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Asymptotic and validity problems for Vlasov-type equations

by

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triste in un treno all'alba: aver veduto
fuori la luce incerta: aver sentito
nel corpo rotto la malinconia
vergine e aspra dell'aria pungente.

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improvvisa è più dolce: a me vicino
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Abstract

In this thesis we study some asymptotic and validity problems concerning Vlasov-type equations. In the first part of the work we focus on the Landau damping and the long-time behavior of solutions of the Vlasov-HMF and Vlasov-Poisson equations. We do it by looking at the scattering problem, where the asymptotic datum is fixed, in the style of the work [17, Bibliog. Part I]. In the second part of the thesis we focus on validity problems for kinetic equations with topological interaction. This interaction does not depend on the metric distance but rather on the proximity rank among the agents and, in the last decade, has been widely used to describe biological systems that exhibit collective behaviors.

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Introduction

This thesis deals with the mathematical theory of various partial differential equations arising as models of phenomena belonging to two different branches of physics: plasma physics and the physics of complex systems. These models share the feature that they all come from kinetic theory and they use a statistical description of the system.

Most of the PDE models studied in this work can be englobed in a unique definition, giving them the name of Vlasov-type equations. They read as

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + \nabla_v \cdot \Big(\mathcal{K}[f](t,x,v) f(t,x,v)\Big) = 0, \tag{1}$$

where $f_t = f(t,x,v) : \mathbb{R} \times X \times \mathbb{R}^d$ is the distribution function of the agents in the system having position x in the configuration space X and velocity $v \in \mathbb{R}^d$ at time $t \in \mathbb{R}$. The functional $\mathcal{K}[f] : \mathbb{R} \times X \times \mathbb{R}^d \to \mathbb{R}$ will be the nonlocal mean-field interaction. This is obtained by averaging the individual interactions between agents with the distribution given by f. In the following we will specify its definition from time to time.

Concerning these equations, we focus on two main aspects:

- the asymptotic behavior of solutions of Vlasov-type equations coming from plasma physics and, in particular, the phenomenon of Landau damping;
- the rigorous derivation of Vlasov-type equations from microscopic particle systems modeling collective phenomena with agents interacting via "topological" interaction, a type of irregular interaction presenting jump discontinuities.

This work is organized into two parts, each concerning one of these two topics.

The configuration space X will depend on the problem addressed. For the plasma physics problems of the first part of the thesis we work on the d-dimensional torus, i.e. the periodic box $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$, while in the case of collective dynamics problems of the second part we work in the d-dimensional Euclidean space $\mathbb{R}^d$.

Part I - Scattering approach to the Landau damping

In the first part of this work, we consider the Vlasov-Poisson equation, the main kinetic equation used to describe collisionless plasmas of electrons. In this case

$$\mathcal{K}[f](t,x) = -\nabla_x \Big(\int_{\mathbb{T}^d \times \mathbb{R}^d} W(x-y) f(t,y,v) \, \mathrm{d}y \, \mathrm{d}v \Big) \tag{2}$$

where W is the fundamental solution of the Laplace operator in $\mathbb{T}^d$, i.e. the Coulomb potential on $\mathbb{T}^3$ if $d = 3$. We consider stationary regular solutions depending only on the velocities and we perturb them in suitable functional spaces with high regularity.

Already in the '40s, L. Landau in [26, Bibliog. Part I], considered an analytic stationary state η which satisfies a precise stability condition, and noticed that the perturbed solution of the linearized equation around η relaxes asymptotically towards a new equilibrium causing the electric field to decay exponentially. This phenomenon is now called Landau damping.

In this thesis, we focus on the nonlinear Landau damping for (1), (2) and on a scattering approach for the existence of damped solutions introduced by E. Caglioti and C. Maffei in [17, Bibliog. Part I]. The goal is to understand the relationship between the backward scattering result and the forward result for the Cauchy problem, developed by C. Mouhot and C. Villani in [36, Bibliog. Part I].

The analysis will concentrate especially on some resonances present in the equation and experimentally observed, called plasma echoes. Let us briefly explain what they are, considering the one-dimensional case.

Referring to equation (1), (2), passing to Fourier transform and considering the equation for the density $\rho(t,x) = \int f(t,x,v) \, \mathrm{d}v$, we will deal with a mode-by-mode equation of the form

$$\widehat{\rho}_t(n) = \text{linear terms} + \sum_{k \neq 0} \int_t^{+\infty} \widehat{\rho}_s(k) \frac{n}{k} (s-t) \widehat{h}_s(n-k, nt-ks) \, \mathrm{d}s, \tag{3}$$

where $k, n \in \mathbb{Z}$ and $\widehat{\rho}$ denotes the Fourier transform of ρ (see (1.7) for the notations) and $h(t,x,v) = f(t, x+vt, v)$.

The modes $\widehat{h}$ should decay with a rate that depends on the regularity of f, but notice that, when $nt \approx ks$ in (3), the corresponding term has no decay and thus we expect that, if $\frac{n}{k} > 1$, at times $\tau = \frac{nt}{k}$ the density $\widehat{\rho}_\tau(k)$ would strongly influence $\widehat{\rho}_t(n)$. This effect is called plasma echo and has been experimentally observed in the '60s by J. H. Malmberg et al. in [32, Bibliog. Part I].

From a mathematical point of view, in the Eulerian approach, the aim is to obtain global in-time regularity estimates on the solutions using norms quantifying the decay of its Fourier transform. Plasma echoes make it challenging to close the a priori estimates using the cited norms. In general the validity of the damping depends on the regularity setting, and in particular on the choice of data with analytic or Gevrey regularity, an intermediate class between analytic and C^∞ functions.

To study this issue, after reviewing some results on the Landau damping in Chapter 1, we present two works.

Comparative study for Vlasov-HMF equation In Chapter 2, we consider a simplification of the Vlasov equation, called Vlasov-HMF (Hamiltonian mean-field model) model. In this case $d = 1$ and

$$\mathcal{K}[f](t,x) = -\partial_x \Big(\int_{\mathbb{T}^1 \times \mathbb{R}} \cos(x-y) f(t,y,v) \, \mathrm{d}y \, \mathrm{d}v \Big).$$

This approximated model has been widely studied in the last decades being a handy reduction of the Vlasov-Poisson equation, in which the singularity of the kernel is removed by replacing it with a cosine function. It can be easily implemented numerically to study the features of a long-range interaction (see [1, 5, 18, Bibliog. Part I]). Furthermore, as we will see, it is also a useful testing ground from a mathematical point of view for studying issues about long-time behavior of solutions.

However, in this case the resonances due to plasma echoes are few: in fact, the equation verified by the given density ρ is not the one (3) but

$$\widehat{\rho}_t(n) = \text{linear terms} + \sum_{k=\pm 1} \int_t^{+\infty} \rho_s(k)(s-t) h_s(n-k, nt-ks) \, \mathrm{d}s, \quad n = \pm 1.$$

So we have that $nt - ks = 0$ only when $s = \pm t$. This simplifies the treatment.

For this model, we adapt the Eulerian forward techniques to the backward problem to make a comparison in the case of analytic solutions. What results is that in the backward approach the a priori estimates on the solutions of the equation are greatly simplified by the exponential decay guaranteed by the analytic regularity. In the backward case, this also allows to provide a very precise rate that describes the evolution of the regularity of the solution over time.

We also prove a nonperturbative result, through a more accurate analysis of nonlinear terms.

Backward approach for the Vlasov-Poisson equation with Gevrey data In Chapter 3, we focus on the scattering problem for the one-dimensional Vlasov-Poisson equation given by (1), (2). We consider analytic and Gevrey asymptotic data, the latter case not covered in [17, Bibliog. Part I].

In this work, a function $f \in L^2(\mathbb{T}^d \times \mathbb{R}^d)$ belongs to the $1/\gamma$-Gevrey class with regularity parameter λ if

$$\|f\|^2_{\lambda;\gamma;\sigma} := \sum_{n \in \mathbb{Z}^d} \int_{\mathbb{R}^d} e^{2\lambda \langle n,\xi \rangle^\gamma} \langle n,\xi \rangle^{2\sigma} \, |\widehat{f}(n,\xi)|^2 \, \mathrm{d}\xi < +\infty, \tag{4}$$

where $\widehat{f}$ is the Fourier transform of f, $\langle n,\xi \rangle := (1+|\xi|^2+|\eta|^2)^{1/2}$ and $\sigma > 0$ is fixed.

Notice that, if (4) holds with γ=1, then the function f has analytic regularity. In general, this class of functions has some properties - among which the existence of Gevrey functions with compact support - which make their use in the study of evolutive PDEs very convenient (see [20, 42, 19, Bibliog. Part I]).

Also in this Chapter we use forward techniques for the backward perturbative problem, allowing, in the analytic case, to overcome the plasma echoes mechanism with a simple proof. Moreover, we extend the proof to asymptotic data of $1/\gamma$-Gevrey regularity with

$\gamma > 1/3$, and we recover the 3-Gevrey threshold for the existence of damped solutions that was found in the case of the Cauchy problem by [36, 9, Bibliog. Part I].

Nonetheless, our scattering approach makes it clear that the plasma echoes mechanism is a secondary linear effect. This fact allows us to formally argue that the linear part of the backward equation is ill-posed for data with Gevrey regularity less than $1/3$.

Part II - Mean-field limit and propagation of chaos for particle systems with topological interaction

The second part of the thesis deals with the rigorous derivation of effective kinetic equations from deterministic and stochastic particle dynamics at the microscopic level.

In Chapter 4, we review some validity results for systems with regular interactions in the mean-field scaling. This is the so-called mean-field limit, i.e. we consider the interaction intensity scaling with $1/N$ and the density of particles diverging with their number N, to obtain an effective kinetic equation in the limit $N \to \infty$.

In Chapters 5 and 6, we focus on the derivation of kinetic equations for models coming from the physics of complex systems, in particular from the area of collective dynamics, focusing on interactions that are called "topological". In these topological models, the strength of the interaction between two agents x_i and x_j is a function of the proximity rank $R(x_i, x_j)$ of x_j with respect to x_i:

$$R(x_i, x_j) \coloneqq \sum_{k \neq i} \mathcal{X}\{|x_i - x_k| \leqslant |x_i - x_j|\}, \tag{5}$$

where $\mathcal{X}\{|x_i - x_k| \leqslant r\}$ is the characteristic function of the set $\{|x_i - x_k| \leqslant r\}$. So $R(x_i, x_j)$ counts the number of agents at distance less than or equal to $|x_i - x_j|$ from x_i.

This type of interaction is widely used in biophysics to describe the collective behavior of flocks of birds, fish schools and swarms. From a mathematical point of view it presents some nontrivial difficulties since the methods usually adopted in validity problems fail due to the exotic properties of the topological interaction: it is not Lipschitz continuous having jump-like discontinuities and it is not a pair interaction since it depends on the state of the other agents.

We present two results that investigate these two aspects.

Mean-field limit for a topological Cucker-Smale model In Chapter 5, we study the following mean-field system

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} K\Big(\frac{R(x_i, x_j)}{N}\Big)(v_j(t) - v_i(t)), \end{cases} \qquad i = 1, \dots, N \tag{6}$$

where $(x_i(t), v_i(t)) \in \mathbb{R}^d \times \mathbb{R}^d$, $R(x_i, x_j)$ is the proximity rank in (5) and $K : [0,1] \to \mathbb{R}^+$ is a regular nondecreasing function.

This is the so-called topological Cucker-Smale model introduced by J. Haskovec in [32, Bibliog. Part II]. The interaction is such that neighboring birds tend to align their velocities but with weights given by $K(R(x_i, x_j)/N)$. It can be proved that under suitable assumptions on the initial datum, this model exhibits asymptotic consensus in the velocities.

Here we want to prove a mean-field limit result for this topological model. We do this by studying the empirical measure related to (6):

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(t)}\delta_{v_i(t)},$$

where δ is the Dirac delta measure.

Having the interaction function a discontinuity of jump-type, there are several problems in the study of this model. In particular, the dynamics of N particles does not fall within the classical Cauchy-Lipschitz theory and it is not clear whether and when the dynamics is well-defined.

Furthermore, at least formally, we expect that in the limit $N \to \infty$ we recover a Vlasov-type equation (1) with

$$\mathcal{K}[f](t,x,v) = \int K(M[\rho](x, |x-y|))(v-w) f(t,y,w)\,\mathrm{d}y\,\mathrm{d}w, \tag{7}$$

where

$$M[\rho](t,x) = \int_{|x'-x|\leqslant r} \rho(t,x')\,\mathrm{d}x' \tag{8}$$

and $\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v$. Nevertheless, the rigorous proof of the mean-field limit is not trivial, since it does not fit into the so-called Dobrushin theory (see Chapter 4), which requires a Lipschitz-type interaction.

Hence, in this Chapter we prove the following results:

- the N-particle dynamics is well-defined, except for a set of measure zero;
- if f_0 is bounded, there exists a unique weak solution f_t of the topological Cucker-Smale equation (1) (7) with initial datum f_0, which is bounded;
- μ_t^N weakly converge to f_t, provided this is true at time 0.

Propagation of chaos for a jump process In Chapter 6, a stochastic process describing alignment via topological interaction is studied.

In this model, particles $\{(x_i, v_i)\}_{i=1}^N$ go freely, namely following the trajectories $x_i + v_i t$. At some random time dictated by a Poisson process of intensity N, a particle (say i) is chosen with probability $\frac{1}{N}$ and a partner particle (say j) with probability $\pi_{i,j}$ equal to

$$\pi_{i,j} := \frac{K\Big(R(x_i,x_j)/N\Big)}{\sum_{s=1}^{N-1} K(\frac{s}{N-1})},$$

where $K : \mathbb{R}^d \to \mathbb{R}$ is a regular function and $R(x_i, x_j)$ is the proximity rank defined in (5). Then the transition $(v_i, v_j) \to (v_j, v_j)$ is performed. After that, the system goes freely with the new velocities and so on.

In [7, Bibliog. Part II], the authors derived formally that the kinetic equation expected to be valid in the limit $N \to \infty$ is

$$\Big(\partial_t + v \cdot \nabla_x\Big) f(t,x,v) = -f(t,x,v) + \rho(t,x) \int K(M[\rho](x,|x-y|)) f(t,y,v)\,\mathrm{d}y \quad (9)$$

where $M[\rho](x,r)$ is defined as in (8).

We underline that equation (9) cannot be reduced to the Vlasov-type equations in (1) and it is more correct to see it as a Boltzmann-type equation coming from a collisional stochastic model with a gain and loss term.

The rigorous derivation of the kinetic equation from the N-particle system has been done in a later work [21, Bibliog. Part II]. The authors prove the result, comparing the N-particle and the limit processes using the BBGKY hierarchies, assuming that the interaction function K is real analytic.

To avoid this assumption, we present here a more natural proof that improves the previous result, using a classical coupling technique instead of the hierarchies and assuming K to be only Lipschitz continuous.

Part I

Scattering approach to the Landau Damping

Chapter 1

A brief overview on some results about Landau damping

In this first Chapter we begin the study of the asymptotic behavior of some specific solutions of the Vlasov-Poisson equation.

After having recalled the basic properties of the Vlasov-Poisson equation, we briefly review two important results about the theory of Landau damping: the linearized result carried out by L. Landau in his pioneering work [26] and the proof of the nonlinear damping for the one-dimensional Vlasov-Poisson equation, obtained by E. Caglioti and C. Maffei in [17] using the scattering approach.

1.1 Vlasov-Poisson equation: basic properties

In the kinetic theory of plasmas, the Vlasov-Poisson equation is a nonlinear partial differential equation that describes the time evolution of the distribution function of electrons in a collisionless globally neutral plasma. It was first introduced by A. Vlasov in 1938 in [43].

Denoting $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ the d-dimensional torus, the Vlasov-Poisson equation reads as

$$\begin{cases} \partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + \mathcal{F}[f](t,x) \cdot \nabla_v f(t,x,v) = 0, \\ \mathcal{F}[f](t,x) = -\nabla_x V[f](t,x), \quad \Delta_x V[f](t,x) = -\rho(t,x) + \bar{\rho}, \\ \rho(t,x) = \displaystyle\int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v. \end{cases} \tag{1.1}$$

Here $f(t,x,v) : \mathbb{R} \times \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}^+$ is the distribution function of electrons having at time $t \in \mathbb{R}$ position $x \in \mathbb{T}^d$ and velocity $v \in \mathbb{R}$.

Due to the compatibility condition for the Poisson equation in (1.1) we are subtracting to ρ its mean over $\mathbb{T}^d$ called $\bar{\rho}$. This is physically justified by saying that we are considering as a fixed background the density of ions and that, since the system has to be globally neutral, this is equal to the mean density of electrons.

Equation (1.1) can be seen as a nonlinear Liouville equation where $\mathcal{F}[f](t,x)$ is the mean-field force generated by the spatial density $\rho(t,x)$ of electrons. Notice that

$$V[f](t,x) = \int_{\mathbb{T}^d\times\mathbb{R}^d} W(x-y)f(t,y,w)\,\mathrm{d}y\,\mathrm{d}w,$$

where W is the fundamental solution of the Laplace operator on $\mathbb{T}^d$.

The study of the Cauchy problem for (1.1) has produced a huge literature. We mention here the works by Arsen'ev on the existence of weak solutions and of classical solutions for short times ([2], [3]) and the theory of Pfaffelmoser [40] and Lions-Perthame [29] for the existence of global classical solutions. See [21] or [15] for a review.

Let $f(t,x,v)$ be a regular solution of equation (1.1) and let us consider the flow $\Phi_t(x,v) = (X(t,x,v),V(t,x,v))$, defined in the phase space by the following characteristics

$$\begin{cases} \dot{X}(t,x,v) = V(t,x,v) \\ \dot{V}(t,x,v) = \mathcal{F}[f](t,X(t,x,v)) \\ X(0,x,v) = x \quad V(0,x,v) = v. \end{cases} \tag{1.2}$$

Then, it is easy to see that f is conserved along the flow, i.e.

$$f(t,X(t,x,v),V(t,x,v)) = f(0,x,v)$$

and so, denoting $f_0(x,v) = f(0,x,v)$,

$$f(t,x,v) = f_0((\Phi_t)^{-1}(x,v)). \tag{1.3}$$

Clearly this is only a representation formula, since to determine the flow $\Phi_t(x,v)$ and solve (1.2), one must already know the solution f.

As first properties of equation (1.1), we have the classical conserved quantities:

$$\mathcal{M}[f] = \int f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v,$$
$$\mathcal{P}[f] = \int vf(t,x,v)\,\mathrm{d}x\,\mathrm{d}v,$$
$$\mathcal{E}[f] = \mathcal{T}[f] + \mathcal{V}[f],$$

where

$$\mathcal{T}[f] = \frac{1}{2}\int |v|^2 f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \tag{1.4}$$
$$\mathcal{V}[f] = \frac{1}{2}\int W(x-x')\rho(x,t)\rho(x',t)\,\mathrm{d}x\,\mathrm{d}x'.$$

Moreover, as a consequence of (1.3), given an arbitrarily regular function $G:\mathbb{R}\to\mathbb{R}$, the quantity

$$\int G(f(t,x,v))\,\mathrm{d}x\,\mathrm{d}v$$

is conserved. In particular, considering the entropy function $G(x) = x\log(x)$, we get

$$\int f(t,x,v)\log(f(t,x,v))\,\mathrm{d}x\,\mathrm{d}v$$

is constant. Hence the entropy of the system is preserved.

1.1.1 Stationary solutions

The Vlasov-Poisson equation (1.1) admits infinitely many stationary states. Indeed, as can be easily seen, any spatially homogeneous distribution function $\eta(v)$ is a stationary solution of equation (1.1). This follows from

$$F[\eta](x,t) = \nabla_x \Big(\int_{\mathbb{T}^d \times \mathbb{R}^d} W(x-y)\eta(w)\, \mathrm{d}y\, \mathrm{d}w \Big) = 0.$$

It is natural to study the stability of these equilibria, and in this first part of the thesis we will address some aspects of this broad problem. For now, we mention an important theorem of C. Marchioro and M. Pulvirenti (see [33, 35]) which proves that a homogeneous function $\eta(|v|)$ nonincreasing in $|v|$ is an orbitally stable solution of the Vlasov-Poisson equation.

Theorem 1.1. *Let $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$ and consider an homogeneous solution $\eta(|v|)$ such that $\mathcal{M}[\eta] + \mathcal{T}[\eta] < +\infty$ (see (1.4)), where $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing function. Given $\varepsilon > 0$, let $f_0(x,v)$ an initial datum such that*

$$\int |f_0(x,v) - \eta(v)|(1+v^2)\, \mathrm{d}x\, \mathrm{d}v < \varepsilon,$$

then

$$\sup_{t\in\mathbb{R}} \int |f(t,x,v) - \eta(v)|\, \mathrm{d}x\, \mathrm{d}v < \delta(\varepsilon),$$

where $f(t,x,v)$ is solution[1] *of (1.1) with initial datum f_0 and $\delta(\varepsilon) \to 0$ if $\varepsilon \to 0$.*

We don't give the proof of this result, which follows from rearrangement techniques.

Beyond the homogeneous ones, there exist other types of stationary solutions. In 1951, I. B. Bernstein, J. M. Greene e M. D. Kruskal in [14] proved the existence of stationary solutions of (1.1) $g(x,v)$ which are not spatially homogeneous. These stationary states are called BGK waves, since, being the Vlasov equation invariant under Galilean transformations, if $g(x,v)$ is a stationary solution, then the wave $g(t,x,v) := g(x-ct, v-c)$ solves the equation (1.1) for any $c \in \mathbb{R}$.

We observe that, given the Vlasov equation (1.1), any state $g(x,v)$, obtained composing a smooth function G with the mesoscopic energy of the system, i.e.

$$g(x,v) = G\big(E(x,v)\big), \quad \text{where} \quad E(x,v) = \frac{v^2}{2} + V[g](x),$$

gives rise to a stationary solution, provided that the following compatibility condition is verified

$$\int W(x-y) G\Big(\frac{v^2}{2} + V[g](y)\Big)\, \mathrm{d}y\, \mathrm{d}v = V[g](x).$$

For a proof of the existence of BGK solutions, we refer to section (2.2) of the next Chapter, where in Remark (2.2) this is done in the case of the HMF approximation of the Vlasov-Poisson equation.

