LTV stochastic systems stabilization with large and variable input delay

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Abstract

In this paper we propose a solution to the state-feedback and output-feedback stabilization problem for linear time-varying stochastic systems affected by arbitrarily large and variable input delay. It is proved that under the proposed controller the underlying stochastic process is exponentially centered and mean square bounded. The solution is given through a set of delay differential equations with cardinality proportional to the delay bound. The predictor is based on the semigroup generated by the closed-loop system in absence of delay, and its computation is described by a numerically reliable and robust method. In the deterministic case this method generates the same optimal trajectories as in the delay-less case.

Key words: stochastic systems; time-varying systems; input delay; state-feedback and output-feedback stabilization.

1 Introduction

The control problem of linear systems with input delay has been thoroughly studied in recent years both in the Lyapunov-Krasovskii framework and in the predictor feedback approach (Fridman, 2014; Kharitonov, 2015; Zhou, 2014b; Karafyllis and Krstic, 2017). In this paper we consider linear time-varying stochastic systems (LTVSS) with large and known input delays for which comparatively fewer results exist. Although the problem of stability conditions for time-varying systems with delays has been studied in the linear deterministic case (e.g. Krstic (2010); Cai et al. (2017); Sanz et al. (2019)) and in the more general context of nonlinear systems in several recent contributions (e.g. Bekiaris-Liberis and Krstic (2012); Mazenc and Malisoff (2016) and the references therein for the deterministic case and Li et al. (2020); Zhou and Luo (2018) for the stochastic case), the problem of designing controllers for LTVSS in presence of arbitrarily large input delays is still, to the best of our knowledge, unresolved. In Niu et al. (2009) and Li et al. (2009) it is possible to stabilize in probability/almost surely the state of the system (e.g. Niu et al. (2009) and Li et al. (2009)). However, *(i)* the noise process is only multiplicative and it allows to stabilize to zero the system in probability/almost surely (this is not possible in our case) *(ii)* the presence of an undelayed control input term simplifies the theoretical analysis. Ai et al. (2016) solve an output feedback stabilization problem for a class of stochastic feedforward nonlinear systems with time-varying input delay, and still differs from the proposed works because of the framework (point *(i)*) and the employed theoretical analysis. A relevant work which solves the stochastic control problem for the linear-quadratic case is Zhang and Xu (2016) where the noise is again multiplicative with respect to the state and control. For the case of unknown delays the most popular approach is emulation in which Lyapunov-Krasovskii functionals are used to derive robust stability conditions of basic feedback control designs for delay-less systems Malisoff and Zhang (2015); Mazenc et al. (2008). However, emulation provides conservative estimates of the maximum delay and it is thus unsuited to large delays. In order to compensate large delays, either constant or time-varying, the control design needs to use information about the delays as in the classical reduction model approach of Artstein (1982) (see also Mazenc et al. (2014) for time-varying systems). The reduction model approach is able to compensate for arbitrarily long input delays, but the computation of the input signal involves distributed terms and

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the solution of integral equations that are computationally challenging and potentially not robust (see Mondié and Michiels (2003)). In the approaches based on sequential predictors (originally proposed in Germani et al. (2002) for observers and in Besançon et al. (2007) for predictors), distributed terms are replaced by dynamic equations coupled by delayed correction terms that predict the system dynamics at different time points in the future (see also Najafi et al. (2013); Zhou et al. (2017); Mazenc and Malisoff (2017)). This is the approach that we pursue in this paper for dealing with large delays. This paper builds also over the approach based on closed-loop predictors, also known as pseudo-predictors, of Cacace et al. (2014); Zhou (2014a), where the core idea is to use the exponential of the closed-loop matrix as a finite-dimensional predictor. For systems that are exponentially unstable, the resulting feedback stabilizes the system up to a certain delay. In this work we adopt a different predictor, in the form of an observer with a delayed correction term that has the important advantage of allowing a cascaded structured that can cope with arbitrarily large delays, analogously to the approaches mentioned above and to Cacace et al. (2016); Najafi et al. (2013). The predictor has the same structure as in Cacace et al. (2016), where it was used in the deterministic time-invariant setting, and it makes use of the closed-loop dynamics in the observer gain. Our work extends Zhou (2014a) to large delays – also in exponentially unstable case – and to stochastic systems, and it provides also the computation of the closed-loop state transition matrix. In fact, a crucial step in the stochastic time-varying case is the computation of the closed-loop semigroup over a delay interval. We describe a numerically robust computation of the semigroup that, together with the cascaded observer-predictor, makes up the complete controller structure. This work can be seen also as the extension to the time-varying case of our recent papers Cacace et al. (2019) and Cacace et al. (2020) that are focused on the time-invariant case for linear systems with nonlinear diffusions and additive noise with large delays, respectively. A feature of the proposed approach is that in the deterministic case it recovers optimal solution in the LQR sense, that is, the same trajectories and value of the cost function.
The stabilization problem is defined in Section 2 and Section 3 provides the main results for the case of state-feedback. The robust implementation of the closed-loop semigroup is described in Section 4. The extension to output feedback control and time-varying delays is carried out in Section 5. Finally, Section 6 provides a numerical example and Section 7 gives some conclusions.

Notation. $\mathrm{tr}(M)$ is the trace of a square matrix M. $M > 0$ denotes a positive definite matrix. $\|x\|$ denotes the Euclidean norm for $x \in \mathbb{R}^n$ and $\|M\|$ the operator norm. On a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, $\mathbb{E}[\cdot]$ denotes the expectation, and $L^2(\Omega; \mathbb{R}^n)$ denotes the linear space of square integrable random vectors of $\mathbb{R}^n$ endowed with the norm $\|x\|_{L_2} = \left(\mathbb{E}\left[\|x\|^2\right]\right)^{\frac{1}{2}}$. Given a time-varying dynamical matrix A, then the semigroup or state transition matrix of A is denoted by $\Phi_A : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and it is such that for any $t, s \in \mathbb{R}_+$, $\frac{\partial \Phi_A}{\partial t}(t,s) = A(t)\Phi_A(t,s)$ and $\Phi_A(t,t) = I$. Furthermore, a state transition matrix Φ is uniformly exponentially stable (UES) with rate λ iff there exist $c > 0$ and $\lambda > 0$ such that $\|\Phi(t,t_0)\| \le c\,\mathrm{e}^{-\lambda(t-t_0)}$ for $t \ge t_0$.

