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Concentrated Euler flows and point vortex model

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Abstract: *This paper is an improvement of previous results on concentrated Euler flows and their connection with the point vortex model. Precisely, we study the time evolution of an incompressible two dimensional Euler fluid when the initial vorticity is concentrated in N disjoint regions of diameter ϵ. We show that the evolved vorticity is concentrated in N regions of diameter d, $d \leq b\ \epsilon^{\alpha}$ (b independent of ϵ) for any $\alpha < 1/2$. The connection is obtained as $\epsilon \to 0$.*

1 – Introduction and main result

In the present paper we study the time evolution of an incompressible inviscid fluid with a planar symmetry, governed by the Euler equation, that in $\mathbb{R}^2$ in term of vorticity reads:

$$\partial_t \omega(x,t) + (u \cdot \nabla)\omega(x,t) = 0 \quad , \quad x = (x_1, x_2) \ , \tag{1.1}$$

$$\nabla \cdot u(x,t) = 0 \ , \tag{1.2}$$

$$\omega(x,0) = \omega_0 \ , \tag{1.3}$$

where $\omega(x,t) = \partial_1 u_2 - \partial_2 u_1$ is the vorticity and $u = (u_1, u_2)$ denotes the velocity field. We assume that u decays at infinity and so we can reconstruct the velocity by means of ω as

$$u(x,t) = \int dy\ K(x-y)\ \omega(y,t) \ , \tag{1.4}$$

$$K = \nabla^{\perp} G \quad , \quad \nabla^{\perp} = (\partial_2, -\partial_1) \quad , \quad G(x) = -\frac{1}{2\pi} \log |x| \ . \tag{1.5}$$

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Eq. (1.1) means that the vorticity remains constant along the particle paths which are the characteristic of the Euler Equations:

$$\omega(x(x_0),t) = \omega(x_0,t) \ , \tag{1.6}$$

where $(x(x_0),t)$ is the trajectory of the fluid particle, initially in x_0, that satisfies

$$\frac{d}{dt}x(x_0,t) = u(x(x_0,t)) \ . \tag{1.7}$$

It is possible to extend the Euler Equation and to consider initial data with weak regularity, assuming directly equations(1.6),(1.7); a formal integration by parts on (1.1) gives:

$$\frac{d}{dt}\omega[f] = \omega[u\cdot\nabla f] + \omega[\partial_t f] \ , \tag{1.8}$$

where $f(x,t)$ is a bounded smooth function and

$$\omega[f] = \int dx\ \omega(x,t)f(x,t) \ . \tag{1.9}$$

It is well-known that there exists a unique solution $\omega(x,t)\in L_1\cap L_\infty$ to the initial value problem associated to (1.8) provided that $\omega_0\in L_1\cap L_\infty$. Moreover the divergence free condition (1.2) implies that the time evolution preserves the Lebesgue measure in $\mathbb{R}^2$.

In the present paper we study the fluid when the initial vorticity of finite global intensity is concentrated in N small disjoint regions of the plane $\Lambda_i(0)$ of diameter 2ϵ around the points z_i. As well known in the literature, this system can be approximated by a system of N differential equations in two dimensions, called point vortex model:

$$\dot{z}_i(t) = -\nabla_i^\perp \frac{1}{2\pi}\sum_{j=1;j\neq i}^{N} a_j \log|z_i(t)-z_j(t)| \ \ ; \ \ z_i(0)=z_i \ \in\mathbb{R}^2 \ . \tag{1.10}$$

This model has been introduced by Helmholtz [9] as a particular "solution" of the Euler equation and investigated by several authors; for a review see [20, 21] and references quoted in.

Actually the connection between the fluid mechanics and this model is more complex, as we will see. The more natural, but not correct, way to study this connection, would be consider the time evolution of each particle of fluid (*i.e.* the characteristics of the Euler equation) and to investigate the limit as the initial vorticity is very concentrated (*i.e.* $\epsilon\to 0$). Actually the modulus of the velocity of each trajectory (and indeed its length) becomes infinite as $\epsilon\to 0$, since each particle turns fast. Hence the point vortex model may approximate the Euler flow at most in *average*. The difficulties arise in computing the velocity field produced by the particles very closed to a tagged one (a sort of self-interaction). The textbooks on the argument (see for instance [3]) neglect this contribution for symmetry reason. Obviously, if the initial support of the vorticity is the union of circles, the self-interactions are zero, but the time evolution destroys this symmetry. We can neglect this term for more deep reasons, as we will see. In conclusion, we do not follow the motion of each particle of the fluid, but the time evolution of the center of the vorticity supported in each small region. These are the quantities approximated by the point vortex model.