[1] Here we are assuming that a given solution exists. Clearly it depends on the aforementioned results about the well-posedness theory.

Furthermore, we mention that in [27], Z. Lin and C. Zeng proved the existence of solutions of BGK type for the Vlasov-Poisson equation in any small neighborhood of a homogeneous stationary solution $\eta(v)$ with low Sobolev regularity. As will be clear from the next section, this result implies that Landau Damping doesn't hold around homogeneous solutions belonging to these functional spaces.

1.2 Linear Landau Damping

In 1946, L. Landau in [26], linearizing the Vlasov-Poisson equation around a suitable analytic spatially homogeneous equilibrium $\eta(v)$, predicted the existence of damped solutions near the given stationary regular state, proving that the electric field of the plasma decays exponentially, so that the flow governed by the mean-field force is asymptotically free.

This decay of the electric field would have been experimentally observed only eighteen years later by J. H. Malmberg and C. B. Wharton in [31]. A similar phenomenon also occurs in the dynamics of galaxies and it was observed by the astrophysicist D. Lynden-Bell in [30].

Here we give an idea of the linear Landau damping, a classical result in the literature. We consider solutions of the form

$$f(t,x,v) = \eta(v) + \epsilon \bar{f}(t,x,v), \tag{1.5}$$

where $\bar{f}$ will be the perturbation and $\epsilon > 0$ its size.

We start by writing the linearized Vlasov-Poisson equation around the homogeneous equilibrium $\eta(v)$. Let $\bar{f}(t,x,v)$ as in (1.5), replacing it in equation (1.1) and by linearizing it, i.e. neglecting the terms of $O(\epsilon^2)$, we obtain the linearized Vlasov equation

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + \mathcal{F}[f](t,x) \cdot \nabla_v \eta(v) = 0, \tag{1.6}$$

where from now on, we drop the sign over the perturbation and denote it by $f(t,x,v)$.

Landau showed that it is possible to exactly solve this linearized equation. For this purpose, let $\widehat{f}_t$ be the Fourier transform of f_t in both positions and velocities, i.e

$$\widehat{f}_t(n,\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d \times \mathbb{R}^d} e^{-\mathrm{i} n \cdot x} e^{-\mathrm{i}\xi \cdot v} f(t,x,v)\, \mathrm{d}x\, \mathrm{d}v, \tag{1.7}$$

with $n \in \mathbb{Z}^d$ and $\xi \in \mathbb{R}^d$.

Using the Duhamel formula and applying the Fourier transform to (1.6), we get

$$\widehat{f}_t(n,\xi) = \widehat{f}_0(n,\xi + nt) - \widehat{W}(n) \int_0^t \widehat{f}_s(n,0) \widehat{\eta}(\xi + n(t-s)) [n \cdot (\xi + n(t-s))]\, \mathrm{d}s. \tag{1.8}$$

Since $\widehat{\rho}_t(n) = \widehat{f}_t(n,0)$, setting $\xi = 0$ in (1.8), we obtain a closed equation in $\widehat{\rho}_t(n)$ for $n \neq 0$

$$\widehat{\rho}_t(n) = \widehat{f}_0(n,nt) - \widehat{W}(n) \int_0^t \widehat{\rho}_s(n) \widehat{\eta}(n(t-s)) |n|^2 (t-s)\, \mathrm{d}s. \tag{1.9}$$

We observe instead that for $n = 0$, $\widehat{\rho}_t(0) = \bar{\rho}$ is a conserved quantity.

Equation (1.9) is a Volterra integral equation of the form

$$\widehat{\rho}_t(n) = \widehat{f}_0(n, tn) + \int_0^t j_n(t-s)\widehat{\rho}_s(n)\,\mathrm{d}s, \tag{1.10}$$

where j is the following kernel

$$j_n(t) \coloneqq -\widehat{W}(n)\widehat{\eta}(nt)|n|^2 t.$$

Fixed n, the (1.10) is of the form

$$\alpha(t) = \beta(t) + \int_0^t j(t-s)\alpha(s)\,\mathrm{d}s, \tag{1.11}$$

for $\alpha, \beta \in L^1(\mathbb{R})$. This integral equation can be solved by using the Laplace transform

$$\mathcal{L}[\alpha](\sigma) = \int_0^{+\infty} e^{-\sigma t}\alpha(t)\,\mathrm{d}t, \quad \sigma \in \mathbb{C},$$

well-defined for $\Re(\sigma) \geqslant 0$. By applying the Laplace transform to the equation (1.11) we obtain that

$$\mathcal{L}[\alpha](\sigma) = \mathcal{L}[\beta](\sigma) + \mathcal{L}[j](\sigma)\mathcal{L}[\alpha](\sigma)$$

and so

$$\mathcal{L}[\alpha](\sigma) = \frac{\mathcal{L}[\beta](\sigma)}{1 - \mathcal{L}[j](\sigma)},$$

which is well defined, provided that $\mathcal{L}[\beta]$ and $\mathcal{L}[j]$ are, and that $\mathcal{L}[j](\sigma) \neq 1$.

To reconstruct $\alpha(t)$ from its Laplace transform it is necessary to integrate $\mathcal{L}[\alpha]$ on a path of the complex plane suitably chosen. In this regard we refer the reader to the next two chapters and the reference [37]. We state the following lemma, the proof of which is similar to the one given in Lemma (3.1).

Lemma 1.1. *Let $j(t)$ be an integral kernel defined for $t \geqslant 0$, such that*

$$|j(t)| \leqslant c_0 e^{-\lambda_0 t}$$

with positive constants c_0 and λ_0. Given $\kappa > 0$, suppose further that $|\mathcal{L}[j](\sigma) - 1| \geqslant \kappa$ for each $\sigma \in \mathbb{C}$ such that $\Re\sigma \geqslant 0$.

Moreover, let $\beta(t)$ be an analytic function such that $|\beta(t)| \leqslant c_1 e^{-\lambda t}$, where c_1 and λ are positive constants and let $\alpha(t)$ be a solution of the equation

$$\alpha(t) = \beta(t) + \int_0^t j(t-s)\alpha(s)\,\mathrm{d}s.$$

Then, for $\lambda' < \min\{\lambda_0, \lambda\}$,

$$|\alpha(t)| \leqslant c e^{-\lambda' t}.$$

Let's apply the Lemma (1.1) to our equation (1.9): the kernel $j_n(t)$ decays exponentially provided that $\eta(v)$ is an analytic stationary solution. Similarly $\widehat{f}_0(n, nt)$ decays exponentially if the initial data f_0 is analytic. We just have to require the following stability condition, sometimes called Penrose condition ([39]),

$$\inf_{n \in \mathbb{Z}^d,\, \Re\sigma \geqslant 0} |\mathcal{L}[j_n](\sigma) - 1| \geqslant \kappa > 0.$$

This condition is implicit in Landau's work [26] and is due to G. Backus, who was the first to give a complete treatment of the linearized equation in [4].

We therefore observe that, by Lemma (1.1), the solution $\widehat{\rho}_t(n)$ of the integral equation with $n \neq 0$ tends exponentially to zero and therefore $\rho(t,x)$ converges weakly to the value $\bar{\rho}$, so that the force field vanishes.

1.3 Scattering approach for the Landau Damping

Since the work of Landau, the damping phenomenon has been extensively studied and understood, but the extension from the linear to the true nonlinear case has proved to be particularly difficult for the mathematical theory.

Only in 1998, E. Caglioti and C. Maffei in [17] gave a first proof in the case where the domain is the one-dimensional torus $\mathbb{T}^1$. Subsequently, a proof with less restrictive hypotheses was given in [24].

The idea of the proof in [17] is the following one: let $(x,v) \in \mathbb{T}^1 \times \mathbb{R}$, as seen by the linearized analysis we expect the distribution function $f(t,x,v)$ to weakly converge to a homogeneous equilibrium, so that for large times the self-consistent force in (1.1) would be zero and f_t would evolve similarly to the free evolute of a suitable asymptotic density $\omega(x,v)$, i.e. for large times

$$f(t,x,v) \approx \omega(x-vt,v).$$

It is therefore reasonable to consider the scattering problem where, instead of studying the evolution of a given initial datum, one looks for a solution of the equation (1.1) with the asymptotic condition

$$\lim_{t\to+\infty} \|f(t,x,v) - \omega(x-vt,v)\|_{L^\infty(\mathbb{T}^1\times\mathbb{R})} = 0. \tag{1.12}$$

Then, if a solution of the scattering problem (1.12) exists, it is not difficult to prove that it weakly converges to a homogeneous equilibrium, given by the spatial mean of ω.

Indeed, taking a test function $\phi(x,v) \in C_c(\mathbb{T}^1 \times \mathbb{R})$, we have that

$$\int \omega(x-vt,v)\phi(x,v)\,\mathrm{d}x\,\mathrm{d}v = \int \omega(x,v)\phi(x+vt,v)\,\mathrm{d}x\,\mathrm{d}v.$$

Moreover

$$\frac{1}{2\pi}\int_{\mathbb{T}^1\times\mathbb{R}} \phi(x+vt,v)e^{-inx}e^{-\mathrm{i}\xi v}\,\mathrm{d}x\,\mathrm{d}v = \widehat{\phi}(n,\xi+nt),$$

where $\widehat{\phi}$ is the Fourier transform of ϕ as in (1.7). Hence, by Plancherel theorem,

$$\int \omega(x,v)\phi(x+vt,v)\,\mathrm{d}x\,\mathrm{d}v = \int \widehat{\omega}(0,\xi)\widehat{\phi}(0,\xi)\,\mathrm{d}\xi + \sum_{n\neq 0}\int \widehat{\omega}(n,\xi)\widehat{\phi}(n,\xi+nt)\,\mathrm{d}\xi.$$

The first term gives the spatial mean of ω while the second term goes to zero as t goes to infinity, by dominated convergence.

From now on, let

$$B(x) = -\frac{x}{2\pi} + \frac{1}{2}, \quad x \in [0,2\pi)$$

and periodically extended in $\mathbb{R}$, the fundamental solution of the problem

$$\partial_x B(x) = \delta(x) - \frac{1}{2\pi}$$

with $x \in \mathbb{T}^1$, where the constant $\frac{1}{2\pi}$ has been added to make the system globally neutral.

We want to prove the existence of solutions to the backward problem (1.12) in the following sense. Let $\tau \geqslant 0$ and $\omega(x, v)$ be a regular function. A weak formulation of the Vlasov-Poisson equation (1.1) with asymptotic condition (1.12) is given requiring that

$$f(t, x, v) = \omega((\Phi_t[f])^{-1}(x, v)) \tag{1.13}$$

for $t \geqslant \tau$, where $\Phi_t[f](x, v) = (X[f](t, x, v), V[f](t, x, v))$ solves

$$\begin{cases} \dot{X}[f] = V[f] \\ \dot{V}[f] = \mathcal{F}[f](t, X[f], V[f]) \end{cases}$$

with asymptotic conditions

$$\begin{cases} \lim_{t\to\infty} X[f] - V[f]t = x \\ \lim_{t\to\infty} V[f] = v \end{cases}$$

and the force field must verify

$$\mathcal{F}[f](t, x) = \int_{\mathbb{T}^1} B(x - y) f(t, y, v) \,\mathrm{d}y \,\mathrm{d}v.$$

For the thesis of the theorem to be true, the asymptotic datum ω must belong to a suitable space of functions: we say that $\omega \in S_{\lambda,c_1,c_2}$ if $\omega \geqslant 0$ and there exist positive constants λ, c_1, c_2 such that

$$|\widehat{\omega}(n, \xi)| \leqslant \frac{c_1}{1 + n^2} e^{-\lambda|\xi|}.$$

and

$$\omega(x, v) \leqslant \frac{c_2}{1 + v^4}.$$

We will follow an iterative strategy. Given $\omega \in S_{\lambda,c_1,c_2}$, we solve the sequence of linear problems

$$\partial_t f^{(n)}(t, x, v) + v f^{(n)}(t, x, v) + F^{(n-1)}(t, x)\partial_v f^{(n)}(t, x, v) = 0, \tag{1.14}$$

$$\|f^{(n)}(t, x, v) - \omega(x - vt, v)\|_{L^\infty(\mathbb{T}^1\times\mathbb{R})} \to 0$$

for $n \geqslant 1$. Here the force fields are given by

$$F^{(n)}(t, x) = \int_{\mathbb{T}^1} B(x - y)\rho^{(n)}(t, y) \,\mathrm{d}y, \quad F^{(0)} = 0. \tag{1.15}$$

Then the aim is to prove that the sequence $\{f^{(n)}\}$ converges to a solution of the Vlasov-Poisson equation in the sense of (1.13).

Before doing this, we state a lemma that ensures the well-posedness of the linear problem in (1.14). For a given field $F(t, x)$ we define

$$\|F\|_{\lambda,\tau} = \sup_{t\geqslant\tau} e^{\lambda t} \|F(t, \cdot)\|_{L^\infty(\mathbb{T}^1)}.$$

Lemma 1.2. *Let $\tau \geqslant 0$ e $F(t,x) \in C(\mathbb{T}^1 \times [\tau, +\infty))$ a Lipschitz force field with Lipschitz constant L_F. Moreover suppose that $\|F\|_{\lambda,\tau} < \infty$. Then, if $\lambda > \sqrt{L_F}$ and $t \geqslant \tau$, there exists a unique solution of the flow $\Phi_t(x,v) = (X(t,x,v), V(t,x,v))$ that solves*

$$\begin{cases} \dot{X} = V \\ \dot{Y} = F(t, X, V) \end{cases}$$

with asymptotic conditions $\lim_{t\to\infty}(X - Vt, V) = (x, v)$.

Moreover the flow $\Phi_t(x,v)$ is Holder-continuous in the asymptotic datum and defining

$$f(t,x,v) = \omega((\Phi_t)^{-1}(x,v))$$

for $\omega \in S_{\lambda,c_1,c_2}$, it holds that f is a weak solution of

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) + F(t,x)\partial_v f(t,x,v) = 0, \tag{1.16}$$

with asymptotic condition given by (1.12).

Thanks to the previous lemma, given $F(t,x)$ a Lipschitz field with Lipschitz constant L_F such that $\|F\|_{\lambda,\tau} < \infty$ and $\omega \in S_{\lambda,c_1,c_2}$, we can define the operator

$$\mathscr{F}(F)(t,x) = \int_{\mathbb{T}^1\times\mathbb{R}} B(x-y) f(t,y,v)\,\mathrm{d}y\,\mathrm{d}v \tag{1.17}$$

where f solves the linear problem (1.16).

We observe that, thanks to this definition, equation (1.15) can be rewritten as $F^{(n)} = \mathscr{F}(F^{(n-1)})$. Moreover, in the first step of the iterations, $F^{(0)} = 0$ and $f^{(1)}(t,x,v) = \omega(x - vt, v)$ so that

$$\widehat{\rho^{(1)}}_t(n) = \hat{\omega}(n, nt).$$

Hence

$$|\mathscr{F}(0)(t,x)| \leqslant \sum_{n\neq 0} \frac{1}{|n|}|\omega(n,nt)| \leqslant \sum_{n\neq 0} c_1 \frac{1}{|n|}\frac{1}{1+n^2} e^{-\lambda|n|t} \leqslant 4c_1 e^{-\lambda t},$$

that is

$$\|\mathscr{F}(0)\|_{\lambda,\tau} \leqslant 4c_1.$$

A fundamental result for the proof of the theorem is the following proposition, which guarantees the contractivity of the operator $\mathscr{F}$ in the norm $\|\cdot\|_{\lambda,\tau}$.

Proposition 1.1. *Let $\omega \in S_{\lambda,c_1,c_2}$ with $\lambda \geqslant 15\sqrt{c_2}$, for $t > \tau$, $F(t,x)$ be a Lipschitz field with Lipschitz constant $L_F \leqslant 24c_2$ such that $\|F\|_{\lambda,\tau} < \infty$. Then it holds that*

1. $|\mathscr{F}(F)(t,x) - \mathscr{F}(F)(t,x')| \leqslant 16c_2|x - x'|$,
2. $\|\mathscr{F}(F_1) - \mathscr{F}(F_2)\|_{\lambda,\tau} \leqslant \frac{1}{2}\|F_1 - F_2\|_{\lambda,\tau}$,
3. $\|\mathscr{F}(F)\|_{\lambda,\tau} \leqslant 4c_1 + \frac{1}{2}\|F\|_{\lambda,\tau}$.

Thanks to this Proposition and to (1.14) we obtain the convergence of the fields defined in (1.15). Hence we have the following main theorem.

Theorem 1.2. *Let $\omega \in S_{\lambda,c_1,c_2}$ such that $\lambda \geqslant 15\sqrt{c_2}$ and τ sufficiently large, then for $t > \tau$ there exists a weak solution of the Vlasov-Poisson equation in the sense of* (1.13). *Moreover the solution verifies the asymptotic condition*

$$\lim_{t\to\infty} \|f(t,x,v) - \omega(x - vt, v)\|_{L^\infty(\mathbb{T}^1\times\mathbb{R})} = 0,$$

so that the electric field exponentially decays.

The presented scattering result is of nonperturbative type and provides the solution that behaves asymptotically like ω, but does not allow to characterize the initial data for which the Landau damping occurs. In 2009, C. Villani and C. Mouhot in [36], introducing new mathematical techniques, solve the Cauchy problem for the nonlinear Vlasov-Poisson equation, with suitable analytic and Gevrey initial data, proving the existence of $\omega(x, v)$ such that

$$\lim_{t\to+\infty} (f(t,x,v) - \omega(x - vt, v)) = 0. \tag{1.18}$$

A substantial analogy exists between the Landau damping in plasma physics and the inviscid damping for the two-dimensional Euler equation (see [25, 41, 38, 34]). In fact in [8] the damping near the Couette flow has been proved using different techniques, this gives rise to a new simpler proof of the Landau damping result in [9] (see also the recent result in [22] for a more elementary proof). We refer the reader to [7] for a review of the state of the art.

In the next two chapters we want to use the Eulerian techniques developed for the Cauchy problem, to better understand the advantages and possible limitations of the backward approach introduced in [17]. The results of the analysis will strongly depend on the regularity of the asymptotic datum. In particular in the case of analytic regularity the scattering approach demonstrates its effectiveness, since, as we have seen in this section and as will be clear from the next chapters, the decay due to analytic regularity greatly facilitates the proof of Landau damping.

We conclude by saying that, for what concerns the damping with Sobolev data, as shown by Lin and Zeng ([27], [28]), for very low regularities Landau damping cannot occur. Although, in the case of the Vlasov-HMF equation with sufficiently high Sobolev regularity, Faou and Rousset in [18] have succeeded in proving the damping with a polynomial rate. Moreover, in the case of the Kuramoto model, a scattering result with Sobolev regularity has been proved in [10].

However, the Vlasov-Poisson equation differs greatly from its HMF approximation and the Kuramoto model. Indeed, a Landau damping result for the full Vlasov-Poisson equation with general Sobolev data is still missing, although Bedrossian in [6] has given a negative answer to the possibility of a straightforward extension to this setting of Mouhot and Villani's work in [36]. This is mainly due to the analysis of the aforementioned plasma echoes which, as we shall see in Chapter 3, is particularly challenging in the case of equation (1.1).

Chapter 2

Backward vs Forward approach for the Vlasov-HMF model

joint work with D. Benedetto and E. Caglioti ([12])

In this Chapter we present a work where we study the differences between the backward and forward approaches for the Landau damping in the Vlasov-HMF equation. We adapt the forward techniques to the backward problem to make a comparison in the case of analytic solutions. In particular, we discuss the different ways the two approaches overcome the difficulties due to the presence of the "echoes", i.e. resonances at certain times between the Fourier modes of the solution. This highlights a simplified structure of the norms used in the backward approach. Moreover, we also give a nonperturbative result, i.e. without requiring the solution to be a small perturbation of a stationary state.

2.1 The Vlasov-HMF model and the scattering problem

We recall here the Vlasov-Poisson equation in the HMF approximation which, in the spatially periodic case, reads as

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) + \mathcal{F}[f](t,x)\partial_v f(t,x,v) = 0, \tag{2.1}$$

where

$$\mathcal{F}[f](t,x) = -\partial_x \left(\int_{\mathbb{T}^1\times\mathbb{R}} \cos(x-y) f(t,y,v)\, \mathrm{d}y\, \mathrm{d}v \right) \tag{2.2}$$

is the mean-field force. Here $f(t,x,v)$ is the normalized density of electrons with position $x \in \mathbb{T}^1$ and velocity $v \in \mathbb{R}$, in a collisionless electrically neutral plasma.

We consider solutions of (2.1) which are small perturbations of a spatially homogeneous solution η, i.e.

$$f(t,x,v) = \eta(v) + \varepsilon r(t,x,v), \tag{2.3}$$

and we assume η is an analytic function of the velocities. The equation verified by the perturbation r is

$$\partial_t r(t,x,v) + v\partial_x r(t,x,v) + \mathcal{F}[r](t,x)\partial_v\big(\eta(v) + \varepsilon r(t,x,v)\big) = 0,$$

where the operator $\mathcal{F}$ is defined in (2.2).

To state the asymptotic behavior as in (1.18) of the previous Chapter, we define $h(t,x,v) = r(t,x+vt,v)$, which verifies the following equation:

$$\partial_t h = \{\psi[h], \eta + \varepsilon h\}, \tag{2.4}$$

where ψ is the potential field generated by the perturbation, evaluated along the free flow

$$\psi[h](t,x,v) = \int_{\mathbb{T}^1\times\mathbb{R}} \cos(x-y+(v-u)t)h(t,y,u)\,\mathrm{d}y\,\mathrm{d}u \tag{2.5}$$

and where $\{,\}$ is the Poisson bracket.