2 Problem statement and preliminaries

On a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, we study the stabilization problem of linear-time varying systems with variable input delay in the Itô formalism of stochastic differential equation (SDE)

$$\mathrm{d}x(t) = \big(A(t)x(t) + B(t)u(\psi(t))\big)\mathrm{d}t + F(t)\,\mathrm{d}W_t, \quad (1)$$
$$\mathrm{d}y(t) = C(t)x(t)\mathrm{d}t + G(t)\,\mathrm{d}V_t, \quad (2)$$

where the state $x(t) \in \mathbb{R}^n$, and $\mathbb{E}[x(0)]$ and $\|x(0)\|_{L_2}$ are finite. The map ψ is Borel measurable, and the control $u : [-\delta, T] \to \mathbb{R}^p$ is $\mathcal{F}_t$-adapted such that $\int_{-\delta}^{T} \|u(s)\|^2_{L_2}\mathrm{d}s < +\infty$, for any $T > 0$. Moreover, $W_t \in \mathbb{R}^d$ and $V_t \in \mathbb{R}^\ell$ are $\mathcal{F}_t$-adapted standard Wiener processes mutually independent. The matrices $A(t)$, $B(t)$, $C(t)$, $F(t)$, $G(t)$ are of appropriate size piece-wise continuous in t and uniformly bounded. We made the following hypotheses.

Assumption 1 (a) *The couple* $(A(t), B(t))$ *is uniformly controllable;* (b) *the couple* $(C(t), A(t))$ *is uniformly observable.*

The couple $(A(t), B(t))$ is uniformly controllable if the inequalities (A.3) hold true, whilst the couple $(C(t), A(t))$ is uniformly observable if the dual version of (A.3) holds true (Cheng (1979)).

Assumption 2 *The map* ψ *can be expressed as* $\psi(t) = t - \delta(t)$, *with the delay function* $\delta : \mathbb{R}_+ \to [0, \bar{\delta}]$ *Borel measurable. Moreover, there exists and it is known the inverse function* ψ^{-1}.

Definition 1 *A stochastic process* $\{\xi(t)\}_{t\ge 0}$ *is said to be* exponentially centered with rate λ *if there exists* $\lambda > 0$ *such that given* ξ_0 *there exists* $c > 0$ *such that* $\|\mathbb{E}[\xi(t)]\| \le c\,\mathrm{e}^{-\lambda t}$, *and* mean square bounded *if there exist* $\kappa > 0$ *such that* $\sup_{t\ge 0} \|\xi(t)\|_{L_2} < \kappa$.

We briefly comment upon the requirements of Definition 1. Since system (1) is affected by additive noise, it is impossible with any control to achieve $\mathbb{E}[\|x\|^2] \to 0$ or any other type of convergence *to zero* that involves the norm of the state. Thus, the aim of the control is to have $\mathbb{E}[\|x\|^2] < \kappa$. However, in a control problem it is obviously desirable that the expected value of the state asymptotically or exponentially approaches the origin as its equilibrium point, a requirement that corresponds to exponentially centered solutions. Moreover, when $W_t = V_t = 0$, that is, the system is deterministic, an exponentially centered solution corresponds to an exponentially stabile solution.

Lemma 1 *Let us consider the stochastic system*

$$\mathrm{d}\xi(t) = (A(t)\xi(t) + v(t))\,\mathrm{d}t + F(t)\,\mathrm{d}W_t, \quad \|\xi(t_0)\|_{L_2} < \infty, \tag{3}$$

with the state transition matrix Φ_A UES with rate $\lambda > 0$ and the process $\{v(t)\}$ exponentially centered with rate $\nu > 0$ and mean square bounded. Thus, the process $\{\xi(t)\}$ is exponentially centered with rate $\min\{\lambda, \nu\}$ and mean square bounded;

Proof. By writing explicitly the unique solution to (3), and the assumption $\|v(t)\|_{L_2} < \kappa$, we can write

$$\|\xi(t)\|_{L_2} \leq \|\Phi_A(t,t_0)\|\|\xi(t_0)\|_{L_2} + \kappa \int_{t_0}^{t} \|\Phi_A(t,s)\|\,\mathrm{d}s + \left(\int_{t_0}^{t} \|\Phi_A(t,s)F(s)\|^2\,\mathrm{d}s\right)^{\frac{1}{2}}. \tag{4}$$

Since by assumption Φ_A is UES with rate λ, then from (4) it follows that $\{\xi(t)\}$ is mean square bounded. The solution $\{\xi(t)\}$ is exponentially centered with rate $\min\{\lambda, \nu\}$ since $v(t)$ is exponentially centered with rate ν and it obeys to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\xi(t)] = A(t)\mathbb{E}[\xi(t)] + \mathbb{E}[v(t)]. \tag{5}$$

3 State-feedback with constant input delay

In this section we focus on the state-feedback stabilization problem with constant input delay. In Section 5 we shall extend the results to the output-feedback stabilization problem with time-varying input delay. With $\psi(t) = t - \delta$, $\delta > 0$, the SDE (1) becomes

$$\mathrm{d}x(t) = \big(A(t)x(t) + B(t)u(t-\delta)\big)\mathrm{d}t + F(t)\,\mathrm{d}W_t, \tag{6}$$

3.1 Basic predictor

Let us define $\tilde{A}(t) = A(t) - B(t)K(t)$, where $K : \mathbb{R}_+ \to \mathbb{R}^p \times \mathbb{R}^n$ is a gain sequence. Moreover, let $M(t,s) = K(t)\Phi_{\widetilde{A}}(t, t-s)B(t-s)$, with $t \geq s \geq 0$ and let

$$\gamma_M(\lambda, t, \Delta) = \int_0^{\Delta} \|M(t,\tau)\| e^{\lambda\tau} d\tau. \tag{7}$$

Theorem 2 *Consider the process $\{x(t)\}$ solution to* (6) *with the Assumption 1*(a)*. Let $K : \mathbb{R}_+ \to \mathbb{R}^p \times \mathbb{R}^n$ be a control gain such that the closed-loop state transition matrix $\Phi_{\widetilde{A}}$ of $\widetilde{A}(t) = A(t) - B(t)K(t)$ is UES with rate $\lambda > 0$. For $t \geq -\delta$ consider the control law*

$$u(t) = -K(t+\delta)\theta(t) \tag{8}$$

$$\dot{\theta}(t) = \widetilde{A}(t+\delta)\theta(t) + B(t+\delta)K(t+\delta) \cdot \Phi_{\widetilde{A}}(t+\delta, t)\,(x(t) - \theta(t-\delta)), \tag{9}$$

where $\theta(\tau) = \Phi_{\widetilde{A}}(\tau+\delta, 0)\theta_0$ for $\tau \in [-\delta, 0]$ with $\theta_0 = x_0$. If $\sup_{t\geq\delta} \gamma_M(\lambda, t, \delta) < 1$, then the process $\{x(t)\}$ is (a) *exponentially centered with rate λ and* (b) *mean square bounded.*

