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Optimal Investment-consumption for Partially Observed Jump-diffusions

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Abstract. We deal with an optimal consumption-investment problem under restricted information in a financial market where the risky asset price follows a non-Markovian geometric jump-diffusion process. We assume that agents acting in the market have access only to the information flow generated by the stock price and that their individual preferences are modeled through a power utility. We solve the problem with a two steps procedure. First, by using filtering results we reduce the partial information problem to a full information one involving only observable processes. Next, by using dynamic programming, we characterize the value process and the optimal-consumption strategy in terms of solution to a backward stochastic differential equation.

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1. Introduction

In this paper we study an extension of the classical Merton optimal investment-consumption problem to a partially observable financial market in which asset prices follow geometric jump-diffusions. A single agent manages his portfolio by investing in a bond and in the stock asset S_t and chooses a portfolio-consumption strategy in order to maximize on a finite horizon his total expected utility from consumption and terminal wealth. The agent's information is described by the natural filtration of the stock price process,$\{\underline{\mathrm{F}}_t^S\}_{t\in[0,T]}$, hence his decisions must be adapted to $\{\underline{\mathrm{F}}_t^S\}_{t\in[0,T]}$ and this leads to a utility maximization problem under restricted information.

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Utility maximization problems in a full information setting have been largely studied in the literature by using different approaches, such as convex duality methods, stochastic control techniques based on the Hamilton–Jacobi–Bellman equation or backward stochastic differential equations (see for example [2, 8, 11, 14, 17, 20, 21, 25] and references therein). Portfolio selection problems with partial information have been studied among others in [16, 23, 24] in a continuous setting, in [1, 18] for jump-diffusions and in [6] in the case where the risky asset follows a Markov pure jump process. In [1] it is assumed that investors are only able to observe the stock price process and not the Markov chain which drives the jump intensity. In [18] a default model is studied where investors only observe asset prices and default times, while the drift of the asset price dynamics and the default intensities are not directly observable.

The contribution of this note consists in solving the utility maximization problem with intermediate consumption under partial information in a general jump-diffusion setting. More precisely, we do not assume Markovianity of the asset price dynamics and we work with a jump component described by a general integer-valued measure.

The outline of the paper is as follows. In Section 2, we describe the market model and the optimal investment-consumption problem under restricted information. In Section 3, by projection on the information flow we reduce the partial observable problem to a full information one and we give a representation theorem for $\underline{F}_t^S$-martingales. In Section 4, we formulate the full information problem (with respect to the filtration $\{\underline{F}_t^S\}_{t\in[0,T]}$) as a stochastic control problem. The special form of the power utility leads to a factorization of the associated value process into a part depending on the current wealth and the so-called opportunity process J_t ([21, 22]) around which our analysis is built. In Section 5, by using dynamic programming we show that J_t solves a backward stochastic differential equation and we provide a feedback formula for the optimal consumption in terms of J_t. We discuss the particular case of bounded investment strategies and finally we characterize the opportunity process in the case of non constrained strategies via a sequence of solutions of Lipschitz BSDEs. We conclude the section providing a verification result and giving as application a simplified model where the risky asset dynamics is driven by two independent point processes whose intensities are not directly observed by investors.

2. The market model and problem formulation

In this paper, we consider a complete filtered probability space $(\Omega, \{\underline{F}_t\}_{t\in[0,T]}, P)$ $\Longleftarrow$ text changed
endowed with a Brownian motion W_t with values in $\mathbb{R}$ and a Poisson random measure $N(dt, d\zeta)$ independent of W_t. Here T is a fixed final time. The financial market consists of a nonrisky asset, with price process normalized to unity, and one risky asset with logreturn process Y_t given by the following jump-diffusion

process

$$dY_t = b_t dt + \sigma_t dW_t + \int_Z K(t;\zeta)N(dt, d\zeta), \quad Y_0 = 0. \tag{2.1}$$

The mean measure of $N(dt, d\zeta)$ is denoted by $\nu(d\zeta)\ dt$ with $\nu(d\zeta)$ a σ-finite measure on a measurable space $(Z, \underline{Z})$. The coefficients b_t and σ_t are progressive $\underline{F}_t$-adapted processes with $\sigma_t > 0$ P-a.s. $\forall t \in [0, T]$, and $K(t;\zeta)$ is an $\mathbb{R}$-valued $(P, \underline{F}_t)$-predictable process joint measurable w.r.t. $(t, \zeta) \in [0, T] \times Z$. We also assume some requirements for (2.1) to be well defined

$$\mathbb{E}\int_0^T |b_t|dt < \infty \quad \mathbb{E}\int_0^T \sigma_t^2 dt < \infty \quad \mathbb{E}\int_0^T \int_Z |K(t;\zeta)|\nu(d\zeta)dt < \infty \tag{2.2}$$

and which entail that Y_t has finite first moment. The price S_t of the risky asset follows a geometric jump-diffusion process given by

$$S_t = S_0 e^{Y_t} \quad S_0 \in \mathbb{R}^+.$$

From Itô's formula we get that S_t solves the following differential equation

$$dS_t = S_{t^-}\left\{\mu_t dt + \sigma_t dW_t + \int_Z \widetilde{K}(t;\zeta)N(dt, d\zeta)\right\}$$

where

$$\mu_t = b_t + \frac{1}{2}\sigma_t^2, \quad \widetilde{K}(t;\zeta) = e^{K(t;\zeta)} - 1.$$

We are interested in solving an optimal portfolio problem for an agent who has access only to the observable flow generated by asset prices

$$\underline{F}_t^S = \sigma\{S_s; s \leq t\} = \underline{F}_t^Y = \sigma\{Y_s; s \leq t\} \subseteq \underline{F}_t.$$

We shall call this situation the case of partial information to distinguish it from the case of full information where investors observe the whole filtration $\{\underline{F}_t\}_{t\in[0,T]}$. We assume that $\{\underline{F}_t^S\}_{t\in[0,T]}$ satisfies the usual conditions of right-continuity and completeness.

The investor starts with initial capital $z_0 > 0$, invests at any time $t \in [0, T]$ the fraction θ_t of the wealth Z_t in stock S_t and also consumes at the rate $C_t Z_t$. We consider both cases of utility from terminal wealth only and with intermediate consumption. As in [21] and [22], to unify the notations we introduce the measure $\mu(dt)$ on $[0, T]$ by $\mu(dt) = 0$ in the case without consumption and $\mu(dt) = dt$ in the case with consumption and assume the convention $C_T = 1$ (which means that all the remaining wealth is consumed at time T).

Because the agent's information is described by the filtration $\{\underline{F}_t^S\}_{t\in[0,T]}$ the decisions (θ_t, C_t) must be adapted to $\underline{F}_t^S$. By considering $\underline{F}_t^S$-predictable, self-financing trading strategies, the dynamics of the wealth process controlled by the investment-consumption process $(\theta_t,\, C_t)$ evolves according with

$$dZ_t = Z_{t^-}\left(\theta_t \frac{dS_t}{S_{t^-}} - C_t\mu(dt)\right), \quad Z_0 = z_0. \tag{2.3}$$

The solution process Z_t to (2.3) of course depends on the chosen strategy (θ, C). To be precise we should therefore denote the process Z_t by $Z_t^{\theta,C}$ but sometimes we will suppress θ, C.

For an agent with power utility

$$U(x) = \frac{x^\alpha}{\alpha} \quad 0 < \alpha < 1$$

the objective is to maximize over a suitable class of strategies $\underline{\mathrm{A}}$ either the expected utility from terminal wealth

$$\sup_{(\theta,C)\in\underline{\mathrm{A}}} \mathbb{E}\Big[U(Z_T^{\theta,C})\Big]$$

and with intermediate consumption

$$\sup_{(\theta,C)\in\underline{\mathrm{A}}} \mathbb{E}\bigg[\int_0^T U(C_t Z_t^{\theta,C})\, dt + U(Z_T^{\theta,C})\bigg].$$

Defining $\mu^0(dt) = \mu(dt) + \delta_{\{T\}}(dt)$, where δ_a denotes the Dirac measure at the point a, both the cases can be written as

$$\sup_{(\theta,C)\in\underline{\mathrm{A}}} \mathbb{E}\bigg[\int_0^T U(C_t Z_t^{\theta,C})\mu^0(dt)\bigg]. \tag{2.4}$$

Let us come back to the market model. We introduce the discrete random measure ([4],[13]) associated to the jump component of Y_t

$$m(dt, dx) = \sum_{s:\Delta Y_s \neq 0} \delta_{\{s,\Delta Y_s\}}(dt, dx) \tag{2.5}$$

and observe that for any real-valued function $f(x)$ the following equality holds

$$\int_0^t \int_Z f(K(s;\zeta))\mathbb{I}_{\{K(s;\zeta)\neq 0\}}(s,\zeta)N(ds, d\zeta) = \int_0^t \int_{\mathbb{R}} f(x)m(ds, dx). \tag{2.6}$$

We recall Proposition 2.2 in [5] which provides the $(P, \underline{F}_t)$-local characteristics of $m(dt, dx)$ in terms of the measure $\nu(d\zeta)$.

Proposition 2.1. *Let* $\forall t \in [0,T]$, $\forall A \in \underline{B}(\mathbb{R})$ (*where* $\underline{B}(\mathbb{R})$ *denotes the family of Borel sets of* $\mathbb{R}$)

$$D_t^A(\omega) = \{\zeta \in Z : K(t,\omega;\zeta) \in A \setminus \{0\}\} \subseteq D_t(\omega) = \{\zeta \in Z : K(t,\omega;\zeta) \neq 0\}.$$

Under the assumption

$$\mathbb{E}\int_0^T \nu(D_s)\, ds < \infty \tag{2.7}$$

the $(P, \underline{F}_t)$*-predictable projection of* m *is given by*

$$m^p(dt, dx) = \lambda_t \Phi_t(dx)dt$$

where λ_t *is a non-negative* $\underline{F}_t$*-predictable process and* $\Phi_t(dx)$ *is an* $\underline{F}_t$*-predictable process taking values in the space of probability measures over* $(\mathbb{R}, \underline{B}(\mathbb{R}))$ *and they* *satisfy* $\forall A \in \underline{B}(\mathbb{R})$

$$m^p(dt, A) = \lambda_t \Phi_t(A)dt = \nu(D_t^A)dt. \tag{2.8}$$

In particular $\lambda_t = \nu(D_t)$ *provides the* $(P, \underline{F}_t)$*-predictable intensity of the point process* $N_t = m((0,t], \mathbb{R})$ *which counts the total number of jumps of* Y *until time* t.