Recalling (1.18) and (2.3), we study the damping problem by setting $\omega(x,v) = \eta(v) + \varepsilon h_\infty(x,v)$, i.e. by searching for a solution for (2.4) such that

$$\lim_{t\to+\infty} \|h(t,x,v) - h_\infty(x,v)\|_{L^\infty(\mathbb{T}^1\times\mathbb{R})} = 0$$

where h_∞ is a mean-zero analytic datum with $\|h_\infty\|_\lambda < +\infty$ for some $\lambda > 0$.

Firstly, we study the evolution in the time interval $[0,T]$ considering the following problem:

$$\begin{cases} \partial_t h^T(t,x,v) = \{\psi[h^T], \eta + \varepsilon h^T\} & 0 \leqslant t \leqslant T, \\ h^T(T,x,v) = h_\infty(x,v). \end{cases} \tag{2.6}$$

Then, we show that, for $T \to +\infty$, h^T converges to a solution h, which solves the asymptotic problem.

We work in Fourier transform in $\mathbb{T}^1 \times \mathbb{R}$, using the following notation:

$$\widehat{g}_t(n,\xi) = \frac{1}{2\pi}\int_{\mathbb{T}^1\times\mathbb{R}} e^{-inx}e^{-iv\xi} g(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \tag{2.7}$$

with $n \in \mathbb{Z}$ and $\xi \in \mathbb{R}$. In Fourier space the system is

$$\partial_t \widehat{h^T}_t(n,\xi) = \delta_{n,\pm1} n \frac{\mathrm{i}}{2}\zeta_t^T(n)\widetilde{\eta'}(\xi - nt) - \varepsilon \sum_{k=\pm1} k \frac{\zeta_t^T(k)}{2}\widehat{h^T}_t(n-k,\xi-kt)(\xi-nt), \tag{2.8}$$

where $\widetilde{\eta'}$ is the Fourier transform of η' in the velocity and $\zeta_t^T(n)$ for $n = \pm1$ is the electric field:

$$\zeta_t^T(n) = \widehat{h^T}_t(n,nt). \tag{2.9}$$

Integrating equation (2.8) between $[t,T]$ and putting $\xi = nt$, we get an equation for ζ^T:

$$\begin{aligned} \zeta_t^T(n) = \widehat{h^T}_T(n,nt) - \frac{\mathrm{i}}{2}n\int_t^T \zeta_s^T(n)\widetilde{\eta'}(n(t-s))\,\mathrm{d}s \\ - \frac{\varepsilon}{2}\sum_{k=\pm1}\int_t^T \zeta_s^T(k)\widehat{h^T}_s(n-k,nt-ks)kn(s-t)\,\mathrm{d}s. \end{aligned} \tag{2.10}$$

To give a priori estimates, it is convenient to consider $(\zeta^T(\pm1), h^T)$ as a coupled system, where (2.9) is a consequence of the uniqueness.

We need to control the loss of analytic regularity of the solutions. For this reason we use techniques inspired by the abstract Cauchy-Kovalevskaya theory (see [16]), adapted to these kinds of problems in [11].

A key point in Landau damping problems is the decay of the electric field. To show this we define the norm of the electric field ζ^T as

$$M_{\lambda,T}[\zeta^T] = \sup_{t\in[0,T]} e^{\lambda t}|\zeta_t^T(1)| = \sup_{t\in[0,T]} e^{\lambda t}|\zeta_t^T(-1)|. \tag{2.11}$$

We also define a norm that quantifies the analyticity of a function g of the phase space:

$$\|g\|_\mu = \sup_{n,\xi} e^{\mu\langle n,\xi\rangle}|\widehat{g}(n,\xi)|, \tag{2.12}$$

where $\mu > 0$ is a parameter and $\langle n,\xi\rangle = (1+n^2+\xi^2)^{\frac{1}{2}}$.

To take into account the decay of the analytic regularity, we define the weighted-in-time analytic norm of the solution $h^T(t,x,v)$ as

$$N_{\lambda,T}[h^T] = \sup_{(\mu,t)\in D_{\lambda,T}} \alpha_\delta^T(\mu,t)^{1/2}\|h^T(t)\|_\mu, \tag{2.13}$$

where

$$D_{\lambda,T} = \{(\mu,t)\in[0,\lambda)\times[0,T], \alpha_\delta^T(\mu,t) > 0\} \tag{2.14}$$

and $\alpha_\delta^T(\mu,t) = \lambda-\mu-a_{T,\delta}(t)$. The function $a_{T,\delta}(t)$ is the unique solution of the following ordinary differential equation

$$\begin{cases} \dot{a}_{T,\delta}(t) = -\delta e^{-a_{T,\delta}(t)t}(1+t) \quad \text{if} \quad 0\leqslant t\leqslant T \\ a_{T,\delta}(T) = 0, \end{cases} \tag{2.15}$$

and measures the loss of analytic regularity of the solutions with respect to the final datum, as in (2.23) below: it is 0 at time T, and it is maximum at $t=0$. In view of the limit $T\to+\infty$, we need the following lemma.

Lemma 2.1. *For $\delta > 0$ the unique solution of the backward Cauchy problem (2.15) is positive and decreasing in time, and verifies*

$$a_{T,\delta}(0)\leqslant C(\delta),$$

with $C(\delta)\to 0$ when δ goes to zero. The solution $a_{\infty,\delta}(t)$ with initial datum

$$a_{\infty,\delta}(0) = \lim_{T\to+\infty} a_{T,\delta}(0)$$

is positive in $[0,+\infty)$ and

$$\lim_{t\to+\infty} a_{\infty,\delta}(t) = 0.$$

As a consequence, given $\lambda > 0$, we can choose δ sufficiently small such that there exist $\mu\in(0,\lambda)$ for which for any $T>0$, $[0,\mu]\times[0,T]\subset D_{\lambda,T}$.

Proof. Here we omit the symbol δ from $a_{T,\delta}$. Since $a_T(t)$ is decreasing, we have, for any $\bar{t} \in [0,T]$,

$$a_T(0) = a_T(\bar{t}) + \delta \int_0^{\bar{t}} e^{-a_T(s)s}(1+s)\,\mathrm{d}s \leqslant a_T(\bar{t}) + \delta \int_0^{\bar{t}} e^{-a_T(\bar{t})s}(1+s)\,\mathrm{d}s$$
$$\leqslant a_T(\bar{t}) + \delta \frac{1}{a_T(\bar{t})} + \delta \frac{1}{a_T^2(\bar{t})}.$$

If $\delta \leqslant 1$, the minimum of $x + \delta/x + \delta/x^2$, for $x > 0$, is less than $c_1\delta^{1/3}$ and is reached in $x < c_2\delta^{1/3}$. Then, if $a_T(0) \geqslant \max(c_1, c_2)\delta^{1/3}$, the right-hand side reach the minimum for some $\bar{t}$, and then $a_T(0) \leqslant c_1\delta^{1/3}$. This implies that $a_T(0) \leqslant \max(c_1, c_2)\delta^{1/3}$.

For any $t < T$, a_T is uniformly bounded and is increasing in T, so it converges to a positive function $a_\infty(t)$. For any time interval in $[0, +\infty)$, by dominated convergence in the integral formulation of (2.15), we get that $a_\infty(t)$ solves the differential equation with initial datum $a_\infty(0)$.

Now we prove that $\lim_{t\to+\infty} a_\infty(t) = 0$. First notice that given $b > 0$ there exists $b_0 > 0$ such that the solution of

$$\dot{a} = -\delta e^{-ta}(1+t)$$

with initial datum b_0 exists for all times and $a(t) \geqslant b$ for all time. To prove this, we choose $b_0 > b + \delta(1/b + 1/b^2)$ and consider the first time τ such that $a(\tau) = b$. Until τ,

$$b_0 - a(t) = \delta \int_0^t e^{-as}(1+s)\,\mathrm{d}s \leqslant \delta\left(\frac{1}{b} + \frac{1}{b^2}\right).$$

Then $\tau = +\infty$.

Let $a(0)$ be the initial datum of a generic solution $a(t)$. Set

$$\bar{a} = \inf\{a(0) | \lim_{t\to+\infty} a(t) \geqslant 0\},$$

and let $\bar{a}(t)$ the solution with initial datum $\bar{a}$. It is easy to prove that $\bar{a}(t) \to 0$, otherwise $\bar{a}$ is not the infimum. We conclude the proof by noticing that $a_\infty(0) \leqslant \bar{a}$, then $a_\infty(t)$ is dominated by $\bar{a}(t)$ which is a vanishing function.

□

We define $\mathcal{B}_{\lambda,T}$ the space of function $h(t,x,v)$, defined for $t \in [0,T]$, with $N_{\lambda,T}[h] < +\infty$, and $\mathcal{B}_{\lambda,\infty}$ as the space of functions $h(t,x,v)$ with $t \in [0,+\infty)$ such that $N_{\lambda,\infty}[h] < +\infty$, where $N_{\lambda,\infty}[h]$ is defined in the region $D_{\lambda,\infty} = \{(\mu,t) \in [0,\lambda) \times [0,+\infty), \alpha_\delta^\infty(\mu,t) > 0\}$ with $\alpha_\delta^\infty(\mu,t) = \lambda - \mu - a_{\infty,\delta}(t)$.

2.1.1 Estimates for ζ^T

As we show more accurately in the following lemma, eq. (2.10) for the field ζ^T has the structure of a Volterra equation. In order to invert the term of order one in the equation, we use the following classical result about the theory of Volterra operators.

Theorem 2.1 ([23], p. 45). *Given a Volterra equation of the form $f(t) + j * f(t) = g(t)$, where*

$$j * f(t) = \int_0^t j(t-s)f(s)\,\mathrm{d}s$$

with $j \in L^1(\mathbb{R}_+)$. The resolvent kernel r, i.e. the unique solution of the equation

$$r + j * r = j,$$

belongs to $L^1(\mathbb{R}_+)$ if and only if

$$\mathcal{L}[j](\sigma) \neq -1 \quad \text{for} \quad \Re\sigma \geqslant 0,$$

where

$$\mathcal{L}[j](\sigma) = \int_0^{+\infty} e^{-\sigma t} j(t)\,\mathrm{d}t$$

is the Laplace transform of j. The solution f is then given by $f(t) = g(t) - r * g(t)$.

We can now state the inversion lemma. We set

$$j_t(n) \equiv \mathrm{i}\frac{n}{2}\tilde{\eta}'(nt), \tag{2.16}$$

and

$$H_\varepsilon^T(t) = \widehat{h^T}_T(n, nt) - \frac{\varepsilon}{2}\sum_{k=\pm 1}\int_t^T \zeta_s^T(k)\widehat{h^T}_s(n-k, nt-ks)kn(s-t)\,\mathrm{d}s. \tag{2.17}$$

Lemma 2.2. *Let $\lambda > 0$ with $\|h_\infty\|_\lambda < +\infty$ and $\|\eta\|_\lambda < +\infty$. Assume that*

$$\mathcal{L}[j(1)](\sigma) \neq 1, \quad \Re\sigma \geqslant 0$$

then

$$M_{\lambda,T}[\zeta^T] \leqslant C_\lambda M_{\lambda,T}[H_\varepsilon^T].$$

We notice that the condition on the Laplace transform is fulfilled also by $j(-1)$ since $\overline{j(1)} = j(-1)$.

Proof. Let us define $\phi_\lambda(t) = e^{\lambda(T-t)}\zeta_{T-t}^T(1)$, $F_\varepsilon(t) = e^{\lambda(T-t)}H_\varepsilon^T(T-t)$. Multiplying by $e^{\lambda t}$, (2.10) can be rewritten as

$$\phi_\lambda(t) + \jmath_\lambda * \phi_\lambda(t) = F_\varepsilon(t), \tag{2.18}$$

for $t \in [0, T]$, where $\jmath_\lambda(t) = -e^{-\lambda t}j_t(-1)$. We notice that $\jmath_\lambda \in L^1(\mathbb{R}_+)$ and if $\Re\sigma \geqslant 0$

$$\mathcal{L}[\jmath_\lambda](\sigma) = -\mathcal{L}[j(-1)](\sigma + \lambda) \neq -1.$$

Then, from Theorem (2.1), the resolvent kernel r_λ related to $\jmath_\lambda$ belongs to $L^1(\mathbb{R}_+)$. Convolving with r_λ in (2.18), we get

$$\phi_\lambda(t) = F_\varepsilon(t) - \int_0^t r_\lambda(t-s)F_\varepsilon(s)\,\mathrm{d}s.$$

Taking the absolute values, it holds

$$M_{\lambda,T}[\zeta^T] = \sup_{t\in[0,T]} |\phi_\lambda(t)| \leqslant M_{\lambda,T}[H_\varepsilon^T] + \|r_\lambda\|_{L^1(\mathbb{R}_+)} M_{\lambda,T}[H_\varepsilon^T]$$

and the thesis follows with $C_\lambda = 1 + \|r_\lambda\|_{L^1(\mathbb{R}_+)}$. □

We now state the main estimate of this section.

Proposition 2.1. *Let $\zeta^T(\pm 1)$ solution of (2.10) and suppose $N_{\lambda,T}[h^T] < +\infty$. Then, under the hypothesis of Lemma (2.2), we have*

$$M_{\lambda,T}[\zeta^T] \leqslant C_\lambda \|h_\infty\|_\lambda + \varepsilon \frac{C_\lambda}{\lambda^2 \sqrt{\lambda - a_{\infty,\delta}(0)}} M_{\lambda,T}[\zeta^T] N_{\lambda,T}[h^T]. \tag{2.19}$$

Proof. From Lemma (2.2) we need only to estimate $N_{\lambda,T}[H^T_\varepsilon]$. Being $h^T(T,x,v) = h_\infty(x,v)$, from (2.17) we have

$$\begin{aligned} e^{\lambda t}|H^T_\varepsilon(t)| &\leqslant \|h_\infty\|_\lambda \\ &+ \varepsilon M_{\lambda,T}[\zeta^T] N_{\lambda,T}[h^T] \sum_{k=\pm 1} \int_t^T \frac{e^{-\lambda(s-t)-\mu'\langle n-k, nt-ks\rangle}}{\alpha^T(\mu',s)^{1/2}} (s-t)\,\mathrm{d}s, \end{aligned} \tag{2.20}$$

for any $\mu' < \lambda - a_{T,\delta}(s)$. Then, by choosing $\mu' = 0$, and using that $a_{T,\delta}(s) \leqslant a_{T,\delta}(0) < a_{\infty,\delta}(0)$ we get

$$e^{\lambda t}|H^T_\varepsilon(t)| \leqslant \|h_\infty\|_\lambda + \varepsilon \frac{M_{\lambda,T}[\zeta^T] N_{\lambda,T}[h^T]}{(\lambda - a_{\infty,\delta}(0))^{1/2}} \int_t^T e^{-\lambda(s-t)}(s-t)\,\mathrm{d}s.$$

□

2.1.2 Estimates for h^T

Now we turn to give a Cauchy-Kovalevskaya estimate on h^T. Due to the loss of analytic regularity in time, it is crucial to use the weighted norm introduced in (2.13).

Proposition 2.2. *Let h^T a solution of (2.6) and assume $M_{\lambda,T}[\zeta^T] < +\infty$ then the following estimate holds:*

$$N_{\lambda,T}[h^T] \leqslant C\|h_\infty\|_\lambda + \frac{C}{\delta} M_{\lambda,T}[\zeta^T] \|\eta\|_\lambda + \varepsilon \frac{C}{\delta} M_{\lambda,T}[\zeta^T] N_{\lambda,T}[h^T]. \tag{2.21}$$

Proof. Fixing $\mu < \lambda - a_{T,\delta}(t)$, from (2.8) we get

$$e^{\mu\langle n,\xi\rangle}|\widehat{h^T}_t(n,\xi)| \leqslant \|h_\infty\|_\lambda + e^{\mu\langle n,\xi\rangle}|D^T_t(n,\xi)| + e^{\mu\langle n,\xi\rangle}|E^T_t(n,\xi)| \tag{2.22}$$

where

$$D^T_t(n,\xi) = \delta_{n,\pm 1} \frac{\mathrm{i}}{2} n \int_t^T \zeta^T_s(n) \widetilde{\eta}'(\xi - ns)\,\mathrm{d}s$$

and

$$E^T_t(n,\xi) \equiv \frac{\varepsilon}{2} \sum_{k=\pm 1} \int_t^T \zeta^T_s(k) \widehat{h^T}_s(n-k, \xi - ks) k(\xi - ns)\,\mathrm{d}s.$$

We estimate separately the two terms. As regards $E^T_t(n,\xi)$, since

$$e^{\mu\langle n,\xi\rangle} \leqslant e^{\mu\langle n-k,\xi-ks\rangle} e^{\mu\langle k,ks\rangle},$$

by the triangular inequality and taking

$$\mu(s) = \frac{\lambda + \mu - a_{T,\delta}(s)}{2},$$

i.e. the middle point between μ and $\lambda - a_{T,\delta}(s)$, we have

$$e^{\mu\langle n,\xi\rangle}|E_t^T(n,\xi)| \leqslant \sum_{k=\pm 1} M_{\lambda,T}[\zeta^T]\times$$
$$\times \int_t^T e^{-(\lambda-\mu)s}\|h^T(s)\|_{\mu(s)}e^{-(\mu(s)-\mu)\langle n-k,\xi-ks\rangle}|\xi - ns|\,\mathrm{d}s,$$

where we have also used that $\langle \pm 1, \pm s\rangle \leqslant C + s$. Noting that

$$e^{-(\mu(s)-\mu)\langle n-k,\xi-ks\rangle}|\xi - ns| \leqslant \frac{2(1+s)}{\lambda - \mu - a_{T,\delta}(s)},$$

we get

$$e^{\mu\langle n,\xi\rangle}|E_t^T(n,\xi)| \leqslant \frac{C}{\delta}M_{\lambda,T}[\zeta^T]N_{\lambda,T}[h^T]\int_t^T \frac{\delta e^{-(\lambda-\mu)s}(1+s)}{\alpha^T(\mu,s)^{3/2}}\,\mathrm{d}s. \tag{2.23}$$

Being $\lambda - \mu > a_{T,\delta}(s)$ and using the definition of $a_{T,\delta}$ in (2.15)

$$\frac{e^{-(\lambda-\mu)s}(1+s)}{\alpha^T(\mu,s)^{3/2}} \leqslant -\frac{2}{\delta}\frac{\mathrm{d}}{\mathrm{d}s}\alpha^T(\mu,s)^{-1/2}$$

and then

$$e^{\mu\langle n,\xi\rangle}|E_t(n,\xi)| \leqslant \frac{C}{\delta}M_{\lambda,T}[\zeta^T]N_{\lambda,T}[h^T]\frac{1}{\alpha^T(\mu,t)^{1/2}}.$$

As regards $D_t^T(n,\xi)$, for $\mu < \lambda - a_{T,\delta}(t)$,

$$e^{\mu\langle n,\xi\rangle}|D_t^T(n,\xi)| \leqslant CM_{\lambda,T}[\zeta^T]\|\eta\|_\lambda \int_t^T e^{-(\lambda-\mu)s}e^{-(\lambda-\mu)\langle\xi-ns\rangle}\langle\xi - ns\rangle\,\mathrm{d}s$$
$$\leqslant \frac{C}{\delta}M_{\lambda,T}[\zeta^T]\|\eta\|_\lambda \int_t^T \frac{\delta e^{-a_{T,\delta}(s)s}(1+s)}{\alpha^T(\mu,s)}\,\mathrm{d}s$$

where in the last inequality we have used that $\lambda - \mu > \lambda - \mu - a_{T,\delta}(s) = \alpha^T(\mu,s)$ and also that $\lambda - \mu > a_{T,\delta}(s)$. Computing the integral, we get

$$e^{\mu\langle n,\xi\rangle}|D_t^T(n,\xi)| \leqslant \frac{C}{\delta}M_{\lambda,T}[\zeta^T]\|\eta\|_\lambda \ln\Big(\frac{\alpha^T(\mu,T)}{\alpha^T(\mu,t)}\Big).$$

We conclude the proof multiplying (2.22) by $\alpha^T(\mu,t)^{1/2}$, and taking the supremum over $D_{\lambda,T}$. □

2.1.3 The backward result

Theorem 2.2. *Let $h_\infty \in L^1(\mathbb{T}^1\times\mathbb{R})$ analytic such that $\|h_\infty\|_\lambda < +\infty$ with $\lambda > 0$. Consider $\eta \in L^1(\mathbb{R})$ analytic such that $\|\eta\|_\lambda < +\infty$. Moreover, assume*

$$\mathcal{L}[j(1)](\sigma) \neq 1, \quad \Re\sigma \geqslant 0,$$

with $j(1)$ as in (2.16). Then, for small values of ε, there exists a unique solution $h_t = h(t,x,v)$ of (2.4) with $N_{\lambda,\infty}[h] < +\infty$ such that

$$\lim_{t\to+\infty}\|h_t - h_\infty\|_{L^\infty(\mathbb{T}^1\times\mathbb{R})} = 0$$

with exponential rate.

