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Lucas decomposition and extrapolation methods for the evaluation of infinite integrals involving the product of three Bessel functions of arbitrary order

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ABSTRACT

A method to efficiently evaluate integrals containing the product of three Bessel functions of the first kind and of any non negative real order is presented. The numerical problem is particularly challenging since standard integration techniques are completely unsuccessful in integrating such anomalously oscillating (and possibly slow decaying) functions. The proposed method is based on a decomposition of such a product into a sum of functions which asymptotically approach sinusoidal functions, for which integration schemes based on integration then summation procedures followed by extrapolation methods can be applied. Different extrapolation procedures are compared in order to identify the most efficient extrapolation strategy. Several numerical examples and comparisons with known analytic results are provided to show the robustness and the accuracy of the proposed approach and to help in identifying the most efficient and reliable extrapolation scheme. Particular attention is given to a class of integrals emerging in many fields of physics, in particular in quantum mechanics and particle physics, showing that the proposed method can be even more efficient (sometimes hundred of times faster) than the available analytical formulas.

1. Introduction

The problem at hand consists in the efficient and accurate evaluation of integrals of the form

$$\mathcal{I} = \int_0^\infty f(x) \mathrm{J}_\mu(ax) \mathrm{J}_\nu(bx) \mathrm{J}_\xi(cx)\, \mathrm{d}x \tag{1}$$

where $f(x)$ is a non-oscillating function and $\mathrm{J}_\sigma(\cdot)$ is the Bessel function of the first kind and of non-negative real order σ, and a, b, and c are real parameters. Such integrals, which involve a triple product of Bessel functions (in the following denoted as TBFIs), occur in a variety of physical problems such as in electromagnetics [1–3], diffraction theory [4], geophysics [5], fluid dynamics [6], light–matter interaction [7], quantum mechanics and particle physics [8–11], as well as in cosmological studies [12,13]. More precisely, in the latter applications triple products of *spherical* Bessel functions of integer order are involved, which, however, as is well known, can be reduced to a triple product of half-integer-order Bessel functions. In many of the mentioned studies, analytical expressions have been derived, which however can be applied only to specific cases, with particular values of the Bessel functions orders and of their arguments and for very few examples of the function $f(x)$. Other analytical expressions for particular cases of TBFIs have been derived starting from the seminal work of Bailey [14] and developing different techniques through the years [15–22]. In any case,

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most of the analytical expressions (attainable for very few cases) are very complicated or have to be obtained by recursion. This implies that, in general, a reliable numerical quadrature procedure is mandatory: however, it is well known that such a problem can be very challenging since the oscillatory behavior of the integrand can be highly irregular because of the presence of asymptotically oscillating factors with different periods. Since standard integration techniques are in general unsuccessful in dealing with infinite integrals with an oscillatory (and possibly slow decaying) integrand, the anomalous oscillations of the products of Bessel functions make such conventional routines even less applicable [23].

From a numerical point of view, in [24,25] an algorithm has been presented for evaluating infinite integrals containing products of an arbitrary number of Bessel functions of the first kind. Basically, the interval of integration is split into a finite and an infinite part. The former is computed using Gauss–Legendre rules while the latter is approximated using suitable asymptotic expansions which are analytically integrated in terms of the upper incomplete Gamma function. However, because of the last procedure, the class of $f(x)$ functions are limited to polynomials or functions with simple exponential or rational factors.

In [26], Lucas and Stone have considered the simpler problem of evaluating integrals of the form

$$I_1 = \int_0^\infty f(x)\, \mathrm{J}_\sigma(px)\, \mathrm{d}x \tag{2}$$

where *only one* Bessel function is considered. Such integrals can be evaluated through an integration, summation and extrapolation (ISE) procedure in which the integral is evaluated as a sum of a series of partial integrals over finite subintervals which are alternating in sign. The computation of the integral in (2) is thus reduced to finding the limit of a sequence of partial sums. However, this sequence oscillates about the exact value of the integral and usually converges slowly. The idea is then to look for a transformation which leads to a new sequence which converges faster. In practice, the sought limit is found using an extrapolation algorithm that speeds up the convergence and in the literature many variants have been suggested [27]. Such an integration then summation procedure, followed by extrapolation, known as ISE method (or partition-extrapolation methods) has been the subject of intensive research (see the book [28] and the comprehensive papers [29,30] and references therein) and, in particular, the ISE method has resulted to be very effective to evaluate integrals as (2) [26,29,31].

Unfortunately, an ISE method cannot be applied to integrals such as (1) since, due to the product of the Bessel functions, the resulting series is not of alternating sign, and the extrapolation procedure does not improve the convergence of the original sequence [32].

In [32] Lucas described a method to evaluate integrals with the product of *two* Bessel functions by rewriting such a product as the sum of two oscillating functions (which is known as Lucas decomposition) which asymptotically behave as sinusoids. The sought integral is thus written as a sum of two integrals of the form (2) and for each of them the ISE method can efficiently be applied. The adopted extrapolation procedure was the modified W transform of Sidi (known as mW transform) [33]. In [34] Lucas procedure has recently been reviewed by considering the Weighted-Averages method as the extrapolation method [35].

In the present paper we present a generalization of the method proposed by Lucas (and suggested in the last part of [34]) to evaluate integrals containing the product of *three* Bessel functions. In particular, following Lucas, we express the product of three Bessel functions as a sum of four asymptotically sinusoidal functions so that the evaluation of the original integral is reduced to the evaluation of four integrals of the form (2) for which the ISE method can be applied. As concerns the extrapolation procedure, we compare different methods, i.e., different variants of the Levin–Sidi method and the generalized weighted-averages (GWA) method [35] (which has been shown to be particularly efficient in treating Sommerfeld-like integrals). Since each extrapolation method requires the evaluation of partial integrals over finite intervals, we adopt the double exponential (DE) rules [36] to adaptively perform such integrations.

After presenting all the components of the suggested algorithm, we provide extensive numerical examples to show the robustness, accuracy, and efficiency of the proposed approach and we try to identify the most efficient and reliable extrapolation scheme for the considered class of integrals.

2. Decomposition of the integral

To generalize the Lucas decomposition in [32] we start by considering the asymptotic expressions of the Bessel function of first kind for large x, i.e.,

$$\mathrm{J}_\sigma(rx) \simeq \sqrt{\frac{2}{\pi r x}} \cos\left(rx - \sigma\frac{\pi}{2} - \frac{\pi}{4}\right) \tag{3}$$

We then use the trigonometric identity

$$\cos(ax)\cos(bx)\cos(cx) = \frac{1}{4}\left[\cos\left(u_1 x\right) + \cos\left(u_2 x\right) + \cos\left(u_3 x\right) + \cos\left(u_4 x\right)\right] \tag{4}$$

where

$$u_m = \begin{cases} a+b+c\,, & m=1\\ -\ a+b+c\,, & m=2\\ a-b+c\,, & m=3\\ a+b-c\,, & m=4 \end{cases} \tag{5}$$

Taking into account that

$$\begin{aligned}
\cos\left(u_1 x\right) &= \cos(ax)\cos(bx)\cos(cx) - \sin(ax)\sin(bx)\cos(cx) - \sin(ax)\cos(bx)\sin(cx) - \cos(ax)\sin(bx)\sin(cx)\\
\cos\left(u_2 x\right) &= \cos(ax)\cos(bx)\cos(cx) + \sin(ax)\sin(bx)\cos(cx) + \sin(ax)\cos(bx)\sin(cx) - \cos(ax)\sin(bx)\sin(cx)\\
\cos\left(u_3 x\right) &= \cos(ax)\cos(bx)\cos(cx) + \sin(ax)\sin(bx)\cos(cx) - \sin(ax)\cos(bx)\sin(cx) + \cos(ax)\sin(bx)\sin(cx)\\
\cos\left(u_4 x\right) &= \cos(ax)\cos(bx)\cos(cx) - \sin(ax)\sin(bx)\cos(cx) + \sin(ax)\cos(bx)\sin(cx) + \cos(ax)\sin(bx)\sin(cx)
\end{aligned} \tag{6}$$

and the asymptotic expressions of the Bessel function of second kind for large x, i.e.,

$$\mathrm{Y}_\sigma(rx) \simeq \sqrt{\frac{2}{\pi r x}}\,\sin\left(rx - \sigma\frac{\pi}{2} - \frac{\pi}{4}\right) \tag{7}$$

we can define the functions

$$\begin{aligned}
h_1(x) &= \left[\mathrm{J}_\mu(ax)\mathrm{J}_\nu(bx)\mathrm{J}_\xi(cx) - \mathrm{Y}_\mu(ax)\mathrm{Y}_\nu(bx)\mathrm{J}_\xi(cx) - \mathrm{Y}_\mu(ax)\mathrm{J}_\nu(bx)\mathrm{Y}_\xi(cx) - \mathrm{J}_\mu(ax)\mathrm{Y}_\nu(bx)\mathrm{Y}_\xi(cx)\right]\\
h_2(x) &= \left[\mathrm{J}_\mu(ax)\mathrm{J}_\nu(bx)\mathrm{J}_\xi(cx) + \mathrm{Y}_\mu(ax)\mathrm{Y}_\nu(bx)\mathrm{J}_\xi(cx) + \mathrm{Y}_\mu(ax)\mathrm{J}_\nu(bx)\mathrm{Y}_\xi(cx) - \mathrm{J}_\mu(ax)\mathrm{Y}_\nu(bx)\mathrm{Y}_\xi(cx)\right]\\
h_3(x) &= \left[\mathrm{J}_\mu(ax)\mathrm{J}_\nu(bx)\mathrm{J}_\xi(cx) + \mathrm{Y}_\mu(ax)\mathrm{Y}_\nu(bx)\mathrm{J}_\xi(cx) - \mathrm{Y}_\mu(ax)\mathrm{J}_\nu(bx)\mathrm{Y}_\xi(cx) + \mathrm{J}_\mu(ax)\mathrm{Y}_\nu(bx)\mathrm{Y}_\xi(cx)\right]\\
h_4(x) &= \left[\mathrm{J}_\mu(ax)\mathrm{J}_\nu(bx)\mathrm{J}_\xi(cx) - \mathrm{Y}_\mu(ax)\mathrm{Y}_\nu(bx)\mathrm{J}_\xi(cx) + \mathrm{Y}_\mu(ax)\mathrm{J}_\nu(bx)\mathrm{Y}_\xi(cx) + \mathrm{J}_\mu(ax)\mathrm{Y}_\nu(bx)\mathrm{Y}_\xi(cx)\right]
\end{aligned} \tag{8}$$