Remark 2.2. Equation (2.8) can be also written as

$$m^p(dt, dx) = \lambda_t \Phi_t(dx)dt = \int_{D_t} \delta_{K(t;\zeta)}(dx)\nu(d\zeta)dt.$$

Let us observe that the local characteristics $(\lambda_t, \Phi_t(dx))$ of $m(dt, dx)$ are not observable by investors since the process $K(t;\zeta)$ is not $\underline{F}_t^S$-adapted.

The $(P, \underline{F}_t)$-semimartingale structure of the risky asset S_t is described in the following proposition.

Proposition 2.3. *Under* (2.2), (2.7) *and in addition*

$$\mathbb{E}\int_0^T \int_Z |\widetilde{K}(t;\zeta)|\nu(d\zeta) < \infty \tag{2.9}$$

S_t *is a* $(P, \underline{F}_t)$*-semimartingale with the decomposition*

$$S_t = S_0 + M_t^S + A_t^S$$

where

$$A_t^S = \int_0^t S_r \mu_r dr + \int_0^t \int_{\mathbb{R}} S_{r^-}(e^x - 1)\lambda_r \Phi_r(dx)dr$$

is a process with finite variation paths, and

$$M_t^S = \int_0^t S_r \sigma_r dW_r + \int_0^t \int_{\mathbb{R}} S_{r^-}(e^x - 1)(m(dr, dx) - \lambda_r \Phi_r(dx)dr)$$

is a $(P, \underline{F}_t)$*-local martingale.*

Proof. Under (2.2), (2.7) and (2.9), the process

$$\int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_Z \widetilde{K}(s;\zeta)N(ds, d\zeta)$$
$$= \int_0^t \left\{\mu_s + \int_Z \widetilde{K}(s;\zeta)\nu(d\zeta)\right\}ds + \int_0^t \sigma_s dW_s + \int_0^t \int_Z \widetilde{K}(s;\zeta)(N(ds, d\zeta) - \nu(d\zeta)ds)$$

is a $(P, \underline{F}_t)$-semimartingale, hence S_t is a semimartingale being the Doléans-Dade exponential of a semimartingale. The expressions of the processes A_t^S and M_t^S follow by Equation (2.6). □

3. Reduction to an optimization problem with complete information

To solve the utility maximization problem under partial information we first reduce it to a full information one involving only $\underline{\mathrm{F}}_t^S$-adapted processes. To this aim we need to compute the $(P, \underline{\mathrm{F}}_t^S)$-predictable projection of the integer-valued measure $m(dt, dx)$.

From now on we will denote by $\widehat{R}_t$ the $(P, \underline{\mathrm{F}}_t^S)$-optional projection of a generic process R_t, satisfying $\mathbb{E}|R_t| < \infty$ $\forall t \in [0,T]$, defined as the unique optional process (in a P-indistinguishable sense) such that for each $\underline{\mathrm{F}}_t^S$-stopping time τ, $\widehat{R}_\tau = \mathbb{E}[R_\tau | \underline{\mathrm{F}}_\tau^S]$ P-a.s. on $\{\tau < \infty\}$.

Remark 3.1. We recall two well-known facts: for every $(P, \underline{\mathrm{F}}_t)$-martingale m_t, the projection $\widehat{m}_t$ is a $(P, \underline{\mathrm{F}}_t^S)$-martingale and that for any progressively measurable process Ψ_t with $\mathbb{E}\int_0^T |\Psi_t| dt < \infty$

$$\widehat{\int_0^t \Psi_s ds} - \int_0^t \widehat{\Psi}_s ds$$

is a $(P, \underline{\mathrm{F}}_t^S)$-martingale. Note that this implies that $E\int_0^T \Psi_t dt = \mathbb{E}\int_0^T \widehat{\Psi}_t dt$.

Let us denote by $\underline{\mathrm{P}}(\underline{\mathrm{F}}_t^S)$ the $\underline{\mathrm{F}}_t^S$-predictable σ-field on $(0,T] \times \Omega$.

Proposition 3.2. *Let us assume* (2.7). *The* $(P, \underline{F}_t^S)$*-predictable projection,* $\nu^p(dt, dx)$*, of* $m(dt,dx)$ *is given by* $\nu^p(dt,dx) = \nu_t^p(dx)dt$*, where* $\nu_t^p(dx)$ *is a measure-valued* $\underline{F}_t^S$*-predictable process satisfying* $\nu_t^p(dx) = \widehat{(\lambda_t \Phi_t)}(dx)$*,* $dP \times dt$*-a.e. More precisely, for each* $H(t,x)$*,* $\underline{P}(\underline{F}_t^S)$*-measurable*

$$\begin{aligned}\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} H(t,x)\ \nu_t^p(dx)dt\right] &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} H(t,x)\widehat{(\lambda_t \Phi_t)}(dx)dt\right] \\ &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} H(t,x)\ m(dt,dx)\right].\end{aligned}$$

Proof. By definition of $(P, \underline{\mathrm{F}}_t)$-predictable projection of the integer-valued measure $m(dt,dx)$ it follows that, for each $H(t,x)$ $(P, \underline{\mathrm{F}}_t)$-predictable process jointly measurable w.r.t. $(t,x) \in [0,T] \times \mathbb{R}$, verifying the condition ⟸ disp

$$\mathbb{E}\int_0^T \int_{\mathbb{R}} |H(r,x)| \lambda_r \Phi_r(dx) dr < \infty,$$

the process

$$m_t = \int_0^t \int_{\mathbb{R}} H(r,x)(m(dr,dx) - \lambda_r \Phi_r(dx)dr) \tag{3.1}$$

is a $(P, \underline{F}_t)$-martingale. Let us now consider in (3.1) a process $H(t,x)$ which is $(P, \underline{F}_t^S)$-predictable. By Remark 3.1 we get that

$$\int_0^t \int_{\mathbb{R}} H(r,x) m(dr,dx) - \mathbb{E}\left[\int_0^t \int_{\mathbb{R}} H(r,x)\lambda_r \Phi_r(dx)dr | \underline{F}_t^S\right]$$

is a $(P, \underline{F}_t^S)$-martingale, and

$$\int_0^t \int_{\mathbb{R}} H(r,x) m(dr,dx) - \int_0^t \int_{\mathbb{R}} H(r,x) \widehat{\lambda_r \Phi_r}(dx)dr$$

is a $(P, \underline{F}_t^S)$-martingale. In particular, for any $A \in \underline{B}(\mathbb{R})$

$$m((0,t],A) - \int_0^t \widehat{\nu(D_s^A)} ds = m((0,t],A) - \int_0^t \int_A \widehat{\lambda_s \Phi_s}(dx)ds$$

is a $(P, \underline{F}_t^S)$-martingale. Hence, since $\widehat{\nu(D_t^A)}$ is a progressively measurable process, it provides the $(P, \underline{F}_t^S)$-intensity of $N_t(A) = m((0,t],A)$ and as in [4, Theorem T13] one can find a $(P, \underline{F}_t^S)$-intensity, λ_t^A, that is predictable. It suffices define λ_t^A, for any $A \in \underline{B}(\mathbb{R})$, as the Radon–Nikodym derivatives of $P(d\omega)\widehat{\nu(D_t^A)}(\omega)dt$ w.r.t. $P(d\omega)dt$ on $\underline{P}(\underline{F}_t^S)$. □

Throughout the paper we denote by $m^S(dt,dx)$ the $(P, \underline{F}_t^S)$-compensated martingale random measure

$$m^S(dt,dx) = m(dt,dx) - \nu_t^p(dx)dt$$

and we recall that, for any $H(t,x)$, jointly measurable process, $\underline{F}_t^S$-predictable such that

$$\mathbb{E}\int_0^T\int_Z |H(t,x)|\nu_t^p(dx)dt < \infty \quad \left(\text{resp.} \int_0^T\int_Z |H(t,x)|\nu_t^p(dx)dt < \infty \quad P\text{-a.s.}\right)$$

the process $\int_0^T \int_Z H(t,x) m^S(dt,dx)$ is a $(P, \underline{F}_t^S)$-martingale (resp. local-martingale).

Next, assuming

$$\mathbb{E}\int_0^T \frac{|b_t|}{\sigma_t} dt < \infty, \tag{3.2}$$

and the volatility σ_t to be $\underline{F}_t^S$-adapted, we introduce the innovation process

$$I_t := W_t + \int_0^t \frac{1}{\sigma_s}(b_s - \widehat{b}_s)ds.$$

By extending classical results in filtering theory ([19]) to our frame we have the following

Proposition 3.3. *Let σ_t to be $\underline{F}_t^S$-adapted. The random process $\{I_t\}_{t\in[0,T]}$ is a $(P, \underline{F}_t^S)$-Wiener process.*

Proof. By Equation (2.6) we get that $\int_Z K(t;\zeta)N(dt,d\zeta) = \int_{\mathbb{R}} x\, m(dt,dx)$. Hence, taking into account Equation (2.1), we have

$$dI_t = \frac{1}{\sigma_t}\left\{dY_t - \widehat{b}_t - \int_{\mathbb{R}} x\, m(dt,dx)\right\},$$

which implies that I_t is an $\underline{\mathrm{F}}_t^S$-adapted process. We now compute the following conditional expectation, $\forall s \leq t$

$$\mathbb{E}\left[I_t - I_s \mid \underline{\mathrm{F}}_s^S\right] = \mathbb{E}\left[\int_s^t \left\{\frac{b_u}{\sigma_u} - \frac{\widehat{b}_u}{\sigma_u}\right\} du \mid \underline{\mathrm{F}}_s^S\right] + E[W_t - W_s \mid \underline{\mathrm{F}}_s^S].$$

Since, the first term of the right-hand side vanishes because of the properties of the conditional expectation and the second one vanishes because W_t is an $\underline{\mathrm{F}}_t$-Brownian motion and $\underline{\mathrm{F}}_t^S \subseteq \underline{\mathrm{F}}_t$ we get that I_t is a $(P, \underline{\mathrm{F}}_t^S)$-martingale. Finally, the thesis follows by the Lévy Theorem. □

Taking into account (2.6), Propositions 3.2 and 3.3, we are able to give the $(P, \underline{\mathrm{F}}_t^S)$-decompositions of the semimartingales Y_t and S_t

$$Y_t = Y_0 + \int_0^t \left\{\widehat{b}_s + \int_{\mathbb{R}} x\nu^p(dx)\right\}ds + \int_0^t \sigma_s dI_s + \int_0^t \int_{\mathbb{R}} x\, m^S(ds,dx) \tag{3.3}$$

$$\begin{aligned} S_t = S_0 &+ \int_0^t S_s\left\{\widehat{\mu}_s + \int_{\mathbb{R}} (e^x - 1)\nu_s^p(dx)\right\}ds \\ &+ \int_0^t S_s\sigma_s dI_s + \int_0^t \int_{\mathbb{R}} S_{s^-}(e^x - 1)m^S(ds,dx). \end{aligned} \tag{3.4}$$

Remark 3.4. Let us observe that by Proposition 3.2 and assumptions (2.9) we get that

$$\begin{aligned} \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} |e^x - 1|\nu_t^p(dx)ds\right] &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} |e^x - 1|\widehat{(\lambda_t\Phi_t)}(dx)dt\right] \\ = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} |e^x - 1|\lambda_t\Phi_t(dx)dt\right] &= \mathbb{E}\int_0^T \int_Z |\widetilde{K}(t;\zeta)|\nu(d\zeta) < \infty. \end{aligned}$$

By virtue of (3.4) the wealth process Z_t induced by the investment-consumption strategy (θ_t, C_t), satisfies

$$dZ_t = Z_{t^-}\left(\theta_t\widehat{\mu}_t dt - C_t\mu(dt) + \theta_t\sigma_t dI_t + \theta_t\int_{\mathbb{R}} (e^x - 1)m(dt,dx)\right).$$

Then the utility maximization problem defined in (2.4) can be now treated as a full information problem since all the processes involved are adapted to the observable flow $\{\underline{\mathrm{F}}_t^S\}_{t\in[0,T]}$.