Proof. For every T we get the unique solution h^T of (2.6) using the following iterative procedure. For $j \geqslant 0$ and $0 \leqslant t \leqslant T$ let

$$\begin{aligned}\partial_t \widehat{h}_t^{(j+1),T}(n,\xi) &= \delta_{n,\pm 1} n \frac{\mathrm{i}}{2} \zeta_t^{(j),T}(n) \widetilde{\eta}'(\xi - nt) \\ &\quad - \varepsilon \sum_{k=\pm 1} k \frac{\zeta_t^{(j),T}(k)}{2} \widehat{h}_t^{(j+1),T}(n-k, \xi - kt)(\xi - nt),\end{aligned} \tag{2.24}$$

where $\zeta_t^{(j),T}(1)$ is defined by

$$\begin{aligned}\zeta_t^{(j),T}(1) &= \widehat{h}_\infty(1,t) - \frac{\mathrm{i}}{2} \int_t^T \zeta_s^{(j),T}(1) \widetilde{\eta}'(t-s)\,\mathrm{d}s \\ &\quad - \frac{\varepsilon}{2} \sum_{k=\pm 1} \int_t^T \zeta_s^{(j),T}(k) \widehat{h}_s^{(j),T}(n-k, t-ks) k(s-t)\,\mathrm{d}s,\end{aligned}$$

where $\zeta^{(j),T}(-1) = \overline{\zeta^{(j),T}(1)}$ and with initial step $h^{(0),T}(t,x,v) = h_\infty(x,v)$.
Then $h^{(j),T}$ verifies the same bounds of the a priori estimates in (2.19) and (2.21):

$$M_{\lambda,T}[\zeta^{(j),T}] \leqslant C \|h_\infty\|_\lambda + \varepsilon C M_{\lambda,T}[\zeta^{(j),T}] N_{\lambda,T}[h^{(j),T}]$$

and

$$\begin{aligned}N_{\lambda,T}[h^{(j+1),T}] &\leqslant C\|h_\infty\|_\lambda + C M_{\lambda,T}[\zeta^{(j),T}]\Big(\|\eta\|_\lambda + \varepsilon N_{\lambda,T}[h^{(j+1),T}]\Big) \\ &\leqslant C\|h_\infty\|_\lambda + \varepsilon C M_{\lambda,T}[\zeta^{(j),T}]\Big(N_{\lambda,T}[h^{(j),T}] + N_{\lambda,T}[h^{(j+1),T}]\Big),\end{aligned}$$

where we have used (2.19) in the last inequality and where C is a generic constant depending on λ and δ. Since $N_{\lambda,T}[h^{(0),T}] \leqslant C\|h_\infty\|_\lambda$, taking $\varepsilon\|h_\infty\|_\lambda$ sufficiently small, we get that $M_{\lambda,T}[\zeta^{(j),T}]$ and $N_{\lambda,T}[h^{(j+1),T}]$ are uniformly bounded in $j \geqslant 0$. Then, taking $\delta' > \delta$ in Lemma (2.1), the time derivative of $h^{(j),T}$ is uniformly bounded in $N_{\lambda,T}[\cdot]$. Hence there exists a subsequence $h^{(j_k),T}$ which converge to a function $h^T \in \mathcal{B}_{\lambda,T}$, while $\zeta^{(j_k),T}(\pm 1)$ converge to a function $\zeta^T(\pm 1)$ such that $M_{\lambda,T}[\zeta^T] < +\infty$. Then $h_n^T(t,nt) = \zeta_t^T(n)$ for $n = \pm 1$ and it is a solution of the nonlinear problem (2.6).
We now extend $h^T(t,x,v) = h_\infty(x,v)$ for $t \geqslant T$ and we consider the sequence of solutions $\{h^T\}$, with $h^T \in \mathcal{B}_{\lambda,\infty}$. We can see that h^T fulfills the Cauchy property as a function of T in $\mathcal{B}_{\lambda',\infty}$ with $\lambda > \lambda' > a_{\delta,\infty}(0)$. In fact, fixed T^*, taking $T' \geqslant T \geqslant T^*$, we have for $t \leqslant T$

$$\begin{aligned}&\widehat{h^{T'}}_t(n,\xi) - \widehat{h^T}_t(n,\xi) = \delta_{n,\pm 1} n \frac{\mathrm{i}}{2} \int_t^T \Big(\zeta_s^T(n) - \zeta_s^{T'}(n)\Big) \widetilde{\eta}'(\xi - ns)\,\mathrm{d}s \\ &- \varepsilon \sum_{k=\pm 1} k \int_t^T \frac{\Big(\zeta_s^T(k) - \zeta_s^{T'}(k)\Big)}{2} \widehat{h^T}_s(n-k, \xi - ks)(\xi - ns)\,\mathrm{d}s \\ &- \varepsilon \sum_{k=\pm 1} k \int_t^T \frac{\zeta_s^{T'}(k)}{2} \Big(\widehat{h^T}_s(n-k,\xi-ks) - \widehat{h^{T'}}_s(n-k,\xi-ks)\Big)(\xi - ns)\,\mathrm{d}s \\ &+ \delta_{n,\pm 1} n \frac{\mathrm{i}}{2} \int_T^{T'} \zeta_s^{T'}(n) \widetilde{\eta}'(\xi - ns)\,\mathrm{d}s \\ &+ \varepsilon \sum_{k=\pm 1} k \int_T^{T'} \frac{\zeta_s^{T'}(k)}{2} \widehat{h^{T'}}_s(n-k, \xi - ks)(\xi - ns)\,\mathrm{d}s\end{aligned}$$

and an analogous of equation (2.10) holds for $\zeta^T - \zeta^{T'}$. Doing estimates in the style of (2.19) and (2.21), we get

$$M_{\lambda',T}[\zeta^{T'} - \zeta^T] \leqslant \varepsilon C M_{\lambda',T}[\zeta^{T'} - \zeta^T] + \varepsilon C N_{\lambda',\infty}[h^{T'} - h^T] + \varepsilon \frac{C}{\lambda'^2} e^{-(\lambda-\lambda')T*}$$

and

$$\begin{aligned} N_{\lambda',\infty}[h^{T'} - h^T] \leqslant C M_{\lambda',T}[\zeta^{T'} - \zeta^T] + \varepsilon C M_{\lambda',T}[\zeta^{T'} - \zeta^T] \\ + \varepsilon C N_{\lambda',\infty}[h^{T'} - h^T] + \frac{(1+\varepsilon)C}{\min\{1, \lambda-\lambda'\}^3} e^{-\frac{(\lambda-\lambda')}{2}T*}. \end{aligned} \tag{2.25}$$

Hence, using again the smallness of ε, we conclude that

$$\lim_{T*\to+\infty} \sup_{T'\geqslant T\geqslant T*} N_{\lambda',\infty}[h^{T'} - h^T] = 0.$$

Being uniformly bounded in $\mathcal{B}_{\lambda,\infty}$, the sequence $\{h^T\}$ converge to a function $h \in \mathcal{B}_{\lambda,\infty}$ and, passing to the limit by dominated convergence in the integral formulation, $h(t,x,v)$ is solution of the nonlinear equation (2.4) in $[0,+\infty)$. So, taking $\bar{\mu} < \lambda - a_{\infty,\delta}(0)$, we have that $\|h(t,x,v) - h_\infty(x,v)\|_{\bar{\mu}} \to 0$.
We get the uniqueness of the solutions with a similar procedure. Let $g(t,x,v)$ and $h(t,x,v)$ be two solutions of (2.4) with the same asymptotic datum h_∞. Proceeding as before, we can prove that they verify the estimates (2.19) and (2.21). Hence, denoting ζ_h the electric field associated to h, we get

$$\max\Big(N_{\lambda,\infty}[h], M_{\lambda,\infty}[\zeta_h]\Big) \leqslant C\|h_\infty\|_\lambda$$

and analogously for $g(t,x,v)$. Estimating $N_{\lambda,\infty}[g-h]$, we obtain the same estimates as in (2.25) without the rest terms:

$$A \equiv \max\Big(N_{\lambda,\infty}[g-h], M_{\lambda,\infty}[\zeta_g - \zeta_h]\Big) \leqslant C(\varepsilon)A.$$

Using the smallness on ε as before, we have $C(\varepsilon) < 1$, from which the uniqueness follows.

We remark that in [17], in the case of the scattering problem for the Vlasov-Poisson equation, the uniqueness is guaranteed for a wider class of solutions, not necessarily analytic. □

2.2 Nonperturbative regime

Using the backward approach for large times it is possible to construct solutions without perturbating around the homogeneous equilibrium $\eta(v)$, in the style of [17]. The price to pay is that the analytic estimates hold only in $[\tau,+\infty)$ for τ large enough.

Fixed an analytic asymptotic state $\omega(x,v)$, consider (2.1) and write

$$f(t,x,v) = \bar{\omega}(v) + g(t,x,v),$$

where $\bar{\omega}$ is the mean of $\omega(x,v)$ with respect to the x variable. Then $h(t,x,v) = g(t,x+vt,v)$ verifies the equation

$$\partial_t h = \{\psi[h], \bar{\omega} + h\}$$

where ψ is defined as in (2.5). For $T \geqslant \tau$, let us consider the following sequence of problems

$$\begin{cases} \partial_t h^T = \{\psi[h^T], \bar{\omega} + h^T\} & \tau \leqslant t \leqslant T, \\ h^T(T, x, v) = (\omega - \bar{\omega})(x, v). \end{cases}$$

We introduce the weighted norm

$$Q_{\lambda,T}[h^T] = \sup_{(\mu,t)\in\Omega_{\lambda,T}} \theta^T(\mu,t)^{1/2} \|h^T(t)\|_\mu,$$

with the weight $\theta^T(\mu, t) = (\lambda - \mu - \Delta a_T(s))$, where $\Delta = \lambda'/a_\infty(\tau)$, $\lambda' < \lambda$ and $a_T(s)$ is defined as in (2.15) putting $\delta = 1$. Notice now that Δ is a diverging quantity for sufficiently large τ. Here $\Omega_{\lambda,T} = \{(\mu, t) \in [0, \lambda) \times [\tau, T], \theta^T(\mu, t) > 0\}$ and, as in the previous case, we can give the analogous definitions for $Q_{\lambda,\infty}[\cdot]$, θ^∞ and $\Omega_{\lambda,\infty}$.
We define $\zeta^T_t(n) = \widehat{h^T}_t(n, nt)$, $n = \pm 1$, then ζ^T verifies the following equation:

$$\zeta^T_t(n) = \int_t^T \zeta^T_s(n) j_n(t-s)\,\mathrm{d}s + W^T(t), \tag{2.26}$$

where we have defined

$$W^T(t) \equiv \widehat{\omega}(n, nT) - \frac{1}{2}\sum_{k=\pm 1} \int_t^T \zeta^T_s(k) \widehat{h^T}_s(n-k, nt-ks) kn(t-s)\,\mathrm{d}s$$

and

$$j_t(n) = \mathrm{i}\frac{n}{2}\tilde{\omega}'_0(nt). \tag{2.27}$$

As in (2.11) we denote

$$P_{\lambda,T}[\zeta^T] = \sup_{t\in[\tau,T]} e^{\lambda t}|\zeta^T_t(1)| = \sup_{t\in[\tau,T]} e^{\lambda t}|\zeta^T_t(-1)|.$$

We can now state the following theorem.

Theorem 2.3. *Let $\omega \in L^1(\mathbb{T}^1 \times \mathbb{R})$ analytic such that $\|\omega\|_\lambda < +\infty$ and assume that*

$$\mathcal{L}[j(1)](\sigma) \neq 1, \quad \Re\sigma \geqslant 0, \tag{2.28}$$

with $j(1)$ as in (2.27). Then, for sufficiently large τ, there exists a unique solution $h(t, x, v)$ of

$$\partial_t h = \{\psi[h], \bar{\omega} + h\} \quad \text{if} \quad \tau \leqslant t < +\infty,$$

with $Q_{\lambda,\infty}[h] < +\infty$ such that

$$\lim_{t\to+\infty} \|h_t - (\omega - \bar{\omega})\|_{L^\infty(\mathbb{T}^1\times\mathbb{R})} = 0$$

with exponential rate.

Proof of Theorem (2.3). The proof goes in the same way of Theorem (2.2) but instead of using the smallness of ε, we can use the size of Δ. Indeed as in Proposition (2.2) we can estimate h^T in $[\tau, T]$ where h^T verifies the equation

$$\widehat{h^T}_t(n, \xi) = D_t(n, \xi) - \sum_{k=\pm 1} \int_0^t k \frac{\zeta^T_s(k)}{2} \widehat{h^T}_s(n-k, \xi - ks)(\xi - ns)\,\mathrm{d}s$$

with

$$D_t(n,\xi) = \delta_{n,\pm 1}\frac{\mathrm{i}}{2}n\int_t^T \zeta_s^T(n)\widetilde{\omega_0'}(\xi - ns)\,\mathrm{d}s.$$

We first treat the case $n \neq \pm 1$. As in (2.23) and using $\lambda - \mu > \Delta a_T(s) > a_T(s)$ we have

$$\begin{aligned} e^{\mu\langle n,\xi\rangle}|\widehat{h^T}_t(n,\xi)| &\leqslant \|\omega\|_\lambda + CP_{\lambda,T}[\zeta^T]Q_{\lambda,T}[h^T]\int_t^T \frac{e^{-(\lambda-\mu)s}(1+s)}{\Theta^T(\mu,s)^{3/2}}\,\mathrm{d}s \\ &\leqslant \|\omega\|_\lambda + C\frac{P_{\lambda,T}[\zeta^T]Q_{\lambda,T}[h^T]}{\Delta}\int_t^T \frac{\Delta e^{-a_T(s)s}(1+s)}{\Theta^T(\mu,s)^{3/2}}\,\mathrm{d}s \end{aligned}$$

and thus, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta^T(\mu,t)^{-1/2} = -\frac{\Delta}{2}\frac{e^{-a_T(s)s}(1+s)}{\Theta^T(\mu,t)^{3/2}}$$

we get

$$e^{\mu\langle n,\xi\rangle}|\widehat{h^T}_t(n,\xi)| \leqslant \|\omega\|_\lambda + C\frac{P_{\lambda,T}[\zeta^T]Q_{\lambda,T}[h^T]}{\Delta\Theta^T(\mu,s)^{1/2}}. \tag{2.29}$$

Now we estimate $D_t(n,\xi)$, $n = \pm 1$. Take $\mu < \lambda - \Delta a_T(t)$, hence $\lambda - \mu > (\lambda - \mu - \Delta a_T(s))/2$, so we get

$$\begin{aligned} e^{\mu\langle n,\xi\rangle}|D_t(n,\xi)| &\leqslant CP_{\lambda,T}[\zeta^T]\|\omega\|_\lambda\int_t^T e^{-(\lambda-\mu)s}e^{-(\lambda-\mu)\langle\xi-ns\rangle}\langle\xi-ns\rangle\,\mathrm{d}s \\ &\leqslant CP_{\lambda,T}[\zeta^T]\|\omega\|_\lambda\int_t^T \frac{e^{-a_T(s)s}(1+s)}{\Theta^T(\mu,s)}\,\mathrm{d}s \\ &\leqslant C\frac{P_{\lambda,T}[\zeta^T]\|\omega\|_\lambda}{\Delta}\ln\Big(\frac{\Theta^T(\mu,T)}{\Theta^T(\mu,t)}\Big). \end{aligned} \tag{2.30}$$

Hence, multiplying by $\Theta^T(\mu,t)^{1/2}$ in (2.29) and (2.30) we get

$$Q_{\lambda,T}[h^T] \leqslant C\|\omega\|_\lambda + \frac{C}{\Delta}P_{\lambda,T}[\zeta^T]\|\omega\|_\lambda + \frac{C}{\Delta}P_{\lambda,T}[\zeta^T]Q_{\lambda,T}[h^T].$$

Regarding ζ^T in (2.26), using (2.28) and (2.1) we have

$$P_{\lambda,T}[\zeta^T] \leqslant C_\lambda P_{\lambda,T}[W^T].$$

We need better estimates than that in (2.20). We get them by splitting the two modes $k = \pm 1$ in

$$\sum_{k=\pm 1}\int_t^T \zeta_s^T(k)\widehat{h^T}_s(1-k, t-ks)k(t-s)\,\mathrm{d}s = B_1 + B_{-1}. \tag{2.31}$$

If $k = -1$, for $\mu' < \lambda - \Delta a_\infty(\tau) = \lambda - \lambda'$, we get

$$\begin{aligned} e^{\lambda t}|B_{-1}| &\leqslant P_{\lambda,T}[\zeta^T]\int_t^T e^{-\lambda(s-t)}\frac{Q_{\lambda,T}[h^T]}{\Theta(\mu',s)^{1/2}}e^{-\mu'(t+s)}(s-t)\,\mathrm{d}s \\ &\leqslant P_{\lambda,T}[\zeta^T]Q_{\lambda,T}[h^T]\frac{e^{-2\mu'\tau}}{(\lambda-\mu'-\lambda')^{1/2}}\int_t^T e^{-\lambda(s-t)}(s-t)\,\mathrm{d}s \\ &\leqslant CP_{\lambda,T}[\zeta^T]Q_{\lambda,T}[h^T]\frac{\sqrt{\tau}}{\lambda^2}e^{-(\lambda-\lambda')\tau}, \end{aligned}$$

where we have taken the infimum on $\mu' \in [0, \lambda - \lambda']$ in the last inequality. In the other case, using that $\omega - \bar{\omega}$ has mean zero in the x variable, we have

$$\widehat{h^T}_s(0, t-s) = \sum_{k=\pm 1} \int_s^T \zeta_l^T(k) \widehat{h^T}_l(-k, t-s-kl) k(t-s) \, \mathrm{d}l. \tag{2.32}$$

Replacing (2.32) in (2.31) we obtain

$$\begin{aligned} e^{\lambda t}|B_1| &\leqslant P_{\lambda,T}[\zeta^T] \int_t^T e^{-\lambda(s-t)}(s-t)|\widehat{h^T}_s(0,t-s)| \, \mathrm{d}s \\ &\leqslant C P_{\lambda,T}[\zeta^T] \frac{Q_{\lambda,T}[h^T]}{\lambda - \lambda'} \int_t^T e^{-\lambda(s-t)}(s-t)^2 \int_s^T e^{-\lambda l} \, \mathrm{d}l \\ &\leqslant C \frac{P_{\lambda,T}[\zeta^T] Q_{\lambda,T}[h^T]}{\lambda^3(\lambda - \lambda')} e^{-\lambda \tau}. \end{aligned}$$

Hence

$$P_{\lambda,T}[\zeta^T] \leqslant \|\omega\|_\lambda + C P_{\lambda,T}[\zeta^T] Q_{\lambda,T}[h^T] \left(\frac{\sqrt{\tau}}{\lambda^2} e^{-(\lambda-\lambda')\tau} + \frac{e^{-\lambda\tau} P_{\lambda,T}[\zeta^T]}{\lambda^3(\lambda - \lambda')} \right)$$

and we can reason as in the proof of the main theorem avoiding to use the smallness of ε. □

Remark 2.1. We notice that in this setting we have obtained an Eulerian analog of the scattering result in [17], in the special case of the HMF model. In [17] Caglioti and Maffei, using the Lagrangian description of the flow, obtain the damping result for the Vlasov-Poisson equation, by a fixed point technique, considering an asymptotic state ω with $\|\omega\|_\lambda < +\infty$ such that

$$\omega(x, v) \leqslant \frac{M}{(1+v^2)^2}$$

for some $M > 0$ and $\lambda \geqslant C\sqrt{M}$, with C some purely numerical constant. Here we show that such class of final data fulfills condition (2.28), if $\lambda > \pi\sqrt{M}$. Indeed, taking $n = 1$ in (2.26) and multiplying by $e^{\lambda t}$ we get as in (2.18)

$$\phi_\lambda^T(t) + \phi_\lambda^T * \jmath_\lambda(t) = e^{\lambda(T-t)} W^T(T-t)$$

with $\jmath_\lambda(t) = -e^{-\lambda t} j_{-1}(t)$ and $\phi_\lambda^T(t) = e^{\lambda(T-t)} \zeta_1^T(T-t)$. So it is sufficient to notice that, since $|\widetilde{\omega_0}| \leqslant M\pi^2$, we have

$$|\mathcal{L}[\jmath(\pm 1)](\sigma)| \leqslant M\pi^2 \int_0^{+\infty} e^{-\Re\sigma t} e^{-\lambda t} t \, \mathrm{d}t \leqslant \pi^2 \frac{M}{\lambda^2} < 1, \quad \Re\sigma \geqslant 0$$

hence (2.1) holds.

Remark 2.2. The nonperturbative scattering result in Theorem (2.3) allows the choice of asymptotic states ω within a distance of $O(1)$ from a given homogeneous state $\eta(v)$. This fact poses a significant difference with respect to the forward perturbative results where, as we show in the next section (2.3), given an equilibrium $\eta(v)$ which verify some stability properties, there exists an $\varepsilon_0 > 0$ such that every initial data in an analytic neighborhood of η of $O(\varepsilon)$ with $\varepsilon < \varepsilon_0$ verifies the Landau damping.

Actually, solutions of the backward and forward problems are of a different type. Indeed, in the case of the attractive HMF model [1], it is easy to find nonhomogeneous BGK stationary solutions $\omega(x,v)$ of the HMF that can be chosen as scattering asymptotic datum for the HMF, i.e. such that there exists a solution $f_\omega(t,x,v)$ such that

$$\lim_{t\to+\infty} \|f_\omega(t,x,v) - \omega(x-vt,v)\|_{L^\infty(\mathbb{T}^1\times\mathbb{R})} = 0.$$

This solution f_ω could never be a Landau Damping solution because it is not close, in a strong norm, say L^1, to its weak asymptotic limit $\eta(v)$ which is given by the average in x of $\omega(x,v)$. Indeed at the same L^1 distance from η there exists a BGK stationary solution of the HMF model.

We give an example of such BGK solution, which can be constructed using that any function of the mean-field energy is an equilibrium. In this example we consider the attractive HMF model with

$$\mathcal{F}[f](t,x) = \partial_x\left(\int_{\mathbb{T}^1\times\mathbb{R}} \cos(x-y) f(t,y,v)\,\mathrm{d}y\,\mathrm{d}v\right)$$

in (2.1) and we choose, for $\beta,\nu>0$ to be fixed,

$$\omega_{\beta,\nu}(x,v) = \frac{e^{-\beta H_\nu(x,v)}}{\mathcal{Z}},$$

where $H_\nu(x,v) = \frac{v^2}{2} - \nu\cos x$ and $\mathcal{Z}$ is the normalizing constant. Using the simple structure of the potential, we have that $\omega_\nu(x,v)$ is a stationary solution of the attractive HMF model, provided that the following compatibility condition is fulfilled:

$$\Omega_\beta(\nu) \equiv \int \omega_{\beta,\nu}(x,v)\cos x\,\mathrm{d}x\,\mathrm{d}v = \nu.$$

By Taylor expansion $\Omega_\beta(\nu) = \beta\nu/2 + o(\beta\nu)$ as $\nu\to 0$, while $\Omega_\beta(\nu)\to 1$ if $\nu\to+\infty$. Hence for $\beta>2$ there exists at least one value $\bar\nu$ such that $\Omega_\beta(\bar\nu)=\bar\nu$.

