}$ is the space of bounded gauge transformation which preserve the boundary conditions of Definition 3.13.*

Furthermore, there exists a uniform constant $C > 0$ depending only on λ_0, κ and $p_1, \ldots, p_n$ with the following significance. For every $(x_0, \tau) \in \mathcal{P}$ define ζ by (4.10) and set $(A, \Phi) = c(x_0, \tau)$.

(i) *The error $\Psi(x_0, \tau) = *F_A - d_A\Phi$ is supported on $Ann_{j,int}$ and $Ann_{j,ext}$. Moreover, define*

$$(4.15) \qquad \Psi_\zeta = 4\pi\, d_2 \left(\sum_{h=1}^{3} \zeta_h\, o_h \right).$$

Then Ψ_ζ is supported on $Ann_{j,ext}$ and we have estimates

$$|\Psi(x_0, \tau) - \Psi_\zeta| \leq C \qquad and \qquad \rho_j^2 |\Psi_\zeta| \leq \frac{C}{\sqrt{\lambda}}.$$

(ii) *On every ball $B_1(q_j)$, $j = 1, \ldots, k$, we have*

$$(4.16) \qquad \left(\lambda_j^{-2} + \rho_j^2 \right) |d_A\Phi| \leq C.$$

(iii) *$|\Phi| \geq \frac{1}{2}$ over U_{ext}.*

Proof. The first part of the Proposition is a simple restatement of the construction of the pair $c(x_0, \tau)$ for $(x_0, \tau) \in \mathcal{P}$. We have to verify the estimates in (i), (ii) and (iii).

From the construction of $c(x_0, \tau)$ recall that

$$\begin{cases} c(x_0, \tau) = c_j^0(x_0) + \chi_{int}^j\, O(\lambda_j^{-2} \rho_j^{-3}) & \text{over } Ann_{j,int} \\[2mm] c(x_0, \tau) = c_j^0(x_0) + \chi_{ext}^j\, O(\lambda_j^{-1} + \rho_j) + 4\pi k \sum_{h=1}^{3} \zeta_h\, o_h & \text{over } Ann_{j,ext} \end{cases}$$

with $c_j^0(x_0)$ defined explicitly in (4.4). Observe also that if (A, Φ) and $(A, \Phi) + (a, \psi)$ both solve the Bogomolny equation and χ is a smooth function, then

$$*F_{A+\chi a} - d_{A+\chi a}(\Phi + \chi\psi) = *(d\chi \wedge a) - (d\chi)\psi + \chi(\chi - 1)(*[a, a] - [a, \psi]).$$

A direct computation using Definition 4.1.(ii) therefore yields

$$(4.17) \qquad \begin{cases} |\Psi(x_0, \tau)| \leq CN^4 & \text{over } Ann_{j,int}, \\[1mm] |\Psi(x_0, \tau) - \Psi_\zeta| \leq C & \text{over } Ann_{j,ext}. \end{cases}$$

In order to control $|\Psi_\zeta|$, we use the fact that G_{q_j} is harmonic outside of q_j to obtain $|d_2 o_h| \leq C\rho_j^{-2} |\nabla \chi_{ext}^j|$. The estimate now follows from the properties of χ_{ext}^j in Definition 4.1, the definition of δ_j and (4.10). This concludes the proof of (i).

We prove (4.16) separately in different regions.

1. On the ball $\rho_j \leq \frac{\delta_j}{2N}$, $c(x_0, \tau)$ is gauge equivalent to a translation of a PS monopole rescaled by λ_j. The quantity $(1 + \rho^2)|d_A\Phi|$ is scale invariant and therefore Lemma 2.12 and the fact that $|x_0^j| < \kappa$ imply the estimate.

2. On the annulus Ann_j, $c(x_0, \tau)$ coincides with the modified Dirac monopole $c_j^0(x_0)$:

$$\left(\lambda_j^{-2} + \rho_j^2 \right) |d_A\Phi| \leq C \left(1 + N\lambda_j^{-\frac{1}{2}} \right)$$

follows directly from the definition (4.4) of $c_0^j(x_0)$.

3. We deduce the estimate on the annulus $Ann_{j,int}$ from the previous two and Lemma 2.14. Write $c(x_0, \tau) = c_j^0(x_0) + \chi_j^{int}(a, \psi)$, where $(a, \psi) = O\left(\lambda_j^{-2} \rho_j^{-3} \right)$. In Lemma 2.14 we did not calculate the decay of the covariant derivative of (a, ψ), but we can argue as follows. Observe that

$$d_{A+\chi a}(\Phi + \chi\psi) = d_A\Phi + \chi\, d_{A+a}(\Phi + \psi) + d\chi \wedge \psi + \chi(\chi - 1)[a, \psi]$$

where $(A, \Phi) = c_0^j(x_0)$ and $\chi = \chi_{\mathrm{int}}^j$. Since $(A+a, \Phi+\psi)$ is a translation of the PS monopole, we deduce that

$$\rho_j^2 |d_A \Phi| \leq C(1 + N^2 \lambda_j^{-1} + N^4 \lambda_j^{-2}).$$

4. Finally, on the annulus $B_1(q_j) \setminus B_{N\delta_j}$ write

$$c(x_0, \tau) = c_j^0(x_0) + \chi_{\mathrm{ext}}^j(a, \psi) + 4\pi k \sum_{h=1}^{3} \zeta_h \, o_h,$$

where $(a, \psi) = O(\rho_j + \lambda_j^{-1})$ and $(\nabla a, \nabla \psi) = O(1)$. A direct computation using (4.12) yields

$$\rho_j^2 |d_A \Phi| = O(1 + \lambda_j^{-\frac{1}{2}}).$$

Finally, the statement in (iii) is a consequence of the following lemma. $\qquad \square$

Lemma 4.18. *If* $(A, \Phi) = c_{ext}'(x_0) + 4\pi \sum_{h=1}^{3} \zeta_h \, o_h$ *then, possibly after taking* λ_0 *larger and* κ *smaller if necessary, there exists* $\frac{1}{\lambda_j} < \delta(\lambda_j) < \frac{\sqrt{2}}{\lambda_j}$ *such that*

$$(4.19) \qquad\qquad\qquad\qquad |\Phi| \geq \frac{1}{2}$$

outside of $\bigcup_{j=1}^{k} B_{\delta(\lambda_j)}(q_j)$.

Proof. We prove the lemma controlling the size of $|\Phi|$ through each step of the construction of the pair $c(x_0, \tau)$.

First, by the definition (4.6) of $c_{\mathrm{ext}}(x_0)$, the function $\langle \Phi_{\mathrm{ext}}(x_0), \hat{\sigma} \rangle$ is harmonic outside of the points $\{p_1, \ldots, p_n, q_1, \ldots, q_k\}$. One can argue as in Lemma 3.10 to show that there exists $\lambda_0 > 0$ such that if $\lambda_j > \lambda_0$ then $\langle \Phi_{\mathrm{ext}}(x_0), \hat{\sigma} \rangle \geq 1$ outside of $\bigcup_{j=1}^{k} B_{\delta(\lambda_j)}(q_j)$ for some $\frac{1}{\lambda_j} < \delta(\lambda_j) \leq \frac{1+\sqrt{1+\kappa}}{2\lambda_j}$.

Secondly, picking λ_0 even larger if necessary (depending on N), one can make sure that the term of order $O\left(\frac{\rho_j}{d} + \rho_j^2 + \frac{1}{\lambda_j}\right)$ multiplied by the cut-off function χ_{ext}^j in (4.7) is smaller than $\frac{1}{4}$.

Finally, by (4.12), Definition 3.8.(iii) and the fact that $|\zeta| \leq \frac{C\kappa}{\lambda}$, we can choose κ small enough (depending on K of Definition 3.8) so that $\left| 4\pi \sum_{h=1}^{3} \zeta_h \, o_h \right| \leq \frac{1}{4}$. $\qquad \square$

5. The linearised equation

Having constructed a family of approximate solutions to the Bogomolny equation, our aim is now to find $\xi = \xi(x_0, \tau)$ for every $(x_0, \tau) \in \mathcal{P}$ with the appropriate decay at infinity and at the singularities $p_i \in S$ such that $c(x_0, \tau) + \xi$ is a solution to the Bogomolny equation (1.1). Hence we look for a solution ξ of the equation

$$(5.1) \qquad\qquad\qquad\qquad d_2 \xi + \xi \cdot \xi + \Psi(x_0, \tau) = 0,$$

where d_2 is the linearisation (2.2) of the Bogomolny equation at $c(x_0, \tau)$.

In this section, the technical core of the paper, we study the linearised equation $d_2 \xi = f$. This is the crucial step to solve (5.1). The strategy we adopt to understand the invertibility properties of d_2 is to first solve the equation separately on U_j and U_{ext}, in the latter case only modulo obstructions. A global solution of the linearised equation modulo obstructions is then obtained from the local right inverses of d_2 by a simple iteration. As the main technical tool we will employ a range of weighted Sobolev spaces to carry out the analysis. For technical reasons we will have to distinguish the high mass and large distance limit when studying the equation $d_2 \xi = f$ over U_{ext}.

Due to the gauge invariance of the Bogomolny equation, d_2 is not elliptic. We will look for a solution of the form $\xi = d_2^* u$ for a 1–form u with values in V. The Weitzenböck formula

$$(5.2) \qquad\qquad\qquad d_2 d_2^* u = \nabla_A^* \nabla_A u - \mathrm{ad}^2(\Phi) u + *[\Psi, u]$$

can be deduced from [8, Lemma 18]; here $\Psi = \Psi(x_0, \tau)$. At times it will be convenient to pair a V–valued 1–form u with the zero section of V and consider the V–valued form of mixed degree $(u, 0)$. Observe that $d_2^* u = D^*(u, 0)$, where D is the Dirac operator (2.4), and the equation $d_2 d_2^* u = f$ is equivalent to

$$(5.3) \qquad\qquad DD^*(u, 0) = (f, *[\Psi, *u]).$$

We will make use of the Weitzenböck formulas [8, Lemma 18]

$$(5.4) \qquad DD^* = \nabla_A^* \nabla_A - \mathrm{ad}(\Phi)^2 + \Psi \qquad \text{and} \qquad D^*D = DD^* + 2d_A\Phi.$$

5.1. **The linearised equation on U_j.** There are no obstructions to the invertibility of the operator $d_2 d_2^*$ over U_j. The only issue is instead the fact that by (4.16) the curvature term $d_A\Phi$ blows up as $\lambda_j \to \infty$. In view of the Weitzenböck formula (5.4) for D^*D, the norm of the inverse of the operator $d_2 d_2^*\colon W^{2,2} \to L^2$ cannot be uniformly bounded. Following a standard approach in gluing problems, we resolve this difficulty introducing appropriate weighted Sobolev spaces.

