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Stochastic Predictor-Based Leader-Following Control with Input and Communication Delays

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ABSTRACT
We consider the leader-following control problem on connected directed graphs for stochastic linear agents in the presence of communications and actuator delays. We propose to use a distributed protocol for detecting the distance of agents from the leader and we show that by suitably using this information it is possible to solve efficiently the leader-following control problem by means of predictors, thus recovering results for the single-agent case. The proposed predictor and controller are easy to design and the delay bound that guarantees stability can be computed from closed-form expressions without resorting to LMIs.

KEYWORDS
Leader-following control; predictor feedback; Kalman-Bucy filtering; stochastic systems; time-delay systems.

1. Introduction

This work is devoted to the stochastic leader-following control problem over static networks in the presence of input and communicating delays by means of a stabilizing distributed predictor-based control law. In recent years the distributed consensus problem and the leader-following control problem for multi-agent systems communicating over a network have received considerable interest. The problem has been solved for linear agent dynamics with fixed or switching network topology – see for example Dong and Hu (2016); Hong, Hu, and Gao (2006); Z. Li, Wen, Duan, and Ren (2015); Ni and Cheng (2010); Olfati-Saber, Fax, and Murray (2007); W. Ren and Beard (2005); X. Wang, Li, and Shi (2013) – and extended to output-feedback consensus – Kim, Shim, and Seo (2010); Lv, Li, Duan, and Chen (2016); J. Wang, Lanzon, and Petersen (2015) –, measurement noise – Cheng, Hou, and Tan (2014); Cheng, Hou, Tan, and Wang (2011); Cheng, Wang, Ren, Hou, and Tan (2016); Hu and Feng (2010); T. Li and Zhang (2009, 2010) – and nonlinear systems – Battilotti and Califano (2019); Ding and Li (2016a); Z. Li, Ren, Liu, and Fu (2012); Yu, Chen, Cao, and Kurths (2009).

On the other hand, the feedback control problem of deterministic linear systems affected by input and output delays has been studied in recent years, starting from the well-known approaches based on finite spectrum assignment Manitius and Olbrot

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(1979); Olbrot (1978), up to reduction approaches Artstein (1982) in which the control is defined through an integral over the delay interval. These approaches have been extended beyond linear systems by Krstic (2010) and they can compensate, in principle, constant delays of any magnitude. Yet, they require a careful implementation and are typically quite computationally intensive. Recently, approaches based on an ODE-PDE cascade have been proposed to deal with large and/or time-varying delays (see for example Krstic and Smyshlyaev (2008); Sanz, Garcia, and Krstic (2019) and the references therein). Instead, for relatively small delay (that is, not arbitrarily large) another class of approaches based on control Lyapunov functionals that typically lead to control gains computed through LMIs was devised Fridman (2014). Approximate finite-dimensional predictors define a third class, that contains approaches like truncated predictor feedback Zhou, Lin, and Duan (2012) and closed-loop predictors Cacace, Conte, Germani, and Palombo (2016). Predictors in this class are typically simple to implement and have an upper bound for the maximum tolerable delay. However, it is possible to use a cascade of predictors to compensate for arbitrarily large delays. For example, in Cacace, Conte, Germani, and Palombo (2016) a cascade of predictors in the form of delay differential equations is shown to be able to compensate any delay and to generate the same trajectory as the (undelayed) optimal control. The case of stochastic systems with input and output delays is more recent and fewer approaches exist. Several works investigate the control problem for linear stochastic systems with input and state delays in the stochastic H_∞ framework Hinrichsen and Pritchard (1998), see for example H. Li, Chen, Zhou, and Lin (2009). In Gershon, Fridman, and Shaked (2017) a predictor-based control is applied for the first time to linear systems with state multiplicative noise. In Cacace, Conte, and Germani (2016) a closed-loop predictor containing the exponential of the closed-loop matrix is proven to be able to compensate small delays with guaranteed delay bounds for systems with additive noise. In Cacace, Germani, Manes, and Papi (2019) the same predictor is applied to the control problem of systems with nonlinear diffusions. All of these work deal with small delays. In Cacace, Germani, Manes, and Papi (2021) and Cacace, d'Angelo, and Germani (2021) the stability results of cascaded closed-loop predictors for deterministic systems are extended to stochastic linear systems (possibly time-varying) with additive noise and arbitrarily large time-varying input delay, under suitable hypothesis on the delay function.

In a distributed context the presence of delays cannot be ignored. Whereas actuator delays can be dealt with in a similar way as for single systems – C. Wang and Ding (2016); C. Wang, Zuo, Lin, and Ding (2017) –, communication delays represent a serious problem for the stability of consensus and are more difficult to manage. Many works in the recent literature study the consensus problem with measurement noise and communication delays of first or second-order multi-agent systems – Liu, Liu, Xie, and Zhang (2011); Olfati-Saber and Murray (2004); H. Ren and Deng (2017); Zhang, Li, Zhao, and Huo (2018); Zhu and Cheng (2010); Zong, Li, and Zhang (2019). In detail, Liu et al. (2011) solve the mean square consensus problem of single-integrator systems with measurement noise and communication delays under strongly connected and balanced digraphs, Olfati-Saber and Murray (2004) consider first-order integrators under switching topology and communication delay, H. Ren and Deng (2017) solve the consensus problem for a tracking problem on integrators, Zhang et al. (2018) consider stochastic single integrators with delay, Zong et al. (2019) consider a network of integrators with communication delays and additive, as well as multiplicative, noise and Zhu and Cheng (2010) provide an approach for second-order systems with multiple and time-varying delays. For general linear systems, Z. Wang, Zhang, Fu, and Zhang

(2017) solve the problem of delay through a predictor that involves an integral term. Since this approach results in significant complexity of the implementation, which is particularly critical for agents without large computing resources, Zhou and Lin (2014) propose a truncated predictor approach for the case of deterministic agents, that however requires that the open-loop dynamics is not exponentially unstable. The approach in C. Wang, Zuo, Qi, and Ding (2018) addresses general linear systems with possibly nonlinear disturbance and communications delay. This approach is based on a novel extended state predictor, and the solution is found through LMIs.

In this work we address general stochastic linear systems with actuator and communication delays over a digraph. The improvement with respect to existing proposals is that we design a computationally cheap controller that does not involve distributed terms, it is amenable of a direct and constructive design and provides a non conservative delay bound, which is easy to compute without resorting to LMIs.

We emphasize that the presence of communication delays poses significant challenges to the solution of distributed estimation and consensus problems, since, loosely speaking, the delay accumulates over network paths and the synchronization of information coming from neighboring nodes is lost. The approach pursued in this paper is based on the following elements:

- a distributed stratification algorithm to redirect the flow of information from the leader to the followers, thus avoiding network loops that are particularly detrimental in presence of delays;
- a new consensus filter to estimate the state of the leader, that exploits the modified structure of the communication graph;
- the application of existing results on finite-dimensional filters and controllers for single systems affected by input and output delays.

In this way, we are able to recover the same performance as in the case of single agents with input and output delays – Cacace, Conte, Germani, and Palombo (2016); Cacace, Germani, and Manes (2014) –, and the proposed protocol may also reduce redundant communications among agents, thus improving the overall efficiency of the solution of the leader-following problem. To the best of our knowledge, this is the first approach for the leader-following problem with fully stochastic linear agents of general type in presence of communication delays.

The paper is organized as follows: Section 2 proposes the stochastic problem setting and a few preliminaries. Section 3 introduces the algorithm that constraints the communication topology. Section 4 introduces a distributed state estimator for the modified graph structure based on Kalman-Bucy filtering in the case of no communication delay. Section 5 describes the predictor-based controller for stabilizing the delayed leader-tracking error estimate for the multi-agent system. Numerical simulations in Section 6 show that the proposed approach provides non conservative delay bounds and conclusions are drawn in Section 7.

Notation. The symbol $\otimes$ is the Kronecker product, $\sigma(M)$ is the spectrum of the square matrix M and $\mathrm{tr}(M)$ its trace. Moreover, $\mu(M) = \max_i \Re\{\sigma_i(M)\}$ is the spectral radius, and if $\mu(M) < 0$, M is said to be Hurwitz stable. $M > 0$ denotes a positive definite matrix, and the matrix I_n is the identity of size n. The operators $\mathrm{row}_i(A_i)$, $\mathrm{col}_i(A_i)$, and $\mathrm{diag}_i(A_i)$ yield respectively the horizontal, vertical and diagonal composition of matrices A_i. $\mathrm{st}(M)$ denotes the vertical stack of the columns of M. $\|x\|$ denotes the Euclidean norm for $x \in \mathbb{R}^n$ and $\|M\|$ the operator norm. On a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, $\mathbb{E}[\cdot]$ denotes the expectation, and $L^2(\Omega; \mathbb{R}^n)$ denotes the linear space of square integrable random vectors of $\mathbb{R}^n$ endowed with the norm $\|x\|_{L_2}^2 = \mathbb{E}\left[\|x\|^2\right]$. $L_T^2([0,T] \times \Omega; \mathbb{R}^n)$ is the linear space of $\mathbb{R}^n$-valued stochastic processes in $[0,T]$, such that $x \in L_T^2([0,T] \times \Omega; \mathbb{R}^n)$ if $|x|_T < \infty$ where $|x|_T^2 = \int_0^T \|x(\tau)\|_{L_2}^2 \, d\tau$. Time dependence is made explicit in the presence of delays.