It is immediate to check that

$$\mathrm{J}_\mu(ax)\mathrm{J}_\nu(bx)\mathrm{J}_\xi(cx) = \frac{1}{4}\left[h_1(x) + h_2(x) + h_3(x) + h_4(x)\right] \tag{9}$$

and therefore

$$\mathcal{I} = \frac{1}{4}\sum_{m=1}^{4} H_m \tag{10}$$

where

$$H_m = \int_0^\infty f_m(x)\,\mathrm{d}x\,, \qquad f_m(x) = f(x)\,h_m(x)\,, \qquad m = 1,\dots,4 \tag{11}$$

By considering the asymptotic expressions (3) and (7) we thus have

$$\begin{aligned}
h_1(x) &\simeq \sqrt{\frac{8}{abc\pi^3 x^3}}\,\cos\left[u_1 x - (\mu+\nu+\xi)\frac{\pi}{2} - \frac{3\pi}{4}\right]\\
h_2(x) &\simeq \sqrt{\frac{8}{abc\pi^3 x^3}}\,\cos\left[u_2 x - (-\mu+\nu+\xi)\frac{\pi}{2} - \frac{\pi}{4}\right]\\
h_3(x) &\simeq \sqrt{\frac{8}{abc\pi^3 x^3}}\,\cos\left[u_3 x - (\mu-\nu+\xi)\frac{\pi}{2} - \frac{\pi}{4}\right]\\
h_4(x) &\simeq \sqrt{\frac{8}{abc\pi^3 x^3}}\,\cos\left[u_4 x - (\mu+\nu-\xi)\frac{\pi}{2} - \frac{\pi}{4}\right]
\end{aligned} \tag{12}$$

Therefore, for $u_m \neq 0$ ($m = 1,\dots,4$), all the h_m functions asymptotically approach cosine functions, in a similar manner to the Bessel function $\mathrm{J}_\sigma(x)$. In such cases the integration of h_m can efficiently be performed through the ISE method used to evaluate integrals of the form (2). On the other hand, if $u_m = 0$, the relevant h_m function asymptotically approaches a monotonically decreasing function and the relevant integral can efficiently be computed by a simple adaptive routine.

3. Application of the ISE method

When applying the generalized Lucas decomposition (9) for the evaluation of the integral in (10) through (11) care should be payed to the fact the Bessel functions of second kind $\mathrm{Y}_\sigma(x)$ in (8) are singular at $x = 0$ and for $\mu, \nu, \xi > 0$, definite integrals with left endpoint $x = 0$ that involve the functions h_m do not exist. Moreover, for small x, the behavior of $h_m(x)$ is dominated by the product of the Bessel functions of second kind which may have very large magnitudes and the sum in (10) can give rise to catastrophic cancellation. To avoid this problem, the integral (1) is split in two parts, i.e.,

$$\mathcal{I} = I_0 + I_\mathrm{d} \tag{13}$$

where

$$\begin{aligned}
I_0 &= \int_0^{x_\mathrm{max}} f(x)\mathrm{J}_\mu(ax)\mathrm{J}_\nu(bx)\mathrm{J}_\xi(cx)\,\mathrm{d}x\\
I_\mathrm{d} &= \int_{x_\mathrm{max}}^\infty f(x)\mathrm{J}_\mu(ax)\mathrm{J}_\nu(bx)\mathrm{J}_\xi(cx)\,\mathrm{d}x
\end{aligned} \tag{14}$$

and only the integral I_d is evaluated through the decomposition (9) so that

$$\mathcal{I} = I_0 + \frac{1}{4}\sum_{m=1}^{4} I_m \tag{15}$$

where

$$I_m = \int_{x_{\max}}^{\infty} f_m(x)\,\mathrm{d}x\,, \qquad m = 1,\ldots,4 \tag{16}$$

Since the oscillatory behavior of $\mathrm{Y}_\sigma(x)$ starts after its first zero, it is convenient to choose the point $x_{\max}$ as the largest of the first zeros of $\mathrm{Y}_\mu(ax)$, $\mathrm{Y}_\nu(bx)$, and $\mathrm{Y}_\xi(cx)$. The zeros of $\mathrm{Y}_\sigma(x)$ when σ is an integer or a half integer are tabulated in [37]. In general, a good approximation for the first zero of $\mathrm{Y}_\sigma(x)$ (denoted as $y_{\sigma,1}$) for any $\sigma > 1$ can be found in [38] or [39, 10.21.40] which gives

$$y_{\sigma,1} \simeq \sigma + 0.9315768\sigma^{1/3} + 0.260351\sigma^{-1/3} + 0.011976\sigma^{-1} - 0.00602\sigma^{-5/3} - 0.0012\sigma^{-7/3} \tag{17}$$

We therefore select

$$x_{\max} = \max\left(\frac{y_{\mu,1}}{a}, \frac{y_{\nu,1}}{b}, \frac{y_{\xi,1}}{c}\right) \tag{18}$$

Up to $x_{\max}$, the original integral (i.e., the integral I_0) can accurately be computed through an adaptive method. In fact, independently of the values of the orders μ, ν, and ξ, the product $\mathrm{J}_\mu(ax)\,\mathrm{J}_\nu(bx)\,\mathrm{J}_\xi(cx)$ on $[0, x_{\max}]$ will only have a finite number of oscillations and an adaptive rule will perfectly work.

3.1. Extrapolation method

At this point, a particular extrapolation procedure needs to be adopted to evaluate the integrals (16). In what follows we illustrate two very effective methods introduced in the literature, i.e., the generalized weighted averages method and the Levin–Sidi extrapolation method in order to identify the most effective one.

To briefly and roughly introduce the extrapolation procedures, let us then consider the integrals I_m in (16) which can be seen as the limits of a sequence of partial sums and, in particular,

$$I_m = \lim_{n\to\infty} P_n^m, \qquad m = 1,\ldots,4 \tag{19}$$

with

$$P_n^m = \sum_{i=0}^{n} p_i^m, \qquad m = 1,\ldots,4 \tag{20}$$

where p_i^m are the *partial integrals*

$$p_i^m = \int_{x_{i-1}^m}^{x_i^m} f_m(x)\,\mathrm{d}x, \qquad m = 1,\ldots,4 \tag{21}$$

where $x_i^m > x_{i-1}^m$ are suitable break points and $x_{-1}^m = x_{\max}$.

3.1.1. The Levin–Sidi method

The Levin–Sidi method is based on the general Levin (GL) transformation used to accelerate the rate of convergence of a sequence of partial sums for which (19) holds [40]. In what follows we momentarily omit the superscript m which distinguishes the four integrals in (19)–(21). According to the GL transformation [40], given a sequence of partial sums $\{P_n, P_{n+1}\ldots, P_{n+k}\}$ with elements as in (20)–(21), a transformed sequence $P_n^{(k)}$ (where k indicates the order of the transformation) is constructed such that $P_n^{(k)}$ is much closer to the limit than the element P_{n+k}. Although the original GL transformation was non-recursive, it was observed that the recursive W-algorithm of Sidi could be used to efficiently compute it [29]. In particular, by introducing the divided difference operator δ^k, defined by

$$\delta^{k+1}\left(p_n\right) = \frac{\delta^k\left(p_{n+1}\right) - \delta^k\left(p_n\right)}{\dfrac{1}{x_{n+k+1}} - \dfrac{1}{x_n}}, \qquad n \geq 0, k \geq 0 \tag{22}$$

with $\delta^0\left(p_n\right) = p_n$, it can be shown that

$$P_n^{(k)} = \frac{A_n^{(k)}}{B_n^{(k)}} = \frac{\delta^k\left(\dfrac{P_n}{w_n}\right)}{\delta^k\left(\dfrac{1}{w_n}\right)} \tag{23}$$

where w_n is the remainder estimate whose particular choice gives rise to different acceleration properties and will be defined below. In our problems the acceleration starts directly from $n = 0$ and the accelerated sequence elements are therefore

$$P_0^{(k)} = \frac{A_0^{(k)}}{B_0^{(k)}} \tag{24}$$

where both the numerator and the denominator in (24) obey the same three-terms recurrence formula

$$R_{k-i}^{(i)} = \frac{R_{k-i+1}^{(i-1)} - R_{k-i}^{(i-1)}}{\dfrac{1}{x_k} - \dfrac{1}{x_{k-i}}}, \qquad k \geq 1, 1 \leq i \leq k \tag{25}$$

with starting values $A_0^{(0)} = P_0/w_0$ and $B_0^{(0)} = 1/w_0$.