For each $j = 1, \ldots, k$ consider the weight function $w_j = \sqrt{\lambda_j^{-2} + \rho_j^2}$. By abuse of notation, we won't distinguish between the globally defined function w_j on $\mathbb{R}^3$ and a fixed smooth increasing function on $(\mathbb{R}^2 \times \mathbb{S}^1) \setminus S$ with the properties $w_j \leq 1$ and

$$(5.5) \qquad\qquad w_j = \begin{cases} \sqrt{\lambda_j^{-2} + \rho_j^2} & \text{if } \rho_j \leq \frac{1}{2}, \\ 1 & \text{if } \rho_j \geq 1. \end{cases}$$

By scaling, we will work on $\mathbb{R}^3$ endowed with the weight function $w = \sqrt{1 + \rho^2}$. On the trivial $SO(3)$–bundle $\mathbb{R}^3 \times \mathfrak{su}(2)$ we fix a pair (A, Φ) which coincides with the monopole (A_{x_0}, Φ_{x_0}) of Lemma 2.14 if $\rho \leq (2N)^{-1}\sqrt{\lambda_j}$ and with the reducible pair induced by a charge 1 Euclidean Dirac monopole of unit mass when $\rho \geq N^{-1}\sqrt{\lambda_j}$. In other words, we work with the pair obtained from $c_j'(x_0)$ by scaling, but the estimates of Proposition 5.8 below will only depend on the curvature control (4.16) and the fact that A is a smooth metric connection.

By Kato's inequality, standard results about functions can be extended to $\mathfrak{su}_2$–valued forms and their covariant derivatives. In addition to the Euclidean Sobolev inequality $\|u\|_{L^6} \leq C_{Sob}\|\nabla u\|_{L^2}$ we will make use of the following Hardy-type inequality, whose proof is obtained by a simple integration by parts [12, Lemma 2].

Lemma 5.6. *For all $\delta \in (-1, 0)$ and $u \in C_0^\infty(\mathbb{R}^3; \mathfrak{su}_2)$*

$$\int w^{-2\delta-3} |u|^2 \leq \frac{1}{\delta^2} \int w^{-2\delta-1} |\nabla_A u|^2.$$

Definition 5.7. For all $\delta \in \mathbb{R}$ and all smooth forms $u \in \Omega(\mathbb{R}^3; \mathfrak{su}_2)$ with compact support define:

$$\|u\|_{L^2_{w,\delta}} = \|w^{-\delta-\frac{3}{2}} u\|_{L^2}, \qquad \|u\|_{W^{1,2}_{w,\delta}}^2 = \|u\|_{L^2_{w,\delta}}^2 + \|\nabla_A u\|_{L^2_{w,\delta-1}}^2 + \|[\Phi, u]\|_{L^2_{w,\delta-1}}^2.$$

Define spaces $L^2_{w,\delta}$ and $W^{1,2}_{w,\delta}$ as the completion of C_0^∞ with respect to these norms. Finally, we say that $u \in W^{2,2}_{w,\delta}$ if $u \in W^{1,2}_{w,\delta}$ and

$$\|\nabla_A(D^* u)\|_{L^2_{w,\delta-2}} + \|[\Phi, D^* u]\|_{L^2_{w,\delta-2}} < \infty.$$

Proposition 5.8. *For all $-1 < \delta < 0$ there exist $\varepsilon > 0$ and $C > 0$ such that if $\|w\Psi\|_{L^3} < \varepsilon$ then the following holds. For all $f \in L^2_{w,\delta-2}$ there exists a unique solution $u \in W^{2,2}_{w,\delta}$ to $d_2 d_2^* u = f$. Moreover,*

$$\|u\|_{W^{2,2}_{w,\delta}} \leq C\|f\|_{L^2_{w,\delta-2}}.$$

Proof. By an approximation argument, we can assume that $f \in C_0^\infty$. In view of the Weitzenböck formula (5.2), the solution u can be found by variational methods. Indeed, by Hölder's inequality

$$|\langle *[\Psi, u], u \rangle_{L^2}| \leq \|w\Psi\|_{L^3} \|w^{-1}u\|_{L^2} \|u\|_{L^6} \leq 2C_{Sob} \|w\Psi\|_{L^3} \|\nabla_A u\|_{L^2}^2.$$

The last inequality follows from Lemma 5.6 with $\delta = -\frac{1}{2}$ and the Sobolev inequality. Thus $\|d_2^* u\|_{L^2}$ is a norm on $W_{w,\frac{1}{2}}^{1,2}$ provided that $\|w\Psi\|_{L^3} < \frac{1}{2C_{Sob}}$. Moreover, since $f \in C_0^\infty$ the functional $\langle f, u \rangle_{L^2}$ is continuous on $W_{w,\frac{1}{2}}^{1,2}$. Hence there exists a unique solution u to $d_2 d_2^* u = f$, which, by standard elliptic regularity, lies in $W_{w,\frac{1}{2}}^{1,2} \cap C_{loc}^\infty$. We have to prove that $u \in W_{w,\delta}^{2,2}$.

In order to justify the integrations by parts below, observe that $|u| = O(\rho^{-1})$ as $\rho \to \infty$. This is because $d_2 d_2^* u = 0$ outside of the support of f and (A, Φ) is reducible outside of a large compact set; thus $u = u_D + u_T$ with u_D harmonic and u_T exponentially decaying due to the non-vanishing of $|\Phi|$ at infinity, *cf.* [9, Lemma 7.10 and Remark 7.11]. A first integration by parts yields

$$\|u\|_{L_{w,\delta}^2} \|f\|_{L_{w,\delta-2}^2} \geq \int \langle f, u \rangle \, w^{-2\delta-1} = \int \langle *[\Psi, u], u \rangle \, w^{-2\delta-1}$$
$$+ \int (|\nabla_A u|^2 + |[\Phi, u]|^2) w^{-2\delta-1} + (1+2\delta)|\delta| \int |u|^2 w^{2\delta-3}$$

As before, we control the term involving the error Ψ as follows:

$$\left| \langle *[\Psi, u], u \, w^{-2\delta-1} \rangle_{L^2} \right| \leq \|w\Psi\|_{L^3} \|u\|_{L_{w,\delta}^2} \|w^{-\delta-\frac{1}{2}} u\|_{L^6} \leq C \|w\Psi\|_{L^3} \|u\|_{W_{w,\delta-1}^{1,2}}^2$$

using Lemma 5.6, the Sobolev inequality and the fact that $\nabla_A(w^{-\delta-\frac{1}{2}} u) \in L^2$ if $u \in W_{w,\delta-1}^{1,2}$. Thus

$$\|u\|_{W_{w,\delta}^{1,2}} \leq C \|f\|_{L_{w,\delta-2}^2}$$

provided that $\|w\Psi\|_{L^3}$ is sufficiently small.

Set $\xi = d_2^* u = D^*(u, 0)$. Since $\| *[\Psi, *u]\|_{L_{w,\delta-2}^2} \leq \|w\Psi\|_{L^3} \|w^{-\delta-\frac{1}{2}} u\|_{L^6}$, the Sobolev inequality and (5.3) imply that $\|D\xi\|_{L_{w,\delta-2}^2}$ is controlled by $\|f\|_{L_{w,\delta-2}^2}$. We will conclude the proof of the Proposition by establishing the a priori estimate

$$\|\nabla_A \xi\|_{L_{w,\delta-2}^2} + \|[\Phi, \xi]\|_{L_{w,\delta-2}^2} \leq C \left(\|D\xi\|_{L_{w,\delta-2}^2} + \|\xi\|_{L_{w,\delta-1}^2} \right).$$

Integrate the Weitzenböck formula (5.4) for D^*D against $w^{-2\delta+1}\xi$:

$$\int \left(|\nabla_A \xi|^2 + |[\Phi, \xi]|^2 \right) w^{-2\delta+1} \leq c_1 \int |\xi|^2 w^{-2\delta-1} + c_2 \int \langle D\xi, \xi \rangle \, w^{-2\delta}$$
$$+ \int w^{-2\delta+1} |D\xi|^2 + \int w^{-2\delta+1} |\Psi| \, |\xi|^2$$
$$+ \int w^{-2\delta+1} |d_A \Phi| \, |\xi|^2,$$

using $|\nabla w| \leq 1$. By Proposition 4.14.(ii) $w^2 |d_A \Phi|$ is uniformly bounded. The term involving Ψ is controlled as before using the smallness of $\|w\Psi\|_{L^3}$. $\qquad\square$

5.2. **The linearised equation on U_{ext}: the high mass case.** We move on to study the equation $d_2 d_2^* u = f$ over the exterior region U_{ext}. The background is now the pair $c'_{\text{ext}}(x_0) + 4\pi k \sum_{h=1}^3 \zeta_h \, o_h$ of (4.7) and (4.13). Recall that this is a reducible solution to the Bogomolny equation on the complement of $\bigcup_{j=1}^k B_{2N\delta_j}(q_j)$.

If u is a section of the reducible $SO(3)$–bundle $\underline{\mathbb{R}} \oplus M$ of (3.11), we write $u = u_D + u_T$. By Lemma 4.18

$$4|[\Phi, u]|^2 \geq |u_T|^2 \tag{5.9}$$

ouside of small neighbourhoods of $\{q_1, \ldots, q_k\}$. By Fourier analysis with respect to the circle variable t we can further decompose $u_D = \Pi_0 u_D + \Pi_\perp u_D$ into $\mathbb{S}^1$–invariant and oscillatory part. On each circle $\{z\} \times \mathbb{S}^1_t$ one has the Poincaré inequality

$$\int_{\mathbb{S}^1} |\nabla(\Pi_\perp u_D)|^2 \geq \int_{\mathbb{S}^1} |\Pi_\perp u_D|^2. \tag{5.10}$$

The inequalities (5.9) and (5.10) suggest that, via the Weitzenböck formula (5.2), we have extremely good control of the off-diagonal and oscillatory piece of u in terms of $d_2 d_2^* u$. In order to control the $\mathbb{S}^1$–invariant diagonal piece $\Pi_0 u_D$ we introduce appropriate weighted spaces. The choice of weight functions is different in the two distinct situations (A) and (B) of Section 3, *i.e.* the high mass and large distance case, respectively.