The last part of this section is devoted to derive a martingale representation theorem for $(P, \underline{\mathrm{F}}_t^S)$-martingales. Let us observe that from Proposition 3.3 it follows that

$$\underline{\mathrm{F}}_t^I \vee \underline{\mathrm{F}}_t^m \subseteq \underline{\mathrm{F}}_t^S$$

where $\underline{F}_t^m = \sigma\{m((0,s]\times A); s\le t, A\in \underline{B}(\mathbb{R})\}$, and in general this inclusion holds in a strict sense. From now on we will assume a stronger condition than (3.2), that is

$$\mathbb{E}\int_0^T \left(\frac{b_s}{\sigma_s}\right)^2 ds < \infty \quad P\text{-a.s.} \tag{3.5}$$

and we consider the positive local martingale defined as the Doléans-Dade exponential of the $(P,\underline{F}_t)$-martingale $-\int_0^t \frac{b_s}{\sigma_s} dW_s$,

$$L_t = \operatorname{Exp}\left(-\int_0^t \frac{b_s}{\sigma_s} dW_s\right) = \exp\left\{-\int_0^t \frac{b_s}{\sigma_s} dW_s - \frac{1}{2}\int_0^t \left(\frac{b_s}{\sigma_s}\right)^2 ds\right\}.$$

We shall make the usual standing assumption

Assumption A: L_t is a $(P,\underline{F}_t)$-martingale, that is $\mathbb{E}[L_T]=1$.

Under this last assumption we can define on $\underline{F}_T$ a probability measure Q equivalent to P such that

$$\frac{dQ}{dP}|_{\underline{F}_T} = L_T. \tag{3.6}$$

By Girsanov theorem the process

$$\widetilde{W}_t := W_t + \int_0^t \frac{b_s}{\sigma_s} ds$$

is a $(Q,\underline{F}_t)$-Wiener process, moreover since by the definition of I_t the following equality is fulfilled

$$\widetilde{W}_t = I_t + \int_0^t \frac{\widehat{b}_s}{\sigma_s} ds \tag{3.7}$$

it turns out that the process $\widetilde{W}_t$ is $\underline{F}_t^S$-adapted, and as a consequence

$$\widehat{L}_t = \mathbb{E}[L_t|\underline{F}_t^S] = \frac{dQ}{dP}|_{\underline{F}_t^S} = \operatorname{Exp}\left(-\int_0^t \frac{\widehat{b}_s}{\sigma_s} dI_s\right). \tag{3.8}$$

Let us notice that, by Jensen's inequality and (3.5)

$$\mathbb{E}\int_0^T \frac{(\widehat{b}_t)^2}{\sigma_t^2} dt \le \mathbb{E}\int_0^T \frac{\widehat{b_t^2}}{\sigma_t^2} dt = \mathbb{E}\int_0^T \left(\frac{b_t}{\sigma_t}\right)^2 dt < \infty.$$

In order to derive a representation theorem for $(P,\underline{F}_t^S)$-martingales we need an additional assumption on σ_t. Since σ_t is $\underline{F}_t^S$-adapted and $\underline{F}_t^S = \underline{F}_t^Y$ there exists for each $t\in[0,T]$ a Borel measurable $H_t : D_{\mathbb{R}}[0,T] \to (0,+\infty)$ such that $\sigma_t = H_t(Y_{\cdot\wedge t})$ P-a.s. Here $D_{\mathbb{R}}[0,T]$ denotes the space of càdlàg $\mathbb{R}$-valued paths endowed with the Skorokhod metric, and we assume that H_t satisfies a global Lipschitz condition on $D_{\mathbb{R}}[0,T]$.

We summarize below all the conditions introduced in this section that we shall use from now on

Assumptions B: Assumption A, (2.2), (2.7), (2.9), (3.5) and assume σ_t to be $\underline{F}_t^S$-adapted and such that H_t satisfies a global Lipschitz condition on $D_{\mathbb{R}}[0,T]$.

Lemma 3.5. *Under Assumptions B, the filtration* $\underline{F}_t^S$ *coincides with the filtration generated by* $\widetilde{W}_t$ *and the jump measure* $m(dt,dx)$*, that is*

$$\underline{F}_t^S = \underline{F}_t^{\widetilde{W}} \vee \underline{F}_t^m.$$

Proof. Since $\widetilde{W}_t$ and $m(dt,dx)$ are $\underline{F}_t^S$-adapted we have that $\underline{F}_t^{\widetilde{W}} \vee \underline{F}_t^m \subseteq \underline{F}_t^S$. To prove the converse, let us observe that, taking into account (3.3) and (3.7), the process Y_t solves under the probability Q, defined by (3.6), the following equation driven by $\widetilde{W}_t$ and $m(dt,dx)$

$$dY_t = \sigma_t d\widetilde{W}_t + \int_{\mathbb{R}} x\ m(dt,dx). \tag{3.9}$$

Finally, since $\sigma_t = H_t(Y_{\cdot\wedge t})$ P-a.s. and $H_t : D_{\mathbb{R}}[0,T] \to (0,+\infty)$ satisfies a global Lipschitz condition on $D_{\mathbb{R}}[0,T]$, the stochastic functional differential equation (3.9) has a unique strong solution $\underline{F}_t^{\widetilde{W}} \vee \underline{F}_t^m$-adapted, hence $\underline{F}_t^S = \underline{F}_t^Y \subseteq \underline{F}_t^{\widetilde{W}} \vee \underline{F}_t^m$, and this concludes the proof. □

Finally we are able to prove the announced martingale representation theorem, which extend to a non-Markovian case Proposition 2.6 in [7].

Proposition 3.6. *Under Assumptions B, every* $(P,\underline{F}_t^S)$*-local-martingale* M_t *admits the decomposition*

$$M_t = M_0 + \int_0^t \int_{\mathbb{R}} \eta(t,x) m^S(ds,dx) + \int_0^t \psi_s dI_s$$

where $\eta(t,x)$ *a* $\underline{F}_t^S$*-predictable process and* ψ_t *a* $\underline{F}_t^S$*-adapted process such that*

$$\int_0^T \int_{\mathbb{R}} |\eta(t,x)| \nu_t^p(dx)dt < \infty, \qquad \int_0^T \psi_t^2 dt < \infty \quad P\text{-a.s.}$$

Proof. Let Q be the probability measure defined on $\underline{F}_T$ by (3.6). Notice that $\int_0^T \nu_t^p(\mathbb{R})dt < \infty$ P-a.s. since $\int_0^T \nu_t^p(\mathbb{R})dt = \int_0^T \widehat{\lambda}_t dt$ P-a.s. and, by (2.7), $\mathbb{E}\int_0^T \widehat{\lambda}_t dt = \mathbb{E}\int_0^T \lambda_t dt < \infty$. Hence, recalling that $\underline{F}_t^S = \underline{F}_t^{\widetilde{W}} \vee \underline{F}_t^m$ we can apply Remark 3.2 in [3] which states that for any $\widetilde{M}_t$, $(Q,\underline{F}_t^S)$- local-martingale, there exist two $\underline{F}_t^S$-adapted processes $\widetilde{\phi}(t,x)$ predictable and $\widetilde{\psi}_t$ such that

$$\widetilde{M}_t = \widetilde{M}_0 + \int_0^t \int_{\mathbb{R}} \widetilde{\eta}(s,x) m^S(ds,dx) + \int_0^t \widetilde{\psi}_s d\widetilde{W}_s$$

with

$$\int_0^T \int_{\mathbb{R}} |\widetilde{\eta}(t,x)| \nu_t^p(dx) < \infty, \qquad \int_0^T \widetilde{\psi}_t^2 dt < \infty \quad Q\text{-a.s.}$$

Let M_t be a $(P,\underline{F}_t^S)$-local martingale, by Kallianpur–Striebel formula $\widetilde{M}_t = M_t \widehat{L}_t^{-1}$ is a $(Q,\underline{F}_t^S)$-local martingale, where $\widehat{L}_t$ is defined in (3.8). We can write $M_t = \widetilde{M}_t \widehat{L}_t$

and by the product rule we deduce

$$dM_t = \widetilde{M}_{t^-} d\widehat{L}_t + \widehat{L}_{t^-} d\widetilde{M}_t + d\langle \widetilde{M}^c, \widehat{L^c}\rangle_t + d\Big(\sum_{s\le t} \Delta\widetilde{M}_s \Delta\widehat{L}_s\Big)$$
$$= \widehat{L}_t\Big(\widetilde{\psi}_t - \frac{\widehat{b}_t}{\sigma_t}\widetilde{M}_t\Big)dI_t + \int_{\mathbb{R}} \widehat{L}_{t^-}\widetilde{\eta}(t,x)m^S(dt,dx)$$

which gives the martingale representation for M_t with $\psi_t = \widehat{L}_t\widetilde{\psi}_t - \frac{\widehat{b}_t}{\sigma_t}M_t$ and $\eta(t,x) = \widehat{L}_{t^-}\widetilde{\eta}(t,x)$. □