The first rigorous proof of this connection have been given for short times [18]. In that paper the authors study the time evolution of the support of the vorticity. Initially, by assumption, this support is composed by N disjoint small regions and in that paper it is proved that the growth of the diameters is bounded. As a consequence, for small times these regions remain disjoint; the other steps in the proof are simple. The main tool is the control of moment of inertia of each blob of vorticity with respect to its center.

Secondly, it has been studied the motion of a single concentrated blob of vorticity which moves in a bounded domain of the plane. It has been proved globally in time that the main part to the vorticity converges as the concentration increases to the generalization of the point vortex model in presence of borders [23]. In that paper the main tool is the conservation of the energy of the system. This result has been extended by using a similar technique to two vortices of the same sign in [17].

It has been reasonable to prove this connection for N vortices globally in time. The first step has been made in the case of point vortices of the same sign. In [11] it has been proved the convergence globally in time to the point model vortex, by proving that the vorticity far from point vortices becomes in the limit negligible.

When the point vortices have different signs the problem is more complicated for two reasons: in some case the equation (1.10) has not a global solution, *i.e.* two different vortices can collapse in the same point in finite time, and the right hand side of the equation becomes infinite; moreover the positive and the negative vorticities must remain separated, otherwise the system becomes very unstable. On the first point, the collapses may happen but are exceptional ([6] and [20]). We overcome the second point proving that the initial localization holds globally in time.

Precisely we consider an initial datum of the form:

$$\omega_\epsilon(x,0) = \sum_{i=1}^{N} \omega_{\epsilon;i}(x,0) \ , \tag{1.11}$$

where $\omega_{\epsilon;i}(x,0)$ is a function with a definite sign supported in a region $\Lambda_{\epsilon;i}$ such that

$$\Lambda_{\epsilon;i} = \text{supp}\ \omega_{\epsilon;i} \subset \Sigma(z_i|\epsilon)\ ;\ \Sigma(z_i|\epsilon) \cap \Sigma(z_j|\epsilon) = 0 \quad if \quad i \neq j\ , \tag{1.12}$$

for ϵ small enough. Here $\Sigma(z|r)$ denotes a circle of center z and radius r.

We denote by

$$\int dx\ \omega_{\epsilon;i}(x,0) = a_i \in \mathbb{R}\ , \tag{1.13}$$

the vortex intensity (independent of ϵ) and we assume

$$|\omega_{\epsilon;i}(x,0)| \leq M\epsilon^{-\gamma}\ , \quad M > 0\ , \gamma > 0\ . \tag{1.14}$$

We discuss the following

PROPOSITION 1.1. *Denote by $\omega_{\epsilon;i}(x,t)$ the time evolution of $\omega_{\epsilon;i}(x,0)$ according the Euler equation. Then for any fixed $T > 0$ there exists $C(\alpha,T)$ such that for any $0 \leq t \leq T$*

$$\text{supp}\ \omega_{\epsilon;i}(x,t) \subset \Sigma(z_i(t)|d) \quad where\ d = C(\alpha,T)\ \epsilon^{\alpha}\ , \tag{1.15}$$

and $z_i(t)$ is the solution of the ordinary differential system (1.10) *provided such a solution exists up to the time T.*

This proposition states that the blobs of vorticity remain localized until the time T.

A Corollary states that for any continuos bounded function $f(x)$

$$\lim_{\epsilon \to 0} \int dx\ \omega_{\epsilon;i}(x,t)\ f(x) = \sum_{i=1}^{N} a_i\ f(z_i(t))\ , \tag{1.16}$$

and

$$\omega_{\epsilon;i}(x,0) \longrightarrow_{\epsilon \to 0} \sum_{i=1}^{N} a_i\ \delta(z_i(t))\ . \tag{1.17}$$

weakly in the sense of measures, where $\delta(\cdot)$ denotes the Dirac measure. This last statement gives a rigorous justification of the point vortex model.