(A) If we are constructing monopoles in the high mass limit $v \to \infty$ and $q_1, \ldots, q_k, S$ are contained in a fixed set $B_{R_0} \times \mathbb{S}^1 \subset \mathbb{R}^2 \times \mathbb{S}^1$, the framework of [9] applies and some care is needed only to check that the constants are uniform as $v \to \infty$. We will use the weight function $\omega = \sqrt{1 + |z|^2}$ and let all constants depend on R_0 without further notice.

(B) If instead $n \leq 2(k-1)$ and we allow $d \to \infty$, then the error is concentrated around k points $q_1, \ldots, q_k$ moving off to infinity and we would like to replace $\sqrt{1 + |z|^2}$ with a weight function which is uniformly bounded above and below in a neighbourhood of each q_j but maintains the same behaviour $O(|z|)$ at large distances.

We begin with the high mass case (A) and explain how to extend the results to case (B) in a second step. Thus set $\omega = \sqrt{1 + |z|^2}$ and consider additional weight functions $\hat{\rho}_j, \hat{\rho}_i$ defined in a neighbourhood of the point q_j and $p_i \in S$, respectively. The weight function $\hat{\rho}_j$ is a fixed smooth increasing function with $\hat{\rho}_j \leq 1$ and

$$\hat{\rho}_j = \begin{cases} \rho_j & \text{if } \rho_j \leq \frac{1}{2}, \\ 1 & \text{if } \rho_j \geq 1. \end{cases} \tag{5.11}$$

The function $\hat{\rho}_i$ is defined in a similar way, but the transition between ρ_i and 1 takes place on the annulus $B_{2\sigma}(p_i) \setminus B_\sigma(p_i)$, where $\sigma > 0$ is chosen so that the balls $B_{2\sigma}(p_i)$ are all disjoint. Constants will be allowed to depend on σ without further notice. We denote by U_σ the complement of the union $\bigcup_{i=1}^n B_\sigma(p_i) \cup \bigcup_{j=1}^k B_{\frac{1}{2}}(q_j)$. In the definition below, we introduce the relevant Sobolev norms.

Definition 5.12. Given a triple $(\delta_1, \delta_2, \delta_3)$ and a smooth compactly supported form $u \in \Omega(\underline{\mathbb{R}} \oplus M)$ define $\|u\|_{L^2_{(\delta_1, \delta_2, \delta_3)}}$ as the maximum of the semi-norms:

$$\left\| \omega^{-\delta_1 - 1} u \right\|_{L^2(U_\sigma)}, \qquad \|\hat{\rho}_i^{-\delta_2 - \frac{3}{2}} u\|_{L^2(B_{2\sigma}(p_i))}, \qquad \|\hat{\rho}_j^{-\delta_3 - \frac{3}{2}} u\|_{L^2(B_1(q_j))}.$$

Given $\delta > 0$, set $\underline{\delta} = (-\delta, \delta, -\delta)$ and for each $m \in \mathbb{Z}$ let $\underline{\delta} + m$ denote the triple $\underline{\delta} + (m, m, m)$. For a smooth compactly supported form $u \in \Omega(\underline{\mathbb{R}} \oplus M)$ we say that

(1) $u \in L^2_{\underline{\delta}-2}$ if the corresponding norm is finite;

(2) $u \in W^{1,2}_{\underline{\delta}-1}$ if $u \in L^2_{\underline{\delta}-1}$ and $\nabla_A u, [\Phi, u] \in L^2_{\underline{\delta}-2}$;

(3) $u \in W^{2,2}_{\underline{\delta}}$ if $D^* u \in W^{1,2}_{\underline{\delta}-1}$ and $u \in L^2_{(\delta, -\delta, -\delta)}$.

The space $W^{m,2}_{\underline{\delta}-2+m}$ is defined as the completion of C_0^∞ with respect to the corresponding norm. By convention $W^{0,2}_{\underline{\delta}-2} = L^2_{\underline{\delta}-2}$.

We stress two aspects of this definition, referring to [9] and the rest of the section for further details. On one side, we distinguished the points q_j from the singularities p_i. More precisely, around each of the singularities p_i we imposed a stronger norm. As we will see later, this choice is necessary to control the non-linearities of (5.1). Secondly, observe that if $u \in W_\delta^{2,2}$ then the transversal and oscillatory part u_T and $\Pi_\perp u_D$ have a stronger decay and lie in L_δ^2. The reason for the odd definition of $W_\delta^{2,2}$ is to include diagonal sections which have non-zero limits over the punctures and at infinity. This is necessary to ensure the surjectivity of the operator $d_2 d_2^*$.

The main result of the sub-section is the following proposition.

Proposition 5.13. *For all $0 < \delta < \frac{1}{2}$ there exists ε and C with the following significance. Suppose that $\|\hat{\rho}_j \Psi|_{B_1(q_j)}\|_{L^3} < \varepsilon$ for all $j = 1, \ldots, k$. Then for all $f \in L_{\underline{\delta}-2}^2$ such that $\int \langle f, \hat{\sigma} \otimes dx_h \rangle = 0$ for $h = 1, 2, 3$ there exists a unique solution $\xi \in W_{\underline{\delta}-1}^{1,2}$ to $d_2 \xi = f$ of the form $\xi = d_2^* u$ with $\int \langle u, \hat{\sigma} \otimes dx_h \rangle \omega^{-2(\delta+1)} = 0$. Moreover,*

$$\|\xi\|_{W_{\underline{\delta}-1}^{1,2}} \leq C \|f\|_{L_{\underline{\delta}-2}^2}.$$

The proof is given in two steps. First we prove the existence of a weak solution.

Lemma 5.14. *For all $0 < \delta \leq \frac{1}{2}$ there exists $\varepsilon > 0$ and C such that the following holds.*

Suppose that $\|\hat{\rho}_j \Psi|_{B_1(q_j)}\|_{L^3} < \varepsilon$ for all $j = 1, \ldots, k$. Let $f \in L_{\underline{\delta}-2}^2$ be a $(\mathbb{R} \oplus M)$–valued 1–form satisfying $\int \langle f, \hat{\sigma} \otimes dx_h \rangle = 0$ for $h = 1, 2, 3$. Then there exists a unique weak solution u to $d_2 d_2^* u = f$ with

$$\int \langle u, \hat{\sigma} \otimes dx_h \rangle \omega^{-2(\delta+1)} = 0 \qquad and \qquad \|d_2^* u\|_{L^2} \leq C \|f\|_{L_{\underline{\delta}-2}^2}.$$

Proof. The first claim is that $\|d_2^* u\|_{L^2}^2$ is uniformly equivalent to $\|\nabla_A u\|_{L^2}^2 + \|[\Phi, u]\|_{L^2}^2$ for all $u \in C_0^\infty$ provided that $\|\hat{\rho}_j \Psi|_{B_1(q_j)}\|_{L^3}$ is sufficiently small. Indeed, integrate by parts the Weitzenböck formula (5.2) for $d_2 d_2^*$ and use the Sobolev and Hardy inequalities to control

$$|\langle \Psi \cdot u, u \rangle_{L^2}| \leq \|\hat{\rho}_j \Psi\|_{L^3} \|\hat{\rho}_j^{-1} u\|_{L^2} \|u\|_{L^6} \leq C \|\hat{\rho}_j \Psi\|_{L^3} \|\nabla_A u\|_{L^2}^2.$$

The second observation is that $\|\nabla_A u\|_{L^2}^2 + \|[\Phi, u]\|_{L^2}^2$ is a norm on the space of smooth compactly supported forms with $\int \langle u, \hat{\sigma} \rangle \omega^{-2(\delta+1)} = 0$. More precisely, we are going to show that there exists a uniform constant C such that

$$(5.15) \qquad \|u\|_{L_{(\delta, -\frac{1}{2}, -\frac{1}{2})}^2}^2 \leq C \int |\nabla_A u|^2 + |[\Phi, u]|^2.$$

Indeed, set $B = B_1(q_j)$ and let χ be a smooth cut-off function supported in B with $\chi \equiv 1$ in $\frac{1}{2}B$. Lemma 5.6 applied to χu with $\delta = -\frac{1}{2}$ yields

$$\|\hat{\rho}_j^{-1} u\|_{L^2(\frac{1}{2}B)} \leq C \left(\|\nabla_A u\|_{L^2} + \|u\|_{L^2(B \setminus \frac{1}{2}B)} \right)$$

with a uniform constant $C > 0$. In the same way we can control the norm $\|\hat{\rho}_i^{-1} u\|_{L^2}$ in a punctured neighbourhood of p_i in terms of $\|\nabla_A u\|_{L^2}$ and the L^2–norm of u on an annulus around p_i. Thus (5.15) will follow once we establish the weighted Poincaré inequality

$$\int \omega^{-2(\delta+1)} |u|^2 \leq C \int |\nabla_A u|^2 + |[\Phi, u]|^2.$$

Decompose $u = \Pi_0 u_D + \Pi_\perp u_D + u_T$. Since $\omega \geq 1$, the estimate for u_T and $\Pi_\perp u_D$ follows from (5.9) and (5.10), respectively. If $u = \Pi_0 u_D \in C_0^\infty(\mathbb{R}^2)$ and $\int u \omega^{-2(\delta+1)} = 0$, the stated weighted Poincaré inequality is proved in [1, Corollary 8.4].

Finally, notice that, for δ in the range specified, $L^2_{\underline{\delta}-2}$ is contained in the dual of $L^2_{(\delta,-\frac{1}{2},-\frac{1}{2})}$ and therefore the existence of a weak solution u to $d_2 d_2^* u = f$ follows by variational methods. $\qquad\square$

It remains to prove that $\xi \in W^{1,2}_{\underline{\delta}-1}$. This is a consequence of the following a priori estimate.