4. The optimal investment-consumption problem

In this section we focus on formulating the $\underline{\mathrm{F}}^S_t$-optimal investment-consumption problem as a stochastic control problem. We begin by recalling that the wealth process Z_t satisfies

$$dZ_t = Z_{t^-}\Big(\theta_t \frac{dS_t}{S_{t^-}} - C_t\mu(dt)\Big) \tag{4.1}$$
$$= Z_{t^-}\Big\{\theta_t\widehat{\mu}_t dt - C_t\mu(dt) + \theta_t\sigma_t dI_t + \theta_t\int_{\mathbb{R}}(e^x-1)m(dt,dx)\Big\}.$$

The set of admissible strategies $\underline{\mathrm{A}}$ consists of all the pairs (θ_t, C_t), where θ_t is an $\mathbb{R}$-valued, $\underline{\mathrm{F}}^S_t$-predictable process and C_t a non-negative $\underline{\mathrm{F}}^S_t$-adapted process such that $C_T = 1$ and

$$\int_0^T \Big\{|\theta_t\widehat{\mu}_t - C_t| + \theta_t^2\sigma_t^2 + |\theta_t|\int_{\mathbb{R}}|e^x-1|\nu^p_t(dx)\Big\}dt < \infty \quad P\text{-a.s.} \tag{4.2}$$
$$\forall x\in\mathbb{R} \quad 1+\theta_t(e^x-1) > 0 \quad dP\times dt\text{-a.e.} \tag{4.3}$$

Proposition 4.1. *Let $\{\theta_t, C_t\}_{t\in[0,T]}$ be an admissible strategy. Then the wealth equation has a unique positive solution $Z^{\theta,C}_t$ given by*

$$Z^{\theta,C}_t = z_0 e^{\int_0^t\int_{\mathbb{R}}\log(1+\theta_s(e^x-1))m(ds,dx) + \int_0^t\theta_s\sigma_s dI_s + \int_0^t(\theta_s\widehat{\mu}_s - \frac{1}{2}\theta_s^2\sigma_s^2)ds - \int_0^t C_s\mu(ds)}. \tag{4.4}$$

Proof. Equation (4.1) can be written as $dZ_t = Z_{t^-}dM^{\theta,C}_t$, where from (4.2)

$$M^{\theta,C}_t := \int_0^t\{\theta_s\widehat{\mu}_s + \theta_s\int_{\mathbb{R}}(e^x-1)\nu^p_s(dx)\}ds - \int_0^t C_s\mu(ds)$$
$$+ \int_0^t\theta_s\sigma_s dI_s + \int_0^t\theta_s\int_{\mathbb{R}}(e^x-1)m^S(ds,dx)$$

is a $(P,\underline{\mathrm{F}}^S_t)$-semimartingale. By the Doléans-Dade Theorem we get that there exists a unique semimartingale $Z^{\theta,C}_t$ given by

$$Z^{\theta,C}_t = z_0\, e^{M^{\theta,C}_t - \frac{1}{2}<(M^{\theta,C})^c>_t}\Pi_{s\le t}(1+\Delta M^{\theta,C}_s)e^{-\Delta M^{\theta,C}_s}.$$

Moreover, $Z_t^{\theta,C} > 0$ if and only if $1 + \Delta M_s^{\theta,C} = 1 + \int_{\mathbb{R}} \theta_s(e^x - 1) m(\{s\}, dx) > 0$ $\forall s \leq t$, and this condition is implied by (4.3). Finally, by standard computation we derive expression (4.4). □

Remark 4.2. Let us observe that the pair $(\theta_t, C_t) = (0, 0)$, $\forall t \in [0, T)$, is an admissible strategy whose associated wealth is given by $Z_t^{0,0} = z_0$.

Remark 4.3. For any $(\theta, C) \in \underline{\mathrm{A}}$, the following inequality is fulfilled

$$\int_{\mathbb{R}} |(1 + \theta_t(e^x - 1))^\alpha - 1| \nu_t^p(dx) \leq \int_{\mathbb{R}} |\theta_t||e^x - 1| \nu_t^p(dx) < \infty \quad P\text{-a.s.} \tag{4.5}$$

As a consequence

$$\begin{aligned} M_t(\alpha) := & \int_0^t \int_{\mathbb{R}} \{[1 + \theta_s(e^x - 1)]^\alpha - 1\} m^S(ds, dx) + \int_0^t \alpha \theta_s \sigma_s dI_s \\ & + \int_0^t \alpha(\theta_s \widehat{\mu}_s ds - C_s \mu(ds)) + \int_0^t \int_{\mathbb{R}} \{[1 + \theta_s(e^x - 1)]^\alpha - 1\} \nu_s^p(dx) ds \end{aligned}$$

is a $(P, \underline{\mathrm{F}}_t^S)$-semimartingale and by (4.4), using standard computations, we have

$$Z_t^\alpha = z_0^\alpha e^{\frac{1}{2}\alpha(\alpha - 1) \int_0^t \theta_s^2 \sigma_s^2 ds} \operatorname{Exp}(M_t(\alpha)) \tag{4.6}$$

where we recall Exp denotes the Doléans-Dade exponential.

From now on we shall furthermore assume that

$$\sup_{(\theta,C) \in \underline{\mathrm{A}}} \mathbb{E}\left[\int_0^T (C_t Z_t)^\alpha \mu^0(dt)\right] < \infty.$$

As usual in stochastic control frame we introduce the associated value process which gives a dynamic extension of the optimization problem (2.4) to each initial time $t \in [0, T]$. For any $t \in [0, T]$, $(\bar{\theta}, \bar{C}) \in \underline{\mathrm{A}}$, let us consider the set of strategies coinciding with $(\bar{\theta}, \bar{C})$ until time t

$$\underline{\mathrm{A}}_t(\bar{\theta}, \bar{C}) := \{(\theta, C) \in \underline{\mathrm{A}} : (\theta_s, C_s) = (\bar{\theta}_s, \bar{C}_s), s \leq t\}$$

and define the value process as

$$V_t(\bar{\theta}, \bar{C}) = \operatorname{ess} \sup_{(\theta,C) \in \underline{\mathrm{A}}_t(\bar{\theta}, \bar{C})} \mathbb{E}\left[\int_t^T \frac{(C_s Z_s)^\alpha}{\alpha} \mu^0(ds) \mid \underline{\mathrm{F}}_t^S\right].$$

From the dynamic programming principle ([10]) $\forall (\bar{\theta}, \bar{C}) \in \underline{\mathrm{A}}$

$$V_t(\bar{\theta}, \bar{C}) + \int_0^t \frac{(\bar{C}_s Z_s^{\bar{\theta}, \bar{C}})^\alpha}{\alpha} \mu(ds)$$

is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale and $(\theta^*, C^*) \in \underline{\mathrm{A}}$ is optimal for problem (2.4) if and only if

$$V_t(\theta^*, C^*) + \int_0^t \frac{(C_s^* Z_s^{\theta^*, C^*})^\alpha}{\alpha} \mu(ds)$$

is a $(P, \underline{\mathrm{F}}_t^S)$-martingale. By Equation (4.4) we get that, for any $(\bar\theta, \bar C) \in \underline{\mathrm{A}}$

$$V_t(\bar\theta, \bar C) = \frac{(Z_t^{\bar\theta,\bar C})^\alpha}{\alpha} J_t$$

where the càdlàg process J_t does not depend on $(\bar\theta, \bar C)$ and is defined as

$$J_t = \text{ess} \sup_{(\theta,C)\in\underline{\mathrm{A}}_t} \mathbb{E}\Big[\int_t^T \frac{(C_s Z_s)^\alpha}{Z_t^\alpha} \mu^0(ds) \mid \underline{\mathrm{F}}_t^S\Big], \tag{4.7}$$

here $\underline{\mathrm{A}}_t$ denotes the set of admissible strategies over $[t, T]$. The process J_t is the so-called opportunity process and it is a suitable tool to derive results about the optimal investment-consumption strategy. In particular, the Bellman optimality principle can be stated as follows.

Proposition 4.4. *The following properties hold true:*

(i) $\{J_t\}_{t\in[0,T]}$ *is the smallest càdlàg* $\underline{F}_t^S$*-adapted process s.t.* $J_T = 1$ *and* $\forall(\theta, C) \in \underline{A}$, $(Z_t^{\theta,C})^\alpha J_t + \int_0^t (C_s Z_s^{\theta,C})^\alpha \mu(ds)$ *is a* $(P, \underline{F}_t^S)$*-supermartingale.*
(ii) $(\theta^*, C^*) \in \underline{A}$ *is an optimal investment-consumption strategy if and only if* $(Z_t^{\theta^*,C^*})^\alpha J_t + \int_0^t (C_s^* Z_s^{\theta^*,C^*})^\alpha \mu(ds)$ *is a* $(P, \underline{F}_t^S)$*-martingale.*

We give now some other properties of the process J_t.

Proposition 4.5. $\forall t \in [0, T]$, $J_t \geq 1$, P*-a.s. and* $\sup_{t\in[0,T]} \mathbb{E}[J_t] \leq J_0 < \infty$.

Proof. Since $(\theta_t, C_t) = (0, 0)$ $\forall t \in [0, T)$ is an admissible strategy, by (4.7) we get that $J_t \geq 1$ and, from Proposition 4.4, J_t is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale. Then $\mathbb{E}(J_t) \leq J_0$, where $J_0 = \frac{\alpha}{z_0^\alpha} \sup_{(\theta,C)\in\underline{\mathrm{A}}} \mathbb{E}\Big[\int_0^T U(C_t Z_t)\mu^0(dt)\Big] < \infty$. □

5. A BSDE approach

In this section, we address the problem of characterizing dynamically the opportunity process J_t. In all this section we make the class of hypotheses summarized in **Assumptions B**. First, let us fix some notations

- $\underline{\mathrm{S}}^p, 1 \leq p \leq +\infty$, denotes the space of $\mathbb{R}$-valued $\underline{\mathrm{F}}_t^S$-adapted stochastic processes $\{H_t\}_{t\in[0,T]}$ with $\|H\|_{\underline{\mathrm{S}}^p} = \| \sup_{t\in[0,T]} \mid H_t \mid \|_{L^p} < \infty$.
- $\underline{\mathrm{L}}^2_{\nu^p}$ ($\underline{\mathrm{L}}^1_{\nu^p,\mathrm{loc}}$) denotes the space of $\mathbb{R}$-valued $\underline{\mathrm{F}}_t^S$-predictable processes $\{U(t, x)\}_{t\in[0,T]}$ indexed by x with

$$\mathbb{E}\int_0^T\int_{\mathbb{R}} | U(t, x) |^2 \nu_t^p(dx)dt < \infty$$

$$\bigg(\text{resp.} \int_0^T\int_{\mathbb{R}} | U(t, x) |^2 \; \nu_t^p(dx)dt < \infty, P\text{-a.s.}\bigg).$$

