HIGHER ORBITAL INTEGRALS, RHO NUMBERS AND INDEX THEORY

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Abstract. Let G be a connected, linear real reductive group. We give sufficient conditions ensuring the well-definedness of the delocalized eta invariant $\eta_g(D_X)$ associated to a Dirac operator D_X on a cocompact G-proper manifold X and to the orbital integral τ_g defined by a semisimple element $g \in G$. Along the way, we give a detailed account of the large time behaviour of the heat kernel and of its short time behaviour near the fixed point set of g. We prove that such a delocalized eta invariant enters as the boundary correction term in an index theorem computing the pairing between the index class and the 0-degree cyclic cocycle defined by τ_g on a G-proper manifold with boundary. More importantly, we also prove a higher version of such a theorem, for the pairing of the index class and the higher cyclic cocycles defined by the higher orbital integral Φ^P_g associated to a cuspidal parabolic subgroup $P < G$ with Langlands decomposition $P = MAN$ and a semisimple element $g \in M$. We employ these results in order to define (higher) rho numbers associated to G-invariant positive scalar curvature metrics.

Contents

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1. Introduction

This article is a contribution to higher index theory on G-proper manifolds, with G a connected, linear real reductive group. Before considering the case of G-proper manifolds with G a Lie group, it is worth spending a few words on the pivotal example of Galois Γ-coverings. Let Γ be a finitely generated discrete group and let $\Gamma \to X \to X/\Gamma$ be a Galois covering; let D be a Γ-equivariant Dirac-type operator on a spin^c manifold X, acting on the sections of a Γ-equivariant vector bundle E. We initially assume that X/Γ is a smooth compact manifold *without* boundary. We also assume that Γ is Gromov hyperbolic or of polynomial growth with respect to a word metric. The seminal work of Connes and Moscovici [6] allows to define a pairing between $HC^*(\mathbb{C}\Gamma, \langle e\rangle)$, the cyclic cohomology of $\mathbb{C}\Gamma$ localized at the unit element, and the K-theory of the Roe C^*-algebra $K_*(C^*(X,E)^\Gamma)$. In particular, by applying this pairing to the index class associated to D, $\mathrm{Ind}(D) \in K_*(C^*(X,E)^\Gamma)$, it is possible to define higher indices associated to D, parametrized by the elements in $HC^*(\mathbb{C}\Gamma, \langle e\rangle)$, a group which is in fact isomorphic to $H^*(\Gamma,\mathbb{C})$. The higher index formula of Connes-Moscovici provides a geometric formula for these higher indices, with very interesting geometric applications to higher signatures and higher $\widehat{A}$-genera. One can also pair the index class with $HC^*(\mathbb{C}\Gamma, \langle x\rangle)$, the cyclic cohomology localized at the conjugacy class $\langle x\rangle$, but this pairing turns out to be identically zero.

Despite these interesting geometric applications the higher index invariants we have just introduced will be inadequate whenever we have a geometric situation in which the index class vanishes. This is the case, for example, if D is the spin Dirac operator associated to a metric of positive scalar curvature. Thus, in geometric questions involving, for example, the moduli space of metrics of positive scalar curvature $\mathcal{R}^+(M)/\mathrm{Diffeo}(M)$ on a manifold M, one is led to consider *secondary* invariants. One such invariant is the delocalized eta invariant of Lott, $\eta_{\langle x\rangle}(D)$, associated to an invertible Dirac operator D and, initially, to a finite conjugacy class $\langle x\rangle$ in Γ, see [28]. This invariant was extended to conjugacy classes of polynomial growth in [39] and then, much more generally, to arbitrary conjugacy classes of Gromov hyperbolic groups in the deep work of Puschnigg, see [41].

The geometric interest for such an invariant stems from its connection with the Atiyah-Patodi-Singer index class on a Galois covering with boundary $\Gamma \to Y \to Y/\Gamma$, once we assume the boundary operator associated to D to be invertible. Thanks to the higher Atiyah-Patodi-Singer index formula of Leichtnam and Piazza and Wahl, see [23, 24, 47], one can prove that the pairing of the Atiyah-Patodi-Singer index class with the cyclic cocycle $[\tau_{\langle x\rangle}] \in HC^0(\mathbb{C}\Gamma, \langle x\rangle)$,

$$\tau_{\langle x\rangle}\Big(\sum_{\gamma} \alpha_\gamma \gamma\Big) := \sum_{\gamma\in\langle x\rangle} \alpha_\gamma \gamma \tag{1.1}$$

is well defined if Γ is Gromov hyperbolic or of polynomial growth and precisely equal to the delocalized eta invariant of Lott; in formulae

$$\langle \mathrm{Ind}(D), [\tau_{\langle x\rangle}]\rangle = -\frac{1}{2}\eta_{\langle x\rangle}(D_\partial) \tag{1.2}$$

We call the formula appearing in (1.2) a 0-degree *delocalized APS index formula.* Recent work of Chen-Wang-Xie-Yu [4], Sheagan [42] and Piazza-Schick-Zenobi [40] extend this pairing to all elements in $HC^*(\mathbb{C}\Gamma, \langle x\rangle)$, x any element in a Gromov hyperbolic or polynomial growth group, obtaining correspondingly *higher delocalized Atiyah-Patodi-Singer index theorems* for Gromov hyperbolic groups. These express the pairing between $\tau \in HC^*(\mathbb{C}\Gamma, \langle x\rangle)$ and the Atiyah-Patodi-Singer index class $\mathrm{Ind}(D)$ in terms of certain *higher rho numbers*, secondary invariants of Dirac operators that are particularly useful in studying, for example, metrics of positive scalar curvature. See [40] for these geometric applications. Notice that these higher rho numbers appear here for the boundary operator D_∂ on ∂Y but they can be defined for any Galois covering $\Gamma \to X \to X/\Gamma$, with X without boundary.

We now turn to a G-proper manifold, initially without boundary, with compact quotient and a G-equivariant Dirac operator D acting on the sections of a G-equivariant twisted spinor bundle E. Here G is a unimodular Lie group. Let $\mathcal{L}^c_G(X,E)$ be the algebra of G-equivariant smoothing operators of G-compact support. There is a compactly supported index class $\mathrm{Ind}_c(D) \in K_*(\mathcal{L}^c_G(X,E))$ and a homomorphism $H^*_{\mathrm{diff}}(G) \to HC^*(\mathcal{L}^c_G(X,E))$, from the differentiable cohomology of G, $H^*_{\mathrm{diff}}(G)$, to the cyclic cohomology of $\mathcal{L}^c_G(X,E)$, $HC^*(\mathcal{L}^c_G(X,E))$. We think to these cyclic cocycles in $HC^*(\mathcal{L}^c_G(X,E))$ as localized at the identity element of G. The higher index theorem of Pflaum-Posthuma-Tang [35] gives a formula for the pairing of $\mathrm{Ind}_c(D)$ with the cyclic cocycle τ_φ^M associated to $[\varphi] \in H^*_{\mathrm{diff}}(G)$. Under additional assumptions on the group G, satisfied for example by a *connected, linear real reductive group*, this result was improved to a C^*-index theorem in [36]. In the latter work a key role is played by a suitable dense holomorphically closed subalgebra $\mathcal{L}^\infty_{G,s}(X,E)$ (Definition 4.3) of the Roe algebra $C^*(X,E)^G$ and by a smooth index class $\mathrm{Ind}_\infty(D) \in K_*\big(\mathcal{L}^\infty_{G,s}(X,E)\big)$. Applications were given once again to higher signatures and higher $\widehat{A}$-genera. This index theorem was extended to G-proper manifolds with boundary in the recent paper [37]; this is a higher APS index theorem on G-proper manifolds for cyclic cocyles *localized at the identity element.*

Let us now go back to a G-proper manifold without boundary X. In contrast with the free case, in the proper case we also have *delocalized* index theorems associated to delocalized cyclic cocycles. Let g be a semisimple element and set $Z := Z_G(g)$. First of all we have the orbital integral $\tau_g : C^\infty_c(G) \to \mathbb{C}$ associated to such a g:

$$\tau_g(f) := \int_{G/Z} f(xgx^{-1})d(xZ)\,. \tag{1.3}$$

Assume G is a connected, linear real reductive group. Then τ_g extends to the Lafforgue Schwartz algebra $\mathcal{L}_s(G)$ (Definition 4.3), a dense holomorphically closed subalgebra of C^*_rG, where it defines a 0-degree cyclic cocycle $[\tau_g] \in HC^0(\mathcal{L}_s(G))$. The orbital integral (1.3) also defines a 0-degree cyclic cocycle τ^X_g on $\mathcal{L}^c_G(X,E)$:

$$\tau^X_g(T_\kappa) := \int_{G/Z}\int_X c(hgh^{-1}x)\mathrm{tr}(hgh^{-1}\kappa(hg^{-1}h^{-1}x,x))dx\, d(hZ), \tag{1.4}$$

where κ denotes the kernel of an element $T_\kappa \in \mathcal{L}^c_G(X,E)$ and c is a cut-off function for the action of G on X and tr denotes the vector-bundle fiberwise trace.

The Lafforgue algebra $\mathcal{L}_s(G)$ defines an algebra of smoothing operators $\mathcal{L}^\infty_{G,s}(X,E)$ which is a dense and holomorphically closed subalgebra of the Roe algebra $C^*(X,E)^G$; moreover τ^X_g extends from $\mathcal{L}^c_G(X,E)$ to $\mathcal{L}^\infty_{G,s}(X,E)$ and defines on this algebra a 0-degree cyclic cocycle $[\tau^X_g] \in HC^0(\mathcal{L}^\infty_{G,s}(X,E))$. We prove the following index formula for any semi-simple element g,

$$\langle \mathrm{Ind}_\infty(D), \tau^X_g\rangle = \int_{X^g} c^g \mathrm{AS}_g(D). \tag{1.5}$$

In [19], Hochs and Wang partially developed the above formula by considering a splitting Dirac operator, c.f. (3.6). We give a more detailed approach to (1.5) in this paper. For more on the subtleties involved in the proof of this index formula we refer the reader to Section 5.

This formula establishes a 0-degree delocalized index theorem. In Equation (1.5), X^g is the fixed point submanifold associated to the g-action on X; c^g is a compactly supported cutoff function on X^g for the action of Z on X^g. Below, we recall the explicit differential form expression for $\mathrm{AS}_g(D)$. Consider the following curvature forms.

- Since we are assuming that E is a G-equivariant twisted spinor bundle on X, we can write
$$E = \mathcal{E} \otimes W,$$
where $\mathcal{E}$ is the spinor bundle associated to the Spin^c-structure on X and W is an auxiliary G-equivariant vector bundle on X.
We define R^W to be the curvature form of the Hermitian connection on W;
- $R^{\mathcal{N}}$, the curvature form associated to the Hermitian connection on $\mathcal{N}_{X^g} \otimes \mathbb{C}$, where $\mathcal{N}_{X^g}$ is the normal bundle of the g-fixed point submanifold X^g in X;
- R^L, the curvature form associated to the Hermitian connection on $L_{\det}|_{X^g}$ ($L_{\det}$ is the determinant line bundle of the Spin^c-structure on X and $L_{\det}|_{X^g}$ is its restriction to X^g);
- R_{X^g}, the Riemannian curvature form associated to the Levi-Civita connection on the tangent bundle of X^g.

The $\mathrm{AS}_g(D)$ in Equation (1.5) is, by definition, the following expression:

$$\mathrm{AS}_g(D) := \frac{\widehat{A}\big(\frac{R_{X^g}}{2\pi i}\big)\,\mathrm{tr}\left(g\exp(\frac{R^W}{2\pi i})\right)\exp(\mathrm{tr}(\frac{R^L}{2\pi i}))}{\det\left(1 - g\exp(-\frac{R^{\mathcal{N}}}{2\pi i})\right)^{\frac{1}{2}}}. \tag{1.6}$$

This will also be denoted by $\mathrm{AS}_g(X,E)$ or, if there is no confusion on the vector bundle E, simply by $\mathrm{AS}_g(X)$.

One might wonder if there is a *higher* delocalized index theorem; the answer is affirmative and it is work of Song-Tang and Hochs-Song-Tang as we shall now explain. Let $P < G$ be a cuspidal parabolic subgroup and $P = MAN$ its Langlands decomposition. Let $g \in M$ be a semisimple element. Song and Tang in [44] have defined a higher delocalized cyclic cocycle $[\Phi_g^P]$ on the Lafforgue Schwartz algebra $\mathcal{L}_s(G)$. For $m = \dim(A)$, Φ_g^P is an m-cyclic cocycle on $\mathcal{L}_s(G)$ generalizing the orbital integral, Equation (1.3). When g runs over all semisimple elements of the maximal torus T of M and P runs over all cuspidal parabolic subgroups of G, the pairing between Φ_g^P and $K_*(\mathcal{L}_s(G))$ detects all elements of $K_*(\mathcal{L}_s(G)) = K_*(C_r^*(G))$, c.f. [44, Theorem III.]. When the metric on a G proper cocompact manifold X without boundary is slice compatible (i.e. there is a slice Z_0 equipped with a K action such that $X = G \times_K Z_0$ and the metric on X is obtained from a K-invariant metric on Z_0 and G-invariant metric on G/K), the pairing between Φ_g^P and the K-theory index $\mathrm{Ind}_\infty(D)$ is computed by Hochs-Song-Tang in [16]. More precisely, the index pairing has the following topological formula,

$$\langle \mathrm{Ind}_\infty(D), \Phi_g^P \rangle = \int_{(X/AN)^g} c^g_{X/AN} \mathrm{AS}(X/AN)_g \tag{1.7}$$

In the above formula, X/AN is the quotient of X with respect to the $AN < G$ action; the property of the Langlands decomposition implies that the group M acts on the quotient X/AN; $(X/AN)^g$ is the fixed point submanifold of the g action on X/AN; $c^g_{X/AN}$ is a smooth compactly supported function on $(X/AN)^g$ defined in the same way as c^g in Equation (1.5) ; the action of the Dirac operator D on the AN-invariant sections of E defines a Dirac type operator $D_{X/AN}$ on X/AN; the characteristic classes are defined in a similar but slightly different way from those in Equation (1.6) for the M action on X/AN (see Equation (9.12) for its explicit expression).

We can finally state the main goals of this article:

(i) state and prove a (higher) delocalized Atiyah-Patodi-Singer index theorem on G-proper manifolds with boundary;

(ii) define consequently (higher) rho numbers on G-proper manifolds without boundary and study their properties.

We state now in detail our main results; these deal both with cocompact G-proper manifolds with boundary and without boundary.

We begin by stating the results on manifolds with boundary; these are obtained by combining the results of Hochs, Song and Tang, the use of (an extension of) Melrose's b-calculus to the present context developed in [37] and relative K-theory and relative cyclic cohomology techniques, as developed in [9, 34, 37].

First we give an improvement, through the above techniques, of a result already proved in [18]. Let G be a connected, linear real reductive group. Let g be a semisimple element. Let Y_0 be a G-proper manifold with boundary with compact quotient and let Y be the associated manifold with cylindrical ends. We fix a G-equivariant slice-compatible metric. Let D_0 be an equivariant Dirac operator on Y_0, acting on the sections of an equivariant vector bundle E_0. We denote by D and E the associated objects on Y.

1.8. **Theorem.** *(0-degree delocalized APS on G-proper manifolds)*
We assume that $D_{\partial Y}$ is L^2-invertible. Then:

1) *there exists a dense holomorphically closed subalgebra $\mathcal{L}^\infty_{G,s}(Y,E)$ of the Roe algebra $C^*(Y_0 \subset Y, E)^G$ and a smooth index class $\mathrm{Ind}_\infty(D) \in K_0(\mathcal{L}^\infty_{G,s}(Y,E)) \cong K_0(C^*(Y_0 \subset Y, E)^G)$;*

2) *the delocalized eta invariant*

$$\eta_g(D_{\partial Y}) := \frac{1}{\sqrt{\pi}} \int_0^\infty \tau_g^{\partial Y}(D_{\partial Y} \exp(-tD^2_{\partial Y}) \frac{dt}{\sqrt{t}}$$

exists and for the pairing of the index class $\mathrm{Ind}_\infty(D)$ with the 0-cocycle $\tau_g^Y \in HC^0(\mathcal{L}^\infty_{G,s}(Y,E))$ defined by the orbital integral τ_g the following delocalized 0-degree index formula holds:

$$\langle \tau_g^Y, \mathrm{Ind}_\infty(D) \rangle = \int_{(Y_0)^g} c^g \mathrm{AS}_g(D_0) - \frac{1}{2}\eta_g(D_{\partial Y}), \tag{1.9}$$

where the integrand $c^g \mathrm{AS}_g(D_0)$ in the above integral is defined as that in Equation (1.5).

This theorem is an improvement with respect to [18] because we do *not* assume that $G/Z_G(g)$ is compact. Moreover, the two proofs are different.

Consider now Y/AN, an M-proper manifold, which has a slice decomposition given by $M \times_{K\cap M} Z =: Y_M$. The following theorem is one of the main results of this paper:

1.10. **Theorem.** *Suppose that the metric is slice compatible (Definition 3.5). Assume that $D_{\partial Y}$ is L^2-invertible and consider the higher index $\langle \Phi^P_{Y,g}, \mathrm{Ind}_\infty(D)\rangle$. The following formula holds:*

$$\langle \Phi^P_{Y,g}, \mathrm{Ind}_\infty(D)\rangle = \int_{(Y_0/AN)_g} c^g_{Y_0/AN} \mathrm{AS}(Y_0/AN)_g - \frac{1}{2}\eta_g(D_{\partial Y_M}) \tag{1.11}$$

with

$$\eta_g(D_{\partial Y_M}) = \frac{1}{\sqrt{\pi}}\int_0^\infty \tau_g^{\partial Y_M}(D_{\partial Y_M}\exp(-tD^2_{\partial Y_M}))\frac{dt}{\sqrt{t}},$$

where the integrand $c^g_{Y_0/AN}\mathrm{AS}(Y_0/AN)_g$ is defined in the same way as that in Equation (1.7). We refer to Equation (9.12) for its explicit expression. We regard $\eta_g(D_{\partial Y_M})$ as a higher delocalized eta invariant associated to P and g.

1.12. **Remark.** We observe that in Theorem 1.10, when P is not maximal cuspidal parabolic subgroup (see Definition 6.3), the pairing $\langle \Phi^P_{Y,g}, \mathrm{Ind}_\infty(D)\rangle$ vanishes, similarly to [16, Theorem 2.1]; moreover, each term on the right side of Equation (1.11) vanishes because of the existence of extra symmetry (see Remark 9.14).

In these two theorems the well-definedness of the (higher) delocalized eta invariant for the boundary operator is a consequence of the delocalized Atiyah-Patodi-Singer index theorem. In Section 5 we pass to the general case of a cocompact G proper manifold without boundary (thus not necessarily equal to the boundary of a cocompact G proper manifold with boundary). Some discussion about this question was given in [17, Proposition 4.1 and Section 4.4]. We make a very detailed study on the most general hypothesis under which the delocalized eta invariant is well defined and establish the following result:

1.13. **Theorem.** *Let $(X, \mathbf{h})$ be a cocompact G-proper manifold without boundary and let D be a G-equivariant Dirac-type operator (associated to a unitary Clifford action and a Clifford connection). Let $g \in G$ be a semi-simple element. If D is L^2-invertible, then the integral*

$$\frac{1}{\sqrt{\pi}}\int_0^\infty \tau_g^X(D\exp(-tD^2))\frac{dt}{\sqrt{t}} \tag{1.14}$$

converges.

The proof, which is rather involved, is divided into two parts: the small t and the large t integrability. The former is based on a very detailed study of the Schwartz kernel of $D\exp(-tD^2)$ and the way it behaves near the fixed-point set of g; the latter is based on a detailed study of the large time behavior of the heat kernel. In this paper, we have focused our study on the operator D whose associated boundary operator $D_{\partial Y}$ is invertible. We study the case when $D_{\partial Y}$ is not invertible in [38]. Our analysis of the short time behaviour also gives, as a byproduct, a rigorous proof of the Hoch-Wang index formula in [19]. We end the paper by defining (higher) rho-invariants associated to metrics of positive scalar curvature and by studying their properties.

This article uses quite some background knowledge from diverse fields of mathematics such as noncommutative geometry, analysis on manifolds with boundaries and representation theory. For the benefit of the reader we give a quick, but certainly incomplete, guide to the literature for the necessary background in these subjects

- The basic tools from noncommutative geometry needed for this paper are cyclic cohomology and its pairing with K-theory, as can be found in [5, Ch. III]. For manifolds with boundary the formalism of *relative* cyclic cohomology is particularly useful, for which we refer to [27, 34]
- The basic ingredient from the theory of representations of reductive Lie groups for this paper is the so-called Harish—Chandra algebra of functions on such groups [21]. The connection between representation theory and noncommutative geometry is rooted in the Connes—Kasparov conjecture proved by Wassermann in [48], see also [22].

- A standard reference for index theory on manifolds with boundary using the so-called b-calculus is the book by Melrose [32]. Important for the construction of the relative cyclic cocycles of this paper is the construction of the b-trace and the so-called "defect-formula"; these two topics are discussed in detail in [32]. Quick introductions to the b-calculus are given in the Appendix of [33] and also in the surveys [10, 29, 30]. This material is adapted to the present context of G-proper manifolds in Section 4B and Section 6 of [37].

The paper is organized as follows. In Section 3 we give a few geometric preliminaries; these will play a major role throughout the paper. Section 4 is devoted to a proof of the 0-degree delocalized Atiyah-Patodi-Singer, using relative cyclic 0-cocycles associated to orbital integrals. In Section 5 we prove Theorem 1.13 above (existence of the delocalized eta invariant in the non-bounding case); crucial in the proof is the study of the large time behaviour of the heat kernel, as an element in the algebra of integral operators associated to the Lafforgue algebra, and of the short time behaviour near a fixed point-set; the latter can also be used to give a detailed proof of the index formula (1.5). In Section 6 we recall the higher cyclic cocycles Φ_g^P introduced in [44] and we introduce the cyclic cocycles they define on G-proper manifolds; we also introduce the relative version of these cocycles on manifolds with boundary. In Section 7 and Section 8 we explain an approach to a general higher delocalized APS index theorem, using again the interplay between cyclic cocycles and relative cyclic cocycles associated to Φ_g^P. In Section 8 we do prove such a theorem in the slice compatible case by adapting to manifolds with boundary a reduction procedure due to Hochs, Song and Tang [16]. Finally in Section 10 we introduce (higher) rho numbers and study their properties.

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2. Notations

We use the following notations throughout this paper:

- G = connected, linear real reductive group with maximal compact subgroup K;
- Y_0 = cocompact proper G-manifold with boundary;
- $\mathbf{h}_0$ a G-invariant metric on Y_0, product-type near the boundary;
- Z_0 = a K-slice of Y_0 so that Y_0 is diffeomorphic to $G \times_K Z_0$; and abusing notation, we will write $Y_0 = G \times_K Z_0$;
- Y = the G-proper b-manifold associated to Y_0;
- $\mathbf{h}$ the G-invariant b-metric on Y associated to $\mathbf{h}_0$;
- Z = the b-manifold associated to the slice Z_0; Z is a K-slice of Y so that $Y = G \times_K Z$;
- X = cocompact proper G-manifold without boundary;
- S = a K-slice of X so that $X = G \times_K S$;
- E = G-equivariant twisted spinor bundle so that $E = \mathcal{E} \otimes W$, where $\mathcal{E}$ is the spinor bundle associated Spin^c-structure and W is an auxiliary G-equivariant vector bundle;
- $C_c^\infty(G)$ = space of compactly supported smooth functions on G;
- $\mathcal{L}_s(G)$ = space of Lafforgue's Schwartz functions on G associated to the norm ν_s;
- $\mathcal{L}_G^c(X)$ = the space of G-equivariant G-compactly supported smoothing operators;
- ${}^b\mathcal{L}_G^c(Y)$ = the space of G-equivariant G-compactly supported b-smoothing operators in the b-calculus with ϵ-bounds;
- $\mathcal{L}_{G,s}^\infty(X)$ = the space of G-Lafforgue Schwartz smoothing operators associated to the norm ν_s;
- $\mathcal{L}_{G,s}^\infty(Y)$ = the space of G-Lafforgue Schwartz residual smoothing operators in the b-calculus associated to ν_s with ϵ-bounds ;

- ${}^b\mathcal{L}^\infty_{G,s}(Y)$= the space of G-Lafforgue Schwartz b-smoothing operators in the b-calculus associated to ν_s with ϵ-bounds;
- $C^*_r(G)$=reduced group C^*-algebra of G;
- τ_g = orbital integral on $\mathcal{L}_s(G)$ associated to the conjugacy class of g;
- τ^X_g = trace on $\mathcal{L}^\infty_{G,s}(X)$ associated to the orbital integral τ_g;
- τ^Y_g = trace on $\mathcal{L}^\infty_{G,s}(Y)$ associated to the orbital integral τ_g;
- Φ^P_g=cyclic cocycle on the Harish-Chandra Schwartz algebra $\mathcal{L}_s(G)$ associated with a cuspidal parabolic subgroup P and a semisimple element g;
- $\Phi^P_{Y,g}$=cyclic cocycle on $\mathcal{L}^c_G(Y)$ and $\mathcal{L}^\infty_{G,s}(Y)$.

3. Geometric preliminaries.

Let Y_0 be a manifold with boundary, G a connected linear real reductive Lie group acting properly and cocompactly on Y_0. We denote by X the boundary of Y_0. There exists a collar neighbourhood U of the boundary ∂Y_0, $U \cong [0,2] \times \partial Y_0$, which is G-invariant and such that the action of G on U is of product type. We assume that Y_0 is endowed with a G-invariant metric $\mathbf{h}_0$ which is of product type near the boundary. We let $(Y_0, \mathbf{h}_0)$ be the resulting Riemannian manifold with boundary; in the collar neighborhood $U \cong [0,2] \times \partial Y_0$ the metric $\mathbf{h}_0$ can be written, through the above isomorphism, as $dt^2 + \mathbf{h}_X$, with $\mathbf{h}_X$ a G-invariant Riemannian metric on $X = \partial Y_0$. We denote by c_0 a cut-off function for the action of G on Y_0; since the action is cocompact, this is a compactly supported smooth function. We consider the associated manifold with cylindrical ends $\widehat{Y} := Y_0 \cup_{\partial Y_0} ((-\infty, 0] \times \partial Y_0)$, endowed with the extended metric $\widehat{\mathbf{h}}$ and the extended G-action. We denote by $(Y, \mathbf{h})$ the b-manifold associated to $(\widehat{Y}, \widehat{\mathbf{h}})$. We shall often treat $(\widehat{Y}, \widehat{\mathbf{h}})$ and $(Y, \mathbf{h})$ as the same object. We denote by c the obvious extension of the cut-off function c_0 for the action of G on Y_0 (constant along the cylindrical end); this is a cut-off function of the extended action of G on Y. If x is a boundary defining function for the cocompact G-manifold Y_0, then the b-metric $\mathbf{h}$ has the following product-structure near the boundary X:
$$\frac{dx^2}{x^2} + \mathbf{h}_X$$
We remark at this point that our arguments will actually apply to the more general case of *exact* b-metrics, or, equivalently, manifolds with asymptotic cylindrical ends. We shall not insist on this point.

In this article we shall be interested in the case in which Y_0 admits a G-invariant Spin^c-structure. Let E_0 be the G-equivariant spinor bundle associated to a Spin^c-structure on Y_0 twisted by an auxiliary G-equivariant vector bundle. In particular, E_0 has a G-equivariant Cliff(TY_0)-module structure. Let ∇^{E_0} be a *Clifford connection* on E_0, that is

$$\left[\nabla^{E_0}_V, c(W)\right] = c(\nabla^{TY_0}_V W), \quad V, W \in C^\infty(Y_0, TY_0) \tag{3.1}$$

where c denotes the Clifford action and ∇^{TY_0} is the Levi-Civita connection. The Dirac operator associated to the Clifford connection is given by the following composition

$$D_{Y_0}\colon C^\infty(Y_0, E_0) \xrightarrow{\nabla^{E_0}} C^\infty(Y_0, T^*Y_0 \otimes E_0) \cong C^\infty(Y_0, TY_0 \otimes E_0) \xrightarrow{c} C^\infty(Y_0, E_0). \tag{3.2}$$

3.3. **Assumption.** Suppose that G is a connected reductive Lie group with maximal compact subgroup K. We denote by
$$\mathfrak{r}\colon = \dim G/K \pmod 2, \quad \mathfrak{t}\colon = \dim Y_0 \pmod 2.$$
The index class
$$\mathrm{Ind}_\infty(D_{Y_0}) \in K_{\mathfrak{t}}\left(\mathcal{L}^\infty_{G,s}(Y, E)\right) \cong K_{\mathfrak{t}}(C^*_r G).$$
Recall that $K_{\mathfrak{r}+1}(C^*_r G) = 0$, see [22]. We assume that
$$\dim G/K = \dim Y_0 \pmod 2.$$
Otherwise, the index class is written as
$$\mathrm{Ind}_\infty(D_{Y_0}) \in K_{\mathfrak{t}}(C^*_r G) = K_{\mathfrak{r}+1}(C^*_r G) = 0.$$
Without loss of generality, we can further assume
$$\dim G/K = \dim Y_0 = 0 \pmod 2;$$

otherwise, we can simply replace G (and Y_0) by $G \times \mathbb{R}$ (respectively $Y_0 \times \mathbb{R}$). To sum up, we assume that

(1) the symmetric space G/K;
(2) the G-manifold Y_0.

are all even dimensional.