2. Problem Formulation and Preliminaries

In a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}$ right-continuous with F_0 and containing all $\mathbb{P}$-null sets we consider a group of $N+1$ agents consisting of N followers denoted by indices $k \in \mathcal{I} := \{1, \ldots, N\}$ and one leader indexed by 0. Let $W_k^{(1)} \in \mathbb{R}^{d_1}$ and $W_k^{(2)} \in \mathbb{R}^{d_2}$ be $2N$ independent standard Wiener processes defined on the aforementioned probability space, and moreover for $i = \{1, 2\}$, the noise $W_k^{(i)}$ is independent of $W_j^{(i)}$ for $j \neq k$. Each agent is a linear stochastic system in the form

$$\mathrm{d}x_k(t) = (Ax_k(t) + Bu_k(t - \delta_u))\,\mathrm{d}t + F\,\mathrm{d}W_k^{(1)}(t) \tag{1}$$

$$\mathrm{d}y_k(t) = Cx_k(t)\,\mathrm{d}t + G\,\mathrm{d}W_k^{(2)}(t), \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the state of the system, $x_k(0) \in L^2(\Omega; \mathbb{R}^n)$ independent from $W_k^{(1)}$ and $W_k^{(2)}$, $y_k \in \mathbb{R}^q$ is the output, $u_k \in \mathbb{R}^p$ is the control input affected by a known actuator delay δ_u, $u_k(\tau) = \psi_k(\tau)$ for $\tau \in [-\delta_u, 0)$, $\psi_k \in L_{\delta_u}^2([-\delta_u, 0]; \mathbb{R}^p)$. The matrices A, B, C, F, G are of appropriate dimensions, with $FF^\top$ and $GG^\top$ positive definite. The information available to each agent is its own instantaneous output $\mathrm{d}y_k(t)$, the output of its neighbors, affected by a communication delay δ_o, and the estimates of the leader tracking error of its neighbors with the leader, that are also delayed (see Section 5.1). Moreover, we shall see that neighboring nodes also exchange the covariance matrices of the filter (as discussed in the subsequent Remarks 3 and 4).

In the leader-following framework it is reasonable to assume that the leader has no neighbors and the leader's control input is zero, Ding and Li (2016b); Z. Li et al. (2015); Ni and Cheng (2010); H. Ren and Deng (2017); C. Wang et al. (2018), that is,

$$\mathrm{d}x_0(t) = Ax_0(t)\mathrm{d}t + F\,\mathrm{d}W_0^{(1)}(t) \tag{3}$$

$$\mathrm{d}y_0(t) = Cx_0(t)\,\mathrm{d}t + G\,\mathrm{d}W_0^{(2)}(t). \tag{4}$$

The communication connections among agents are described by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_0, v_1, \ldots, v_N\}$ represents the agents and $\mathcal{E}$ represents the connections among the agents, $(v_k, v_j) \in \mathcal{E}$ represents the communication from the j-th agent to the k-th agent, but not vice versa. The associated adjacency matrix of $\mathcal{G}$ is denoted by $\mathcal{A} = [a_{kj}]_{(N+1)\times(N+1)} \in \mathbb{R}^{(N+1)\times(N+1)}$. If there is a connection from agent j to agent k, $a_{kj} = 1$; otherwise $a_{kj} = 0$. The Laplacian matrix $\mathcal{L} = [\ell_{kj}]_{(N+1)\times(N+1)}$ associated with $\mathcal{A}$ is defined by $\ell_{kk} = \sum_{j=0}^N a_{kj}$ and $\ell_{kj} = -a_{kj}$ when $k \neq j$. The set of neighbors of agent k is defined as $\mathcal{N}^k = \{j : \ \ell_{kj} = -1\}$. In order to track the leader's state each agent makes use of information from its neighbors, but this information is affected by a known communication delay δ_o.

Assumption 1. When $j \in \mathcal{N}^k$ (or equivalently $a_{kj} = 1$) agent j can send information to agent k, with a known communication delay δ_o.

Assumption 2. The couples (A, C) and (A, B) are observable and controllable, respectively.

Assumption 3. The communication topology $\mathcal{G}$ contains a directed acyclic graph (DAG) with the leader as root.

We define the leader-tracking error η_k of node k as

$$\eta_k := x_k - x_0. \tag{5}$$

The process $\{\eta_k\}$ satisfies

$$\mathrm{d}\eta_k(t) = (A\eta_k(t) + Bu_k(t - \delta_u))\,\mathrm{d}t + F\,\mathrm{d}\widetilde{W}_k^{(1)}(t), \tag{6}$$

where $\widetilde{W}_k^{(1)} = W_k^{(1)} - W_0^{(1)}$.

Definition 2.1. We say that the process $\{\xi(t)\}_{t \geq t_0}$ is

- *exponentially centered with rate* α: if there exists $\kappa > 0$ such that $\|\mathbb{E}[\xi(t)]\| \leq \kappa e^{-\alpha t}$ for any initial condition;
- *mean square bounded*: if there exists $\kappa > 0$ such that $\|\xi(t)\|_{L_2} < \kappa$ for any $t \geq 0$ and any initial condition.

The problem considered in this paper is the following.
Goal: Given α, design a local output-feedback control law and determine bounds $\delta_u^*(\alpha)$ and $\delta_o^*(\alpha)$ such that, when $\delta_u < \delta_u^*(\alpha)$ and $\delta_o < \delta_o^*(\alpha)$, $\forall k \in \mathcal{I}$ the leader tracking error η_k is exponentially centered with rate α and mean square bounded.

Remark 1. We note that the function $\|\eta_k(t)\|_{L_2}$ cannot converge to zero due to the noise processes in (1) and (3).

Similarly to C. Wang et al. (2018), let $\{z_k\}$ be the process

$$\mathrm{d}z_k = C\eta_k\,\mathrm{d}t + G\,\mathrm{d}W_k^{(2)}. \tag{7}$$

The fictitious output process (7) is not available. Nevertheless, it enjoys the following two properties that are immediate to prove (see C. Wang et al. (2018) for details).

Lemma 2.2. *For any* $k, j \in \mathcal{I}$, $\mathrm{d}z_k - \mathrm{d}z_j = \mathrm{d}y_k - \mathrm{d}y_j$.

Lemma 2.3. *For any* $k \in \mathcal{I}$,

$$\mathrm{d}\tilde{z}_k := \sum_{j=0}^{N} \ell_{kj}\mathrm{d}z_j = \sum_{j=0}^{N} a_{kj}(\mathrm{d}y_k - \mathrm{d}y_j). \tag{8}$$

Lemma 2.3 states that, even though $\mathrm{d}z_k$ is not available at agent $k \in \mathcal{I}$, the term $\mathrm{d}\tilde{z}_k$ can be computed from the difference $\mathrm{d}y_k - \mathrm{d}y_j$ between neighboring nodes. Finally, we note that $\eta_0 = 0$ and $z_0 = GW_0^{(2)}$.

3. Link Weighting for the Leader-Following Problem

In this section we introduce a simple distributed algorithm to restrict the communication among nodes so that the associated topology becomes a DAG with the leader as the root. This is obtained by replacing the Laplacian entries ℓ_{kj} with the new entries $\bar{\ell}_{kj}$ defined by the following (distributed) algorithm.

Algorithm 1.

Step 1) For each agent $k \in \{0, \ldots, N\}$, set

$$d_k = \begin{cases} 0 \text{ if } k = 0 \\ \infty \text{ otherwise} \end{cases} \tag{9}$$

Step 2) Each agent $k \in \{1, \ldots, N\}$, at each step, sends d_k to the neighbors and update the value of d_k as

$$d_k = \min_{j \in \mathcal{N}^k} \{d_j\} + 1 \tag{10}$$

$$\bar{\ell}_{kj} = \begin{cases} \ell_{kj} \text{ if } d_j < d_k \\ 0 \text{ otherwise} \end{cases} \tag{11}$$

$$\bar{\ell}_{kk} = - \sum_{j \in \mathcal{N}^k} \bar{\ell}_{kj}. \tag{12}$$

The idea behind the algorithm is the following. If $\bar{d}_k$ denotes the distance of the agent k from the leader, expressed as the minimum number of graph edges from the leader to agent k, the variable d_k converges to $\bar{d}_k$ in exactly $\bar{d}_k$ steps. All incoming information from agents that have the same distance or are farther away from the leader is suppressed by setting $\ell_{kj} = 0$ whenever $d_j \geq d_k$. Consequently, the resulting graph has no oriented cycles and the information reaching agent k originates from agents that are closer than k to the leader (or, equivalently, $\mathcal{N}^k$ contains only agents j that satisfy $d_j < d_k$). It is immediate to prove the following.