Remark 1 *The gain sequence $\{K(t)\}$ is such that, with $u(t) = -K(t)x(t)$ and $F(t) \equiv 0$, the process $\{x(t)\}$ is exponentially centered with rate λ. A possible choice of K is therefore given by*

$$K(t) = \frac{1}{2}B^\top(t)H_\lambda^{-1}(t+\Delta, t), \tag{10}$$

where the map H_λ is given by (A.2)*. See the Appendix for more details.*

Proof. (b) Let $v(t) = x(t) - \theta(t-\delta)$ for $t \geq 0$, then the process $\{x(t)\}$ obeys the following SDE for $t \geq 0$:

$$\mathrm{d}x(t) = \Big(\widetilde{A}(t)x(t) + BK(t)v(t)\Big)\,\mathrm{d}t + F(t)\,\mathrm{d}W_t. \tag{11}$$

Moreover, we have for $t \geq \delta$

$$\dot{\theta}(t) = \widetilde{A}(t+\delta)\theta(t) + B(t+\delta)K(t+\delta)\Phi_{\widetilde{A}}(t+\delta, t)v(t), \tag{12}$$

$$\mathrm{d}v(t) = \Big(\widetilde{A}(t)v(t) + B(t)\varphi(t)\Big)\,\mathrm{d}t + F(t)\,\mathrm{d}W_t, \tag{13}$$

where $\varphi(t) = K(t)\Big(v(t) - \Phi_{\widetilde{A}}(t, t-\delta)v(t-\delta)\Big)$. By integrating equation (13) in $[t-\delta, t]$, we can write for $t \geq 2\delta$

$$v(t) = \Phi_{\widetilde{A}}(t, t-\delta)v(t-\delta) + \int_{t-\delta}^{t} \Phi_{\widetilde{A}}(t,s)B(s)\varphi(s)\,\mathrm{d}s + + \int_{t-\delta}^{t} \Phi_{\widetilde{A}}(t,s)F(s)\,\mathrm{d}W_s, \tag{14}$$

and thus, with a change of variable, we have for $t \geq 2\delta$,

$$\varphi(t) = \int_0^{\delta} K(t)\Phi_{\widetilde{A}}(t, t-s)B(t-s)\varphi(t-s)\,\mathrm{d}s + - \int_0^{\delta} K(t)\Phi_{\widetilde{A}}(t, t-s)F(t-s)\,\mathrm{d}W_{t-s}. \tag{15}$$

By the definition of M above (7) and by letting

$$N(t,s) = K(t)\Phi_{\widetilde{A}}(t,t-s)F(t-s), \tag{16}$$

$$\gamma_N(t,\Delta) = \left(\int_0^{\Delta} \|N(t,\tau)\|^2 \,\mathrm{d}\tau\right)^{1/2}, \tag{17}$$

we can write

$$\varphi(t) = \int_0^{\delta} M(t,s)\varphi(t-s)\,\mathrm{d}s - \int_0^{\delta} N(t,s)\,\mathrm{d}W_{t-s}, \tag{18}$$

and estimate the L_2 norm of $\varphi(t)$, by using triangular inequality, Ito isometry [1], for $t \geq 2\delta$ as follows

$$\|\varphi(t)\|_{L_2} \leq \int_0^{\delta} \|M(t,s)\|\,\|\varphi(t-s)\|_{L_2}\mathrm{d}s + \left(\int_0^{\delta} \|N(t,s)\|^2\,\mathrm{d}s\right)^{\frac{1}{2}}. \tag{19}$$

Moreover, by the definitions of γ_M and γ_N in (7) and (17), we can write

$$\|\varphi(t)\|_{L_2} \leq \gamma_M(0,t,\delta) \sup_{s\in[t-\delta,t]} \|\varphi(s)\|_{L_2} + \gamma_N(t,\delta) \tag{20}$$

By taking the supremum on both sides in $[t-\delta,t]$ we have for $t \geq 3\delta$,

$$\sup_{s\in[t-\delta,t]} \|\varphi(s)\|_{L_2} \leq \sup_{s\in[t-\delta,t]} \gamma_M(0,s,\delta) \sup_{s\in[t-2\delta,t]} \|\varphi_k(s)\|_{L_2} + \sup_{s\in[t-\delta,t]} \gamma_N(s,\delta), \tag{21}$$

where we note that condition $\sup_{t\geq\delta}\gamma_M(\lambda,t,\delta) < 1$ implies $\sup_{t\geq\delta}\gamma_M(0,t,\delta) < 1$ since λ is positive. Moreover, for $i \geq 1$, by defining

$$w_i = \sup_{s\in[i\delta,(i+1)\delta]} \|\varphi_k(s)\|_{L_2},$$
$$\gamma_i^M = \sup_{s\in[i\delta,(i+1)\delta]} \gamma_M(0,s,\delta), \quad \gamma_i^N = \sup_{s\in[i\delta,(i+1)\delta]} \gamma_N(s,\delta),$$

we can rewrite inequality (21) as

$$w_{i+1} \leq \gamma_{i+1}^M \max(w_i, w_{i+1}) + \gamma_{i+1}^N. \tag{22}$$

Furthermore, because of the uniform exponential stability of $\Phi_{\widetilde{A}}$, both γ_i^M and γ_i^N are uniformly bounded in $i \geq 1$ from which it follows that

$$\sup_{i\geq 1} w_i \leq \max\{w_1, b\}, \qquad b := \frac{\sup_i \gamma_i^N}{1-\sup_i \gamma_i^M} \tag{23}$$

with b finite, since $\sup_i \gamma_i^M < 1$ by hypothesis. Indeed, it is easy to prove that $w_i > b$ implies $w_{i+1} < w_i$ (because $w_{i+1} \geq w_i$ would imply $w_{i+1} \leq b < w_i$, a contradiction) and that $w_i \leq b$ implies $w_{i+1} \leq b$ (because $w_{i+1} > b$ would imply $w_{i+1} \leq b$, a contradiction). Thus, if $w_1 > b$ then $\sup_i\{w_i\} \leq w_1$ and if $w_1 \leq b$ then $\sup_i\{w_i\} \leq b$. This proves $\sup_i\{w_i\} \leq \max\{w_1, b\}$ from which it follows that φ is mean square bounded with bound $\bar{b} = \max\{w_1, b\}$. It is now easy to prove that v is mean square bounded, too. In fact, from (13) it descends that for $t \geq 3\delta$

$$v(t) = \Phi_{\widetilde{A}}(t,\delta)v(\delta) + \int_{\delta}^{t} \Phi_{\widetilde{A}}(t,s)B(s)\varphi(s)\,\mathrm{d}s + \int_{\delta}^{t} \Phi_{\widetilde{A}}(t,s)F(s)\,\mathrm{d}W_s, \tag{24}$$

and consequently, for

$$\|v(t)\|_{L_2} \leq \|\Phi_{\widetilde{A}}(t,\delta)\|\,\|v(\delta)\|_{L_2} + \bar{b}\int_{\delta}^{t} \|\Phi_{\widetilde{A}}(t,s)B(s)\|\,\mathrm{d}s + \left(\int_{\delta}^{t} \|\Phi_{\widetilde{A}}(t,s)F(s)\|^2\,\mathrm{d}s\right)^{\frac{1}{2}}, \tag{25}$$

where all terms on the right-hand side are uniformly bounded in t. Thus, we conclude that $\{v(t)\}$ is mean square bounded. Finally, by considering equation (11), the mean square boundedness of $\{x(t)\}$ descends with similar steps of (25).