The above procedure can be applied to each of the integrals I_m in (16) so that with N partial integrals evaluations (21) we have

$$I_m^{N,\mathrm{GLS}} = P_{m,0}^{(N)} = \frac{A_{m,0}^{(N)}}{B_{m,0}^{(N)}} \tag{26}$$

In (26) the subscript m distinguishes the approximations of the integrals I_m which are constructed through different partial integrals p_i^m (with different integrand functions $f_m(x)$ and different break points x_n^m).

As mentioned above, the properties of the GL transformation strongly depend on the choice of the remainder estimates w_n. For example, using $w_n = 1/x_n$ leads to the Richardson extrapolation [41], while $w_n = p_n$ or $w_n = x_n p_n$ leads to the t or the u-transformations, respectively [40]. On the other hand the choice $w_n = p_{n+1}$ results in the mW transformation of Sidi [33]. It is worth noting that analytical remainder estimates can be used if the asymptotic behavior of the integrand function is known. In particular, if the function $f(x)$ can be expressed as

$$f(x) = g(x)\psi(x) \tag{27}$$

where $\psi(x)$ is a periodic function with period $2Q$ (with Q real positive parameter) for which

$$\psi(x+Q) = -\psi(x) \tag{28}$$

and the function $g(x)$ asymptotically behaves as

$$g(x) \simeq A x^{\lambda}\, \mathrm{e}^{-\alpha x} \tag{29}$$

where λ and α are real nonnegative parameters which describe the algebraic and exponential decays, respectively, the analytical remainder estimate is

$$w_n = (-1)^{n+1}\, x_n^{\lambda}\, \mathrm{e}^{-nQ\alpha} \tag{30}$$

With the choice (30), the GL transformation results in the W transformation of Sidi [42].

Finally, it should be pointed out that attention should be paid to the choice of the break points x_n^m for the evaluation of the partial integrals in (21). One possibility is to use the Sidi partition for which the break points are equally spaced and separated by the asymptotic half-period of the corresponding functions h_m starting from the lower integration limit $x_{\max}$, i.e,

$$x_n^m = x_{\max} + nQ_m = x_{\max} + \frac{n\pi}{|u_m|}, \qquad m = 1, \ldots, 4 \tag{31}$$

However, Michalski noted that it would be much more efficient to use a modified Sidi partition for which the lower integration limit is placed at the first zero x_{h_m} of $h_m(x)$ which is larger than $x_{\max}$ [31]. This means that the integrals (16) need to be split as

$$I_m = \int_{x_{\max}}^{x_{h_m}} f_m(x)\,\mathrm{d}x + \int_{x_{h_m}}^{\infty} f_m(x)\,\mathrm{d}x, \qquad m = 1, \ldots, 4 \tag{32}$$

The first integral in (32) is evaluated through a conventional quadrature strategy while the second integral is evaluated through the GL transformation (26) with break points

$$x_n^m = x_{h_m} + nQ_m = x_{h_m} + \frac{n\pi}{|u_m|}, \qquad m = 1, \ldots, 4 \tag{33}$$

The determination of the first zero x_{h_m} of $h_m(x)$ larger than $x_{\max}$ can be performed following the procedure suggested in [31] and, in particular, bracketing it evaluating the function $h_m(x)$ at $x_{\max}$, by taking steps of size $Q_m/4$ until a change in the sign of the function $h_m(x)$ is detected, and then apply the derivative-free Brent method [43].

Obviously, the ISE method should be applied if $u_m \neq 0$ otherwise the relevant h_m function asymptotically approaches a monotonically decreasing (and non oscillating) function for which an adaptive rule will be perfectly fine to evaluate the relevant I_m integral.

3.1.2. The generalized weighted averages method

The generalized weighted-averages (GWA) method [35] is one of the most powerful extrapolation methods for integrals with Bessel function kernels and has been developed based on a previous version [29,44] to treat Sommerfeld integral tails arising in the context of electromagnetics of planar stratified media further improving the efficiency and robustness with respect to the original version. Let us consider the integral

$$I = \int_{x_0}^{\infty} g(x)\psi(x)\,\mathrm{d}x \tag{34}$$

where $\psi(x)$ and $g(x)$ are the functions defined in (28)–(29) and x_0 is a nonnegative real parameter. Based on the GWA method, the estimate of the integral (34) through N partial integrals is

$$I^{N,\mathrm{GWA}} = \frac{\sum_{n=1}^{N} w_n P_n}{\sum_{n=1}^{N} w_n} \tag{35}$$

where the break points x_n are selected as

$$x_n = x_0 + nQ \tag{36}$$

and the weights w_n are given by

$$w_n = \binom{N-1}{n-1} \mathrm{e}^{-\alpha x_n} \left(x_n\right)^{N-2-\lambda} \tag{37}$$

Now we can observe that, for $u_m \neq 0$, the integrals I_m in (16) are asymptotically of the form (34) so that we can estimate them through

$$I_m^{N,\mathrm{GWA}} = \frac{\sum_{n=1}^{N} w_n^m P_n^m}{\sum_{n=1}^{N} w_n^m} \tag{38}$$

with (20)–(21), the Sidi partition (31) and the weights

$$w_n^m = \binom{N-1}{n-1} \mathrm{e}^{-\alpha x_n^m} \left(x_n^m\right)^{N-\eta-1/2}, \qquad m = 1, \dots, 4 \tag{39}$$

where now η describes the algebraic behavior at infinity of the function $f(x)$ in (1), so that $\eta = \lambda - 3/2$.

As mentioned above in connection with the GL transformation, if $u_m = 0$, the relevant h_m function asymptotically approaches a monotonically decreasing (and non oscillating) function: in this case an adaptive rule will be perfectly fine to evaluate the relevant I_m integral.

3.2. Quadrature: the double exponential rule

Although in principle any quadrature scheme can be adopted, the double exponential (DE) rules presented in [36] are simple, efficient, versatile, and can be implemented in a progressive way to check the convergence of the integration [31]. Nodes and weights are easily generated and in the adaptive scheme the function values calculated at the previous step are reused. Moreover, the DE rules can also easily handle integrable singularities by splitting the integration range placing the singularities at the endpoints, since the method is inherently almost insensitive to the endpoint behavior of the integrand function (besides, it should be noted that particular care should be given to the implementation of the algorithm to avoid large errors due to the loss of significant digits close to the singularities, as remarked in [45]). The method basically consists in mapping the interval of integration to the whole real axis in such a way that the Jacobian of the transformation tends to 0 double exponentially at $\pm\infty$ and a simple trapezoidal rule with equal mesh size is then applied [36].

For the evaluation of the partial integrals to be calculated in the extrapolation procedure, we adopted the *tanh–sinh* rule proposed in [36] for which the general integral on a finite interval can be approximated as

$$\int_{x_1}^{x_2} f(x)\,\mathrm{d}x \simeq \lambda d \left\{ \left.\frac{\mathrm{d}g}{\mathrm{d}t}\right|_{t=0} f(\gamma) + \sum_{m=1}^{M} w_m \left[f\left(x_1 + \chi\delta_m\right) + f\left(x_2 - \chi\delta_m\right) \right] \right\} \tag{40}$$

where

$$\chi = \frac{x_2 - x_1}{2}, \qquad \gamma = \frac{x_2 + x_1}{2} \tag{41}$$

d is the mesh size, while

$$w_m = 2 \left.\frac{\mathrm{d}g}{\mathrm{d}t}\right|_{t=md} \frac{\delta_m}{1+q_m}, \qquad \delta_m = 2\frac{q_m}{1+q_m} \tag{42}$$

and

$$q_m = \mathrm{e}^{-2g(md)} \tag{43}$$

The tanh–sinh rule is defined by the choice of the function $g(t)$, i.e.,

$$g(t) = \sinh(t) \tag{44}$$

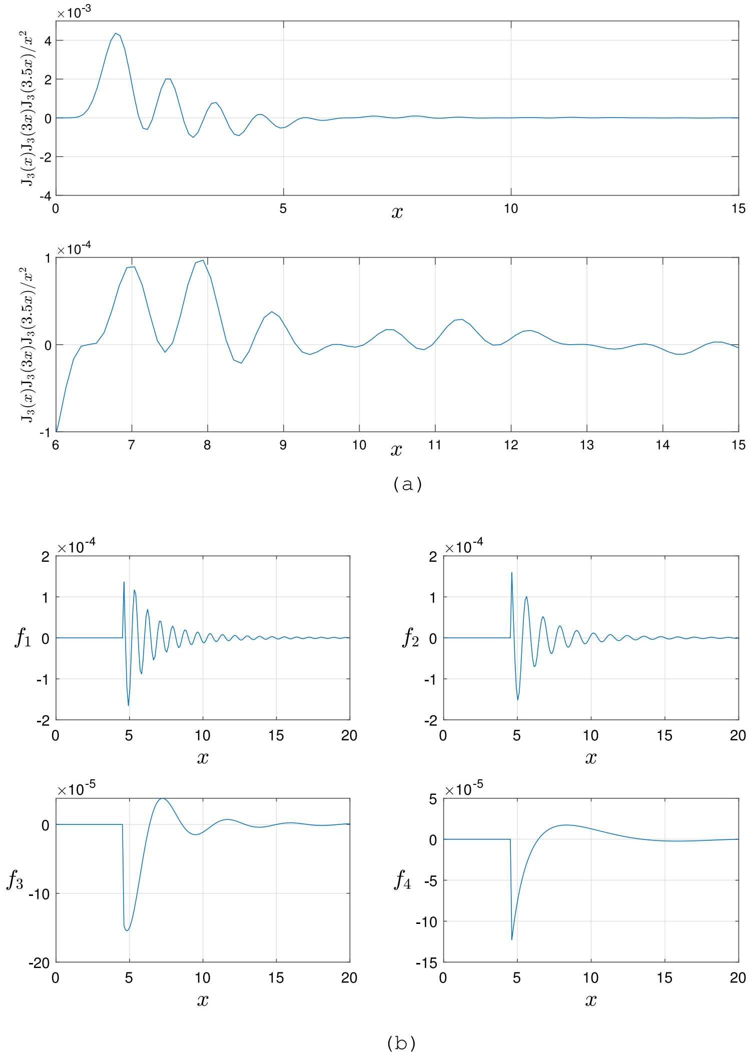


Fig. 1. (*a*) Behavior of the integrand function for the integral in (48) with $r = 3$ $a = 1$, $b = 3$, $c = 3.5$ and (*b*) behavior of the relevant integrand functions $f_m(x)$.