- $\underline{\mathrm{L}}^2$ ($\underline{\mathrm{L}}^2_{\mathrm{loc}}$) denotes the space of $\mathbb{R}$-valued $\underline{\mathrm{F}}^S_t$-adapted processes $\{R_t\}_{t\in[0,T]}$ with

$$\mathbb{E}\int_0^T \mid R_t \mid^2 dt < \infty \quad \left(\text{resp. } \int_0^T \mid R_t \mid^2 dt < \infty \quad P\text{-a.s.}\right).$$

From Proposition 4.4, since $(\theta_t, C_t) = (0, 0) \in \underline{\mathrm{A}}$, the process $\{J_t\}_{t\in[0,T]}$, is a $(P, \underline{\mathrm{F}}^S_t)$-supermartingale and it admits a unique Doob–Meyer decomposition

$$J_t = m^J_t - A_t$$

with m^J_t a $(P, \underline{\mathrm{F}}^S_t)$-local martingale and A_t a nondecreasing $(P, \underline{\mathrm{F}}^S_t)$-predictable process with $A_0 = 0$. By the martingale representation result (Proposition 3.6) there exist $\Gamma(t, x) \in \underline{\mathrm{L}}^1_{\nu^p,\mathrm{loc}}$ and $R_t \in \underline{\mathrm{L}}^2_{\mathrm{loc}}$ such that

$$m^J_t = \int_0^t \int_{\mathbb{R}} \Gamma(s, x) m^S(ds, dx) + \int_0^t R_s dI_s. \tag{5.1}$$

Theorem 5.1. *If there exists an optimal strategy* $(\theta^*, C^*) \in \underline{A}$ *for the utility maximization problem* (2.4)*, the process* $\{J_t, \Gamma(t, x), R_t\}_{t\in[0,T]}$ *solves the following BSDE*

$$\begin{aligned} J_t = 1 &- \int_t^T \int_{\mathbb{R}} \Gamma(s, x) m^S(ds, dx) - \int_t^T R_s dI_s \\ &+ \int_t^T \operatorname{ess} \sup_{(\theta,C)\in\underline{A}} \Big\{ f(s, J, \Gamma, R, \theta) ds + (C^\alpha_s - \alpha C_s J_s)\mu(ds) \Big\} \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} f(t, y, u, r, \theta) = &\int_{\mathbb{R}} \big(y + u(t, x)\big)\big[\{1 + \theta_t(e^x - 1)\}^\alpha - 1\big]\nu^p_t(dx) \\ &+ \alpha\theta_t\sigma_t r + \left\{\alpha\theta_t\widehat{\mu}_t + \frac{\alpha(\alpha - 1)}{2}\sigma^2_t\theta^2_t\right\} y. \end{aligned} \tag{5.3}$$

Moreover, the optimal strategy (θ^*, C^*) *realizes the essential supremum in* (5.2) *and* $C^*_t = (J_t)^{\frac{1}{\alpha-1}}$*, P-a.s..*

Proof. For any $(\theta, C) \in \underline{\mathrm{A}}$ we apply the product rule to compute $(Z^{\theta,C}_t)^\alpha J_t$

$$\begin{aligned} (Z^{\theta,C}_t)^\alpha J_t = z^\alpha_0 J_0 &+ \int_0^t J_{s^-} d(Z^{\theta,C}_s)^\alpha + \int_0^t (Z^{\theta,C}_{s^-})^\alpha dJ_s \\ &+ \sum_{s\le t} \Delta(Z^{\theta,C}_s)^\alpha \Delta J_s + d\left\langle Z^{\theta,C}, J\right\rangle_t. \end{aligned} \tag{5.4}$$

Since by (5.1) and (4.6)

$$\begin{aligned} \Delta J_s &= \int_{\mathbb{R}} \Gamma(s, x) m(\{s\}, dx), \\ \Delta(Z^{\theta,C}_s)^\alpha &= (Z^{\theta,C}_{s^-})^\alpha \int_{\mathbb{R}} [\{1 + \theta_s(e^x - 1)\}^\alpha - 1] m(\{s\}, dx), \end{aligned}$$

we get that (5.4) becomes

$$\begin{aligned} d\big((Z_t^{\theta,C})^\alpha J_t\big) &= (Z_{t^-}^{\theta,C})^\alpha dm_t^J \\ &+ (Z_{t^-}^{\theta,C})^\alpha J_{t^-}\left\{\frac{\alpha(\alpha-1)}{2}\sigma_t^2\theta_t^2 dt + dM_t(\alpha)\right\} - (Z_{t^-}^{\theta,C})^\alpha dA_t \\ &+ \int_{\mathbb{R}} \big(J_{t^-} + \Gamma(t,x)\big)(Z_{t^-}^{\theta,C})^\alpha \big[\{1+\theta_t(e^x-1)\}^\alpha - 1\big] m(dt,dx) \\ &+ \alpha\theta_t\sigma_t R_t (Z_t^{\theta,C})^\alpha dt. \end{aligned}$$

Then, taking into account Equation (4.6)

$$\begin{aligned} &d\big((Z_t^{\theta,C})^\alpha J_t\big) \\ &= dM_t^J - (Z_{t^-}^{\theta,C})^\alpha \bigg[dA_t - \int_{\mathbb{R}}\big(J_t + \Gamma(t,x)\big)[\{1+\theta_t(e^x-1)\}^\alpha - 1]\nu_t^p(dx)dt \\ &\qquad - \frac{\alpha(\alpha-1)}{2}\sigma_t^2\theta_t^2 J_t dt - \alpha J_t(\theta_t\widehat{\mu}_t dt - C_t\mu(dt)) - \alpha\theta_t\sigma_t R_t dt\bigg] \end{aligned}$$

with

$$\begin{aligned} M_t^J = M_0^J &+ \int_0^t \int_{\mathbb{R}} (Z_{s^-}^{\theta,C})^\alpha \Gamma(s,x)\{1+\theta_s(e^x-1)\}^\alpha m^S(ds,dx) \\ &+ \int_0^t \int_{\mathbb{R}} (Z_{s^-}^{\theta,C})^\alpha J_{s^-}\big[\{1+\theta_s(e^x-1)\}^\alpha - 1\big] m^S(ds,dx) \\ &+ \alpha\int_0^t \theta_s\sigma_s (Z_s^{\theta,C})^\alpha J_s dI_s. \end{aligned} \tag{5.5}$$

Based on the above derivations, we obtain

$$\begin{aligned} &d\big((Z_t^{\theta,C})^\alpha J_t\big) + C_t^\alpha (Z_t^{\theta,C})^\alpha \mu(dt) \\ &\quad = dM_t^J - (Z_{t^-}^{\theta,C})^\alpha [dA_t - f(t,J,\Gamma,R,\theta)dt - (C_t^\alpha - \alpha C_t J_t)\mu(dt)] \end{aligned} \tag{5.6}$$

with $f(t,y,u,r,\theta)$ given by (5.3). Since, by the Bellman optimality principle (Proposition 4.4), $\forall(\theta,C)\in\underline{\mathrm{A}}$

$$(Z_t^{\theta,C})^\alpha J_t + \int_0^t C_r^\alpha (Z_r^{\theta,C})^\alpha \mu(dr) \tag{5.7}$$

is a $(P,\underline{\mathrm{F}}_t^S)$-supermartingale it follows that (5.5) is a $(P,\underline{\mathrm{F}}_t^S)$- local martingale and $dA_t - f(t,J,\Gamma,R,\theta)dt - (C_t^\alpha - \alpha C_t J_t)\mu(dt) \geq 0$, which in turn implies

$$dA_t \geq \operatorname*{ess\ sup}_{(\theta,C)\in\underline{\mathrm{A}}} [f(t,J,\Gamma,R,\theta)dt + (C_t^\alpha - \alpha C_t J_t)\mu(dt)].$$

On the other hand, again by the Bellman optimality principle, $(\theta^*,C^*)\in\underline{\mathrm{A}}$ is an optimal strategy if and only if the associated process given in (5.7) by replacing

(θ, C) by (θ^*, C^*) is a $(P, \underline{\mathrm{F}}_t^S)$-martingale. Thus if and only if

$$dA_t = f(t, J, \Gamma, R, \theta^*)dt + \{(C_t^*)^\alpha - \alpha C_t^* J_t\}\mu(dt)$$
$$= \operatorname{ess} \sup_{(\theta,C)\in\underline{\mathrm{A}}} [f(t, J, \Gamma, R, \theta) + (C_t^\alpha - \alpha C_t J_t)\mu(dt)].$$

To conclude the proof, let us notice that since the essential supremum of $C_t^\alpha - \alpha C_t J_t$ is attained in $(J_t)^{\frac{1}{\alpha-1}}$ this implies that $C_t^* = (J_t)^{\frac{1}{\alpha-1}}$, P-a.s. □

Remark 5.2. Conditions for existence of optimal strategies can be found in [14] for the case of terminal wealth and [15] for the case with consumption.

Remark 5.3. By Proposition 4.5, $\forall t \in [0, T]$ $J_t \geq 1$ P-a.s., thus if (θ^*, C^*) is an optimal investment-consumption strategy then $C_t^* = (J_t)^{\frac{1}{\alpha-1}}$, which in turn implies that $\forall t \in [0, T]$, $0 \leq C_t^* \leq 1$, P-a.s.

We now study the utility maximization problem defined in (2.4) over the subset $\underline{\mathrm{A}}^k \subset \underline{\mathrm{A}}$ of admissible strategies, $(\theta, C) \in \underline{\mathrm{A}}$, such that θ is uniformly bounded by k, with $k \geq 1$. In this frame the process J_t is replaced by

$$J_t^k = \operatorname{ess} \sup_{(\theta,C)\in\underline{\mathrm{A}}_t^k} \mathbb{E}\left[\int_t^T \frac{(C_s Z_s)^\alpha}{Z_t^\alpha} ds + \frac{Z_T^\alpha}{Z_t^\alpha} \mid \underline{\mathrm{F}}_t^S\right], \tag{5.8}$$

here $\underline{\mathrm{A}}_t^k$ denotes the set of admissible strategies $\underline{\mathrm{A}}^k$ over $[t, T]$. We introduce for any $(\theta, C) \in \underline{\mathrm{A}}$ the process

$$\xi_t^{\theta,C} := \mathbb{E}\left[\int_t^T \frac{(C_s Z_s)^\alpha}{Z_t^\alpha} ds + \frac{Z_T^\alpha}{Z_t^\alpha} \mid \underline{\mathrm{F}}_t^S\right].$$

Proposition 5.4. *Let us assume* $\forall t \in [0, T]$

$$|\widetilde{K}(t;\zeta)| \leq c, \quad \lambda_t = \nu(D_t) \leq c, \quad |b_t| \leq c, \quad \sigma_t \leq c \quad P\text{-a.s.} \tag{5.9}$$

with c positive constant. Then, for any $(\theta, C) \in \underline{A}^k$, $\xi_t^{\theta,C}$ *is uniformly bounded on t by a constant independent of* (θ, C).