The Proposition has been proved in [19] for any $\alpha < 1/300$ and any $\gamma < 8/3$ (see also [20]). The rapidity of the convergence has been improved in [12] proving the result for any $\alpha < 1/3$ and any positive γ. In the present paper we obtain:

THEOREM 1.2 (aa). *We prove the Proposition and the Corollary for any $\alpha < 1/2$ and any positive γ.*

This result gives a rigorous bound on the evolution of concentrated vorticity and it improves the known property of localization . We believe that fact is interesting in itself, independently of the connection with the point vortex model.

2 – Proof

The proof is a mixture of the previous proofs and some new ideas. First we study the motion of a single blob of vorticity in a Lipschitz external field. Then we observe that the other vortices produce a regular field and this concludes the proof.

Let us state the reduced problem. We consider a single blob of unitary vorticity moving in an external, divergence-free, uniformly bounded, time dependent vector field $F(x,t)$, satisfying the Lipschitz condition

$$|F(x,t) - F(y,t)| \leq L|x-y| \ , \ L > 0 \ . \tag{2.1}$$

Equation (1.7) becomes

$$\frac{d}{dt}x(x_0,t) = u(x(x_0,t)) + F(x,t) \ , \tag{2.2}$$

while eq. (1.6) remains unchanged. The Euler equation in weak form reads

$$\frac{d}{dt}\omega[f] = \omega[(u+F)\cdot\nabla f] + \omega[\partial_t f] \ . \tag{2.3}$$

Define the center of vorticity as

$$B_\epsilon(t) = \int x \ \omega_\epsilon(x,t) \ dx. \tag{2.4}$$

We prove

THEOREM 2.1. *Suppose that*

$$\text{supp } \omega_\epsilon(x,0) \subset \Sigma(x^*|\epsilon) \quad , \tag{2.5}$$

and

$$|\omega_\epsilon(x,0)| \leq const \ \epsilon^{-\gamma} \ , \tag{2.6}$$

where from now on const denotes a constant independent of ϵ.

$$\int dx \ \omega_\epsilon(x,t) = 1 \ . \tag{2.7}$$

Then for any $\alpha < 1/2$ *there exists* $C(\alpha,T) > 0$ *such that for* $0 \leq t \leq T$

$$\text{supp } \omega_\epsilon(x,t) \subset \Sigma(B_\epsilon(t)|d), \tag{2.8}$$

where

$$d = C(\alpha,T) \ \epsilon^\alpha \quad . \tag{2.9}$$

Moreover

$$|B_\epsilon(t) - B(t)| \to_{\epsilon\to 0} 0 \textit{ at least as } \epsilon \textit{ , uniformly in } \ t \in [0,T] \tag{2.10}$$

where $B(t)$ is the solution of the ordinary differential equation

$$\frac{d}{dt}B(t) = F(B(t),t) \quad , \quad B(0) = x^* \ . \tag{2.11}$$

Proof. The strategy of the proof is similar to that one used in [12] plus a sharper estimate on the radial field produced by a blob of vorticity. First we prove that the main part of the vorticity remains close to $B_\epsilon(t)$. Then we study the radial velocity field on the farthest part of the blob of vorticity and we give a bound on it that vanishes as $\epsilon \to 0$. From now on, without lack of generality, we assume $\epsilon < 1$.

We introduce the moment of inertia I_ϵ with respect of B_ϵ:

$$I_\epsilon(t) = \int dx \ |x - B_\epsilon(t)|^2 \ \omega_\epsilon(x,t) \ . \tag{2.12}$$

We study the growth in time of $B_\epsilon(t)$ and $I_\epsilon(t)$ by using eq. (2.3):

$$\frac{d}{dt}B_\epsilon(t) = \int dx \ F(x,t) \ \omega_\epsilon(x,t) \ , \tag{2.13}$$

$$\frac{d}{dt}I_\epsilon(t) = 2\int dx \ (x - B_\epsilon(t)) \cdot F(x,t) \ \omega_\epsilon(x,t) \ , \tag{2.14}$$

where we have taken into account the antisymmetry of K.