When implementing the rule (40) the value of m is increased until a certain truncation error is achieved. Next, formula (40) is used to determine a higher-level estimate by halving the mesh size d (and thus doubling the number of points). Higher levels are thus generated until a specified tolerance is reached [46]. The initial mesh size is empirically set to $d = 1.5$ [31].

The DE rule can also be used to evaluate the integrals I_m when $u_m = 0$ and the relevant integrand function is not oscillating. In this case, the general integral on a semi-finite interval can be approximated as [36]

$$\int_{x_0}^{\infty} f(x)\,\mathrm{d}x \simeq d \sum_{m=-M_1}^{M_2} w_m f\left(x_0 + \delta_m\right) \tag{45}$$

where

$$w_m = \left(\mathrm{e}^{-md} + \mathrm{e}^{md}\right)\delta_m\,, \qquad \delta_m = \mathrm{e}^{\left(\mathrm{e}^{-md} + \mathrm{e}^{md}\right)} \tag{46}$$

However, if the integrand function is exponentially decaying at infinity, a mixed double–single exponential rule is preferable [36] for which the quadrature rule (45) applies with

$$w_m = \left(1 + \mathrm{e}^{-md}\right)\delta_m\,, \qquad \delta_m = \mathrm{e}^{\left(md - \mathrm{e}^{-md}\right)} \tag{47}$$

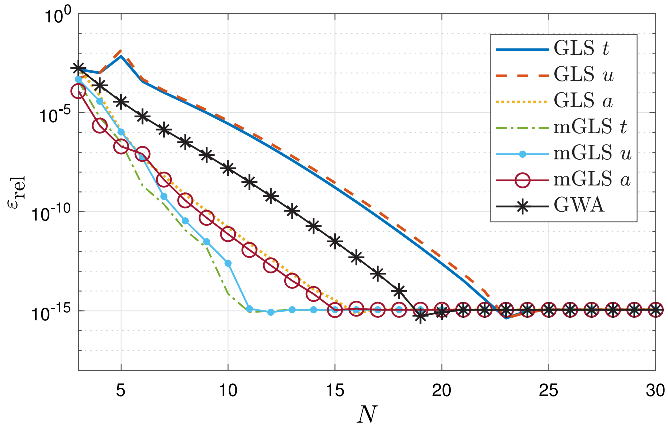


Fig. 2. Relative error $\varepsilon_{\rm rel}$ between the exact result in (50) for $I^{\rm ex1}(3; 1, 3, 3.5)$ and the integral calculated with the proposed approach as a function of the number N of the partial integrals (by assuming the same number N for each I_m integral) for different extrapolation strategies.

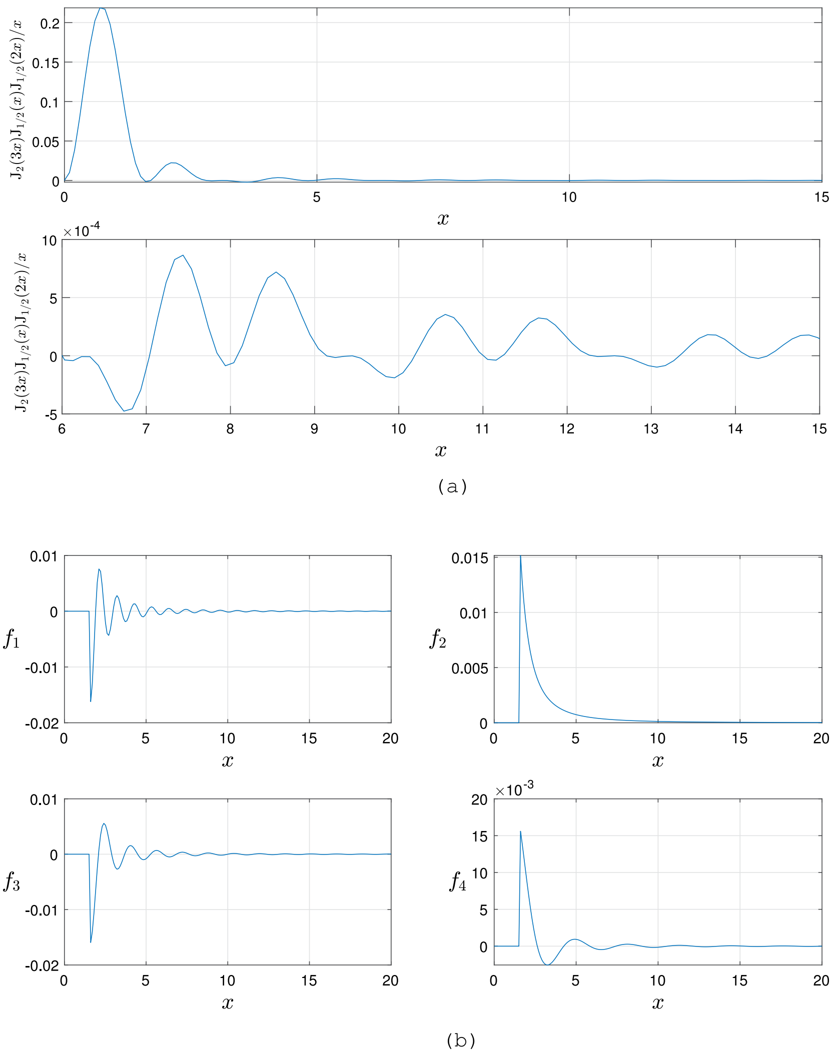


Fig. 3. (a) Behavior of the integrand function for the integral in (55) with $r = 2$, $t = 1/2$ $a = 3$, $b = 1$, $c = 2$ and (b) behavior of the relevant integrand functions $f_m(x)$.

Table 1
Number of PIEs for the calculation of $I^{\mathrm{ex1}}(3,1,3,3.5)$ with different extrapolation methods.

Extrapolation method	N_1	N_2	N_3	N_4	N_{tot}	Computation time
GLS d	24	20	13	14	71	9.3 s
GLS t	28	23	13	14	78	9.6 s
GLS u	28	23	13	15	79	9.7 s
GLS a	18	20	19	18	75	9.0 s
mGLS d	15	12	13	14	54	8.1 s
mGLS t	15	13	13	14	55	7.9 s
mGLS u	15	13	13	14	55	7.9 s
mGLS a	16	18	19	19	72	9.9 s
GWA	24	23	16	14	77	9.7 s

4. Numerical results

To assess the numerical accuracy of the proposed method, some illustrative examples are presented in this Section. We start with an integral which is known in a closed form, i.e.,

$$I^{\mathrm{ex1}}(r,a,b,c)=\int_0^\infty x^{1-r}\mathrm{J}_r(ax)\,\mathrm{J}_r(bx)\,\mathrm{J}_r(cx)\,\mathrm{d}x \tag{48}$$

In [47, 6.578.9] the result of (48) is given as

$$I^{\mathrm{ex1}}(r,a,b,c)=\begin{cases}\dfrac{2^{r-1}\left\{\left[c^2-(a-b)^2\right]\left[(a+b)^2-c^2\right]\right\}^{r-1/2}}{4^{2r-1}(abc)^r\,\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}, & |a-b|<c<a+b\\[2ex] 0, & c\le|a-b|\ \text{or}\ c\ge a+b\end{cases} \tag{49}$$

For example, it results

$$I^{\mathrm{ex1}}(3,1,3,3.5)=\int_0^\infty \frac{\mathrm{J}_3(x)\,\mathrm{J}_3(3x)\,\mathrm{J}_3(3.5x)}{x^2}\,\mathrm{d}x=0.00304967832719117\,(\dots) \tag{50}$$

In Fig. 1(a) the behavior of the integrand function is reported as a function of x in the interval $[0,15]$ and in a narrower interval $[6,15]$. As it can be seen, oscillations occur very irregularly. Based on the proposed approach, the oscillating part of the integrand function can be decomposed as in (9) and the most challenging part of the problem is reduced to the integration of the four functions f_m in (16). For the considered values it results $x_{\max}\simeq 4.527$ and the behavior of the functions f_m are reported in Fig. 1(b) in the interval $[x_{\max},20]$, where it can be verified that they are regularly oscillating, as expected.