(a) For $t > 2\delta$, let $E_{\varphi}(t) = \mathbb{E}[\varphi(t)]$, then from (18) we have

$$E_{\varphi}(t) = \int_0^{\delta} M(t,s)E_{\varphi}(t-s)\,\mathrm{d}s. \tag{26}$$

Moreover, by setting $E_{\varphi}^{\lambda}(t) = \mathrm{e}^{\lambda t}\,\mathbb{E}[\varphi(t)]$, we have

$$E_{\varphi}^{\lambda}(t) = \int_0^{\delta} M(t,s)\,\mathrm{e}^{\lambda s}\,E_{\varphi}^{\lambda}(t-s)\,\mathrm{d}s, \tag{27}$$

and thus

$$\|E_{\varphi}^{\lambda}(t)\| \leq \int_0^{\delta} \|M(t,s)\|\,\mathrm{e}^{\lambda s}\,\|E_{\varphi}^{\lambda}(t-s)\|\,\mathrm{d}s \leq \gamma_M(\lambda,t,\delta) \sup_{s\in[t-\delta,t]} \|E_{\varphi}^{\lambda}(s)\|. \tag{28}$$

Since $\sup_{t\geq\delta}\gamma_M(\lambda,t,\delta) < 1$, by proceeding as in (20), we conclude that $E_{\varphi}^{\lambda}(t)$ is uniformly bounded. It follows that $\{\varphi(t)\}$ is exponentially centered with rate λ. Consequently, it is easy to see from (13) that also $\{v(t)\}$

[1] all the processes are square-integrable random variables for any $t \geq 0$.

is exponentially centered with rate λ and consequently, from (11), the same holds for $\{x(t)\}$.

■

By comparing the predictor (9) with the one obtained in the classical reduction approach, namely

$$\theta(t) = \Phi_A(t, t-\mathrm{d})x(t-\delta) + \int_{t-\delta}^{t} \Phi_A(t,s)B(s)u(s-\delta)\,\mathrm{d}s, \tag{29}$$

we observe that the implementation of (9) is much cheaper, due to the absence of distributed terms, and more robust thanks to the presence of a correction term. We also remark that the initialization $\theta_0 = x_0$ is not mandatory, and Theorem 2 holds with any bounded pre-shape function $\theta(\tau)$, $\tau \in [-\delta, 0]$.

In the deterministic case when the control law $u(t) = -K(t)x(t)$ is optimal with respect to some criteria and it generates an UES state transition matrix then the predictor of Theorem 2 preserves its properties in presence of delay in the following sense.

Corollary 3 *Let $F(t) = 0$ for any $t \geq 0$ and $K : \mathbb{R}_+ \to \mathbb{R}^p \times \mathbb{R}^n$ be an optimal control gain such that the closed-loop state transition matrix $\Phi_{\widetilde{A}}$ of $\widetilde{A}(t) = A(t) - B(t)K(t)$ is UES with rate $\lambda^* > 0$. If $\sup_{t\geq\delta} \gamma_M(\lambda, t, \delta) < 1$ for any $\lambda > \lambda^*$ then* (a) *the trajectory $x(t)$ solution to the closed-loop system* (6) *with input delay under the control law* (8)–(9) *and the initialization $\theta(\tau) = \Phi_{\widetilde{A}}(\tau + \delta, 0)x_0$ for $\tau \in [-\delta, 0]$ is the same as the optimal trajectory without delay;* (b) *if $\theta(\tau)$ for $\tau \in [-\delta, 0]$ is chosen arbitrarily, $x(t)$ converges exponentially to the optimal trajectory without delay.*

Proof (sketch). The optimal trajectory is $x^o(t) = \Phi(t, t_0)x_0^o$. *(a)* $v(t) = 0$ for $t \in [0, \delta]$ and exponentially centered with rate $\lambda > \lambda^*$. From (11) with $F(t) \equiv 0$ it follows $x(t) = \Phi(t, t_0)x_0^o = x^o(t)$. *(b)* $v(t)$ is not null in $[0, \delta]$ but still exponentially centered, thus $\|x(t) - x^o(t)\|$ converges to 0 with rate $\lambda - \lambda^*$.

■

3.2 Modular predictor

Let us set $\tilde{\gamma}_M(\lambda, \delta) = \sup_{t\geq\delta} \gamma_M(\lambda, t, \delta)$ and define $\delta_{\max}$ such that $\tilde{\gamma}_M(0, \delta_{\max}) = 1$, or, if $\tilde{\gamma}_M(0, \delta) < 1$ for any $\delta > 0$, $\delta_{\max} = +\infty$. Clearly, $\delta_{\max}$ exists and is unique since $\delta \mapsto \tilde{\gamma}_M(\lambda, \delta)$ is continuous and monotonically increasing for any $\lambda > 0$. This section is devoted to the case when the delay affecting the input of the system (6) is larger than the maximum delay, namely $\delta > \delta_{\max}$. Thus, in the case $\delta > \delta_{\max}$, we shall design a *modular predictor* that allows to achieve the same results of Theorem 2 at the expenses of the memory of the controller.

Definition 2 *Given $\delta, \delta^* > 0$, a* delay partition *$\mathcal{P}_{\delta,\delta^*}$ is a set $\{\delta_j\}$, $j = 1, \ldots, m$, $\delta^* \geq \delta_j > 0$ and $\sum_{j=1}^{m} \delta_j = \delta$. An* equi-partition *is such that $\delta_j = \tilde{\delta} = \delta/m$.*

In the sequel we consider equi-partitions, and denote $d_j = j\tilde{\delta}$, for $j = 0, \ldots, m$ (hence $d_0 = 0$).

Definition 3 *Given system* (6) *and a gain K such that the state transition matrix $\Phi_{\widetilde{A}}$ of $\widetilde{A}(t) = A(t) - B(t)K(t)$ is UES with rate λ, then we say that the delay partition $\mathcal{P}_{\delta,\delta^*}$ is* λ-feasible *if $\tilde{\gamma}_M(\lambda, \delta^*) < 1$.*

To the purpose of designing a λ-feasible equi-partition one can proceed as follows.