Remark 2.3. In section (2.1) we have proved exponential damping of solutions of the HMF model in the scattering setting in the perturbative case, while in this section we prove the result for τ large. These two sections could have been partially joined by considering as a smallness parameter $\epsilon = e^{-\lambda\tau}$. However, given the different nature of the problems faced, we believe it is clearer to derive the two results separately.

2.3 The Cauchy problem

In this section, instead of fixing an asymptotic condition, we study the Cauchy problem for equation (2.1), with initial condition at time zero. We refer to the last section (2.4) for the discussion of the differences and advantages of the backward approach compared to this. Putting (2.8) in integral form we get

$$\begin{aligned}\widehat{h}_t(n,\xi) = \widehat{h}_0(n,\xi) + \delta_{n,\pm1} n\frac{\mathrm{i}}{2}\int_0^t \zeta_s(n)\widetilde{\eta}'(\xi-ns)\,\mathrm{d}s \\ -\frac{\varepsilon}{2}\sum_{k=\pm1} k\int_0^t \zeta_s(k)\widehat{h}_s(n-k,\xi-ks)(\xi-ns)\,\mathrm{d}s,\end{aligned} \tag{2.33}$$

[1] Except this paragraph, the choice of an attractive or repulsive potential is indifferent in this Chapter.

and taking $\xi = nt$ for $n = \pm 1$ in (2.33), we obtain the equation for the electric field:

$$\begin{aligned}\zeta_t(n) &= \widehat{h}_0(n, nt) + n\frac{\mathrm{i}}{2}\int_0^t \zeta_s(n)\widetilde{\eta}'(n(t-s))\,\mathrm{d}s \\ &\quad - \frac{\varepsilon}{2}\sum_{k=\pm1} kn\int_0^t \zeta_s(k)\widehat{h}_s(n-k, nt-ks)(t-s)\,\mathrm{d}s.\end{aligned} \tag{2.34}$$

We introduce the weight $A^{\lambda,p}(n,\xi) = e^{\lambda\langle n,\xi\rangle}\langle n,\xi\rangle^p$ and the corresponding analytic norm of a generic function f as

$$\|f\|_{\lambda,p} = \sup_{n,\xi} A^{\lambda,p}(n,\xi)|\widehat{f}(n,\xi)|.$$

In the following we take a mean-zero initial datum h_0 such that $\|h_0\|_{\lambda_0,p} < +\infty$, for some λ_0 and p to be fixed.

As done before, we want to study the coupled system $(\zeta(\pm1), h)$. For this purpose, we define the norm of the electric field ζ as

$$J^p_{\lambda_0}[\zeta] = \sup_{\beta(\lambda,t)>0} e^{\lambda t}\langle t\rangle^p\,|\zeta_t(\pm1)|. \tag{2.35}$$

Here

$$\beta(\lambda, t) = \lambda_0 - \lambda - \delta\arctan(t) \tag{2.36}$$

with $\delta < 2\lambda_0/\pi$ measures the loss of analytic regularity with respect to λ_0.

We remark that the choice of the $\arctan$ function is not mandatory, contrary to the backward case previously described, in which the regularity decay is more precisely prescribed by the structure of the estimates.

We define a weighted-in-time norm on h with two terms:

$$K^{3,p+1}_{\lambda_0,q}[h] = \mathcal{K}^3[h] + K^{p+1}_q[h], \tag{2.37}$$

where

$$\mathcal{K}^3[h] = \sup_{\beta(\lambda,t)>0} \|h(t)\|_{\lambda,3}$$

and

$$K^{p+1}_q[h] = \sup_{\beta(\lambda,t)>0} \beta(\lambda,t)^{1/2}\frac{\|h(t)\|_{\lambda,p+1}}{\langle t\rangle^q}.$$

The occurrence of the last term is in the spirit of the abstract Cauchy-Kovalevskaya theorem, while the term $\mathcal{K}^3$ is due to the treatment of the two echoes term in the equation for $\zeta(\pm1)$, as we show in Prop. (2.3).

2.3.1 Estimates for ζ

In the sequel, for $\gamma > \lambda_0$, it is useful to introduce the quantity

$$j_t(n) = \mathrm{i}\frac{n}{2}\widetilde{\eta}'(nt)e^{\lambda_0 t} \tag{2.38}$$

and define

$$G_\varepsilon(t) \equiv \widehat{h}_0(n, nt) - \frac{\varepsilon}{2}\sum_{k=\pm1}\int_0^t \zeta_s(k)\widehat{h}_s(n-k, nt-ks)kn(t-s)\ \mathrm{d}s. \tag{2.39}$$

Lemma 2.3. *Let $\eta(v)$ analytic such that $\|\eta'\|_\gamma < +\infty$ with $\gamma > \lambda_0$. If*

$$\mathcal{L}[j(1)](\sigma) \neq 1 \quad \text{for} \quad \Re\sigma \geqslant 0$$

then

$$J^p_{\lambda_0}[\zeta] \leqslant C(\gamma, \lambda_0) J^p_{\lambda_0}[G_\varepsilon].$$

Proof. Assume $p = 0$ and take $\lambda > 0$ such that $\beta(\lambda, t) > 0$ then

$$e^{\lambda t}\zeta_t(1) = \int_0^t \jmath_\lambda(t-s) e^{\lambda s}\zeta_s(1)\,\mathrm{d}s + e^{\lambda t} G_\varepsilon(t)$$

with $\jmath_\lambda(t) \equiv e^{-(\lambda_0-\lambda)t} j_t(1)$. From Theorem (2.1), since $\jmath_\lambda \in L^1(\mathbb{R}_+)$ for $\gamma > \lambda_0$ and

$$\mathcal{L}[\jmath_\lambda](\sigma) = \mathcal{L}[j(1)](\sigma + \lambda_0 - \lambda) \neq 1 \quad \text{for} \quad \Re\sigma \geqslant 0,$$

there exists a unique resolvent kernel r_λ associated to $\jmath_\lambda$ with $r_\lambda \in L^1(\mathbb{R}_+)$. Doing the convolution with r_λ, we get

$$e^{\lambda t}\zeta_t = \int_0^t r_\lambda(t-s) e^{\lambda s} G_\varepsilon(s)\,\mathrm{d}s + e^{\lambda t} G_\varepsilon(t).$$

Taking the absolute value, we obtain

$$e^{\lambda t}|\zeta_t| \leqslant (1 + \|r_\lambda\|_{L^1(\mathbb{R}_+)}) J^0_{\lambda_0}[G_\varepsilon], \tag{2.40}$$

and we get the thesis for $p = 0$ taking the supremum over $\beta(\lambda, t) > 0$.

Let us give the proof in the case $p = 1$, which is not difficult to extend to the general one.

$$te^{\lambda t}\zeta_t = \int_0^t \jmath_\lambda(t-s) s e^{\lambda s}\zeta_s\,\mathrm{d}s + Z_\varepsilon(t)$$

with

$$Z_\varepsilon(t) \equiv \int_0^t \jmath_\lambda(t-s)(t-s) e^{\lambda s}\zeta_s\,\mathrm{d}s + te^{\lambda t} G_\varepsilon(t).$$

Using (2.40), we get

$$J^1_{\lambda_0}[\zeta] \leqslant C(\gamma, \lambda_0) \sup_{\beta(\lambda,t)>0} |Z^\varepsilon(t)|$$

and

$$|Z^\varepsilon(t)| \leqslant C(\gamma, \lambda_0) J^0_{\lambda_0}[\zeta] + J^1_{\lambda_0}[G_\varepsilon] \leqslant C(\gamma, \lambda_0) J^1_{\lambda_0}[G_\varepsilon],$$

using again (2.40). □

Proposition 2.3. *In the hypothesis of the previous lemma, let $p \geqslant q + 3$ with $q \geqslant 3$ fixed. Given $h(t, x, v)$ such that $K^{3,p+1}_{\lambda_0,q}[h] < +\infty$ we have*

$$J^p_{\lambda_0}[\zeta] \leqslant C + \varepsilon C J^p_{\lambda_0}[\zeta] K^{3,p+1}_{\lambda_0,q}[h].$$

Proof. From the previous lemma, we only need to estimate $J^p_{\lambda_0}[G_\varepsilon]$. Multiplying by $e^{\lambda t}\langle t\rangle^p$ in (2.39) and using $\langle t\rangle^p \leqslant C\Big(\langle t-s\rangle^p + \langle s\rangle^p\Big)$ we have

$$e^{\lambda t}\langle t\rangle^p \,|G_\varepsilon[\zeta(1)](t)| \leqslant \|h(0)\|_{\lambda_0,p} + \varepsilon(I_1+I_2)$$

where

$$\begin{aligned} I_1 &= \int_0^t z_{\lambda,p}(s)e^{\lambda(t-s)}\Big(|\widehat{h}_s(0,t-s)|(t-s) + |\widehat{h}_s(2,t+s)|(t-s)\Big)\,\mathrm{d}s \\ &\leqslant \int_0^t z_{\lambda,p}(s)\|h(s)\|_{\lambda,3}\left(\frac{1}{\langle t-s\rangle^2} + \frac{1}{\langle t+s\rangle^2}\right)\mathrm{d}s \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_0^t z_{\lambda,p}(s)e^{\lambda(t-s)}\frac{\langle t-s\rangle^p}{\langle s\rangle^p}\Big(|\widehat{h}_s(0,t-s)|(t-s) + |\widehat{h}_s(2,t+s)|(t+s)\Big)\,\mathrm{d}s \\ &\leqslant \int_0^t z_{\lambda,p}(s)\|h(s)\|_{\lambda,p+1}\frac{1}{\langle s\rangle^p}\,\mathrm{d}s. \end{aligned}$$

Thus we obtain,

$$I_1 \leqslant J^p_{\lambda_0}[\zeta]\mathcal{K}^3[h]\int_0^t\left(\frac{1}{\langle t-s\rangle^2} + \frac{1}{\langle t+s\rangle^2}\right)\mathrm{d}s \leqslant CJ^p_{\lambda_0}[\zeta]\mathcal{K}^3[h]$$

while, if $p-q \geqslant 2$,

$$I_2 \leqslant J^p_{\lambda_0}[\zeta]K^{p+1}_q[h]\int_0^t \frac{1}{\langle s\rangle^{p-q}\,\beta^{1/2}(\lambda,s)}\,\mathrm{d}s \leqslant CJ^p_{\lambda_0}[\zeta]K^{p+1}_q[h]$$

and this concludes the proof. □

2.3.2 Estimates for h

We start by showing how to split the term with $|\xi - ns|$ in (2.33).

Lemma 2.4. *Let $\xi \in \mathbb{R}$, $p \in \mathbb{N}$, $n \in \mathbb{Z}$ and $\lambda > 0$ then*

$$A^{\lambda,p}(n,\xi)|\xi-ns| \leqslant C\left(A^{\lambda,p+1}(n-k,\xi-ks)A^{\lambda,1}(1,s) + A^{\lambda,1}(n-k,\xi-ks)A^{\lambda,p+1}(1,s)\right) \tag{2.41}$$

with $k = \pm 1$.

Proof. We notice that

$$|\xi-ns| = |\xi-ks+(k-n)s| \leqslant \langle s\rangle\langle n-k,\xi-ks\rangle\,.$$

Using the triangular inequality

$$\langle n,\xi\rangle \leqslant \langle n-k,\xi-ks\rangle + \langle k,ks\rangle\,,$$

the fact that

$$\Big(\langle n-k,\xi-ks\rangle + \langle k,ks\rangle\Big)^p \leqslant C\big(\langle n-k,\xi-ks\rangle^p + \langle k,ks\rangle^p\big)$$

and $k=\pm 1$, we get (2.41). □

We now turn to estimate equation (2.33). As usual, we define

$$D_t(n,\xi) = \delta_{n,\pm 1} n \frac{\mathrm{i}}{2} \int_0^t \zeta_s(n) \widetilde{\eta'}(\xi - ns)\,\mathrm{d}s.$$

Lemma 2.5. *Given $\zeta_t(\pm 1)$, for $\lambda, q \geqslant 0$ we have*

$$\begin{aligned} \|h(t)\|_{\lambda,q} &\leqslant \|h_0\|_{\lambda_0,q} + \|D(t)\|_{\lambda,q} \\ &\quad + \varepsilon \int_0^t z_{\lambda,q+1}(s)\|h(s)\|_{\lambda,1} + z_{\lambda,1}(s)\|h(s)\|_{\lambda,q+1}\,\mathrm{d}s. \end{aligned} \tag{2.42}$$

Proof. Multiplying by $A^{\lambda,q}(n,\xi)$ in (2.33) and using (2.41), we get

$$\begin{aligned} A^{\lambda,q}(n,\xi)|h_t(n,\xi)| &\leqslant \|h_0\|_{\lambda_0,q} + A^{\lambda,q}(n,\xi)|D_t(n,\xi)| \\ &\quad + \varepsilon \sum_{k=\pm 1} \int_0^t A^{\lambda,1}(1,s)|\zeta_s(k)|A^{\lambda,q+1}(n-k,\xi-ks)|h_s(n-k,\xi-ks)|\,\mathrm{d}s \\ &\quad + \varepsilon \sum_{k=\pm 1} \int_0^t A^{\lambda,q+1}(1,s)|\zeta_s(k)|A^{\lambda,1}(n-k,\xi-ks)|h_s(n-k,\xi-ks)|\,\mathrm{d}s. \end{aligned}$$

Since $e^{\lambda\langle 1,s\rangle}\langle 1,s\rangle^q \leqslant Ce^{\lambda s}\langle s\rangle^q$, after taking the supremum over n,ξ we obtain the thesis. ☐

Proposition 2.4. *Let $p \geqslant q+3$ with $q \geqslant 3$ fixed. Given $\zeta(\pm 1)$ such that $J^p_{\lambda_0}[\zeta] < +\infty$ we have*

$$K^{3,p+1}_{\lambda_0,q}[h] \leqslant C\|h_0\|_{\lambda_0,p} + CJ^p_{\lambda_0}[\zeta]\|\eta'\| + \varepsilon C\Big(1 + \frac{1}{\delta}\Big) J^p_{\lambda_0}[\zeta] K^{3,p+1}_{\lambda_0,q}[h].$$

Proof. We first estimate the term of order one in (2.33). If $m \geqslant p$,

$$\begin{aligned} A^{\lambda,m}(n,\xi)|D_t(n,\xi)| &\leqslant CJ^p_{\lambda_0}[\zeta]\|\eta'\| \int_0^t e^{-(\gamma-\lambda)\langle\xi-ns\rangle}\langle s\rangle^{m-p}\langle\xi-ns\rangle^p\,\mathrm{d}s \\ &\leqslant CJ^p_{\lambda_0}[\zeta]\|\eta'\|\langle t\rangle^{m-p} \end{aligned} \tag{2.43}$$

where we have used that $A^{\lambda,q}(n,\xi) \leqslant CA^{\lambda,q}(n,\xi-ns)A^{\lambda,q}(n,ns)$ and the hypothesis on η'.

Now, since the norm (2.37) is composed of two parts, we start giving an estimate of the $\mathcal{K}^3$ norm. Using the result in (2.42) we obtain

$$\|h(t)\|_{\lambda,3} \leqslant \|h(0)\|_{\lambda_0,p} + \|D(t)\|_{\lambda,3} + \varepsilon J^p_{\lambda_0}[\zeta] \int_0^t \frac{\mathcal{K}^3[h]}{\langle s\rangle^{p-4}} + \frac{K^{p+1}_q[h]}{\langle s\rangle^{p-1-q}}\,\mathrm{d}s.$$

Using (2.43), we get

$$\mathcal{K}^3[h] \leqslant \|h(0)\|_{\lambda_0,p} + CJ^p_{\lambda_0}[\zeta]\|\eta'\|_\gamma + \varepsilon CJ^p_{\lambda_0}[\zeta]K^{3,p+1}_{\lambda_0,q}[h].$$

Next, we focus on K^{p+1}_q. Using (2.42) with $p+1$, we get

$$\|h(t)\|_{\lambda,p+1} \leqslant C\|h(0)\|_{\lambda_0,p} + \|D(t)\|_{\lambda,p+1} + \varepsilon J^p_{\lambda_0}[\zeta](A_1 + A_2),$$

where

$$A_1 = \int_0^t \langle s\rangle^2 \|h(s)\|_{\lambda,1}\,\mathrm{d}s \leqslant C\langle t\rangle^3 \mathcal{K}^3[h], \quad A_2 = \int_0^t \frac{\|h(s)\|_{\lambda,p+2}}{\langle s\rangle^{p-1}}\,\mathrm{d}s.$$

For what concern A_2 we take

$$\lambda'(s) = \frac{\lambda_0 - \delta\arctan(s) - \lambda}{2}$$

then

$$\|h(s)\|_{\lambda,p+2} \leqslant \frac{\|h(s)\|_{\lambda',p+1}}{\lambda'-\lambda}$$

and we get the bound

$$A_2 \leqslant C\int_0^t \frac{1}{\langle s\rangle^{p-q-1}}\frac{K_q^{p+1}[h]}{\beta^{3/2}(\lambda,s)}\,\mathrm{d}s \leqslant \frac{C}{\delta}\frac{K_q^{p+1}[h]}{\beta^{1/2}(\lambda,t)}$$

where we have used that $p \geqslant q+3$ and the fact that the integral is exactly computable by

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta^{-1/2}(\lambda,t) = \frac{\delta}{2}\frac{1}{\beta^{3/2}(\lambda,t)\langle t\rangle^2}.$$

Then we get, using $q \geqslant 3$,

$$\frac{\beta(\lambda,t)^{1/2}}{\langle t\rangle^q}\varepsilon J^p_{\lambda_0}[\zeta](A_1+A_2) \leqslant \varepsilon J^p_{\lambda_0}[\zeta]\Big(C\mathcal{K}^3[h] + \frac{C}{\delta}K_q^{p+1}[h]\Big). \tag{2.44}$$

It remains to estimate the term of order one $D_t(n,\xi)$. Using (2.43), we obtain

$$\frac{\beta(\lambda,t)^{1/2}}{\langle t\rangle^q}\|D(t)\|_{\lambda,p+1} \leqslant CJ^p_{\lambda_0}[\zeta]\|\eta'\|. \tag{2.45}$$

Collecting the terms in (2.44) and (2.45) we conclude the proof. □

2.3.3 The forward result

Theorem 2.4. *Let us fix $p \geqslant q+3$ with $q \geqslant 3$ and consider $h_0(x,v) \in L^1(\mathbb{T}^1\times\mathbb{R})$ a mean-zero analytic initial perturbation such that $\|h_0\|_{\lambda_0,p} < +\infty$ for some $\lambda_0 > 0$. Let $\eta(v) \in L^1(\mathbb{R})$ analytic such that $\|\eta'\|_\gamma < +\infty$ with $\lambda_0 < \gamma$. Moreover, assume*

$$\mathcal{L}[j(1)](\sigma) \neq 1 \quad \textit{if} \quad \Re\sigma \geqslant 0,$$

with $j(1)$ as in (2.38). Then there exists a unique solution $h_t = h(t,x,v)$ of (2.4) with initial datum h_0 such that $K^{3,p+1}_{\lambda_0,q}[h] < +\infty$ and exist h_∞ with $\|h_\infty\|_{\bar\lambda,p} < +\infty$ for $\bar\lambda < \lambda_0 - \delta\pi/2$ such that

$$\lim_{t\to\infty}\|h_t - h_\infty\|_{L^\infty(\mathbb{T}^1\times\mathbb{R})} = 0$$

with exponential rate.

Proof. The proof is analogous to the first part of Theorem (2.2). By a standard iterative procedure as in (2.24) and using the smallness of the parameter ε, we get the existence of the unique solution h in the class of functions such that $K_q^{3,p+1} < +\infty$. Then the damping property follows from the estimate

$$\|\partial_t h(t)\|_{0,p} \leqslant Ce^{-\bar\lambda t}$$

with $\bar\lambda < \lambda_0 - \delta\pi/2$. It follows that $h(t) \to h_\infty$ with exponential rate. □

2.4 Backward vs forward

In the scattering problem, the decay of the analytic regularity, in the spirit of the abstract Cauchy-Kovalevskaya theorem, is more difficult to establish (compare the definition of $\alpha^T(\mu, t)$ in (2.14), (2.15) with that of $\beta(\lambda, t)$ in (2.36)). Despite this fact, the scattering approach is easier. In particular, the bound on the norm (2.11) guarantees that for any $t \geqslant 0$

$$|\zeta_t(\pm 1)| \leqslant ce^{-\lambda t},$$

while the bound on the norm (2.35) guarantees an estimate with a time correction: for any $t \geqslant 0$ and $\lambda < \lambda_0 - \delta \arctan t$

$$|\zeta_t(\pm 1)| \leqslant ce^{-\lambda t}/\langle t\rangle^p.$$

More in general, the norm on h in (2.12), (2.13) is simpler than that in (2.37), in which we have to introduce algebraic weights like $\langle t\rangle^q$ in order to obtain closed estimates.

This technical issue is mainly due to the different treatment of the plasma echoes, the resonances which occur in (2.10) and (2.34) when $nt = ks$, i.e. when $n = k = \pm 1$, and $t = s$. In the a-priori estimate of $\zeta_{\pm 1}$ in Proposition (2.1), there are no difficulties and we control the resonant terms, those with $k = n$, in the same way as the nonresonant ones, those with $k = -n$. In Proposition (2.3), the echoes force us to introduce the additional term $\mathcal{K}^3$ in the norm of h. Note also that, in Theorem (2.3), we perform a more subtle control of the echoes in (2.31), with an estimate in two-time steps, by using (2.32) and the mean zero of $\omega - \bar{\omega}$. In this way, we obtain the backward nonperturbative result.

The main reason for this different behavior is that the solution $h(t)$, with asymptotic datum h_∞, gains regularity as t increases, thanks to the damping properties of the free flow, while the solution $h(t)$, with initial datum h_0, loses regularity as t increases. This property, together with the hypothesis of analytical regularity, allows to tackle the problem without having to study resonances coming from the echo terms.