For any cuspidal parabolic subgroup $P = MAN$ with $m = \dim(A)$, the m-cyclic cocycle introduced below in (6.1), Φ^P_g, is an m-cyclic cocycle on $\mathcal{L}_s(G)$. The pairing

$$\langle \Phi^P_{Y,g}, \mathrm{Ind}_\infty(D)\rangle$$

is automatically zero if m is odd. Thus it suffices to consider those cuspidal parabolic subgroups such that $m = \dim(A)$ is even. In particular, for the maximal parabolic subgroup $P = MAN$, it is always this case under our assumption.

Slice compatible metric on Y_0: Recall Abels' slice theorem [1] that for any proper G-space W, there exists a G-equivariant map $f\colon W \to G/K$ and thus W has a global K-slice $V = f^{-1}(eK)$ such that $W \cong G \times_K V$. In our case, let U be a collar neighbourhood U of ∂Y_0 with $U \cong [0,2] \times \partial Y_0$. Applying Abels' slice theorem, we have a G-equivariant map $f_0\colon \partial Y_0 \to G/K$. Then we define $f_U\colon U \to G/K$ by

$$f_U(t,y) = f_0(y), \quad (t,y) \in U \cong [0,2] \times \partial Y_0.$$

By [1, Lemma 1.11] we can extend f_U to a G-equivariant map $f\colon Y_0 \to G/K$ such that $f|_U = f_U$. We can choose a global K-slice $Z_0 = f^{-1}(eK)$ such that

$$Y_0 = G \times_K Z_0.$$

By the construction, we have that Z_0 is a K-manifold with boundary $\partial Z_0 = f_0^{-1}(eK) \subset \partial Y_0$. We denote $S = \partial Z_0$. Consequently, $Y \cong G \times_K Z$ with Z the b-manifold associated to Z_0 and the boundary

$$X = \partial Y_0 \cong G \times_K S.$$

Choose a K-invariant inner product on the Lie algebra $\mathfrak{g}$ of G, so that we have the so called Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ its orthogonal complement. We have an isomorphism

$$TY_0 \cong G \times_K (\mathfrak{p} \oplus TZ_0). \tag{3.4}$$

Here we abuse the notation $\mathfrak{p}$ to denote the trivial vector bundle $Z_0 \times \mathfrak{p} \to Z_0$.

3.5. **Definition.** (slice compatible metrics) Given a slice Z_0, we shall say that a G-invariant metric on Y_0 is slice compatible with Z_0 if it is constructed from a K-invariant metric on Z_0 and a K-invariant metric on $\mathfrak{p}$ via the above isomorphism $TY_0 \cong G \times_K (\mathfrak{p} \oplus TZ_0)$. We shall say that the G-invariant metric $\mathbf{h}_0$ on Y_0 is slice-compatible if there is a slice Z_0 such that it is slice compatible with Z_0.

When Y_0 has a G-equivariant Spin^c-structure, we can construct its spinor bundle as follow. We can assume, up to the passage to double covers, that the adjoint representation $\mathrm{Ad} : K \to SO(\mathfrak{p})$ admits a lift $\widetilde{\mathrm{Ad}} : K \to \mathrm{Spin}(\mathfrak{p})$. We then obtain a G-invariant Spin^c-structure $P^{G/K} := G \times_K \mathrm{Spin}(\mathfrak{p}) \to G/K$. Assume now that Z_0 admits a K-invariant Spin^c-structure. Then, proceeding as in [13, 14], we obtain a G-invariant Spin^c-structure on M. Because

$$0 \to G \times_K \mathfrak{p} \to TY_0 \to G \times_K TZ_0 \to 0,$$

the two out of three lemma of Spin^c-structure shows that every G-invariant Spin^c-structure on Y_0 is induced from a K-invariant Spin^c-structure on Z_0. The vector bundle E_0 induces a K-equivariant vector bundle E_{Z_0} such that

$$E_0 \cong G \times_K (S_{\mathfrak{p}} \otimes E_{Z_0})$$

and E_{Z_0} admits a K-equivariant Cliff(TZ_0)-module structure and $S_{\mathfrak{p}}$ is the spinor bundle along the $\mathfrak{p}$ direction. We call the above a *slice compatible Spin^c-structure*. Throughout this paper, we only need to assume that the Spin^c-structure on Y_0 is slice compatible in section 8.

We decompose

$$L^2(Y_0, E_0) \cong \left[L^2(G) \otimes S_{\mathfrak{p}} \otimes L^2(Z_0, E_{Z_0})\right]^K .$$

We can define a *split Dirac operator* D_{split} by the following formula

$$D_{\text{split}} = D_{G,K}\hat{\otimes}1 + 1\hat{\otimes}D_{Z_0}, \tag{3.6}$$

where $D_{G,K}$ is the Spin^c-Dirac operator on $L^2(G)\otimes S_{\mathfrak{p}}$, and D_{Z_0} is a K-equivariant Dirac operator on E_{Z_0}, and $\hat{\otimes}$ means the graded tensor product. We point out here that the two Dirac operators D_{Y_0} and D_{split} are different in general. We would like to thank Jean-Michel Bismut and Xiaonan Ma for pointing out this difference to us. It was incorrectly stated in [15] and used in the articles[1] [11, 16–19] that these two Dirac operators are identical. In the appendix, we consider the example where $Y_0 = G$ and give an explicit formula for the two Dirac operators, showing in particular that their difference is different from zero. Nevertheless, as the operators have the same principal symbol they give the same index class provided the G proper manifold has no boundary; thus in this case

$$\text{Ind}_\infty(D_{Y_0}) = \text{Ind}_\infty(D_{\text{split}}) \in K_0(C^*_rG).$$

The split Dirac operator D_{split} has been extensively studied in [16–18]. It is important to point out that the connection used in the definition of D_{split} might not be the Clifford connection (see (3.1)); consequently, the short time behaviour of the heat kernel is ill behaved for D^2_{split}. The main goal of this paper is to study delocalized APS index theory, where the choice of connection is even more crucial, given that it affects the invertibility properties of the boundary operator. Hence, in this paper it is crucial that we work with the Dirac operator introduced in Equation (3.2).

4. Delocalized traces and the APS index formula

In this section we want to tackle the case of a delocalized APS index theorem on G-proper manifolds with boundary, that is, 0-degree delocalized cyclic cocycles. Similar results have been discussed in [17, 18]. Our treatment is different, centred around the interplay between absolute and relative cyclic cohomology and the b-calculus; moreover our treatment allows us to get sharper results compared to [18] in the case of a connected linear real reductive group[2] G. More precisely, in Theorem 1.8 we only assume that g is a semisimple element of G to obtain the index formula (1.9), while in [18, Theorem 2.1], the authors require that G/Z_g is compact[3].

4.1. **Orbital integrals and associated cyclic 0-cocycles in the closed case.** Consider first a cocompact G-proper manifold *without* boundary X. We know that there exists a compact submanifold $S \subset X$ on which the G-action restricts to an action of a maximal compact subgroup $K \subset G$, so that the natural map

$$G \times_K S \to X, \quad [g, x] \mapsto g \cdot x,$$

is a diffeomorphism. This decomposition of X induces an isomorphism

$$\mathcal{L}^c_G(X) \cong \left(C^\infty_c(G)\hat{\otimes}\Psi^{-\infty}(S)\right)^{K\times K} \tag{4.1}$$

and more generally

$$\mathcal{L}^c_G(X, E) \cong \left(C^\infty_c(G)\hat{\otimes}\Psi^{-\infty}(S, E|_S)\right)^{K\times K}, \tag{4.2}$$

in the presence of a G-equivariant vector bundle E. (We shall often expunge the vector bundle E from the notation.) On the left hand side we have the smoothing operators defined by G-invariant smooth kernels of G-compact support; on the right side we use the (unique) completion of the algebraic tensor product between the two algebras given that they are both nuclear.

[1] The list of references using this wrong property might be incomplete.

[2] In [18], the authors consider general locally compact topological groups.

[3] Notice that for the numeric g-index associated to D a formula is proved in [17] under the same hypothesis given here; on the other hand, the g-index is proved to be equal to the pairing of an index class with a 0-degree cyclic cocycle (this is the number we consider in the present article) only under the additional assumption that G/Z_g is compact.

4.3. **Definition.** For $s \in [0, \infty)$, define the Lafforgue algebra $\mathcal{L}_s(G)$ to be the completion of $C_c(G)$ with respect to the norm ν_s defined as follows

$$\nu_s(f) := \sup_{g \in G} \left\{ (1 + \|g\|)^s \cdot \Xi^{-1}(g) \cdot |f(g)| \right\},$$

where $\Xi(g)$ denotes Harish-Chandra's spherical function.

The family of Banach algebras $\{\mathcal{L}_s(G)\}_{s \geq 0}$ satisfies the following properties [22]:

(1) For every $s \in [0, \infty)$, $\mathcal{L}_s(G)$ is a dense subalgebra of $C^*_r(G)$ stable under holomorphic calculus.
(2) For $0 \leq s_1 < s_2$, $\|f\|_{s_1} \leq \|f\|_{s_2}$, $\forall f \in \mathcal{L}_{s_2}(G)$. Hence, $\mathcal{L}_{s_2}(G) \subset \mathcal{L}_{s_1}(G)$. Define

$$\mathcal{L}(G) := \cap_{s \geq 0} \mathcal{L}_s(G).$$

(3) For any semisimple element $x \in G$, there exists $d_0 > 0$, such that $\forall s > d_0$, the orbital integral[4]

$$\int_{G/Z_x} f(gxg^{-1})\, dgZ_x$$

is convergent for all $f \in \mathcal{L}_s(G)$.

We observe that Harish-Chandra's Schwartz algebra is contained in the algebra $\mathcal{L}_s(G)$ for any $s > 0$.

Throughout the paper, for a fixed $x \in G$, we work with a Lafforgue Schwartz algebra $\mathcal{L}_s(G)$ for a sufficiently large s so that the orbital integral is a well defined continuous linear functional on $\mathcal{L}_s(G)$. The existence of such an s is confirmed by above Property (3). We refer the reader to [44, Appendix, A.4.&A.5.] for details.

Let us consider now a cocompact G-proper manifold without boundary $X \cong G \times_K S$. The algebras $\mathcal{L}^\infty_{G,s}(X)$ and $\mathcal{L}^\infty_G(X)$ are defined as

$$\mathcal{L}^\infty_{G,s}(X) := \left(\mathcal{L}_s(G) \hat{\otimes} \Psi^{-\infty}(S) \right)^{K \times K}, \ \mathcal{L}^\infty_G(X) := \left(\mathcal{L}(G) \hat{\otimes} \Psi^{-\infty}(S) \right)^{K \times K}.$$

There are similar algebras when we consider a G-equivariant vector bundle E on X; however, for notational simplicity, we shall expunge the vector bundle E from the notation. For

$$\widetilde{k} \in \mathcal{L}^\infty_G(X) := \left(\mathcal{L}(G) \hat{\otimes} \Psi^{-\infty}(S) \right)^{K \times K}$$

or

$$\widetilde{k} \in \mathcal{L}^\infty_{G,s}(X) := \left(\mathcal{L}_s(G) \hat{\otimes} \Psi^{-\infty}(S) \right)^{K \times K}$$

we consider the bounded operator $T_{\widetilde{k}}$ on $L^2(X)$ given by

$$(T_{\widetilde{k}} e)(gs) = \int_G \int_S g\widetilde{k}(g^{-1}g', s, s') g'^{-1} e(g's') ds' dg'. \tag{4.4}$$

The operator $T_{\widetilde{k}}$ is an integral operator with G-equivariant Schwartz kernel κ given by

$$\kappa(gs, g's') = g\widetilde{k}(g^{-1}g', s, s') g'^{-1}.$$

The map $\widetilde{k} \to T_{\widetilde{k}}$ is injective; moreover

$$T_{\widetilde{k}} \circ T_{\widetilde{k}'} = T_{\widetilde{k} * \widetilde{k}'}$$

so that its image is a subalgebra of the G-equivariant bounded operators on $L^2(X)$. Following an established abuse of notation we shall not distinguish between these two algebras, thus identifying a smooth kernel with the bounded operators it defines.

Following [37], there are isomorphisms that associate to a smooth G-equivariant kernel A on $X \times X$ a map $\Phi_A : G \to \Psi^{-\infty}(S)$, with Φ_A equivariant with respect to the natural $K \times K$ action on $\Psi^{-\infty}(S)$ and the action $\alpha(k_1, k_2)(g) := k_1 g k_2^{-1}$ on G. More precisely, by [37, Prop. 1.7], there are isomorphisms

$$\mathcal{L}^c_G(X) \cong \left\{ \Phi : G \to \Psi^{-\infty}(S), \text{ smooth, compactly supported and } K \times K \text{ invariant} \right\}, \tag{4.5}$$

$$\mathcal{L}^\infty_{G,s}(X) \cong \left\{ \Phi : G \to \Psi^{-\infty}(S), \ K \times K \text{ invariant and } g \mapsto v_s(\|\Phi(g)\|_\alpha) \text{ bounded} \right\} \tag{4.6}$$

[4]We refer the reader to [44, Appendix Proposition A.4.] for the proof of this property.

with α a multi-index indexing derivatives with respect to the spacial variables of S, $\| \ \|_\alpha$ denoting the associated well-known seminorm on $\Psi^{-\infty}(S)$ and $v_s(\ )$, denoting the Lafforgue norm. The image of the product by convolution of A and B on the left hand side is equal to $\Phi_A * \Phi_B$ on the right side, with

$$(\Phi_A * \Phi_B)(g) = \int_G \Phi_A(gh^{-1}) \circ \Phi_B(h) dh \,. \tag{4.7}$$

Following again [20] we define for $T = T_{\widetilde{k}}$

$$\tau_g^X(T) := \int_{G/Z_g} \int_X c(hgh^{-1}x) \mathrm{tr}(hgh^{-1}\kappa(hg^{-1}h^{-1}x, x)) dx \, d(hZ) \tag{4.8}$$

with c a cut-off function for the action of G on X and tr denoting the vector-bundle fiberwise trace. There are equivalent ways to write the right-hand-side, see [20, Lemma 3.2]; for example if c_G is a cutoff function for the action of Z_g on G by right multiplication and $c^g(x) = \int_G c_G(h) c(hgx) dh$ then the right side of (4.8) can be written as

$$\int_X c^g(x) \mathrm{tr}(\kappa(x, gx) g) dx. \tag{4.9}$$

It is proved in [20, Lemma 3.4] that τ_g^X defines a continuous trace

$$\tau_g^X : \mathcal{L}^\infty_{G,s}(X) \to \mathbb{C} \,. \tag{4.10}$$

In fact, the two traces (4.8) and (4.10) are related by a homomorphism of integration along the slice S, $\mathrm{Tr}_S : \mathcal{L}^\infty_{G,s}(X) \to \mathcal{L}_s(G)$, such that

$$\tau_g^X = \tau_g \circ \mathrm{Tr}_S$$

(see [20, Section 3.3]). We see that Tr_S associates to Φ the function

$$G \ni g \to \mathrm{Tr}(\Phi(g)) \,.$$

Using [37, Lemma 1.24] and the well known inequality $|\,\mathrm{Tr}(T)| \le \|T\|_1$ for a smoothing operator on a smooth compact manifold[5], we see that $\mathrm{Tr}_S : \mathcal{L}^\infty_{G,s}(X) \to \mathcal{L}_s(G)$ so defined is a *continuous* map. (Even though [37, Lemma 1.24] uses a slightly different algebra on G instead of $\mathcal{L}_s(G)$, the proof showing that the trace norm $A \mapsto ||A||_1, \ A \in \Psi^{-\infty}(S)$ is continuous for the Fréchèt topology on $\Psi^{-\infty}(S)$ applies verbatim to show continuity in our case.) Exactly the same results hold for the algebra $\mathcal{L}^\infty_{G,s}(X)$.

Let now D be an equivariant Dirac operator, of product type near the boundary. We shall make the following assumption:

$$\text{the boundary operator } \ D_{\partial Y} \ \text{ is } \ L^2\text{-invertible.} \tag{4.11}$$

The following Theorems sharpen the corresponding results in [37], (see in particular Subsection 5.3, Proposition 5.27, Proposition 5.33 and Theorem 5.42 there).

4.12. **Theorem.** *Let D be as above and let Q^σ be a symbolic b-parametrix for D. The Connes-Skandalis projector*

$$P^b_Q := \begin{pmatrix} {}^bS_+^2 & {}^bS_+(I + {}^bS_+)Q^b \\ {}^bS_- D^+ & I - {}^bS_-^2 \end{pmatrix} \tag{4.13}$$

associated to a true b-parametrix $Q^b = Q^\sigma - Q'$ with remainders ${}^bS^\pm$ in $\mathcal{L}^\infty_{G,s}(Y_0)$ is a 2×2 matrix with entries in $\mathcal{L}^\infty_{G,s}(Y)$ [6]. We thus have a well-defined smooth index class

$$\mathrm{Ind}_\infty(D) := [P^b_Q] - [e_1] \in K_0(\mathcal{L}^\infty_{G,s}(Y)) \equiv K_0(C^*(Y_0 \subset Y)^G) \quad \textit{with} \quad e_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.14}$$

4.15. **Theorem.**

[5] with $\| \ \|_1$ denoting the trace norm

[6] really in a slightly extended algebra because of the identity appearing in the right lower corner of the matrix

1) *The Connes-Moscovici projector* $V(D)$,

$$V(D) := \begin{pmatrix} e^{-D^-D^+} & e^{-\frac{1}{2}D^-D^+}\left(\frac{I-e^{-D^-D^+}}{D^-D^+}\right)D^- \\ e^{-\frac{1}{2}D^+D^-}D^+ & I-e^{-D^+D^-} \end{pmatrix},$$

is a 2×2 *matrix with entries in* ${}^b\mathcal{L}^{\infty}_{G,s}(Y_0)$*; the Connes-Moscovici projector* $V(D^{\rm cyl})$ *is a* 2×2 *matrix with entries in* ${}^b\mathcal{L}^{\infty}_{G,s,\mathbb{R}}(\mathrm{cyl}(\partial Y))$*. These two projectors define a smooth relative index class* $\mathrm{Ind}_\infty(D,D_\partial)\in K_0({}^b\mathcal{L}^{\infty}_{G,s}(Y_0),{}^b\mathcal{L}^{\infty}_{G,s,\mathbb{R}}(\mathrm{cyl}(\partial Y)))$.

2) *The projector* $V^b(D)$ *obtained by improving the parametrix* $Q := \frac{I-\exp(-\frac{1}{2}D^-D^+)}{D^-D^+}D^-$ *defining* $V(D)$ *to a true b-parametrix* Q^b *is a* 2×2 *matrix with entries in* $\mathcal{L}^{\infty}_{G,s}(Y)$ *and defines the same smooth index class in* $K_0(\mathcal{L}^{\infty}_{G,s}(Y))$ *as the Connes-Skandalis projector of Theorem 4.12*

3) *The class* $\mathrm{Ind}_\infty(D)$ *is sent to the class* $\mathrm{Ind}_\infty(D,D_\partial)$ *through the excision isomorphism* $\alpha_{\rm exc}$.

Summarizing: using the Connes-Moscovici projector(s) we have smooth index classes $\mathrm{Ind}_\infty(D)\in K_0(\mathcal{L}^{\infty}_{G,s}(Y))$ and $\mathrm{Ind}_\infty(D,D_\partial)\in K_0({}^b\mathcal{L}^{\infty}_{G,s}(Y),{}^b\mathcal{L}^{\infty}_{G,s,\mathbb{R}}(\mathrm{cyl}(\partial Y)))$, with the first one sent to the second one by the excision isomorphism $\alpha_{\rm exc}$.

Proof. The proof proceeds as in [37]. Recall, in particular, that the relative index class is defined by the triple

$$(V(D),e_1,q_t)\,,\quad t\in[1,+\infty]\,,\quad \text{with } q_t := \begin{cases} V(tD_{\rm cyl}) & \text{if} \quad t\in[1,+\infty) \\ e_1 & \text{if} \quad t=\infty \end{cases} \tag{4.16}$$

and with $e_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}$. The well-definedness of this class is a consequence of the large time behaviour of the heat kernel on b-manifolds for invertible operators, treated in detail in the next section. Notice that most of the arguments given in [37] use the closure under holomorphic calculus of certain algebras. This applies unchanged whether we use pseudodifferential operators based on the Lafforgue algebra $\mathcal{L}_s(G)$ or the rapid decay algebra $H^\infty_L(G)$. The only exception is the analogue of Lemma 2.7 in [37] that we state now explicitly, in the present case, for the benefit of the reader:

4.17. **Lemma.** *Let* X *be a cocompact* G*-proper manifold without boundary with a slice* S*. Consider* $\Psi^{-\infty}_{G,c}(X)$ *and its extension* $\mathcal{L}^{\infty}_{G,s}(X) := (\mathcal{L}_s(G)\hat{\otimes}\Psi^{-\infty}_c(S))^{K\times K}$*, an algebra of smoothing operators. Then the composition* $\Psi^0_{G,c}(X)\times\Psi^{-\infty}_{G,c}(X)\to\Psi^{-\infty}_{G,c}(X)$ *extends to a continuous map*

$$\Psi^0_{G,c}(X)\times\mathcal{L}^{\infty}_{G,s}(X)\to\mathcal{L}^{\infty}_{G,s}(X).$$

Proof of the Lemma
The proof is a variation of the corresponding one for $(H^\infty_L(G)\hat{\otimes}\Psi^{-\infty}_c(S))^{K\times K}$ given in [37, Lemma 2.7]. Recall that $\Psi^0_{G,c}(X) = \left(\Psi^0_{G,c}(G)\hat{\otimes}\Psi^0_c(S)\right)^{K\times K}$. As $\Psi^{-\infty}_c(S)$ is an ideal of $\Psi^0_c(S)$, we are reduced to prove that the product

$$\Psi^0_c(G)\times\mathcal{L}_c(G)\to\mathcal{L}_c(G),\ (A,f)\mapsto A*f$$

is well-defined and continuous with respect to f.

A general element A of $\Psi^0_c(G)$ can be written as

$$A=\mathrm{Op}(a)+K,$$

where $K\in\Psi^{-\infty}_{G,c}(G)\cong C^\infty_c(G)$ and $\mathrm{Op}(a)$ is the operator corresponding to a symbol $a\in S^0(\mathfrak{g}^*)$ of order zero with property

$$|D^\alpha_\xi(a)|\le C_\alpha(1+|\xi|)^{-|\alpha|}.$$

It follows from the inclusion $C^\infty_c(G)\subset\mathcal{L}_s(G)$ that the product $(K,f)\mapsto K*f$ is well-defined and continuous for $K\in\Psi^{-\infty}_{G,c}(G)$. We are left to show that the composition $(\mathrm{Op}(a),f)\mapsto\mathrm{Op}(a)(f)$ belongs to

$\mathcal{L}(G)$ and is continuous. Recall that $\mathrm{Op}(a)(f)$ has the following expression

$$\mathrm{Op}(a)(f)(g) := \int_G \int_{\mathfrak{g}^*} \chi(h^{-1}) e^{i\langle \xi, \exp^{-1}(h^{-1})\rangle} a(\xi) f(hg) d\xi dh.$$

Let $\Xi(g)$ be Harish-Chandra's spherical function as before. We have

$$\begin{aligned}
&(1+\|g\|)^s\, \Xi(g)^{-1}\big(\mathrm{Op}(a)(f)(g)\big)\\
=&\int_G \int_{\mathfrak{g}^*} \chi(h^{-1}) e^{i\langle \xi, \exp^{-1}(h^{-1})\rangle} a(\xi) f(hg)\,(1+\|g\|)^s\, \Xi(g)^{-1} d\xi dh\\
=&\int_G \int_{\mathfrak{g}^*} \chi(h^{-1}) e^{i\langle \xi, \exp^{-1}(h^{-1})\rangle} a(\xi) \frac{\Xi(hg)}{\Xi(g)} \Xi(hg)^{-1} f(hg)\,(1+\|g\|)^s\, d\xi dh
\end{aligned}$$

By [21, Lemma 12.5], for h^{-1} in the support of the function χ, the support of which is compact, there is a constant $C>0$ such that

$$\left|\frac{\Xi(hg)}{\Xi(g)}\right| < C,\ \forall g\in G, h\in \mathrm{supp}(\chi).$$

Peetre's inequality gives,

$$(1+\|g\|)^s \le \big(1+\|h^{-1}\|\big)^s \cdot (1+\|hg\|)^s.$$

It follows from the above two inequalities that we have that

$$\|(1+\|g\|)^s\, \Xi(g)^{-1}\big(\mathrm{Op}(a)(f)(g)\big)\|$$

is bounded by

$$C\Big\|\int_G \int_{\mathfrak{g}^*} \chi(h^{-1})) e^{i\langle \xi, \exp^{-1}(h^{-1})\rangle} a(\xi)(1+\|h\|))^s d\xi dh\Big\| \times \sup(1+\|hg\|)^s \big\|\Xi(hg)^{-1} f(hg)\big\|.$$

The integral

$$\Big\|\int_G \int_{\mathfrak{g}^*} \chi(h^{-1})) e^{i\langle \xi, \exp^{-1}(h^{-1})\rangle} a(\xi)(1+\|h\|)^s d\xi dh\Big\|$$

is finite as the integration of h is over the support of χ, which is compact.

Summarizing the above estimates, we have proved that there is a constant $\widetilde{C}>0$ such that

$$\||1+\|g\||^s \Xi(g)^{-1}\big(\mathrm{Op}(a)(f)(g)\big)\| \le \widetilde{C} \sup(1+\|hg\|)^s \big\|\Xi(hg)^{-1} f(hg)\big\|.$$

This proves that $\mathrm{Op}(a)(f)$ belongs to the Lafforgue algebra $\mathcal{L}_s(G)$, and the map $(\mathrm{Op}(a), f)\mapsto \mathrm{Op}(a)(f)$ is continuous with respect to f. This proves the Lemma. ☐

4.2. **0-degree (relative) cyclic cocycles associated to orbital integrals.** Let us now pass to cyclic cocycles associated to orbital integrals. Consider for the time being a compact b-manifold Y endowed with a b-metric $\mathbf{h}_Y$ which has product structure $\mathbf{h}_Y = dx^2/x^2 + \mathbf{h}_{\partial Y}$ near the boundary. We use the associated volume form in order to trivialize the relevant half-density bundles. In the b-calculus with ϵ-bounds we consider

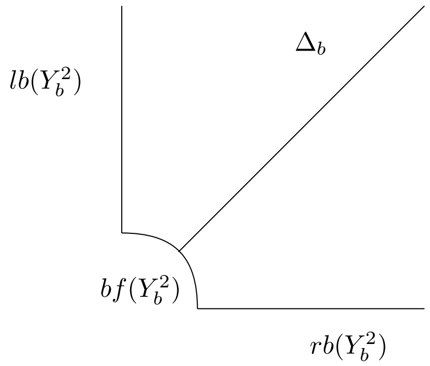
$$\mathrm{Ker}\left({}^b\Psi^{-\infty,\epsilon}(Y) \xrightarrow{I_Y} {}^b\Psi^{-\infty,\epsilon}_{\mathbb{R}}(\partial Y\times\mathbb{R})\right)$$

the kernel of the (surjective) indicial homomorphism.