Proof. Firstly, we observe that assumptions (5.9) imply

$$\widehat{\lambda_t \Phi_t}(\mathbb{R}) = \mathbb{E}[\lambda_t | \underline{\mathrm{F}}_t^S] \leq c \quad P\text{-a.s.}$$

and since $\nu_t^p(dx) = \widehat{\lambda_t \Phi_t}(dx)$, $dP \times dt$-a.e.

$$\int_{\mathbb{R}} |e^x - 1| \nu_t^p(dx) = \int_{\mathbb{R}} |e^x - 1| \widehat{\lambda_t \Phi_t}(dx) = \mathbb{E}\left[\int_Z \widetilde{K}(t;\zeta)\nu(d\zeta) | \underline{\mathrm{F}}_t^S\right] \leq c^2 \quad P\text{-a.s.} \tag{5.10}$$

$\forall (\theta, C) \in \underline{\mathrm{A}}^k$ let us consider the probability measure $P^{\theta,\alpha}$ defined on $\underline{\mathrm{F}}_T^S$ as

$$\frac{dP^{\theta,\alpha}}{dP}\Big|_{\underline{\mathrm{F}}_T^S} = L_T^\theta = \operatorname{Exp}(M^{\theta,\alpha})_T$$

with

$$M_t^{\theta,\alpha} := \int_0^t \alpha\theta_s\sigma_s dI_s + \int_0^t \int_{\mathbb{R}} [(1+\theta_s(e^x-1))^\alpha - 1] m^S(ds,dx)$$

By the Doléans-Dade exponential formula for all $t \le s \le T$

$$\frac{Z_s^\alpha}{Z_t^\alpha} = \frac{L_s^\theta}{L_t^\theta} \exp\left\{\alpha \int_t^s \left[\left(\theta_r\, \widehat{\mu}_r + \frac{\alpha-1}{2}\theta_r^2\sigma_r^2\right)dr - C_r\mu(dr)\right] + \alpha \int_t^s \int_Z [(1+\theta_r(e^x-1))^\alpha - 1]\nu_r^p(dx)dr\right\}$$

and, taking into account (4.5), we get

$$\mathbb{E}\left[\frac{Z_s^\alpha}{Z_t^\alpha}|\underline{\mathrm{F}}_t^S\right] \le \mathbb{E}^{\theta,\alpha}\left[\exp\left\{\alpha\int_t^s |\theta_r\widehat{\mu}_r|dr + \alpha\int_t^s\int_Z |\theta_r||e^x-1|\nu_r^p(dx)dr\right\}|\underline{\mathrm{F}}_t^S\right]$$

where $\mathbb{E}^{\theta,\alpha}$ denotes the expectation w.r.t. $P^{\theta,\alpha}$. Finally, by (5.9) and (5.10),

$$\mathbb{E}\left[\frac{Z_s^\alpha}{Z_t^\alpha}|\underline{\mathrm{F}}_t^S\right] \le e^{c(k)(s-t)} \quad P\text{-a.s.}$$

with $c(k)$ a suitable positive constant independent of (θ, C), which in turn implies that $\forall t \in [0,T]$

$$\xi_t^{\theta,C} = \mathbb{E}\left[\int_t^T \frac{(C_sZ_s)^\alpha}{Z_t^\alpha}\mu^0(ds) \mid \underline{\mathrm{F}}_t^S\right] \le (k+1)e^{c(k)T} \quad P\text{-a.s.} \qquad \square$$

Lemma 5.5. *Under* (5.9), $\forall(\theta,C) \in \underline{A}^k$*, the process* $\{\xi_t^{\theta,C}, \Gamma^{\theta,C}(t,x),\, R_t^{\theta,C}\}_{t\in[0,T]}$ *is the unique solution in* $\underline{S}^2 \times \underline{L}^2_{\nu^p} \times \underline{L}^2$ *to the BSDE*

$$\xi_t^{\theta,C} = 1 - \int_t^T \int_{\mathbb{R}} \Gamma^{\theta,C}(s,x)m^S(ds,dx) - \int_t^T R_s^{\theta,C}dI_s + \int_t^T \left[f(s,\xi^{\theta,C},\Gamma^{\theta,C},R^{\theta,C},\theta)ds + (C_s^\alpha - \alpha C_s\xi_s^{\theta,C})\mu(ds)\right] \tag{5.11}$$

with $f(s,y,u,r,\theta)$ *given in* (5.3).

Proof. As in [2] we consider the space $L(\mathbb{R},\nu^p)$ of measurable functions $u(x)$ with the topology of convergence in measure and define for $u, \widetilde{u} \in L(\mathbb{R},\nu^p)$,

$$\|u-\widetilde{u}\|_t = \left(\int_{\mathbb{R}} |u(x)-\widetilde{u}(x)|^2\nu_t^p(dx)\right)^{\frac{1}{2}}. \tag{5.12}$$

By (5.9), $\forall(\theta,C) \in \underline{\mathrm{A}}^k$, $u(x) \in L(\mathbb{R},\nu^p)$ and $y \in \mathbb{R}$ there exists a positive constant $d(k)$, independent of (θ, C), such that

$$\int_{\mathbb{R}} \big(y+u(x)\big)\big[\{1+\theta_t(e^x-1)\}^\alpha - 1\big]\nu_t^p(dx) \le |\theta_t|\int_{\mathbb{R}}\big\{|y|+|u(x)|\big\}|e^x-1|\nu_t^p(dx) \le d(k)\{|y|+\|u\|_t\} \quad P\text{-a.s.} \tag{5.13}$$

Observing that the generator of BSDE (5.11) is given by

$$g(t,y,u,r,\theta,C) = \int_{\mathbb{R}} \big(y+u(x)\big)\big[\{1+\theta_t(e^x-1)\}^\alpha - 1\big]\nu_t^p(dx) \qquad (5.14)$$

$$+\alpha\theta_t\sigma_t r + C_t^\alpha + \left\{\alpha(\theta_t\widehat{\mu}_t - C_t) + \frac{\alpha(\alpha-1)}{2}\sigma_t^2\theta_t^2\right\}y$$

in the case with intermediate consumption (by (5.14) without the part in C_t if there is no intermediate consumption), it follows that it is uniformly Lipschitz in (y,u,r). By classical results (see for instance Proposition 3.2 in [2]) there exists a unique solution $(\widetilde{\xi},\widetilde{\Gamma},\widetilde{\mathbb{R}}) \in \underline{\mathrm{S}}^2 \times \underline{\mathrm{L}}^2_{\nu^p} \times \underline{\mathrm{L}}^2$ to BSDE (5.11) and following the same computations as in the proof of Theorem 5.1 we get that

$$d\big((Z_t^{\theta,C})^\alpha\widetilde{\xi}_t\big) + C_t^\alpha (Z_t^{\theta,C})^\alpha \mu(dt) = dM_t^{\widetilde{\xi}}$$

where

$$dM_t^{\widetilde{\xi}} = \int_{\mathbb{R}} (Z_{t^-}^{\theta,C})^\alpha \widetilde{\Gamma}(t,x)\{1+\theta_t(e^x-1)\}^\alpha m^S(dt,dx)$$

$$+\int_{\mathbb{R}} (Z_{t^-}^{\theta,C})^\alpha \widetilde{\xi}_{t^-}\big[\{1+\theta_t(e^x-1)\}^\alpha - 1\big]m^S(dt,dx) + \alpha\theta_t\sigma_t(Z_t^{\theta,C})^\alpha\widetilde{\xi}_t dI_t.$$

Equation (4.6) and conditions (5.9) imply that $\forall(\theta,C) \in \underline{\mathrm{A}}^k$

$$\sup_{t\in[0,T]} (Z_t^{\theta,C})^\alpha \le e^{d(N_T+|I_T|+T)} \quad P\text{-a.s.}$$

where $N_t = m((0,t],\mathbb{R})$ and d is a suitable positive constant. Now, the intensity λ_t of the point process N_t is bounded by c, hence for any constant b, $\mathbb{E}[e^{bN_T}] \le e^{(e^b-1)c}$. This entails that $\forall(\theta,C) \in \underline{\mathrm{A}}^k$, $(Z_t^{\theta,C})^\alpha$ belongs to $\underline{\mathrm{S}}^p$, for any $p \ge 1$. Therefore $M_t^{\widetilde{\xi}}$ is a $(P,\underline{\mathrm{F}}_t^S)$-uniformly integrable martingale, whose t-time value is the $\underline{\mathrm{F}}_t^S$-conditional expectation of its terminal value, which implies that $\widetilde{\xi}_t = \xi_t^{\theta,C}$. □

Now we are in a position to solve the investment-consumption problem in the case of bounded strategies.

Proposition 5.6. *Under* (5.9)*, the following hold:*

- $(J_t^k, \Gamma^k(t,x), R_t^k) \in \underline{S}^2 \times \underline{L}^2_{\nu^p} \times \underline{L}^2$ *is the unique solution to BSDE*

$$J_t^k = 1 - \int_t^T \int_{\mathbb{R}} \Gamma^k(s,x) m^S(ds,dx) - \int_t^T R_s^k dI_s \qquad (5.15)$$

$$+\int_t^T \operatorname*{ess\,sup}_{(\theta,C)\in\underline{A}^k} [f(s,J^k,\Gamma^k,R^k,\theta)ds + (C_s^\alpha - \alpha C_s J_s^k)\mu(ds)]$$

  with $f(s,y,u,r,\theta)$ *given in* (5.3).
- *There exists an optimal strategy* $(\theta^k, C^k) \in \underline{A}^k$ *for* (5.8).
- *A strategy* $(\theta^k, C^k) \in \underline{A}^k$ *is optimal if and only if it attains the essential supremum in* (5.15).