Lemma 5.16. *For all $0 < \delta < \frac{1}{2}$ there exists $C > 0$ such that for all smooth compactly supported ξ*

$$\|\xi\|_{W^{1,2}_{\underline{\delta}-1}} \leq C \left(\|D\xi\|_{L^2_{\underline{\delta}-2}} + \|\xi\|_{L^2} \right).$$

Proof. The estimate is equation (7.8) in the proof of [9, Proposition 7.7]. The fact that the constant C is uniform independently of the mass of the monopole is proved in [9, Lemma 8.11]. For the convenience of the reader we summarise the main aspects of the argument.

The first step is to prove the weighted elliptic estimate

$$(5.17) \qquad \|\xi\|_{W^{1,2}_{\underline{\delta}-1}} \leq C \left(\|D\xi\|_{L^2_{\underline{\delta}-2}} + \|\xi\|_{L^2_{\underline{\delta}-1}} \right).$$

As in the proof of Proposition 5.8, the main tool is the Weitzenböck formula (5.4) for the operator D^*D. The constant C thus depends on appropriate norms of $d_A\Phi$ and Ψ. Using a partition of unity we can always reduce to prove the estimate under the additional assumption that ξ is supported in a specific given region.

(1) If ξ is supported in $B_1(q_j)$, as in Proposition 5.8, integrate by parts the Weitzenböck formula and use Proposition 4.14 to show that $|\hat\rho_j^2\Psi|$ and $\hat\rho_j^2|d_A\Phi|$ are uniformly bounded.

(2) Assume now that $\xi \in C^\infty_0(U_\sigma)$. Since $\Psi \equiv 0$ on U_σ, the existence of a uniform constant C follows from the fact that $\omega|d_A\Phi|$ is uniformly bounded. In order to check this last statement, recall that $\Phi'_{\text{ext}}(x_0)$ is a sum of Green's functions and their derivatives. By Lemma 2.8.(ii), for any $p = (z_0, t_0) \in \mathbb{R}^2 \times \mathbb{S}^1$

$$|\nabla G_p| + |z - z_0|\,|\nabla^2 G_p| \leq \frac{C}{|z - z_0|}$$

for all (z,t) such that $|z - z_0| > 2$. We conclude that there exists C depending only on σ and R_0 such that $\omega|d_A\Phi| \leq C$.

(3) Finally, suppose that ξ is supported on $B_{2\sigma}(p_i)$. In view of the Weitzenböck formula, we have to check that $\hat\rho_i^2|d_A\Phi|$ is uniformly bounded. This follows from (3.4) and the fact that the modifications of Φ_{ext} in (4.6) and (4.13) introduce smooth terms controlled by d^{-1} in a neighbourhood of p_i.

In order to conclude the proof of the Lemma we have to improve (5.17) to the required estimate, *i.e.* replace the $L^2_{\underline{\delta}-1}$–norm of ξ in the RHS with its L^2–norm. We distinguish diagonal and off-diagonal part. If $\xi = \xi_D$ the statement is deduced from standard theory for the Laplacian in weighted Sobolev spaces, *cf.* part (1) in the proof of [9, Proposition 7.7]. Now suppose that $\xi = \xi_T$. If ξ is supported on U_σ we apply word by word the argument of part (2) in the proof of [9, Proposition 7.7]. Indeed, the argument only relies on (5.9) and the fact that $\omega \to \infty$ as $|z| \to \infty$. Similarly, the proof of (8.12) in [9, Lemma 8.11], case (3), yields the result when $\xi = \xi_T$ is supported on $B_{2\sigma}(p_i)$. $\qquad\square$

Proposition 5.13 follows from the two lemmas, since we can control $\|D\xi\|_{L^2_{\underline{\delta}-2}}$ in terms of f as in the proof of Proposition 5.8.

5.3. **The linearised equation on U_{ext}: the large distance case.** We come to the task of adapting the analysis to deal with the situation in which the points $q_1, \ldots, q_k$ move off to infinity. The first step is to define an adapted weight function.

By the assumption $d > d_0 = 5$ of Definition 3.8, the set $B_2(z_j) \times \mathbb{S}^1$ does not contain any of the points $q_1, \ldots, q_k, p_1, \ldots, p_n$ other than q_j. By taking d_0 larger, we can also assume that there exists $R_0 > 0$ such that that the ball $B_{R_0}(0) \subset \mathbb{R}^2$ is disjoint from $B_2(z_j)$ for all j and $B_{R_0} \times \mathbb{S}^1$ contains all the singularities $p_1, \ldots, p_n$. Set $z_0 = 0$.

In addition to the condition (iii) in Definition 3.8, we will need an extra assumption.

Assumption 5.18. There exists $K' > 1$ such that
$$\overline{d} = \max\left\{|z_j - z_h|, |z_j - m_i| \text{ for all } j, h = 1, \ldots k, j \neq h, i = 1, \ldots n\right\} \leq K'd.$$

The assumption clearly implies Definition 3.8.(iii). Moreover, once the centre of mass of $q_1, \ldots, q_k$ is fixed, this extra requirement is vacuous when $k \leq 2$.

Fix a cover $\{\Omega_j\}_{j=0}^k$ of $\mathbb{R}^2 \times \mathbb{S}^1$ such that Ω_j is an open neighbourhood of the set

(5.19) $$\{(z,t) \in \mathbb{R}^2 \times \mathbb{S}^1 \text{ such that } |z - z_j| \leq |z - z_h| \text{ for all } h = 0, \ldots, k\}$$

for all $j = 0, \ldots, k$. Let $\chi_0, \ldots, \chi_k$ be a partition of unity subordinate to this cover and set $\omega_j(z,t) = \sqrt{1 + |z - z_j|^2}$. We want to define a global weight function ω such that:

(5.20a) $$\frac{1}{C_1}\omega_j \leq \omega \leq C_1\omega_j \quad \text{on } \Omega_j \quad \text{and} \quad \omega \leq C_1\omega_j \quad \text{on } \mathbb{R}^2 \times \mathbb{S}^1$$

(5.20b) $$|\nabla\omega| \leq C_2, \quad \text{and} \quad |\omega \triangle \omega| \leq C_3$$

Given $z_1, \ldots, z_k \in \mathbb{C}$, rescale by d around $z_0 = 0$. By Assumption 5.18 $z_1, \ldots, z_k$ are mapped to a collection of k points in $\mathbb{C}$ such that the maximum and the minimum of the mutual distances are uniformly bounded above and below. Fix a function $\tilde{r}(z)$ which is a smoothing of $\min_{j=0,\ldots,k}\{|z - d^{-1}z_j|\}$ outside of $z_0, \ldots, z_k$. Since the distance function on $\mathbb{R}^2$ satisfies $|\nabla r| = 1$ and $r\triangle r = -1$ outside of the origin, $\|\nabla\tilde{r}\|_{L^\infty}$ and $\|\tilde{r}\triangle\tilde{r}\|_{L^\infty}$ are bounded. Now set

(5.21) $$\omega(z,t) = \sqrt{1 + d^2\,\tilde{r}^2(d^{-1}z)}.$$

Then (5.20b) holds with constants depending only on $\|\nabla\tilde{r}\|_{L^\infty}$ and $\|\tilde{r}\triangle\tilde{r}\|_{L^\infty}$.

Define weighted Sobolev spaces $W^{m,2}_{\underline{\delta}-2+m}$, $m = 0, 1, 2$, as in Definition 5.12 using weight functions $\hat{\rho}_i, \hat{\rho}_j$ and ω. It will also be useful to consider the space $L^2_{\underline{\delta}-2,j}$ defined as in Definition 5.12.(1) using the weight function ω_j instead of ω. By (5.20a), $\chi_j f \in L^2_{\underline{\delta}-2,j}$ for all $f \in L^2_{\underline{\delta}-2}$.

We study the equation $d_2 d_2^* u = f$ in these newly defined spaces. When we restrict to the off-diagonal component, only minor modifications to the proof of Proposition 5.13 are necessary to show that $d_2 d_2^*\colon W^{2,2}_{\underline{\delta}} \to L^2_{\underline{\delta}-2}$ is an isomorphism.

Proposition 5.22. *For all $0 < \delta < \frac{1}{2}$ there exists ε and C with the following significance. Suppose that $\|\hat{\rho}_j\Psi|_{B_1(q_j)}\|_{L^3} < \varepsilon$ for all $j = 1, \ldots, k$. Then for all $f = f_T \in L^2_{\underline{\delta}-2}$ there exists a unique solution $u = u_T \in W^{2,2}_{\underline{\delta}}$ to $d_2 d_2^* u = f$. Moreover,*
$$\|u\|_{W^{2,2}_{\underline{\delta}}} \leq C\|f\|_{L^2_{\underline{\delta}-2}}.$$

Proof. Given that $\omega \geq 1$, $\omega \to \infty$ as $|z| \to \infty$ and (5.20b) holds, the precise definition of ω is only used in Lemma 5.16 to show that $\omega|d_A\Phi|$ is uniformly bounded on the exterior domain U_σ.

Therefore we only have to explain why this quantity remains bounded. Recall that the Higgs field is a sum of Green's function and their derivatives. Then, by Lemma 2.8.(ii) and (5.20a)

$$\omega \left(|\nabla G_{q_j}| + |\nabla^2 G_{q_j}| \right) \leq C \frac{\omega_j}{|z - z_j|} \leq C \qquad \text{and} \qquad \omega |dG_{p_i}| \leq C \frac{\omega_0}{|z - m_i|} \leq C_i$$

if $|z - z_j| > 2$ and $|z - m_i| > 2$, respectively, for a constant C_i depending only on $|m_i|$. $\qquad \square$

On the diagonal component there is an additional technical difficulty arising from the following finite dimensional family of functions on which the Laplacian is not well-behaved.

Definition 5.23. For all $j = 0, 1, \ldots, k$ let ψ_j be a smooth cut-off function with $\psi_j \equiv 0$ if $|z - z_j| \leq 1$ and $\psi_j \equiv 1$ if $|z - z_j| \geq 2$. Define functions $v_j = -\frac{1}{4\pi^2} \psi_j \log |z - z_j|$.

The following two properties of v_j are of easy verification.
   (i) There exists a constant $C > 0$ such that $\|\nabla v_j\|_{L^\infty} + \|\nabla^2 v_j\|_{L^\infty} \leq C$;
   (ii) $\int_{\mathbb{R}^2 \times \mathbb{S}^1} \triangle v_j = 1$.