Proposition 3.1. *Under Assumption 3, the graph associated to the transformed Laplacian matrix $\tilde{\mathcal{L}}$ is a DAG with the leader as root.*

In practice, a distributed controller that implements Algorithm 1 at node k only needs to compute and send d_k to the neighbors and to disregard any communication from neighbors j such that $d_j \geq d_k$. Consequently, the communications arriving at the controller k originates from nodes j that are closer than k to the leader. In the sequel, we assume that the links in the communication topology have been changed according to Algorithm 1. The next propositions are a standard consequence of the fact that $\tilde{\mathcal{L}}$ is a DAG (see Kahn (1962)).

Proposition 3.2. *The transformed Laplacian matrix $\tilde{\mathcal{L}}$ has the following structure:*

$$\tilde{\mathcal{L}} = \begin{bmatrix} 0 & 0_{1\times N} \\ \mathcal{L}_2 & \mathcal{L}_1 \end{bmatrix} \tag{13}$$

where $\mathcal{L}_1 \in \mathbb{R}^{N\times N}$ and $\mathcal{L}_2 \in \mathbb{R}^{N\times 1}$. Furthermore, $\mathcal{L}_1$ is a non-singular matrix with *positive eigenvalues.*

Proposition 3.3. *Any DAG admits a topological ordering such that the associated Laplacian matrix is triangular.*

Remark 2. The idea behind Algorithm 1 is that the leader-following control problem differs from a the classic consensus problem in that the flow of information should be oriented from the leader to the followers, whereas in the consensus problem all the agents have the same role. We claim that the presence of loops in the topology has no effect for instantaneous communications, but it is actually detrimental in presence of delays. For example, the behavior of an agent having the root as neighbor can be influenced by other neighbors that are farther away from the leader and communicate outdated estimates of the leader's behavior. This makes the network much more sensitive to delays and the design of the local controllers more complicated and less robust.

4. Leader-Tracking Error Estimation in the Undelayed Case

The aim of this Section is to design an estimator of the leader-tracking error variable $\eta_k = x_k - x_0$ in the undelayed case, namely $\delta_u = \delta_o = 0$[1]. The estimate at node k is denoted with $\hat{\eta}_k$ and the corresponding estimation error is $\hat{\varepsilon}_k = \eta_k - \hat{\eta}_k$. Notice that, by definition, $\eta_0 \equiv 0$, thus we can set $\hat{\eta}_0 \equiv 0$ and $\hat{\varepsilon}_0 = 0$. Thus, in the case $\delta_u = 0$, the available measurement process $\{\tilde{z}_k\}$ can be expressed as

$$\mathrm{d}\tilde{z}_k = \sum_{j=0}^{N} \ell_{kj} \mathrm{d}z_j = \left(C_k \eta_k + \sum_{j \in \mathcal{N}^k} \ell_{kj} C \hat{\eta}_j + \sum_{j \in \mathcal{N}^k} \ell_{kj} C \hat{\varepsilon}_j \right) \mathrm{d}t + G\, \mathrm{d}\widetilde{W}_k^{(2)}, \tag{14}$$

where $C_k = \ell_{kk} C$, $\widetilde{W}_k^{(2)} = \sum_{j=0}^{N} \ell_{kj} W_j^{(2)}$. We remark again that in accordance to Lemma 2.3, the quantity $\tilde{z}_k$ is available to the agent k, and the reason to use it is that it involves the leader-tracking error η_k. For each $k \in \mathcal{I}$, the process $\{\widetilde{W}_k^{(2)}\}$ is a Wiener process, since it is a linear combination (with coefficients ℓ_{kj}) of the independent Wiener processes $\{W_j^{(2)}\}$, and it has covariance matrix given by

$$\mathbb{E}\left[\widetilde{W}_k^{(2)} \left(\widetilde{W}_k^{(2)} \right)^\top \right] = \sum_{j=0}^{N} \sum_{i=0}^{N} \ell_{kj} \ell_{ki} \mathbb{E}\left[W_j^{(2)} W_i^{(2)\top} \right] = \tilde{\ell}_k \, I_{d_2} \, t, \tag{15}$$

where $\tilde{\ell}_k := \sum_{j=0}^{N} \ell_{kj}^2 > 0$. From equations (6) and (14) it follows that η_k can be estimated by a Kalman-Bucy like filter, and, in order to derive an implementable filter, we approximate the measurement error process $\sum_{j \in \mathcal{N}^k} \ell_{kj} C \hat{\varepsilon}_j \mathrm{d}t + G\, \mathrm{d}\widetilde{W}_k^{(2)}$ with a white process having power spectral density

$$\tilde{R}_k = \sum_{j \in \mathcal{N}^k} \ell_{kj}^2 C P_j C^\top + \tilde{\ell}_k G G^\top, \tag{16}$$

[1]We note that the input delay does not play any role in the estimation phase.

where $P_j = \mathbb{E}[\hat{\varepsilon}_j \hat{\varepsilon}_j^\top]$ is the covariance of the estimation error of the neighbors $j \in \mathcal{N}^k$. Each agent k computes and sends to the agents that have it as neighbor: (i) its own estimate $\hat{\eta}_k$; (ii) its present measurement $\mathrm{d}y_k$; (iii) the present value of its covariance of the estimation error P_k. These variable are computed according to the following filtering algorithm

$$\mathrm{d}\hat{\eta}_k = (A\hat{\eta}_k + Bu_k)\,\mathrm{d}t + L_k\Big(\mathrm{d}\tilde{z}_k - C\sum_{j=0}^{N} \ell_{kj}\hat{\eta}_j\,\mathrm{d}t\Big) \tag{17}$$

$$\mathrm{d}\tilde{z}_k = \sum_{j=0}^{N} a_{kj}(\mathrm{d}y_k - \mathrm{d}y_j) \tag{18}$$

$$L_k = P_k C_k^\top \tilde{R}_k^{-1} \tag{19}$$

$$\dot{P}_k = AP_k + P_k A^\top + Q - L_k C_k P_k, \tag{20}$$

with $\hat{\eta}_k(0) = \mathbb{E}[\eta_k(0)]$, $P_k(0) = \mathbb{E}[\eta_k(0)\eta_k^\top(0)]$, $Q = 2FF^\top$. It is easy to see that the proposed filter (17) can be also written as

$$\mathrm{d}\hat{\eta}_k = (A\hat{\eta}_k + Bu_k)\,\mathrm{d}t + L_k \sum_{j=0}^{N} \ell_{kj}\left(\mathrm{d}z_j - C\hat{\eta}_j\,\mathrm{d}t\right), \tag{21}$$

where, as remarked in Lemma 2.3, even though the fictitious output $\mathrm{d}z_j$ is not available, the quantity $\sum_{j=0}^{N} \ell_{kj}\mathrm{d}z_j$ can be computed. Thus, the version (21) of the filter can also be implemented. We note that the implementation of the proposed filter (17)–(21) requires to exchange the measurements $\mathrm{d}y_j$, the estimates $\hat{\eta}_j$ and the error covariances P_j among neighboring nodes.

By defining $L = \mathrm{diag}(L_1, \dots, L_N)$, $\hat{\eta} = \mathrm{col}(\hat{\eta}_1, \dots, \hat{\eta}_N)$, $u = \mathrm{col}(u_1, \dots, u_N)$, $z = \mathrm{col}(z_1, \dots, z_N)$, from (21) we can write the aggregate system of the filters of all followers as

$$\mathrm{d}\hat{\eta} = [(I_N \otimes A) - L\,(\mathcal{L}_1 \otimes C)]\,\hat{\eta}\,\mathrm{d}t + (I_N \otimes B)\,u\,\mathrm{d}t + L(\mathcal{L}_2 \otimes I_n)\,\mathrm{d}z. \tag{22}$$

In the Theorem below, we prove internal stability of the stationary solution of the global filter introduced above (which exists by Khasminskii (2011)), uniform boundedness of the estimation error, and consistency of the estimator.