- Find the gain $K : \mathbb{R}_+ \to \mathbb{R}^p \times \mathbb{R}^n$ with the associated rate of stability $\lambda > 0$ of the state transition matrix $\Phi_{\widetilde{A}}$, where $\widetilde{A}(t) = A(t) - B(t)K(t)$;
- Choose a small $\epsilon \in (0, 1)$ and find $\delta^* : \tilde{\gamma}_M(\lambda, \Delta) = 1 - \epsilon$;
- Compute the number of predictors $m = \lceil \delta/\delta^* \rceil$;
- Compute $\tilde{\delta} = \delta_j = \delta/m$.

Since $\delta \mapsto \tilde{\gamma}_M(\lambda, \delta)$ is continuous and monotonically increasing, δ^* defined above satisfies $\delta^* \geq \tilde{\delta} > 0$. With the equi-partition described above let us denote $\widetilde{\Phi}_{\tilde{\delta}}(t_j) = \Phi_{\widetilde{A}}(t_{j-1}, t_j)$, where we set from now on $t_j = t - d_j + \delta$. The modular predictor consists of the following chain of m predictors for $j = 1, \ldots, m$ and $t \geq 0$

$$\begin{aligned}\dot{\theta}_j(t) &= A(t_{j-1})\theta_j(t) - B(t_{j-1})K(t_{j-1})\theta_1(t - d_{j-1}) \\ &\quad + B(t_{j-1})K(t_{j-1})\widetilde{\Phi}_{\tilde{\delta}}(t_j)(\theta_{j+1}(t) - \theta_j(t - \tilde{\delta})),\end{aligned} \tag{30}$$

where $\theta_j(\tau) = \Phi_{\widetilde{A}}(\tau + \delta - d_{j-1}, 0)\theta_0$ for $\tau \in [-\delta, 0)$ with $\theta_0 = x_0$ and $\theta_{m+1}(t) := x(t)$. The idea is that each $\theta_j(t)$ predicts $x(t_{j-1})$, thus $\theta_1(t)$ predicts $x(t + \delta)$.

Theorem 4 *Consider the process $\{x(t)\}$ solution to* (6) *with the Assumption 1*(a)*. Let $K : \mathbb{R}_+ \to \mathbb{R}^p \times \mathbb{R}^n$ be a control gain such that the closed-loop state transition matrix $\Phi_{\widetilde{A}}$ of $\widetilde{A}(t) = A(t) - B(t)K(t)$ is UES with rate $\lambda > 0$. For a λ-feasible equi-partition $\mathcal{P}_{\delta,\delta^*}$ and the control law for $t \geq -\delta$*

$$u(t) = -K(t + \delta)\theta_1(t) \tag{31}$$

where $\theta_1(t)$ is defined by (30) *with $\theta_j(\tau) = \Phi_{\widetilde{A}}(\tau + \delta - d_{j-1}, 0)\theta_0$ for $\tau \in [-\delta, 0)$ and $j = 1, \ldots, m$, $\theta_0 = x_0$ and $\theta_{m+1}(t) := x(t)$, the process $\{x(t)\}$ is exponentially centered with rate λ and mean square bounded.*

Proof (sketch). Let $v_j(t) = x(t) - \theta_j(t_{j-1})$ be the prediction error of the j-th predictor. v_m is exponentially centered with rate λ and mean square bounded by

Theorem 2. The proof is concluded by backward induction. By induction hypothesis $v_{j+1}(t-\tilde{\delta})$ is exponentially centered with rate λ and mean square bounded, and by Theorem 2 we conclude that v_j is exponentially centered with rate λ and mean square bounded for all $j = 1, \dots, m$. Finally, since the dynamics of $x(t)$ can be written as

$$\mathrm{d}x(t) = \left(\widetilde{A}(t)x(t) + B(t)K(t)v_1(t)\right)\mathrm{d}t + F(t)\mathrm{d}W_t,$$

the thesis follows by Lemma 1.

4 Robust numerical solution to the state transition matrix differential equation

In this section we give a simple tool to obtain the state transition matrix Φ_A as the numerical solution of a coupled matrix differential equations. Given $\dot{x}(t) = A(t)x(t)$, $x(t_0) = x_0$, one has

$$x(t) = \Phi_A(t, t_0)x_0 \tag{32}$$

where the state transition matrix $\Phi_A(t, t_0)$ obeys

$$\dot{\Phi}_A(t, t_0) = A\Phi_A(t, t_0) \tag{33}$$
$$\Phi_A(t_0, t_0) = I. \tag{34}$$

The state transition matrix of the predictor an interval $\delta > 0$ is defined for $t \geq t_0 + \delta$ as

$$\widetilde{\Phi}_\delta(t) = \Phi_A(t, t-\delta), \tag{35}$$

with $\widetilde{\Phi}_\delta(t_0+\delta) = \Phi_A(t_0+\delta, t_0)$. The state transition matrix property implies

$$\widetilde{\Phi}_\delta(t) = \Phi_A(t, t_0)\Phi_A^{-1}(t-\delta, t_0) \tag{36}$$

and one has

$$\dot{\Phi}_A(t, t_0) = A\Phi_A(t, t_0) = \dot{\widetilde{\Phi}}_\delta(t)\Phi_A(t-\delta, t_0) \tag{37}$$
$$+ \widetilde{\Phi}_\delta(t)A(t-\delta)\Phi_A(t-\delta, t_0). \tag{38}$$

Thus,

$$A\Phi_A(t, t_0)\Phi_A^{-1}(t-\delta, t_0) = \dot{\widetilde{\Phi}}_\delta(t) + \widetilde{\Phi}_\delta(t)A(t-\delta), \tag{39}$$

or, finally, by using (36), for $t \geq t_0 + \delta$

$$\dot{\widetilde{\Phi}}_\delta(t) = A(t)\widetilde{\Phi}_\delta(t) - \widetilde{\Phi}_\delta(t)A(t-\delta), \tag{40}$$
$$\widetilde{\Phi}_\delta(t_0+\delta) = \Phi_A(t_0+\delta, t_0) \tag{41}$$

4.1 Robust implementation

For large values of $t - t_0$ the integration of the matrix differential equation (40) with the initial condition (41) is not reliable due to the propagation of numerical errors. From now on, let us assume $t_0 = 0$. A safe implementation can be obtained by re-initializing periodically $\widetilde{\Phi}_\delta(t)$ with the value of $\Phi(t, t-\delta)$. The latter term is obtained by integrating the state transition matrix equation (36) with initial condition $\Phi(t-\delta, t-\delta) = I$.