Given $h \neq j$, set $u = v_j - v_h$. By (ii) $\triangle u$ has mean value zero and (i) implies that $\|\triangle u\|_{L^2_{\underline{\delta}-2}} \leq C$ for a uniform constant C. However, restricting to the annulus $2 \leq |z - z_j| \leq \frac{1}{2}|z_j - z_h|$,

$$\int |\nabla u|^2 \geq c_1 \log |z_j - z_h| - \frac{c_2}{|z_j - z_h|^2} \xrightarrow{d \to \infty} \infty$$

and this fact explains the special role of these functions.

Definition 5.24.   (i) Let W be the finite dimensional subspace of 1–forms with values in $\underline{\mathbb{R}} \oplus M$

$$W = \left\{ \sum_{h=1}^3 \sum_{j=0}^k \alpha_{h,j} \, d_2^* \left(v_j \, \hat{\sigma} \otimes dx_h \right) \text{ such that } \sum_{j=0}^k \alpha_{h,j} = 0 \text{ for all } h = 1, 2, 3 \right\}.$$

   Define a norm on W by declaring $d_2^* (v_j \, \hat{\sigma} \otimes dx_h)$ an orthonormal system.
   (ii) Given $f \in L^2_{\underline{\delta}-2}$ with $\int \langle f, \hat{\sigma} \otimes dx_h \rangle = 0$, denote by $\alpha(f)$ the element of W defined by $\alpha_{h,j} = \int \langle \chi_j f, \hat{\sigma} \otimes dx_h \rangle$.

Notice that the inclusion $L^2_{\underline{\delta}-2} \hookrightarrow L^1$ is continuous. Indeed, since $\delta > 0$, (5.20a) implies that $\int \omega^{-2(\delta+1)} \leq C_1 \sum_{j=0}^k \int \omega_j^{-2(\delta+1)} < +\infty$. In particular, there exists a constant $C > 0$ such that

$$(5.25) \qquad\qquad\qquad |\alpha(f)| \leq C\|f\|_{L^2_{\underline{\delta}-2}}.$$

Proposition 5.26. *For all $0 < \delta < \frac{1}{2}$ there exists a constant $C > 0$ with the following significance.*
For all $f = f_D \in L^2_{\underline{\delta}-2}$ such that $\int \langle f, \hat{\sigma} \otimes dx_h \rangle = 0$ for $h = 1, 2, 3$ there exists $\xi = \xi_D \in W^{1,2}_{\underline{\delta}-1}$ such that $d_2 \xi = f - d_2 \alpha(f)$. Moreover,

$$\|\xi\|_{W^{1,2}_{\underline{\delta}-1}} \leq C\|f\|_{L^2_{\underline{\delta}-2}}.$$

Proof. The idea is to write $f = \sum_{j=0}^k \chi_j f$ and use Proposition 5.13 to find a solution to $d_2\xi = f$ of the form $\xi = \sum_{j=0}^k d_2^* u_j$.

Set $\alpha_{h,j} = \int \langle \chi_j f, \hat{\sigma} \otimes dx_h \rangle$. The 1–form $f_j = \chi_j f - \sum_{h=1}^3 \alpha_{h,j} \, d_2 d_2^* (v_j \otimes dx_h)$ is now orthogonal to constant forms, *i.e.* $\int \langle f_j, \hat{\sigma} \otimes dx_h \rangle = 0$ for all $h = 1, 2, 3$. Moreover, $\|f_j\|_{L^2_{\underline{\delta}-2,j}} \leq C\|f\|_{L^2_{\underline{\delta}-2}}$ by (5.20a) and property (i) after Definition 5.23. We can therefore apply Proposition 5.13: there exists u_j, unique up to the addition of a constant, with the following properties:
   (i) u_j is defined on $(\mathbb{R}^2 \times \mathbb{S}^1) \setminus S$ if $j = 0$ and on $(\mathbb{R}^2 \times \mathbb{S}^1) \setminus \{q_j\}$ otherwise;
   (ii) $d_2 d_2^* u_j = f_j$;

(iii) $\|\omega_j^\delta d_2^* u_j\|_{L^2} + \|\omega_j^{\delta+1} \nabla(d_2^* u_j)\|_{L^2} \leq C\|f\|_{L^2_{\underline{\delta}-2}}$;

(iv) $\|u_0|_{B_{2\sigma}(p_i)}\|_{W^{2,2}_{\underline{\delta}}} \leq C\|f\|_{L^2_{\underline{\delta}-2}}$ for all $i = 1, \ldots, n$;

(v) $\|u_j|_{B_1(q_j)}\|_{W^{2,2}_{\underline{\delta}}} \leq C\|f\|_{L^2_{\underline{\delta}-2}}$ for $j \neq 0$.

Set $\xi = \sum_{j=0}^k d_2^* u_j$. We have to show that $\xi \in W^{1,2}_{\underline{\delta}-1}$. Restrict first to the exterior domain U_σ. The fact that $\xi \in W^{1,2}_{\underline{\delta}-1}$ follows immediately from the second statement in (5.20a) and (iii) above.

Next, consider the ball $B_1(q_j)$. Since $\omega \geq 1$, (iii) implies that $d_2^* u_h|_{B_1(q_j)} \in W^{1,2}$ if h and j are distinct and neither equal to 0 and similarly $d_2^* u_h|_{B_{2\sigma}(p_i)} \in W^{1,2}$ if $h \neq 0$. Then the Sobolev embedding $W^{1,2} \hookrightarrow L^6$ implies that

$$\|d_2^* u_h\|_{W^{1,2}_{\underline{\delta}-1}(B_1(q_j))} \leq \|\hat{\rho}_j^{\delta-\frac{1}{2}}\|_{L^3} \|d_2^* u_h\|_{L^6(B_1(q_j))} + \|\hat{\rho}_j^{\delta+\frac{1}{2}}\|_{L^\infty} \|\nabla(d_2^* u_h)\|_{L^2(B_1(q_j))} \leq C\|f\|_{L^2_{\underline{\delta}-2}}.$$

The norms of $\hat{\rho}_j$ here are bounded because $\delta \in (0, \frac{1}{2})$. Up to changing δ into $-\delta$, the same argument yields a similar estimate on $B_{2\sigma}(p_i)$. Together with (iv) and (v) above this concludes the proof. $\quad\square$

5.4. **Solving the linearised equation modulo obstructions.** With these technical details out of the way, we combine Propositions 5.8, 5.13, 5.22 and 5.26 to solve the equation $d_2\xi = f$ modulo obstructions.

Fix $0 < \delta < \frac{1}{2}$. We define weighted Sobolev spaces as in Definition 5.12, with the difference that over $B_1(q_j)$ we replace $\hat{\rho}_j$ of (5.11) with the smooth weight function w_j of (5.5). Moreover, the weight function ω is defined differently in the two situations:

(A) $S \cup \{q_1, \ldots, q_k\} \subset B_{R_0} \times \mathbb{S}^1$ for some $R_0 > 0$;
(B) $d \to \infty$ and $S, q_1, \ldots, q_k$ satisfy Assumption 5.18 for some $K' > 0$.

With these modifications and distinctions understood, the $W^{m,2}_{\underline{\delta}+m-2}$–norm coincides with the $W^{m,2}_{w,\delta}$–norm of Definition 5.7 over U_j; over U_{ext}, the spaces $W^{m,2}_{\underline{\delta}+m-2}$ are equivalent to the ones used in Proposition 5.13 in case (A) and to those introduced in Propositions 5.22 and 5.26 in case (B). In the latter case, consider also the finite dimensional space W introduced in Definition 5.24.

It is necessary to introduce cut-off functions $\gamma_j, \gamma_{\text{ext}}, \beta_j, \beta_{\text{ext}}$ with some specific properties. Let γ_j be a smooth function supported in $B_{2\delta_j}(q_j)$ and such that $\gamma_j \equiv 1$ when $\rho_j \leq \frac{\delta_j}{2}$. Then $|\nabla \gamma_j| \leq \frac{2}{\delta_j}$. Define γ_{ext} by $\gamma_{\text{ext}} = 1 - \gamma_j$ if $\rho \leq 2\delta_j$ and $\gamma_{\text{ext}} \equiv 1$ otherwise. The cut-off functions β_j and β_{ext} are defined in [7, Lemma 7.2.10]: β_j is a smooth function such that $\beta_j \equiv 1$ on $B_{2\delta_j}(q_j)$, $\beta_j \equiv 0$ if $\rho_j \geq N\delta_j$ and

$$(5.27) \qquad\qquad\qquad |\nabla \beta_j| \leq \frac{C}{\log N} \frac{1}{\rho_j}.$$

Similarly, $\beta_{\text{ext}} \equiv 1$ outside of $\bigcup_{j=1}^k B_{\frac{\delta_j}{2}}(q_j)$, $\beta_{\text{ext}} \equiv 0$ on $B_{N^{-1}\delta_j}(q_j)$ and $|\nabla \beta_{\text{ext}}| \leq \frac{C}{\log N} \frac{1}{\rho_j}$.

Recall also that in Definition 4.11 we distinguished smooth sections $o_h \in \Omega(V)$. They are supported on U_{ext} and under the identification $V|_{U_{\text{ext}}} \simeq \mathbb{R} \oplus M$ have only diagonal component. The crucial property of o_h, $h = 1, 2, 3$, is given by the following lemma.