Chapter 3

On the scattering approach for the Vlasov-Poisson equation: analytic and Gevrey data

joint work with D. Benedetto and E. Caglioti ([13])

In this Chapter, we apply the Eulerian techniques to the backward Landau damping problem for the Vlasov-Poisson equation. We cover the cases of asymptotic data with analytic and $1/\gamma$-Gevrey regularity, with $\gamma > 1/3$ (see (4) in the Introduction). The asymptotic regime allows us to provide a simplified proof in the perturbative setting with analytic regularity. In the Gevrey case, we recover the 3-Gevrey threshold. This is due to the resonance terms that, in our formulation, are hidden in a linear term of the equation.

3.1 The framework

We recall that, in the spatially periodic case, the one-dimensional Vlasov-Poisson equation reads as

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) + \mathcal{F}[f](t,x)\partial_v f(t,x,v) = 0, \tag{3.1}$$

where

$$\mathcal{F}[f](t,x) = -\partial_x \left(\int_{\mathbb{T}^1\times\mathbb{R}} W(x-y) f(t,y,v)\,\mathrm{d}y\,\mathrm{d}v \right) \tag{3.2}$$

is the mean-field force. Here $W(x)$ is the fundamental solution of the Laplace operator on $\mathbb{T}^1$.

In the previous Chapter, we compared the scattering approach and the forward approach in the case of the Vlasov-HMF (Hamiltonian mean-field) equation. Indeed, concerning the backward Landau damping problem in the perturbative setting with analytic regularity, we showed that no estimates are needed to suppress the echoes, i.e. resonances at certain times between the Fourier modes of the solution, allowing a simple proof of the result.

Here we extend the study of the backward problem to the Vlasov-Poisson equation, also exploring the case of asymptotic data with Gevrey regularity. In this case we recover the 3-Gevrey threshold, also obtained in [9, 22] for the Cauchy problem.

However, in the perturbative setting, the asymptotic regime allows us to observe that:

- in the case of analytic regularity, as in [17, 12], the echo terms are irrelevant, greatly facilitating the proof;
- in the case of Gevrey regularity, the echo terms cannot be neglected and lead to the 3-Gevrey regularity threshold, but it is possible to isolate naturally the reaction term (using the terminology in [36, 9]), i.e. the term of the equation where the echo mechanism is revealed. This allows us to highlight how the echo mechanism is inherent in a linear part of the equation (see also [44] where the echo chains are defined as a secondarily linear effect).

For this reason we divide the proof of the result into the two cases in which the asymptotic datum has analytic or Gevrey regularity. This is also done because in the former case simpler norms than the Gevrey case are used and there is no need to use the energy method in the estimates for the distribution function.

3.2 Notations and conventions

As in the previous Chapter, we introduce the Fourier transform in $\mathbb{T}^1 \times \mathbb{R}$, using the following notation:

$$\widehat{g}(n,\xi) = \frac{1}{2\pi}\int_{\mathbb{T}^1\times\mathbb{R}} e^{-\mathrm{i}nx}e^{-\mathrm{i}v\xi}g(x,v)\,\mathrm{d}x\,\mathrm{d}v$$

with $n \in \mathbb{Z}$ and $\xi \in \mathbb{R}$.

In order to quantify the regularity of the solutions, we will use L^2-type Gevrey norms

$$\|g\|^2_{\mathcal{G}^{\lambda;\gamma,\sigma}} = \sum_n \int e^{2\lambda\langle n,\xi\rangle^\gamma}\langle n,\xi\rangle^{2\sigma}\left(|\widehat{g}(n,\xi)|^2 + |\partial_\xi\widehat{g}(n,\xi)|^2\right)\mathrm{d}\xi, \tag{3.3}$$

with $\lambda, \sigma > 0$ and $0 < \gamma < 1$. In the analytic case, we will work with L^∞-type analytic norms

$$\|g\|_{\lambda;\sigma} = \sup_{n,\xi} e^{\lambda\langle n,\xi\rangle}\langle n,\xi\rangle^\sigma\,|\widehat{g}(n,\xi)| \tag{3.4}$$

while, when working with the spatial density ρ_t, we will use

$$\|\rho_t\|_{\lambda;\gamma,\sigma,\alpha} = \sup_n e^{\lambda\langle n,nt\rangle^\gamma}\langle n,nt\rangle^\sigma\,\frac{|\widehat{\rho}_t(n)|}{|n|^\alpha}, \tag{3.5}$$

where $5/12 < \alpha < 1/2$. We consider solutions $f_t(x,v) = f(t,x,v)$ of (3.1) which are small perturbations of a spatially homogeneous solution η, i.e.

$$f_t(x,v) = \eta(v) + \varepsilon r_t(x,v),$$

and we assume η to be an analytic function of the velocities. The equation verified by the perturbation r is

$$\partial_t r_t(x,v) + v\partial_x r_t(x,v) + \mathcal{F}[r](t,x)\partial_v\big(\eta(v) + \varepsilon r_t(x,v)\big) = 0,$$

where the operator $\mathcal{F}$ is defined in (3.2).

Let $h_t(x,v) = r_t(x+vt,v)$, then it verifies the following equation:

$$\partial_t h = \{\psi[h], \eta + \varepsilon h\}, \tag{3.6}$$

where ψ is the potential field generated by the perturbation, evaluated along the free flow

$$\psi[h](t,x,v) = \int_{\mathbb{T}^1\times\mathbb{R}} W(x-y+(v-u)t)h_t(y,u)\,\mathrm{d}y\,\mathrm{d}u$$

and where $\{,\}$ is the Poisson bracket.

We study the damping problem by searching for a solution for (3.6) such that

$$\lim_{t\to+\infty} \|h_t(x,v) - h_\infty(x,v)\|_{L^\infty(\mathbb{T}^1\times\mathbb{R})} = 0$$

where h_∞ is a mean-zero analytic datum with $\|h_\infty\|_{\lambda;\sigma} < +\infty$ for some $\lambda > 0, \sigma > 0$ as in (3.4).

Firstly, we study the evolution in the time interval $[0,T]$ considering the following problem:

$$\begin{cases} \partial_t h_t^T = \{\psi[h_t^T], \eta + \varepsilon h_t^T\} & 0 \leqslant t \leqslant T, \\ h_T^T(x,v) = h_\infty(x,v). \end{cases} \tag{3.7}$$

Then, we show that, for $T\to+\infty$, h^T converges to a solution h, which solves the asymptotic problem.

The system (3.7) in Fourier space reads as

$$\partial_t \widehat{h_t^T}(n,\xi) = \mathrm{i}\frac{\widehat{\rho_t^T}(n)}{n}\widetilde{\eta}'(\xi-nt) - \varepsilon\sum_{k\neq 0}\frac{\widehat{\rho_t^T}(k)}{k}\widehat{h_t^T}(n-k,\xi-kt)(\xi-nt), \tag{3.8}$$

where $\widetilde{\eta}'$ is the Fourier transform of η' in the velocity and $\widehat{\rho_t^T}(n) = \widehat{h_t^T}(n,nt)$ is the Fourier transform of the spatial density.
Integrating equation (3.8) between $[t,T]$ and putting $\xi = nt$, we get an equation for ρ^T:

$$\begin{aligned} \widehat{\rho_t^T}(n) = \widehat{h_\infty}(n,nt) &- \int_t^T \widehat{\rho_s^T}(n)\widehat{\eta}(n(t-s))(s-t)\,\mathrm{d}s \\ &- \varepsilon\sum_{k\neq 0}\int_t^T \widehat{\rho_s^T}(k)\frac{n(s-t)}{k}\widehat{h_s^T}(n-k,nt-ks)\,\mathrm{d}s. \end{aligned} \tag{3.9}$$

As usual, we are going to give a priori estimates on the coupled system (ρ^T, h^T).

3.3 The analytic case

In this section we give a proof of the scattering result for an analytic asymptotic state $h_\infty(x,v)$ such that $\|h_\infty\|_{\lambda;\sigma} < +\infty$, for a given $\lambda,\sigma > 0$. As we know from [17, 12], the asymptotic regime facilitates the analysis quite well in the analytic setting.

To do this, we define the following norm which quantifies the decaying of the spatial density

$$M_{\lambda,T}[\rho^T] = \sup_{(\mu,t)\in D_{\lambda,T}} e^{\frac{\lambda}{2}t} \|\rho_t^T\|_{\mu;\sigma},$$

where, with little abuse of notation, we denoted by $\|\rho_t^T\|_{\mu;\sigma}$ the quantity $\|\rho_t^T\|_{\mu;1,\sigma,1}$.

Here

$$D_{\lambda,T} = \{(\mu,t) \in [0,\lambda/2) \times [0,T], \mu < \alpha_T(t)\}$$

and $\alpha_T(t) = \lambda/2 - C_\alpha(e^{-\frac{\lambda}{4}t} - e^{-\frac{\lambda}{4}T})$, with C_α such that $\lambda/2 - C_\alpha > 0$.

To take into account the decay of the analytic regularity, we define the weighted-in-time analytic norm of $(h_t^T - h_\infty)$ as

$$N_{\lambda,T}[h_t^T - h_\infty] = \sup_{(\mu,t)\in D_{\lambda,T}} (\alpha_T(t) - \mu)^{1/2} e^{\frac{\lambda}{4}t} \|h_t^T - h_\infty\|_{\mu;\sigma}. \tag{3.10}$$

3.3.1 Analytic a priori estimates for ρ^T

As we show more accurately in the following proposition, eq. (3.9) for the field ρ^T has the structure of a Volterra equation.

We set

$$j_n(t) \equiv t\widehat{\eta}(-nt) \tag{3.11}$$

and

$$H_\varepsilon^T(t) = \widehat{h_\infty}(n, nt) - \varepsilon \sum_{k\neq 0} \int_t^T \widehat{\rho_s^T}(k) \frac{n(s-t)}{k} \widehat{h_s^T}(n-k, nt-ks)\,\mathrm{d}s. \tag{3.12}$$

In order to invert the term of order one in the equation, we use the following result.

Lemma 3.1. *Let $\lambda > 0$ with $\|h_\infty\|_{\lambda;\sigma} < +\infty$ and $\|\eta\|_{\lambda;\sigma} < +\infty$. Assume that*

$$\inf_{n\in\mathbb{Z};\Re\sigma\geqslant 0} |\mathcal{L}[j_n](\sigma) + 1| \geqslant \kappa > 0, \tag{3.13}$$

where

$$\mathcal{L}[j](\sigma) = \int_0^{+\infty} e^{-\sigma t} j(t)\,\mathrm{d}t$$

is the Fourier-Laplace transform of j. Then

$$M_{\lambda,T}[\rho^T] \leqslant C M_{\lambda,T}[H_\varepsilon^T].$$

Proof. Let us define $\phi_n(t) = \rho_{T-t}^T(n)$, $F_\varepsilon(t) = H_\varepsilon^T(T-t)$. Then, (3.9) can be rewritten as

$$\phi_n(t) + j_n * \phi_n(t) = F_\varepsilon(t), \tag{3.14}$$

for $t \in [0,T]$, where

$$j * f(t) = \int_0^t j(t-s) f(s)\,\mathrm{d}s$$

is the convolution between j and f.

Taking the Laplace transform on both sides, we get

$$\mathcal{L}[\phi_n](\sigma) = \frac{\mathcal{L}[F_\varepsilon](\sigma)}{1+\mathcal{L}[j_n](\sigma)} = \mathcal{L}[F_\varepsilon](\sigma) - \frac{\mathcal{L}[j_n](\sigma)}{1+\mathcal{L}[j_n](\sigma)}\mathcal{L}[F_\varepsilon](\sigma),$$

provided that $1+\mathcal{L}[j_n](\sigma) \neq 0$, which is guaranteed by (3.13). Moreover, taking the inverse Laplace transform, we get that $\phi_n(t) = F_\varepsilon(t) - r_n * F_\varepsilon(t)$ where

$$r_n(t) = \frac{1}{2\pi \mathrm{i}}\int_{\sigma_0-\mathrm{i}\infty}^{\sigma_0+\mathrm{i}\infty} e^{\sigma t}\frac{\mathcal{L}[j_n](\sigma)}{1+\mathcal{L}[j_n](\sigma)}\,\mathrm{d}\sigma, \tag{3.15}$$

with σ_0 sufficiently large. Then, it is not difficult to show that

$$\frac{\mathcal{L}[j_n](\sigma)}{1+\mathcal{L}[j_n](\sigma)}$$

is holomorphic in $\Re\sigma \geqslant -\mu$, for any $0<\mu<\lambda$ and that, integrating by parts, it verifies

$$\left|\frac{\mathcal{L}[j_n](\sigma)}{1+\mathcal{L}[j_n](\sigma)}\right| \leqslant \frac{C}{1+|\Im\sigma|^2}$$

for $\Re\sigma = -\mu$ (see [22] for more details). Then, by contour deformation in (3.15), we get that $|r_n(t)| \leqslant Ce^{-\nu|nt|}$ with $\lambda/2<\nu<\lambda$.
Multiplying by $e^{\mu\langle n,n(T-t)\rangle}\langle n,n(T-t)\rangle^\sigma/|n|$, using the triangular inequality and taking the sup over n, we get

$$\|\rho^T_{T-t}\|_{\mu;\sigma} \leqslant \|H^T_\varepsilon(T-t)\|_{\mu;\sigma} + C\int_0^T e^{-(\nu-\lambda/2)(s-t)}\|\rho^T_s\|_{\mu;\sigma}\,\mathrm{d}s.$$

Multiplying by $e^{\frac{\lambda}{2}(T-t)}$ and taking the sup over $D_{\lambda,T}$, we conclude that

$$M_{\lambda,T}[\rho^T] \leqslant M_{\lambda,T}[H^T_\varepsilon] + C_\lambda M_{\lambda,T}[H^T_\varepsilon].$$

□

We now arrive at the main estimate of this section. In the proof we will notice that the echoes terms will be easily neglected thanks to the decay given by the analytic regularity.

Proposition 3.1. *Let ρ^T solution of* (3.9) *and suppose $N_{\lambda,T}[h^T] < +\infty$. Then, under the hypothesis of Lemma* (3.1)*, we have*

$$M_{\lambda,T}[\rho^T] \leqslant \|h_\infty\|_{\lambda;\sigma} + \varepsilon C_\lambda M_{\lambda,T}[\rho^T]N_{\lambda,T}[h^T-h_\infty] + \varepsilon C_\lambda M_{\lambda,T}[\rho^T]\|h_\infty\|_{\lambda;\sigma}.$$

Proof. From Lemma (3.1) we need only to estimate $M_{\lambda,T}[H^T_\varepsilon]$. From (3.12), we have

$$\begin{aligned} H^T_\varepsilon(t) = \widehat{h_\infty}(n,nt) &- \varepsilon\sum_{k\neq 0}\int_t^T \widehat{\rho^T_s}(k)\frac{n(s-t)}{k}\widehat{(h^T_s-h_\infty)}(n-k,nt-ks)\,\mathrm{d}s \\ &- \varepsilon\sum_{k\neq 0}\int_t^T \widehat{\rho^T_s}(k)\frac{n(s-t)}{k}\widehat{h_\infty}(n-k,nt-ks)\,\mathrm{d}s. \end{aligned}$$

By simple computations, we have

$$
\begin{aligned}
&e^{\mu\langle n,nt\rangle}\langle n,nt\rangle^{\sigma}\frac{|H_{\varepsilon}^{T}(t)|}{|n|}\leqslant e^{-\frac{\lambda}{2}t}\|h_\infty\|_{\lambda;\sigma}\\
&+\varepsilon M_{\lambda,T}[\rho^T]N_{\lambda,T}[h^T-h_\infty]\sum_{k\neq 0}\Big(\frac{1}{\langle n-k\rangle^{\sigma}}+\frac{1}{\langle k\rangle^{\sigma}}\Big)\int_t^T\frac{e^{-\frac{\lambda}{2}s}(s-t)e^{-\frac{\lambda}{4}s}}{(\alpha_T(s)-\mu)^{1/2}}\,\mathrm{d}s\\
&+\varepsilon\sum_{k\neq 0}M_{\lambda,T}[\rho^T]\|h_\infty\|_{\lambda;\sigma}\int_t^T e^{-\frac{\lambda}{2}s}(s-t)e^{-\frac{\lambda}{2}\langle n-k,nt-ks\rangle}\,\mathrm{d}s.
\end{aligned}
$$

Then, taking the sup over n, multiplying by $e^{\frac{\lambda}{2}t}$ and using the definition of $\alpha_T(t)$ we have:

$$
M_{\lambda,T}[H_{\varepsilon}^T]\leqslant\|h_\infty\|_{\lambda;\sigma}+\varepsilon C_\lambda M_{\lambda,T}[\rho^T]N_{\lambda,T}[h^T-h_\infty]+\varepsilon C_\lambda M_{\lambda,T}[\rho^T]\|h_\infty\|_{\lambda;\sigma}.
$$

□

3.3.2 Analytic a priori estimates for h^T-h_∞

Now we turn to give a Cauchy-Kovalevskaya estimate on h^T-h_∞. Due to the loss of analytic regularity in time, it is crucial to use the weighted norm introduced in (3.10).

Proposition 3.2. *Under the hypothesis of the previous Proposition, let h^T a solution of* (3.7) *and assume $M_{\lambda,T}[\zeta^T]<+\infty$ then the following estimate holds:*

$$
N_{\lambda,T}[h^T-h_\infty]\leqslant C_\lambda M_{\lambda,T}\Big(\|\eta\|_\lambda+\varepsilon N_{\lambda,T}[h^T-h_\infty]+\varepsilon\|h_\infty\|_{\lambda;\sigma}\Big).
$$

Proof. From (3.8) we have

$$
(\widehat{h_t^T-h_\infty})(n,\xi)=D_t^T(n,\xi)+\varepsilon E_t^T(n,\xi)+\varepsilon F_t^T(n,\xi),
$$

where

$$
D_t^T(n,\xi)=-\frac{\mathrm{i}}{n}\int_t^T\widehat{\rho_s^T}(n)\widehat{\eta}'(\xi-ns)\,\mathrm{d}s,
$$

$$
E_t^T(n,\xi)=\sum_{k\neq 0}\int_t^T\frac{\widehat{\rho_s^T}(k)}{k}(\widehat{h_s^T-h_\infty})(n-k,\xi-ks)(\xi-ns)\,\mathrm{d}s
$$

and

$$
F_t^T(n,\xi)=\sum_{k\neq 0}\int_t^T\frac{\widehat{\rho_s^T}(k)}{k}\widehat{h_\infty}(n-k,\xi-ks)(\xi-ns)\,\mathrm{d}s.
$$

Fixing $\mu<\alpha_T(t)$, we estimate separately the three terms. As regards D_t, by the triangular inequality we get

$$
e^{\mu\langle n,\xi\rangle}\langle n,\xi\rangle^{\sigma}|D_t^T(n,\xi)|\leqslant CM_{\lambda,T}[\rho^T]\|\eta\|_{\lambda;\sigma}\int_t^T e^{-\frac{\lambda}{2}s}\,\mathrm{d}s.
$$

Taking the supremum over n,ξ, multiplying by $e^{\frac{\lambda}{4}t}$ and taking the supremum over $D_{\lambda,T}$, we arrive at

$$
N_{\lambda,T}[D^T]\leqslant CM_{\lambda,T}[\rho^T]\|\eta\|_{\lambda;\sigma}.
$$

Concerning E^T, using that

$$\frac{\langle n,\xi\rangle^\sigma}{\langle k,ks\rangle^\sigma\,\langle n-k,\xi-ks\rangle^\sigma}\leqslant\Big(\frac{1}{\langle n-k\rangle^\sigma}+\frac{1}{\langle k\rangle^\sigma}\Big),\tag{3.16}$$

we have

$$e^{\mu\langle n,\xi\rangle}\langle n,\xi\rangle^\sigma\,|E_t^T(n,\xi)|\leqslant\sum_{k\neq 0}\Big(\frac{1}{\langle n-k\rangle^\sigma}+\frac{1}{\langle k\rangle^\sigma}\Big)M_{\lambda,T}[\rho^T]\times$$
$$\times\int_t^T e^{-\frac{\lambda}{2}s}\|h_s^T-h_\infty\|_{\mu_s;\sigma}e^{-(\mu_s-\mu)\langle n-k,\xi-ks\rangle}|\xi-ns|\,\mathrm{d}s.$$

Taking $\mu_s=(\alpha_T(s)+\mu)/2$, i.e. the middle point between μ and $a_{T,\delta}(s)$, we have

$$e^{-(\mu(s)-\mu)\langle n-k,\xi-ks\rangle}|\xi-ns|\leqslant\frac{2(1+s)}{\alpha_T(s)-\mu}.$$

Hence

$$e^{\mu\langle n,\xi\rangle}\langle n,\xi\rangle^\sigma\,|E_t^T(n,\xi)|\leqslant CM_{\lambda,T}[\rho^T]N_{\lambda,T}[h^T-h_\infty]\int_t^T\frac{e^{-\frac{3\lambda}{4}s}(1+s)}{(\alpha_T(s)-\mu)^{3/2}}\,\mathrm{d}s.$$

Taking the supremum over n,ξ, multiplying by $e^{\frac{\lambda}{4}t}$ and that, from the definition of α_T

$$\frac{e^{-\frac{\lambda}{4}s}}{(\alpha_T(s)-\mu)^{3/2}}=-\frac{2}{C_\alpha}\frac{\mathrm{d}}{\mathrm{d}s}(\alpha_T(s)-\mu)^{-1/2},$$

we get

$$N_{\lambda,T}[E^T]\leqslant C_\lambda M_{\lambda,T}[\rho^T]N_{\lambda,T}[h^T-h_\infty].$$

We need only to estimate F^T. In this case we have

$$e^{\mu\langle n,\xi\rangle}\langle n,\xi\rangle^\sigma\,|F_t^T(n,\xi)|\leqslant\sum_{k\neq 0}M_{\lambda,T}[\rho^T]\int_t^T e^{-\frac{\lambda}{2}s}\|h_\infty\|_{\lambda;\sigma}e^{-\frac{\lambda}{2}\langle n-k,\xi-ks\rangle}|\xi-ns|\,\mathrm{d}s$$

and we use the extra-decay in h_∞ to control the derivative and the sum in k. In the end, taking the supremum over n,ξ and multiplying by $e^{\frac{\lambda}{4}t}$, we conclude

$$N_{\lambda,T}[F^T]\leqslant C_\lambda M_{\lambda,T}[\rho^T]\|h_\infty\|_{\lambda;\sigma}.$$

By collecting the estimates we get the thesis. □

3.4 The Gevrey case

In this section, we give a priori estimates for the Gevrey regularity setting. It is worth noticing that the norms that will be introduced are more general than the ones introduced in the previous paragraph, hence the following estimates include also the analytic case.