Theorem 4.1. *Under the Assumptions 1, 2, 3 and with the communications topology induced by Algorithm 1, when $\delta_o = 0$ the filters (17)–(20) for each follower $k \in \mathcal{I}$ applied to the system (6) with the measurements (18) are asymptotically stationary and internally stable. Moreover, the steady state estimation error $\hat{\bar{\varepsilon}}_k$ satisfies $\|\hat{\bar{\varepsilon}}_k\|_{L_2}^2 \leq \mathrm{tr}\{\bar{P}_k\}$, $\forall k \in \mathcal{I}$, with $\bar{P}_k$ steady state value of P_k.*

The proof is given in Appendix B.

Remark 3. Note that the computation of the filter parameters can be performed by each agent in a distributed fashion. In fact, the filter (17)–(20) of agent k relies on: $\mathrm{d}\tilde{z}_k$, which can be computed from the difference $\mathrm{d}y_k - \mathrm{d}y_j$ among neighboring nodes; the delayed estimates $\hat{\eta}_j(t - \delta_o)$ associated with the neighboring nodes; and the gain L_k, which depends on $\tilde{R}_k$ computed through the P_j of neighboring nodes.

Since the considered system is time-invariant, a steady state value for the P_j's is eventually reached. Thus, each node exchanges the covariance matrix P_j with its neighbours up to the moment when such a steady state value is achieved.

When $\delta_u = \delta_o = 0$ the output feedback that makes the leader-tracking error variable mean square bounded can be easily designed by using $\hat{\eta}_k$.

Corollary 4.2. *If $\delta_u = 0$ and $\delta_o = 0$, given any $\alpha > 0$ and gain K such that $\mu(\widetilde{A}) < -\alpha$, $\widetilde{A} := A - BK$, with the control law, $k \in \mathcal{I}$,*

$$u_k = -K\hat{\eta}_k, \tag{23}$$

with $\hat{\eta}_k$ defined in (17)*, then the consensus processes η_k defined in* (6) *are exponentially centered with rate α and mean square bounded.*

5. Leader-Following Control with Delays

5.1. Leader-tracking error estimate with communication delays

Let us now consider the output feedback leader-following control problem for a stochastic multi-agent system subject to input and communication delays in the form (1)–(2). In presence of a communications delay δ_o each agent receives from its neighbors j, for $t \geq \delta_o$,

$$\mathrm{d}\bar{y}_j(t) = Cx_j(t-\delta_o)\mathrm{d}t + G\mathrm{d}W_j^{(2)}(t-\delta_o). \tag{24}$$

In this equation, the differential $\mathrm{d}W_j^{(2)}(t-\delta_o) \doteq \mathrm{d}\left(\int_0^{t-\delta_o} \mathrm{d}W_j^{(2)}(s)\right)$, $t \geq \delta_o$. With the initial condition $\bar{y}_j(\delta_o) = y(0)$, it is easy to check that, for $t \geq \delta_o$,

$$\begin{aligned} \bar{y}_j(t) =& \bar{y}_j(\delta_o) + \int_{\delta_o}^{t} Cx(s-\delta_o)\,\mathrm{d}s + \int_{\delta_o}^{t} G\mathrm{d}W_j^{(2)}(s-\delta_o) \\ =& y(0) + \int_0^{t-\delta_o} Cx(\tau)\,\mathrm{d}\tau + \int_0^{t-\delta_o} G\mathrm{d}W_j^{(2)}(\tau) = y_j(t-\delta_o). \end{aligned} \tag{25}$$

We shall solve the problem by designing: (i) a filter to estimate the leader-tracking error variable $\eta_k(t)$ from $\mathrm{d}\bar{y}_j(t)$; (ii) a predictor of $\eta_k(t+\delta_u)$ from an estimate of $\eta_k(t)$. Clearly, an alternative solution is to use the filter in point (i) to estimate $\eta_k(t+\delta_u)$ from $\mathrm{d}\bar{y}_j(t)$, but the solution with two modules is more flexible and works for a larger total delay.

Let us set $\bar{A}_k = A - \bar{L}_k C_k$, where $\bar{L}_k$ is the stationary value of the filter gain L_k, then consider the following estimation algorithm, for $t \geq \delta_o$,

$$\begin{aligned} \mathrm{d}\hat{\eta}_k(t) =& A\hat{\eta}_k(t)\mathrm{d}t + Bu_k(t-\delta_u)\mathrm{d}t \\ &+ e^{\bar{A}_k\delta_o}\bar{L}_k\left(\mathrm{d}\tilde{z}_k(t-\delta_o) - C\sum_{j=0}^{N}\ell_{kj}\hat{\eta}_j(t-\delta_o)\,\mathrm{d}t\right) \end{aligned} \tag{26}$$

$$\mathrm{d}\tilde{z}_k(t-\delta_o) = \sum_{j=0}^{N} a_{kj}(\mathrm{d}\bar{y}_k(t) - \mathrm{d}\bar{y}_j(t)) \tag{27}$$

$$\bar{L}_k = \bar{P}_k C_k^\top \tilde{R}_k^{-1} \tag{28}$$

$$\bar{A}_k = A - \bar{L}_k C_k \tag{29}$$

$$\tilde{R}_k = \sum_{\substack{j=1 \\ j\neq k}}^{N} \ell_{kj}^2 C\bar{P}_j C^\top + \tilde{\ell}_k G G^\top \tag{30}$$

$$0 = A\bar{P}_k + \bar{P}_k A^\top + Q - \bar{L}_k C_k \bar{P}_k, \tag{31}$$

with $\hat{\eta}_k(\tau) = 0$ for $\tau \leq \delta_o$, $\tilde{z}_k(0) = 0$.

Theorem 5.1. *Under Assumptions 1, 2 and 3, and with the communications topology induced by Algorithm 1, if $\alpha_k \in (0, -\mu(\bar{A}_k))$, $k \in \mathcal{I}$, are such that*

$$\gamma_o(\alpha_k, \delta_o) := \int_0^{\delta_o} \|C_k e^{\bar{A}_k\theta}\bar{L}_k\| e^{\alpha_k\theta}\,\mathrm{d}\theta < 1, \tag{32}$$

then the filters (26)–(31) *applied to the system* (6) *with the measurements* (27) *are asymptotically stationary and internally stable. Moreover, $\forall k \in \mathcal{I}$ the estimation error $\hat{\varepsilon}_k$ is exponentially centered with rate α_k and mean square bounded.*
As a consequence, with the control law, $k \in \mathcal{I}$, (23)*, with $\hat{\eta}_k$ defined in* (26)*, then the consensus processes η_k defined in* (6) *are exponentially centered with rate α and mean square bounded.*

The proof, reported in Appendix C, is inspired to the one used in Cacace, Conte, d'Angelo, Germani, and Palombo (2022) for the case of single systems with measurement delays.

We stress that the delay bound for δ_o in (32) is sufficient, that is, it can be less than the actual maximum allowable delay for the stability of the filters.

5.2. Leader-following control with input delays

The predictor-based control is a feedback from the prediction $\theta_k(t)$ of $\eta_k(t+\delta_u)$ from the estimate $\hat{\eta}_k(t)$. The prediction θ_k is thus $\delta_o + \delta_u$ time units ahead of the available information.

Theorem 5.2. *If the hypotheses of Theorem 5.1 are satisfied by some choice $\{\alpha_k\}$, given any set of gains K_k such that $\widetilde{A}_k := A - BK_k$ with $\mu(\widetilde{A}_k) < -\alpha_k$, consider the* *control law for* $t \geq 0$

$$u_k(t) = -K_k\theta_k(t) \tag{33}$$
$$\dot{\theta}_k(t) = \widetilde{A}_k\theta_k(t) + BK_k e^{\widetilde{A}_k\delta_u}\left(\hat{\eta}_k(t) - \theta_k(t-\delta_u)\right), \quad t \geq \delta_o \tag{34}$$
$$\theta_k(t) = 0, \quad t \in [-\delta_u,\, \delta_o] \tag{35}$$

with $\hat{\eta}_k(t)$ *defined by* (26)–(31)*. If* $\forall k \in \mathcal{I}$

$$\gamma_u(\alpha_k, \delta_u) := \int_0^{\delta_u} \|K_k e^{\widetilde{A}_k\theta} B\| e^{\alpha_k\theta}\, \mathrm{d}\theta < 1, \tag{36}$$

then under the control law (33)–(34)*,* $\forall k \in \mathcal{I}$ *the processes* η_k *defined in* (6) *are exponentially centered with rate* α_k *and mean square bounded.*

Remark 4. Note that a distributed computation of the control parameters by each agent is possible. In fact, the control law (33) of agent k relies on K_k, which can be computed by the agent with standard eigenvalues assignment such that the condition $\mu(\widetilde{A}_k) < -\alpha_k$ is satisfied, and on the process θ_k which depends on the known delay and on the estimate $\hat{\eta}_k$ provided by the filter (the filter parameters are also computed in a distributed fashion, see Remark 3).