Lemma 5.28. *For all $h, l = 1, 2, 3$, $\langle d_2 o_h, \hat{\sigma} \otimes dx_l \rangle_{L^2} = \delta_{hl}$.*

Proof. Since $c(x_0, \tau)$ is abelian on U_{ext}, $d_2 o_h \oplus d_1^* o_h = \slashed{D} o_h$, where $\slashed{D}$ is the Dirac operator of $\mathbb{R}^2 \times \mathbb{S}^1$. Since the Clifford multiplication by dx_l commutes with $\slashed{D}$, it is enough to prove that

$$\langle \slashed{D} o_4, \hat{\sigma} \otimes dx_h \rangle_{L^2} = 0, \qquad \langle \slashed{D} o_4, \hat{\sigma} \rangle_{L^2} = 1.$$

This follows by direct calculation using the definition $o_4 = -\frac{1}{2\pi k} \sum_{j=1}^{k} \left(\chi_{\text{ext}}^j \, dG_{q_j}, 0 \right) \hat{\sigma}$. For example,

$$-\langle \slashed{D}(\chi \, dG, 0), \hat{\sigma} \rangle_{L^2} = \int_B d(\chi * dG) = \int_{\partial B} *dG = 2\pi$$

implies the second identity. $\qquad \square$

We refer to $\text{span}\{d_2 o_h \mid h = 1, 2, 3\} \subset L^2_{\underline{\delta}-2}$ as the *obstruction space*. Define $\pi \colon L^2_{\underline{\delta}-2} \to L^2_{\underline{\delta}-2}$ by

$$(5.29) \qquad \pi(f) = f - \sum_{h=1}^{4} \langle f, \gamma_{\text{ext}} \, \hat{\sigma} \otimes dx_h \rangle_{L^2} \, d_2 o_h.$$

Lemma 5.30. *There exists a constant C such that*

$$\|\pi(f)\|_{L^2_{\underline{\delta}-2}} \leq C \left(N^{-2} \lambda \right)^{\frac{1-\delta}{2}} \|f\|_{L^2_{\underline{\delta}-2}}.$$

Furthermore, if f is supported on the union of the annuli $Ann_{j,int} \cup Ann_j \cup Ann_{j,ext}$ for $j = 1, \ldots, k$ then the estimate can be improved to

$$\|\pi(f)\|_{L^2_{\underline{\delta}-2}} \leq C \|f\|_{L^2_{\underline{\delta}-2}}.$$

Proof. Recall that $L^2_{\underline{\delta}-2} \hookrightarrow L^1$ is continuous. Moreover, if f is supported on $\text{Ann}_{j,\text{int}} \cup \text{Ann}_j \cup \text{Ann}_{j\text{ext}}$

$$\|f\|_{L^1} \leq C \left(N^{-2} \lambda \right)^{-\frac{1-\delta}{2}} \|f\|_{L^2_{\underline{\delta}-2}}$$

Therefore, it is enough to estimate $\|d_2 o_h\|_{L^2_{\underline{\delta}-2}}$. From Definition 4.11 $|d_2 o_h| \leq C \sum_{j=1}^{k} \rho_j^{-2} |\nabla \chi_{\text{ext}}^j|$ and, since $\nabla \chi_{\text{ext}}^j$ is supported in the region where $w_j \sim \rho_j$ uniformly, we conclude

$$(5.31) \qquad \|d_2 o_h\|_{L^2_{\underline{\delta}-2}} \leq C \left(N^{-2} \lambda \right)^{\frac{1-\delta}{2}}. \qquad \square$$

The following theorem yields the solution to the linear problem modulo obstruction.

Theorem 5.32. *Fix $0 < \delta < \frac{1}{2}$.*

(A) *Fix $R_0 > 0$ such that $S \cup \{q_1, \ldots, q_k\} \subset B_{R_0} \times \mathbb{S}^1$. There exist $\varepsilon > 0$, $N_0 > 2$ and C with the following significance. Suppose that $\|w_j \Psi|_{B_1(q_j)}\|_{L^3} < \varepsilon$ for all $j = 1, \ldots, k$ and $N > N_0$. Then there exists a map $Q \colon im\,\pi \subset L^2_{\underline{\delta}-2} \to W^{1,2}_{\underline{\delta}-1}$ such that $\pi \circ d_2 \circ Q(f) = f$ and*

$$\|Qf\|_{W^{1,2}_{\underline{\delta}-1}} \leq C \|f\|_{L^2_{\underline{\delta}-2}}.$$

(B) *Suppose that $S, q_1, \ldots, q_k$ satisfy Assumption 5.18 for some $K' > 0$. There exist $\varepsilon > 0$, $N_0 > 2$ and $C > 0$ with the following significance. Suppose that $\|w_j \Psi|_{B_1(q_j)}\|_{L^3} < \varepsilon$ for all $j = 1, \ldots, k$ and $N > N_0$. Then there exist a map $Q = (Q_1, Q_2)$, where*

$$Q_1 \colon im\,\pi \subset L^2_{\underline{\delta}-2} \to W^{1,2}_{\underline{\delta}-1}, \qquad Q_2 \colon im\,\pi \subset L^2_{\underline{\delta}-2} \to W,$$

such that $\pi \circ d_2 \circ Q_1(f) + \pi \circ d_2\big(\beta_{ext} Q_2(f)\big) = f$. Moreover,

$$\|Q_1(f)\|_{W^{1,2}_{\underline{\delta}-1}} + |Q_2(f)| \leq C \|f\|_{L^2_{\underline{\delta}-2}}.$$

Proof. We prove the statement in (B). The statement in (A) follows in a similar way using Proposition 5.13 instead of Propositions 5.22 and 5.26.

By abuse of notation, regard d_2 as the operator $d_2 \colon W^{1,2}_{\underline{\delta}-1} \oplus W \to L^2_{\underline{\delta}-2}$ defined by

$$d_2(\xi, \eta) = d_2(\xi + \beta_{\text{ext}} \eta).$$

For $f \in L^2_{\underline{\delta}-2}$ with $f = \pi(f)$, write $f = \sum_{j=1}^k \gamma_j f + \gamma_{\text{ext}} f$ and define maps $Q'_1 \colon \operatorname{im} \pi \to W^{1,2}_{\underline{\delta}-1}$ and $Q'_2 \colon \operatorname{im} \pi \to W$ as follows: $Q'_2(f) = \alpha(\gamma_{\text{ext}} f)$, where α is the map of Definition 5.24.(ii), while

$$Q'_1(f) = \sum_{j=1}^k \beta_j\, \xi_j + \beta_{\text{ext}}\, \xi_{\text{ext}}.$$

Here $\xi_j = d_2^* u_j$ for the solution u_j to $d_2 d_2^* u_j = \gamma_j f$ of Proposition 5.8 and ξ_{ext} is the solution to $d_2 \xi_{\text{ext}} = \gamma_{\text{ext}} f - \alpha(\gamma_{\text{ext}} f)$ obtained combining Propositions 5.22 and 5.26. In particular, using (5.25) and (5.27), we deduce the existence of a constant $C > 0$ such that

$$\|Q'_1(f)\|_{W^{1,2}_{\underline{\delta}-1}} + |Q'_2(f)| \leq C \|f\|_{L^2_{\underline{\delta}-2}}.$$

Set $Q' = (Q'_1, Q'_2)$. We have $f - d_2 Q'(f) = \sum_{j=1}^k \nabla\beta_j \cdot \xi_j + \nabla\beta_{\text{ext}} \cdot \xi_{\text{ext}} + \nabla\beta_{\text{ext}} \cdot \alpha(\gamma_{\text{ext}} f)$. By (5.27) and Hölder's inequality

$$\|f - d_2 Q'(f)\|_{L^2_{\underline{\delta}-2}} \leq \frac{C}{\log N}\|Q'_1(f)\|_{W^{1,2}_{\underline{\delta}-1}} + C\frac{\lambda^{-\frac{1+\delta}{2}}}{\log N}\|Q'_2(f)\|_{L^\infty} \leq \frac{C}{\log N}\|f\|_{L^2_{\underline{\delta}-2}}$$

because $\|\eta\|_{L^\infty} \leq C|\eta|$ for all $\eta \in W$. Since $f - d_2 Q'(f)$ is supported in $\bigcup_{j=1}^k \operatorname{Ann}_j$, Lemma 5.30 yields $\|f - \pi \circ d_2 \circ Q'(f)\|_{L^2_{\underline{\delta}-2}} \leq \frac{C}{\log N}\|f\|_{L^2_{\underline{\delta}-2}}$ and if N is sufficiently large we can iterate. $\qquad\square$

6. Deformation

In this section we complete the construction of a family of solutions to the Bogomolny equation by deforming the approximate solutions $c(x_0, \tau)$ given by the pre-gluing map of Proposition 4.14. Using the projection π of (5.29), we split the non-linear equation (5.1) into an infinite dimensional and a finite dimensional equation. We solve the infinite dimensional equation first. The following lemma, an immediate consequence of the contraction mapping principle, is a quantitative version of the Implict Function Theorem adapted to case (B) in Theorem 5.32. The statement in case (A) is obtained by setting $W = \{0\}$ (and therefore $\eta = 0$).

Lemma 6.1. *Given $c(x_0, \tau)$ for some $(x_0, \tau) \in \mathcal{P}$, let $\Psi \colon W^{1,2}_{\underline{\delta}-1} \oplus W \to L^2_{\underline{\delta}-2}$ be the smooth map*

$$\Psi(\xi, \eta) = d_2(\xi + \beta_{ext}\eta) + (\xi + \beta_{ext}\eta) \cdot (\xi + \beta_{ext}\eta) + \Psi(x_0, \tau).$$

Suppose that the following conditions hold.

(i) *There exists a projection $\pi \colon L^2_{\underline{\delta}-2} \to L^2_{\underline{\delta}-2}$ such that the map $\pi \circ d_2 \colon W^{1,2}_{\underline{\delta}-1} \oplus W \to \operatorname{im} \pi$ admits a right inverse $Q = (Q_1, Q_2)$ with*

$$\|Q_1(f)\|_{W^{1,2}_{\underline{\delta}-1}} + |Q_2(f)| \leq C\|f\|_{L^2_{\underline{\delta}-2}}$$

for all $f \in L^2_{\underline{\delta}-2}$ with $\pi(f) = f$.

(ii) *There exists $q > 0$ such that*

$$\left\|\pi\big((\xi, \eta) \cdot (\xi, \eta)\big) - \pi\big((\xi', \eta') \cdot (\xi', \eta')\big)\right\|_{L^2_{\underline{\delta}-2}} \leq q\left(\|\xi + \xi'\|_{W^{1,2}_{\underline{\delta}-1}} + |\eta + \eta'|\right)\left(\|\xi - \xi'\|_{W^{1,2}_{\underline{\delta}-1}} + |\eta - \eta'|\right).$$

Here $(\xi, \eta) \cdot (\xi', \eta') = (\xi + \beta_{ext}\eta) \cdot (\xi' + \beta_{ext}\eta')$.