It is well known, see Melrose's book [32], that if $\epsilon<1$, then

$$\mathrm{Ker}(I_Y)\subset \Psi^{-\infty,\epsilon}(Y)$$

with $\Psi^{-\infty,\epsilon}(Y)$ denoting the *residual operators*; these are smoothing kernels on the b-stretched product Y^2_b that vanish at order ϵ on *all* boundary faces, that is, the left boundary $lb(Y^2_b)$, the right boundary $rb(Y^2_b)$ and the front face $bf(Y^2_b)$. See the figure below. In fact $\mathrm{Ker}(I_Y) = \rho_{bf(Y^2_b)}{}^b\Psi^{-\infty,\epsilon}(Y)$ with $\rho_{bf(Y^2_b)}$ a boundary defining function for the front face, and by definition $\rho_{bf(Y^2_b)}{}^b\Psi^{-\infty,\epsilon}(Y)\subset \Psi^{-\infty,\epsilon}(Y)$ if $\epsilon<1$.



Because of this vanishing, the residual operators are trace class on L^2_b and the trace is obtained by integration over the lifted diagonal Δ_b (because of the extra vanishing the integral with respect to the b-volume form, which near the boundary can be written as $\frac{dx}{x} dvol_{\partial X}$, is absolutely convergent). Put it differently, this algebra of operators behaves very much as the smoothing operators on a smooth compact manifold without boundary.

Consider now the G-proper case and

$$\mathcal{L}^\infty_{G,s}(Y) := \mathrm{Ker}\left(I : {}^b\mathcal{L}^\infty_{G,s}(Y) \to {}^b\mathcal{L}^\infty_{G,s,\mathbb{R}}(\mathrm{cyl}\,\partial Y_0)\right) . \tag{4.18}$$

For exactly the same reason as above this algebra behaves very much like $\left(\mathcal{L}_s(G)\hat{\otimes}\Psi^{-\infty}(S)\right)^{K\times K}$, with S now a slice for the action of G on a cocompact G-proper manifold *without* boundary X. This is in fact a great advantage of our method and allows us to avoid completely *ad hoc* arguments.

We thus have a trace-homomorphism

$$\tau^Y_g : \mathcal{L}^\infty_{G,s}(Y) \to \mathbb{C}\,. \tag{4.19}$$

exactly as in the closed case:

$$\tau^Y_g(T) := \int_{G/Z_g}\int_Y c(hgh^{-1}x)\mathrm{tr}(hgh^{-1}\kappa(hg^{-1}h^{-1}x, x))dx\, d(hZ) \tag{4.20}$$

with dx denoting now the b-volume form associated to the b-metric $\mathbf{h}$. The proof of the well-definedness of (4.20) now proceeds as in the case of manifold without boundary, using the fact that κ is residual and therefore vanishing of order ϵ at all boundary faces. In fact: there is a homomorphism of integration along the slice S, $\mathrm{Tr}_S : \mathcal{L}^\infty_{G,s}(Y, E) \to \mathcal{L}_s(G)$, which is continuous and such that $\tau^Y_g = \tau_g \circ \mathrm{Tr}_S$.

The trace τ^Y_g defines a cyclic 0-cocycle on the algebra $\mathcal{L}^\infty_{G,s}(Y)$. Using the pairing between K-theory and cyclic cohomology, denoted $\langle\, ,\, \rangle$, we have in our case

$$\langle\cdot\,,\cdot\rangle : HC^0(\mathcal{L}^\infty_{G,s}(Y)) \times K_0(\mathcal{L}^\infty_{G,s}(Y)) \to \mathbb{C} \tag{4.21}$$

and thus a homomorphism

$$\langle\tau^Y_g, \cdot\rangle : K_0(\mathcal{L}^\infty_{G,s}(Y)) \to \mathbb{C}. \tag{4.22}$$

4.23. **Definition.** Let D be a G-equivariant operator on Y as in section 3. Assume that the induced boundary operator is L^2-invertible, so that there is a well-defined index class $\mathrm{Ind}_\infty(D) \in K_0(\mathcal{L}^\infty_{G,s}(Y))$. The g-index of D is, by definition, the number $\langle\tau^Y_g, \mathrm{Ind}_\infty(D)\rangle$.

Our goal in this section is to give a formula for $\langle\tau^Y_g, \mathrm{Ind}_\infty(D)\rangle$.

Following the relative cyclic cohomology approach in [9, 34, 37] we want to find a relative cyclic 0-cocycle $(\tau^{Y,r}_g, \sigma_g)$ verifying

$$\langle\tau^Y_g, \mathrm{Ind}_\infty(D)\rangle = \langle(\tau^{Y,r}_g, \sigma_g), \mathrm{Ind}_\infty(D, D_\partial)\rangle. \tag{4.24}$$

(The r on the right side stands for *regularized.*)

4.25. **Proposition.** *Let X be a cocompact G-proper manifold without boundary. Define the following 1-cochain on* ${}^b\mathcal{L}^\infty_{G,s,\mathbb{R}}(\mathrm{cyl}(X))$

$$\sigma^X_g(A_0,A_1)=\frac{i}{2\pi}\int_{\mathbb{R}}\tau^X_g(\partial_\lambda I(A_0,\lambda)\circ I(A_1,\lambda))d\lambda\,, \tag{4.26}$$

where the indicial family of $A\in {}^b\mathcal{L}^\infty_{G,s,\mathbb{R}}(\mathrm{cyl}(X))$, *denoted* $I(A,\lambda)$, *appears. Then* $\sigma^X_g(\,,\,)$ *is well-defined and a cyclic 1-cocycle.*

Proof. Fourier transform identifies ${}^b\mathcal{L}^\infty_{G,s,\mathbb{R}}(\mathrm{cyl}(X))$ with holomorphic families

$$\{\mathbb{R}\times i(-\epsilon,\epsilon)\ni\lambda\to\mathcal{L}^\infty_{G,s}(\partial X)\}$$

with values in the Fréchet algebra $\mathcal{L}^\infty_{G,s}(X)$, rapidly decreasing in $\mathrm{Re}\lambda$. (Recall that we do not write ϵ in the notation, but elements in our algebras are built from b-operators of order $-\infty$ in the calculus with ϵ-bounds.) It is then immediate that the integral is absolutely convergent and depends continuously on A_0, A_1. The fact that it is a cyclic 1-cocycle follows from the tracial property of τ^X_g and integration by parts in λ. □

Let Y be now a b-manifold and let ∂Y be its boundary. Let $\tau^{Y,r}_g$ be the functional on ${}^b\mathcal{L}^c_G(Y)$:

$$\tau^{Y,r}_g(T):=\int_{G/Z_g}{}^b\!\!\int_Y c(hgh^{-1}y)\mathrm{tr}(hgh^{-1}\kappa(hg^{-1}h^{-1}y,y))dy\,d(hZ)$$

where Melrose's b-integral has been used, dy denotes the b-density associated to the b-metric $\mathbf{h}$ and where we recall that the cut-off function c_0 on Y_0 is extended constantly along the cylinder to define c. This is the regularization of τ^Y_g on a b-manifold, for the time being on kernels of G-compact support (we shall deal with the extension of $\tau^{Y,r}_g$ on all of ${}^b\mathcal{L}^\infty_{G,s}(Y)$ momentarily). Observe that

$$\tau^{Y,r}_g=\tau_g\circ{}^b\mathrm{Tr}_S$$

with ${}^b\mathrm{Tr}_S:{}^b\mathcal{L}^c_G(Y)\to C^\infty_c(G)$ denoting b-integration along the slice S. More precisely, as in the closed case, we have an isomorphism

$${}^b\mathcal{L}^c_G(Y)\cong\left\{\Phi:G\to{}^b\Psi^{-\infty,\epsilon}(S),\ \text{smooth, compactly supported and}\ \ K\times K\ \text{invariant}\right\}$$

and ${}^b\,\mathrm{Tr}_S$ associates to Φ the function $G\ni\gamma\to{}^b\,\mathrm{Tr}(\Phi(\gamma))$. The continuity of this map will be treated more generally in the proof of Proposition 4.28 below.

4.27. **Proposition.** *The pair* $(\tau^{Y,r}_g,\sigma^{\partial Y}_g)$ *defines a relative 0-cocycle for* ${}^b\mathcal{L}^c_G(Y)\xrightarrow{I}{}^b\mathcal{L}^c_{G,\mathbb{R}}(\mathrm{cyl}(\partial Y))$.

Proof. As we are dealing with a 0-cochain, it suffices to show that for the Hochschild b-differential of $\tau^{Y,r}_g$ the following formula holds:

$$(b\tau^{Y,r}_g)(A_0,A_1)=\sigma^{\partial Y}_g(I(A_0),I(A_1))$$

where we recall that

$$\sigma^{\partial Y}_g(I(A_0),I(A_1))=\frac{i}{2\pi}\int_{\mathbb{R}}\tau^{\partial Y}_g(\partial_\lambda I(A_0,\lambda)\circ I(A_1,\lambda))d\lambda\,.$$

The left hand side $(b\tau^{Y,r}_g)(A_0,A_1)$ is equal to $\tau^{Y,r}_g[A_0,A_1]$; if A_i, $i=0,1$, corresponds to $\Phi_i:G\to{}^b\Psi^{-\infty}(S)$ then $\tau^{Y,r}_g[A_0,A_1]$ is equal to

$$\int_{G/Z_g}\int_G{}^b\,\mathrm{Tr}(\Phi_0(h_1gh_1^{-1}h^{-1})\circ\Phi_1(h)dhd(h_1Z_g)-\int_{G/Z_g}\int_G{}^b\,\mathrm{Tr}(\Phi_1h_1gh_1^{-1}h^{-1})\circ\Phi_0(h)dhd(h_1Z_g)\,.$$

Changing the order of integration and with a suitable change of coordinates in the second summand, using the unimodularity of G, we can rewrite the above expression as

$$\int_{G/Z_g}\int_G{}^b\,\mathrm{Tr}[\Phi_0(h_1gx^{-1}h_1^{-1}),\Phi_1(h)]dhd(h_1Z_g)\,.$$

Now we can apply Melrose's formula for the b-trace of a commutator and get

$$\frac{i}{2\pi}\int_{G/Z_g}\int_G\int_{\mathbb{R}}\mathrm{Tr}\left(\partial_\lambda I(\Phi_0(h_1gh_1^{-1}h^{-1}),\lambda)\circ I(\Phi_1(h),\lambda)\right)d\lambda dhd(h_1Z_g)\,.$$

We can interchange the order of integration without problems here (rapid decay in λ and compact support in G) and so we conclude reverting to $I(A_0,\lambda)$ and $I(A_1,\lambda)$ that

$$\begin{aligned}\tau_g^{Y,r}[A_0,A_1] &= \frac{i}{2\pi}\tau_g \int_{\mathbb{R}} \mathrm{Tr}_{\partial S}(\partial_\lambda I(A_0,\lambda)\circ I(A_1,\lambda))d\lambda \\ &= \frac{i}{2\pi}\int_{\mathbb{R}} \tau_g^{\partial Y}(\partial_\lambda I(A_0,\lambda)\circ I(A_1,\lambda))d\lambda\end{aligned}$$

This completes the proof. □

4.28. **Proposition.** *The pair $(\tau_g^{Y,r},\sigma_g^{\partial Y})$ extends continuously to a relative 0-cocycle for*

$${}^b\mathcal{L}^\infty_{G,s}(Y) \xrightarrow{I} {}^b\mathcal{L}^\infty_{G,s,\mathbb{R}}(\mathrm{cyl}(\partial Y)).$$

Moreover, the following formula holds:

$$\langle \tau_g^Y, \mathrm{Ind}_\infty(D)\rangle = \langle (\tau_g^{Y,r},\sigma_g^{\partial Y}), \mathrm{Ind}_\infty(D,D_\partial)\rangle\,. \tag{4.29}$$

Proof. We already know that $\sigma_g^{\partial Y}$ extends to a 1-cocycle on $\mathcal{L}^\infty_{G,s,\mathbb{R}}(\mathrm{cyl}(\partial Y))$. If we could show that $\tau_g^{Y,r}$ extends to ${}^b\mathcal{L}^\infty_{G,s}(Y)$ then by density and continuity we would get the first statement of the proposition. The second would then follow as usual from the fact that

$$\tau_g^{Y,r}|_{\mathcal{L}^\infty_{G,s}(Y)} = \tau_g^Y$$

given that on residual operators the b-integral equals the ordinary integral.

As in [37] we have an isomorphism

$${}^b\mathcal{L}^\infty_{G,s}(Y) \cong \left\{\Phi : G\to {}^b\Psi^{-\infty,\epsilon}(S),\ K\times K \text{ invariant and } g\mapsto v_s(\|\Phi(g)\|_\alpha)\ \text{ bounded}\right\}$$

where $\|\ \|_\alpha$ are now the C^∞ seminorms of a smooth kernel on the b-stretched product.
We want to show that the map

$${}^b\mathcal{L}^\infty_{G,s}(Y) \ni \Phi \longrightarrow {}^b\,\mathrm{Tr}(\Phi(\cdot)) \in \mathcal{L}_s(G) \tag{4.30}$$

is continuous. Here we can use a Proposition of Lesch-Moscovici-Pflaum [27, Proposition 2.6], expressing the b-trace of a b-smoothing operator on a compact b-manifold in terms of the trace of two residual operators, see for example Proposition 7.1 in [37]. Using this Proposition and the arguments in [37, Lemma 7.8], directly inspired in turn on those of [9], we prove the continuity of the map

$${}^b\Psi^{-\infty,\epsilon}(S) \ni T \longrightarrow {}^b\,\mathrm{Tr}(T) \in \mathbb{C}$$

and thus of the map (4.30). □

4.3. **The 0-degree delocalized APS index theorem on G-proper manifolds.** We now apply formula (4.29) to the index class associated to the Connes-Moscovici projector

$$V(D) = \begin{pmatrix} e^{-D^-D^+} & e^{-\frac12 D^-D^+}\left(\frac{I-e^{-D^-D^+}}{D^-D^+}\right)D^- \\ e^{-\frac12 D^+D^-}D^+ & I-e^{-D^+D^-}\end{pmatrix} \tag{4.31}$$

On the left hand-side of (4.29) we have

$$\langle \tau_g^M, V^b(D)\rangle$$

with $V^b(D)$ the modified Connes-Moscovici projector. Indeed, recall that the Connes-Moscovici projector is simply the Connes-Skandalis projector for the choice of parametrix

$$Q := \frac{I-\exp(-\frac12 D^-D^+)}{D^-D^+}D^-$$

This produces the remainders $I-QD^+ = \exp(-\frac12 D^-D^+)$, $I-D^+Q = \exp(-\frac12 D^+D^-)$ and $V^b(D)$ is obtained by improving this parametrix to a true parametrix in the b-calculus, that is, an inverse modulo residual operators.

On the right hand-side we have the pairing of the relative cocycle $(\tau_g^{Y,r}, \sigma_g^{\partial Y})$ with the relative class

$$(V(D), e_1, q_t)\,, \quad t \in [1, +\infty]\,, \quad \text{with } q_t := \begin{cases} V(tD_{\text{cyl}}) & \text{if } \; t \in [1, +\infty) \\ e_1 & \text{if } \; t = \infty \end{cases} \tag{4.32}$$

and with $e_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Let us concentrate first on the right hand-side. By definition of relative pairing we have:

$$\langle (\tau_g^{Y,r}, \sigma_g^{\partial Y}), (V(D), e_1, q_t) \rangle = \tau_g^{Y,r}(e^{-D^- D^+}) - \tau_g^{Y,r}(e^{-D^+ D^-}) + \int_1^\infty \sigma_g^{\partial Y}([\dot{q}_t, q_t], q_t) dt\,. \tag{4.33}$$

4.34. **Proposition.** *The term* $\int_1^\infty \sigma_g^{\partial Y}([\dot{q}_t, q_t], q_t)dt$*, with* $q_t := V(tD_{\text{cyl}})$ *is equal to*

$$-\frac{1}{\sqrt{\pi}} \int_1^\infty \tau_g^{\partial Y}(D_{\partial Y} \exp(-t^2 D_{\partial Y}^2)) dt\,.$$

Proof. By definition

$$\sigma_g^{\partial Y}([\dot{q}_t, q_t], q_t)dt = \frac{i}{2\pi} \int_{\mathbb{R}} \tau_g^{\partial Y}(\partial_\lambda(I([\dot{q}_t, q_t], \lambda)) \circ I(q_t, \lambda)) d\lambda.$$

We can integrate by parts in λ on the right side, given that the two terms are rapidly decreasing in λ. Thus, taking into account the multiplicative constants, we want to prove that

$$\int_{\mathbb{R}} \tau_g^{\partial Y}((I([\dot{q}_t, q_t], \lambda) \circ \partial_\lambda(I(q_t, \lambda)) d\lambda = 2i\sqrt{\pi}\, \tau_g^{\partial Y}(D_{\partial Y} \exp(-t^2 D_{\partial Y}^2))\,. \tag{4.35}$$

We need to write down the indicial family of q_t. By definition

$$q_t = \begin{pmatrix} e^{-t^2 D_{\text{cyl}}^- D_{\text{cyl}}^+} & e^{-\frac{t^2}{2} D_{\text{cyl}}^- D_{\text{cyl}}^+} \left(\frac{I - e^{-t^2 D_{\text{cyl}}^- D_{\text{cyl}}^+}}{t^2 D_{\text{cyl}}^- D_{\text{cyl}}^+} \right) t D_{\text{cyl}}^- \\ e^{-\frac{t^2}{2} D_{\text{cyl}}^+ D_{\text{cyl}}^-} t D^+ & I - e^{-t^2 D_{\text{cyl}}^+ D_{\text{cyl}}^-} \end{pmatrix}$$

and thus, denoting $D_{\partial Y}$ by B we have

$$I(q_t, \lambda) = \begin{pmatrix} e^{-t^2(\lambda^2 + B^2)} & e^{-\frac{t^2}{2}(\lambda^2 + B^2)} \left(\frac{I - e^{-t^2(\lambda^2 + B^2)}}{t^2(\lambda^2 + B^2)} \right) t(-i\lambda + B) \\ e^{-\frac{t^2}{2}(\lambda^2 + B^2)} t(i\lambda + B) & I - e^{-t^2(\lambda^2 + B^2)} \end{pmatrix}.$$

We set

$$p(t, \lambda) := I(q_t, \lambda)\,.$$

We must compute

$$\int_{\mathbb{R}} \tau_g^{\partial Y} \left((\partial_t p(t, \lambda) \circ p(t, \lambda) - p(t, \lambda) \circ \partial_t p(t, \lambda)) \circ \partial_\lambda p(t, \lambda) \right) d\lambda$$

and show that it equals

$$\frac{i\sqrt{\pi}}{\sqrt{t}} \tau_g^{\partial Y}(D_{\partial Y} \exp(-t D_{\partial Y}^2)).$$

This is a complicated computation; however, we remark that all operators appearing in the 2×2 matrices

$$\partial_t p(t, \lambda)\,, \quad p(t, \lambda) \quad \text{and} \quad \partial_\lambda p(t, \lambda)$$

are given by operators obtained by functional calculus for B and thus, in particular, they commute. This means that we can deal with this computation in a formal way and we can use e.g. *Mathematica* to see that $\partial_t p(t, \lambda)$ is equal to

$$\begin{pmatrix} -2t\left(B^2 + \lambda^2\right) e^{-t^2\left(B^2 + \lambda^2\right)} & \frac{e^{-\frac{3}{2}t^2\left(B^2+\lambda^2\right)}\left(-e^{t^2\left(B^2+\lambda^2\right)}\left(B^2t^2+\lambda^2t^2+1\right)+3B^2t^2+3\lambda^2t^2+1\right)}{t^2(B+i\lambda^2)} \\ -e^{-\frac{1}{2}t^2\left(B^2+\lambda^2\right)(B+i\lambda)}\left(B^2t^2 + \lambda^2 t^2 - 1\right) & 2t\left(B^2 + \lambda^2\right) e^{-t^2\left(B^2 + \lambda^2\right)} \end{pmatrix},$$

as well as that $\partial_\lambda p(t,\lambda)$ is given by

$$\begin{pmatrix} -2\lambda t^2 e^{-t^2(B^2+\lambda^2)} & \frac{e^{\frac{1}{2}(-3)t^2(B^2+\lambda^2)}\left(-e^{t^2(B^2+\lambda^2)}(B\lambda t^2+i(\lambda^2t^2+1))+3B\lambda t^2+3i\lambda^2t^2+i\right)}{t(B+i\lambda)^2} \\ -te^{-\frac{1}{2}t^2(B^2+\lambda^2)}\left(B\lambda t^2+i\left(\lambda^2t^2-1\right)\right) & 2\lambda t^2e^{-t^2(B^2+\lambda^2)} \end{pmatrix}.$$

This leads, again with the help of *Mathematica*, to the integrand

$$\mathrm{Tr}_{M_2(\mathbb{C})}\left([\partial_t p(t,\lambda), p(t,\lambda)]\circ\partial_\lambda p(t,\lambda)\right) = 2iBte^{-t^2(B^2+\lambda^2)},$$

and after a Gaussian integration over λ, formula (4.35) does follow. □

4.36. **Remark.** When we scale $D_{\rm cyl}$ with $\sqrt{t}$ instead of t, the formula for the η-term in Proposition 4.34takes the more familiar expression

$$-\frac{1}{2}\left(\frac{1}{\sqrt{\pi}}\int_1^\infty \tau_g^{\partial Y}(D_{\partial Y}\exp(-tD^2_{\partial Y}))\frac{dt}{\sqrt{t}}\right).$$

This convention is more standard in the literature, and we shall adopt this in the rest of the paper.

Thanks to this Proposition we have that

(4.37)
$$\langle(\tau_g^{Y,r},\sigma_g^{\partial Y}),(V(D),e_1,q_t)\rangle = \tau_g^{Y,r}(e^{-D^-D^+}) - \tau_g^{Y,r}(e^{-D^+D^-}) - \frac{1}{2}\int_1^\infty \frac{1}{\sqrt{\pi}}\tau_g^{\partial Y}(D_{\partial Y}\exp(-tD^2_{\partial Y}))\frac{dt}{\sqrt{t}}.$$

As the last step we replace D by sD; in the equality

$$\langle\tau_g^Y, \mathrm{Ind}_\infty(D)\rangle = \langle(\tau_g^{Y,r},\sigma_g^{\partial Y}),\mathrm{Ind}_\infty(D,D_{\partial Y})\rangle$$

the left hand side $\langle\tau_g^Y, \mathrm{Ind}_\infty(D)\rangle$ remains unchanged whereas the right side becomes

$$\tau_g^{Y,r}(e^{-s^2D^-D^+}) - \tau_g^{Y,r}(e^{-s^2D^+D^-}) - \frac{1}{2}\int_s^\infty \frac{1}{\sqrt{\pi}}\tau_g^{\partial Y}(D_{\partial Y}\exp(-tD^2_{\partial Y})\frac{dt}{\sqrt{t}}.$$

Summarizing, for each $s>0$ we have

$$\tau_g^{Y,r}(e^{-s^2D^-D^+}) - \tau_g^{Y,r}(e^{-s^2D^+D^-}) = \langle\tau_g^Y,\mathrm{Ind}_\infty(D)\rangle + \frac{1}{2}\int_s^\infty \frac{1}{\sqrt{\pi}}\tau_g^{\partial Y}(D_{\partial Y}\exp(-tD^2_{\partial Y})\frac{dt}{\sqrt{t}}. \tag{4.38}$$

Now we take the limit as $s\downarrow 0$. It is a general principle, explained in detail in [32], that the Getzler rescaling applies to the heat kernel in the b-context; needless to say, the geometry here is more complicated that in the case of a compact b-manifold endowed with a product b-metric. Still, we shall prove in the next section the following proposition, where all structures are assumed to be product-like near the boundary.

4.39. **Proposition.** *Let $(Y_0,\mathbf{h}_0)$ be a cocompact G-proper manifold. Let $(Y,\mathbf{h})$ be the associated b-manifold. Let D_0 be a G-equivariant Dirac operator defined in (3.2) and let D be the associated b-differential operator. Let g be a semisimple element and let Y_0^g the fixed point set of g. Then the limit $\lim_{s\downarrow 0}\tau_g^{Y,r}(e^{-s^2D^-D^+}) - \tau_g^{Y,r}(e^{-s^2D^+D^-})$ exists and we have*

$$\lim_{s\downarrow 0}\tau_g^{Y,r}(e^{-s^2D^-D^+}) - \tau_g^{Y,r}(e^{-s^2D^+D^-}) = \int_{Y_0^g} c_0^g\mathrm{AS}_g(D_0)$$

with $c_0^g\mathrm{AS}_g(D_0)$ defined in Equation (1.6).

Assuming the last Proposition we can infer that the limit

$$\lim_{s\downarrow 0}\frac{1}{2}\int_s^\infty \frac{1}{\sqrt{\pi}}\tau_g^{\partial Y}(D_{\partial Y}\exp(-tD^2_{\partial Y})\frac{dt}{\sqrt{t}}$$

exists and equals

$$\int_{Y_0^g} c^g\mathrm{AS}_g(D_0) - \langle\tau_g^Y,\mathrm{Ind}_\infty(D)\rangle.$$

We conclude that we have proved the following

4.40. **Theorem.** *(0-degree delocalized APS)*
Let G be connected, linear real reductive group. Let g be a semisimple element. Let Y_0, Y, D, $D_{\partial Y}$ as above. Assume that $D_{\partial Y}$ is L^2-invertible. Then

$$\eta_g(D_{\partial Y}) := \frac{1}{\sqrt{\pi}} \int_0^\infty \tau_g^{\partial Y}(D_{\partial Y} \exp(-tD_{\partial Y}^2)) \frac{dt}{\sqrt{t}}$$

exists and for the pairing of the index class $\mathrm{Ind}_\infty(D) \in K_0(\mathcal{L}^\infty_{G,s}(Y)) \equiv K_0(C^*(Y_0 \subset Y)^G)$ with the 0-cocycle $\tau_g^Y \in HC^0((\mathcal{L}^\infty_{G,s}(Y)))$ the following delocalized 0-degree APS index formula holds:

$$\langle \tau_g^Y, \mathrm{Ind}_\infty(D) \rangle = \int_{Y_0^g} c^g \mathrm{AS}_g(D_0) - \frac{1}{2}\eta_g(D_{\partial Y}),$$

where the integrand $c^g \mathrm{AS}_g(D_0)$ is defined in the same way as the one in Equation (1.6).

5. Delocalized eta invariants for G-proper manifolds

In the previous section we have obtained the well-definedness of $\eta_g(D_{\partial Y})$, with $D_{\partial Y}$ being L^2-invertible, as a byproduct of the proof of the delocalized APS index theorem for 0-degree cocycles. In fact, one can show that $\eta_g(D)$ is well defined on a cocompact G-proper manifold even if D does not arise as a boundary operator. This is the content of the next theorems, partially discussed also in [17, 18]. The main result of this section is the following theorem.

5.1. **Theorem.** *Let $(X, \mathbf{h})$ be a cocompact G-proper manifold without boundary endowed with a G-equivariant Spin^c-structure and let D be the Dirac-type operator defined in (3.2). Let g be a semi-simple element. If D is L^2-invertible, then the integral*

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \tau_g^X(D \exp(-tD^2) \frac{dt}{\sqrt{t}} \tag{5.2}$$

converges.

Proof. We split the proof into two parts.
In Proposition 5.8, we study the large time behavior of the heat kernel and prove the following integral

$$\frac{1}{\sqrt{\pi}} \int_1^\infty \tau_g^X(D \exp(-tD^2)) \frac{dt}{\sqrt{t}}$$

converges.
In Proposition 5.31, we study the small time behavior of the heat kernel and prove the following integral

$$\frac{1}{\sqrt{\pi}} \int_0^1 \tau_g^X(D \exp(-tD^2)) \frac{dt}{\sqrt{t}}$$

converges.
We complete the proof of the theorem by combining the above two results. □

Recall, following [37], that for all $t > 0$

$$\exp(-tD^2) = \frac{1}{2\pi i} \int_\gamma e^{-t\mu}(D^2 - \mu)^{-1} d\mu$$

is an element in the Fréchet algebra $\mathcal{L}^\infty_{G,s}(X, E)$ with any $s > 0$. It was stated in [37], but without a proper proof, that if D is L^2-invertible then $\exp(-tD^2)$ converges exponentially to 0 in $\mathcal{L}^\infty_{G,s}(X, E)$. The corresponding statement for b-manifolds was also stated there. The next two subsections provide detailed proofs of these two results.

5.1. **Large time behaviour on manifolds without boundary.** Assume now that there exists $a > 0$ such that

$$\mathrm{Spec}_{L^2}(D) \cap [-2a, 2a] = \emptyset \,. \tag{5.3}$$

We can and we shall choose γ so that $\mathrm{Spec}_{L^2}(D) \cap \gamma(z) = \emptyset$ and $\mathrm{Re}\,\gamma(z) > a$ for every z. We want to show that $\exp(-tD^2) \in \mathcal{L}^\infty_{G,s}(X)$ is exponentially converging to 0 in $\mathcal{L}^\infty_{G,s}(X)$ as $t \to +\infty$.