Let $t_k = k\delta$, $\mathcal{I}_k = [t_k, t_{k+1}]$. Consider the sequence of matrix functions $M_k : \mathcal{I}_k \to \mathbb{R}^{n\times n}$ defined by

$$\dot{M}_k(t) = A(t)M_k(t), \quad t \in \mathcal{I}_k \tag{42}$$
$$M_k(t_k) = I_n. \tag{43}$$

Clearly,

$$M_k(t_{k+1}) = \Phi(t_{k+1}, t_k) = \widetilde{\Phi}_\delta(t_{k+1}). \tag{44}$$

Or, $\widetilde{\Phi}_\delta(t_k)$ can be computed reliably at t_k by integrating (42)–(43). The remaining values of $\widetilde{\Phi}_\delta(t)$ are obtained by integrating (40) in $\mathcal{I}_k$. To this end consider the sequence of matrix functions $\widetilde{\Phi}_\delta^k : \mathcal{I}_k \to \mathbb{R}^{n\times n}$, $k > 0$, defined as

$$\dot{\widetilde{\Phi}}_\delta^k(t) = A(t)\widetilde{\Phi}_\delta^k(t) - \widetilde{\Phi}_\delta^k(t)A(t-\delta), \quad t \in \mathcal{I}_k \tag{45}$$
$$\widetilde{\Phi}_\delta(t_k) = M_{k-1}(t_k). \tag{46}$$

It follows that

$$\widetilde{\Phi}_\delta^k(t) = \widetilde{\Phi}_\delta(t), \quad t \in \mathcal{I}_k \tag{47}$$
$$\widetilde{\Phi}_\delta^k(t_{k+1}) = \widetilde{\Phi}_\delta(t_{k+1}) = M_k(t_{k+1}). \tag{48}$$

Summarizing, a robust computation of $\widetilde{\Phi}_\delta(t)$ is obtained by iterating the integration of the pair of matrix differential equations (42)–(43) and (45)–(46) over the intervals $\mathcal{I}_k$ (for $k = 0$ it is necessary to integrate (42)–(43) only, since $\widetilde{\Phi}_\delta(t)$ is not defined in $\mathcal{I}_0$).

The robust implementation described above is not affected by the propagation of numerical errors, whilst the computation of the state transition matrix $\widetilde{\Phi}_\delta(t)$ through equation (36) (by integrating (33)) or through direct integration of equation (40) is not a practical solution.

5 Output-feedback and variable delay

5.1 State-feedback with variable delay

We extend here the results of the previous section to the case of time-varying delay. Consider the process

$$\mathrm{d}x(t) = \left(A(t)x(t) + B(t)u(\psi(t))\right)\mathrm{d}t + F(t)\,\mathrm{d}W_t,$$

as in (1), where ψ is introduced at the beginning of Section 2 and it satisfies Assumption 2.

Corollary 5 *Consider the process* $\{x(t)\}$ *solution to* (1) *with the Assumption* 1(a) *and* 2. *Let* $K : \mathbb{R}_+ \to \mathbb{R}^p \times \mathbb{R}^n$ *be a control gain such that the closed-loop state transition matrix* $\Phi_{\widetilde{A}}$ *of* $\widetilde{A}(t) = A(t) - B(t)K(t)$ *is UES with rate* $\lambda > 0$. *For* $t \geq -\bar{\delta}$ *consider the control law*

$$u(t) = -K(\psi^{-1}(t))\theta(\psi^{-1}(t) - \bar{\delta}) \tag{49}$$

$$\begin{aligned}\dot{\theta}(t) &= \widetilde{A}(t+\bar{\delta})\theta(t) \\ &\quad + B(t+\bar{\delta})K(t+\bar{\delta})\Phi_{\widetilde{A}}(t+\bar{\delta},t)\left(x(t) - \theta(t-\bar{\delta})\right),\end{aligned} \tag{50}$$

where $\theta(\tau) = \Phi_{\widetilde{A}}(\tau+\bar{\delta},0)\theta_0$ *for* $\tau \in [-\bar{\delta},0]$ *with* $\theta_0 = x_0$. *If* $\sup_{t\geq\delta}\gamma_M(\lambda,t,\bar{\delta}) < 1$, *then the process* $\{x(t)\}$ *is exponentially centered with rate* λ *and mean square bounded.*

The proof is omitted since it follows the same line of the proof of Theorem 2. We notice that the control law (49) can be computed since $\psi^{-1}(t)$ is known at time $t \geq 0$ and it is causal since $\psi^{-1}(t) - \bar{\delta} \leq t$.

With an equi-partition $\mathcal{P}_{\bar{\delta},\delta^*}$ let us denote $\widetilde{\Phi}_{\bar{\delta}}(t_j) = \Phi_{\widetilde{A}}(t_{j-1},t_j)$, where similarly with the constant delay case $t_j = t - d_j + \bar{\delta}$. The modular predictor consists of the following chain of m predictors for $j = 1,\ldots,m$, for $t \geq 0$

$$\begin{aligned}\dot{\theta}_j(t) =& A(t_{j-1})\theta_j(t) - B(t_{j-1})K(t_{j-1})\theta_1(t-d_{j-1}) \\ &+ B(t_{j-1})K(t_{j-1})\widetilde{\Phi}_{\bar{\delta}}(t_j)(\theta_{j+1}(t) - \theta_j(t-\bar{\delta})),\end{aligned} \tag{51}$$

where $\theta_j(\tau) = \Phi_{\widetilde{A}}(\tau+\bar{\delta}-d_{j-1},0)\theta_0$ for $\tau \in [-\bar{\delta},0)$ with $\theta_0 = x_0$ and $\theta_{m+1}(t) := x(t)$. The idea is that each $\theta_j(t)$ predicts $x(t_{j-1})$, thus $\theta_1(t)$ predicts $x(t+\bar{\delta})$.

Corollary 6 *Consider the process* $\{x(t)\}$ *solution to* (1) *with the Assumption* 1(a) *and* 2. *Let* $K : \mathbb{R}_+ \to \mathbb{R}^p \times \mathbb{R}^n$ *be a control gain such that the closed-loop state transition matrix* $\Phi_{\widetilde{A}}$ *of* $\widetilde{A}(t) = A(t) - B(t)K(t)$ *is UES with rate* $\lambda > 0$. *Take a* λ*-feasible equi-partition* $\mathcal{P}_{\bar{\delta},\delta^*}$. *For* $t \geq -\bar{\delta}$ *consider the control law*

$$u(t) = -K(\psi^{-1}(t))\theta_1(\psi^{-1}(t) - \bar{\delta}), \tag{52}$$

where $\theta_1(t)$ *is defined by* (51) *with* $\theta_j(\tau) = \Phi_{\widetilde{A}}(\tau + \bar{\delta} - d_{j-1},0)\theta_0$ *for* $\tau \in [-\bar{\delta},0)$ *and* $j = 1\ldots,m$, $\theta_0 = x_0$ *and* $\theta_{m+1}(t) = x(t)$. *Then, the process* $\{x(t)\}$ *is exponentially centered with rate* λ *and mean square bounded.*

5.2 Output-feedback with variable delay

In this section we focus on the output-feedback stabilization problem of system (1)–(2).