The proposed quadrature schemes are always applied by setting a relative error $\varepsilon_{\mathrm{rel}}$: in practice, each integral I_m $(m=1,\dots,4)$ is calculated with a number N_m of partial integral evaluations (PIEs) such that

$$\left|I_m^{N_m}\right|=\varepsilon_{\mathrm{rel}}\left|I_m^{N_m-1}\right| \tag{51}$$

For a fair comparison, the partial integrals are always evaluated through the DE algorithm with machine precision (i.e., $\varepsilon_{\mathrm{rel}}<10^{-15}$). The exact result in (50) is readily reached with machine precision with all the considered extrapolation methods, i.e., the d variant of the general Levin–Sidi transformation with a Sidi partition (GLS d), the t variant of the general Levin–Sidi transformation with a Sidi partition (GLS t), the u variant of the general Levin–Sidi transformation with a Sidi partition (GLS u), the general Levin–Sidi transformation with analytical remainder estimates and Sidi partition (GLS a), the d variant of the general Levin–Sidi transformation with a modified Sidi partition (mGLS d), the t variant of the general Levin–Sidi transformation with a modified Sidi partition (mGLS t), the u variant of the general Levin–Sidi transformation with a modified Sidi partition (mGLS u), the general Levin–Sidi transformation with analytical remainder estimates and modified Sidi partition (mGLS a), and the generalized weighted averages method (GWA).

For the methods which use an analytic remainder estimate (i.e., the GLS a, the mGLS a, and the GWA methods) the parameters to be used can be inferred by a comparison of the integrand function in (48) (decomposed in the sum (10)–(11), where the functions h_m asymptotically behaves as in (12)) with (27)–(29). It is then immediated to check that $\alpha=0$, $\lambda=1-r$ (or $\eta=-r-1/2$), and $Q_m=\pi/|u_m|$, with u_m as in (5), for $m=1,\dots,4$.

To obtain the desired accuracy, the number of partial integrals N_m corresponding to the computation of the integrals I_m $(m=1,\dots,4)$ in (16) (and the total number of PIEs N_{tot}) with relative error $\varepsilon_{\mathrm{rel}}<10^{-15}$ are reported in Table 1 for different extrapolation methods. It can be seen that the mGLS d, t, and u are those which require the lower number of PIEs and perform similarly. However, it should be noted that a lower number of PIEs not necessarily correspond to a lower computation time: in fact, the modified Sidi partition requires the numerical evaluation of the first zeros of the function $h_m(x)$ while the GWA method requires the evaluation of binomial coefficients when calculating the relevant weights. For these reasons in the last column of Table 1 the computation time required for the evaluation of $I^{\mathrm{ex1}}(3,1,3,3.5)$ is reported. All the simulations have been performed through the commercial software MATLAB with a processor 12th Gen Intel(R) Core(TM) i7-1255U 1.70 GHz and since the computation of a single integral is faster than 0.1 s we perform the computation 200 times to have reliable computation times. It can be seen that the

Table 2
Computation time (CT) for the evaluation of $I^{\text{ex1}}(3, 1, 3, c)$ in the interval $1 \le c \le 6$ with 200 sampling points.

Extrapolation method	CT for $\varepsilon_{\text{rel}} \le 10^{-4}$	CT for $\varepsilon_{\text{rel}} \le 10^{-8}$	CT for $\varepsilon_{\text{rel}} \le 10^{-15}$
GLS d	3.9 s	6.0 s	9.7 s
GLS t	3.5 s	5.9 s	9.4 s
GLS u	3.6 s	6.0 s	9.6 s
GLS a	3.2 s	4.8 s	8.7 s
mGLS d	3.9 s	5.3 s	8.5 s
mGLS t	3.4 s	4.9 s	8.4 s
mGLS u	3.6 s	5.2 s	8.5 s
mGLS a	3.7 s	5.2 s	9.4 s
GWA	4.3 s	6.6 s	9.6 s

Table 3
Computation times for the evaluation of the integral $I^{\text{ex1}}(r; a, b, c)$ (in s).

r	a	b	c	$I^{\text{ex1}}(r; a, b, c)$	GLS t	GLS a	mGLS t	GWA
1	1.0	1.0	1.0	$2.75664447710896 \cdot 10^{-1}$	4.8	6.2	5.3	4.9
5	1.0	1.0	1.0	$9.22983641889160 \cdot 10^{-5}$	8.9	7.8	10.0	7.4
10	1.0	1.0	1.0	$3.16133487347087 \cdot 10^{-11}$	19.0	9.8	15.7	10.7
1	10.0	1.0	10.0	$3.17911749835126 \cdot 10^{-2}$	4.5	5.7	4.7	6.8
5	10.0	1.0	10.0	$3.33062999122039 \cdot 10^{-5}$	6.9	6.9	7.7	12.4
10	10.0	1.0	10.0	$4.74745756340809 \cdot 10^{-11}$	17.0	9.8	20.6	14.6
1	10.0	10.0	10.0	$2.75664447710896 \cdot 10^{-2}$	4.9	6.0	5.2	4.9
5	10.0	10.0	10.0	$9.22983641889160 \cdot 10^{-2}$	8.9	7.8	10.1	7.7
10	10.0	10.0	10.0	$3.16133487347087 \cdot 10^{-3}$	19.0	9.6	16.2	10.5

GLS transformations with the modified Sidi partition and with numerical remainder estimates (i.e., the mGLS d, t, and u) are the most efficient extrapolation strategies. While the modified Sidi partition significantly improve the efficiency of the GLS transformations with numerical remainder estimates, the original Sidi partition is more efficient when applied to the GLS with analytical remainder estimates.

In order to show how the relative error ε_{rel} between the exact result in (50) and the integral calculated with the proposed approach depends on the number of PIEs, in Fig. 2 such an error is reported as a function of the number N of the partial integrals (by assuming the same number for each I_m integral) for different extrapolation strategies.

Finally, the integral is perfectly and correctly reproduced by the proposed integration schemes. For example, by considering the integral as a function of the parameter c, when $I^{\text{ex1}}(3, 1, 3, c) \neq 0$, the relative error is always below $\varepsilon = 10^{-15}$, while when $I^{\text{ex1}}(3, 1, 3, c) = 0$ the result of the quadrature scheme is 0 up to the 18-th decimal digit. This example offers us the chance to clarify an issue that may arise when the result of the integral is zero and can be a possible general drawback of the proposed formulation. In fact, all the integrals I_m $(m = 0, 1, 1, \ldots, 4)$ in (15) can be calculated with a precision $\varepsilon = 1 \cdot 10^{-15}$, i.e., up to 15 significant digits. Let us assume that their absolute value is of the order of 10^d. This means that in the worst case, when a significant cancellation of all the addends occurs, one may expect an accuracy up to the $(15 - d)$-th *decimal* digit. For example, in the case $r = 3$, $a = 1$, $b = 3$, and $c = 5$ it results

$$
\begin{aligned}
&I_0 = 0.00017586687105131 58(\ldots) \\
&\frac{1}{4}\sum_{m=1}^{4} I_m = -0.00017586687105131 55(\ldots)
\end{aligned} \tag{52}
$$

and the relevant sum in (15) leads to

$$
I^{\text{ex1}}(3; 1, 3, 5) = \int_0^\infty \frac{\mathrm{J}_3(x)\,\mathrm{J}_3(3x)\,\mathrm{J}_3(5x)}{x^2}\,\mathrm{d}x = I_0 + \frac{1}{4}\sum_{m=1}^{4} I_m = 0.0000000000000000003(\ldots) \tag{53}
$$

The relevant computation time (CT) corresponding to the different extrapolation strategies is reported in Table 2 when 200 sampling points are selected in the interval $1 \le c \le 6$ by setting a relative error ε less than 10^{-4}, 10^{-8}, and 10^{-15}, respectively.

In order to investigate the efficiency of the methods as a function of the order of the Bessel functions and of their arguments, in Table 3 we report the computation time for different extrapolation methods needed to reach the machine accuracy for different values of r, a, b, and c (in order to reliably compare the computation times we have evaluated the relevant integrals 200 times). The GLS t and the mGLS t methods are chosen as representative methods for the GLS strategy with numerical remainder estimate (NRE-GLS methods) since we have checked that only small differences in CTs exist for these methods.

It can be seen that when a low order r of the Bessel functions is involved, the GLS t strategy appears to be the most efficient extrapolation method and we have numerically checked that this happens up to $r = 3$. However, increasing the order of the Bessel functions, the NRE-GLS methods become slower and slower, while the methods with an analytical remainder estimates (ARE methods, i.e., GLS a and GWA) becomes significantly more efficient, regardless of the argument values. In fact, as the order of the Bessel functions increases, the equally spaced zeros approximation used for the interval endpoints in (31) or (33) becomes less and less accurate and badly affects the performance of NRE-GLS methods, as already pointed out in [26]. In fact, in general, the distance

Table 4
Computation times for the evaluation of the integral $I^{\mathrm{ex1}}(r;a,b,c)$ (in s).

r	a	b	c	$I^{\mathrm{ex1}}(r;a,b,c)$	GLS t with exact zeros
1	1.0	1.0	1.0	$2.75664447710896 \cdot 10^{-1}$	6.5
5	1.0	1.0	1.0	$9.22983641889160 \cdot 10^{-5}$	10.9
10	1.0	1.0	1.0	$3.16133487347087 \cdot 10^{-11}$	7.9
1	10.0	1.0	10.0	$3.17911749835126 \cdot 10^{-2}$	15.9
5	10.0	1.0	10.0	$3.33062999122039 \cdot 10^{-5}$	26.6
10	10.0	1.0	10.0	$4.74745756340809 \cdot 10^{-11}$	24.6
1	10.0	10.0	10.0	$2.75664447710896 \cdot 10^{-2}$	6.9
5	10.0	10.0	10.0	$9.22983641889160 \cdot 10^{-2}$	10.6
10	10.0	10.0	10.0	$3.16133487347087 \cdot 10^{-3}$	8.1

between the exact zeros of the h_i functions is largest between the initial zeros, and this separation converges to $\pi/|u_i|$ only for sufficiently large x. On the other hand, such an approximation does not have effects on the ARE methods which thus turn out to be the most efficient extrapolation methods in these cases. One could think to use the exact zeros of the h_i functions in (8) as breaking points for the NRE-GLS methods and we have checked that this allows for a significant lowering of the number of PIEs necessary for achieving a given accuracy. According to our tests, we have found a small advantage in terms of computation time with respect to the ARE methods (i.e., the GLS a and the GWA methods). However, when used with integrals involving only Bessel functions of low order, the numerical determination of the exact zeros makes the NRE-GLS methods very inefficient with respect to all the other methods. The determination of the nth exact zero has been performed with a method similar to that used to determine the initial zero x_{h_m} in (32), and in particular, evaluating the functions $h_m(x)$ at $x_{\max}+3(n-1)Q_m/4$, by taking steps of size $Q_m/4$ until a change in the sign of the function $h_m(x)$ is detected, and then apply the derivative-free Brent method [43]. For completeness, we report in Table 4 the same of Table 3 for the GLS t method using the exact zeros of the h_i functions.