By [37, Proposition 5.21],

$$(D^2 - \mu)^{-1} = B(\mu) + C(\mu)$$

where

- $B(\mu) \in \Psi^{-2}_{G,c}(X)$ is a symbolic parametrix for $D^2 - \mu$ and defines a pseudodifferential operator with parameter of order -2;
- $C(\mu) \in \mathcal{L}^\infty_{G,s}(X)$, and $C(\mu)$ goes to 0 in the Fréchet topology of $\mathcal{L}^\infty_{G,s}(X)$, as $|\mu| \to +\infty$.

In fact, we can improve this result and see easily that $C(\mu)$ is rapidly decreasing in $|\mu|$, with values in $\mathcal{L}^\infty_{G,s}(X)$. Indeed, $(D^2 - \mu)^{-1} = B(\mu)(1 + F(\mu))$ with $1 + F(\mu) = (1 + R(\mu))^{-1}$ and $R(\mu)$ the remainder of the symbolic parametrix $B(\mu)$. By the pseudodifferential calculus with parameter we know that $R(\mu)$ is rapidly decreasing in $|\mu|$ (it is a smoothing operator with parameter). Using the elementary identity

$$1 - (1 + R(\mu))^{-1} = R(\mu)[(1 + R(\mu))^{-1}]$$

we understand that $F(\mu)$ is in fact rapidly decreasing in $|\mu|$ with values in $\mathcal{L}^\infty_{G,s}(X)$. Thus $C(\mu) := B(\mu) \circ F(\mu)$ is also rapidly decreasing in $|\mu|$, with values in $\mathcal{L}^\infty_{G,s}(X)$; here Lemma 4.17 has been used. By the same analysis carried out in [37] and above we also know that for all μ and $k \geq 1$

$$(D^2 - \mu)^{-k} = F(\mu) + G(\mu)$$

with $F(\mu) \in \Psi^{-2k}_{G,c}(X, \Lambda)$, of G-compact support uniformly in μ, and $G(\mu) \in \mathcal{L}^\infty_{G,s}(X)$ and rapidly decreasing in $\mathcal{L}^\infty_{G,s}(X)$ as $|\mu| \to +\infty$.

Let $p_{s,\alpha}$ be a seminorm on the Fréchet algebra $\mathcal{L}^\infty_{G,s}(X) := (\mathcal{L}_s(G) \hat{\otimes} \Psi^{-\infty}(S))^{K \times K}$. This depends on s through the Banach norm on $\mathcal{L}_s(G)$ and on the multiindex α through a seminorm $\| \cdot \|_\alpha$ on the smoothing operators on the slice. Let $|\alpha| = \ell$. Consider now $(D^2 - \mu)^{-k}$ with $k > \max(\dim X, \ell/2)$. Observe that

$$\exp(-tD^2) = \frac{(k-1)!}{2\pi i} t^{1-k} \int_\gamma e^{-t\mu} (D^2 - \mu)^{-k} d\mu$$

We want to show that

$$p_{s,\alpha} \left(\int_\gamma e^{-t\mu} (D^2 - \mu)^{-k} d\mu \right) \to 0 \quad \text{as} \quad t \to +\infty \,.$$

We thus consider the decomposition

$$(D^2 - \mu)^{-k} = F(\mu) + G(\mu)$$

and

$$\int_\gamma e^{-t\mu} F(\mu) d\mu, \quad \int_\gamma e^{-t\mu} G(\mu) d\mu.$$

Consider the first integral. Using the arguments employed by Shubin in [43, Ch. XII, Sect. 11] in order to discuss the properties of complex powers, see also Vassout [45, Ch. 4] and Gilkey[8, Lemma 1.8.1], we see that this first integral is in $\mathcal{L}^c_G(X)$, which is a subalgebra of $\mathcal{L}^\infty_{G,s}(X)$; the integrand, on the other hand, is in $\Psi^{-2k}_{G,c}(X, \Lambda)$ and to such an integrand we can apply the seminorm $p_{s,\alpha}$, given that it is contained in $(\mathcal{L}_s(G) \hat{\otimes} C^{2k}(S \times S))^{K \times K}$. From the properties of the Bochner integral we thus have that

$$p_{s,\alpha} \left(\int_\gamma e^{-t\mu} F(\mu)) d\mu \right) \leq \int_\gamma e^{-t\mu} p_{s,\alpha}(F(\mu)) d\mu$$

Now, $F(\mu)$ is a pseudodifferential operator with parameter of order $(-2k)$ of G-compact support uniformly in μ. We claim that $p_{m,\alpha}(F(\mu))$ can be bounded by a negative power of $|\mu|$. To see this we take the associated element $\Phi_{F(\cdot)} : G \to \Psi^{-2k}(S, \Lambda)$ under the isomorphism (4.5). We can equivalently show that the seminorm $p^\Phi_{s,\alpha}$ defined by v_s and $\| \cdot \|_\alpha$ can be bounded by a negative power of μ when applied to the element $\Phi_{F(\cdot)}$.

Now, $\Phi_{F(\cdot)}$ is a compactly supported function on G with values in $\Psi^{-2k}(S,\Lambda)$. Because of the compactness of the support in G we only need to understand the statement for the elements in $\Psi^{-2k}(S,\Lambda)$ for k large; however this is well known, see for example [8, Lemma 1.7.4]. It is at this point clear that

$$A(t) := \int_\gamma e^{-t\mu} p_{m,\alpha}(F(\mu)) d\mu \tag{5.4}$$

goes to 0 as $t \to +\infty$. Indeed, as we can choose the path γ so that $\mathrm{Re}\gamma(z) > a$ for all z, the result is immediate from the stated properties of $p_{m,\alpha}(F(\mu))$.
Consider now the second term,

$$p_{s,\alpha}\left(\int_\gamma e^{-t\mu} G(\mu) d\mu\right)$$

Here the integrand is already an element in $\mathcal{L}^\infty_{G,s}(X)$, going to 0 as $|\mu|$ goes to $+\infty$. In this case we can bring the seminorm under the sign of integral. Thus we are considering

$$B(t) := \int_\gamma e^{-t\mu} p_{s,\alpha}(G(\mu)) d\mu \tag{5.5}$$

Since we know that $p_{s,\alpha}(G(\mu))$ goes to 0 as $|\mu|$ goes to $+\infty$ (in fact, rapidly), using once again in a crucial way the fact that $\mathrm{Re}\gamma(z) > a$, we can conclude that $B(t)$ goes to 0 as $t \to +\infty$.
Notice that in fact, by elementary manipulations, writing $e^{-t\mu}$ as $e^{-t\mu/2} \circ e^{-t\mu/2}$ we can prove that the convergence is weighted exponential, with weight a (and a as in (5.3)). See [37, Remark 2.11].

We summarize our discussion in the following proposition:

5.6. **Proposition.** *If D is L^2-invertible then*

$$\exp(-tD^2) \to 0 \quad \textit{weighted exponentially in} \quad \mathcal{L}^\infty_{G,s}(Y) \quad \textit{as} \quad t \to +\infty. \tag{5.7}$$

In general for a Schwartz function f on $\mathbb{R}$, the operators $f(tD^2)$and $Df(tD^2)$ both converge to 0 weighted exponentially in $\mathcal{L}^\infty_{G,s}(Y)$ as $t \to +\infty$. In particular, the Connes-Moscovici projector $V(tD) - e_1$ converges to 0 weighted exponentially in $\mathcal{L}^\infty_{G,s}(Y)$ as $t \to +\infty$.

Proof. For the heat kernel we have already given a very detailed proof. For a general Schwartz function f on $\mathbb{R}$, we can directly generalize the above analysis on $\exp(-tD^2)$ via a similar estimate for $D(D^2-\lambda)^{-2k}$ to show that $Df(-tD^2)$ converges to 0 weighted exponentially in $\mathcal{L}^\infty_{G,s}(Y)$ using the following integral formulas

$$f(tD^2) = \frac{1}{2\pi i}\int_\gamma f(t\mu)(D^2-\mu)^{-1} d\mu, \qquad Df(tD^2) = \frac{1}{2\pi i}\int_\gamma f(t\mu) D(D^2-\mu)^{-1} d\mu.$$

The property about the Connes-Moscovici projector follows from the observation that every component of the projector is of the form $f(tD^2)$ or $Df(tD^2)$ for some Schwartz function f on $\mathbb{R}$. ☐

5.8. **Proposition.** *Let $(X, \mathbf{h})$ be a cocompact G-proper manifold without boundary endowed with a G-equivariant Spin^c-structure and let D be the Dirac-type operator defined in (3.2). Let g be a semi-simple element. If D is L^2-invertible, then the integral*

$$\frac{1}{\sqrt{\pi}}\int_1^\infty \tau^X_g(D\exp(-tD^2))\frac{dt}{\sqrt{t}} \tag{5.9}$$

converges.

Proof. Consider the Schwartz function $f(x) = \exp^{-|x|}$ on $\mathbb{R}$. It follows from Proposition 5.6 that the operator $Df(-tD^2) = D\exp(-tD^2)$ converges weighted exponentially in $\mathcal{L}^\infty_{G,s}(Y)$ as $t \to +\infty$. Accordingly, the orbital integral $\tau^X_g(D\exp(-tD^2))$ converges to 0 weighted exponentially as $t \to +\infty$. This decay property assures that integral (5.9) is well defined. ☐

5.2. **Large time behaviour on manifolds with boundary.** We now discuss the case of manifolds with boundary. We consider as before $(Y_0, \mathbf{h}_0)$, a cocompact G proper manifold with boundary X and denote by $(Y, \mathbf{h})$ the associated b-manifold. We denote by Z_0 a slice for Y_0 and by Z the associated b-manifold. Let D be a Dirac operator on $(Y, \mathbf{h})$ and assume that D_∂ is L^2-invertible. The same method as in [37] can be directly generalized to establish that

$$\exp(-tD^2) = \frac{1}{2\pi i}\int_\gamma e^{-t\mu}(D^2-\mu)^{-1}d\mu$$

is an element in ${}^b\mathcal{L}^\infty_{G,s}(Y)$ for every $t>0$ and for every $s>0$.

Assume now that there exists $a>0$ such that

$$\mathrm{Spec}_{L^2}(D)\cap[-2a,2a]=\emptyset\,.$$

We can and we shall choose γ so that $\mathrm{Spec}_{L^2}(D)\cap\gamma(z)=\emptyset$ and $\mathrm{Re}\,\gamma(z)>a$ for every z.

We want to show that $\exp(-tD^2)\in {}^b\mathcal{L}^\infty_{G,s}(Y)$ is exponentially converging to 0 in ${}^b\mathcal{L}^\infty_{G,s}(Y)$ as $t\to+\infty$. By [37, Proposition 5.21],

$$(D^2-\mu)^{-1} = B(\mu) + B(\mu)\circ L(\mu)$$

with

$$B(\mu) = B^\sigma(\mu) - \varphi((I(D^2-\mu)^{-1}I(R^\sigma(\mu)))$$

In these formulae

- $\varphi : {}^b\mathcal{L}^\infty_{G,s,\mathbb{R}}(\mathrm{cyl}(\partial Y))) \to {}^b\mathcal{L}^\infty_{G,s}(Y)$ is a section of the indicial homomorphism $I : {}^b\mathcal{L}^\infty_{G,s}(Y) \to {}^b\mathcal{L}^\infty_{G,s,\mathbb{R}}(\mathrm{cyl}(\partial Y))$; φ is defined by using a suitable cut-off function equal to 1 on the boundary;
- $B^\sigma(\mu)\in {}^b\Psi^{-2}_{G,c}(X)$ is a symbolic parametrix for $D^2-\mu$ and defines a b-pseudodifferential operator with parameter of order -2 (in the small b-calculus);
- $R^\sigma(\mu)\in {}^b\Psi^{-\infty}_{G,c}(X)$ is the remainder of a symbolic parametrix and defines a b-pseudodifferential operator with parameter of order $-\infty$ (always in the small calculus);
- $B(\mu)$ is a true parametrix for D^2 in the calculus with bounds.

Moreover, as proved in [37], $\mu\to\varphi((I(D^2-\mu)^{-1}I(R^\sigma(\mu)))$ is rapidly decreasing when $\mathrm{Re}(\mu)\to+\infty$ as a map with values in ${}^b\mathcal{L}^\infty_{G,s}(Y)$ and $L(\mu)\in {}^b\mathcal{L}^\infty_{G,s}(Y)$ goes to 0, always in Fréchet topology, as $\mathrm{Re}(\mu)\to+\infty$. A similar formula can be written for $(D^2-\mu)^{-k}$ for any $k\geq 1$. We consider, as for the case of closed manifolds, k very large; we fix such a k and adopt the same notation as above. Write now

$$\exp(-tD^2) = \frac{(k-1)!}{2\pi i}t^{1-k}\int_\gamma e^{-t\mu}(D^2-\mu)^{-k}d\mu.$$

We can express the integral on the right side initially as the sum of 2 terms

$$\int_\gamma e^{-t\mu}B_\mu d\mu \;+\; \int_\gamma e^{-t\mu}B_\mu\circ L_\mu d\mu$$

and then, using the analogue of the above expression for $(D^2-\mu)^{-k}$, as the sum of 4 terms:

$$\begin{aligned}\int_\gamma e^{-t\mu}B^\sigma(\mu)d\mu - \int_\gamma e^{-t\mu}\varphi(I(D^2-\mu)^{-k}I(R^\sigma(\mu)))d\mu + \int_\gamma e^{-t\mu}B^\sigma(\mu)\circ L(\mu)d\mu - \\ \int_\gamma e^{-t\mu}\varphi(I(D^2-\mu)^{-k}I(R^\sigma(\mu)))\circ L(\mu)d\mu\end{aligned} \tag{5.10}$$

where now $B^\sigma(\mu)\in {}^b\Psi^{-2k}_{G,c}(X)$ and defines a b-pseudodifferential operator with parameter of order $-2k$. The large-time behaviour of these four integrals can be treated as in the closed case. Indeed, for the first term we know that the Schwartz kernel of $B^\sigma(\mu)$ will be C^{2k} across the b-diagonal and so, keeping in mind that this term if of order $(-2k)$ as a pseudodifferential operator with parameter, we can directly estimate the integral of the relevant seminorm applied to $B^\sigma(\mu)$, as in the closed case, c.f. (5.4). For the second summand it suffices to recall that $\mu\to\varphi(I(D^2-\mu)^{-1}I(R^\sigma_\mu))$ is a rapidly decreasing ${}^b\mathcal{L}^\infty_{G,s}(Y)$-valued map. Similarly, the operators in the third integral define a ${}^b\mathcal{L}^\infty_{G,s}(Y)$-valued map going to 0 in the Fréchet topology as $\mathrm{Re}(\mu)\to+\infty$, whereas the operators in the fourth integral define a ${}^b\mathcal{L}^\infty_{G,s}(Y)$-valued map rapidly decreasing as $\mathrm{Re}(\mu) \to +\infty$. When we apply the seminorms to these families of operators inside the integral and we perform the integration, we obtain expressions in t that clearly go to 0 when $t \to +\infty$; here we use again, crucially, that $\mathrm{Re}\gamma(z) > a > 0$. As this is precisely as in (5.5) we omit the details. In fact, by a simple trick with the countour, we see also in this case that the convergence is weighted exponential.

We summarize our discussion in the following proposition:

5.11. **Proposition.** *If D is L^2-invertible then*

$$\exp(-tD^2) \to 0 \quad \text{weighted exponentially in} \quad {}^b\mathcal{L}^\infty_{G,s}(Y) \quad \text{as} \quad t \to +\infty. \tag{5.12}$$

In general for a Schwartz function f on $\mathbb{R}$, the operators $f(tD^2)$ and $Df(tD^2)$ converge to 0 weighted exponentially in ${}^b\mathcal{L}^\infty_{G,s}(Y)$ as $t \to +\infty$. In particular the Connes-Moscovici projector $V(tD) - e_1$ converges to 0 weighted exponentially in ${}^b\mathcal{L}^\infty_{G,s}(Y)$ as $t \to +\infty$.

5.3. **Small time behavior.** To study the small t convergence of the eta integral, we will need the following properties.

5.13. **Lemma.** *Let X be a cocompact G-proper manifold without boundary with a G-invariant complete Riemannian metric $\mathbf{h}_X$. Let $d_X(-,-)$ be the associated distance function, and $d_G(-,-)$ be the distance function on G/K. Suppose that V is a compact set on X and g is any element in G. There are constants $C_0 > 0, C_1 > 0$ such that*

$$d_X(hgh^{-1}x, x) \geq C_1 d_G(hgh^{-1}e, e) - C_0, \ \forall x \in V, h \in G. \tag{5.14}$$

Proof. The slice theorem gives a fibration structure on $X = (G \times S)/K$ over G/K, i.e. $\pi : X \to G/K$. For every $x \in X$, let T_xX be the tangent space of X at x, and V_x be the kernel of the map $\pi_* : T_xX \to T_{\pi(x)}G/K$, and H_x be the orthogonal complement of V_x in T_xX with respect to the metric $\mathbf{h}_X$. The induced map $\pi_* : H_x \to T_{\pi(x)}G/K$ is a linear isomorphism. Though the map $\pi_* : H_x \to T_{\pi(x)}G/K$ might not be an isometry with respect to the restricted metric $\mathbf{h}_X|_{H_x}$ and the metric $\mathbf{h}_{G/K}$ on $T_{\pi(x)}G/K$, there is a positive constant c_x such that

$$\mathbf{h}_X|_{H_x} \geq c_x(\pi_*)^{-1}\big(\mathbf{h}_{G/K}\big).$$

Running x over every point in the fiber $\pi^{-1}(\pi(x))$, as the fiber is compact, there is a constant $C_1 > 0$ such that

$$\mathbf{h}_X|_{H_y} \geq C_1(\pi_*)^{-1}\big(\mathbf{h}_{G/K}\big), \ \forall y \in \pi^{-1}(\pi(x)).$$

As G acts on G/K transitively and the metric $\mathbf{h}_X$ and $\mathbf{h}_{G/K}$ are G-invariant, the above estimate holds for all $x \in X$, i.e.

$$\mathbf{h}_X|_{H_x} \geq C_1(\pi_*)^{-1}\big(\mathbf{h}_{G/K}\big), \ \forall x \in X.$$

The above comparison of the Riemannian metrics gives the following property on the distance functions by the standard argument on path lengths

$$d_X(hgh^{-1}x, x) \geq C_1 d_G(hgh^{-1}\pi(x), \pi(x)).$$

Choose x_0 to be a point in $\pi^{-1}([e])$, where $[e]$ the point in G/K associated to the coset of the identity element. The above inequality shows

$$d_X(hgh^{-1}x_0, x_0) \geq C_1 d_G(hgh^{-1}e, e). \tag{5.15}$$

We apply the triangle inequality to obtain

$$d_X(hgh^{-1}x_0, hgh^{-1}x) + d_X(hgh^{-1}x, x) + d_X(x, x_0) \geq d_X(hgh^{-1}x_0, x_0). \tag{5.16}$$

As V is compact, there is a finite upper bound $C/2$ such that $d(x, x_0) \leq C/2$ for all $x \in V$. Combining this with Inequalities (5.16) and (5.15), we conclude with the desired inequality, i.e.

$$d_X(hgh^{-1}x, x) \geq C_1 d_G(hgh^{-1}e, e) - C_0.$$

□

5.17. **Remark.** It is obvious that Lemma 5.13 also holds if instead of $(X,\mathbf{h})$, $V\subset X$ and, for (ii), $C(g)$ we consider:
– $(Y_0,\mathbf{h}_0)$, a cocompact G-proper manifold with boundary, with slice-compatible metric, product-type near the boundary;
– $V_0\subset Y_0$ a compact set in Y_0.
We also consider $(Y,\mathbf{h})$, the G-manifold with cylindrical ends associated to $(Y_0,\mathbf{h}_0)$. Here we recall that there is a collar neighbourhood of ∂Y_0 in Y_0 where the action of G is of product type (and thus extendable to a G-proper action on $(Y,\mathbf{h})$). Notice that, consequently, the action of G preserves the decomposition $Y:=(-\infty,0]\times\partial Y_0\cup_{\partial Y_0} Y_0$. Consider the compact subset $V_0\subset Y_0$ and $V_0\cap\partial Y_0$, denoted ∂V_0 if non-empty. Consider finally $V:=(-\infty,0]\times\partial V_0\cup_{\partial V_0} V_0$. Lemma 5.13 also holds if we consider now $(Y,\mathbf{h})$, V. Indeed (5.14) does hold if $x\in V_0\subset Y_0$, as we have just observed, and also holds if $x\in(-\infty,0]\times\partial V_0\subset V\subset Y$ because the action of G is of product type along the cylinder.

The following property follows from [3, Proposition 4.2, (i)] because a cocompact G-proper manifold without boundary has bounded geometry.

5.18. **Lemma.** *Let X be a cocompact G-proper manifold without boundary and let D be a Dirac-type operator (associated to a unitary Clifford action and a Clifford connection) on a spinor bundle $\mathcal{E}$. Let $\kappa_t(x,y)$ be the kernel function of the operator $D\exp(-tD^2)$. There are constant $\alpha,\beta>0$ such that*

$$||\kappa_t(x,y)||\leq\beta t^{-\frac{n+1}{2}}\exp\left(-\alpha\frac{d_X(x,y)^2}{t}\right).$$

where $||\cdot||$ is the operator norm from $\mathcal{E}_x$ to $\mathcal{E}_y$.

5.19. **Remark.** Consider the heat kernel k_t associated to a Dirac-Laplacian on $(Y,\mathbf{h})$, the G-proper manifold with cylindrical ends associated to a cocompact G-proper manifold with boundary $(Y_0,\mathbf{h}_0)$. As $(Y,\mathbf{h})$ is a complete manifold with bounded geometry we can apply again [3, Proposition 4.2, (i)] and conclude that there are constant $\alpha_0,\beta_0>0$ such that

$$||k_t(x,y)||\leq\beta_0 t^{-\frac{n}{2}}\exp\left(-\alpha\frac{d_Y(x,y)^2}{t}\right). \tag{5.20}$$

The following estimate is proved by Harish-Chandra [12, Theorem 6] (see also [19, Lemma 4.4]).

5.21. **Lemma.** *For a semisimple element g, when t is close to 0, the integral*

$$\int_{G/Z_g}\exp\left(-\alpha\frac{d_G(hg^{-1}h^{-1}e,e)^2}{t}\right)$$

is bounded. We will sometimes use $d_G(g)$ for $d_G(ge,e)$ in the following of this article for abbreviation.

5.22. **Lemma.** *For a semisimple $g\in G$, the quotient X^g/Z_g is compact.*

Proof. If g is not conjugate to an element in K (that is, g is not elliptic) then X^g is empty (See Proposition 10.6). It suffices to work with $g=k_0\in K$. By the slice theorem, we assume that $X=G\times_K S$. A point $[(h,x)]$ belongs to X^{k_0} if there is $k\in K$ such that $k_0h=hk^{-1}$ and $kx=x$. Therefore,

$$X^{k_0}=\{[(h,x)]\in G\times_K S|h^{-1}k_0^{-1}h=k,kx=x\}.$$

Let π be the quotient map from $G\times S$ to $G\times_K S$. The space $\tilde{X}^{k_0}:=\{(h,x)\in G\times S|h^{-1}k_0^{-1}h=k,kx=x\}$ as a subset of $G\times S$ is the preimage $\pi^{-1}(X^{k_0})$. As π is a principal K-bundle, $\tilde{X}^{k_0}$ is a manifold. As the Z_{k_0} action on $X\times G$ commutes with the K action, it is sufficient to prove that $Z_{k_0}\backslash\tilde{X}^{k_0}$ is compact to conclude that $Z_{k_0}\backslash X^{k_0}$ is compact.

Consider the right conjugacy action of G on the conjugacy class $C(k_0^{-1})=\{hk_0^{-1}h^{-1}|h\in G\}$. $C(k_0^{-1})$ is a closed submanifold of G, and the G action on $C(k_0^{-1})$ is transitive with the stabilizer group at k_0^{-1} being Z_{k_0}. We have that $C(k_0^{-1})$ is diffeomorphic to $Z_{k_0}\backslash G$ and the map $f:G\to C(k_0^{-1})$ mapping g to $h^{-1}k_0^{-1}h$ is a fibration.

Consider the intersection $C(k_0,K):=C(k_0^{-1})\cap K$, which is a compact subset. Let $F(k_0)$ be $f^{-1}\big(C(k_0^{-1})\cap K\big)$. By the fibration property of f, $Z_{k_0}\backslash F(k_0)$ is homeomorphic to $C(k_0,K):=C(k_0^{-1})\cap K$. For $C(k_0,K)$, we consider the closed space $I_{k_0,K} := \{(k,x)|k \in C(k_0,K), k_0x = x\}$ of $K\times S$. The map $\tilde{f} : I_{k_0,K} \to C(k_0,K)$ mapping $(k,x) \in I_{k_0,K}$ to $k \in C(k_0,K)$ is a continuous surjective map.

Let $F(k_0)_f \times_{\tilde{f}} I_{k_0,K}$ be the fiber product of $F(k_0)$ and $I_{k_0,K}$ over the maps $f|_{F(k_0)}$ and $\tilde{f}$. As f is a fibration, $F(k_0)_f \times_{\tilde{f}} I_{k_0,K}$ can be identified with $\tilde{X}^{k_0} = \{(g,x)|g \in C(k_0,K), x \in S^{f(g)}\}$.

With the above identification, the quotient of $\tilde{X}^{k_0}$ by Z_{k_0} is homeomorphic to

$$Z_{k_0}\backslash \left(F(k_0)_f \times_{\tilde{f}} I_{k_0,K}\right) = (Z_{k_0}\backslash F(k_0))_f \times_{\tilde{f}} I_{k_0,K},$$

which is homeomorphic to $I_{k_0,K}$ as $Z_{k_0}\backslash F(k_0)$ is homeomorphic to $C(k_0,K)$. As a closed subspace of $K\times S$, $I_{k_0,K}$ is compact. We conclude that $Z_{k_0}\backslash\tilde{X}^{k_0}$ is compact. ☐

For an elliptic element g, the fixed point set could be non-empty. To tackle this situation, let c_G be a cut-off function associated to the right Z_g action on G. Following [19, Lemma 3.2], we have the following equivalent expression for $\tau_g^X(D\exp(-tD^2))$,

$$\begin{aligned}
\tau_g^X(D\exp(-tD^2)) &= \int_{G/Z_g}\int_X c(y)\mathrm{tr}(\kappa_t(y,hgh^{-1}y)hgh^{-1})dydh(Z_g)\\
&= \int_G c_G(h)\int_X c(y)\mathrm{tr}(\kappa_t(y,hgh^{-1}y)hgh^{-1})dydh\\
&= \int_X\int_G c_G(h)c(y)\mathrm{tr}(h\kappa_t(h^{-1}y,gh^{-1}y)gh^{-1})dhdy\\
&= \int_X\int_G c_G(h)c(hy)\mathrm{tr}(\kappa_t(y,gy)g)dhdy.
\end{aligned}\tag{5.23}$$

Recall that V is the support of the cut-off function c. We apply Lemma 5.13 to V and conclude that there is a constant $C_0 > 0$ such that

$$d_X(hgh^{-1}x,x) \geq C_1 d_G(hgh^{-1}e,e) - C_0, \forall x\in V, h\in G.$$

We introduce two subsets of the conjugacy class $C(g)$, i.e.

$$K(g) := \{g\in C(g)|d_G(ge,e)\leq 2C_0/C_1\},\ K^c(g) := \{g\in C(g)|d_G(ge,e) > 2C_0/C_1\}.$$

Define a map $\chi : G\to C(g)$ by $\chi(h) := hgh^{-1}$. Let $U(g)$ be the subset of G defined to be the preimage of $K(g)$. And let $U^c(g)$ be the complement of $U(g)$ in G. As g is semisimple, $C(g)$ is a closed subset of G, and therefore the image of $K(g)$ in G/K is a bounded closed subset. Therefore, the image of $K(g)$ in G/K is compact, and $K(g)$ is compact. Observe that $U(g)$ is invariant under the right Z_g action and the quotient $U(g)/Z_g$ is diffeomorphic to $K(g)$. So $U(g)/Z_g$ is also compact. We split the integral (5.23) into two parts,

$$\begin{aligned}
\tau_g^X(D\exp(-tD^2)) &= \int_X\int_G c_G(h)c(hy)\mathrm{tr}(\kappa_t(y,gy)g)dhdy\\
&= \int_X\int_{U(g)} c_G(h)c(hy)\mathrm{tr}(\kappa_t(y,gy)g)dhdy + \int_X\int_{U^c(g)} c_G(h)c(hy)\mathrm{tr}(\kappa_t(y,gy))dhdy.
\end{aligned}$$

5.24. **Lemma.** *Let $\widetilde{W}$ be a closed subset of X. The integral*

$$I^t_{\widetilde{W},U^c}(g) := \int_{\widetilde{W}}\int_{U^c(g)} c_G(h)c(hy)\mathrm{tr}(\kappa_t(y,gy)g)dhdy \tag{5.25}$$

is of exponential decay as $t\to 0$.