Theorem 7 *Consider the process* $\{x(t)\}$ *solution to* (1) *and the measurement equation* (2) *with Assumption* 1 *and* 2. *Let* $K : \mathbb{R}_+ \to \mathbb{R}^p \times \mathbb{R}^n$ *be a control gain such that the closed-loop state transition matrix* $\Phi_{\widetilde{A}}$ *of* $\widetilde{A}(t) = A(t) - B(t)K(t)$ *is UES with rate* $\lambda > 0$. *For* $t \geq -\bar{\delta}$ *consider the control law*

$$u(t) = -K(\psi^{-1}(t))\theta(\psi^{-1}(t) - \bar{\delta}) \tag{53}$$

$$\begin{aligned}\dot{\theta}(t) &= \widetilde{A}(t+\bar{\delta})\theta(t) + \\ &\quad + B(t+\bar{\delta})K(t+\bar{\delta})\Phi_{\widetilde{A}}(t+\bar{\delta},t)\left(\hat{x}(t) - \theta(t-\bar{\delta})\right)\end{aligned} \tag{54}$$

$$\begin{aligned}\dot{\hat{x}}(t) &= A(t)\hat{x}(t) - B(t)K(t)\theta(t-\bar{\delta}) \\ &\quad + L(t)\left(y(t) - C\hat{x}(t)\right)\end{aligned} \tag{55}$$

$$L(t) = P(t)C(t)^\top R(t)^{-1} \tag{56}$$

$$\begin{aligned}\dot{P}(t) &= A(t)P(t) + P(t)A^\top(t) + Q(t) \\ &\quad - L(t)C(t)P(t),\end{aligned} \tag{57}$$

where $\theta(\tau) = \Phi_{\widetilde{A}}(\tau+\bar{\delta},0)\theta_0$ *for* $\tau \in [-\bar{\delta},0]$ *with* $\theta_0 = \hat{x}(0) = \mathbb{E}[x_0]$, $P(0) = \mathbb{E}[(x(0) - \mathbb{E}[x_0])(x(0) - \mathbb{E}[x_0])^\top]$, $Q(t) = F(t)F^\top(t)$, $R(t) = G(t)G^\top(t)$. *If* $\sup_{t\geq\bar{\delta}}\gamma_M(\lambda,t,\bar{\delta}) < 1$, *then the process* $\{x(t)\}$ *is exponentially centered with rate* λ *and mean square bounded.*

Proof. We note that condition $\sup_{t\geq\bar{\delta}}\gamma_M(\lambda,t,\bar{\delta}) < 1$ implies $\sup_{t\geq\bar{\delta}}\gamma_M(0,t,\bar{\delta}) < 1$ since λ is positive. For $t \geq 0$, the closed-loop dynamics are given by

$$\mathrm{d}x(t) = \left(A(t)x(t) - B(t)K(t)\theta(t-\bar{\delta})\right)\mathrm{d}t + F(t)\mathrm{d}W_t$$

$$\dot{\hat{x}}(t) = A(t)x(t) - B(t)K(t)\theta(t-\bar{\delta}) + L(t)\left(y(t) - C(t)\hat{x}(t)\right)$$

and for $t \geq \bar{\delta}$

$$\begin{aligned}\dot{\theta}(t) &= \widetilde{A}(t+\bar{\delta})\theta(t) \\ &\quad + B(t+\bar{\delta})K(t+\bar{\delta})\Phi_{\widetilde{A}}(t+\bar{\delta},t)\left(v(t) - \varepsilon(t)\right),\end{aligned} \tag{58}$$

where $\varepsilon(t) = x(t) - \hat{x}(t)$ is the estimation error of the filter, and $v(t) = x(t) - \theta(t-\bar{\delta})$ is the prediction error of the controller for $t \geq 0$. Thus, we have for $t \geq \bar{\delta}$

$$\begin{aligned}\mathrm{d}v(t) &= \Big(A(t)v(t) - B(t)K(t)\Phi_{\widetilde{A}}(t+\bar{\delta},t)v(t-\bar{\delta})\Big)\mathrm{d}t \\ &\quad + B(t)K(t)\Phi_{\widetilde{A}}(t+\bar{\delta},t)\varepsilon(t)\mathrm{d}t + F(t)\mathrm{d}W_t.\end{aligned} \tag{59}$$

In view of Assumption 1 and $\hat{x}(0) = \mathbb{E}[x_0]$ the term $B(t)K(t)\Phi_{\widetilde{A}}(t+\bar{\delta},t)\varepsilon(t)$ is zero-mean and mean square bounded, and following the proof of Theorem 2 the process $\{v(t)\}$ is exponentially centered with rate λ and mean square bounded. The closed-loop dynamics of $\{x(t)\}$ is still given by (11), and the thesis follows by Lemma 1.

■

From Corollary 6 and Theorem 7, we derive the following final result.

Corollary 8 *Consider $\{x(t)\}$ solution to* (1) *and the measurement equation* (2) *with Assumption* 1 *and* 2. *Let $K : \mathbb{R}_+ \to \mathbb{R}^p \times \mathbb{R}^n$ be a control gain such that $\Phi_{\widetilde{A}}$ of $\widetilde{A}(t) = A(t) - B(t)K(t)$ is UES with rate $\lambda > 0$. For a λ-feasible equi-partition $\mathcal{P}_{\delta,\delta^*}$ and the control law for $t \geq -\bar{\delta}$*

$$u(t) = -K(\psi^{-1}(t))\theta_1(\psi^{-1}(t) - \bar{\delta}) \tag{60}$$

where $\theta_1(t)$ is defined by (30) *with $\theta_j(\tau) = \Phi_{\widetilde{A}}(\tau + \bar{\delta} - d_{j-1}, 0)\theta_0$ for $\tau \in [-\bar{\delta}, 0)$ and $j = 1 \ldots, m$, $\theta_0 = \mathbb{E}[x_0]$, $\theta_{m+1}(t) = \hat{x}(t)$,*

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) - B(t)K(t)\theta_1(t-\bar{\delta}) + L(t)\left(y(t) - C\hat{x}(t)\right), \tag{61}$$

with $\hat{x}(0) = \mathbb{E}[x_0]$, where $L(t)$ is given by (56)*–*(57) *with $P(0) = \mathbb{E}[(x(0) - \mathbb{E}[x_0])(x(0) - \mathbb{E}[x_0])^\top]$, the process $\{x(t)\}$ is exponentially centered with rate λ and mean square bounded.*