A second example of analytical evaluation of an integral of the class (1) is the following

$$I^{\mathrm{ex2}}(r,t;a,b,c) = \int_0^\infty x^{1-r} \mathrm{J}_r(ax)\,\mathrm{J}_t(bx)\,\mathrm{J}_t(cx)\,\mathrm{d}x \tag{54}$$

In [47, 6.578.8] the result of (54) is given as

$$I^{\mathrm{ex2}}(r,t;a,b,c) = \begin{cases} \sqrt{\dfrac{2}{\pi^3}}\,\dfrac{(bc)^{r-1}}{a^r}\,(\sinh u)^{r-\frac{1}{2}}\,\sin[(r-t)\pi]\,\mathrm{e}^{\mathrm{j}\left(r-\frac{1}{2}\right)\pi}\,Q_{t-\frac{1}{2}}^{\frac{1}{2}-r}(\cosh u)\,, & a > b+c \\ \sqrt{\dfrac{1}{2\pi}}\,\dfrac{(bc)^{r-1}}{a^r}\,(\sin v)^{r-\frac{1}{2}}\,P_{t-\frac{1}{2}}^{\frac{1}{2}-r}(\cos v)\,, & |b-c| \le a \le b+c \\ 0\,, & 0 < a < |b-c| \end{cases} \tag{55}$$

where $P_\alpha^\beta(\cdot)$ and $Q_\alpha^\beta(\cdot)$ are the associated Legendre functions of first and second kind, respectively, defined in terms of Gauss hypergeometric functions ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ [47, 8.702-8.703] and where

$$\begin{aligned} 2bc\cosh u &= a^2 - b^2 - c^2\,, \\ 2bc\cos v &= b^2 + c^2 - a^2\,. \end{aligned} \tag{56}$$

The result (55) is valid provided $b > 0$, $c > 0$, $r > -1/2$, and $t > -1$.

For example, it results

$$I^{\mathrm{ex2}}\left(2,\frac{1}{2},3,1,2\right) = \int_0^\infty \frac{\mathrm{J}_2(3x)\,\mathrm{J}_{1/2}(x)\,\mathrm{J}_{1/2}(2x)}{x}\,\mathrm{d}x = 0.188628080701505(\ldots) \tag{57}$$

In Fig. 3(a) the behavior of the integrand function is reported as a function of x in the interval $[0,15]$ and in a narrower interval $[6,15]$. As before, the oscillating part of the integrand function can be decomposed as in (9) and we need to integrate the four functions f_m in (16). For the considered values it results $x_{\max} \simeq \pi/2$ and the behavior of the functions f_m are reported in Fig. 3(b) in the interval $[x_{\max}, 20]$, where it can be verified that they are regularly oscillating, as expected, except for f_2: in fact, since for the chosen values it results $u_2 = 0$, f_2 is a monotonically decreasing (and non oscillating) function. Once the proposed quadrature scheme is applied with relative error $\varepsilon_{\mathrm{rel}} = 10^{-15}$ the exact result in (50) is readily reached with machine precision with all the considered extrapolation methods. To obtain this goal, the number of partial integrals N_m corresponding to the computation of the integrals I_m $(m = 1,3,4)$ in (16) are reported in Table 5 for different extrapolation methods. As before, the computation time required for the evaluation of 100 integrals $I^{\mathrm{ex2}}\left(2,\frac{1}{2},3,1,2\right)$ is also reported in the last column. It can be seen that the GLS t and GLS u with the original and modified Sidi partitions are those which require the lowest and comparable number of PIEs. However, the modified Sidi partition is computationally more cumbersome, so that eventually the GLS t and GLS u are the most efficient extrapolation strategies with a slight advantage with respect to the GWA method.

The integral is perfectly and correctly reproduced by the proposed integration scheme. In particular, by considering the integral as a function of the parameter a, the computation time corresponding to the different extrapolation strategies used to obtain a given accuracy is reported in Table 6 when 200 sampling points are selected in the interval $1 \le a \le 10$.

Table 5
Number of PIEs for the calculation of $I^{\text{ex2}}\left(2, \frac{1}{2}, 3, 1, 2\right)$ with different extrapolation methods. The number N_2 of partial integrals corresponding to the integration of the function h_2 is missing since such an integration does not require the application of an extrapolation method (the function h_2 is a monotonically decreasing function).

Extrapolation method	N_1	N_3	N_4	N_{tot}	Computation time
GLS d	13	14	14	41	1.8 s
GLS t	13	13	14	40	1.7 s
GLS u	13	14	15	42	1.7 s
GLS a	20	20	20	60	2.4 s
mGLS d	12	13	13	38	2.3 s
mGLS t	12	13	14	39	2.1 s
mGLS u	13	14	15	42	2.2 s
mGLS a	20	20	20	60	2.8 s
GWA	18	16	14	48	2.0 s

Table 6
Computation time (CT) for the evaluation of $I^{\text{ex2}}\left(2, \frac{1}{2}, a, 1, 2\right)$ in the interval $1 \leq a \leq 10$ with 200 sampling points.

Extrapolation method	CT for $\epsilon_{\text{rel}} \leq 10^{-4}$	CT for $\epsilon_{\text{rel}} \leq 10^{-8}$	CT for $\epsilon_{\text{rel}} \leq 10^{-15}$
GLS d	3.0 s	4.0 s	5.3 s
GLS t	2.6 s	3.6 s	5.1 s
GLS u	2.7 s	3.7 s	5.3 s
GLS a	2.8 s	4.0 s	6.8 s
mGLS d	3.3 s	4.1 s	5.4 s
mGLS t	2.9 s	3.7 s	5.2 s
mGLS u	3.0 s	3.8 s	5.3 s
mGLS a	3.1 s	4.0 s	6.9 s
GWA	3.2 s	4.3 s	6.0 s

Table 7
Computation time (CT) for the evaluation of $I^{\text{ex2}}(r, 1/2, 3, 1, 1/2)$ in the interval $0 \leq r \leq 10$ with 200 sampling points.

Extrapolation method	CT for $\epsilon_{\text{rel}} \leq 10^{-4}$	CT for $\epsilon_{\text{rel}} \leq 10^{-8}$	CT for $\epsilon_{\text{rel}} \leq 10^{-15}$
GLS d	3.3 s	4.8 s	6.9 s
GLS t	3.1 s	4.7 s	7.0 s
GLS u	3.3 s	4.8 s	7.2 s
GLS a	3.0 s	4.1 s	7.0 s
mGLS d	3.6 s	5.2 s	7.7 s
mGLS t	3.5 s	5.1 s	7.5 s
mGLS u	3.5 s	5.2 s	7.8 s
mGLS a	3.3 s	4.3 s	7.2 s
GWA	3.3 s	4.6 s	6.8 s

Table 8
Computation time (CT) for the evaluation of $I^{\text{ex2}}(2, t, 1, 1, 1/2)$ in the interval $0 \leq t \leq 10$ with 200 sampling points.

Extrapolation method	CT for $\epsilon_{\text{rel}} \leq 10^{-4}$	CT for $\epsilon_{\text{rel}} \leq 10^{-8}$	CT for $\epsilon_{\text{rel}} \leq 10^{-15}$
GLS d	5.1 s	8.0 s	13.8 s
GLS t	4.9 s	8.1 s	13.9 s
GLS u	5.0 s	8.3 s	14.2 s
GLS a	3.9 s	5.6 s	10.5 s
mGLS d	5.4 s	8.1 s	13.7 s
mGLS t	5.2 s	8.0 s	13.7 s
mGLS u	5.1 s	8.2 s	13.7 s
mGLS a	4.4 s	5.8 s	10.3 s
GWA	4.8 s	6.9 s	9.9 s

The same applies for the integrals $I^{\text{ex2}}(r, 1/2, 3, 1, 1/2)$ and $I^{\text{ex2}}(2, t, 1, 1, 1/2)$ considered as functions of the (real) orders r and t, respectively. It is worth noting that for $r = 0$, the integral in (54) is an improper integral that, however, can easily be integrated by the proposed algorithm. The relevant computation time corresponding to the different extrapolation strategies used to obtain a given accuracy is reported in Tables 7 and 8 when 200 sampling points are selected in the interval $0 \leq r \leq 10$ and $0 \leq t \leq 10$, respectively.

We have noted that by increasing the order t of the Bessel functions a larger number of PIEs is needed to reach machine accuracy precision. In fact, for example, let us consider the particular integral

$$I^{\text{ex2}}(2, 10, 1, 1, 1/2) = \int_0^{\infty} \frac{\mathrm{J}_2(x)\,\mathrm{J}_{10}(x)\,\mathrm{J}_{10}\left(\frac{x}{2}\right)}{x}\,\mathrm{d}x = -0.0012483776971016481\,(\ldots) \tag{58}$$

Table 9
Number of PIEs for the calculation of $I^{\mathrm{ex2}}(2, 10, 1, 1, 1/2)$ with different extrapolation methods.