Proof. It is sufficient to prove the property for $\widetilde{W} = X$. We rewrite the integral (5.25) by applying the change of variable $x = hy$.

$$\begin{aligned}
I^t_{U^c}(g) &:= \int_X\int_{U^c(g)} c_G(h)c(x)\mathrm{tr}(\kappa_t(x,hgh^{-1}x)hgh^{-1})dhdx\\
&= \int_{U^c(g)} c_G(h)\int_V c(x)\mathrm{tr}(\kappa_t(x,hgh^{-1}x)hgh^{-1})dhdx.
\end{aligned}$$

By Lemma 5.18, we have the following estimate for $|\mathrm{tr}(\kappa_t(x, hgh^{-1}x)hgh^{-1})|$,

$$||\kappa_t(x, hgh^{-1}x)|| \leq \beta t^{-\frac{n+1}{2}} \exp\left(-\alpha \frac{d_X^2(x, hgh^{-1}x)}{t}\right),$$

$$|\mathrm{tr}(\kappa_t(x, hgh^{-1}x)hgh^{-1})| \leq \tilde{\beta} t^{-\frac{n+1}{2}} \exp\left(-\alpha \frac{d_X^2(x, hgh^{-1}x)}{t}\right), \qquad \forall x \in X.$$

Applying Lemma 5.13, we get

$$d_X(x, hgh^{-1}x) \geq C_1 d_G(hgh^{-1}e, e) - C_0.$$

By the definition of $U^c(g)$, for $h \in U^c(g)$, we have

$$d_X(x, hgh^{-1}x) \geq C_0,\ d_X(x, hgh^{-1}x) \geq \frac{C_1}{2} d_G(hgh^{-1}e, e),$$

and accordingly

$$d_X(x, hgh^{-1}x) \geq \frac{C_0}{2} + \frac{C_1 d_G(hgh^{-1}e, e)}{4},\ d_X(x, hgh^{-1}x)^2 \geq \frac{C_0^2}{4} + \frac{C_1^2 d_G(hgh^{-1}e, e)^2}{16}.$$

Hence, we have the estimate

$$|\mathrm{tr}(\kappa_t(x, hgh^{-1}x)hgh^{-1})| \leq \tilde{\beta} t^{-\frac{n+1}{2}} \exp\left(-\frac{\alpha C_0^2}{4t}\right) \exp\left(-\alpha C_1^2 \frac{d_G^2(hgh^{-1}e, e)}{16t}\right).$$

As V is compact, the integral

$$\int_V c(x)dx$$

is finite. And $I^t_{U^c}(g)$ can be bounded by

$$\begin{aligned}|I^t_{U^c}(g)| \leq& \gamma t^{-\frac{n+1}{2}} \exp\left(-\frac{\alpha C_0^2}{4t}\right) \int_{U^c(g)} c(g) \exp\left(-\alpha C_1^2 \frac{d_G^2(hgh^{-1}e, e)}{16t}\right) dh \\ \leq& \gamma t^{-\frac{n+1}{2}} \exp\left(-\frac{\alpha C_0^2}{4t}\right) \int_{G/Z_g} \left(-\alpha C_1^2 \frac{d_G^2(hgh^{-1}e, e)}{16t}\right) d(hZ_g).\end{aligned}$$

This implies the exponential decay property of $I^t_{U^c}(g)$. ☐

Let W be an open subset of X containing the g fixed point submanifold X^g.

5.26. **Lemma.** *The integral*

$$I^t_{W^c,U}(g) := \int_{W^c} \int_{U(g)} c_G(h)c(hy)\mathrm{tr}(\kappa_t(y, gy)g)dhdy$$

converges exponentially to 0 as $t \to 0$.

Proof. Because $U(g)/Z_g$ is compact and c_G is the cut-off function for the Z_g-action on G, we know that c_G is compactly supported on $U(g)$. Moreover, since c is compactly supported, the properness of the G action on X implies that the function

$$\tilde{c}(y) := \int_{U(g)} c_G(h)c(hy)dh$$

is also compactly supported. Let $\tilde{V}$ be the support of $\tilde{c}$. And $I^t_{W^c,U}(g)$ can be expressed as

$$\int_{W^c \cap \tilde{V}} \tilde{c}(y)\mathrm{tr}(\kappa_t(y, gy)gdy.$$

Using Lemma 5.18, we have the estimate

$$|\mathrm{tr}(\kappa_t(y, gy)g)| \leq \tilde{\beta} t^{-\frac{n+1}{2}} \exp\left(-\alpha \frac{d_X^2(y, gy)}{t}\right) \forall y \in X.$$

As $W^c \cap \tilde{V}$ is compact and does not contain any g fixed point, there is a positive ϵ_0 such that

$$d_X(y, gy) \geq \epsilon_0,\ \forall y \in W^c \cap \tilde{V}.$$

We can bound $I^t_{W^c,U}(g)$ using the above estimate about $d(y, gy)$,

$$|I^t_{W^c,U}(g)| \leq \tilde{\beta} t^{-\frac{n+1}{2}} \exp\left(-\alpha \frac{\epsilon_0^2}{t}\right) \int_{W^c \cap \tilde{V}} |\tilde{c}(y)| dy.$$

This gives the exponential decay property of $I^t_{W^c,U}(g)$. □

We can summarize Lemma 5.24 and 5.26 in the following result.

5.27. **Proposition.** *Given any open set W containing the g fixed point submanifold X^g, then the integral*

$$I^t_{W^c}(g) := \int_{W^c} \int_G c_G(h) c(hy) \mathrm{tr}(\kappa_t(y, gy)g) dh dy = I^t_{W^c,U}(g) + I^t_{W^c,U^c}(g).$$

decay exponentially to zero at $t \to 0$. When W is assumed to be g invariant, $I^t_{W^c}(g)$ has the following expression under change of coordinates

$$I^t_{W^c}(g) = \int_{W^c} \int_G c_G(h) c(hgx) \mathrm{tr}(\kappa_t(x, gx)g) dh dx.$$

Let W be a sufficiently small neighborhood of X^g. Using the Z_g-invariant tubular neighborhood theorem, we can identify W to be an r-ball bundle $N(r)$ in the normal bundle N of X^g in X with a sufficiently small radius r. Define c_g, a smooth function on X, by

$$c_g(x) := \int_G c_G(h) c(hgx) dh.$$

5.28. **Lemma.** *The function c_g is a cut-off function for the Z_g-action on X. In particular, for any $x \in X$,*

(1) the integral

$$\int_{Z_g} c_g(hx) = 1;$$

(2) Let $q : X \to X/Z_g$ be the quotient map. The intersection of the support of c_g with $q^{-1}(V')$ is compact for any compact subset V' in X/Z_g.

5.29. **Remark.** Part (1) of Lemma 5.28 is proved in [19, Lemma 4.11].

Proof. For part (1), compute that

$$\int_{Z_g} c_g(zx) dz = \int_{Z_g} \int_G c_G(h) c(hgzx)\ dh dz = \int_{Z_g} \int_G c_G(h' z'^{-1}) c(h'x)\ dh dz = 1$$

where we substitute $h' = hgz$ and $z' = gz$.

For part (2), suppose that V' is a compact subset in X/Z_g. The intersection of the support of c_g with $q^{-1}(V')$ consists is the closure of the following set,

$$Q_g(V') := \{x \in X : c_g(x) \neq 0, q(x) \in V'\}.$$

Recall that

$$c_g(x) = \int_G c_G(h) c(hgx) dh$$

Hence, $c_g(x) \neq 0$ if and only if there is some h such

$$c_G(h) \neq 0, \text{ and } c(hgx) \neq 0.$$

Using this characterization of $c_g(x) \neq 0$, we have

$$Q_g(V') = \{x \in X : \exists h \in G, c_G(h) \neq 0, c(hgx) \neq 0, q(x) \in V'\}.$$

Observe that g commutes with Z_g. So g acts on the quotient X/Z_g, and the map $q : X \to X/Z_g$ is g-equivariant. Hence, substituting $x' = gx$, we obtain the following expression,

$$Q_g(V') = g^{-1}\left(\{x' \in X : \exists h \in G, c_G(h) \neq 0, c(hx') \neq 0, q(x') \in g^{-1}(V')\}\right).$$

As V' is compact in X/Z_g, $g^{-1}(V')$ is also compact. We can assume without loss of generality that $q^{-1}\big(g^{-1}(V')\big)$ is of the form

$$q^{-1}\big(g^{-1}(V')\big) = Z_g V'',$$

where V'' is a compact subset of X.

Consider the set $G(V'') := \{h | \exists x' \in V'', c(hx') \neq 0\}$. Using the following properties

- the set V'' is compact,
- the support of c is compact,
- the G action on X is proper,

we conclude that $G(V'')$ is relatively compact.

Consider the set $P_g(V'') := \{h | c(hx') \neq 0, \exists x' \in Z_g V''\}$. We have the property that $P_g(V'') = G(V'')Z_g$. Hence, we conclude from the relative compactness of $G(V'')$ that $P_g(V'')/Z_g$ is relatively compact.

Consider the set $R_g(V'') := \{h | \exists x' \in Z_g V'', c_G(h) \neq 0, c(hx') \neq 0\}$. $R_g(V'')$ is the intersection of $P_g(V'')$ with the support of c_G. As c_G is a cutoff function of the right Z_g action on G, we conclude from the relative compactness of $P_g(V'')/Z_g$ that $R_g(V'')$ is compact.

Observe that the set

$$g(Q_g(V)) = \{x' \in X : \exists h \in G, c_G(h) \neq 0, c(hx') \neq 0, x' \in qx' \in g^{-1}(V')\}$$

is a subset of

$$\{x' \in X : \exists h \in R_g(V''), c(gx') \neq 0\}.$$

As $R_g(V'')$ is compact, we conclude from the compactness of the support of c and properness of the G action that $g(Q_g(V'))$ is relatively compact, and therefore $Q_g(V')$ is also relatively compact.

□

The integral

$$I^t_{N(r)}(g) := \int_{N(r)} \int_G c_G(h) c(hgy) \mathrm{tr}(\kappa_t(y, gy)g) dh dy$$

can be expressed by

$$\int_{N(r)} c_g(y) \mathrm{tr}(\kappa_t(y, gy)g) dy.$$

By Lemma 5.28, we know that the set X^g/Z_g is compact. Thus $N(r)/Z_g$ is relatively compact. It follows from Lemma 5.28 that c_g has relative compact support in $N(r)$ and is a smooth function on $N(r)$ with bounded derivatives. This allows us to use the method and estimates in the proof of [53, Equation (2.2)] to prove the following lemma.

5.30. **Lemma.** *The integral $I^t_{N(r)}(g)$ satisfies*

$$\lim_{t \to 0} \frac{|I^t_{N(r)}(g)|}{\sqrt{t}} < +\infty.$$

Proof. We use the local asymptotic expansion [53, Equation (2.16)]

$$g\kappa_t(y, g) = \frac{e^{-\frac{d_X(y, gy)^2}{4t}}}{(4\pi t)^{\frac{n}{2}}} dg \left[\sum_i (((dg - I)y)_i e_i) \left(\sum_{i=0}^{[n/2]+2} U_i t^i + o(t^{[n/2]+2}) \right) + \left(\sum_{i=0}^{[n/2]+2} V_i t^i + o(t^{[n/2]+2}) \right) \right].$$

In [53], the author assumed to work with an isometry group action on a closed manifold M. For our case, as the support c_g in $N(r)$ is relatively compact and g is contained in the group Z_g which acts properly on $N(r)$, the author's analysis in [53, Section 2] in the tubular neighborhood $N(r)$ is also valid in our case near $\mathrm{supp}(c_g) \cap N(r)$. We refer to [53, Section 2] for the explanation of the above formula.

We use the above formula to compute $c_g(y)\mathrm{tr}(\kappa_t(y, gy)g)$,

$$c_g(y)\mathrm{tr}(\kappa_t(y, gy)g) =$$
$$\frac{e^{-\frac{d(y, gy)^2}{4t}}}{(4\pi t)^{\frac{n}{2}}} dg \left[\sum_i (((dg - I)y)_i e_i) \left(\sum_{i=0}^{[n/2]+2} c_g(y) U_i t^i + o(t^{[n/2]+2}) \right) + \left(\sum_{i=0}^{[n/2]+2} c_g(y) V_i t^i + o(t^{[n/2]+2}) \right) \right].$$

Apply [53, Lemma 2.17] to the integral of each term in the above expansion of $c_g(y)\mathrm{tr}(\kappa_t(y, gy)g)$. We can conclude the estimate in the lemma following the same argument as the one for [53, Equation (2.2)], that is,

$$\lim_{t\to 0^+} \frac{1}{\sqrt{t}} |c_g(y)\mathrm{tr}(\kappa_t(y, gy)g)| < C$$

for some constant $C > 0$. □

Combing Proposition 5.27 and Lemma 5.30, we have reached the following proposition.

5.31. **Proposition.** *Let X be a cocompact G-proper manifold without boundary and let D be a Dirac-type operator defined in (3.2). Let g be a semi-simple element in G. The integral*

$$\frac{1}{\sqrt{\pi}} \int_0^1 \tau_g^X(D\exp(-tD^2)\frac{dt}{\sqrt{t}}$$

converges.

5.4. **Short time limit and the Atiyah-Segal integrand.** If we replace $D\exp(-sD^2)$ by $\exp(-sD^2)$, a similar argument as in Theorem 5.1 proves the topological formula stated in [19, Theorem 2.8] for the pairing between τ_g and the index element $\mathrm{ind}_G(D)$ for a Dirac type operator D on a G-proper cocompact manifold X without boundary, as we shall now explain. We know that

$$\langle \mathrm{Ind}_G(D), \tau_g\rangle = \langle \mathrm{Ind}(D), \tau_g^X\rangle = \tau_g^X(e^{-s^2D^-D^+}) - \tau_g^X(e^{-s^2D^+D^-}).$$

We compute the left hand side by taking the limit as $s \downarrow 0$ and using the following

5.32. **Theorem.** *Let (X, h) be a cocompact G-proper manifold without boundary. Assume that X, G/K, and the slice S are all even dimensional. Let D be a G-equivariant Dirac operator defined in (3.2). Let g be a semi-simple element and let X^g be the fixed point set of g. Then the limit $\lim_{s\downarrow 0} \tau_g(e^{-s^2D^-D^+})-\tau_g(e^{-s^2D^+D^-})$ exists and we have*

$$\lim_{s\downarrow 0} \tau_g^X(e^{-s^2D^-D^+}) - \tau_g^X(e^{-s^2D^+D^-}) = \int_{X^g} c^g \mathrm{AS}_g(D)$$

with $c^g\mathrm{AS}_g(D)$ defined in Equation (1.6). Consequently we obtain the following formula

$$\langle \mathrm{Ind}(D), \tau_g^X\rangle = \int_{X^g} c^g \mathrm{AS}_g(D).$$

Proof. Let κ_s be the kernel of the heat operator $\exp(-sD^2)$. Observe that in term of supertrace, $\tau_g^X(e^{-s^2D^-D^+})-\tau_g^X(e^{-s^2D^+D^-})$ can be written as the integral

$$J(s) := \int_{G/Z_g}\int_X c(hgh^{-1}x)\,\mathrm{str}(hgh^{-1}\kappa_{s^2}(hg^{-1}h^{-1}x, x))dxd(hZ). \tag{5.33}$$

By a similar computation as in Equation (5.23), we can express $J(s)$ defined by Equation (5.33) as follows,

$$J(s) = \int_X\int_G c_G(h)c(hy)\mathrm{str}(\kappa_{s^2}(y, gy)g)dydh,$$

where c_G is the cut-off function associated to the right Z_g action on G. We can follow the exactly same strategy as in Section 5.3 to study the limit of $J(s)$ as $s \to 0$. Below is a brief outline of steps.

(1) We observe that the analysis on the small time behavior the operator $D\exp(-sD^2)$ in Section 5.3 also holds for the operator $\exp(-sD^2)$. In particular, similar to Lemma 5.18, there are constant $\alpha_0, \beta_0 > 0$ such that

$$||\kappa_s(x, y)|| \le \beta_0 s^{-\frac{n}{2}} \exp\left(-\alpha \frac{d_X(x, y)^2}{s}\right).$$

(2) Similar to Proposition 5.27, using the above Gaussian estimate of the heat kernel $\kappa_t(x, y)$, we prove that given any Z_g-invariant open set W containing the g fixed point submanifold X^g, the integral

$$J_{W^c}(s) := \int_{W^c}\int_G c_G(h)c(hy)\,\mathrm{str}(\kappa_{s^2}(y, gy)g)dhdy = \int_{W^c}\int_G c_G(h)c(hgx)\,\mathrm{str}(\kappa_{s^2}(x, gx)g)dhdx$$

decays exponentially to zero as $s \to 0$. The last equality holds because of the invariance property of W.

(3) Following Lemma 5.30, we choose W to be an r-ball bundle $N(r)$ in the normal bundle N of X^g in X with a sufficiently small radius r. For the integral

$$J_{N(r)}(s) := \int_{N(r)} \int_G c_G(h)c(hy)\,\mathrm{str}(\kappa_{s^2}(y, gy)g)dhdy,$$

we follow the local analysis as in the proofs [2, Theorem 6.16] and [19, Proposition 4.12] to get

$$\lim_{s\to 0} J_{N(r)}(s) = \int_{X^g} c^g\,\mathrm{AS}_g(D).$$

Combining the above (1)-(3), we reach the desired equality

$$\lim_{s\downarrow 0} \tau^X_g(e^{-s^2D^-D^+}) - \tau^X_g(e^{-s^2D^+D^-}) = \int_{X^g} c^g\mathrm{AS}_g(D).$$

□

5.34. **Remark.** Theorem 5.32 was originally stated by Hochs and Wang [19, Theorem 2.8] with a stronger assumption on the group element g. Our strategy of proof is similar to the one in [19]. However, our proof does not use the inequality (4.6) in [19] for the heat kernel $\widetilde{\kappa}^{G,K}_t$, the proof of which is not clear to us. Instead, we use the global Gaussian estimates for the heat kernel e^{-tD^2}, c.f. Lemma 5.18, in order to establish, as a crucial step, the exponential decay property of $\tau_g\big(\exp(-tD^2)\big)$ outside a neighborhood W of the fixed point submanifold X^g as in Proposition 5.27. Notice also that in [19] the split Dirac operator (3.6) is used throughout and we know that this operator is *not* associated to a Clifford connection.

We are now in the position to prove Proposition 4.39 that we restate here for the benefit of the reader: *Let $(Y_0, \mathbf{h}_0)$ be a cocompact G-proper Riemannian manifold. Let $(Y, \mathbf{h})$ be the associated b-manifold. Let D_0 be a G-equivariant Dirac operator defined in (3.2) and let D be the associated b-differential operator. Let g be a semisimple element and let Y^g_0 the fixed point set of g. Then the limit $\lim_{s\downarrow 0} \tau^{Y,r}_g(e^{-s^2D^-D^+}) - \tau^{Y,r}_g(e^{-s^2D^+D^-})$ exists and we have*

$$\lim_{s\downarrow 0} \tau^{Y,r}_g(e^{-s^2D^-D^+}) - \tau^{Y,r}_g(e^{-s^2D^+D^-}) = \int_{Y^g_0} c^g_0\mathrm{AS}_g(D_0)$$

with $c^g_0\mathrm{AS}_g(D_0)$ defined in Equation (1.6).

Proof. Let κ_t be the kernel of the heat operator $\exp(-tD^2)$. By definition

$$\tau^{Y,r}_g(e^{-s^2D^-D^+}) - \tau^{Y,r}_g(e^{-s^2D^+D^-})$$

is equal to

$$ {}^bJ(s) := \int_{G/Z_g} {}^b\!\!\int_Y c(hgh^{-1}y)\,\mathrm{str}(hgh^{-1}\kappa_{s^2}(hg^{-1}h^{-1}y, y))dyd(hZ) \tag{5.35}$$

with dy denoting the b-density associated to the b-metric $\mathbf{h}$. We claim that the following expression holds:

$$ {}^bJ(s) = {}^b\!\!\int_Y \int_G c_G(h)c(hy)\mathrm{str}(\kappa_{s^2}(y, gy)g)dhdy \tag{5.36}$$

where c_G is the cut-off function associated to the right Z_g action on G. In order to prove (5.36) we make a preliminary remark. The spaces of b-pseudodifferential operators ${}^b\mathcal{L}^c_G(Y, E)$ and ${}^b\mathcal{L}^\infty_{G,s}(Y, E)$ have been defined in terms of the slice theorem, as projective tensor products. We could have defined these operators directly, in terms of a b-stretched product $Y \times_b Y$. Proceeding exactly as in [23, Chapter 4] we see that $Y \times_b Y$ inherits an action of $G \times G$; in particular, it makes sense to consider $L^*_{(h,h')}\exp(-tD^2)$ which is a smoothing b-kernel (not G-invariant unless $h = h'$). Coming back to the claim, we can rewrite the right side of (5.35) as

$$\int_{G/Z_g} {}^b\!\!\int_Y c(y)\,\mathrm{str}(\kappa_{s^2}(y, hgh^{-1}y)hgh^{-1})dyd(hZ)$$

which is in turn equal to

$$\int_G c_G(h) \int_Y^b c(y)\,\mathrm{str}(\kappa_{s^2}(y, hgh^{-1}y)hgh^{-1})dydh. \tag{5.37}$$

Consider the smoothing b-kernel

$$\Theta_g(h) := L^*_{e,hgh^{-1}}(\exp(-(sD)^2));$$

we have that

$$\int_G c_G(h) \int_Y^b c(y)\,\mathrm{str}(\kappa_{s^2}(y, hgh^{-1}y)hgh^{-1})dydh = \int_G c_G(h) \int_Y^b c(y)\,\mathrm{str}(\kappa(\Theta_g(h))(y,y)dydh. \tag{5.38}$$

Write $\int_Y^b c(y)\,\mathrm{str}(\kappa(\Theta_g(h))(y,y)dy$ as

$$\int_{Y_0} c_0(y_0)\,\mathrm{str}(\kappa(\Theta_g(h))(y_0,y_0)dy_0 + \int^b_{(-\infty,0]_t\times\partial Y_0} c_0(x)\,\mathrm{str}(\kappa(\Theta_g(h))(t,x,t,x)dtdx.$$

Consider the function $F_g(h)$ on Y, so defined:
- $F_g(h)(y) := c_0(y_0)\,\mathrm{str}(\kappa(\Theta_g(h))(y_0,y_0)$ if $y = y_0 \in Y_0$;
- $F_g(h)(y) := -t\frac{d}{dt}\,(c_0(x)\,\mathrm{str}(\kappa(\Theta_g(h))(t,x,t,x))$ if $y = (t,x) \in (-\infty,0]_t \times \partial Y_0$.

The function $F_g(h)$ is discontinuous at ∂Y_0, a set of measure 0, but it is otherwise smooth. By [27, Prop. 2.6] we have that

$$\int_G c_G(h) \int_Y^b c(y)\,\mathrm{str}(\kappa(\Theta_g(h))(y,y)dydh = \int_G \int_Y c_G(h)F_g(h)dydh.$$

We know that the double integral on the right side is absolutely convergent; we can now proceed as in [19], use Fubini's theorem and interchange the two integrals in the right side of the above formula, obtaining

$$\int_Y \int_G c_G(h)F_g(h)dydh\,.$$

Going back to the b-integral we obtaining finally that (5.37) is equal to

$$\int_Y^b \int_G c_G(h)c(y)\,\mathrm{str}(\kappa_{s^2}(y, hgh^{-1}y)hgh^{-1})dydh$$

which is easily seen to be equal to

$$\int_Y^b \int_G c_G(h)c(hy)\mathrm{str}(k_{s^2}(y, gy)g)dhdy$$

as claimed. Summarizing:

$$\tau_g^{Y,r}(e^{-s^2D^-D^+}) - \tau_g^{Y,r}(e^{-s^2D^+D^-}) = \int_Y^b \int_G c_G(h)c(hy)\mathrm{str}(k_{s^2}(y, gy)g)dhdy\,. \tag{5.39}$$

We now consider the fixed point set $Y_0^g \subset Y_0$ and the associated $Y^g \subset Y$. We fix a Z_g-invariant open set W_0 containing Y_0^g and the associated $W \subset Y$ containing Y^g. We choose W_0 to be an r-ball bundle $N_0(r)$ in the normal bundle N_0 of Y_0^g in Y_0 with a sufficiently small radius r. We can then decompose the right side of (5.39) as:

$$\int_{W^c}^b \int_G c_G(h)c(hy)\mathrm{str}(k_{s^2}(y, gy)g)dhdy + \int_W^b \int_G c_G(h)c(hy)\mathrm{str}(k_{s^2}(y, gy)g)dhdy \tag{5.40}$$

Recall that if Z is a b-manifold obtained from a manifold with boundary Z_0 and ϕ is a b-density obtained as an extension from Z_0 of a density ϕ_0 on Z_0, then

$$\int_Z^b \phi = \int_{Z_0} \phi_0\,.$$

In particular if c_0 is a cut-off function for the G action on Y_0 and c is obtained by extending constantly along the cylindrical end, then

$$\int_Y^b c(y)d\mathrm{vol}_b = \int_{Y_0} c_0(y_0)d\mathrm{vol}\,.$$

Thanks to Remark 5.19 we know that there are constant $\alpha_0, \beta_0 > 0$ such that

$$||k_t(x,y)|| \leq \beta_0 t^{-\frac{n}{2}} \exp\left(-\alpha \frac{d_Y(x,y)^2}{t}\right).$$

Using this global Gaussian estimate and proceeding exactly as in the closed case we find that the first integral in (5.40) converges exponentially to 0 (here we use the above remark about the cut-off function c; it is not compactly supported but its b-integral is equal to the integral of a compactly supported function). We are left with the second integral in (5.40). In this case Y_0^g is compact and $W_0 := N_0(r)$ is relatively compact; put it differently, Y^g and W are obtained attaching cylinders to compact (respectively relatively compact) spaces. We can thus analyze

$$\int^b_{N(r)} \int_G c_G(h) c(hy) \mathrm{str}(k_{s^2}(y, gy)g) dh dy$$

using Getzler rescaling and find that the limit as $s \downarrow 0$ of this double integral is equal to

$$\int^b_{Y^g} c^g \mathrm{AS}_g(D).$$

As all the structures are product-like near the boundary we see that

$$\int^b_{Y^g} c^g \mathrm{AS}_g(D) = \int_{Y_0^g} c_0^g \mathrm{AS}_g(D_0)$$

which is what we wanted to prove. □

6. Higher delocalized cyclic cocycles

In this section we shall introduce higher delocalized cyclic cochains.

Let $K < G$ be a maximal compact subgroup and let $P < G$, $P = MAN$, be a cuspidal parabolic subgroup of G. By the Iwasawa decomposition $G = KMAN$ we can write an element $g \in G$ as

$$g = \kappa(g)\mu(g)e^{H(g)}n \in KMAN = G.$$

Let $\dim(A) = m$. By choosing coordinates of the Lie algebra $\mathfrak{a}$ of A, consider the function

$$H = (H_1, \dots, H_m) : G \to \mathfrak{a}.$$

We define the following cyclic cochain on the algebra $\mathcal{L}_s(G)$, the Lafforgue Schwartz algebra. For $f_0, ..., f_m \in \mathcal{L}_s(G)$ and a semi-simple element $g \in M$, define Φ_g^P by the following integral,

$$\begin{aligned} &\Phi_g^P(f_0, f_1, \dots, f_m) \\ &:= \int_{h\in M/Z_M(g)} \int_{KN} \int_{G^{\times m}} \sum_{\tau \in S_m} \mathrm{sgn}(\tau) \cdot H_{\tau(1)}(g_1 ... g_m k) H_{\tau(2)}(g_2 ... g_m k) \dots H_{\tau(m)}(g_m, k) \\ &\quad f_0\big(khgh^{-1}nk^{-1}(g_1 \dots g_m)^{-1}\big) f_1(g_1) \dots f_m(g_m) dg_1 \cdots dg_m dk dn dh, \end{aligned} \tag{6.1}$$

where $Z_M(g)$ is the centralizer of g in M. The following result is proved in the work of Song-Tang, [44, Theorem 3.5].