Proof of Theorem 5.2. The delay differential equation (34) defines a stochastic process for all $t \geq -\delta_u$ (see Theorem 1.5 of Khasminskii (2011)). Let us define the prediction error v_k,

$$v_k(t) = \eta_k(t) - \theta_k(t-\delta_u), \quad t \geq 0. \tag{37}$$

When $u_k(t) = -K_k\theta_k(t)$, by replacing (34) and (6) in (37) we obtain the following stochastic differential equation for $t \geq \max\{2\delta_u, \delta_o\}$:

$$\begin{aligned}
\mathrm{d}v_k(t) =& \mathrm{d}\eta_k(t) - \dot{\theta}_k(t-\delta_u)\mathrm{d}t \\
=& A\eta_k(t)\mathrm{d}t - BK_k\theta_k(t-\delta_u)\mathrm{d}t + F\,\mathrm{d}\widetilde{W}_k^{(1)}(t) \\
& - (A - BK_k)\theta_k(t-\delta_u)\mathrm{d}t - BK_k e^{\widetilde{A}_k\delta_u}\left(\hat{\eta}_k(t-\delta_u) - \theta_k(t-2\delta_u)\right)\mathrm{d}t \\
=& Av_k(t)\mathrm{d}t + F\,\mathrm{d}\widetilde{W}_k^{(1)}(t) - BK_k e^{\widetilde{A}_k\delta_u} \\
& \cdot \left(\hat{\eta}_k(t-\delta_u) - \eta_k(t-\delta_u) + \eta_k(t-\delta_u) - \theta_k(t-2\delta_u)\right)\mathrm{d}t \\
=& Av_k(t)\mathrm{d}t - BK_k e^{\widetilde{A}_k\delta_u}\left(v_k(t-\delta_u) - \hat{\varepsilon}_k(t-\delta_u)\right)\mathrm{d}t + F\,\mathrm{d}\widetilde{W}_k^{(1)}(t).
\end{aligned} \tag{38}$$

Since $\hat{\varepsilon}_k$ is exponentially centered with rate α and mean square bounded, Theorem 1 in Cacace, Germani, et al. (2021) guarantees in the hypothesis (36) that the same holds for v_k. But since

$$\mathrm{d}\eta_k(t) = \widetilde{A}_k\eta_k(t)\mathrm{d}t + v_k(t)\mathrm{d}t + \mathrm{d}\widetilde{W}_k^{(1)}(t), \tag{39}$$

the thesis follows. □

Notice that $\{\alpha_k\}$ is not used in the implementation of the filter and controller, but it is the lower bound for the convergence rate to 0 of the expected value of $\hat{\varepsilon}_k$ and η_k and it is not part of the design. Theorems 5.1–5.2 state that $\forall k$ α_k cannot be larger than the spectral abscissa of $\bar{A}_k$ and $\widetilde{A}_k$, which is intuitive, and that they also depend on the input and output delay via (32), (36). The gain K_k is computed through standard eigenvalue assignment algorithms. The dynamics of the leader-tracking error variable is both exponentially centered and mean square bounded when $\gamma_u(0, \delta_u) < 1$. The function γ_u is monotonically increasing with δ_u, thus a delay bound is guaranteed to exist (it may be infinite if $\gamma(0, \infty) < 1$) and it is trivial to compute. In general, this bound increases with the norm of K_k, thus a slower rate of convergence to 0 of the mean of η_k corresponds to a larger delay. Conversely, given δ_u, it is rather easy to check whether there exist gains K_k that make the closed-loop dynamics of the leader-tracking error variable exponentially centered and mean square bounded. For example, in the scalar input case, K_k is uniquely determined by the choice of the eigenvalues of $\widetilde{A}_k$, thus the largest delay bound is obtained with the smallest gain K_k that moves all the eigenvalues of $\widetilde{A}_k$ in the left-hand part of the complex plane. Notice that the condition $\gamma_u(0, \delta_u) < 1$ is only sufficient, however for scalar inputs it is sometimes a necessary condition Cacace et al. (2014). Finally, when the total delay exceeds the largest delay bound, it is possible to resort to a chain of predictors (see Cacace, Germani, et al. (2021)).

6. Numerical Simulations

6.1. Interconnected kinematic planar systems

In this Section, an example is used to demonstrate the potential applications of the proposed approach. Suppose a network of five kinematic planar systems are subject to the connection topology with Laplacian $\mathcal{L}$ specified by the DAG $\mathcal{G}_1$ shown in Fig. 1.

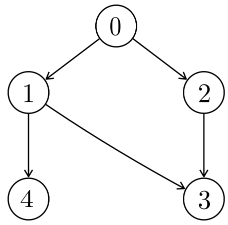

Figure 1. Communication digraph $\mathcal{G}_1$.

The communication graph in Fig. 1 shows that only the followers indexed by 1 and 2 can get access to the leader and the communication topology contains a directed spanning tree. The dynamics of the i-th agent is described by (1)–(2), with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \ F = \begin{pmatrix} 0 & 0 \\ 0.02 & 0 \\ 0 & 0 \\ 0 & 0.02 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ G = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.05 \end{pmatrix}, \tag{40}$$

with $x_k(t) \in \mathbb{R}^4$, $u_k(t) \in \mathbb{R}^2$, $y_k(t) \in \mathbb{R}^2$. The noise processes $W_k^{(1)}(t), W_k^{(2)} \in \mathbb{R}^2$, with $k = 0, 1, \dots, 4$, are assumed to be independent standard Wiener processes. The initial states of the agents are chosen, in a uniformly random fashion, in the interval $[0.5, 8.5]$ for all four components of each agent's state, and $u(\tau) = \begin{pmatrix} 0 & 0 \end{pmatrix}^T, \forall \tau \in [-\delta_u, 0]$ and $y(\tau) = \begin{pmatrix} 0 & 0 \end{pmatrix}^T, \forall \tau \in [-\delta_o, 0]$. A feedback gain K moving the eigenvalues of each agent's dynamics to $\{-2, -2.2, -2.4, -2.6\}$ is found to be

$$K = \begin{pmatrix} 5.63 & 4.76 & -0.28 & -0.12 \\ -0.26 & -0.11 & 4.89 & 4.44 \end{pmatrix}. \tag{41}$$

With this gain, the total delay bound of $\delta_o + \delta_u$ to have exponentially centered agent processes obtained from the condition $\gamma(0, \delta_{\max}) = 1$ is $\delta_{\max} = 0.41s$. Fig. 2 shows the empirical value of the mean square leader-tracking error process $\mathbb{E}\left[\|\eta_k\|^2\right]$, for $k = 1, \dots, 4$, averaged over 100 simulations with constant delays $\delta_u = 0.2s$ and $\delta_o = 0.15s$. Even though $\delta = \delta_u + \delta_o = 0.35$ is close to the delay bound $\delta_{\max}$, the values of $\mathbb{E}\|\eta_k\|^2$ quickly converge to a bounded steady-state value. This value is slightly different across agents as a consequence of the network topology.

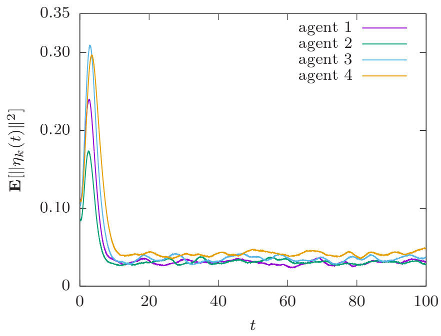

Figure 2. Evolution of $\mathbb{E}\|\eta_k\|^2$ obtained averaging over 100 realizations, $k = 1, 2, 3, 4$, as a function of time.