We will see, that these equations impose a sort of localization. Actually using the Lipschitz condition on F and the fact that

$$\int dx \ (x - B_\epsilon(t)) \cdot F(B_\epsilon(t),t) \ \omega_\epsilon(x,t) \ = 0 \ , \tag{2.15}$$

we have:

$$|\frac{d}{dt}I_\epsilon(t)| \leq 2L\int dx \ (x - B_\epsilon(t))^2 \ \omega_\epsilon(x,t) \ , \tag{2.16}$$

from which

$$I_\epsilon(t) \leq I_\epsilon(0)\exp(2 \ L \ t) \leq const \ \epsilon^2 \ . \tag{2.17}$$

Therefore

$$\lim_{\epsilon\to 0} I_\epsilon(t) = 0 \ \ \textit{at least as } \epsilon^2 \textit{ uniformly in } t \in [0,T] \ . \tag{2.18}$$

We prove now eq. (2.10). The integral equations give:

$$|B(t)-B_\epsilon(t)| \le |x^*-B_\epsilon(0)| + \int_0^t ds \left| F(B(s),s) - \int dx\ F(x,s)\ \omega_\epsilon(x,s) \right| \tag{2.19}$$

$$\le |x^*-B_\epsilon(0)| + \int_0^t ds \left[|F(B(s),s) - F(B_\epsilon(s),s)| \right. \tag{2.20}$$

$$\left. + \left| F(B_\epsilon(s),s) - \int dx\ F(x,s)\ \omega_\epsilon(x,s) \right| \right]$$

$$\le |x^*-B_\epsilon(0)| + \ L \int_0^t ds \ \left[|B(s)-B_\epsilon(s)| + \int dx \| B_\epsilon(s) - x|\ \omega_\epsilon(x,s)| \right] \tag{2.21}$$

$$\le |x^*-B_\epsilon(0)| + \ L \int_0^t ds\ |B(s)-B_\epsilon(s)| + L\ T \sup_{0\le t\le T} \sqrt{I_\epsilon(t)}\ . \tag{2.22}$$

The first and the third terms of the right hand side of this equation are bounded by a quantity proportional to ϵ. So, by use of the Gronwall Lemma we achieve the proof of eq. (2.10).

In the previous steps we have proved that the main part of the vorticity remains concentrated around its center, but a priori small filaments of vorticity could go far. We shall exclude this fact by studying the radial velocity field on the fluid particle which is farthest from the center $B_\epsilon(t)$ and by proving that it vanishes as $\epsilon \to 0$.

The growth of the distance of a fluid particle in $x \in \operatorname{supp}\, \omega_\epsilon(x,t)$ farthest from $B_\epsilon(t)$ reads:

$$\left| \left[u(x,t) + F(x,t) - \frac{d}{dt} B_\epsilon(t) \right] \cdot \frac{x - B_\epsilon(t)}{|x-B_\epsilon(t)|} \right| \tag{2.23}$$

$$\le \left| F(x,t) - \int dy\ \omega_\epsilon(y,t)\ F(y,t) \right| + \left| \frac{x-B_\epsilon(t)}{|x-B_\epsilon(t)|} \ \cdot \int dy K(x-y)\ \omega_\epsilon(y,t) \right| \tag{2.24}$$

$$= \left| \int dy\ \omega_\epsilon(y,t)\ [F(x,t)-F(y,t)] \right| + \left| \frac{x-B_\epsilon(t)}{|x-B_\epsilon(t)|} \cdot \int dy\ K(x-y)\ \omega_\epsilon(y,t) \right| . \tag{2.25}$$

To bound the first term of the right hand side, due the external field, is easy by using the Lipschitz condition:

$$\le \text{const } R \quad , \quad R = |x - B_\epsilon(t)|\ . \tag{2.26}$$

Now we study the second term. In this point there is an improvement with respect to [12]. We divide the integration region into two parts: a circle of radius $R/2$ centered in $B_\epsilon(t)$ denoted by A_1 and an annulus $A_2 = \Sigma(B_\epsilon(t)|R) - \Sigma(B_\epsilon(t)|R/2)$. We evaluate the contribution of the integration in A_1, *i.e.*:

$$H_1 = \frac{x-B_\epsilon(t)}{|x-B_\epsilon(t)|} \ \cdot \int_{A_1} dy\ K(x-y)\ \omega_\epsilon(y,t)\ . \tag{2.27}$$

Denote by $x' = x - B_\epsilon(t)$ and $y' = y - B_\epsilon(t)$, we have:

$$H_1 = \frac{1}{2\pi} \int_{|y'| \le R/2} dy' \left[\frac{x'}{|x'|} \cdot \frac{(x'-y')^\perp}{|x'-y'|^2} \right] \omega_\epsilon(y' + B_\epsilon(t)) \ , \tag{2.28}$$

where $x^\perp = (x_2, -x_1)$. Using the fact that $x' \cdot (x'-y')^\perp = -x' \cdot y'^\perp$ we can write:

$$H_1 = -\frac{1}{2\pi} \int_{|y'| \le R/2} dy' \ \frac{x' \cdot y'^\perp}{|x'||x'-y'|^2} \ \omega_\epsilon(y' + B_\epsilon(t)) \ . \tag{2.29}$$