6.2. **Proposition.** *Φ_g^P is a cyclic cocycle on the Harish-Chandra Schwartz algebra $\mathcal{L}_s(G)$.*

6.3. **Definition.** We say that a cuspidal parabolic subgroup $P = MAN$ is *maximal* if one of the following equivalent conditions holds

- the dimension of A is minimal;
- the rank of M is maximal;
- the rank of M equals the rank of K.

In [16, Corollary 6.3], the authors showed that Φ_g^P is trivial in the cyclic cohomology of $\mathcal{L}_s(G)$ unless P is maximal.

Let Y_0 be an **even dimensional** cocompact G-proper manifold with boundary and let Y be the associated b-manifold. We shall now use Φ^P_g in order to define a cyclic cocycle $\Phi^P_{Y,g}$ on the algebra $\mathcal{L}^\infty_{G,s}(Y)$; subsequently we shall use $\Phi^P_{Y,g}$ in order to define a relative cyclic cocycle $(\Phi^{r,P}_{Y,g}, \sigma^P_{\partial Y,g})$ for the indicial homomorphism ${}^b\mathcal{L}^\infty_{G,s}(Y) \xrightarrow{I} {}^b\mathcal{L}^c_{G,s,\mathbb{R}}(\mathrm{cyl}(\partial Y))$. These algebras involve the choice of an ϵ strictly smaller than half of the width of the spectral gap for $D_{\partial Y}$. We fix such an ϵ and we choose it in any case smaller than 1.

Recall, see [37], that if S is a slice for the action of G on Y, then we have an identification

$$\mathcal{L}^c_G(Y) \cong \left\{F : G \to \rho_{bf}\, {}^b\Psi^{-\infty,\epsilon}(S),\ K \times K \text{ equivariant, continuous and of compact support in } G\right\}.$$

Recall that $\rho_{bf}\, {}^b\Psi^{-\infty}(S) \subset \Psi^{-\infty,\epsilon}(S)$ if $\epsilon < 1$, which is in turn contained in the trace class operators on $L^2_b(S)$.

We have

$${}^b\mathcal{L}^c_G(Y) \cong \left\{F : G \to {}^b\Psi^{-\infty,\epsilon}(S),\ K \times K \text{ equivariant, continuous and of compact support in } G\right\}.$$

Finally, by Fourier transform we have an injection

$${}^b\mathcal{L}^c_{G,\mathbb{R}}(\mathrm{cyl}(\partial Y)) \hookrightarrow \mathcal{S}(\mathbb{R}, \mathcal{L}^c_G(\partial Y)).$$

6.4. **Definition.** For a semisimple element $g \in M$ and $A_0, ..., A_m \in \mathcal{L}^c_G(Y)$, define a cochain $\Phi^P_{Y,g}$ on $\mathcal{L}^c_G(Y)$ by

$$\begin{aligned}&\Phi^P_{Y,g}(A_0, A_1, \dots, A_m)\\ &:= \int_{h \in M/Z_M(g)} \int_{KN} \int_{G^{\times m}} \sum_{\tau \in S_m} \mathrm{sgn}(\tau) \cdot H_{\tau(1)}(g_1 ... g_m k) H_{\tau(2)}(g_2 ... g_m k) \dots H_{\tau(m)}(g_m, k)\\ &\quad \mathrm{Tr}\left(A_0\big(khgh^{-1}nk^{-1}(g_1 \dots g_m)^{-1}\big) \circ A_1(g_1) \cdots \circ A_m(g_m)\right) dg_1 \cdots dg_m dk dn dh.\end{aligned}$$

Proceeding as in Song-Tang one can prove that this is in fact a cyclic cocycle.

Using always Φ^P_g on $\mathcal{L}_s(G)$ we shall now define a *relative* cyclic cocycle $(\Phi^{r,P}_{Y,g}, \sigma^P_{\partial Y,g})$ for the homomorphism ${}^b\mathcal{L}^c_G(Y) \xrightarrow{I} {}^b\mathcal{L}^c_{G,\mathbb{R}}(\mathrm{cyl}(\partial Y))$.

6.5. **Definition.** For a semisimple element $g \in M$ and $A_0, ..., A_m \in {}^b\mathcal{L}^c_G(Y)$, define a cochain $\Phi^{r,P}_{Y,g}$ on ${}^b\mathcal{L}^c_G(M)$ by

$$\begin{aligned}&\Phi^{r,P}_{Y,g}(A_0, A_1, \dots, A_m)\\ &:= \int_{h \in M/Z_M(g)} \int_{KN} \int_{G^{\times m}} \sum_{\tau \in S_m} \mathrm{sgn}(\tau) \cdot H_{\tau(1)}(g_1 ... g_m k) H_{\tau(2)}(g_2 ... g_m k) \dots H_{\tau(m)}(g_m, k)\\ &\quad {}^b\mathrm{Tr}\left(A_0\big(khgh^{-1}nk^{-1}(g_1 \dots g_m)^{-1}\big) \circ A_1(g_1) \cdots \circ A_m(g_m)\right) dg_1 \cdots dg_m dk dn dh.\end{aligned}$$

For $B_0, ..., B_{m+1} \in {}^b\mathcal{L}^c_{G,\mathbb{R}}(\mathrm{cyl}(Y))$, define a cochain $\sigma^P_{\partial Y,g}$ on ${}^b\mathcal{L}^c_{G,\mathbb{R}}(\mathrm{cyl}(Y))$ by

$$\begin{aligned}&\sigma^P_{\partial Y,g}(B_0, ..., B_{m+1})\\ &:= \int_{h \in M/Z_M(g)} \int_{KN} \int_{G^{\times m+1}} \int_{\mathbb{R}} \sum_{\tau \in S_m} \mathrm{sgn}(\tau) \cdot H_{\tau(1)}(g_1 ... g_m k) H_{\tau(2)}(g_2 ... g_m k) \dots H_{\tau(m)}(g_m, k)\\ &\quad \mathrm{Tr}\left(\hat{B}_0\big(khgh^{-1}nk^{-1}(g_1 \dots g_m g_{m+1})^{-1}, \lambda\big) \circ \hat{B}_1(g_1, \lambda) \circ \cdots \circ \hat{B}_m(g_m, \lambda) \circ \frac{\partial \hat{B}_{m+1}(g_{m+1}, \lambda)}{\partial \lambda}\right)\\ &\qquad dg_1 \cdots dg_{m+1} dk dn dh d\lambda,\end{aligned}$$

where we have used the Fourier transform

$${}^b\mathcal{L}^c_{G,s,\mathbb{R}}(\mathrm{cyl}(Y)) \ni A \longrightarrow \widehat{A} \in \mathscr{S}(\mathbb{R}, \mathcal{L}^c_G(Y)).$$

6.6. **Proposition.** *Let Y_0 be a proper G manifold with boundary and Y be the associated b-manifold. For a semisimple element $g \in M$, we have the following identities:*

$$\begin{pmatrix} (b+B) & -I^* \\ 0 & -(b+B) \end{pmatrix} \begin{pmatrix} \Phi^{r,P}_{Y,g} \\ \sigma^P_{\partial Y,g} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consequently, the pair $(\Phi^{r,P}_{Y,g}, \sigma^P_{\partial Y,g})$ *defines a relative cyclic cocycle for the homomorphism* ${}^b\mathcal{L}^c_G(Y) \xrightarrow{I} {}^b\mathcal{L}^c_{G,\mathbb{R}}(\mathrm{cyl}(\partial Y))$.

Proof. The proof proceeds analogous to the proof of [37, Prop. 6.7.]. For the Hochschild differential, we compute

$$
\begin{aligned}
&b\Phi^{r,P}_{Y,g}(A_0,\ldots,A_{m+1})\\
&= \int_{h\in M/Z_M(x)}\int_{KN}\int_{G^{\times(m+1)}}\Bigg({}^b\mathrm{Tr}\Big(A_0\big(khgh^{-1}nk^{-1}(g_1\ldots g_m)^{-1}(g')^{-1}\big)\circ A_1(g')\circ A_2(g_1)\circ\cdots\circ A_{m+1}(g_m)\Big)\\
&+\sum_{i=1}^m(-1)^i\Big[{}^b\mathrm{Tr}\Big(A_0\big(khgh^{-1}nk^{-1}(g_1\ldots g_m)^{-1}\big)\circ A_1(g_1)\circ\ldots\circ A_i(g_i(g')^{-1})\circ A_{i+1}(g')\circ\cdots\circ A_{m+1}(g_m)\Big)\Big]\\
&+(-1)^{m+1}\Big[{}^b\mathrm{Tr}\Big(A_{m+1}\big(khgh^{-1}nk^{-1}(g_1\ldots g_m)^{-1}(g')^{-1}\big)\circ A_0(g')\circ A_2(g_1)\circ\cdots\circ A_m(g_m)\Big)\Big]\Bigg)\\
&\qquad\times\sum_{\tau\in S_m}\mathrm{sgn}(\tau)\cdot H_{\tau(1)}(g_1...g_mk)H_{\tau(2)}(g_2...g_mk)\ldots H_{\tau(m)}(g_m,k)dg_1\cdots dg_m dkdndhdg'\\
&=(-1)^{m+1}\int_{h\in M/Z_M(x)}\int_{KN}\int_{G^{\times(m+1)}}{}^b\mathrm{Tr}\Big(\big[A_1(g_1)\circ\cdots\circ A_{m+1}(g_{m+1}),A_0\big(khgh^{-1}nk^{-1}(g_1\ldots g_{m+1})^{-1}\big)\big]\\
&\qquad\times\sum_{\tau\in S_m}\mathrm{sgn}(\tau)\cdot H_{\tau(1)}(g_1...g_mk)H_{\tau(2)}(g_2...g_mk)\ldots H_{\tau(m)}(g_m,k)dg_1\cdots dg_{m+1}dkdndh\\
&=I^*\sigma^P_{\partial Y,g}(A_0,\ldots,A_{m+1}).
\end{aligned}
$$

In this computation we have used the fact that Φ^P_g is a cyclic cocycle and Melrose's formula for the b-trace. To show that $B\Phi^{r,P}_{Y,g}=0$ we write out the differential

$$
B\Phi^{r,P}_{Y,g}(A_0,\ldots,A_{m-1})=\sum_{i=0}^{m-1}(-1)^{m-1)i}\Phi^{r,P}_{Y,g}(1,A_i,\ldots,A_{m-1},A_0,\ldots,A_{i-1}),
$$

where 1 is the delta function at the unit. Ignoring the sign, we can write the i'th term of this expression as

$$
\begin{aligned}
&\int_{h\in M/Z_M(x)}\int_{KN}\int_{G^{\times m}}\sum_{\tau\in S_m}\mathrm{sgn}(\tau)\cdot H_{\tau(1)}(g_1...g_mk)H_{\tau(2)}(g_2...g_mk)\ldots H_{\tau(m)}(g_mk)\\
&\delta_e\big(khgh^{-1}nk^{-1}(g_1\ldots g_m)^{-1}\big)^b\mathrm{Tr}\Big(A_i(g_1)\cdots\circ A_m(g_{m-i+1})\circ A_0(g_{m-i})\circ A_{i-1}(g_m)\Big)dg_1\cdots dg_m dkdndh\\
&=\int_{h\in M/Z_M(x)}\int_{KN}\int_{G^{\times m}}\sum_{\tau\in S_m}\mathrm{sgn}(\tau)\cdot H_{\tau(1)}(khgh^{-1}n)H_{\tau(2)}(g_2...g_mk)\ldots H_{\tau(m)}(g_m,k)\\
&{}^b\mathrm{Tr}\Big(A_i(khgh^{-1}nk(g_2\cdots g_m)^{-1})\cdots\circ A_m(g_{m-i+1})\circ A_0(g_{m-i})\circ A_{i-1}(g_m)\Big)dg_2\cdots dg_m dkdndh\\
&=0,
\end{aligned}
$$

because $H(khgh^{-1}n)=0$. This shows that $B\Phi^{r,P}_{Y,g}=0$ and injectivity of I^* show that $(b+B)\sigma^P_{\partial Y,g}=0$. ☐

Notation: we shall often omit the parabolic subgroup P from the notation, thus denoting by Φ_g the cyclic cocycle on $\mathcal{L}_s(G)$, by $\Phi_{Y,g}$ the cyclic cocycle on $\mathcal{L}^c_G(Y)$ and by $(\Phi^r_{Y,g},\sigma_{\partial Y,g})$ the relative cyclic cocycle for ${}^b\mathcal{L}^c_G(Y)\xrightarrow{I}{}^b\mathcal{L}^c_{G,\mathbb{R}}(\mathrm{cyl}(\partial Y))$.

7. Toward a general higher APS index formula

7.1. **Proposition.** *Assume that G is a connected, linear real reductive group. Then:*

1 *the cyclic cocycle $\Phi^P_{Y,g}$ extends continuously from $\mathcal{L}^c_G(Y)$ to $\mathcal{L}^\infty_{G,s}(Y)$;*

2 *the relative cyclic cocycle $(\Phi^{r,P}_{Y,g},\sigma_{\partial Y,g})$ extends continuously from the pair ${}^b\mathcal{L}^c_G(Y)\xrightarrow{I}{}^b\mathcal{L}^c_{G,\mathbb{R}}(\mathrm{cyl}(\partial Y))$ to the pair ${}^b\mathcal{L}^\infty_{G,s}(Y)\xrightarrow{I}{}^b\mathcal{L}^\infty_{G,s,\mathbb{R}}(\mathrm{cyl}(\partial Y))$.*

Proof. The proof of the first statement is analogous to the considerations in [16] depending crucially on the inequality proved in [44, Thm A.5]:

$$\left|\Phi_g^P(f_0,\dots,f_m)\right| \le C\nu_{d_0+T_0+1}(f_0)\cdots\nu_{d_0+T_0+1}(f_m), \qquad f_0,\dots,f_m\in\mathcal{L}_s(G),$$

where

$$\nu_t(f) := \sup_{g\in G}|(1+||g||)^t\Xi(g)^{-1}f(g)|,$$

and T_0 and d_0 as in [44]. Given $A\in\mathcal{L}^c_G(Y)$, we introduce the norm

$$|||A|||_t := \sup_{g\in G}|(1+||g||)^t\Xi(g)^{-1}||A(g)||_b,$$

where$||P||_b^2 := \|\chi P\|_1^2 + \|\phi[\mathcal{V},P]\|_1^2 + \|[\mathcal{V},P]\|_1^2 + \|[\phi,P]\|^2 + \|P\|^2$ for $P\in {}^b\Psi^{-\infty,\epsilon}(S)+\Psi^{-\infty,\epsilon}(S)$ (c.f. [37, Definition 7.3]). Because $\mathcal{L}^\infty_{G,s}(Y)$ lies inside the norm-completion of $\mathcal{L}^c_G(Y)$ with respect to $|||\ |||_s$, it suffices to estimate the cocycle $\Phi^P_{Y,g}$ in one of these norms. For this we use the inequality

$$|{}^b\operatorname{Tr}(P_0P_1\cdots P_k)| \le C||P_0||_b\cdots||P_k||_b,$$

c.f. [9, Lemma 6.4.] to obtain

$$\left|\Phi^P_{Y,g}(A_0,\dots,A_m)\right| \le C|||A_0|||_{d_0+T_0+1}\cdots|||A_m|||_{d_0+T_0+1}.$$

This proves the first claim. For the second claim we proceed as in [37, Proposition 7.12]: we use the usual trace inequality $|\operatorname{Tr}(AB)|\le||A||_1||B||_1$ together with the estimate $|H_i(gk)|\le C_iL(g)$ of [44, Proposition A.2] to find

$$\begin{aligned}&|\sigma^P_{\partial Y,g}(B_0,...,B_{m+1})|\\ &\le C\int_{h\in M/Z_M(g)}\int_{KN}\int_{G^{\times(m+1)}}\tilde f_0(khgh^{-1}nk^{-1}(g_1\dots g_mg_{m+1})^{-1},\lambda)\tilde f_1(g_1,\lambda)\cdots\\ &\qquad\qquad \tilde f_{m+1}(g_{m+1},\lambda)dg_1\cdots dg_{m+1}dkdndhd\lambda,\end{aligned}$$

where

$$\begin{aligned}\tilde f_0(g,\lambda) &:= ||\hat B_0(g,\lambda)||_1,\\ \tilde f_i(g,\lambda) &:= ||\hat B_i(g,\lambda)||_1(1+L(g))^i, \qquad i=1,\dots,m,\\ \tilde f_{m+1}(g,\lambda) &:= ||\frac{\partial\hat B_{m+1}(g,\lambda)}{\partial\lambda}||_1\end{aligned}$$

By continuity of the map $||\ ||_1:\mathcal{L}^\infty_{G,s}(\partial Y)\to\mathcal{L}_s(G)$, c.f. §4.1, we see that $\tilde f_j\in\mathcal{L}_s(G)$, for all $j=1,\dots,m+1$. We can therefore rewrite the right side of the equality above as

$$\int_{h\in M/Z_M(g)}\int_{KN}\int_{\mathbb{R}}F(khgh^{-1}nk^{-1},\lambda)dkdndhd\lambda,$$

with $F:=\tilde f_0*\dots*\tilde f_{m+1}$. Convergence of this integral now follows as in [44, Theorem A.5]. □

7.2. **Definition.** Let $p_t = V(tD_{\rm cyl})$ and $c_m=(-1)^{\frac m2}\frac{m!}{(\frac m2)!}$. Fix a cuspidal parabolic subgroup $P=MAN<G$ and $g\in M$ a semisimple element. Let

$$\eta^P_g(t) := 2c_m\sum_{i=0}^m\sigma^P_{\partial Y,g}(p_t,...,[\dot p_t,p_t],...,p_t)$$

We define the higher eta invariant associated to Φ^P_g and the boundary operator D_∂ as

$$\eta^P_g(D_\partial) := \lim_{\epsilon\downarrow 0}\int_\epsilon^{1/\epsilon}\eta^P_g(t)dt \equiv \lim_{\epsilon\downarrow 0}\int_\epsilon^{1/\epsilon}2c_m\sum_{i=0}^m\sigma^P_{\partial Y,g}(p_t,...,[\dot p_t,p_t],...,p_t)dt \tag{7.3}$$

if this limit exists.

7.4. **Theorem.** *Let $s \in (0,1]$. For the index pairing $\langle \mathrm{Ind}_\infty(D), [\Phi^P_{Y,g}]\rangle$ the following formula holds:*

$$c_m \langle \mathrm{Ind}_\infty(D), [\Phi^P_{Y,g}]\rangle = \Phi^{r,P}_{Y,g}(V(sD),\dots,V(sD)) - \frac{1}{2}\int_s^\infty \eta^P_g(t)dt$$

where part of the statement is that the t-integral converges at $+\infty$.

Proof. One first establishes the equality

$$\langle \mathrm{Ind}_\infty(D), [\Phi^P_{Y,g}]\rangle = \langle \mathrm{Ind}_\infty(D, D_\partial), [\Phi^{r,P}_{Y,g}, \sigma^P_{\partial Y,g}]\rangle \,, \tag{7.5}$$

exactly as in [34, Theorem 9.7] and [37, Theorem 7.17]. By definition of relative pairing and by the very definition of our relative index class, that is

$$\mathrm{Ind}_\infty(D, D_\partial) := [V(D), e_1, q_t]\,, \quad t\in[1,+\infty]\,, \quad \text{with } q_t := \begin{cases} V(tD_{\rm cyl}) & \text{if } \; t\in[1,+\infty) \\ e_1 & \text{if } \; t=\infty \end{cases}$$

we then obtain from (7.5) the following formula

$$c_m \langle \mathrm{Ind}_\infty(D), [\Phi^P_{Y,g}]\rangle = \Phi^{r,P}_{Y,g}(V(D),\dots,V(D)) - \frac{1}{2}\int_1^\infty \eta^P_g(t)dt$$

The formula we want to prove is obtained by rescaling D to sD. □

7.6. **Remark.** We would like to take the limit as $s\downarrow 0$ in Theorem 7.4 and obtain directly a higher delocalized APS index formula as the sum of a geometric term and the higher delocalized eta invariant. Unfortunately at the moment it is quite unclear how to study the limit as $s\downarrow 0$ of $\Phi^{r,P}_{Y,g}(V(sD),\dots,V(sD))$ which is why in the next section we give a treatment of the higher APS index formula corresponding to Φ^P_g through reduction, as in the closed case treated by Hochs, Song and Tang in [16]. Notice that even for a cocompact G-proper manifold *without* boundary X it is a difficult problem to study the limit

$$\lim_{s\to 0} \Phi^P_{X,g}(V(sD),\dots,V(sD))$$

with $V(D)$ the (symmetrized) Connes-Moscovici projector.

8. Reduction

Suppose that Y is a smooth G-proper manifold with boundary. Let $P = MAN$ be a cuspidal parabolic subgroup. Since the nilpotent subgroup N acts freely on X, we can consider $Y_{MA} := Y/N$ which is a smooth manifold with a proper MA-action. In addition, the abelian group A also acts freely on Y_{MA}, we define

$$Y_M = Y/(AN) = Y_{MA}/A$$

which is a smooth manifold with a proper M-action. The following equation of the index pairings was proved [16, Proposition 4.8] for a smooth G-proper manifold X without boundary,

$$\langle \Phi_{X,g}, \mathrm{Ind}_\infty(D)\rangle = \langle \Phi_{X_{MA},g}, \mathrm{Ind}_\infty(D_{X_{MA}})\rangle\,.$$

Moreover, if the metric is slice compatible (Definition 3.5), we have [16, Lemma 5.3]

$$\langle \Phi_{X_{MA},g}, \mathrm{Ind}_\infty(D_{X_{MA}})\rangle = \langle \Phi_{X_M,g}, \mathrm{Ind}_\infty(D_{X_M})\rangle.$$

These two equations together give a cohomological formula for $\langle \Phi_{X,g}, \mathrm{Ind}_\infty(D)\rangle$, with geometric information coming from $X_{M,g}$. In this section, we generalize the above computation for $\langle \Phi_{X,g}, \mathrm{Ind}_\infty(D)\rangle$ to a G-proper manifold Y with boundary. In particular, we will need to study the invertibility of the Dirac operators on $\partial(Y/N)$ and $\partial(Y_M)$.

8.1. **First reduction for manifolds without boundary.** Recall that
$$\mathcal{L}^\infty_{G,s}(X,E)\cong\left(\mathcal{L}_s(G)\hat{\otimes}\Psi^{-\infty}(S,E|_S)\right)^{K\times K},\quad \mathcal{L}^\infty_{G,s}(X_{MA},E|_{X_{MA}})\cong\left(\mathcal{L}_s(MA)\hat{\otimes}\Psi^{-\infty}(S,E|_S)\right)^{(K\cap M)\times(K\cap M)}.$$

8.1. **Definition.** For any $f\in\mathcal{L}_s(G)$, Harish-Chandra [12] defines a map f^N on MA by
$$f(ma)\colon=\int_N f(man)dn.$$
Then for any $k\in\mathcal{L}^\infty_{G,s}(X,E)$, and for all $m\in M, a\in A, y,y'\in S$, put
$$k^N(ma,y,y'):=\int_N k(nma,y,y')dn\in\operatorname{Hom}\left(E|_y,E|_{y'}\right). \tag{8.2}$$

8.3. **Lemma.** *The map defined in (8.2) is a multiplicative map:*
$$\phi^N\colon\mathcal{L}^\infty_{G,s}(X,E)\ni k\longrightarrow k^N\in\mathcal{L}^\infty_{G,s}(X_{MA},E|_{X_{MA}})$$
Moreover, the map ϕ^N is continuous with respect to Fréchet topologies.

Proof. Harish-Chandra proved in [12][Lemma 22] that (8.2) defines a continuous linear map
$$\mathcal{L}_s(G)\ni k\longrightarrow k^N\in\mathcal{L}_s(MA).$$
It remains to show that it is multiplicative. Suppose that
$$\kappa_1,\kappa_2\in\mathcal{L}^\infty_{G,s}(X,E).$$
Because Lafforgue's Schwartz algebra is closed under convolution,
$$\kappa_1^N\star\kappa_2^N\in\mathcal{L}^\infty_{G,s}(X_{MA},E|_{X_{MA}}).$$
By definition and the Iwasawa decomposition $G=KNMA$,
$$\begin{aligned}(\kappa_1\star\kappa_2)^N(ma)&=\int_G\int_N\kappa_1(nmag'^{-1})\kappa_2(g')\,dndg'\\&=\int_K\int_M\int_A\int_N\int_N\kappa_1(nmaa'^{-1}m'^{-1}n'^{-1}k'^{-1})\kappa_2(k'n'm'a')\,dk'dm'da'dn'dn\end{aligned}\tag{8.4}$$
Since the kernels $\kappa_i, i=1,2$ are $K\times K$-equivariant, we have
$$\kappa_1(k'gk)=k'\cdot\kappa_1(g)\cdot k,$$
where the k,k' on the right-hand side denotes the K-action on $\Psi^{-\infty}(S)$. A similar equation holds for κ_2. The last equation in (8.4) becomes
$$\int_M\int_A\int_N\int_N\kappa_1(nmaa'^{-1}m'^{-1}n'^{-1})\kappa_2(n'm'a')\,dm'da'dn'dn$$
Recall that MA normalizes N, that is,
$$nmaa'^{-1}m'^{-1}n'^{-1}=n''maa'^{-1}m'^{-1}$$
for some $n''\in N$. We conclude that
$$\begin{aligned}(\kappa_1\star\kappa_2)^N(ma)&=\int_M\int_A\int_N\int_N\kappa_1(n''maa'^{-1}m'^{-1})\kappa_2(n'm'a')\,dm'da'dn'dn''\\&=\int_M\int_A\kappa_1^N(maa'^{-1}m'^{-1})\kappa_2^N(m'a')\,dm'da'\\&=\left(\kappa_1^N\star\kappa_2^N\right)(ma).\end{aligned}$$
□

8.5. **Remark.** By Lemma 8.3, the map ϕ^N induces a map
$$\phi^N_*\colon K^*\left(\mathcal{L}^\infty_{G,s}(X,E)\right)\cong K^*\left(C^*_r(G)\right)\to K^*\left(\mathcal{L}^\infty_{G,s}(X_{MA},E|_{X_{MA}})\right)\cong K^*\left(C^*_r(MA)\right).$$

We consider $C^\infty(X,E)$ and $C^\infty(X,E)^{N,c}$, the smooth N-invariant sections of E with compact support in $X/N\cong X_{MA}$. Let D be a Dirac operator on X and $k_t\in\mathcal{L}^\infty_{G,s}(X,E)$ be the Schwartz kernel of $\exp(-tD^2)$. Since $k^N_t\in\mathcal{L}^\infty_{G,s}(X_{MA},E|_{X_{MA}})$, we can regard it as an operator on $C^\infty(X,E)^{N,c}$.

8.6. **Lemma.** *For all $\sigma \in C_c^\infty(X,E)$ and $s \in C^\infty(X,E)^{N,c}$ we have*

$$(k_t^*\sigma, s)_{L^2(X,E)} = (\sigma, k_t^N s)_{L^2(X,E)},$$

where k_t^* is the adjoint kernel of k_t. Moreover,

$$\frac{d}{dt}\left(\sigma, k_t^N s\right)_{L^2(X,E)} = \left(\sigma, -D^2\exp(-tD^2)s\right)_{L^2(X,E)}$$

Proof. This lemma is proved in [16, Lemma 4.1 and Lemma 4.2]. We have included it here for the reader's convenience. □

8.7. **Lemma.** *Let $D_{X_{MA}}$ be the Dirac operator on X_{MA} induced from D. Then the Schwartz kernel of $\exp(-tD^2_{X_{MA}})$ is k_t^N.*

Proof. Since $X_{MA} = X/N$, we can identify $C^\infty(X,E)^{N,c} \cong C^\infty(X_{MA}, E|_{X_{MA}})$. Under such an identification, the restriction D on N-invariant sections equals $D_{X_{MA}}$. By Lemma 8.6, we have that

$$\frac{d}{dt}\left(\sigma, k_t^N s\right)_{L^2(X,E)} = \left(\sigma, -D^2_{X_{MA}} k_t^N s\right)_{L^2(X,E)}$$

and

$$\lim_{t\downarrow 0}\left(\sigma, k_t^N s\right)_{L^2(X,E)} = (\sigma, s)_{L^2(X,E)}\,.$$

Using the uniqueness of the heat equation with usual initial data, we conclude that k_t^N is the Schwartz kernel of $\exp(-tD^2_{X_{MA}})$. □

8.8. **Remark.** Lemma 8.7 is stated in [16, Lemma 4.5] for the wrong operator $D_{X_{MA}}$ introduced in [16, Eq. (4.10)]. It is the operator $D_{X_{MA}}$ introduced in Lemma 8.7 that carries the right property for the development. It is not hard to see that $D_{X_{MA}}$ is a twisted Dirac operator on X_{MA}.