Extrapolation method	N_1	N_2	N_3	N_4	N_{tot}	Computation time
GLS *d*	44	46	24	47	161	10.7 s
GLS *t*	46	48	27	48	169	10.9 s
GLS *u*	46	48	27	49	170	10.9 s
GLS *a*	16	19	20	17	72	5.2 s
mGLS *d*	41	26	33	91	191	12.4 s
mGLS *t*	43	28	35	92	198	12.6 s
mGLS *u*	44	28	35	93	200	12.7 s
mGLS *a*	16	18	20	16	70	5.2 s
GWA	29	19	19	25	92	5.9 s

Once the proposed quadrature scheme is applied with relative error $\varepsilon_{\mathrm{rel}} = 10^{-15}$ the exact result in (58) is reached with machine precision with all the considered extrapolation methods. However, the number of partial integrals N_m ($m = 1, \ldots, 4$) strongly depends on the particular extrapolation method and are reported in Table 9. Again, the computation time required for the evaluation of 100 integrals $I^{\mathrm{ex2}}(2, 10, 1, 1, 1/2)$ is also reported in the last column.

On the other hand, when the order t of the Bessel functions is further increased (larger than $t > 20$), only the ARE extrapolation methods (i.e., GLS *a* and GWA) are able to reach the desired precision. Moreover, as it can be seen in Table 9, all the NRE GLS algorithms require a huge number of PIEs. The problem lies, as in the case of Example 1 previously considered (results in Tables 3 and 4), in the choice of the breaking points ((31) or (33)): in fact, using the exact zeros of the h_i functions in (8) as breaking points for the NRE-GLS methods dramatically lowers the number of PIEs (in particular, for the case of Table 9, it would result $N_1 = 16$, $N_2 = 19$, $N_3 = 15$, $N_4 = 38$, with $N_{\mathrm{tot}} = 88$). However, the additional computation time needed for the determination of such exact zeros does not make the GLS methods with exact zeros more efficient than the ARE extrapolation methods (for the case of Table 9 such a computation time is 6.7 s), although it is significantly faster than all the other NRE-GLS methods and there is no difficultly in reaching the desired accuracy when Bessel functions of large order are involved. Therefore, in this case the ARE methods definitely outperform all the other extrapolation strategies making the proposed quadrature procedure even faster than the direct calculation through the relevant associated Legendre functions by means of built-in functions of some mathematical software (like e.g., Matlab) (which involves the calculation of Gauss hypergeometric functions).

As a final example, we consider a class of integrals which have been extensively studied in the field of nuclear physics, particle physics, astrophysics, and cosmology, i.e.,

$$I^{\mathrm{ex3}}(\mu, \nu, \xi; a, b, c) = \int_0^\infty x^2 j_\mu(ax)\, j_\nu(bx)\, j_\xi(cx)\, \mathrm{d}x \tag{59}$$

Integrals (59) enter in phenomenological calculations in high-energy physics using the Regge formalism [17], in the calculation of Green's functions in pion–nucleon scattering [48], in the evaluation of partial wave integrals occurring in the theory of relativistic (e, 2e) collisions [49], in the computation of the redshift-space matter power spectrum in standard perturbation theory using effective field theory [50], or in the computation of the galaxy 3PCF and 4PCF covariance matrices [51], just to name few examples.

Since the spherical Bessel function $j_\sigma(x)$ can be written in terms of a Bessel function as

$$j_\sigma(x) = \sqrt{\frac{\pi}{2x}}\, \mathrm{J}_{\sigma+1/2}(x) \tag{60}$$

we have

$$I^{\mathrm{ex3}}(\mu, \nu, \xi; a, b, c) = \sqrt{\frac{\pi^3}{8abc}} \int_0^\infty x^{1/2} \mathrm{J}_{\mu+1/2}(ax)\, \mathrm{J}_{\nu+1/2}(bx)\, \mathrm{J}_{\xi+1/2}(cx)\, \mathrm{d}x \tag{61}$$

Integrals (59) or (61) can be evaluated analytically, although the relevant expressions may appear very complicated, involving 3j and 6j Wigner symbols (related to the Clebsch–Gordan and Racah's W coefficients, respectively, [52]) and/or various combinations of associated Legendre functions, generalized hypergeometric functions, or Appell functions of two variables [15–17,19]. In order to assess the accuracy and efficiency of the proposed method in Table 10 we have compared the results presented in [19] pointing out the computation time for different extrapolation methods needed to reach the accuracy of the provided exact results (to compare the computation times we have evaluated the relevant integrals 200 times).

In these cases, where a low order of Bessel functions is involved, the mGLS *t* strategy appears to be the most efficient extrapolation method. It should be noted that in one case (i.e., $\mu = \nu = \xi = 4$ and $a = 0.03$, $b = 1.0$, and $c = 1.02$) the GLS *t* method does not manage to reach the desired accuracy (and the relevant computation time is not reported, but indicated with the symbol –). This could be explained by observing that while the h_i functions ($i = 1, 2, 3$) are correctly oscillating starting from $x_{\max} = 206.59$, the h_4 function is not. However, h_4 is correctly oscillating starting from $x_{h_4} = 569.52$. To clarify this point, in Fig. 4 the behavior of the h_4 function is reported together with the choice of the breaking points as in (31) (GLS *t* method) and in (33) (mGLS *t* method).

We have considered the same integral, but changing significantly the orders of the Bessel functions to test the efficiency and accuracy of the proposed method against the analytical results. We have thus used the analytic expression derived in [19], which

Table 10
Computation times for the evaluation of the integral $I^{\text{ex3}}(\mu,\nu,\xi;a,b,c)$ (in s).

μ	ν	ξ	a	b	c	$I^{\text{ex3}}(\mu,\nu,\xi;a,b,c)$	GLS t	GLS a	mGLS t	GWA
0	0	0	1.0	2.0	1.5	0.261799387799149	3.3	5.5	3.3	3.6
0	1	1	1.0	2.0	1.5	0.229074464324255	5.6	6.0	4.1	4.5
1	1	0	1.0	2.0	1.5	0.179987079111915	4.5	5.4	3.8	4.9
3	2	1	1.0	2.0	1.5	0.112875419641919	9.4	8.0	6.9	6.9
4	4	4	1.0	2.0	1.5	−0.0456849264424663	9.0	8.6	9.9	8.1
0	0	0	1.0	5.0	5.5	0.0285599332144526	3.6	5.3	3.3	4.4
4	4	4	1.0	5.0	5.5	−0.00802387523708195	14.2	8.0	7.0	12.8
0	0	0	0.06	1.0	1.05	12.4666375142	7.1	6.1	2.3	6.2
4	4	4	0.06	1.0	1.05	−1.51165380809695	7.8	7.5	6.5	11.2
0	0	0	0.03	1.0	1.02	25.6666066469756	2.6	4.1	2.6	5.2
4	4	4	0.03	1.0	1.02	−10.9755226591807	–	6.9	6.4	13.1

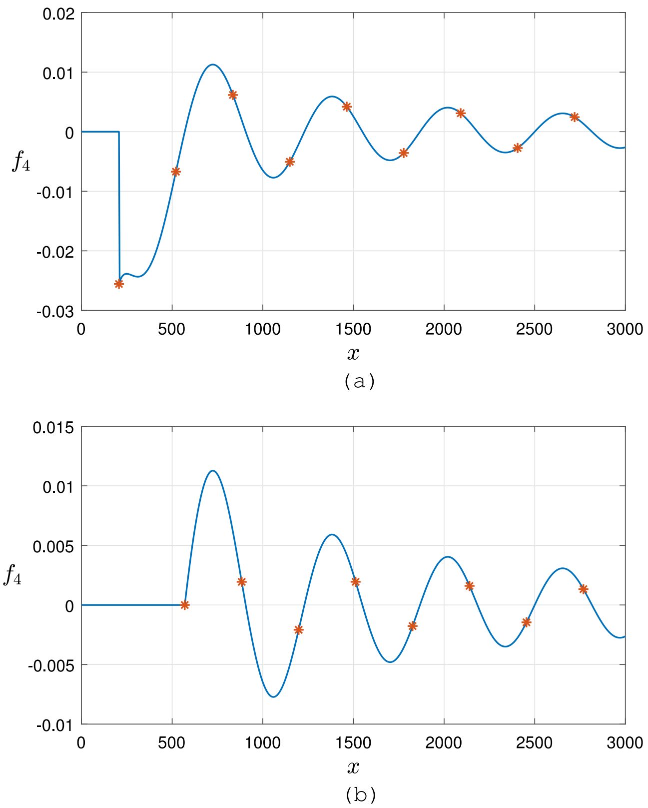


Fig. 4. Behavior of the function h_4 relevant to the integral $I^{\text{ex3}}(4,4,4;0.03,1,1.02)$ and choice of the breaking points according to (31) (a) and (33) (b).

reads

$$I^{\text{ex3}}(\mu,\nu,\xi;a,b,c)=\frac{\pi\beta(\Delta)}{4abc}\,\mathrm{i}^{\mu+\nu-\xi}(2\xi+1)^{1/2}\left(\frac{a}{c}\right)^{\xi}\begin{pmatrix}\mu & \nu & \xi\\ 0 & 0 & 0\end{pmatrix}^{-1}\sum_{L=0}^{\xi}\binom{2\xi}{2L}^{1/2}\left(\frac{b}{a}\right)^{L}$$
$$\sum_{l=|\mu-\xi+L|}^{\mu+\xi-L}(2l+1)\begin{pmatrix}\mu & \xi-L & l\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\nu & L & l\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}\mu & \nu & \xi\\ L & \xi-L & l\end{Bmatrix}P_l(\Delta) \tag{62}$$

where

$$\begin{pmatrix}j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix} \tag{63}$$

and

$$\begin{Bmatrix}j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{Bmatrix} \tag{64}$$

Table 11
Computation times for the evaluation of the integral $I^{\mathrm{ex3}}(\mu,\nu,\xi;a,b,c)$ (in s).