Remark 2 *When the filters of Theorem 7 and Corollary 8 are not initialized in $\mathbb{E}[x_0]$, then the λ exponential stability of the mean process $\{\mathbb{E}[x(t)]\}$ requires that $\Phi_{\bar{A}}$, with $\bar{A}(t) = A(t) - L(t)C(t)$, is UES with rate $\alpha \geq \lambda$. When $\alpha < \lambda$ the process $\{x(t)\}$ is exponentially centered with rate α and mean square bounded. Finally, by giving up the minimum variance property it is possible to design the filter with an arbitrary prescribed rate of convergence for any initial condition $\hat{x}(0)$ (see Stocks and Medvedev (2006)).*

6 Numerical example

We consider state feedback with the control law (52) of Corollary 6 with two predictors and the robust numerical implementation of the state transition matrix $\Phi_{\widetilde{A}}$ in Section 4. In particular, the equations (51) of the predictor with $m = 2$ specify in

$$\begin{aligned}
\dot{\theta}_1(t) &= A(t+\delta)\theta_1(t) - B(t+\delta)K(t+\delta)\theta_1(t) + \\
&\quad B(t+\delta)K(t+\delta)\Phi_{\widetilde{A}}(t+\delta, t+\tilde{\delta})\left[\theta_2(t) - \theta_1(t-\tilde{\delta})\right] \\
\dot{\theta}_2(t) &= A(t+\tilde{\delta})\theta_2(t) - B(t+\tilde{\delta})K(t+\tilde{\delta})\theta_1(t-\tilde{\delta}) + \\
&\quad B(t+\tilde{\delta})K(t+\tilde{\delta})\Phi_{\widetilde{A}}(t+\tilde{\delta}, t)\left[x(t) - \theta_2(t-\tilde{\delta})\right]
\end{aligned}$$

where $\theta_1(\tau) = \Phi_{\widetilde{A}}(\tau + \bar{\delta}, 0)\theta_0$, $\theta_2(\tau) = \Phi_{\widetilde{A}}(\tau + \tilde{\delta}, 0)\theta_0$ for $\tau \in [-\bar{\delta}, 0)$ with $\theta_0 = x_0$ and $\theta_{m+1}(t) = x(t)$. The model (1)–(2) is characterized by

$$A(t) = \begin{bmatrix} 0 & \frac{1}{4}\cos(t) \\ 0 & 1 + \frac{1}{2}\sin(t) \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ 1 + \frac{1}{2}\cos(t) \end{bmatrix}$$

$$F(t) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad G(t) = \frac{1}{2}, \quad C(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The delay function $\delta(t)$ is such that $\max\{\delta(t)\} = \bar{\delta} = 0.4$ and the gain K is chosen with the standard linear-quadratic technique. Figure 1 shows that the process $\{x(t)\}$ is exponentially centered and mean square bounded, where the expectation is estimated as the average over 10^3 independent simulations of noise sequences and initial conditions.

7 Conclusions

We conclude by remarking that the results presented in this paper are new also in the deterministic case, since predictors for linear time-varying systems with large delays in the form of DDEs are not available. The design and implementation of the method proposed here are straightforward. Future extensions include the dual case of LTVSS with large measurements delays, the presence of mutliplicative state noise as well as the extension of the closed-loop predictor approach to nonlinear systems.

A Stability of time-varying linear systems with prescribed rate

For $t \geq 0$ consider the time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{A.1}$$

where $x(t) \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, and $A(t)$, $B(t)$ of appropriate size piece-wise continuous in t. Let us define

$$\begin{aligned}
H(t, t_0) &= \int_{t_0}^{t} \Phi_A(t,s)B(s)B^\top(s)\Phi_A^\top(t,s)\mathrm{d}s \\
H_\alpha(t, t_0) &= \int_{t_0}^{t} \mathrm{e}^{4\alpha(t_0 - s)}\Phi_A(t,s)B(s)B^\top(s)\Phi_A^\top(t,s)\mathrm{d}s
\end{aligned} \tag{A.2}$$

for any $\alpha > 0$. A control law with prescribed rate of uniform exponential stability can be obtained as first proposed in Ikeda et al. (1975) with a state change $x \mapsto x\mathrm{e}^{\alpha t}$. This leads to the following.

Lemma 9 (Cheng (1979)) *Consider the linear time-varying system described by* (A.1). *If there exist $\Delta > 0$ and $h_M \geq h_m > 0$ such that for any $t \geq 0$*

$$0 \leq h_m\, I \leq H(t+\Delta, t) \leq h_M\, I, \tag{A.3}$$

then for any $\alpha > 0$, the linear time-varying state-feedback control law

$$u(t) = -\frac{1}{2}B^\top(t)H_\alpha^{-1}(t+\Delta, t)x(t) \tag{A.4}$$

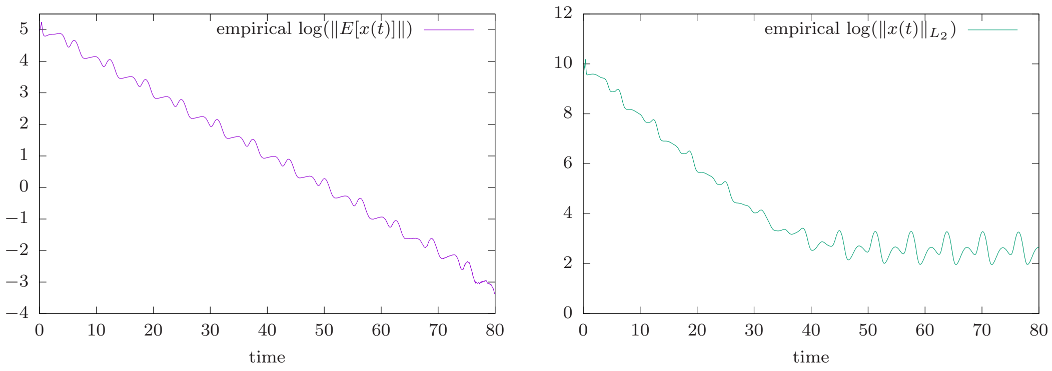

Fig. 1. Empirical value of $\|\mathbb{E}[x(t)]\|$ in logarithmic scale (left). Empirical value of $\sqrt{\mathbb{E}[||x(t)||^2]} = ||x(t)||_{L_2}$ in logarithmic scale (right). The expected value is estimated by averaging over 10^3 Monte Carlo runs.

is such that the null solution of the closed-loop system

$$\dot{x}(t) = \left(A(t) - \frac{1}{2} B(t) B^\top(t) H_\alpha^{-1}(t+\Delta, t) \right) x(t) \tag{A.5}$$

is UES for $t \geq 0$ with rate grater than α.

We notice that (A.4) requires the knowledge at each $t \geq 0$ of $H_\alpha(t+\Delta, t)$ and thus the knowledge at each $t \geq 0$ of $\Phi_A(t+\Delta, t+s)$ for $s \in [0, \Delta]$. We remark that Φ_A can be computed as in Section 4.

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