8.9. **Remark.** As in [16, Lemma 4.6], one can similarly show that $V(tD)^N = V(tD_{X_{MA}})$, where V denotes the Connes-Moscovici projection.

8.10. **Lemma.** *If the Dirac operator D on X is invertible, then the corresponding Dirac operator $D_{X_{MA}}$ is invertible as well.*

Proof. Suppose that

$$\mathrm{Spec}(D) \cap (-\delta, \delta) = \emptyset$$

for some $\delta > 0$. We have shown in Part II (we should move the proof to Part I later) that the Schwartz kernel

$$e^{\frac{t\delta}{2}} \cdot k_t \in \mathcal{L}^\infty_{G,s}(X,E)$$

converges to zero in Frechet topology as $t \to \infty$. Now the key observation is that the map

$$\mathcal{L}^\infty_{G,s}(X,E) \ni k_t \longrightarrow k_t^N \in \mathcal{L}^\infty_{G,s}(X_{MA}, E|_{X_{MA}})$$

is continuous by Lemma 8.3. Thus, k_t^N, which is the Schwartz kernel of $e^{-tD^2_{X_{MA}}}$ by Lemma 8.7, converges to zero faster than $e^{\frac{-t\delta}{2}}$ as $t \to \infty$. It follows that the operator $e^{\frac{t\delta}{2} - tD^2_{X_{MA}}}$ is uniformly bounded. Thus, we can find a constant C such that for $t \gg 0$, and any $s \in L^2(X_{MA}, E|_{X_{MA}})$,

$$\langle e^{-tD^2_{X_{MA}}} s, s\rangle \le e^{-\frac{t\delta}{3}} \cdot \|s\|^2. \tag{8.11}$$

We denote by E_λ the spectrum measure associated to the self-adjoint operator $D^2_{X_{MA}}$, that is

$$E_\lambda \colon \lambda \to \text{projection on } L^2(X_{MA}, E|_{X_{MA}}), \quad \lambda \in \mathrm{Spec}(D^2_{X_{MA}}) \subseteq [0, +\infty).$$

For any $s \in L^2(X_{MA}, E|_{X_{MA}})$,

$$\begin{aligned}\langle e^{-tD^2_{X_{MA}}} s, s\rangle &= \int_0^\infty e^{-t\lambda}\, d\langle E_\lambda s, s\rangle \\ &\ge \int_0^{\frac{\delta}{4}} e^{-t\lambda}\, d\langle E_\lambda s, s\rangle \ge e^{-\frac{t\delta}{4}} \cdot \int_0^{\frac{\delta}{4}} d\langle E_\lambda s, s\rangle\end{aligned}$$

If $\mathrm{Spec}(D^2_{X_{MA}}) \cap \left[0, \frac{\delta}{4}\right) \neq \emptyset$, then

$$P := \int_0^{\frac{\delta}{4}} dE_\lambda$$

defines a non-zero projection on $L^2(X_{MA}, E|_{X_{MA}})$. If we take $s_0 \in \mathrm{Image}\,(P)$, then

$$\langle e^{-tD^2_{X_{MA}}} s_0, s_0 \rangle \geq e^{-\frac{t\delta}{4}} \cdot \|s_0\|^2,$$

which contradicts to (8.11). This completes the proof. □

8.2. **First reduction for manifold with boundary.** Suppose that Y is a smooth G-proper manifold with boundary, denoted ∂Y. Let

$$g \in \mathcal{L}^\infty_{G,s}(Y, E)\,; \quad k \in {}^b\mathcal{L}^\infty_{G,s}(Y, E)\,; \quad k_{\mathbb{R}} \in {}^b\mathcal{L}^\infty_{G,s,\mathbb{R}}(\mathrm{cyl}(\partial Y), p^*E_{\partial Y}),$$

where for the sake of clarity we have now included the bundles in the notation; $p : \mathbb{R} \times \partial Y \to \partial Y$ is the obvious projection. We see these kernels as functions on G with values in pseudodifferential operators on the slice S satisfying a $K \times K$-equivariance. We can then define

$$g^N \in \mathcal{L}^\infty_{G,s}(Y_{MA}, E|_{Y_{MA}})\,; \quad k^N \in {}^b\mathcal{L}^\infty_{G,s}(Y_{MA}, E|_{Y_{MA}})\,; \quad k^N_{\mathbb{R}} \in {}^b\mathcal{L}^\infty_{G,s,\mathbb{R}}(\mathrm{cyl}(\partial Y_{MA}), p^*E|_{\partial Y_{MA}}).$$

These are functions on MA with values in psudodifferential operators on the slice Z satisfying a $(K \cap M) \times (K \cap M)$-equivariance. See [16], Section 4.1. We can of course extend this map to $M_{n\times n}(\mathcal{L}^\infty_{G,s}(Y, E))$, $M_{n\times n}({}^b\mathcal{L}^\infty_{G,s}(Y, E))$ and $M_{n\times n}({}^b\mathcal{L}^\infty_{G,s,\mathbb{R}}(\mathrm{cyl}(\partial Y), p^*E_{\partial Y}))$. By an argument similar to that given in Lemma 8.3, we have that the following three maps

1. $\mathcal{L}^\infty_{G,s}(Y, E) \ni g \longrightarrow g^N \in \mathcal{L}^\infty_{G,s}(Y_{MA}, E|_{Y_{MA}})$
2. ${}^b\mathcal{L}^\infty_{G,s}(Y, E) \ni k \longrightarrow k^N \in {}^b\mathcal{L}^\infty_{G,s}(Y_{MA}, E|_{Y_{MA}})$
3. ${}^b\mathcal{L}^\infty_{G,s,\mathbb{R}}(\mathrm{cyl}(\partial Y), p^*E_{\partial Y}) \ni k_{\mathbb{R}} \longrightarrow k^N_{\mathbb{R}} \in {}^b\mathcal{L}^\infty_{G,s,\mathbb{R}}(\mathrm{cyl}(\partial Y_{MA}), p^*E|_{\partial Y_{MA}})$

are all multiplicative maps and continuous with respect to Fréchet topologies. As before, we have the following result:

8.12. **Proposition.** *We have that*

- k^N_t *is equal to the Schwartz kernel of* $\exp(-tD^2_{Y_{MA}})$*; an analogous result holds for* $k^N_{\mathbb{R}}$*.*
- $V(tD)^N = V(tD_{Y_{MA}})$*.*
- *If the boundary operator of D is L^2-invertible, then* $({}^bV(tD))^N = {}^bV(D_{Y_{MA}})$*.*

We shall now consider various index classes. We consider Y_{MA} and Y_M. The manifold Y_{MA} is a product, i.e.

$$Y_{MA} = Y_M \times A,$$

where M acts properly on Y_M and trivially on A, and A acts properly and freely on A and trivially on Y_M.

8.13. **Proposition.** *Assume that the metric on Y is slice compatible and that the boundary operator of D is L^2-invertible. Then the boundary operator of $D_{Y_{MA}}$ and of D_{Y_M} are also L^2-invertible. Consequently there are well defined index classes:*

$$\mathrm{Ind}_\infty(D_{Y_{MA}}) \in K_0(\mathcal{L}^\infty_{G,s}(Y_{MA}, E|_{Y_{MA}}))\,, \quad \mathrm{Ind}_\infty(D_{Y_M}) \in K_0(\mathcal{L}^\infty_{G,s}(Y_M, E_M)) \tag{8.14}$$

with E_M as in (9.2) below.

Proof. If $D_{\partial Y}$ is L^2-invertible, then we can use Lemma 8.10 in order to see directly that the boundary operator of $D_{Y_{MA}}$ is also L^2-invertible. Next, as the metric is slice compatible, we have that the induced metric on $Y_{MA} = Y_M \times A$ is a product metric. It follows that the Dirac operator $D_{Y_{MA}}$ decomposes as

$$D_{Y_{MA}} = D_{Y_M} \hat{\otimes} 1 + 1 \hat{\otimes} D_A, \tag{8.15}$$

where we use the graded tensor products. This means that

$$D^2_{Y_{MA}} = D^2_{Y_M} \otimes 1 + 1 \otimes D^2_A$$

As $D^2_{Y_{MA}}$ is L^2-invertible and D^2_A is *not* L^2-invertible (A is isomorphic to $\mathbb{R}^n$), we see that $D^2_{Y_M}$ and thus D_{Y_M} must be L^2-invertible. □

Consider now the cyclic cocycle $\Phi_{MA,g}$ on $\mathcal{L}_s(MA)$, see [16, Section 3.1], and the associated cyclic cocycle on $\mathcal{L}^\infty_{G,s}(Y_{MA}, E|_{Y_{MA}})$, denoted $\Phi_{Y_{MA},g}$. Consider the index class $\mathrm{Ind}_\infty(D) \equiv [^bV(D)]$ and the index class $\mathrm{Ind}_\infty(D_{Y_{MA}}) \equiv [^bV(D_{Y_{MA}})]$, which is well defined because of Proposition 8.13.

8.16. **Proposition.** *The following equality holds*

$$\langle \Phi_{Y,g}, \mathrm{Ind}_\infty(D)\rangle = \langle \Phi_{Y_{MA},g}, \mathrm{Ind}_\infty(D_{Y_{MA}})\rangle \,. \tag{8.17}$$

Proof. We must prove that

$$\langle \Phi_{Y,g}, [^bV(D)]\rangle = \langle \Phi_{Y_{MA},g}, [^bV(D_{Y_{MA}})]\rangle \,. \tag{8.18}$$

This follows combining [16, Proposition 3.2], which clearly holds for the algebra of residual operators $\mathcal{L}^\infty_{G,s}(Y,E)$, and Proposition 8.12 above. □

8.3. **Second reduction.** The Lie group M is of equal rank and acts properly on Y_M. Consider the orbital integral τ^M_g on $\mathcal{L}_s(M)$ associated to a semisimple element g in M. As in subsection 4.2, we can associated the orbital integral τ^M_g a 0-cocycle on Y_M, denoted $\Phi_{Y_M,g}$. More precisely, $\Phi_{Y_M,g}$ is a cyclic 0-cocycle on the algebra $\mathcal{L}^\infty_{G,s}(Y_M, E_M)$, with E_M as in (9.2).

8.19. **Proposition.** *If the metric on Y is slice compatible and the boundary operator on Y is L^2-invertible, then the following equality holds:*

$$\langle \Phi_{Y_{MA},g}, \mathrm{Ind}_\infty(D_{Y_{MA}})\rangle = \langle \Phi_{Y_M,g}, \mathrm{Ind}_\infty(D_{Y_M})\rangle \tag{8.20}$$

Proof. We prove this identity in two steps.
Step I: Recall that $\mathrm{Ind}(D_{Y_{MA}})$ is an K-theory element of the algebra $\mathcal{L}^\infty_{MA,s}(Y_{MA}, E_{MA})$. Recall that the manifold Y_{MA} is a product, i.e. $Y_{MA} = Y_M \times A$, where M acts properly on Y_M and trivially on A, and A acts properly and freely on A and trivially on Y_M. With this decomposition, we can write the algebra $\mathcal{L}^\infty_{MA,s}(Y_{MA}, E_{MA})$ as follows,

$$\mathcal{L}^\infty_{MA,s}(Y_{MA}, E_{MA}) = \mathcal{L}^\infty_{M,s}(Y_M, E_M)\hat{\otimes}\mathcal{L}_s(A, \mathrm{End}(\mathcal{S}_A)), \tag{8.21}$$

where $\mathcal{S}_A$ is the space of spinors on $\mathfrak{a}$, the Lie algebra of A, and $\mathrm{End}(\mathcal{S}_A)$ is the algebra of endomorphisms on $\mathcal{S}_A$. Now, we consider the index elements $\mathrm{Ind}_\infty(D_{Y_{MA}})$, $\mathrm{Ind}_\infty(D_{Y_M})$, and $\mathrm{Ind}_\infty(D_A)$, i.e.

$$\mathrm{Ind}_\infty(D_{Y_{MA}}) \in K_*(\mathcal{L}^\infty_{MA,s}(Y_{MA}, E_{MA})),\ \mathrm{Ind}_\infty(D_{Y_M}) \in K_*(\mathcal{L}^\infty_{M,s}(Y_M, E_M)),\ \mathrm{Ind}_\infty(D_A) \in K_*(\mathcal{L}_s(A, \mathrm{End}(\mathcal{S}_A))).$$

We notice that the decomposition Eq. (8.21) of $\mathcal{L}^\infty_{MA,s}(Y_{MA}, E_{MA})$ defines an element

$$\mathrm{Ind}_\infty(D_{Y_M}) \otimes \mathrm{Ind}_\infty(D_A)$$

through the external product

$$K_*(\mathcal{L}^\infty_{M,s}(Y_M, E_M)) \otimes K_*(\mathcal{L}_s(A, \mathrm{End}(\mathcal{S}_A))) \to K_*(\mathcal{L}^\infty_{MA,s}(Y_{MA}, E_{MA})).$$

We claim that

$$\mathrm{Ind}_\infty(D_{Y_{MA}}) = \mathrm{Ind}_\infty(D_{Y_M}) \otimes \mathrm{Ind}_\infty(D_A) \in K_*(\mathcal{L}^\infty_{MA,s}(Y_{MA}, E_{MA})). \tag{8.22}$$

To prove this formula we argue as follows.
Recall that $\mathcal{L}^\infty_{MA,s}(Y_{MA}, E_{MA})$ (respectively $\mathcal{L}^\infty_{M,s}(Y_M, E_M)$ and $\mathcal{L}_s(A, \mathrm{End}(\mathcal{S}_A))$) is a dense subalgebra of the Roe algebra $C^*(Y_{MA}, E_{MA})^{MA}$ (respectively $C^*(Y_M, E_M)^M$ and $C^*_r(A, \mathrm{End}(\mathcal{S}_A))$) closed under holomorphic functional calculus. The image of the smooth index classes in the K-theory of the respective Roe C^*-algebras define the (isomorphic) index classes

$$\mathrm{Ind}(D_{Y_{MA}}) \in K_*(C^*(Y_{MA}, E_{MA})^{MA})\,, \quad \mathrm{Ind}(D_{Y_M}) \in K_*(C^*(Y_M, E_M)^M),\ \mathrm{Ind}(D_A) \in K_*(C^*_r(A, \mathrm{End}(\mathcal{S}_A)))\,.$$

Furthermore, the Roe algebra $C^*(Y_{MA}, E_{MA})^{MA}$ (respectively $C^*(Y_M, E_M)^M$ and $C^*_r(A, \mathrm{End}(\mathcal{S}_A))$) is strongly Morita equivalent to $C^*_r(MA)$ (respectively $C^*_r(M)$ and $C^*_r(A)$) and as explained in [37] we have explicit representatives of the (strongly) Morita equivalent index classes

$$\mathrm{Ind}_{C^*_r(MA)}(D_{Y_{MA}}) \in K_*(C^*_r(MA))\,, \quad \mathrm{Ind}_{C^*_r(M)}(D_{Y_M}) \in K_*(C^*_r(M)) \ \text{ and } \ \mathrm{Ind}_{C^*_r(A)}(D_A) \in K_*(C^*_r(A)). \tag{8.23}$$

Indeed, these index classes are defined in terms of our operators acting on suitable $C^*_r H$-modules, with H one of the above 3 groups. Now, following [25, 49] and [46] we can also express these index classes in terms of unbounded KK-classes associated to a Atiyah-Patodi-Singer boundary condition, denoted here

$$\begin{aligned} \mathrm{Ind}^{\mathrm{APS}}_{C^*_r(MA)}(D_{Y_{MA}}) \in KK_*(\mathbb{C}, C^*_r(MA))\ ,\quad \mathrm{Ind}^{\mathrm{APS}}_{C^*_r(M)}(D_{Y_M}) \in KK_*(\mathbb{C}, C^*_r(M)), \\ \mathrm{Ind}^{\mathrm{APS}}_{C^*_r(A)}(D_A) \in KK_*(\mathbb{C}, C^*_r(A)). \end{aligned} \tag{8.24}$$

Notice that our b-index classes do *not* define KK-elements (the resolvent is not C^*-compact); this is why we need to pass to the APS-index classes defined through the well-known boundary condition. Using formula (8.15) of the Dirac operator $D_{Y_{MA}}$ and proceeding exactly as in [46, Theorem 2.2] we can prove the following identity

$$\mathrm{Ind}^{\mathrm{APS}}_{C^*_r(MA)}(D_{Y_{MA}}) = [\mathrm{Ind}^{\mathrm{APS}}_{C^*_r(M)}(D_{Y_M}) \otimes \mathrm{Ind}^{\mathrm{APS}}_{C^*_r(A)}(D_A)] \in KK_*(\mathbb{C}, C^*_r(MA)) \equiv KK_*(C^*_r(MA))$$

Following the stated isomorphisms of K-theory groups and the compatibility of the various index classes we obtain finally

$$\mathrm{Ind}_\infty(D_{Y_{MA}}) = \mathrm{Ind}_\infty(D_{Y_M}) \otimes \mathrm{Ind}_\infty(D_A) \in K_*(\mathcal{L}^\infty_{MA,s}(Y_{MA}, E_{MA})).$$

This is precisely the claim we wanted to prove.

Step II: Using the product structure (8.21) of the algebra $\mathcal{L}^\infty_{MA,s}(Y_{MA}, E_{MA})$, we compute the cocycle $\Phi_{Y_{MA},g}$ as follows. For any $f_i \otimes g_i \in \mathcal{L}^\infty_{M,s}(Y_{MA}) \otimes \mathcal{L}_s(A, \mathrm{End}(\mathcal{S}_A))$,

$$\begin{aligned} &\Phi_{Y_{MA},g}(f_0 \otimes g_0, \dots f_n \otimes g_n) \\ &= \int_{M/Z_{M,g}} \int_{(MA)^n} \det(a_1, \dots, a_n) \cdot \mathrm{Tr}\Big(f_0(hgh^{-1}(m_1 \dots m_n)^{-1}) g_0((a_1 \cdot \dots \cdot a_n)^{-1}) \\ &\quad f_1(m_1) g_1(a_1) \cdot \dots \cdot f_n(m_n) g_n(a_n)\Big)\, dh da_1 \dots da_n dm_1 \dots dm_n \\ &= \left(\int_{M/Z_{M,g}} \int_{(M)^n} \mathrm{Tr}\Big(f_0(hgh^{-1}(m_1 \dots m_n)^{-1}) f_1(m_1) \cdot \dots \cdot f_n(m_n)\Big)\, dh dm_1 \dots dm_n\right) \\ &\times \left(\int_{(A)^n} \det(a_1, \dots, a_n) \cdot \mathrm{Tr}\Big(g_0((a_1 \cdot \dots \cdot a_n)^{-1}) g_1(a_1) \cdot \dots \cdot g_n(a_n)\Big)\, da_1 \dots da_n\right). \end{aligned} \tag{8.25}$$

In the above equation, we denote

$$\widetilde{\Phi}_{Y_M,g}(f_0, \dots, f_n) = \int_{M/Z_{M,g}} \int_{(M)^n} \mathrm{Tr}\Big(f_0(hgh^{-1}(m_1 \dots m_n)^{-1}) f_1(m_1) \cdot \dots \cdot f_n(m_n)\Big)\, dh dm_1 \dots dm_n$$

and

$$\Phi_{A,e}(g_0, \dots, g_n) = \int_{(A)^n} \det(a_1, \dots, a_n) \cdot \mathrm{Tr}\Big(g_0((a_1 \cdot \dots \cdot a_n)^{-1}) g_1(a_1) \cdot \dots \cdot g_n(a_n)\Big)\, da_1 \dots da_n$$

By (8.22) and (8.25), we conclude that

$$\langle \Phi_{Y_{MA},g}, \mathrm{Ind}_\infty(D_{Y_{MA}})\rangle = \langle \widetilde{\Phi}_{Y_M,g}, \mathrm{Ind}_\infty(D_{Y_M})\rangle \cdot \langle \Phi_{A,e}, \mathrm{Ind}_\infty(D_A)\rangle.$$

By a special case [35, Theorem 4.6] (the case for G = A), we know that

$$\langle \Phi_{A,e}, \mathrm{Ind}_\infty(D_A)\rangle = 1.$$

On the other hand, we can directly check that if $f \star f = f$, then

$$\begin{aligned} \widetilde{\Phi}_{Y_M,g}(f, \dots, f) &= \int_{M/Z_{M,g}} \int_{M^{\times n}} \mathrm{Tr}\Big(f(hgh^{-1}(m_1 \dots m_n)^{-1}) f(m_1) \cdot \dots \cdot f(m_n)\Big)\, dh dm_1 \dots dm_n \\ &= \int_{M/Z_{M,g}} \mathrm{Tr}\big(f(hgh^{-1})\big) dh = \Phi_{Y_M,g}(f). \end{aligned}$$

Thus,

$$\langle \widetilde{\Phi}_{Y_M,g}, \mathrm{Ind}_\infty(D_{Y_M})\rangle = \langle \Phi_{Y_M,g}, \mathrm{Ind}_\infty(D_{Y_M})\rangle.$$

This completes the proof. □

9. An index theorem for higher orbital integral through reduction

We shall now put things together and give a formula for the higher delocalized Atiyah-Patodi-Singer index $\langle \Phi^P_{Y,g}, \mathrm{Ind}_\infty(D)\rangle$.

Let G be connected, linear real reductive, $P = MAN$ a cuspidal parabolic subgroup and $g \in M$ a semisimple element. Let $(Y_0, \mathbf{h}_0)$ be a cocompact G-proper manifold with boundary. We fix a slice Z_0 (and Z) for the G action on Y_0 (and Y), so that

$$Y_0 \cong G \times_K Z_0, \quad Y \cong G \times_K Z.$$

with Z_0 a smooth compact manifold **with** boundary. We assume that $D_{\partial Y}$ is L^2-invertible. Consider Y/AN, an M-proper manifold. Recall that we have the following $K \cap M$-invariant decomposition

$$\begin{aligned}\mathfrak{p} =&(\mathfrak{p}\cap\mathfrak{m}) \oplus \mathfrak{a} \oplus ((\mathfrak{p}\cap\mathfrak{m}) \oplus \mathfrak{a})^\perp \\ \cong&(\mathfrak{p}\cap\mathfrak{m}) \oplus \mathfrak{a} \oplus (\mathfrak{k}/(\mathfrak{k}\cap\mathfrak{m})) .\end{aligned}$$

Accordingly, the spinor bundle

$$S_{\mathfrak{p}} \cong S_{\mathfrak{p}\cap\mathfrak{m}} \otimes S_{\mathfrak{a}} \otimes S_{\mathfrak{k}/(\mathfrak{k}\cap\mathfrak{m})},$$

where $S_{\mathfrak{a}}$ is a vector space of dimension $2^{\lceil \frac{\dim \mathfrak{a}}{2} \rceil}$ on which $M \cap K$ acts trivially. The following facts will play a central role in our study:

(1) We realize Y/N in terms of a slice:

$$Y_{MA} \colon = Y/N \cong AM \times_{K\cap M} Z.$$

Restricting to Y_{MA},

$$E_{MA} \colon = E|_{Y_{MA}} \cong AM \times_{M\cap K} \left(S_{\mathfrak{p}\cap\mathfrak{m}} \otimes S_{\mathfrak{a}} \otimes S_{\mathfrak{k}/(\mathfrak{k}\cap\mathfrak{m})} \otimes E_Z\right).$$

For convenience, we introduce the following

$$\widetilde{E_Z} := E_Z \otimes S_{\mathfrak{k}/(\mathfrak{k}\cap\mathfrak{m})}. \tag{9.1}$$

Thus

$$E_M = M \times_{M\cap K} \left(S_{\mathfrak{p}\cap\mathfrak{m}} \otimes S_{\mathfrak{k}/(\mathfrak{k}\cap\mathfrak{m})} \otimes E_Z\right) \cong M \times_{M\cap K} \left(S_{\mathfrak{p}\cap\mathfrak{m}} \otimes \widetilde{E_Z}\right). \tag{9.2}$$

By the assumption (3.3) on the dimension of Y, we can see that Y_M is even dimensional because both Z and $M/M \cap K$ are even dimensional. The quotient Dirac operator D_{Y_M} on Y_M acts on

$$L^2(Y_M, E_M) \cong \left[L^2(M) \otimes S_{\mathfrak{p}\cap\mathfrak{m}} \otimes L^2(Z, \widetilde{E_Z})\right]^{K\cap M}.$$

We have assumed that E is a G-equivariant twisted spinor bundle on Y. Accordingly, E_M is an M-equivariant twisted spinor bundle on Y_M, and we can write

$$E_M = \mathcal{E}_M \otimes W_M. \tag{9.3}$$

where $\mathcal{E}_M$ is the spinor bundle associated to the induced Spin^c-structure on Y_M and $W_M = M \times_{M\cap M} \widetilde{E}_Z$. It is important to point out that since

$$S_{\mathfrak{k}/(\mathfrak{k}\cap\mathfrak{m})} = S^+_{\mathfrak{k}/(\mathfrak{k}\cap\mathfrak{m})} \oplus S^-_{\mathfrak{k}/(\mathfrak{k}\cap\mathfrak{m})}$$

has a $\mathbb{Z}_2$-grading, the auxiliary M-equivariant vector W_M is equipped with a $\mathbb{Z}_2$-grading as well, denoted by

$$W_M = W_M^+ \oplus W_M^- \tag{9.4}$$

9.5. **Remark.** Unless P is maximal cuspidal parabolic, $K \cap M$ has a lower rank than K. In this case, there exists a $K \cap M$-equivariant isomorphism λ defined by the action of an element in a Cartan subgroup of K but not in $K \cap M$,

$$\lambda : S^+_{\mathfrak{k}/(\mathfrak{k}\cap\mathfrak{m})} \cong S^-_{\mathfrak{k}/(\mathfrak{k}\cap\mathfrak{m})}.$$

Such an isomorphism λ is compatible with the respective connections and metrics. In the constructions in (9.1), (9.2) and (9.3), the tensor products are all graded and grading compatible. Hence, λ gives an M-equivariant isomorphism between the two vector bundles

$$W_M^+ \cong W_M^-,$$

compatible with the respective connections and metrics.

9.6. **Remark.** As W_M in Equation (9.4) is $\mathbb{Z}_2$-graded, the vector bundle E_M in Equation (9.3) can be written as follows with respect to the gradings on $\mathcal{E}_M$ and W_M,

$$E_M = (\mathcal{E}_M^+ \oplus \mathcal{E}_M^-) \otimes W_M^+ \oplus (\mathcal{E}_M^+ \oplus \mathcal{E}_M^-) \otimes W_M^-.$$

Accordingly, the $\mathbb{Z}_2$-grading on E_M gives the following decomposition,

$$E_M = E_M^+ \oplus E_M^-, \qquad E_M^+ = \mathcal{E}_M^+ \otimes W_M^+ \oplus \mathcal{E}_M^- \otimes W_M^-, \qquad E_M^- = \mathcal{E}_M^- \otimes W_M^+ \oplus \mathcal{E}_M^+ \otimes W_M^-. \tag{9.7}$$

Given a linear operator T on E_M, we write T into a 4×4 block matrix,

$$T = \begin{bmatrix} T_{++}^{++} & T_{-+}^{++} & T_{+-}^{++} & T_{--}^{++} \\ T_{++}^{-+} & T_{-+}^{-+} & T_{+-}^{-+} & T_{--}^{-+} \\ T_{++}^{+-} & T_{-+}^{+-} & T_{+-}^{+-} & T_{--}^{+-} \\ T_{++}^{--} & T_{-+}^{--} & T_{+-}^{--} & T_{--}^{--} \end{bmatrix}, \tag{9.8}$$

where $T_{\gamma\delta}^{\alpha\beta}$ is a linear operator from $\mathcal{E}_M^\gamma \otimes W_M^\delta$ to $\mathcal{E}_M^\alpha \otimes W_M^\beta$, for $\alpha, \beta, \gamma, \delta = \pm$. The $\mathbb{Z}_2$-grading, c.f. Equation (9.7), on E_M introduces a supertrace on T as follows,

$$\begin{aligned} \mathrm{Str}(T) &:= \mathrm{tr}(T_{++}^{++}) + \mathrm{tr}(T_{--}^{--}) - \left[\mathrm{tr}(T_{-+}^{-+}) + \mathrm{tr}(T_{+-}^{+-})\right] \\ &= \mathrm{tr}(T_{++}^{++}) - \mathrm{tr}(T_{-+}^{-+}) - \left[\mathrm{tr}(T_{+-}^{+-}) - \mathrm{tr}(T_{--}^{--})\right]. \end{aligned} \tag{9.9}$$

The following theorem is one of the main results of this paper:

9.10. **Theorem.** *Suppose that the metric on Y is slice compatible. Assume that $D_{\partial Y}$ is L^2-invertible and consider the higher index $\langle \Phi_{Y,g}^P, \mathrm{Ind}_\infty(D)\rangle$. The following formula holds:*

$$\langle \Phi_{Y,g}^P, \mathrm{Ind}_\infty(D)\rangle = \int_{(Y_0/AN)^g} c^g_{(Y_0/AN)^g} \mathrm{AS}(Y_0/AN)_g - \frac{1}{2}\eta_g(D_{\partial Y_M}) \tag{9.11}$$

with

$$\eta_g(D_{\partial Y_M}) = \frac{1}{\sqrt{\pi}} \int_0^\infty \tau_g^{\partial Y_M}(D_{\partial Y_M} \exp(-tD^2_{\partial Y_M}))\frac{dt}{\sqrt{t}}.$$

Recall that $c^g_{(Y_0/AN)^g}$ is a compactly supported smooth cutoff function on $(Y_0/AN)^g$ associated to the $Z_{M,g}$ action on $(Y_0/AN)^g$.