In the Gevrey setting, we define the norm for the spatial density as

$$M^\gamma_{\lambda,T}[\rho^T]=\sup_{(\mu,t)\in D_{\lambda,T}}\langle t\rangle^{4/\gamma+6+\delta}\,\|\rho_t^T\|_{\mu;\gamma,\sigma,\alpha},$$

where $\|\cdot\|_{\mu;\gamma,\sigma,\alpha}$ is defined in (3.5), $0 < \delta \ll 1$ and

$$D_{\lambda,T} = \{(\mu,t) \in [0,\lambda/2) \times [0,T], \mu < a_T(t)\},$$

where $a_T(t) = \lambda/2 - C_a[1/\langle t\rangle^\delta - 1/\langle T\rangle^\delta]$ and $C_a > 0$ such that $\lambda/2 - C_a > 0$.

To take into account the decay of the Gevrey regularity, we define the weighted-in-time analytic norm of $h_t^T - h_\infty$ as

$$N_{\lambda,T}^\gamma[h_t^T - h_\infty] = \sup_{(\mu,t)\in D_{\lambda,T}} (a_T(t)-\mu)^{1/4} \langle t\rangle^{2/\gamma+2} \|h_t^T - h_\infty\|_{\mathcal{G}^{\mu;\gamma,\sigma}}, \tag{3.17}$$

where $\|\cdot\|_{\mathcal{G}^{\mu;\gamma,\sigma}}$ as in (3.3).

3.4.1 Gevrey a priori estimates for ζ^T

We now turn to give Gevrey estimates for (3.9). As before, we're going to use Lemma (3.1) to invert the linear part of the equation. We recall the notations in (3.11) and (3.12).

Lemma 3.2. *Let $\gamma < 1$ and assume that h_∞ is a Gevrey function such that $\|h_\infty\|_{\lambda;\gamma,\sigma} = \|h_\infty\|_{\lambda;\gamma,\sigma,0} < +\infty$ (see (3.5)). Assume that η is analytic with $\|\eta\|_{\lambda;\sigma} < +\infty$ and such that*

$$\inf_{n\in\mathbb{Z};\Re\sigma\geqslant 0} |\mathcal{L}[j_n](\sigma)+1| \geqslant \kappa > 0,$$

then

$$M_{\lambda,T}^\gamma[\rho^T] \leqslant C M_{\lambda,T}^\gamma[H_\varepsilon^T].$$

Proof. With the same notations of (3.14), equation (3.9) can be rewritten as

$$\phi_n(t) + j_n * \phi_n(t) = F_\varepsilon(t),$$

for $t \in [0,T]$. Then, from Lemma (3.1), we know that

$$\phi_n(t) = F_\varepsilon(t) - \int_0^t r_n(t-s)F_\varepsilon(s)\,\mathrm{d}s$$

and $|r_n(t)| \leqslant Ce^{-\nu|nt|}$ with $\nu < \lambda$.

Taking $\mu < a_T(T-t)$, multiplying by $e^{\mu\langle n,n(T-t)\rangle^\gamma}\langle n,n(T-t)\rangle^\sigma$ and using the triangular inequalities as in (3.16), we have

$$\|\phi(t)\|_{\mu;\gamma,\sigma,\alpha} \leqslant \|F_\varepsilon(t)\|_{\mu;\gamma,\sigma,\alpha} + \int_0^t e^{-C_\lambda(s-t)}\|\phi(s)\|_{\mu;\gamma,\sigma,\alpha}\,\mathrm{d}s.$$

Therefore, multiplying by $\langle T-t\rangle^{4/\gamma+6+\delta}$ and taking the sup, we get

$$M_{\lambda,T}^\gamma[\rho^T] \leqslant M_{\lambda,T}^\gamma[H_\varepsilon^T] + C_\lambda M_{\lambda,T}^\gamma[H_\varepsilon^T].$$

□

The main result of this paragraph concerns the Gevrey estimate of the nonlinear part of equation (3.9). Here we will discover how the echo mechanism is revealed in a linear term of the equation and we will understand the need for the Gevrey regularity threshold $\gamma > 1/3$.

Proposition 3.3. *Let ρ^T solution of (3.9) and suppose $N^\gamma_{\lambda,T}[h^T] < +\infty$. Then, under the hypothesis of Lemma (3.2), we have*

$$M^\gamma_{\lambda,T}[\rho^T] \leqslant C\|h_\infty\|_{\lambda;\gamma,\sigma} + \varepsilon C M^\gamma_{\lambda,T}[\rho^T] N^\gamma_{\lambda,T}[h^T - h_\infty] + \varepsilon C M^\gamma_{\lambda,T}[\rho^T]\|h_\infty\|_{\lambda;\gamma,\sigma}.$$

Proof. From Lemma (3.2) we need only to estimate $M^\gamma_{\lambda,T}[H^T_\varepsilon]$. From (3.12) we have that

$$H^T_\varepsilon(t) = \widehat{h_\infty}(n,nt) - \varepsilon \sum_{k\neq 0} \int_t^T \widehat{\rho^T_s}(k) \frac{n(s-t)}{k} (\widehat{h^T_s - h_\infty})(n-k, nt-ks)\,\mathrm{d}s$$

$$- \varepsilon \sum_{k\neq 0} \int_t^T \widehat{\rho^T_s}(k) \frac{n(s-t)}{k} \widehat{h_\infty}(n-k, nt-ks)\,\mathrm{d}s \equiv \widehat{h_\infty}(n,nt) + B_1(t) + B_2(t). \quad (3.18)$$

Taking $\mu < a_T(t)$, multiplying by $e^{\mu\langle n,nt\rangle^\gamma}\langle n,nt\rangle^\sigma$ and using the triangular inequalities as in (3.16), we get $\|B_1(t)\|_{\mu;\gamma,\sigma,\alpha}$ is bounded by

$$\varepsilon \sum_{k\neq 0} \int_t^T \frac{e^{-(\mu(s)-\mu)\langle n,nt\rangle^\gamma}\langle n,nt\rangle^\sigma}{\langle n-k,nt-ks\rangle^\sigma\langle k,ks\rangle^\sigma} \frac{|n|^{1-\alpha}(s-t)}{|k|^{1-\alpha}} \|\rho^T_s\|_{\mu(s);\gamma,\sigma}\|h^T_s - h_\infty\|_{\mathcal{G}^{\mu;\gamma,\sigma}}\,\mathrm{d}s. \quad (3.19)$$

Choosing $\mu(s) = \mu + a_T(s) - a_T(t)$, we have that

$$\exp\{-(\mu(s)-\mu)\langle n,nt\rangle^\gamma\} \leqslant \exp\{-a'_T(s)(s-t)|nt|^\gamma\}.$$

From which follows that

$$e^{-(\mu(s)-\mu)\langle n,nt\rangle^\gamma}|n|^{1-\alpha}(s-t) \leqslant \frac{\langle s\rangle^{(1+\delta)(1-\alpha)/\gamma}}{\langle t\rangle^{1-\alpha}(s-t)^{\frac{1-\alpha}{\gamma}-1}}.$$

Hence

$$\begin{aligned}\|B_1(t)\|_{\lambda;\gamma,\sigma} &\leqslant \varepsilon M^\gamma_{\lambda,T}[\rho^T]N^\gamma_{\lambda,T}[h^T - h_\infty]\times\\ &\quad\times \sum_{k\neq 0}\frac{1}{\langle k\rangle^\sigma}\int_t^T \frac{\langle s\rangle^{-4/\gamma-6-\delta}}{\langle t\rangle^{1-\alpha}(s-t)^{\frac{1-\alpha}{\gamma}-1}}\frac{1}{(a_T(s)-\mu)^{1/4}}\,\mathrm{d}s\\ &\leqslant \varepsilon M^\gamma_{\lambda,T}[\rho^T]N^\gamma_{\lambda,T}[h^T - h_\infty]\frac{1}{\langle t\rangle^{1-\alpha}}\int_t^T \frac{\langle s\rangle^{-4/\gamma-6-\delta}}{\langle s\rangle^{3/4}(s-t)^{|\frac{1-\alpha}{\gamma}-3/4|}}\,\mathrm{d}s.\end{aligned}$$

Noting that $\frac{1-\alpha}{\gamma} - 3/4 < 1$ since $1/3 < \gamma \leqslant 1$ and $5/12 < \alpha < 1/2$, multiplying by $\langle t\rangle^{4/\gamma+6+\delta}$ and taking the sup,

$$M^\gamma_{\lambda,T}[B_1] \leqslant C\varepsilon M^\gamma_{\lambda,T}[\rho^T]N^\gamma_{\lambda,T}[h^T - h_\infty].$$

We now give an estimate of $B_2(t)$. As in (3.19), $\|B_2(t)\|_{\mu;\gamma,\sigma,\alpha}$ is bounded by

$$\varepsilon \sum_{k\neq 0}\frac{1}{\langle k\rangle^\sigma}\int_t^T e^{-(\mu(s)-\mu)\langle k,ks\rangle^\gamma}\frac{|n|^{1-\alpha}(s-t)}{|k|^{1-\alpha}}\|\rho^T_s\|_{\mu(s);\gamma,\sigma}e^{-\lambda\langle n-k,nt-ks\rangle^\gamma/2}\|h_\infty\|_{\lambda;\gamma,\sigma}\,\mathrm{d}s.$$

We divide the estimate in two cases: if $|nt-ks|>t/2$ then

$$\exp\{-\frac{\lambda}{2}\langle n-k, nt-ks\rangle^{\gamma}\} \leqslant \exp\{-\frac{\lambda}{2}\langle n-k, t/2\rangle^{\gamma}\},$$

from which is easy to close the estimate, since we have sufficient decay.

If $|nt-ks|<t/2$ then $|k|\leqslant|n|$, since $s>t$. If $k=n$, we get sufficient decay as in the previous case. Hence we reduce to $|k|<|n|$, where we have to overcome the resonances due to plasma echoes. Here, since $\frac{|n|-1/2}{|k|}t<s<\frac{|n|+1/2}{|k|}t$, we get that, choosing $\mu(s)=\mu+a_T(s)-a_T(t)$,

$$\exp\{-(\mu(s)-\mu)\langle k, ks\rangle^{\gamma}\} \leqslant \exp\Big\{-C_a\langle k\rangle^{\gamma}\frac{|n-k|}{|n|}|s|^{\gamma-\delta}\Big\}.$$

It follows that, if $|n-k|\geqslant|n|/2$, then we get sufficient decay one more time. Hence, we focus on $|n-k|<|n|/2$, that means $|k|>|n|/2$. Then, we get

$$\exp\{-(\mu(s)-\mu)\langle k, ks\rangle^{\gamma}\}\frac{(s-t)}{|k|} \leqslant e^{-C_a\langle k\rangle^{\gamma}\frac{|n-k|}{|n|}|s|^{\gamma-\delta}}s\frac{|n-k|}{|n||k|} \leqslant C\frac{|n|^{-2+\frac{1-\gamma}{\gamma-\delta}}}{|n-k|^{\frac{1}{\gamma-\delta}-1}}$$

and we have that $-2+\frac{1-\gamma}{\gamma-\delta}<0$ if $\gamma>1/3$. Hence, we get

$$M^{\gamma}_{\lambda,T}[B_2] \leqslant C\varepsilon M^{\gamma}_{\lambda,T}[\rho^T]\|h_\infty\|_{\mathcal{G}^{\lambda;\gamma,\sigma}}.$$

Collecting the estimates on B_1 and B_2 we get the thesis.

□

3.4.2 A priori Gevrey estimates for h^T

This proof is analogous to the one given in [22]. Here, instead of the generator functions approach, we use the global-in-time norms defined in (3.17).

Given $\mu<a_T(t)$, the Fourier multiplier operator

$$A_\mu(\nabla)=e^{\mu\langle\nabla\rangle^{\gamma}}\langle\nabla\rangle^{\sigma}$$

is defined by

$$(\widehat{A_\mu(\nabla)}h)(n,\xi) \coloneqq e^{\mu\langle n,\xi\rangle^{\gamma}}\langle n,\xi\rangle^{\sigma}\,\widehat{h}(n,\xi).$$

From (3.6), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|h^T_t-h_\infty\|^2_{\mathcal{G}^{\mu;\gamma,\sigma}} &= \frac{\mathrm{d}}{\mathrm{d}t}\|A_\mu(\nabla)(h^T_t-h_\infty)\|^2_2\\ &= 2\Big\langle A_\mu(\nabla)(h^T_t-h_\infty)\Big|A_\mu(\nabla)\{\psi[h^T_t],\eta\}\Big\rangle\\ &+2\varepsilon\Big\langle A_\mu(\nabla)(h^T_t-h_\infty)\Big|A_\mu(\nabla)\{\psi[h^T_t],h_\infty\}\Big\rangle\\ &+2\varepsilon\Big\langle A_\mu(\nabla)(h^T_t-h_\infty)\Big|A_\mu(\nabla)\{\psi[h^T_t],h^T_t-h_\infty\}\Big\rangle,\end{aligned} \tag{3.20}$$

where $\langle\cdot|\cdot\rangle$ is the L^2 scalar product. We rewrite the last term as

$$\begin{aligned}&\Big\langle A_\mu(\nabla)(h^T_t-h_\infty)\Big|A_\mu(\nabla)\{\psi[h^T_t],h^T_t-h_\infty\}-\{\psi[h^T_t],A_\mu(\nabla)(h^T_t-h_\infty)\}\Big\rangle\\ &\qquad+\Big\langle A_\mu(\nabla)(h^T_t-h_\infty)\Big|\{\psi[h^T_t],A_\mu(\nabla)(h^T_t-h_\infty)\}\Big\rangle\end{aligned}$$

and we notice that

$$\left\langle A_\mu(\nabla)(h_t^T - h_\infty)\middle|\{\psi[h_t^T], A_\mu(\nabla)(h_t^T - h_\infty)\}\right\rangle = 0,$$

since $\langle f|\{\psi, g\}\rangle$ is skew-symmetric. We denote the three terms we are going to estimate in (3.20) by

$$A_t^1 = \left\langle A_\mu(\nabla)(h_t^T - h_\infty)\middle|A_\mu(\nabla)\{\psi[h_t^T], \eta\}\right\rangle,$$

$$A_t^2 = \left\langle A_\mu(\nabla)(h_t^T - h_\infty)\middle|A_\mu(\nabla)\{\psi[h_t^T], h_\infty\}\right\rangle,$$

$$A_t^3 = \left\langle A_\mu(\nabla)(h_t^T - h_\infty)\middle|[A_\mu(\nabla), \mathcal{U}\cdot\nabla](h_t^T - h_\infty)\right\rangle,$$

where $[\cdot,\cdot]$ is the commutator and $\mathcal{U} = (-\partial_v\psi, \partial_x\psi)$.

By the Plancherel identity we have that

$$\begin{aligned}
|A_t^1| &\leqslant \sum_n \int \mathrm{d}\xi A_\mu(n,\xi)|(\widehat{h_t^T - h_\infty})(n,\xi)|A_\mu(n,\xi)\frac{|\widehat{\rho_t^T}(n)|}{n}|\widehat{\eta}'(\xi - nt)| \\
&\leqslant C\sum_n A_\mu(n,nt)\frac{|\widehat{\rho_t^T}(n)|}{n}\int \mathrm{d}\xi A_\mu(n,\xi)|(\widehat{h^T - h_\infty})(n,\xi)|A_\mu(\xi - nt)|\widehat{\eta}'(\xi - nt)| \\
&\leqslant C_\lambda\|\eta'\|_{\lambda;\sigma}\left(\sum_n A_\mu^2(n,nt)\frac{|\widehat{\rho_t^T}(n)|^2}{n^2}\right)^{1/2}\left(\sum_n\int \mathrm{d}\xi A_\mu^2(n,\xi)|(\widehat{h^T - h_\infty})(n,\xi)|^2\right)^{1/2},
\end{aligned}$$

where in the second estimate we have used the triangular inequality, while in the third we have used two times the Cauchy-Schwartz inequality (firstly in the ξ variable and then in n).

The estimate on A_t^1 is closed using the definition of the norm $\|\cdot\|_{\lambda;\gamma,\sigma,\alpha}$ and the fact that $\alpha < 1/2$:

$$|A_t^1| \leqslant C_\lambda\|\eta'\|_{\lambda;\sigma}\|\rho_t^T\|_{\lambda;\gamma,\sigma,\alpha}\|h_t^T - h_\infty\|_{\mathcal{G}^{\mu;\gamma,\sigma}}.$$

For A_t^2 we get

$$\begin{aligned}
|A_t^2| &\leqslant \sum_{n,k\neq 0}\int \mathrm{d}\xi|A_\mu(\widehat{h_t^T - h_\infty})(n,\xi)|\frac{|A_\mu(k,kt)\widehat{\rho_t^T}(k)|}{k}|\xi - nt||A_\mu\widehat{h_\infty}(n-k,\xi - kt)| \\
&\leqslant \langle t\rangle\sum_n\Big(\int |A_\mu(\widehat{h_t^T - h_\infty})(n,\xi)|^2\,\mathrm{d}\xi\Big)^{1/2}\times \\
&\times\sum_{k\neq 0}\frac{|A_\mu(k,kt)\widehat{\rho_t^T}(k)|}{k}\Big(\int |A_\mu\widehat{\nabla h_\infty}(n-k,\xi - kt)|^2\,\mathrm{d}\xi\Big)^{1/2} \\
&\leqslant \langle t\rangle\|h_t^T - h_\infty\|_{\mathcal{G}^\mu}\left[\sum_n\left(\sum_{k\neq 0}\frac{|A_\mu(k,kt)\widehat{\rho_t^T}(k)|}{k}\Big(\int |A_\mu\widehat{\nabla h_\infty}(n-k,\xi)|^2\,\mathrm{d}\xi\Big)^{1/2}\right)^2\right]^{1/2}
\end{aligned}$$

and noting that, in the square brackets, we have the L_2-norm of a convolution in n, we get by the Young inequality:

$$|A_2(t)| \leqslant \langle t\rangle C_\lambda\|h_t^T - h_\infty\|_{\mathcal{G}^{\mu;\gamma,\sigma}}\left(\sum_{k\neq 0}\frac{|A_\mu(k,kt)\widehat{\rho_t^T}(k)|^2}{k^2}\right)\|\nabla h_\infty\|_{\lambda;\gamma,\sigma}.$$

Using the definition of the norm $M^{\gamma}_{\lambda,T}$ and the fact that $\alpha < 1/2$ we obtain:

$$|A_2(t)| \leqslant \langle t\rangle C_\lambda \|h^T_t - h_\infty\|_{\mathcal{G}^{\mu;\gamma,\sigma}} \|\rho^T_t\|_{\mu;\gamma,\sigma,\alpha} \|\nabla h_\infty\|_{\lambda;\gamma,\sigma}.$$

We conclude by estimating A^3_t. We have that

$$\begin{aligned}|A^3_t| \leqslant \sum_{n,k\neq 0} \int \mathrm{d}\xi A_\mu(n,\xi) |(\widehat{h^T_t - h_\infty})(n,\xi)| \frac{|\widehat{\rho^T_t}(k)|}{k} \times \\ \times |A_\mu(n,\xi) - A_\mu(n-k,\xi-kt)||\xi - nt||(\widehat{h_t - h_\infty})(n-k,\xi-kt)|.\end{aligned}$$

We split the estimate into two dichotomous cases:

$$\text{(a) } \langle k, kt\rangle \geqslant \langle n,\xi\rangle/2 \quad \text{and} \quad \text{(b) } \langle n-k,\xi-kt\rangle \geqslant \langle n,\xi\rangle/2.$$

In case (a) we have that:

$$\frac{|\xi - nt||A_\mu(n,\xi) + A_\mu(n-k,\xi-kt)|}{A_\mu(k,kt)A_\mu(n-k,\xi-kt)} \leqslant C\langle t\rangle\Big(\frac{1}{\langle n-k\rangle^{\sigma-1}} + \frac{1}{\langle k\rangle^{\sigma-1}}\Big).$$

In case (b) we use that

$$|A_\mu(n,\xi) - A_\mu(n-k,\xi-kt)| \leqslant C\frac{|k,kt|A_\mu(n,\xi)A_\mu(n-k,\xi-kt)}{\langle n,\xi\rangle^{1-\gamma} + \langle n-k,\xi-kt\rangle^{1-\gamma}},$$

gaining that

$$\begin{aligned}\frac{|\xi - nt||A_\mu(n,\xi) - A_\mu(n-k,\xi-kt)|}{A_\mu(k,kt)A_\mu(n-k,\xi-kt)} &\leqslant C\frac{|\xi - nt|\langle k,kt\rangle^{-\sigma+1}}{\langle n,\xi\rangle^{1-\gamma} + \langle n-k,\xi-kt\rangle^{1-\gamma}} \\ &\leqslant C\langle k\rangle^{-\sigma+1}\langle n,\xi\rangle^{\gamma/2}\langle n-k,\xi-kt\rangle^{\gamma/2}.\end{aligned}$$

Then we have, using again that $2ab \leqslant a^2 + b^2$:

$$\begin{aligned}|A^3_t| \leqslant \sum_{n,k\neq 0}\Big(\frac{\langle t\rangle}{\langle n-k\rangle^{\sigma}} + \frac{\langle t\rangle}{\langle k\rangle^{\sigma}}\Big)\|\rho^T_t\|_{\mu;\gamma,\sigma,\alpha}\int \langle n,\xi\rangle^{\gamma} A^2_\mu(n,\xi)|(\widehat{h^T_t - h_\infty})(n,\xi)|^2 \\ + \langle n-k,\xi-kt\rangle^{\gamma}\frac{A^2_\mu(n-k,\xi-nt)}{k^{2-2\alpha}}|(\widehat{h^T_t - h_\infty})(n-k,\xi-kt)|\,\mathrm{d}\xi,\end{aligned}$$

obtaining

$$|A^3(t)| \leqslant \langle t\rangle C_\lambda \|\rho^T_t\|_{\mu;\gamma,\sigma,\alpha}\|h^T_t - h_\infty\|^2_{\mathcal{G}^{\mu;\gamma,\sigma+\gamma/2}}.$$

Moreover $\partial_t\partial_\xi(\widehat{h^T_t - h_\infty})(n,\xi)$ verifies the equation

$$\begin{aligned}\partial_t\partial_\xi(\widehat{h^T_t - h_\infty})(n,\xi) &= \mathrm{i}\frac{\widehat{\rho^T_t}(n)}{n}\partial_\xi\widehat{\eta}'(\xi - nt) \\ &- \varepsilon\sum_{k\neq 0}\frac{\widehat{\rho^T_t}(k)}{k}\partial_\xi(\widehat{h^T_t - h_\infty})(n-k,\xi-kt)(\xi - nt) \\ &- \varepsilon\sum_{k\neq 0}\frac{\widehat{\rho^T_t}(k)}{k}(\widehat{h^T_t - h_\infty})(n-k,\xi-kt) - \varepsilon\sum_{k\neq 0}\frac{\widehat{\rho^T_t}(k)}{k}\partial_\xi\widehat{h_\infty}(n-k,\xi-kt)(\xi - nt) \\ &- \varepsilon\sum_{k\neq 0}\frac{\widehat{\rho^T_t}(k)}{k}\widehat{h_\infty}(n-k,\xi-kt).\end{aligned}$$

Since the new terms appearing don't have derivatives, they are easier to analyze and similar estimates to the previous one hold. We omit the details.