(iii) *The error $\Psi(x_0, \tau)$ satisfies $\|\pi(\Psi(x_0, \tau))\|_{L^2_{\underline{\delta}-2}} \leq \frac{1}{8qC^2}$.*

Then there exists a unique $(\xi, \eta) \in \operatorname{im} Q \subset W^{1,2}_{\underline{\delta}-1} \oplus W$ such that $\pi\big(\Psi(\xi, \eta)\big) = 0$. Moreover,

$$\|\xi\|_{W^{1,2}_{\underline{\delta}-1}} + |\eta| \leq 2C\|\pi\big(\Psi(x_0, \tau)\big)\|_{L^2_{\underline{\delta}-2}}.$$

Theorem 5.32 shows that (i) holds provided $\|w_j\Psi(x_0, \tau)\|_{L^3} < \varepsilon$ in every ball $B_1(q_j)$, $j = 1, \ldots, k$. The next two lemmas imply that (ii) and (iii) are also satisfied if λ is sufficiently large.

Lemma 6.2. *There exists $C > 0$ such that condition (ii) of Lemma 6.1 holds with $q = C\lambda^{\frac{1+\delta}{2}}$.*

Proof. Observe that the product $\cdot$ induced by the Clifford multiplication and the Lie bracket is commutative. In particular, $(\xi, \eta) \cdot (\xi, \eta) - (\xi', \eta') \cdot (\xi', \eta') = (\xi + \xi', \eta + \eta') \cdot (\xi - \xi', \eta - \eta')$. We will show that $\cdot$ defines a continuous map $\left(W^{1,2}_{\underline{\delta}-1} \oplus W\right) \times \left(W^{1,2}_{\underline{\delta}-1} \oplus W\right) \to L^2_{\underline{\delta}-2}$ such that

$$(6.3) \qquad \|(\xi, \eta) \cdot (\xi', \eta')\|_{L^2_{\underline{\delta}-2}} \le C\lambda^\delta \|(\xi, \eta)\|_{W^{1,2}_{\underline{\delta}-1} \oplus W} \|(\xi', \eta')\|_{W^{1,2}_{\underline{\delta}-1} \oplus W}.$$

The lemma then follows combining (6.3) with Lemma 5.30.

The continuity (6.3) of the product follows from Hölder's inequality and the Sobolev embedding $W^{1,2} \hookrightarrow L^6$ along the lines of [9, Lemmas 5.18 and 6.10]. The crucial observation is that, with respect to the decomposition $V \simeq \mathbb{R} \oplus M$ over U_{ext}, there is no $\xi_D \cdot \xi'_D$ term in the product because $\cdot$ is induced by the Lie bracket on $\mathfrak{su}(2)$. Moreover, recall that $\beta_{\text{ext}}\eta$ for $\eta \in W$ has only diagonal component. We now provide some details.

Restrict first to the ball $B_1(q_j)$. If $\xi \in W^{1,2}_{\underline{\delta}-1}$ and $\eta \in W$

$$\int_{B_1(q_j)} w_j^{2\delta+1} |\xi \cdot \beta_{\text{ext}}\eta|^2 \le C\|w_j\eta\|^2_{L^\infty} \int_{B_1(q_j)} w_j^{2\delta-1}|\xi|^2 \le C|\eta|^2 \|\xi\|_{W^{1,2}_{\underline{\delta}-1}}$$

because $w_j \le 1$ and $\|\eta\|_{L^\infty} \le C|\eta|$ by Definition 5.24. On the other hand, in order to estimate the norm $\|\xi \cdot \xi'\|_{L^2_{\underline{\delta}-2}}$ for $\xi, \xi' \in W^{1,2}_{\underline{\delta}-1}$, by Hölder's inequality it is enough to observe that

$$\int_{B_1(q_j)} w_j^{2\delta+1}|\xi|^4 \le \lambda_j^{2\delta} \|w_j^{\delta - \frac{1}{2}}\xi\|_{L^2} \|w_j^{\delta + \frac{1}{2}}\xi\|^3_{L^6} \le C\lambda_j^{2\delta}\|\xi\|^4_{W^{1,2}_{\underline{\delta}-1}}.$$

The last inequality follows from the continuity of the Sobolev embedding $W^{1,2}_{\underline{\delta}-1} \hookrightarrow w_j^{-\delta-\frac{1}{2}}L^6$ on the ball $B_1(q_j)$. The factor of λ_j in the first inequality is due to the fact that $w_j \ge \lambda_j^{-1}$.

This establishes (6.3) on $B_1(q_j)$. The same calculations with $-\delta$ in place of δ yield the desired estimate also on the ball $B_{2\sigma}(p_i)$ for all $i = 1, \ldots, n$, cf. [9, Lemmas 5.18].

On the exterior domain U_σ write $\xi = \xi_D + \xi_T$ with respect to the decomposition $V \simeq \mathbb{R} \oplus M$. Because of the properties (5.20) of the weight function ω, if $\xi \in W^{1,2}_{\underline{\delta}-1}$ then $\omega^\delta \xi_D, \omega^{\delta+1}\xi_T \in W^{1,2}$. Moreover, $W^{1,2} \hookrightarrow L^p$ for all $2 \le p \le 6$ by the Sobolev embedding. Therefore

$$\|\omega^{\delta+1}(\xi \cdot \xi')\|_{L^2} \le \|\xi\|_{L^3}\|\omega^{\delta+1}\xi'_T\|_{L^6} + \|\xi'\|_{L^3}\|\omega^{\delta+1}\xi_T\|_{L^6} \quad \text{and} \quad \|\omega^{\delta+1}(\xi \cdot \eta)\|_{L^2} \le \|\eta\|_{L^\infty}\|\omega^{\delta+1}\xi_T\|_{L^2}$$

by Hölder's inequality. The estimate (6.3) follows. $\qquad\square$

Remark 6.4. The continuity (6.3) of the product $\cdot$ justifies the claim that the map Ψ in Lemma 6.1 is smooth.

Lemma 6.5. *There exists $C > 0$ such that $\|\pi(\Psi(x_0, \tau))\|_{L^2_{\underline{\delta}-2}} \le C\lambda^{-1-\frac{\delta}{2}}$ and $\|w_j\Psi\|_{L^3} \le C\lambda^{-\frac{1}{2}}$ on each ball $B_1(q_j)$, $j = 1, \ldots, k$.*

Proof. By Definition 4.11 and Lemma 5.28, if Ψ_ζ is the component of $\Psi(x_0, \tau)$ defined by (4.15) then $\pi(\Psi_\zeta) = 0$. By Proposition 4.14.(i)

$$(6.6) \qquad \|\Psi(x_0, \tau) - \Psi_\zeta\|_{L^2_{\underline{\delta}-2}} \le C\lambda^{-1-\frac{\delta}{2}}.$$

The second estimate in Lemma 5.30 therefore implies the first statement of the lemma.

Similarly, $\|w_j(\Psi(x_0, \tau) - \Psi_\zeta)\|_{L^3} = O(\lambda^{-1})$ on each ball $B_1(q_j)$, while the estimate $\rho_j^2|\Psi_\zeta| \le \frac{C}{\sqrt{\lambda}}$ in Proposition 4.14.(i) yields $\|w_j\Psi_\zeta\|_{L^3} = O(\lambda^{-\frac{1}{2}})$. $\qquad\square$

6.1. **Existence results.** We have all the ingredients to prove the main result of the paper, an existence theorem for periodic monopoles (with singularities). We begin with a rewriting of the pre-gluing map of Proposition 4.14.

Fix $d_0 \geq 5$, $K > 1$, parameters v, b, the set S of singularities and the centres of non-abelian monopoles $q_1, \ldots, q_k$. For all $N > 2$ we fix $\lambda_0(N)$ sufficiently large and assume that $v, S, q_1, \ldots, q_k$ are (λ_0, d_0, K)–admissible. Moreover, we assume that either:

   (A) There exists $R_0 > 0$ such that $S, \{q_1, \ldots, q_k\} \subset B_{R_0} \times \mathbb{S}^1$; or
   (B) There exist $R_0, K' > 0$ such that $S \subset B_{R_0} \times \mathbb{S}^1$, $q_1, \ldots, q_k \in (\mathbb{R}^2 \times \mathbb{S}^1) \setminus (B_{R_0} \times \mathbb{S}^1)$ and Assumption 5.18 is satisfied.

Finally, for $\kappa \in (0, 1)$ sufficiently small, let $\mathcal{P} = \mathcal{P}_\kappa$ be the set of gluing data of Definition 4.1.

Consider the family $c(x_0, \tau)$ of Proposition 4.14. If κ is sufficiently small and $\lambda_0(N)$ sufficiently large, we can assume that (4.16), (4.17) and (4.19) are satisfied, uniformly for all $(x_0, \tau) \in \mathcal{P}$.

Definition 6.7. Fix a base point $(0, \tau_0) \in \mathcal{P}$ and $\delta > 0$ sufficiently small. Set $k_\infty = 2k - n$ and $q = p_1 + \ldots + p_n$. Let $\mathcal{C}_{\underline{\delta}} = \mathcal{C}_{\underline{\delta}}(p_1, \ldots, p_n, k_\infty, v, b, q)$ be the configuration space of pairs (A, Φ) of the form $(A, \Phi) = c(0, \tau_0) + \xi$ with $\xi \in W^{1,2}_{\underline{\delta}-1} \oplus W$. Here and in the rest of the section we set $W = \{0\}$ when condition (A) above holds.

Observe that once $q_1, \ldots, q_k$ are fixed the weighted Sobolev norms used to define $\mathcal{C}_{\underline{\delta}}$ are equivalent to those used in [9] to define a smooth structure on the moduli space of periodic monopoles. Then the pregluing map of Proposition 4.14 can be considered as a smooth map

$$c \colon \mathcal{P} \to \mathcal{C}_{\underline{\delta}}.$$

Smoothness follows from Lemma 2.14 and the explicit construction of $c(x_0, \tau)$.

The group $\Gamma \simeq SO(2)$ acts on $\mathcal{P}$ by $e^{is} \cdot (x_0, \tau) = (x_0, \tau + s)$ and on $\mathcal{C}_{\underline{\delta}}$ as the gauge transformation $\exp(s\gamma_{\text{ext}} \hat{\sigma})$; the map c is Γ–equivariant.