μ	ν	ξ	a	b	c	$I^{\mathrm{ex3}}(\mu,\nu,\xi;a,b,c)$	Analytic	GLS t	GLS a	mGLS t	GWA
0	0	0	1.0	1.0	1.0	0.78539816339744	0.8	3.6	5.6	3.4	4.8
						0.78539816		1.2	1.4	1.3	1.3
						0.7854		0.7	0.8	0.8	0.7
0	5	5	1.0	1.0	1.0	0.07056311624274	4.9	11.0	10.0	8.0	7.1
						0.070563116		4.5	2.3	3.2	2.6
						0.07056		1.1	1.1	2.2	1.3
6	5	5	1.0	1.0	1.0	0.41113081960316	28.7	13.0	9.4	14.9	9.4
						0.41113082		5.0	2.6	5.5	3.0
						0.4111		1.4	1.1	1.6	1.1
0	10	10	1.0	1.0	1.0	−0.1478344023762	8.9	14.4	7.8	15.9	8.7
						−0.14783440		6.1	2.9	5.7	3.4
						−0.1478		2.0	1.1	2.5	1.2
10	10	10	1.0	1.0	1.0	0.400126273318	100.3	20.6	9.3	18.5	10.4
						0.40012627		9.9	3.7	8.3	4.1
						0.4001		2.0	1.1	2.5	1.2

indicates the 3j and 6j Wigner symbols [52], respectively, $P_l(\cdot)$ indicates the Legendre polynomial of order l, the quantity Δ is defined as

$$\Delta = \frac{a^2+b^2-c^2}{2ab} \tag{65}$$

and the function $\beta(y)$ is

$$\beta(y) = H(1-y)\,H(1+y) \tag{66}$$

where $H(\cdot)$ is the Heaviside unit-step function. It should be noted that the result expressed by (62) is valid under some important restrictions: in particular, the indexes of the spherical Bessel functions must satisfy the triangular condition, i.e.,

$$|\mu-\nu| \le \xi \le \mu+\nu \tag{67}$$

and, moreover, the sum $(\mu+\nu+\xi)$ of the indexes of the spherical Bessel functions must be an even number, i.e.,

$$\mu+\nu+\xi = 2n, \qquad n \in \mathbb{N}. \tag{68}$$

The effect of the function $\beta(\cdot)$ consists in making zero the integral when the triangular condition for the arguments of the Bessel functions is not satisfied, i.e., when $c < |a-b|$ or $c > a+b$. When $c = |a-b|$ or $c = a+b$ a jump discontinuity is present [19].

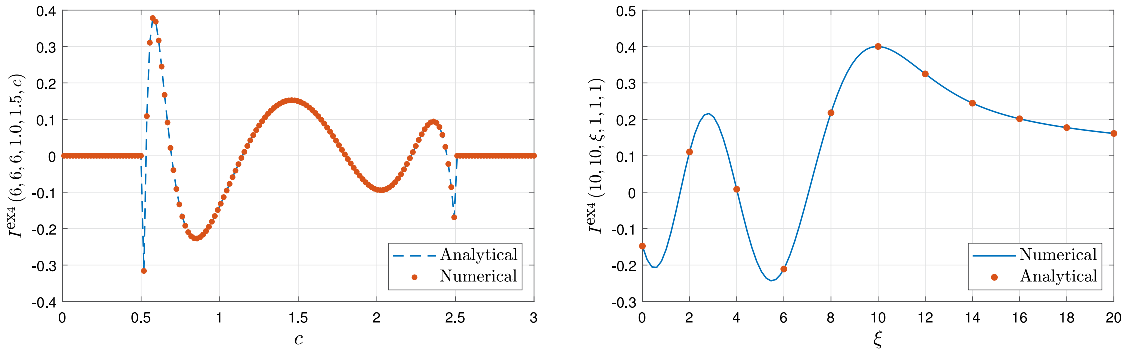
Finally, it is interesting to analyze the efficiency of the proposed method, also in comparison with the analytical expression. In Table 11 we report the results of the integrals $I^{\mathrm{ex3}}(\mu,\nu,\xi;a,b,c)$ when $a=b=c=1$ and different values of μ, ν, and ξ together with the computation time required for 200 evaluations (including the computation time for analytical evaluations through (62)). The computation times are reported for different accuracies. It can clearly be seen that increasing the orders of the involved Bessel functions significantly increases the computation time required to obtain the analytical results. The extrapolation method with analytical remainder estimate (i.e., GLS a and GWA) are in general the most efficient extrapolation strategies and, in the case of low orders of Bessel functions, their computation times are comparable with those of the analytical method for low or moderate accuracies. Moreover, when the sum of the orders are larger than 12, the proposed method becomes more efficient than the analytical formulation even for high accuracy, thus clearly revealing its efficiency.

It is worth noting that the proposed method correctly reproduces the discontinuities of the integral $I^{\mathrm{ex3}}(\mu,\nu,\xi;a,b,c)$ when $c=|a-b|$ or $c=a+b$, as it can be seen in Fig. 5(a) where the integral $I^{\mathrm{ex3}}(6,6,6;1,1.5,c)$ is reported as a function of the argument c (this is one of the cases considered in Figure 2 of [19]). Although not reported, the proposed method can easily reach a relative error smaller than 10^{-11} in the entire range. It is worth noting that the analytical expression requires 20.1 s to cover the entire range with 100 sampling points, while the proposed method with the fastest extrapolation method (in this case the GWA method) requires only 4.1 s (almost 5 times faster than the analytical method) which is reduced to 0.6 s when an accuracy of 5 significant digits is sufficient (in this case the proposed method is more than 300 times faster than the analytical formulation).

Moreover, as pointed out above, the validity of the analytical expression (62) is restricted to conditions (67)–(68). In Fig. 5(b), the value of integral $I^{\mathrm{ex3}}(10,10,\xi;1,1,1)$ is reported as a function of the order ξ and it can easily be seen that the proposed numerical method can simply fill the gap between the allowed values of ξ of the analytical expression (62).

Based also on the previous reported results, the GLS a and GWA methods seem to be the most reliable and efficient extrapolation methods, although in some particular cases (and only when the order of the involved Bessel functions is low) the GLS methods with numerical remainder estimates can be faster.

It is worthy to remark that only few classes of integrals containing three Bessel functions can be solved analytically (basically those considered in the examples shown above), but the proposed method is able to efficiently and accurately evaluate any integral of the type (1) with the condition that $f(x)$ is a non-oscillating function.



(a) (b)

Fig. 5. Comparison between the exact result in (62) and the integral calculated with the proposed approach for the integral $I^{ex3}(6,6,6;1,1.5,c)$ as a function of the argument c (a) and for the integral $I^{ex3}(10,10,\xi;1,1,1)$ as a function of the parameter ξ (b).

Finally, we have already noted in the Introduction that in general standard quadrature routines implemented in commercial software (usually based on adaptive quadrature rules) cannot handle the class of integrals considered in the paper, at least as long as the integrand functions are rapidly oscillating and slowly decaying. Exceptions may occur when the integrand function decays sufficiently fast, as in the case considered in Example 1, with $r \geq 2$. In these cases a black box implementation can be even much faster than the proposed methods. However, such black box implementations are not able to reach convergence when, e.g., $r = 0$ or $r = 1$. Similar considerations hold for the cases in Example 2. As concerns Example 3 (which is of great importance in physics), black box implementations give completely unreliable (and erroneous) results and, in general, several warning messages appear which prevent the systematic use of these packages.

5. Conclusion

A method to efficiently evaluate integrals containing the product of three Bessel functions of the first kind and of any non-negative real order is presented. The proposed method is based on a decomposition of such a product in a sum of at most four functions which asymptotically approach sinusoidal functions for which well-established integration schemes based on partition-extrapolation methods can be applied. In particular, we compare the most powerful extrapolation techniques, i.e., the Levin–Sidi extrapolation method with numerical and analytical remainder estimates (with different partition schemes) and the generalized weighted averages method. Extensive numerical results clearly show the robustness, the accuracy, and the efficiency of the proposed method: in particular, for a class of integrals often encountered in a variety of applications, the proposed method results to be even much faster than the available analytical formulas. The reported examples indicate that the methods with analytical remainder estimates (e.g., GLS a and GWA) are the most efficient and reliable extrapolation methods (especially when Bessel functions of large order are involved) with the GLS a method slightly more efficient for small or moderate accuracy; however, it should be taken into account that the GWA is very simple to program with respect to all the other algorithms. In any case, it should be noted that the methods with analytical remainder estimate are less general since they require the knowledge of the asymptotic behavior of the integrand function. If such an asymptotic behavior cannot be known in advance, the GLS t method with a modified Sidi partition appears to be the most efficient extrapolation scheme, except when Bessel functions of large order are involved: in such a case, it would be much more convenient to use a GLS strategy with the exact zeros of the involved h_i functions.

Data availability

Data will be made available on request.

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