To explain the right-hand side of (1.11) more explicitly, we fix the following data.

- In (9.3), we constructed a M-equivariant, $\mathbb{Z}_2$-graded vector bundle

  $$W_M = W_M^+ \oplus W_M^-.$$

  We define

  $$R^{W_M^\pm} = \text{the curvature form of the Hermitian connection on } W_M^\pm.$$

- $R^{\mathcal{N}}$, the curvature form associated to the Hermitian connection on $\mathcal{N}_{(Y_0/AN)^g} \otimes \mathbb{C}$ ($\mathcal{N}_{(Y_0/AN)^g}$ is the normal bundle of the g-fixed point submanifold $(Y_0/AN)^g$ in Y_0/AN);
- R^L, the curvature form associated to the Hermitian connection on $L_{\det}|_{(Y_0/AN)^g}$ ($L_{\det}$ is the determinant line bundle of the Spin^c-structure on Y_0/AN and $L_{\det}|_{(Y_0/AN)^g}$ is its restriction to $(Y_0/AN)^g$);
- $R_{(Y_0/AN)^g}$, the Riemannian curvature form associated to the Levi-Civita connection on the tangent bundle of $(Y_0/AN)^g$;
- The form $\mathrm{AS}(Y_0/AN)_g$ is then given by the following expression:

$$\frac{\widehat{A}\left(\frac{R_{(Y_0/AN)^g}}{2\pi i}\right)\left[\mathrm{tr}\left(g\exp\left(\frac{R^{W_M^+}}{2\pi i}\right)\right) - \mathrm{tr}\left(g\exp\left(\frac{R^{W_M^-}}{2\pi i}\right)\right)\right]\exp(\mathrm{tr}(\frac{R^L}{2\pi i}))}{\det\left(1 - g\exp(-\frac{R^{\mathcal{N}}}{2\pi i})\right)^{\frac{1}{2}}} \tag{9.12}$$

- Since the auxiliary vector bundle W_M on Y_M is $\mathbb{Z}_2$-graded, the Dirac operator D_{Y_M} is equipped with a $\mathbb{Z}_2$-grading as well. Following the decomposition (9.7) and (9.8), we have the decomposition for D_{Y_M},

$$D_{Y_M} = \begin{bmatrix} 0 & D^+_{Y_M,+} & 0 & 0 \\ D^-_{Y_M,+} & 0 & 0 & 0 \\ 0 & 0 & 0 & D^+_{Y_M,-} \\ 0 & 0 & D^-_{Y_M,-} & 0 \end{bmatrix} = \begin{bmatrix} D_{Y_M,+} & 0 \\ 0 & D_{Y_M,-} \end{bmatrix}$$

$$D_{Y_M,+} = \begin{bmatrix} 0 & D^+_{Y_M,+} \\ D^-_{Y_M,+} & 0 \end{bmatrix}, \qquad D_{Y_M,-} = \begin{bmatrix} 0 & D^+_{Y_M,-} \\ D^-_{Y_M,-} & 0 \end{bmatrix},$$

where

$$D^\alpha_{Y_M,\beta} \colon L^2\left(Y_M, \mathcal{E}^\alpha_{Y_M} \otimes W^\beta_M\right) \to L^2\left(Y_M, \mathcal{E}^{-\alpha}_{Y_M} \otimes W^\beta_M\right), \quad \alpha, \beta = \pm.$$

Accordingly, since ∂Y_M is odd dimensional, the Dirac operator $D_{\partial Y_M}$ is a self-adjoint **even** differential operator on the $\mathbb{Z}_2$-graded spinor bundle $E_{\partial Y_M}$, which is defined as follows,

$$E_{\partial Y_M} := \mathcal{E}_{\partial Y_M} \otimes W_M|_{\partial Y_M}, \qquad E^+_{\partial Y_M} := \mathcal{E}_{\partial Y_M} \otimes W^+_M|_{\partial Y_M}, \qquad E^-_{\partial Y_M} := \mathcal{E}_{\partial Y_M} \otimes W^-_M|_{\partial Y_M}.$$

We can write $D_{\partial Y_M}$ into a 2×2 block diagonal matrix with respect to the grading on $E|_{\partial Y_M}$,

$$\begin{bmatrix} D_{\partial Y_M,+} & 0 \\ 0 & D_{\partial Y_M,-} \end{bmatrix},$$

where $D_{\partial Y_M,+}$ (and $D_{\partial Y_M,-}$) is the Dirac operator on $E^+_{\partial Y_M}$ (and $E^-_{\partial Y_M}$). Following the definition of the supertrace (9.9) and the $\mathbb{Z}_2$-grading of $D_{\partial Y_M}$, we obtain the following expression for the delocalized eta invariant of $D_{\partial Y_M}$:

$$\eta_g(D_{\partial Y_M}) = \eta_g(D_{\partial Y_M,+}) - \eta_g(D_{\partial Y_M,-}). \tag{9.13}$$

This point is slightly different from the 0-degree pairing case, where the operator $D_{\partial Y_M}$ is not graded.

Proof. Denote $\Phi^P_{Y,g}$ briefly by $\Phi_{Y,g}$. By (8.17) we know that

$$\langle \Phi_{Y,g}, \mathrm{Ind}_\infty(D)\rangle = \langle \Phi_{Y_{MA},g}, \mathrm{Ind}_\infty(D_{Y_{MA}})\rangle .$$

On the other hand, by equation (8.20) we have that

$$\langle \Phi_{Y_{MA},g}, \mathrm{Ind}_\infty(D_{Y_{MA}})\rangle = \langle \Phi_{Y_M,g}, \mathrm{Ind}_\infty(D_{Y_M})\rangle$$

so that

$$\langle \Phi_{Y,g}, \mathrm{Ind}_\infty(D)\rangle = \langle \Phi_{Y_M,g}, \mathrm{Ind}_\infty(D_{Y_M})\rangle$$

It suffices to apply now the 0-degree delocalized APS index theorem, Theorem 4.40, to the right side of the above equation and recall that Y/AN is diffeomorphic to $Y_M := M \times_{K\cap M} Z$ and that is obtained by addition of a cylindrical end to Y_0/AN. □

When P is not a maximal parabolic subgroup, we have the following vanishing result:

9.14. **Remark.** If P is not a maximal parabolic subgroup, then $\Phi^P_{Y,g}$ is a trivial class in cyclic cohomology and

$$\langle \Phi^P_{Y,g}, \mathrm{Ind}_\infty(D)\rangle = 0.$$

For the right-hand side of Equation (9.11), we know from Remark 9.5 that

$$W^+_M \cong W^-_M$$

compatible with the respective connections and metrics. Thus, $D_{\partial Y_M,+}$ is unitary equivalent to $D_{\partial Y_M,-}$ under the above isomorphism. By using the expression of $\eta_g(D_{\partial Y_M})$ as $\eta_g(D_{\partial Y_M,+}) - \eta_g(D_{\partial Y_M,-})$, we obtain the following

$$\eta_g(D_{\partial Y_M}) = 0\,.$$

Moreover, the integral

$$\int_{(Y_0/AN)^g} c^g_{(Y_0/AN)^g} \mathrm{AS}(Y_0/AN)_g = 0$$

as follows from the fact that

$$\operatorname{tr}\left(g\exp\left(\frac{R^{W_M^+}}{2\pi i}\right)\right)-\operatorname{tr}\left(g\exp\left(\frac{R^{W_M^-}}{2\pi i}\right)\right)=0,$$

because of the the isomorphism $W_M^+ \cong W_M^-$.

9.15. **Remark.** Let X be a cocompact G-proper manifold without boundary. The pairing $\langle \Phi^P_{X,g}, \operatorname{Ind}_\infty(D)\rangle$ has been computed in [16] via the reduction to the manifold X_M with a proper cocompact M action. Similarly, if Y is a cocompact G-proper manifold with boundary, then we have computed the pairing $\langle \Phi^P_{Y,g}, \operatorname{Ind}_\infty(D)\rangle$ via the reduction to Y_M. Also in this case it would be very interesting to compute this pairing directly, without resuming to the reduced manifold Y_M.

10. Numeric rho invariants on G-proper manifolds

In this section we shall introduce (higher) rho numbers associated to positive scalar curvature (psc) metrics inspired by the recent developments in the discrete group case, [39, 40, 50–52].

10.1. **Rho numbers associated to delocalized 0-cocycles.** We consider a closed G-proper manifold X **without** boundary, G connected, linear real reductive, $g \in G$ a semisimple element, D_X a G-equivariant L^2-invertible Dirac operator defined in (3.2). We know that the following integral is convergent:

$$\eta_g(D_X) := \frac{1}{\sqrt{\pi}}\int_0^\infty \tau_g^X(D_X\exp(-tD_X^2)\frac{dt}{\sqrt{t}}.$$

10.1. **Definition.** Let X be G-equivariantly spin and $D_X \equiv D_{\mathbf{h}}$, the spin Dirac operator associated to a G-equivariant PSC metric $\mathbf{h}$. We know that $D_{\mathbf{h}}$ is L^2-invertible. We define

$$\rho_g(\mathbf{h}) := \eta_g(D_{\mathbf{h}})\,.$$

10.2. **Rho numbers associated to higher delocalized cocycles.** We can generalize the example of the previous subsection and define rho numbers associated the higher cocycles Φ^P_g.

10.2. **Definition.** Let $P = MNA$ be a cuspidal parabolic subgroup, $g \in M$ as above and consider Φ^P_g and $\Phi^P_{X,g}$ (we recall that X is without boundary). Assume that $\mathbf{h}$ is a G-invariant PSC metric on X. In this case we also assume that $\mathbf{h}$ is slice-compatible. Then

$$\rho^P_g(\mathbf{h}) := \eta_g(D_{X_M}) \tag{10.3}$$

(with X_M the reduced manifold associated to X) is well defined.

Notice that as D_X is invertible, we have that D_{X_M} is also invertible, see Proposition 8.13.

10.3. **Bordism properties.** The APS index theorems proved in this article can be used in order to study the bordism properties of these rho invariants.

We assume, unless otherwise stated that we are on a G-proper manifold which is endowed with a G-invariant metric and a G-equivariant spin structure. If needed, we shall assume slice-compatibility of the metric and of the spin structure.

10.4. **Definition.** Consider a cocompact proper G-manifold $(W, \mathbf{h})$, possibly with boundary, endowed with an equivariant spin structure. Let S be the spinor bundle. Let $g \in G$ be semisimple. We shall say that g is *geometrically-simple on W* if

$$\int_{W^g} c^g \mathrm{AS}_g(W,S) = 0\,.$$

with $\mathrm{AS}_g(W,S)$ the usual integrand appearing in the 0-degree delocalized APS-index theorem.

10.5. **Example.** The following proposition provides many examples of geometrically-simple elements g on an arbitrary G-proper manifold W.

10.6. **Proposition.** *If g is non-elliptic, that is, does not conjugate to a compact element, then every element of the conjugacy class $C(g) := \{hgh^{-1} | h \in G\}$ in G does not have any fixed point on W.*

Proof. Without loss of generality, we assume that $W^g \neq \emptyset$. Take

$$x = (g_1, s) \in W = G \times_K S$$

such that

$$gx = (gg_1, s) = (g_1, s) = x.$$

As (g_1, s) and (gg_1, s) correspond to the same point in $G \times_K S$, we can find some $k \in K$ such that

$$gg_1 = g_1 k, \quad ks = s.$$

Thus, $g_1^{-1} g g_1 \in K$, that is, g has to be elliptic, which contradicts to our assumption on g. □

10.7. **Definition.** Let $(Y, \mathbf{h}_0)$ and $(Y, \mathbf{h}_1)$ be two psc metrics. We say that they are G-concordant if there exists a G-invariant metric $\mathbf{h}$ on $Y \times [0,1]$ which is of psc, product-type near the boundary and restricts to $\mathbf{h}_0$ at $Y \times \{0\}$ and to $\mathbf{h}_1$ at $Y \times \{1\}$. If $\mathbf{h}_0$ and $\mathbf{h}_1$ are slice compatible, we also require the metric $\mathbf{h}$ on $Y \times [0,1]$ to be slice compatible.

10.8. **Definition.** Let $(Y_0, \mathbf{h}_0)$ and $(Y_1, \mathbf{h}_1)$ be two G-proper manifolds with psc metrics. We shall say that they are G-psc-bordant if there exists a G-manifold with boundary W with a G-invariant metric $\mathbf{h}$ such that:
(i) $\partial W = Y_0 \sqcup Y_1$;
(ii) $\mathbf{h}$ is product-type near the boundary and of psc;
(iii) $\mathbf{h}$ restricts to $\mathbf{h}_0 \sqcup \mathbf{h}_1$ on $\partial W = Y_0 \sqcup Y_1$.
If $\mathbf{h}_0$ and $\mathbf{h}_1$ are slice compatible, we also require the metric $\mathbf{h}$ to be slice compatible.

10.9. **Proposition.**
1] Assume that the psc metrics $\mathbf{h}_0$ and $\mathbf{h}_1$ on Y are G-concordant. Assume that g is geometrically-simple on $Y \times [0,1]$, Then $\rho_g(\mathbf{h}_0) = \rho_g(\mathbf{h}_1)$.
2] Let $P = MAN < G$ be a cuspidal parabolic subgroup and let $g \in M$ be a semisimple element. Assume that the slice-compatible psc metrics $\mathbf{h}_0$ and $\mathbf{h}_1$ on Y are G-concordant. Assume that g is geometrically-simple on $Y_M \times [0,1]$. Then $\rho_g^P(\mathbf{h}_0) = \rho_g^P(\mathbf{h}_1)$.
3] Let $(Y_0, \mathbf{h}_0)$ and $(Y_1, \mathbf{h}_1)$ be G-psc-bordant through (W, h). Assume that g is geometrically-simple on W. Then $\rho_g(\mathbf{h}_0) = \rho_g(\mathbf{h}_1)$.

Proof. In both cases the proof is an immediate consequence of the geometric-simplicity of $g \in G$ and the relevant delocalized APS index theorems. □

10.10. **Remark.** Examples of geometrically-simple g as in Proposition 10.9 are given by elements in the conjugacy class of a non-elliptic element, c.f. Proposition 10.6. Notice that this applies to *any* bordism entering into Definition 10.7 and Definition 10.8. Thus for such g our (higher) rho invariants are in fact *concordance and bordism invariants.*

10.11. **Remark.** It is a challenge to understand whether these results can be employed in studying the spaces $\mathcal{R}^+_{G,\text{slice}}(Y)$ of slice-compatible G-invariant metrics of psc and the larger space $\mathcal{R}^+_G(Y)$ of G-invariant metrics of psc (if non-empty). It would be also interesting to understand the relationship between the following 3 spaces, especially from the point of view of homotopy theory:
(i) $\mathcal{R}^+_{G,\text{slice}}(Y)$;
(ii) $\mathcal{R}^+_G(Y)$;
(iii) $\mathcal{R}^+_K(S)$, the space of K-invariant metrics of psc on the slice S.
Clearly there is an inclusion $\mathcal{R}^+_{G,\text{slice}}(Y) \hookrightarrow \mathcal{R}^+_G(Y)$; there is also a natural map $\mathcal{R}^+_K(S) \to \mathcal{R}^+_{G,\text{slice}}(Y)$, see [11]. For all these questions it would be interesting to develop a G-equivariant Stolz' sequence and investigate its basic properties. We leave this task to future research.

10.12. **Remark.** If Y_1 and Y_2 are two oriented G-proper manifolds and $\mathbf{f} : Y_1 \to Y_2$ is an oriented G-equivariant homotopy equivalence then we can also define rho invariants associated to f and to a semisimple element g:

$$\rho_g(\mathbf{f}) := \eta_g(D^{\text{sign}}_X + A(\mathbf{f}))$$

with D^{sign}_X the signature operator on $X = Y_1 \cup (-Y_2)$ and $A(\mathbf{f})$ the Hilsum-Skandalis perturbation, as developed by Fukumoto in the G-proper context [7]. Notice however that both the definition of this invariant and the proof of the associated APS index theorem are not a routine extension of the case treated here.

Especially, the large time behaviour of a perturbed heat kernel and the proof of the analogue of Proposition 4.34 are particularly intricate. We plan to report on our results in this direction in a separate article.

Appendix A. Dirac operator on G

In this appendix, we discuss the difference between the Dirac operator, Eq. (3.2), considered in this article and the split Dirac operator operator, Eq. (3.6), on the manifold G with the free proper left G-action.

Suppose that G is a connected reductive Lie group with maximal compact subgroup K. We denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition of the Lie algebra and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ its complexification. Let B be a non-degenerate bilinear symmetric form on $\mathfrak{g}$, which is invariant under the adjoint action of G on $\mathfrak{g}$. We assume B to be negative on $\mathfrak{p}$, and positive on $\mathfrak{k}$. Let $S_{\mathfrak{g}}$ and $S_{\mathfrak{k}}$ be the irreducible $\mathbb{Z}_2$-graded spin representation for $\mathrm{Cliff}(\mathfrak{g}_{\mathbb{C}})$ and $\mathrm{Cliff}(\mathfrak{k}_{\mathbb{C}})$ respectively. Then we have the following decomposition

$$S_{\mathfrak{g}} \cong S_{\mathfrak{k}} \otimes S_{\mathfrak{p}}.$$

The adjoint actions on $\mathfrak{g}$ induce maps

$$\mathrm{ad}^{\mathfrak{g}} \colon \mathfrak{g}_{\mathbb{C}} \to \mathrm{End}(S_{\mathfrak{g}}) \cong \mathrm{Cliff}(\mathfrak{g}_{\mathbb{C}})$$
$$\mathrm{ad}^{\mathfrak{k}} \colon \mathfrak{k}_{\mathbb{C}} \to \mathrm{End}(S_{\mathfrak{k}}) \cong \mathrm{Cliff}(\mathfrak{k}_{\mathbb{C}})$$

defined by the following formulas: for any $X \in \mathfrak{g}_{\mathbb{C}}$,

$$\mathrm{ad}^{\mathfrak{g}}(X) \colon = \frac{1}{4} \sum_{i}^{\dim \mathfrak{g}} c\left([X, e_i]\right) \cdot c(e_i), \quad \mathrm{ad}^{\mathfrak{k}}(X) \colon = \frac{1}{4} \sum_{i}^{\dim \mathfrak{k}} c\left([X, e_i]\right) \cdot c(e_i)$$

where

$$e_1, \cdots e_{\dim \mathfrak{k}}, \quad \text{and} \quad e_{\dim \mathfrak{k}+1}, \cdots, e_{\dim \mathfrak{g}}$$

are the orthonormal bases for $\mathfrak{k}$ and $\mathfrak{p}$ respectively. We have the following relation

$$\mathrm{ad}^{\mathfrak{g}}(X) = \mathrm{ad}^{\mathfrak{k}}(X) + \mathrm{ad}^{\mathfrak{p}}(X), \quad X \in \mathfrak{k}_{\mathbb{C}},$$

where

$$\mathrm{ad}^{\mathfrak{p}}(X) = -\frac{1}{4} \sum_{i,j=\dim \mathfrak{k}+1}^{\dim \mathfrak{g}} B(X, [e_i, e_j]) \cdot c(e_i) \cdot c(e_j) \in \mathrm{Cliff}(\mathfrak{p}), \quad X \in \mathfrak{k}_{\mathbb{C}}$$

By left-trivialization, we can identify the vector field $\mathfrak{X}(G)$ with $C^\infty(G) \otimes \mathfrak{g}$. In particular, for any $X \in \mathfrak{g}$, we can identify it with a left-invariant vector field on G. We define a K-invariant inner product on $\mathfrak{g}$ by

$$\langle \ , \ \rangle = -B|_{\mathfrak{k}} \oplus +B|_{\mathfrak{p}}.$$

Our computation below is similar to those in [31, Chapter 9], while the author considers the pseudo Riemannian metric B, introduced above, instead of a Riemannian metric.

By the Koszul formula, the Levi-Civita connection is determined by

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \left(\langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle \right), \quad X, Y, Z \in \mathfrak{g}.$$

More general, for any vector field V on G, it can be written as

$$V = \sum_{i=1}^{\dim \mathfrak{g}} f_i \cdot V_i$$

where $f_i \in C^\infty(G)$ and $\{V_i\}$ is a basis for $\mathfrak{g}$. Then

$$\nabla_X V = \sum_{i=1}^{\dim \mathfrak{g}} X(f_i) \cdot V_i + \sum_{i=1}^{\dim \mathfrak{g}} f_i \cdot \nabla_X V_i.$$

If $X, Y \in \mathfrak{p}$ and $Z \in \mathfrak{k}$, then

$$\langle [Y, Z], X \rangle = -B([Y, Z], X) = B(Z, [Y, X]) = \langle Z, [Y, X] \rangle = -\langle [X, Y], Z \rangle$$

and

$$\langle [Z, X], Y \rangle = -B([Z, X], Y) = -B(Z, [X, Y]) = -\langle [X, Y], Z \rangle.$$

Thus,

$$\nabla_X Y = \frac{1}{2}[X,Y].$$

Similarly, one can check the following:

- $\nabla_X Y = \frac{1}{2}[X,Y], \quad X, Y \in \mathfrak{k}$.
- $\nabla_X Y = \frac{3}{2}[X,Y], \quad X \in \mathfrak{k}, Y \in \mathfrak{p}$.
- $\nabla_X Y = -\frac{1}{2}[X,Y], \quad X \in \mathfrak{p}, Y \in \mathfrak{k}$.

Therefore, the induced Clifford connection $\nabla^{S_\mathfrak{g}}$ acts on left G-invariant section of $S_\mathfrak{g}$ by the following formula:

$$\nabla_X^{S_\mathfrak{g}} = \frac{1}{8}\sum_{i=1}^{\dim \mathfrak{k}} c\left([X,e_i]\right)\cdot c(e_i) + \frac{3}{8}\sum_{i=\dim \mathfrak{k}+1}^{\dim \mathfrak{g}} c\left([X,e_i]\right)\cdot c(e_i), \quad X \in \mathfrak{k}$$

and

$$\nabla_X^{S_\mathfrak{g}} = -\frac{1}{8}\sum_{i=1}^{\dim \mathfrak{k}} c\left([X,e_i]\right)\cdot c(e_i) + \frac{1}{8}\sum_{i=\dim \mathfrak{k}+1}^{\dim \mathfrak{g}} c\left([X,e_i]\right)\cdot c(e_i), \quad X \in \mathfrak{p}.$$

Note that

$$e_1, \cdots e_{\dim\mathfrak{k}}, \quad \text{and} \quad \sqrt{-1}e_{\dim\mathfrak{k}+1}, \cdots, \sqrt{-1}e_{\dim\mathfrak{g}}$$

give an orthonormal basis for $\mathfrak{g}_\mathbb{C}$ with respect to the inner product $\langle\ ,\ \rangle$. The corresponding Dirac operator on $L^2(G, S_\mathfrak{g})$ is given by

$$D_G = \sum_{i=1}^{\dim \mathfrak{k}} c(e_i)\nabla_{e_i}^{S_\mathfrak{g}} - \sum_{i=\dim \mathfrak{k}+1}^{\dim \mathfrak{g}} c(e_i)\nabla_{e_i}^{S_\mathfrak{g}}.$$

As an element in $\mathcal{U}(\mathfrak{g}_\mathbb{C}) \otimes \mathrm{Cliff}(\mathfrak{g}_\mathbb{C})$,

$$\begin{aligned} D_G = & \sum_{i=1}^{\dim \mathfrak{k}} e_i \otimes c(e_i) - \sum_{i=\dim \mathfrak{k}+1}^{\dim \mathfrak{g}} e_i \otimes c(e_i) \\ & + \frac{1}{8}\sum_{j=1}^{\dim \mathfrak{k}}\sum_{i=1}^{\dim \mathfrak{k}} c\left([e_j,e_i]\right)\cdot c(e_i)c(e_j) + \frac{3}{8}\sum_{j=1}^{\dim \mathfrak{k}}\sum_{i=\dim \mathfrak{k}+1}^{\dim \mathfrak{g}} c\left([e_j,e_i]\right)\cdot c(e_i)c(e_j) \\ & - \frac{1}{8}\sum_{j=\dim \mathfrak{k}+1}^{\dim \mathfrak{g}}\sum_{i=1}^{\dim \mathfrak{k}} c\left([e_j,e_i]\right)\cdot c(e_i)c(e_j) + \frac{1}{8}\sum_{j=\dim \mathfrak{k}+1}^{\dim \mathfrak{g}}\sum_{i=\dim \mathfrak{k}+1}^{\dim \mathfrak{g}} c\left([e_j,e_i]\right)\cdot c(e_i)c(e_j) \end{aligned} \tag{A.1}$$

Let us introduce a *cubic term*

$$\phi^\mathfrak{k} = \frac{1}{4}\sum_{j=1}^{\dim \mathfrak{k}}\sum_{i=1}^{\dim \mathfrak{k}} c\left([e_j,e_i]\right)\cdot c(e_i)c(e_j) = \sum_{i=1}^{\dim \mathfrak{k}} \mathrm{ad}^\mathfrak{k}(e_j)\cdot c(e_j)$$

and a *torsion term* by

$$\Omega = \frac{1}{4}\sum_{j=1}^{\dim \mathfrak{k}}\sum_{i=\dim \mathfrak{k}+1}^{\dim \mathfrak{g}} c\left([e_j,e_i]\right)\cdot c(e_i)c(e_j) = \sum_{i=1}^{\dim \mathfrak{k}} \mathrm{ad}^\mathfrak{p}(e_j)\cdot c(e_j)$$

By straightforward computation, we check that

$$\frac{1}{4}\sum_{j=\dim \mathfrak{k}+1}^{\dim \mathfrak{g}}\sum_{i=1}^{\dim \mathfrak{k}} c\left([e_j,e_i]\right)\cdot c(e_i)c(e_j) = \Omega$$

and

$$\frac{1}{4}\sum_{j=\dim \mathfrak{k}+1}^{\dim \mathfrak{g}}\sum_{i=\dim \mathfrak{k}+1}^{\dim \mathfrak{g}} c\left([e_j,e_i]\right)\cdot c(e_i)c(e_j) = \Omega.$$

Thus, (A.1) can be simplified as follow

$$D_G = \left(\sum_{i=1}^{\dim \mathfrak{k}} e_i \otimes c(e_i) + \frac{1}{2}\phi^{\mathfrak{k}} \right) - \sum_{i=\dim \mathfrak{k}+1}^{\dim \mathfrak{g}} e_i \otimes c(e_i) + \frac{3}{2}\Omega.$$

If we identify

$$L^2(G, S_{\mathfrak{g}}) \cong \left[L^2(G) \otimes S_{\mathfrak{p}} \otimes L^2(K, S_{\mathfrak{k}}) \right]^K,$$

the split Dirac operator is given by

$$\begin{aligned} D_{G,\mathrm{split}} &= D_{G,K} \otimes 1 + 1 \otimes D_K \\ &= - \sum_{i=\dim \mathfrak{k}+1}^{\dim \mathfrak{g}} e_i \otimes c(e_i) + \left(\sum_{i=1}^{\dim \mathfrak{k}} e_i \otimes c(e_i) + \frac{1}{2}\phi^{\mathfrak{k}} \right) \end{aligned}$$

Therefore, the difference between the Dirac operator associated to the Clifford connection and split Dirac operator is given by the torsion term $\frac{3}{2} \cdot \Omega$ (see also [2, Theorem 10.38]).

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