Proof. To prove that J^k is a solution to BSDE (5.15) we follow the same lines of the proof of Theorem 5.1. From Proposition 4.4, since $(\theta_t, C_t) = (0, 0) \in \underline{\mathrm{A}}^k$, the process $\{J_t^k\}_{t\in[0,T]}$ is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale and it admits a unique Doob–Meyer decomposition

$$J_t^k = m_t^{J^k} - A_t^{J^k}$$

with $m_t^{J^k}$ a $(P, \underline{\mathrm{F}}_t^S)$-local martingale and $A_t^{J^k}$ a nondecreasing $(P, \underline{\mathrm{F}}_t^S)$-predictable process with $A_0^{J^k} = 0$. By the martingale representation result there exist $\Gamma^k(t, x) \in \underline{\mathrm{L}}^1_{\nu^p,\mathrm{loc}}$ and $R_t^k \in \underline{\mathrm{L}}^2_{\mathrm{loc}}$ such that

$$m_t^{J^k} = \int_0^t \int_{\mathbb{R}} \Gamma^k(s, x) m^S(ds, dx) + \int_0^t R_s^k dI_s.$$

Again by the Bellman optimality principle (Proposition 4.4)

$$\forall(\theta, C) \in \underline{\mathrm{A}}^k, \quad (Z_t^{\theta,C})^\alpha J_t^k + \int_0^t C_s^\alpha (Z_s^{\theta,C})^\alpha \mu(ds) \tag{5.16}$$

is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale. By applying the product rule and following the same computations as in the proof of Theorem 5.1 (see Equation (5.6)), we get that $\forall(\theta, C) \in \underline{\mathrm{A}}^k$

$$d\big((Z_t^{\theta,C})^\alpha J_t^k\big) + C_t^\alpha (Z_t^{\theta,C})^\alpha \mu(dt) = dM_t^{J^k} - (Z_{t^-}^{\theta,C})^\alpha \left\{ dA_t^{J^k} - dF(t, J, \Gamma, R, \theta, C) \right\}$$

where $dF(t, y, u, r, \theta, C) = f(t, y, u, r, \theta)dt + (C^\alpha - \alpha C y)\mu(dt)$ and M^{J^k} is a $(P, \underline{\mathrm{F}}_t^S)$-local martingale. As a consequence

$$dA_t^{J^k} \geq \operatorname*{ess\,sup}_{(\widetilde{\theta}, \widetilde{C}) \in \underline{\mathrm{A}}^k} dF(t, J^k, \Gamma^k, R^k, \widetilde{\theta}, \widetilde{C})$$

and $(\theta^k, C^k) \in \underline{\mathrm{A}}^k$ is an optimal strategy for the problem (5.8) if and only if

$$dA_t^{J^k} \geq \operatorname*{ess\,sup}_{(\widetilde{\theta}, \widetilde{C}) \in \underline{\mathrm{A}}^k} dF(t, J^k, \Gamma^k, R^k, \widetilde{\theta}, \widetilde{C}) = dF(t, J^k, \Gamma^k, R^k, \theta^k, C^k).$$

Notice that for any fixed $(t, \omega, J^k, \Gamma^k, R^k)$, $F(t, J^k, \Gamma^k, R^k, w_1, w_2)$ is continuous with respect to the pair $(w_1, w_2) \in [-k, k] \times [0, 1]$, since the following inequality holds

$$|\Gamma^k(t, x)||\{1 + w_1(e^x - 1)\}^\alpha - 1| \leq |\Gamma^k(t, x)||w_1||e^x - 1|$$

and, taking into account that $\Gamma^k(t, x)$ and $|e^x - 1| \in \underline{\mathrm{L}}^2_{\nu^p}$, we can apply Lebesgue's Theorem on dominated convergence. Therefore, by a predictable selection theorem we have that there exists $(\theta^k, C^k) \in \underline{\mathrm{A}}^k$ which realizes the essential supremum of $F(t, J^k, \Gamma^k, R^k, \theta, C)$ over $\underline{\mathrm{A}}^k$. Hence $(\theta^k, C^k) \in \underline{\mathrm{A}}^k$ is an optimal strategy for the problem (5.8) and (J^k, Γ^k, R^k) solves BSDE (5.15).

It remains to prove uniqueness of the solutions to BSDE (5.15). It is sufficient to consider the case with intermediate consumption. Notice that the generator of

BSDE (5.15) in such a case can be written as

$$\widetilde{g}(t,y,u,r) = \operatorname{ess}\sup_{(\theta,C)\in\underline{\mathrm{A}}^k} g(t,y,u,r,\theta,C),$$

where $g(t,y,u,r,\theta,C)$ is given in (5.14).

Since we have, $\forall (y,u,r), (\widetilde{y},\widetilde{u},\widetilde{r}) \in \mathbb{R}\times L(\mathbb{R},\nu^p)\times\mathbb{R}$

$$\widetilde{g}(t,y,u,r) \le \operatorname{ess}\sup_{(\theta,C)\in\underline{\mathrm{A}}^k} |g(t,y,u,r,\theta,C) - g(t,\widetilde{y},\widetilde{u},\widetilde{r},\theta,C)| + g(t,\widetilde{y},\widetilde{u},\widetilde{r})$$

by (5.9) and (5.13) we obtain

$$\widetilde{g}(t,y,u,r) - \widetilde{g}(t,\widetilde{y},\widetilde{u},\widetilde{r}) \le L\big(|y-\widetilde{y}| + \|u-\widetilde{u}\|_t + |r-\widetilde{r}|\big)$$

(see (5.12) for the definition of $\|u-\widetilde{u}\|_t$) and by symmetry $g(t,y,u,r)$ is uniformly Lipschitz in (y,u,r).

Applying classical results it follows that $(J^k,\Gamma^k,R^k)\in\underline{\mathrm{S}}^2\times\underline{\mathrm{L}}^2_{\nu^p}\times\underline{\mathrm{L}}^2$ is the unique solution to BSDE (5.15). □

We now come back to the non constrained case and we give a characterization of the value process J_t as the limit of the sequence $\{J^k_t\}_{k\ge1}$. Let us observe that this result does not require the existence of an optimal investment-strategy for the investment-consumption problem (2.4).

Proposition 5.7. *For any $t\in[0,T]$, we have that*

$$J_t = \lim_{k\to\infty} J^k_t \quad P\text{-a.s.}$$

Proof. We follow the same lines of the proof of Theorem 4.1 in [17]. Fix $t\in[0,T]$, since $\underline{\mathrm{A}}^k_t\subset\underline{\mathrm{A}}^{k+1}_t$ $\forall k$, we have that $\{J^k_t\}_{k\ge1}$ is an increasing sequence and we define the random variable

$$J'(t) = \lim_{k\to\infty} J^k_t \quad P\text{-a.s.}$$

Now observing that $\underline{\mathrm{A}}^k_t\subset\underline{\mathrm{A}}_t$ $\forall k$, we get that $J^k_t\le J_t$ and therefore $J'(t)\le J_t$ P-a.s.

Before proving the opposite inequality we first observe that by monotone convergence theorem for conditional expectation, since J^k_t are $\underline{\mathrm{F}}^S_t$-supermartingales $\forall k$, $J'(t)$ is a $\underline{\mathrm{F}}^S_t$-supermartingale, and we can consider its càdlàg version which we denote by J'_t. By the Doob–Meyer decomposition we can write

$$dJ'_t = \int_{\mathbb{R}} \Gamma'(t,x)m^S(dt,dx) + R'_t dI_t - dA'_t$$

with $\Gamma'(t,x)\in L^1_{\nu^p,\mathrm{loc}}$, $R'_t\in L^2_{\mathrm{loc}}$ and A'_t a nondecreasing $(P,\underline{\mathrm{F}}^S_t)$-predictable process. Following the same computations as in Theorem 5.1 (see Equation (5.6)) the product rules gives, $\forall(\theta,C)\in\underline{\mathrm{A}}$

$$\begin{aligned} &d\big((Z^{\theta,C}_t)^\alpha J'_t\big) + C^\alpha_t (Z^{\theta,C}_t)^\alpha \mu(dt) \\ &\quad = dM^{J'}_t - (Z^{\theta,C}_{t^-})^\alpha\left[dA'_t - f(t,J',\Gamma',R',\theta,C)dt - (C^\alpha - \alpha C J'_t)\mu(dt)\right] \end{aligned} \tag{5.17}$$

where $M_t^{J'}$ is a $(P, \underline{\mathrm{F}}_t^S)$-local martingale defined as in (5.5). We now want to prove that $\forall(\theta, C) \in \underline{\mathrm{A}}$

$$(Z_t^{\theta,C})^\alpha J_t' + \int_0^t C_s^\alpha (Z_s^{\theta,C})^\alpha \mu(ds)$$

is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingales. Let $\widetilde{\underline{\mathrm{A}}}$ be the set of uniformly bounded admissible strategies. Since $\forall(\theta, C) \in \widetilde{\underline{\mathrm{A}}}$ there exists $n \geq 1$ such that $(\theta, C) \in \underline{\mathrm{A}}^n$, we have that $(\theta, C) \in \underline{\mathrm{A}}^k$ $\forall k \geq n$, and taking into account Equation (5.16), that

$$(Z_t^{\theta,C})^\alpha J_t^k + \int_0^t C_s^\alpha (Z_s^{\theta,C})^\alpha \mu(ds)$$

is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale. By monotone convergence theorem we derive that

$$\forall(\theta, C) \in \widetilde{\underline{\mathrm{A}}}, \quad (Z_t^{\theta,C})^\alpha J_t' + \int_0^t C_s^\alpha (Z_s^{\theta,C})^\alpha \mu(ds)$$

is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale and by Equation (5.17) we have

$$\forall(\theta, C) \in \widetilde{\underline{\mathrm{A}}}, \quad dA_t' - [f(t, J', \Gamma', R', \theta, C)dt + (C^\alpha - \alpha C J_t')\mu(dt)] \geq 0.$$

Thus

$$dA_t' \geq \operatorname*{ess\,sup}_{(\theta,C)\in\widetilde{\underline{\mathrm{A}}}} [f(t, J', \Gamma', R', \theta, C)dt + (C^\alpha - \alpha C J_t')\mu(dt)].$$

Now, since $\forall(\theta, C) \in \underline{\mathrm{A}}$, $\theta_t = \lim_k \theta_t^k$ with $\theta_t^k = \theta_t 1\!\!1_{|\theta_t| \leq k} \in \widetilde{\underline{\mathrm{A}}}$, we get

$$\begin{aligned}
&\operatorname*{ess\,sup}_{(\theta,C)\in\widetilde{\underline{\mathrm{A}}}} [f(t, J', \Gamma', R', \theta, C)dt + (C^\alpha - \alpha C J_t')\mu(dt)] \\
&\quad = \operatorname*{ess\,sup}_{(\theta,C)\in\underline{\mathrm{A}}} [f(t, J', \Gamma', R', \theta, C)dt + (C^\alpha - \alpha C J_t')\mu(dt)]
\end{aligned}$$

hence $dA_t' \geq \operatorname{ess\,sup}_{(\theta,C)\in\underline{\mathrm{A}}}[f(t, J', \Gamma', R', \theta, C)dt + (C^\alpha - \alpha C J_t')\mu(dt)]$. Again by (5.17)

$$\forall(\theta, C) \in \underline{\mathrm{A}} \quad M_t^{J'} \geq (Z_t^{\theta,C})^\alpha J_t' + \int_0^t C_s^\alpha (Z_s^{\theta,C})^\alpha \mu(ds) \geq 0$$

is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale, since it is a non-negative local martingale. This implies that $(Z_t^{\theta,C})^\alpha J_t' + \int_0^t C_s^\alpha (Z_s^{\theta,C})^\alpha \mu(ds)$ is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale $\forall(\theta, C) \in \underline{\mathrm{A}}$. Finally, by Bellman principle $J_t' \geq J_t$ P-a.s. $\forall t \in [0, T]$ and this concludes the proof. □

We conclude this section by giving a verification result for the general case and providing an example which can be solved using this result.