We observe that

$$\int_{\mathbb{R}^2} dy' \ y'^\perp \ \ \omega_\epsilon(y' + B_\epsilon(t))| = 0 \ , \tag{2.30}$$

so that

$$H_1 = -\frac{1}{2\pi}(H_1' - H_1'') \ , \tag{2.31}$$

where

$$H_1' = \int_{|y'| \le R/2} dy' \ \frac{x' \cdot y'^\perp}{|x'|} \left[\frac{1}{|x'-y'|^2} - \frac{1}{|x'|^2]} \right] \omega_\epsilon(y' + B_\epsilon(t)) \ , \tag{2.32}$$

and

$$H_1'' = \int_{|y'| > R/2} dy' \ \frac{x' \cdot y'^\perp}{|x'|^3} \ \ \omega_\epsilon(y' + B_\epsilon(t)) \ . \tag{2.33}$$

We bound $|H_1'|$.

$$H_1' = \int_{|y'| \le R/2} dy' \ \frac{x' \cdot y'^\perp}{|x'|} \ \frac{y' \cdot (2x' - y')}{|x'-y'|^2 \ |x'|^2} \ \omega_\epsilon(y' + B_\epsilon(t)) \ . \tag{2.34}$$

We note that $|y'| \le R/2$ implies $|2x' - y'| \le |x' - y'| + |x'| \le 3|x' - y'|$, so that

$$|H_1'| \le \frac{6}{R^3} \int_{|y'| \le R/2} dy' \ |y'|^2 \ \omega_\epsilon(y' + B_\epsilon(t)) \ \le \frac{6 \ I_\epsilon(t)}{R^3} \le \frac{\text{const } \epsilon^2}{R^3} \ . \tag{2.35}$$

We study $|H_1''|$. We remember that in (2.33) $|y'| \le R$ because R is the maximal distance of a fluid particle from $B_\epsilon(t)$, so that:

$$|H_1''| \le \text{const } \frac{1}{R} m_t(R/2) \ , \tag{2.36}$$

where we have denoted by $m_t(h)$ the mass of vorticity out of $\Sigma(B_\epsilon|h), h > 0$:

$$m_t(h) = \int_{|x'| > h} dx' \ \omega_\epsilon(x' + B_\epsilon(t)) \ . \tag{2.37}$$

In conclusion we can bound the terms of velocity field that pushes the particle farthest from $B_\epsilon(t)$ due to the fluid in $\Sigma(B_\epsilon(t)|R/2)$ as

$$|H_1| \le \frac{\text{const } \epsilon^2}{R^3} \ + \ \text{const } \frac{1}{R} m_t(R/2) \ . \tag{2.38}$$

Since by (2.17) we have

$$m_t(h) \le \text{const } \frac{\epsilon^2}{h^2} \ , \tag{2.39}$$

the eq. (2.38) becomes:

$$|H_1| \le \frac{\text{const } \epsilon^2}{R^3} \ . \tag{2.40}$$

To complete the proof we must show that velocity field produced by the fluid in A_2 is negligible for small ϵ and $R \ge b \ \epsilon^\alpha$, $b > 0$ large enough. We introduce for any $h > 0$ the following nonnegative function $W_h(r) \in C^\infty(\mathbb{R}^2)$, $r \in \mathbb{R}^2$ depending only on $|r|$, defined as:

$$W_h(r) = 1 \ if \ |r| < h \ , \ = 0 \ if \ |r| > 2R \ , \tag{2.41}$$

such that, for some $C_1 > 0$:

$$|\nabla W_h(r)| < \frac{C_1}{h} \ , \tag{2.42}$$

$$|\nabla W_h(r) - \nabla W_h(r')| < \frac{C_1}{h^2} \ |r - r'| \ . \tag{2.43}$$

Define the quantity:

$$\mu_t(h) = 1 - \int dx \ W_h(x - B_\epsilon(t)) \ \omega_\epsilon(x,t) \ . \tag{2.44}$$

We remark that, if supp $\omega_\epsilon(x,t) \subset \Sigma(B_\epsilon(t)|h)$, then $\mu_t(h) = 0$. Hence we choose $\mu_t(h)$ as a measure of the localization of $\omega_\epsilon(x,t)$ around $B_\epsilon(t)$.