Theorem 6.8. *Fix data as above and $N > N_0$, where N_0 is given by Theorem 5.32. Then there exists $\lambda'_0 \geq \lambda_0(N)$ such that if $v, S, q_1, \ldots, q_k$ are (λ'_0, d_0, K)–admissible then the following holds.*

   (i) *There exists a smooth Γ–equivariant map $c_1 \colon \mathcal{P} \to \mathcal{C}_{\underline{\delta}}$ such that $\pi \circ \Psi \circ c_1 = 0$. Furthermore, c_1 takes the form $c_1(x_0, \tau) = c(x_0, \tau) + \xi(x_0, \tau)$ with $\xi(x_0, \tau) \in W^{1,2}_{\underline{\delta}-1} \oplus W$ and*

$$\|\xi(x_0, \tau)\|_{W^{1,2}_{\underline{\delta}-1} \oplus W} \leq C\lambda^{-1-\frac{\delta}{2}}.$$

   (ii) *There exist smooth Γ–invariant maps $H, h \colon \mathcal{P} \to \mathbb{R}^3$ with*

$$H(x_0, \tau) = -\sum_{j=0}^{k} \frac{x_0^j}{\lambda_j}$$

   and $|h(x_0, \tau)| = O(\lambda^{-\frac{3}{2}})$, such that $c_1(x_0, \tau)$ is a solution to the Bogomolny equation if and only if $H(x_0, \tau) + h(x_0, \tau) = 0$.

   (iii) *Given $(x_0, \tau) \in \mathcal{P}$ and $\zeta \in \mathbb{R}^3$, let $x_0 + \zeta$ denote the k–tuple $x_0 + (\zeta, \ldots, \zeta)$. For all $(x_0, \tau) \in \mathcal{P}_{\frac{\kappa}{2}}$ such that $H(x_0, \tau) = 0$ there exists $\zeta \in \mathbb{R}^3$ such that $|\zeta| = O(\lambda^{-\frac{1}{2}})$ and*

$$H(x_0 + \zeta, \tau) + h(x_0 + \zeta, \tau) = 0.$$

Proof.   (i) Consider the map

$$\pi \circ \Psi \colon \mathcal{P} \times \left(W^{1,2}_{\underline{\delta}-1} \oplus W\right) \to L^2_{\underline{\delta}-2},$$

$(\pi \circ \Psi)(x_0, \tau, \xi) = \pi(d_2\xi + \xi \cdot \xi + \Psi(x_0, \tau))$, which is smooth because of the smoothness of $c \colon \mathcal{P} \to \mathcal{C}_{\underline{\delta}}$, Lemma 5.30 and the continuity of the product $\cdot$ established in (6.3).

Choosing $\lambda_0(N)$ larger if necessary, assume that $\|w_j \Psi(x_0, \tau)\|_{L^3} = O(\lambda^{-\frac{1}{2}}) < \varepsilon$, where ε is given by Theorem 5.32. Fix $N > N_0$ so that Theorem 5.32 holds. Finally, we can choose $\lambda'_0 \geq \lambda_0(N)$ so that

$$\|\pi(\Psi(x_0, \tau))\|_{L^2_{\underline{\delta}-2}} = O(\lambda^{-1-\frac{\delta}{2}}) \leq \frac{1}{2qC^2} = O(\lambda^{-\frac{1}{2}-\frac{\delta}{2}})$$

whenever $\lambda > \lambda'_0$. Here q and C are given by Lemma 6.2 and Theorem 5.32, respectively, and we used Lemma 6.5. Lemma 6.1 therefore implies the existence of the map c_1. The fact that c_1 is smooth follows from the fact that the family of right inverses Q of Theorem 5.32 depends smoothly on (x_0, τ).

(ii) We are left with the three equations

$$\langle d_2^* \xi(x_0, \tau) + \xi(x_0, \tau) \cdot \xi(x_0, \tau) + \Psi(x_0, \tau), \gamma_{\mathrm{ext}} \hat{\sigma} \otimes dx_h \rangle_{L^2} = 0,$$

for $h = 1, 2, 3$. We have to show that these can be written as $H(x_0, \tau) + h(x_0, \tau) = 0$ as claimed. Write $\xi(x_0, \tau) = \xi' + \beta_{\mathrm{ext}} \eta$, with $\xi' \in W^{1,2}_{\underline{\delta}-1}$ and $\eta \in W$. First, we control all the negligible terms:

(a) $\langle d_2 \xi', \gamma_{\mathrm{ext}} \hat{\sigma} \otimes dx_h \rangle_{L^2} = O(\lambda^{-\frac{3}{2}})$. Indeed, integrating by parts

$$\left| \langle d_2 \xi', \gamma_{\mathrm{ext}} \hat{\sigma} \otimes dx_h \rangle_{L^2} \right| \leq \|\xi'\|_{W^{1,2}_{\underline{\delta}-1}} \|w_j^{-\delta+\frac{1}{2}} \nabla \gamma_{\mathrm{ext}}\|_{L^2} = O(\lambda^{-\frac{3}{2}}).$$

(b) $\langle d_2(\beta_{\mathrm{ext}} \eta), \gamma_{\mathrm{ext}} \hat{\sigma} \otimes dx_h \rangle_{L^2} = 0$, because $\gamma_{\mathrm{ext}} \equiv 0$ on the support of $\nabla \beta_{\mathrm{ext}}$, $\beta_{\mathrm{ext}} \equiv 1 \equiv \gamma_{\mathrm{ext}}$ on the support of $d_2 \eta$ and $\langle d_2 \eta, \hat{\sigma} \otimes dx_h \rangle_{L^2} = 0$ by Definition 5.24.

(c) By the continuity of the embedding $L^2_{\underline{\delta}-2} \hookrightarrow L^1$ and (6.3)

$$\left| \langle \xi(x_0, \tau) \cdot \xi(x_0, \tau), \gamma_{\mathrm{ext}} \hat{\sigma} \otimes dx_h \rangle_{L^2} \right| \leq C\lambda^\delta \|\xi(x_0, \tau)\|^2_{W^{1,2}_{\underline{\delta}-1} \oplus W} = O(\lambda^{-2}).$$

(d) Finally, as in the proof of Lemma 5.30,

$$\left| \langle \Psi(x_0, \tau) - \Psi_\zeta, \gamma_{\mathrm{ext}} \hat{\sigma} \otimes dx_h \rangle_{L^2} \right| \leq C\lambda^{-\frac{1-\delta}{2}} \|\Psi(x_0, \tau) - \Psi_\zeta\|_{L^2_{\underline{\delta}-2}} = O(\lambda^{-\frac{3}{2}})$$

because $\Psi(x_0, \tau) - \Psi_\zeta$ is supported on $\bigcup_{j=1}^k (A_{j,\mathrm{int}} \cup A_j \cup A_{j,\mathrm{ext}})$.

On the other hand, by Lemma 5.28

$$\langle \Psi_\zeta, \gamma_{\mathrm{ext}} \hat{\sigma} \otimes dx_h \rangle_{L^2} = -\sum_{j=1}^k \frac{x_0^j}{\lambda_j}.$$

(iii) The claim follows from Brouwer's Fixed Point Theorem by writing

$$\zeta = -\frac{h(x_0 + \zeta, \tau)}{\sum_{j=1}^k \lambda_j^{-1}} = O(\lambda^{-\frac{1}{2}}).$$

In the last equality we used Definition 3.8.(iii). $\square$

In view of [9, Lemma 7.2], if the boundary conditions (including the position of the singularities) are chosen generically then the monopoles constructed in the theorem are necessary irreducible. Indeed, it is shown in [9, Lemma 7.2] that there exists no reducible monopole satisfying the boundary conditions of Definition 3.13 provided that no subset $\{p_{i_1}, \ldots, p_{i_k}\} \subset S$ of cardinality k has centre of mass at the origin. The condition is non-vacuous only when $n \geq k$. In conjunction with [9, Theorem 1.5], Theorem 6.8 then shows that, for generic choices of parameters and provided the mass v is sufficiently large when $n \geq 2(k-1)$, the moduli space $\mathcal{M}_{n,k}$ of charge k $SO(3)$ periodic monopoles with n fixed singularities is a non-empty smooth hyperkähler manifold. Indeed, by [9] singularities of $\mathcal{M}_{n,k}$ can only arise from reducible monopoles.

Corollary 6.9. *Fix integers $k > 0$ and $0 \le n \le 2k$, constants $(v, b) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ and distinct points $p_1, \ldots, p_n, q_1, \ldots, q_k \in \mathbb{R}^2 \times \mathbb{S}^1$ with $q_1 + \ldots + q_k = 0$. If $n \ge k$ assume that no subset $\{p_{i_1}, \ldots, p_{i_k}\}$ has the origin as its centre of mass. Furthermore, assume that conditions (i), (iii) and (iv) of Definition 3.8 are satisfied with $d_0 = 5$ and some $K > 1$. Let $\mathcal{C}_{\underline{\delta}} = \mathcal{C}_{\underline{\delta}}(p_1, \ldots, p_n, k_\infty, v, b, q)$ be the configuration space of Definition 6.7 for some fixed $\delta > 0$ sufficiently small.*

   (i) *Fix $R_0 > 0$ such that $p_1, \ldots, p_n, q_1, \ldots, q_k \in B_{R_0} \times \mathbb{S}^1$. Then there exists $v_0 > 0$ such that if $v > v_0$ the configuration space $\mathcal{C}_{\underline{\delta}}$ contains irreducible monopoles.*

   (ii) *Assume that $0 \le n < 2(k-1)$ and that Assumption 5.18 holds for some $K' > 1$. Then there exists $d_0 > 0$ such that if the minimum distance d of (3.1) satisfies $d > d_0$, then the configuration space $\mathcal{C}_{\underline{\delta}}$ contains irreducible monopoles.*

   (iii) *If $n = 2(k-1)$ and the conditions of (ii) are satisfied, there exist v_0 and d_0 such that if $v > v_0$ and $d > d_0$ then irreducible monopoles exist in $\mathcal{C}_{\underline{\delta}}$.*

The statement in (i) establishes the existence of monopoles for any $0 \le n \le 2k$ in the high mass case. In case (ii) the mass v is arbitrary and we require that the points $q_1, \ldots, q_k$ are widely separated. The statement in (iii) follows from (3.9): when $n = 2(k-1)$, $\lambda_j = O(v)$ for all j as $d \to \infty$. The large distance case considered in (ii) and (iii) is interesting because, in analogy with Taubes's gluing result for Euclidean monopoles, we expect to be able to use Corollary 6.9 to give a description of the asymptotic geometry of the moduli spaces of (singular) periodic monopoles.

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