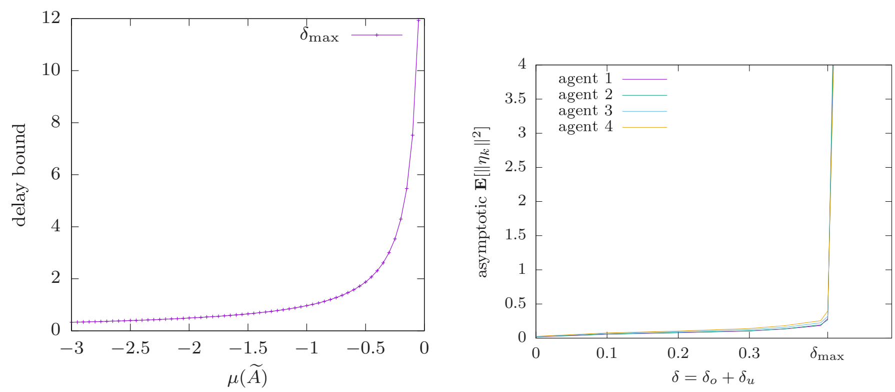

Figure 3. Delay bound for exponentially centered property and mean square boundedness as a function of the most positive eigenvalues assigned to the closed-loop matrix $\widetilde{A}$ (left). Steady-state value of $\mathbb{E}\|\eta_k\|^2$ obtained averaging over 100 realizations, for $k = 1, 2, 3, 4$, as a function of the total delay δ (right)

In Fig. 3 we investigate the total delay bound in order to have exponentially centered and mean square bounded leader-tracking error processes, *i.e.* $\delta_{\max}$, depends on the eigenvalues assigned to $\widetilde{A}$ and therefore on the gain K. We design K so that $\sigma(\widetilde{A}) =$

$\{\bar{\lambda}, \bar{\lambda} - 0.05, \bar{\lambda} \pm 0.05j\}$ with $\bar{\lambda}$ varying between -3 and -0.05. The resulting bounds are plotted in Fig. 3 (left). The plot shows that $\delta_{\max}$ tends to 0 when the eigenvalues of $\widetilde{A}$ move on the left direction of the complex plane. Conversely, the delay bound becomes arbitrarily large when the eigenvalues tend to the imaginary axis. This is a consequence of the fact that $\mu(A) = 0$, thus A can be made exponentially centered with an arbitrarily small K. Thus, in this example, when $\alpha \to 0$ the control law at each agent can be designed to compensate for arbitrary delays.

Finally, we investigate how conservative the delay bounds of Theorem 5.2 are: Fig. 3 (right) shows the steady-state mean square value of the leader-tracking error process $\{\eta_k\}$ as a function of the total delay δ. The plot shows that the predictor is effective exactly up to the theoretical bound $\delta_{\max} = 0.41s$ and that $\mathbb{E}\left[\|\eta_k\|^2\right]$ blows up for any delay exceeding this value.

6.2. Interconnected unstable integrators

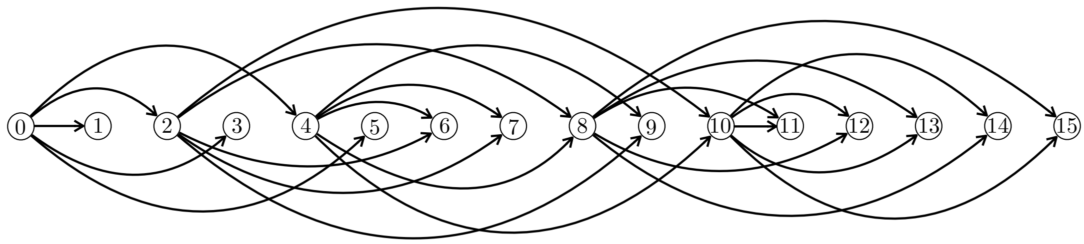

Figure 4. Communication digraph $\mathcal{G}_2$.

We propose here an example where 16 unstable agents are interconnected through the DAG $\mathcal{G}_2$ as shown in Fig. 4. In this case, the dynamics of the i-th agent is simply described by

$$A = \begin{pmatrix} -0.5 & 1 \\ 0 & 0.1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad F = \sigma_x B, C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad G = \sigma_y, \tag{42}$$

with $x_k \in \mathbb{R}^2$, $u_k, y_k \in \mathbb{R}$, and the noise amplitudes $\sigma_x = 0.02$ and $\sigma_y = 0.05$. As before, the noise processes are assumed to be independent standard Wiener processes.

Assuming a uniformly random choice of the initial conditions in the interval $[0.5, 8.5]$ and a feedback gain $K = [2.55, 3.8]$ placing the closed-loop system eigenvalues (that is, the eigenvalues of $\tilde{A}$) to $\{-2, -2.2\}$, the total delay bound is $\delta_{max} = 0.58s$. In this respect, Fig. 5 (left) shows that, in spite of the clear instability of the single agent dynamics, the leader-following result is nonetheless preserved by the control law so long as the total delay stays within the upper bound δ_{max}.

Moreover, Fig. 5 (right) shows that the steady-state value of $\mathbb{E}\|\eta_k\|^2$ obtained averaging over 100 realizations, $k \in \{1, \ldots, 15\}$, with $\delta_o = 0.15s$ and $\delta_u = 0.20s$, increases as the eigenvalue of $\tilde{A}$ with the largest real part is progressively moved closed to the imaginary axis.

Finally, in Fig. 6 we report the trajectories in the plane (x_1, x_2), of the 16 agents initially placed around the point $[5, 5]$, for $\sigma(\widetilde{A}) = \{-2, -2.2\}$ and $t \in [0, 15]$, when $\delta_o = 0.15$ and $\delta_o + \delta_u = 0.50$ (left) and $\delta_o + \delta_u = 0.58 = \delta_{max}$ (the total delay bound). In both cases the trajectories of the followers converge to the leader, that

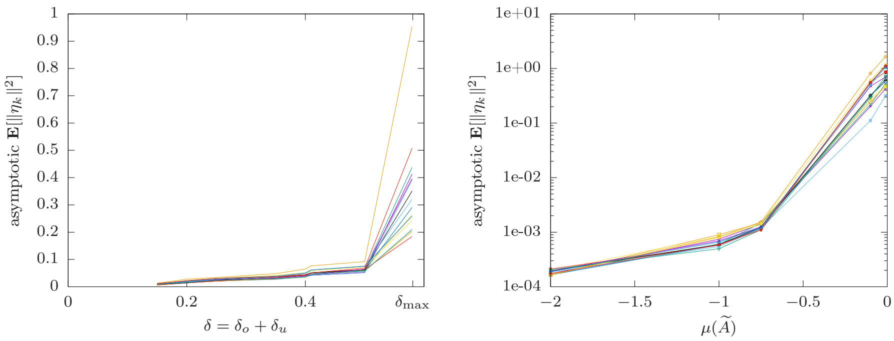

Figure 5. Steady-state value of $\mathbb{E}\|\eta_k\|^2$ obtained averaging over 100 realizations, $k \in \{1, \dots, 15\}$, for the system of Section 6.2 with $\sigma(\widetilde{A}) = \{-2, -2.2\}$, as a function of the total delay δ (left) and as a function of the closed-loop spectral abscissa for $\delta_o = 0.15s$ and $\delta_u = 0.20s$ (right).

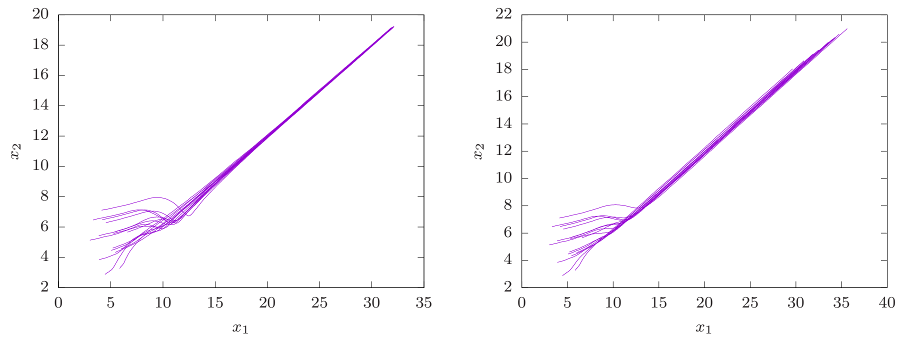

Figure 6. Trajectories of the 16 agents for the system of Section 6.2 with total delay $\delta_o + \delta_u = 0.50$ (left) and $\delta_o + \delta_u = 0.58$ (right). The initial positions are chosen randomly around the point $[5, 5]$ and $t \in [0, 15]$.

moves exponentially fast along the straight line, but the variance of the tracking error $\mathbb{E}\|\eta_k\|^2$ is much larger in the plot on the right, since the followers are disseminated along the trajectory and they follow the leader with a noticeable lag.

6.3. Network of unmanned aerials vehicles

We consider an example adapted from C. Wang et al. (2018) to illustrate the effects of the link weighting mechanism and compare the total delay bounds. A network of unmanned aerials vehicles is composed by one leader and 4 followers with state $x(t) \in \mathbb{R}^2$, input $u_k(t) \in \mathbb{R}^2$, and dynamics, $k = 0, \dots, 4$,

$$\dot{x}_k(t) = Ax_k(t) + Bu_k(t) + \psi(x_k(t)), \tag{43}$$

$$y_k(t) = Cx_k(t), \tag{44}$$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \; \psi(x) = \beta \begin{pmatrix} \sin(x_1) \\ \sin(x_2) \end{pmatrix}. \tag{45}$$

This system is nonlinear, but since in (43) $\psi(x) \in [-\beta, \beta] \times [-\beta, \beta]$ is uniformly bounded with respect to x, we may consider it a zero-mean uniformly distributed disturbance and use the representation (1) with $F = \sqrt{\beta^2/3} I_2$. This approximation

Figure 7. Communication topology for the example in Section 6.3 (left). The graph resulting from weighting the edges according to the algorithm of Section 3 (right).

is legitimate, since the delay bounds of Theorem 5.1 and Theorem 5.2 do not depend on the noise terms but only on the stability properties of the deterministic part of the closed-loop system in presence of delay.