We need to prove that is quantity is negligible for small ϵ when $h = R/2$.

The two quantities $\mu_t(h)$ and $m_t(h)$ are related by the obvious property, that will be used later on:

$$m_t(h) \le \mu_t \left(\frac{h}{2} \right) \ . \tag{2.45}$$

We study the time derivative of $\mu_t(h)$:

$$\begin{aligned} \frac{d}{dt}\mu_t(h) &= -\int dx \nabla W_h(x - B_\epsilon(t)) \cdot \left[u(x,t) + F(x,t) - \frac{d}{dt} B_\epsilon(t) \right] \ \omega_\epsilon(x,t) \\ &= -H_3 - H_4 \ , \end{aligned} \tag{2.46}$$

where

$$H_3 = \int dx\ \omega_\epsilon(x,t)\ \nabla W_h(x - B_\epsilon(t)) \cdot \int dy\ K(x-y)\ \omega_\epsilon(y,t)\ , \tag{2.47}$$

$$H_4 = \int dx\ \omega_\epsilon(x,t)\ \nabla W_h(x - B_\epsilon(t)) \cdot \int dy\ \omega_\epsilon(y,t)[F(x,t) - F(y,t)]\ . \tag{2.48}$$

We estimate $|H_3|$. By the antisymmetry of K it can be written as:

$$\begin{aligned} |H_3| = \frac{1}{2}\bigg| \int dx \int dy\ \omega_\epsilon(x,t)\ \omega_\epsilon(y,t)[\nabla W_h(x - B_\epsilon(t)) - \nabla W_h(y - B_\epsilon(t))] \\ \cdot K(x-y)\bigg|. \end{aligned} \tag{2.49}$$

Obviously it is different from zero only if either $|x - B_\epsilon(t)| > h$ or $|y - B_\epsilon(t)| > h$. We divide the last integral into three parts: $|y - B_\epsilon(t)| \le h/2$, $|x - B_\epsilon(t)| \le h/2$ and otherwise.

In the first case, we repeat the proof which gives (2.40) and this term can be bounded by

$$\frac{\text{const}\ \epsilon^2}{h^4}\ m_t(h)\ . \tag{2.50}$$

The second integral, exchanging x by y, can be bounded in the same way.

Finally we study the integral in the last two cases: either $|x - B_\epsilon(t)| > h$, $|y - B_\epsilon(t)| > h/2$ or $|x - B_\epsilon(t)| > h/2$, $|y - B_\epsilon(t)| > h$. Due to the bound $|K(x)| \le \text{const}\ |x|^{-1}$ and the property (2.43) we have:

$$[\nabla W_h(x - B_\epsilon(t)) - \nabla W_h(y - B_\epsilon(t))] \cdot\ K(x-y) \le \frac{\text{const}}{h^2}\ . \tag{2.51}$$

Then these terms give a contribution smaller than

$$\text{const}\ \frac{\epsilon^2}{h^4}\ m_t(h)\ \ . \tag{2.52}$$

Now we study the term H_4 . We consider two cases: or $|y - B_\epsilon(t)| > h$ or $|y - B_\epsilon(t)| \le h$. In the first case

$$\begin{aligned} &\bigg|\int dx\ \omega_\epsilon(x,t)\ \nabla W_h(x - B_\epsilon(t)) \cdot \int dy\ \omega_\epsilon(y,t)[F(x,t) - F(y,t)]\bigg| \\ &\le \text{const}\ ||F||_\infty \frac{\epsilon^2}{h^3}\ m_t(h)\ . \end{aligned} \tag{2.53}$$

In the second case, by using the Lipschitz condition (2.1),

$$\begin{aligned} &\bigg|\int dx\ \omega_\epsilon(x,t)\ \nabla W_h(x - B_\epsilon(t)) \cdot \int dy\ \omega_\epsilon(y,t)[F(x,t) - F(y,t)]\bigg| \\ &\le \text{const}\ m_t(h). \end{aligned} \tag{2.54}$$

In conclusion

$$\frac{d}{dt}\mu_t(h) \le A(h)\ m_t(h)\ , \tag{2.55}$$

where

$$A(h) = \left[\text{const}\ \frac{\epsilon^2}{h^4} + \text{const}\ \frac{\epsilon^2}{h^3} + \text{const}\right]. \tag{2.56}$$