Proposition 5.8. *Under the assumptions:*

- (i) *there exists a solution* $(\widetilde{J}_t, \widetilde{\Gamma}(t, x), \widetilde{R}_t)$ *to BSDE* (5.2) *such that* $M_t^{\widetilde{J}}$ *defined in* (5.5) *is a* $(P, \underline{F}_t^S)$*-local martingale*
- (ii) *there exists* $(\theta^*, C^*) \in \underline{A}$ *which attains the essential supremum in Equation* (5.2) *with* $(J_t, \Gamma(t, x), R_t)$ *replaced by* $(\widetilde{J}_t, \widetilde{\Gamma}(t, x), \widetilde{R}_t)$

(iii) $\xi_t^{\theta^*,C^*}$ *is the unique solution to BSDE* (5.11) *associated with* (θ^*, C^*).

Then $\widetilde{J}_t = J_t$ *P-a.s. for any* $t \in [0,T]$, *and* (θ^*, C^*) *is an optimal strategy.*

Proof. Let $(\widetilde{J}_t, \widetilde{\Gamma}(t,x), \widetilde{R}_t)$ be a solution to BSDE (5.2), by applying the product rule and following the same computations as in the proof of Theorem 5.1 (see Equation (5.6)), we get that $\forall(\theta, C) \in \underline{\mathrm{A}}$

$$d\big((Z_t^{\theta,C})^\alpha \widetilde{J}_t\big) + C_t^\alpha (Z_t^{\theta,C})^\alpha \mu(dt)$$
$$= dM_t^{\widetilde{J}} - (Z_{t^-}^{\theta,C})^\alpha \left\{ \operatorname*{ess\,sup}_{(\widetilde{\theta},\widetilde{C}) \in \underline{\mathrm{A}}} dF(t, \widetilde{J}, \widetilde{\Gamma}, \widetilde{R}, \widetilde{\theta}, \widetilde{C}) - dF(t, \widetilde{J}, \widetilde{\Gamma}, \widetilde{R}, \theta, C) \right\}$$

where $dF(t,y,u,r,\theta,C) = f(t,y,u,r,\theta)dt + (C^\alpha - \alpha C y)\mu(dt)$ and $M^{\widetilde{J}}$ is a $(P, \underline{\mathrm{F}}_t^S)$-local martingale such that $M_0^{\widetilde{J}} = z_0^\alpha J_0$. Notice now that

$$M_t^{\widetilde{J}} \geq (Z_t^{\theta,C})^\alpha \widetilde{J}_t + \int_0^t C_s^\alpha (Z_s^{\theta,C})^\alpha ds \geq 0$$

and since every non-negative local martingale is a supermartingale the process $M^{\widetilde{J}}$ is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale.

Thus $\forall(\theta, C) \in \underline{\mathrm{A}}$, $(Z_t^{\theta,C})^\alpha \widetilde{J}_t + \int_0^t C_s^\alpha (Z_s^{\theta,C})^\alpha ds$ is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale, and from Bellman principle it yields that $\widetilde{J}_t \geq J_t$ P-a.s. for any $t \in [0,T]$.

To prove the opposite inequality, let us observe that by (ii), $\widetilde{J}_t$ solves BSDE Equation (5.11) associated to $(\theta^*, C^*) \in \underline{\mathrm{A}}$, and by (iii), $\widetilde{J}_t = \xi_t^{\theta^*,C^*} \leq \operatorname{ess\,sup}_{(\widetilde{\theta},\widetilde{C}) \in \underline{\mathrm{A}}} \xi_t^{\theta,C} = J_t$, P-a.s. for any $t \in [0,T]$. Hence $\widetilde{J}_t = J_t$, P-a.s. and (θ^*, C^*) is an optimal strategy. □

Example. We now present a particular model where the risky asset follows a geometric jump-diffusion driven by two independent point processes whose intensities are not directly observed by investors. Let us assume

$$K(t;\zeta) = \sum_{j=1}^{2} K_j(t) \mathbb{1}_{D_j(t)}(\zeta)$$

with $K_1(t) > 0, K_2(t) < 0$ $(P, \underline{\mathrm{F}}_t^S)$-predictable processes and $D_j(t)$, $j = 1,2$, $(P, \underline{\mathrm{F}}_t^S)$-predictable processes taking values in $\underline{\mathrm{Z}}$. In this particular case the logreturn process solves

$$dY_t = b_t dt + \sigma_t dW_t + \sum_{j=1}^{2} K_j(t) N_t^j$$

with $N_t^j = N((0,t), D_j(t))$, $j = 1,2$, independent counting processes with $(P, \underline{\mathrm{F}}_t)$-predictable intensities given by $\lambda_t^j = \nu(D_j(t))$. In this model the agent can observe the processes $K_j(t)$ but not the intensities λ_t^j. As in the general case we assume σ_t

a strictly positive $\underline{\mathrm{F}}_t^S$-adapted process. The integer-valued random measure defined in (2.5) and its $(P, \underline{\mathrm{F}}_t^S)$-predictable dual projection are given by

$$m(dt, dx) = \sum_{j=1}^{2} \delta_{K_j(t)}(dx) N_t^j, \quad \nu^p(dt, dx) = \sum_{j=1}^{2} \delta_{K_j(t)}(dx) \widetilde{\lambda}_t^j dt$$

respectively, where $\widetilde{\lambda}_t^j$, $j = 1, 2$, denote the $(P, \underline{\mathrm{F}}_t^S)$-predictable intensities of N_t^j. From now on we assume $\forall t \in [0, T]$, P-a.s.

$$|b_t| \leq A_2,\ |\sigma_t| \leq A_2,\ A_1 \leq \lambda_t^j \leq A_2,\ A_1 \leq K_j(t) \leq A_2,\ j = 1, 2 \tag{5.18}$$

with A_i, $i = 1, 2$, positive constants. We consider the case with intermediate consumption. The BSDE (5.2) adapted to this particular model is given by

$$\begin{aligned} J_t = 1 - \sum_{j=1}^{2} \int_t^T \Gamma(s, j)(N_t^j - \widetilde{\lambda}_t^j dt) - \int_t^T R_s dI_s \\ + \int_t^T \operatorname*{ess\ sup}_{(\theta, C) \in \underline{\mathrm{A}}} h(s, J, \Gamma(1), \Gamma(2), R, \theta, C) ds \end{aligned} \tag{5.19}$$

where

$$\begin{aligned} h(t, y, u_1, u_2, r, \theta, C) = \sum_{j=1}^{2} \big(y + u_j\big) \big[\{1 + \theta_t(e^{K_j(t)} - 1)\}^{\alpha} - 1\big] \widetilde{\lambda}_t^j \\ + \alpha \theta_t \sigma_t r + C_t^{\alpha} + \left\{ \alpha(\theta_t \widehat{\mu}_t - C_t) + \frac{\alpha(\alpha - 1)}{2} \sigma_t^2 \theta_t^2 \right\} y. \end{aligned}$$

We begin by observing that by (4.3) any admissible trading strategy θ_t necessarily satisfies $\theta_t \in \left(-\frac{1}{e^{K_1(t)}-1}, \frac{1}{e^{K_2(t)}-1}\right)$ for a.e. t and assumption (5.18) yields that admissible investment strategies take values in a compact space. Following similar computations as those performed in the proofs of Lemma 5.5 and Proposition 5.6 we obtain that the generator of the BSDE (5.19) is uniformly Lipschitz in (y, u_1, u_2, r).

From classical results there exists a unique solution, $(\widetilde{J}_t, \widetilde{\Gamma}(t, 1), \widetilde{\Gamma}(t, 2), \widetilde{R}_t) \in \underline{\mathrm{S}}^2 \times \underline{\mathrm{L}}_1^2 \times \underline{\mathrm{L}}_2^2 \times \underline{\mathrm{L}}^2$, to the BSDE (5.19). Here $\underline{\mathrm{L}}_i^2$ denotes the space of $\mathbb{R}$-valued $\underline{\mathrm{F}}_t^S$-predictable processes $\{U(t)\}_{t \in [0,T]}$ such that $\mathbb{E} \int_0^T |U(t)|^2 \widetilde{\lambda}_t^i dt < \infty$.

Finally, we have that for any fixed (t, y, u_1, u_2, r) the essential supremum of $h(t, y, u_1, u_2, r, \theta, C)$ is achieved at $\big(\theta^*(t, y, u_1, u_2, r), C^* = y^{\frac{1}{\alpha-1}}\big)$ where $\theta^*(t, y, u_1, u_2, r)$ is such that $\frac{\partial h}{\partial \theta}|_{\theta = \theta^*} = 0$. Indeed, it is sufficient to observe that $\frac{\partial^2 h}{\partial^2 \theta} < 0$ P-a.s. and that

$$\lim_{\theta \to \frac{-1}{e^{K_1(t)}-1}} \frac{\partial h}{\partial \theta} = +\infty, \quad \lim_{\theta \to \frac{1}{e^{K_2(t)}-1}} \frac{\partial h}{\partial \theta} = -\infty \quad P\text{-a.s.}$$

Proposition 5.8 implies that $\widetilde{J}_t$ coincides with the opportunity process and the unique optimal investment-consumption strategy is given by

$$(\theta_t^*, C_t^*) = \left(\theta^*(t, \widetilde{J}_t, \widetilde{\Gamma}(t,1), \widetilde{\Gamma}(t,2), \widetilde{R}_t), (\widetilde{J}_t)^{\frac{1}{\alpha-1}}\right)$$

with $(\widetilde{J}_t, \widetilde{\Gamma}(t,1), \widetilde{\Gamma}(t,2), \widetilde{R}_t)$ unique solution of BSDE (5.19).

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