The network topology is shown in Figure 7 (left). The agents are strongly connected, but by using the communication links according to the algorithm of of Section 3 the connection is restricted to the DAG in Figure 7 (right). Notice that, by disregarding the link from node 4, the estimation task of node 1 becomes simpler and node 4 is not affected.

The control gain K_k can be chosen identical for all the agents and it is computed by a plain eigenvalue assignment algorithm for $\widetilde{A}_k = A - BK_k$. By choosing $\sigma(\widetilde{A}_k) = \{-0.6 \pm j\}$ we obtain

$$K_k = \begin{pmatrix} 0.8 & -0.4 \\ -0.4 & 0.8 \end{pmatrix} \tag{46}$$

By setting $\alpha_k = -\mu(\widetilde{A}_k) = 0.6$ in (36) we can compute the delay bound $\delta_{\max}$ for $\delta_o + \delta_u$ that ensures a rate of convergence α_k of the leader tracking error η_k to its stead-state value, and we obtain $\delta_{\max} = 0.940$. The simulations confirm that practical consensus, with an average steady state $\|\eta_k(t)\|$ of about 1% of $\|x_k(t)\|$ is obtained when $t > 10$ for $\delta_o + \delta_u$ up to about 0.7, thanks to the robust stability properties of the predictor. This compares favourably with the results reported in C. Wang et al. (2018), where the total delay is $\delta = 0.1$ and practical consensus is reached for $t > 150$. Although the comparison is only partial because the scheme in C. Wang et al. (2018) includes additional modeled disturbances, the proposed method displays better performance in dealing with delays. Besides, the design in C. Wang et al. (2018) requires to solve a systems of LMIs to tune the gain parameters, an explicit formula for the delay bound is not available, and the computation of the parameters depends on the knowledge of the eigenvalues of the Laplacian matrix of the graph. In contrast, our approach relies on a straightforward eigenvalue placement algorithm, is completely distributed and provides an explicit sufficient bound for the total delay.

7. Conclusions

This paper proposes a novel approach to the leader-following control problem. We showed that by suitably layering the network topology it is possible to obtain a stratified decoupling of the estimation error. In this way, a modular design is achieved and each agent may set the control gain independently from the other agents, in analogy with the case of single systems. We claim that in presence of delays this approach is more flexible and robust and it incorporates the intrinsic properties of the flow of information in the leader following context. This additional flexibility can be exploited to deal with more complex settings, for example heterogeneous agent structure, non uniform and time-varying delays, or multiplicative noises that will be the subject of further research. Further developments can include the consideration of unknown delays and the robust stability under the delay estimation error.

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Appendix A.

Lemma A.1. *Let $P_{\hat{\varepsilon}_{kk}} = P_{\hat{\varepsilon}_k} = \mathbb{E}[\hat{\varepsilon}_k \hat{\varepsilon}_k^\top]$ and $P_{\hat{\varepsilon}_{ji}} = \mathbb{E}[\hat{\varepsilon}_j \hat{\varepsilon}_i^\top]$ be the covariance and cross-covariance of the estimation error, then we have*

$$M_k := \sum_{\substack{j,i=1 \\ j\neq k\neq i}}^{N} \ell_{kj}\ell_{ki} C P_{\hat{\varepsilon}_{ji}} C^\top \geq 0. \tag{A1}$$

Proof. We notice that the matrix

$$\Xi_k := \mathbb{E}\left[\mathrm{st}^{-1}\left\{\left(\sum_{j=1}^{k-1} \ell_{kj}\hat{\varepsilon}_j\right)^{[2]}\right\}\right], \tag{A2}$$

is positive semi-definite since it is the covariance matrix of the random vector $\sum_{j=1}^{k-1} \ell_{kj}\hat{\varepsilon}_j$. Moreover, because of Assumption 3 and Algorithm 1, we can use Proposition 3.3 to obtain that the Laplacian $\mathcal{L}$ (and thus $\mathcal{L}_1$) is lower triangular, and it is not difficult to see that $M_k = C\,\Xi_k C^\top$, and the proof is completed. □

Appendix B. Proof of Theorem 4.1

We first prove by induction that equation (20) admits a stationary solution for all $k \in \mathcal{I}$. In fact, as a consequence of Assumption 3 and Algorithm 1, for the agents i having the leader as neighbor, equation (16) becomes $\tilde{R}_i = GG^\top$ which is positive definite. Thus, because of Assumption 2, equation (20) for $k = i$ has a unique positive definite stationary solution $\bar{P}_i$. Due to Proposition 3.3, the covariance $\tilde{R}_k$ depends on the P_j's for $j < k$. Thus, each $\tilde{R}_k$ admits a stationary value $\bar{R}_k$, and, as a consequence, because of Assumption 2, equation (20) has a unique positive definite stationary solution $\bar{P}_k$ satisfying

$$A\bar{P}_k + \bar{P}_k A^\top + Q - \ell_{kk}^2 \bar{P}_k C^\top \bar{R}_k^{-1} C \bar{P}_k = 0. \tag{B1}$$

By noticing that

$$\mathrm{d}z_j - C\hat{\eta}_j \,\mathrm{d}t = C\hat{\varepsilon}_j \,\mathrm{d}t + G\,\mathrm{d}W_j^{(2)}, \tag{B2}$$

we obtain the following error dynamics for the k-th follower

$$\mathrm{d}\hat{\varepsilon}_k = A\hat{\varepsilon}_k \,\mathrm{d}t - L_k \sum_{j=0}^{N} \ell_{kj} \left(C\hat{\varepsilon}_j \mathrm{d}t + G\,\mathrm{d}W_j^{(2)} \right) + F\,\mathrm{d}\widetilde{W}_k^{(1)}. \tag{B3}$$

Let $\bar{L}_k = \bar{P}_k C_k^\top \bar{R}_k^{-1}$ be the stationary value of the gain (19), and set $\bar{L} = \mathrm{diag}(\bar{L}_1, \ldots, \bar{L}_N)$. The aggregate error $\hat{\varepsilon} = \mathrm{col}(\hat{\varepsilon}_1, \ldots, \hat{\varepsilon}_N)$ of the stationary filters (17)–(20) is given by

$$\mathrm{d}\hat{\varepsilon} = \left[(I_N \otimes A) - \bar{L}(\mathcal{L}_1 \otimes C) \right] \hat{\varepsilon} \,\mathrm{d}t + E\,\mathrm{d}N, \tag{B4}$$

with $E = \mathrm{row}(I_N \otimes F, -\bar{L}(\mathcal{L}_2 \otimes G)$, $N = \mathrm{col}(\widetilde{W}^{(1)}, W^{(2)})$, $\widetilde{W}^{(1)} = \mathrm{col}_k(\widetilde{W}_k^{(1)})$ $W^{(2)} = \mathrm{col}_k(W_k^{(2)})$.

Moreover, since the Laplacian $\mathcal{L}$ (and $\mathcal{L}_1$) is triangular, the matrix $\bar{A} := (I_N \otimes A) - \bar{L}(\mathcal{L}_1 \otimes C)$ is block-triangular with entries $\bar{A}_k := A - \ell_{kk}\bar{L}_k C$ (note that $\ell_{kk} = \sigma_k(\mathcal{L}_1)$). $\forall k \in \mathcal{I}$ $\bar{A}_k$ is Hurwitz. In fact from (B1),

$$\begin{aligned} \bar{A}_k P_k + P_k \bar{A}_k &= AP_k + P_k A^\top - 2\ell_{kk}^2 P_k C^\top \bar{R}_k^{-1} C P_k \\ &= -Q. \end{aligned} \tag{B5}$$

Therefore, $\bar{A}$ is Hurwitz and this implies the internal stability of the filter and the uniform boundedness of the error. Finally, let us prove the bound on the covariance matrix of the estimation error. We can rewrite (B3) as

$$\begin{aligned} \mathrm{d}\hat{\varepsilon}_k &= A_k \hat{\varepsilon}_k \,\mathrm{d}t - L_k \sum_{j \in \mathcal{N}^k} \ell_{kj} C\hat{\varepsilon}_j \mathrm{d}t + L_k \sum_{j=0}^{N} \ell_{kj} G\,\mathrm{d}W_j^{(2)} + F\,\mathrm{d}\widetilde{W}_k^{(1)}, \\ \dot{P}_{\hat{\varepsilon}_k} &= A_k P_{\hat{\varepsilon}_k} + P_{\hat{\varepsilon}_k} A_k^\top - L_k \left[\sum_{j,i \in \mathcal{N}^k} \ell_{kj}\ell_{ki} C P_{\hat{\varepsilon}_{ji}} C^\top \right] L_k^\top - \tilde{\ell}_k L_k G G^\top L_k^\top + Q, \end{aligned} \tag{B6}$$

where $P_{\hat{\varepsilon}_k}$ and $P_{\hat{\varepsilon}_{ji}}$ are defined in Lemma A.1, and $A_k = A - L_k C_k$. Since the term $\tilde{\ell}_k L_k G G^\top L_k^\top$ in (B6) is positive semi-definite and because of Lemma A.1, we have the following differential matrix inequality

$$\dot{P}_{\hat{\varepsilon}_k} \leq A_k P_{\hat{\varepsilon}_k} + P_{\hat{\varepsilon}_k} A_k^\top + Q. \tag{B7}$$