Using (2.45), the previous differential inequality gives the integral expression:

$$\mu_t(h) \le \mu_0(h) + A(h)\int_0^t d\tau\ \mu_\tau(h/2)\ , \tag{2.57}$$

where

$$A(h) = \left[\text{const}\ \frac{\epsilon^2}{h^4} + \text{const}\ \frac{\epsilon^2}{h^3} + \text{const}\right]\ . \tag{2.58}$$

We start an iterative procedure

$$\mu_t(h) \le \mu_0(h) + \mu_0(h/2)A(h)\int_0^t d\tau + A(h)A(h/2)\int_0^t dt_1 \int_0^{t_1} d\tau\ \mu_\tau(h/4)\ , \tag{2.59}$$

and so on.

We start from $h = b\ \epsilon^\alpha, \alpha < 1/2,\ b > 1$ and we iterate eq. (2.57) n times. We choose n such that $n \to \infty$ as $\epsilon \to 0$ and in the same time during this limit $A(h2^{-k})$ remains bounded for any integer $k \le n$ and $\mu_0(h2^{-n}) = 0$:

$$n = \text{Integer part of}\ \left[-\frac{1-2\alpha}{3}\log_2 \epsilon\right]\ ,\ (\epsilon < 1)\ . \tag{2.60}$$

Hence

$$h2^{-n} \ge b\ \epsilon^{(1+\alpha)/3}\ , \tag{2.61}$$

and

$$A(h2^{-k}) \le \text{const}\ , \tag{2.62}$$

for any positive integer $k \le n$. After n iteration we obtain

$$\mu_t(h) \le \frac{(\text{const})^n}{n!}\ . \tag{2.63}$$

Using eq. (2.45), the Stirling approximation for $n!$, explicit form (2.60) for n and putting $h = R$, we have that

$$m_t(R/2) \to 0\ as\ \epsilon \to 0 \text{ faster than any power in } \epsilon\ . \tag{2.64}$$

Now we show that a similar bound holds for the velocity field produced by this small vorticity. This field holds

$$\left| \int_{A_2} dy\ K(x-y)\ \omega_\epsilon(y,t) \right| \leq \frac{1}{2\pi} \left| \int_{A_2} dy\ |x-y|^{-1}\ \omega_\epsilon(y,t) \right| \ , \tag{2.65}$$

where $A_2 = \Sigma(B_\epsilon, R) - \Sigma(B_\epsilon, R/2)$.

The integrand is monotonically unbounded as $x \to y$, and so the maximum of the integral is obtained when we rearrange the vorticity mass as close as possible to the singularity:

$$\frac{1}{2\pi} \left| \int_{A_2} dy\ |x-y|^{-1}\ \omega_\epsilon(y,t) \right| \leq \text{const}\ \epsilon^{-\gamma} \int_{\Sigma(O,\eta)} dy\ |y^{-1}| \ , \tag{2.66}$$

where we have used the property (1.14), O denotes the origin and η is such that

$$M\ \epsilon^{-\gamma} \pi \eta^2 = m_t(R/2)\ . \tag{2.67}$$

Using eq. (2.64) we have proved that this velocity field vanishes as $\epsilon \to 0$ faster than any power in ϵ.

We are now able to find a bound on the radial velocity of a fluid particle at distance R from $B_\epsilon(t)$. Using the result on (2.40) and the bound (2.26), we obtain:

$$\left| \frac{d}{dt} R(t) \right| \leq C_1\ R(t) + C_2\ \frac{\epsilon^2}{R(t)^3} + g\ , \tag{2.68}$$

where C_1, C_2 denote two positive constant (depending on T) but independent of ϵ and g denotes terms smaller than any power in ϵ when $R > C_3\ \epsilon^\alpha,\ \alpha < 1/2$.

Hence for $R > C_3\ \epsilon^\alpha$ the last term of the right hand side of the previous equation is negligible and the second one is bounded by $C_1\ C_3^{-3}\ \epsilon^{2-3\alpha}$. The inequality (2.68) by using Gronwall Lemma gives a control of going away. The result (2.9) can be obtained by a proof "ab absurdo". Indeed we choose $C_4 >> C_3$ and an α' such that $\alpha < \alpha' < 1/2$; assume "ab assurdo" that at time $t^*, (0 \leq t^* \leq T)$, $R(t^*) = C_4 \epsilon^\alpha$, going back in time, because of continuity, there is a time $t_1, (0 \leq t_1 < t^*)$ such that for the first time $R(t_1) = C_3 \epsilon^{\alpha'}$. By (2.68) (with α' instead of α) shows that is impossible and hence the support of the vorticity remains in $\Sigma(B_\epsilon(t)|C_4\ \epsilon^\alpha)$. $\square$

We can easily give the proof of the main Theorem and we only sketch it. We denote by R_m the minimal distance between point vortices evolving via (1.10) and we choose $\epsilon \ll R_m$. Initially the vortices are separated and we simulate the influence of other vortices as an external field. The result of this Section states that the vorticities remain separated. We remark that the other vortices produce an external field depending on ϵ, but this dependence is very small and its effects can be controlled. This conclude the proof.

3 – Comments

The main tool used in the present improvement is a good estimate of some cancellations in the vorticity field (see (2.32),(2.33)). In the case of a single vortex these cancellations take into account the conservation of $B_\epsilon(0)$. They have been suggested by the study of the paper [10].

We remark that the growth of the support of the vorticity increases as the time goes by. In the proof this increase is exponential because of the Gronwall Lemma and it is smaller as the point vortices are close to a stationary state.

The present analysis can be extended without serious difficulties to a fluid moving in a region with boundary.

A possible generalization is to study fluids moving in the whole plane, governed by Navier-Stokes equation, when we consider large initial concentrations and the vanishing viscosity limit as in [8, 13, 15].

Until now we have considered a two dimensional system, *i.e.* a fluid in three dimensions with a planar symmetry. A point vorticity in two dimensions means a straight-line in three dimensions. We can study also other symmetries. We introduce cylindrical coordinates (r, θ, z); the fluid has cylindrical symmetry without swirl if the fluid moves in the plane (r, z) independent of θ. A point in this plane means a circle around the symmetry axis in the space. It is well-known that in this plane there are some smooth configurations invariant in form that move in the z direction with a constant speed (see for instance [7, 2]; on the vortex rings see [22]). We can study a concentrated vorticity in the plane (r, z) supported in a small region contained in a circle of radius ϵ (in the three dimensional space appears like a torus or a sort of smoke ring) not far from the symmetry axis. Obviously in general this fluid is not invariant in form for a fixed ϵ, but we can prove that, when the total vorticity vanishes as const./$|\log\epsilon|$, all shapes converge as $\epsilon \to 0$ to an annulus, which moves in the z direction with a constant velocity. In this case a localization holds in a weak form: the main part of the vorticity remains concentrated in a circle of radius const $\epsilon\ |\log\epsilon|$. Unfortunately, we are not able to prove that there is rigorously not vorticity out of it (as it happens in the present paper). The proof given in [4] uses an estimate on the energy of the system that does not allow to employ the technique of the present paper. (For vanishing viscosity see [5]).

We discuss now the case in which the vorticity is far from the symmetry axis. We introduce the new variables $z = x$, $r = r_0 + y$ and we study the motion in the plane (x, y). When $r_0 = const\ |\log\epsilon|$, we can show, by using a technique similar to the smoke ring, that a fluid, initially concentrated in the plane (x, y) in a small region "converges" at time t as $\epsilon \to 0$ to a large annulus and the weak localization holds [16]. In this case also the improvement of the present paper cannot be applied.

On the contrary when $r_0 = \text{const}\ \epsilon^{-\beta}, \beta > 0$, the fluid can be concentrated in many disjoint regions and it "converges" to the point vortex model as $\epsilon \to 0$ [14] in a way that the technique of the present paper can be applied. We could obtain a small improvement.

Finally, when a helicoidal symmetry exists, we can show that there are structures invariant in form which move along cylindrical helices [1], but in this case we do not know not even if the limit $\epsilon \to 0$ exists.

We remark that it exists a different connection between the Euler Equation and the point vortex model, *i.e.* the so called "vortex method". In this case we approximate smooth initial data by a system of N point vortices of intensity $\approx 1/N$ and we evolve this system (which has a finite number of degrees of freedom) via (1.10). We can prove that, if some conditions are fulfilled, as $N \to \infty$ this system converges to the solution of the Euler equation. There is a wide literature on this topic important mainly for numerical purposes, that is out of the aim of the present paper. Some details and references can be found in the books [20, 21].

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