By considering (20) and (B6) by setting $\Lambda_k = \bar{P}_{\hat{\varepsilon}_k} - \bar{P}_k$, we can write

$$\dot{\Lambda}_k \leq \bar{A}_k \Lambda_k + \Lambda_k \bar{A}_k^\top - \bar{P}_k C_k^\top \bar{L}_k^\top = -\Upsilon_k - \bar{P}_k C_k^\top \tilde{R}_k^{-1} C_k \bar{P}_k \leq 0, \tag{B8}$$

where we use the fact that, since the matrix $\bar{A}_k$ is Hurwitz, then there exists $\Upsilon_k > 0$ such that $\bar{A}_k \Lambda_k + \Lambda_k \bar{A}_k^\top = -\Upsilon_k$. Therefore, it follows that $\|\hat{\bar{\varepsilon}}_k\|_{L_2}^2 = \mathrm{tr}\{\bar{P}_{\hat{\varepsilon}_k}\} \leq \mathrm{tr}\{\bar{P}_k\}$. □

Appendix C. Proof of Theorem 5.1

In the case of communication delays equation (B3) of the estimation error $\hat{\varepsilon}_k(t) = \eta_k(t) - \hat{\eta}_k(t)$ becomes, for $t \geq \delta_o$,

$$\begin{aligned} \mathrm{d}\hat{\varepsilon}_k(t) =& A\hat{\varepsilon}_k(t)\,\mathrm{d}t - e^{\bar{A}_k \delta_o} \bar{L}_k C_k \hat{\varepsilon}_k(t-\delta_o) + F\,\mathrm{d}\widetilde{W}_k^{(1)}(t) \\ &- e^{\bar{A}_k \delta_o} \bar{L}_k \sum_{j \in \mathcal{N}^k} \ell_{kj} \Big(C\hat{\varepsilon}_j(t-\delta_o)\mathrm{d}t + G\,\mathrm{d}W_j^{(2)}(t-\delta_o) \Big), \end{aligned} \tag{C1}$$

whereas for $\tau \in [0, \delta_o]$ we have $\hat{\varepsilon}_k(\tau) = \eta_k(\tau)$, since $\hat{\eta}_k(\tau) \equiv 0$ by definition. The solution $\hat{\varepsilon}_k(t)$ admits the following representation, $t \geq \delta_o$

$$\begin{aligned} \hat{\varepsilon}_k(t) =& \int_{t-\delta_o}^{t} e^{\bar{A}_k(t-\tau)} \bar{L}_k C_k \hat{\varepsilon}_k(\tau)\,\mathrm{d}\tau + \int_0^t e^{\bar{A}_k(t-\tau)} F\,\mathrm{d}\widetilde{W}_k^{(1)}(\tau) \\ &- \int_0^{t-\delta_o} e^{\bar{A}_k(t-\tau)} \bar{L}_k C \sum_{j \in \mathcal{N}^k} \ell_{kj} \hat{\varepsilon}_j(\tau)\,\mathrm{d}\tau \\ &- \sum_{j \in \mathcal{N}^k} \ell_{kj} \int_0^{t-\delta_o} e^{\bar{A}_k(t-\tau)} \bar{L}_k G\,\mathrm{d}W_j^{(2)}(\tau) + \vartheta_{k,\delta_o}, \end{aligned} \tag{C2}$$

$$\vartheta_{k,\delta_o} = \eta_k(\delta_o) - \int_0^{\delta_o} e^{\bar{A}_k(\delta_o-\tau)} \bar{L}_k C_k \eta_k(\tau)\,\mathrm{d}\tau + \int_0^{\delta_o} e^{\bar{A}_k(\delta_o-\tau)} F\,\mathrm{d}\widetilde{W}_k^{(1)}(\tau) \tag{C3}$$

as it can be verified by explicit differentiation. Notice that the choice of constant vector ϑ_{k,δ_o} ensures $\hat{\varepsilon}_k(\delta_o) = \eta_k(\delta_o)$. Pre-multiplying by C_k and taking L_2 norm we get

$$
\begin{aligned}
\|C_k\hat{\varepsilon}_k(t)\|_{L_2} \leq & \int_0^{\delta_o} \left\|C_k e^{\bar{A}_k(\theta)}\bar{L}_k\right\| \mathrm{d}\theta \sup_{\tau\in[t-\delta_o,t]} \|C_k\hat{\varepsilon}_k(\tau)\|_{L_2} \\
& + \left(\int_0^t \left\|C_k e^{\bar{A}_k\theta}\right\|^2 \mathrm{d}\theta\right)^{\frac{1}{2}} \left(\|F\|^2 + \sum_{j\in\mathcal{N}^k} |\ell_{kj}|\|\bar{L}_k G\|^2\right)^{\frac{1}{2}} \\
& + \int_0^t \left\|C_k e^{\bar{A}_k(\theta)}\bar{L}_k\right\| \mathrm{d}\theta \sup_{\tau\in[t-\delta_o,t]} \sum_{j\in\mathcal{N}^k} |\ell_{kj}|\|C\hat{\varepsilon}_j(\tau)\|_{L_2} + \phi_{k,\delta_o} \\
& = \gamma_o(0,\delta_o) \sup_{\tau\in[t-\delta_o,t]} \|C_k\hat{\varepsilon}_k(\tau)\|_{L_2} + \beta_k,
\end{aligned} \tag{C4}
$$

where $\phi_{k,\delta_o} = \|C\vartheta_{k,\delta_o}\|_{L_2} < \infty$ depends on the initial conditions in $[0,\delta_o]$. We now proceed inductively on the distance d_k from the leader, as computed in (10). When $d_k = 1$, $\mathcal{N}^k = \{0\}$ and consequently

$$
\beta_k = \left(\int_0^t \left\|C_k e^{\bar{A}_k\theta}\right\|^2 \mathrm{d}\theta\right)^{\frac{1}{2}} \left(\|F\|^2 + \|\bar{L}_k G\|^2\right)^{\frac{1}{2}} + \phi_{k,\delta_o} \; < \; \infty, \tag{C5}
$$

because $\hat{\varepsilon}_0 \equiv 0$. Thus, since $\gamma_o(0,\delta_o) < \gamma_o(\alpha_k,\delta_o) < 1$ we obtain the following bound

$$
\|C_k\hat{\varepsilon}_k(t)\|_{L_2} \leq \frac{\beta_k}{1-\gamma_o(0,\delta_o)} \; < \; \infty. \tag{C6}
$$

We now prove that if $\|C_k\hat{\varepsilon}_j(t)\|_{L_2} \leq \infty$ for the nodes with $d_j < d$ then the same holds for the nodes k with $d_k = d$. In fact, when the Laplacian matrix is transformed according to the procedure described in Section 3 and $d_k = d$ all the nodes j in $\mathcal{N}_k$ have distance $d_j < d$ and it holds by inductive hypothesis that $\|C_k\hat{\varepsilon}_j(t)\|_{L_2} \leq \infty$. This, together with the fact that $\bar{A}_k$ is Hurwitz and the integrals in [0, t] are bounded for $t \to \infty$, implies that $\beta_k < \infty$. Therefore, (C6) holds for all the nodes k with $d_k = d$. Since the depth of the DAG is finite, $\|C_k\hat{\varepsilon}_j(t)\|_{L_2}$ is bounded for all $k \in \mathcal{I}$, but, since (A, C) is observable, is immediate to see from (C1) that $\|C_k\hat{\varepsilon}_k(t)\|_{L_2} < \infty$ implies $\|\hat{\varepsilon}_k(t)\|_{L_2} < \infty$. Finally, it is easily derived that $\hat{\varepsilon}_k$ is exponentially centered with rate α_k by considering the auxiliary variable $\hat{\varepsilon}_k^\alpha = e^{\alpha t}\hat{\varepsilon}_k$, writing the equation of $\mathbb{E}[\hat{\varepsilon}_k^\alpha]$, that does not contain the noise terms, and repeating the same steps as above. □