Putting all togheter, we get

$$\int_t^T |A_s^1|\,\mathrm{d}s \leqslant C_\lambda \|\eta'\|_{\lambda;\sigma} \int_t^T \|\rho_s^T\|_{\lambda;\gamma,\sigma,\alpha} \|h_s^T - h_\infty\|_{\mathcal{G}^{\mu;\gamma,\sigma}}\,\mathrm{d}s$$

$$\leqslant C_\lambda \|\eta'\|_{\lambda;\sigma} M^\gamma_{\lambda,T}[\rho^T] N^\gamma_{\lambda,T}[h^T - h_\infty] \int_t^T \frac{C_a}{\langle s\rangle^{4/\gamma+6+\delta+2/\gamma+2}\,(a_T(s)-\mu)^{1/4}}\,\mathrm{d}s,$$

while

$$\int_t^T |A_s^2|\,\mathrm{d}s \leqslant C_\lambda \|\nabla h_\infty\|_{\lambda;\gamma,\sigma} \int_t^T \langle s\rangle \|\rho_s^T\|_{\lambda;\gamma,\sigma,\alpha} \|h_s^T - h_\infty\|_{\mathcal{G}^{\mu;\gamma,\sigma}}\,\mathrm{d}s$$

$$\leqslant C_\lambda \|\nabla h_\infty\|_{\lambda;\gamma,\sigma} M^\gamma_{\lambda,T}[\rho^T] N^\gamma_{\lambda,T}[h^T - h_\infty] \int_t^T \frac{C_a}{\langle s\rangle^{4/\gamma+5+\delta}} \frac{1}{\langle s\rangle^{4/\gamma+4}\,(a_T(s)-\mu)^{1/4}}\,\mathrm{d}s,$$

and, choosing $\mu(s) = (a_T(s)+\mu)/2$, we have

$$\int_t^T |A_s^3|\,\mathrm{d}s \leqslant C_\lambda \int_t^T \langle s\rangle \|\rho_s^T\|_{\mu;\gamma,\sigma,\alpha} \|h_s^T - h_\infty\|^2_{\mathcal{G}^{\mu;\gamma,\sigma+\gamma/2}}\,\mathrm{d}s$$

$$\leqslant C_\lambda \int_t^T \langle s\rangle \|\rho_s^T\|_{\mu;\gamma,\sigma,\alpha} \frac{\|h_s^T - h_\infty\|^2_{\mathcal{G}^{\mu(s);\gamma,\sigma}}}{(\mu(s)-\mu)}\,\mathrm{d}s$$

$$\leqslant C_\lambda M^\gamma_{\lambda,T}[\rho^T] N^\gamma_{\lambda,T}[h^T - h_\infty]^2 \int_t^T \frac{C_a}{\langle s\rangle^{4/\gamma+5+\delta}} \frac{1}{\langle s\rangle^{4/\gamma+4}\,(a_T(s)-\mu)^{3/2}}\,\mathrm{d}s.$$

Hence, integrating in time (3.20), multiplying by $\langle t\rangle^{4+4/\gamma}\,(a_T(t)-\mu)^{1/2}$, taking the sup and dividing by $N^\gamma_{\lambda,T}[h^T - h_\infty]$, we have proved the following Proposition.

Proposition 3.4. *Let h^T a solution of (3.7) and assume $M^\gamma_{\lambda,T}[\zeta^T] < +\infty$, then the following estimate holds:*

$$N^\gamma_{\lambda,T}[h^T - h_\infty] \leqslant C_\lambda M^\gamma_{\lambda,T}[\rho^T]\Big(\|\eta\|_{\lambda;\sigma} + \varepsilon N^\gamma_{\lambda,T}[h^T - h_\infty] + \varepsilon\|h_\infty\|_{\lambda;\gamma,\sigma}\Big).$$

3.5 The backward result

We can now present the main theorem of this Chapter. Since it is more general, we will state it considering the Gevrey case.

Theorem 3.1. *Let $h_\infty \in L^1(\mathbb{T}^1 \times \mathbb{R})$ of Gevrey regularity such that $\|h_\infty\|_{\lambda;\gamma,\sigma,0} < +\infty$, with $\lambda > 0$, $\sigma > 3$, $\gamma \in (1/3, 1]$. Consider $\eta \in L^1(\mathbb{R})$ analytic such that $\|\eta\|_{\lambda;\sigma} < +\infty$ and assume that*

$$\inf_{n\in\mathbb{Z};\Re\sigma\geqslant 0} |\mathcal{L}[j_n](\sigma)+1| \geqslant \kappa > 0,$$

with j_n as in (3.11). Then, for small values of ε, there exists a unique Gevrey regular solution $h_t(x,v)$ of (3.6) such that

$$\lim_{t\to+\infty} \|h_t(x,v) - h_\infty(x,v)\|_{L^\infty(\mathbb{T}^1\times\mathbb{R})} = 0$$

with exponential rate.

We don't give the proof of this result, which is similar to the one given in Theorem (2.2) of the previous Chapter.

3.6 On the linear part of the equation for $\gamma < 1/3$

Here we want to give a formal argument to show that the linear part of the equation is ill-posed for asymptotic data of Gevrey regularity $0 < \gamma < 1/3$.

Let us consider the equation for the density in (3.18) and assume that $T = +\infty$. Neglecting the Volterra linear term and the nonlinear one, we get

$$\widehat{\rho}_t(n) = \widehat{h_\infty}(n, nt) - \varepsilon \sum_{k \neq 0} \int_t^{+\infty} \widehat{\rho}_s(k) \frac{n(s-t)}{k} \widehat{h_\infty}(n-k, nt-ks)\, \mathrm{d}s.$$

Assume $n > 0$, as we have seen in the previous section, the challenging terms presenting resonances due to plasma echoes are those with $0 < k < n$ in the integral. We restrict our analysis to these modes and in the sum over k we take only the worst term $k = n - 1$.

We get, after a change of variable,

$$\phi_t(n) = \frac{\widehat{h_\infty}(n, nt)}{n} - \varepsilon t \int_1^{+\infty} \phi_{t\tau}(n-1)(\tau - 1)t\, \widehat{h_\infty}(1, t(n - (n-1)\tau))\, \mathrm{d}\tau,$$

where $\phi_t(n) = \widehat{\rho}_t(n)/n$.

We expect that, as $t \to +\infty$, the given function

$$t\, \widehat{h_\infty}\Big(1, t(n - (n-1)\tau)\Big) \rightharpoonup \delta\Big((n-1)(\tau - \frac{n}{n-1})\Big) = \frac{1}{n-1}\delta\Big(\tau - \frac{n}{n-1}\Big).$$

Hence, we want to study the toy model

$$\phi_t(n) = \frac{\widehat{h_\infty}(n, nt)}{n} - \varepsilon \frac{t}{(n-1)^2} \phi_{\frac{n}{n-1}t}(n-1). \tag{3.21}$$

Iterating (3.21), we get the expression

$$\phi_t(n) = \sum_{k=0}^{n-1} (-1)^k (\varepsilon t n)^k \Big[\frac{(n-k)!}{n!}\Big]^3 \frac{\widehat{h_\infty}(n-k, nt)}{n-k}.$$

By Laplace's method and by using that h_∞ is analytic, after some computations we get that

$$|\phi_t(n)| \approx C\varepsilon^n \Big[(n-1)^{n-1} e^{-(n-1)}\Big]^{1/\gamma} \frac{1}{(n!)^3} \approx C\varepsilon^n [(n-1)!]^{1/\gamma - 3},$$

where we used the Stirling formula in the last inequality.

From this it follows that $\phi_t(n)$ is unbounded in n for $\gamma < 1/3$ and this suggests the ill-posedness of the problem in this setting. We note that, differently from what happens in the forward problem, the resonant terms due to the echoes are finite in number for each fixed n. It is not yet clear what this might imply.

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Part II

Mean-field limit and propagation of chaos for topological interactions

Chapter 4

The validity problem in the mean-field scaling

By validity problems we mean those procedures in mathematical physics which justifies the use of a limit effective partial differential equation by a rigorous derivation from a system of N particles in a suitable scaling limit as $N \to +\infty$.

In this Chapter, we mainly focus on a review of the classic works by Dobrushin [23] and Neunzert and Wick [47] on the validity of the Vlasov equation for regular potentials, starting from an N-body system in the mean-field scaling.

In particular, the point of view of Neunzert and Wick in [47] - the first work to give a mean-field limit result - will be very useful in the next Chapter to study topological interactions.

4.1 Mean-field scaling and empirical measures

We focus on a N-particle system governed by the following ODEs[1]

$$\dot{z}_i(t) = \frac{1}{N} \sum_{j=1}^{N} F(z_i - z_j), \quad i = 1, \dots, N, \tag{4.1}$$

where $z_i \in \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a bounded two-body interaction such that F is globally Lipschitz with Lipschitz constant $L_F = \mathrm{Lip}(F)$.

The justification of the $1/N$ scaling in (4.1) is insidious since it is not a priori clear why the magnitude of the single-particle interaction should depend on the number of particles in the system. It can be motivated by considering a dynamic with a different time scaling: if $\bar{z}_i(t)$ is a solution of the N-body problem

$$\dot{\bar{z}}_i(t) = \sum_{j=1}^{N} F(\bar{z}_i - \bar{z}_j), \quad i = 1, \dots, N,$$

[1]Only in this Chapter, not considering mechanical systems but ODEs generally defined by (4.1), the dimension of the space will not be denoted by d as in the rest of the thesis but by n.

then $z_i(t) = \bar{z}_i(\frac{t}{N})$ solves the equation (4.1).

Otherwise one can consider phenomenological examples in which for fixed $N \gg 1$ the term NF is approximately of order 1, justifying both the use of the model (4.1) and the validity problem thanks to the size of the system considered.

For example, let $n = 6$, $z = (x, v)$ and $\bar{z}_i(t) = (\bar{x}_i(t), \bar{v}_i(t)) \in \mathbb{R}^3 \times \mathbb{R}^3$. Considering the Milky Way with all the stars with average mass m and $F(x, v) = (v, -\mathcal{G}mx/|x|^3)$, we have

$$\dot{\bar{x}}_i = \bar{v}_i, \quad \dot{\bar{v}}_i = -\sum_{j\neq i}^{N} \frac{\mathcal{G}m(\bar{x}_i - \bar{x}_j)}{|\bar{x}_i - \bar{x}_j|^3}, \quad 1 \leqslant i \leqslant N$$

where $\mathcal{G} = 6.67 \times 10^{-11}\,\mathrm{N} \cdot \mathrm{m}^2 \cdot \mathrm{kg}^{-2}$ is the gravitational constant.

Writing the equation in adimensional coordinates:

$$(x_i(t), v_i(t)) = \Big(\frac{\bar{x}_i(\tau t)}{L}, \frac{\tau}{L}\bar{x}_i(\tau t)\Big),$$

where $\tau \approx 2.4 \cdot 10^8$ years is the typical temporal dimension related to the rotational period of the sun and L is the typical spatial dimension related to the volume of the galaxy, we get

$$\dot{x}_i = v_i, \quad \dot{v}_i = -\frac{1}{N}\sum_{j\neq i}^{N} N\frac{\mathcal{G}m\tau^2}{L^3}\frac{(x_i - x_j)}{|x_i - x_j|^3}, \quad 1 \leqslant i \leqslant N.$$

Using that $N \approx 2\cdot 10^{11}$, the solar masses $m \approx 10^{31}$ kg and that the Milky Way is $1.7-2\cdot 10^5$ ly in diameter and approximately 10^3 ly thick, we have

$$N\frac{\mathcal{G}m\tau^2}{L^3} = O(1).$$

4.1.1 Empirical measures

The N-particle dynamics in (4.1) lives in the configuration space $\mathbb{R}^{nN}$ which depends on N. It is useful to introduce the notion of empirical measure related to (4.1):

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{z_i(t)}, \tag{4.2}$$

which is an element of $\mathcal{P}(\mathbb{R}^n)$, the space of Borel probability measures on the single-particle configuration space. Notice that μ_t^N is invariant under permutations of $\{z_1(t), \dots, z_N(t)\}$.

It is easy to see that, if $F(0) = 0$, then μ_t^N solves in the weak sense the following Vlasov equation

$$\partial_t f_t + \nabla_z \cdot \Big(\mathcal{F}[f_t]f_t\Big) = 0, \tag{4.3}$$

where $f_t = f(t, z)$ and

$$\mathcal{F}[f_t](z) = \int F(z - w)f_t(\mathrm{d}w)$$

is the mean-field force. Since F is bounded and globally Lipschitz we have that

$$\|\mathcal{F}[f]\|_\infty \leqslant \|F\|_\infty, \quad |\mathcal{F}[f_t](x) - \mathcal{F}[f_t](y)| \leqslant \mathrm{Lip}(F)|x - y|. \tag{4.4}$$

The mean-field limit thus consists in proving that, for large N, the empirical measure μ_t^N, already solution of the kinetic limit equation, closely approximates a given solution f of (4.3), possibly providing an estimate with a convergence rate in N.

In the case of regular potentials, this result is classical and has been given in various independent works in [47], [10] and [23].

In the next sections we will explain the point of view of Dobrushin in [23] and Neunzert-Wick in [47]. Dobrushin's approach relies more on the fact that μ_t^N and f are both solutions of the kinetic equation in the same functional space, on which it is possible to introduce a structure of metric space. Neunzert and Wick were inspired more by the results on the uniform distribution of sequences and the discrepancy theory (see [41, 40, 36]) and their approach is more involved but interesting for the following chapters of the thesis.

4.2 The Dobrushin approach

In 1979 the Russian mathematician Roland L. Dobrushin strengthened the connection between statistical mechanics and optimal transport problems - already noted in works of the '70s about the theory of stochastic fields [24, 15] - using for the first time the Wasserstein (or Kantorovich-Rubinshtein[2]) distance in kinetic theory problems, to prove uniqueness[3] of solutions of the Vlasov equation.

The approach in [23] is to construct a contractive map in the metric space of probability measures equipped with the Wasserstein distance. Thus existence and uniqueness of the solution follow from the usual Banach fixed point theorem. The same approach also allows proving continuity with respect to the initial data, providing the necessary stability estimate from which the mean-field limit follows.

As mentioned before, the tool used to prove stability estimates is the Wasserstein distance on $\mathcal{P}(\mathbb{R}^n)$. Given two measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$, in this Chapter we define it in the following way:

$$\mathcal{W}_1(\mu,\nu) := \inf_{\pi \in \mathcal{C}(\mu,\nu)} \int_{\mathbb{R}^n \times \mathbb{R}^n} \min\{|x-y|, 1\} \, \mathrm{d}\, \pi(x,y),$$

where $\mathcal{C}(\mu,\nu)$ is the set of all possible *couplings* between μ and ν, i.e. the set of Borel probability measures $\pi \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\pi(A \times \mathbb{R}^n) = \mu(A) \quad \text{and} \quad \pi(\mathbb{R}^n \times B) = \nu(B),$$

for all $A, B \in \mathcal{B}$, the σ-algebra of the Borel sets.

Note that, with this definition, the Wasserstein distance between two measures is necessarily finite, having replaced the usual euclidean distance $|x-y|$ between two points $x, y \in \mathbb{R}^n$ with the bounded distance $\min\{|x-y|, 1\}$ (otherwise we would have had to give the definition only for measures with finite first moment).

[2]The attribution of the distance to the name of Wasserstein was given by Dobrushin in [24] after having come into contact with the work of Leonid N. Vaserstein [57]. This distance had already been used in the '40s in the theory of transportation of mass (see [38] and [39]). Only later in [23] this incorrect attribution was recognized.

[3]In [23] the focus is on the proof of uniqueness since existence was already obtained in [10].

We refer to [60] and [61] for the proof that $\mathcal{W}_1$ is indeed a distance and for a discussion of its properties, including the fact that

$$\mu_j \rightharpoonup \mu \iff \mathcal{W}_1(\mu_j, \mu) \to 0, \tag{4.5}$$

where $\mu_j \rightharpoonup \mu$ denotes the weak convergence of the sequence of measures $\{\mu_j\}$ to μ.

Theorem 4.1 ([23]). *It holds that:*

i) Given an initial datum $f_0 \in \mathcal{P}(\mathbb{R}^n)$ and a bounded interaction F that is globally Lipschitz, there exists a unique global weak solution $f \in C\Big([0,+\infty); \mathcal{P}(\mathbb{R}^n)\Big)$ of the Vlasov mean-field equation (4.3).

ii) Solutions of (4.3) are weakly-continuous with respect to the initial datum. It follows that, fixed $T > 0$ and being f_t solution of the Vlasov equation (4.3) with initial datum f_0 and μ_t^N the empirical measure in (4.2) related to the particle system (4.1) with initial datum μ_0^N, for $0 \leqslant t \leqslant T$, it holds

$$\mathcal{W}_1(f_t, \mu_t^N) \leqslant e^{2\max\{\|F\|_\infty, Lip(F)\}T} \mathcal{W}_1(f_0, \mu_0^N).$$

Proof of i) of Theorem 6.1. We divide the proof into several steps.

Step 1 Given $f \in C\Big([0,T]; \mathcal{P}(\mathbb{R}^n)\Big)$ and $z \in \mathbb{R}^n$, we define the flow

$$z^f(t) \equiv z^f(t,z)$$

that solves

$$\begin{cases} \dot{z}^f(t) = \mathcal{F}[f_t](z^f(t)) \\ z^f(0) = z. \end{cases} \tag{4.6}$$

Note that $\mathcal{F}[f_t](x)$ is also continuous in t since in general, given two probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$,

$$|\mathcal{F}[\mu](x) - \mathcal{F}[\nu](x)| \leqslant \max\{\text{Lip}(F), 2\|F\|_\infty\} \mathcal{W}_1(\mu, \nu). \tag{4.7}$$

Indeed, let $\pi \in \mathcal{C}(\mu, \nu)$, from the hypothesis on F we have

$$\begin{aligned} |\mathcal{F}[\mu](x) - \mathcal{F}[\nu](x)| &= \Big| \int_{\mathbb{R}^{2n}} F(x - x_1)\, \mathrm{d}\pi(x_1, x_2) - \int_{\mathbb{R}^{2n}} F(x - x_2)\, \mathrm{d}\pi(x_1, x_2) \Big| \\ &\leqslant \int_{\mathbb{R}^{2n}} |F(x - x_1) - F(x - x_2)|\, \mathrm{d}\pi(x_1, x_2) \\ &\leqslant \max\{\text{Lip}(F), 2\|F\|_\infty\} \int_{\mathbb{R}^{2n}} \min\{|x_1 - x_2|, 1\}\, \mathrm{d}\pi(x_1, x_2). \end{aligned}$$

Taking the infimum between the couplings we get (4.7).

By this and thanks to (4.4), we know that the solution (4.6) is uniquely defined, globally in $[0, T]$.

Step 2 Given a fixed $T > 0$ and $f_0 \in \mathcal{P}(\mathbb{R}^n)$, we define the map

$$\Phi \colon C\Big([0,T]; \mathcal{P}(\mathbb{R}^n)\Big) \to C\Big([0,T]; \mathcal{P}(\mathbb{R}^n)\Big),$$

where $(\Phi f)_t$ is the push-forward of f_0 along the flow $z^f(t,x)$ defined in (4.6), i.e. $(\Phi f)_t$ is defined by the relation

$$\int_{\mathbb{R}^n} \alpha(z)\,\mathrm{d}(\Phi f)_t(z) = \int_{\mathbb{R}^n} \alpha(z^f(t,z))\,\mathrm{d}f_0(z) \tag{4.8}$$

for any function $\alpha \in C_b(\mathbb{R}^n)$.

Thanks to (4.8), $(\Phi f)_t$ is a probability measure for each $t \in [0,T]$. The weak continuity of $(\Phi f)_t$ with respect to time t follows from the continuity of $z^f(t)$. Hence the map Φ is well-defined.

Notice that if $f \in C([0,T];\mathcal{P}(\mathbb{R}^n))$ is a fixed point of the map Φ, then it is a weak solution of the Vlasov equation. Hence, introducing the following distance

$$\overline{\mathcal{W}_1}(f,g) = \int_0^T \mathcal{W}_1(f_t,g_t)\ \mathrm{d}t, \tag{4.9}$$

for $f,g \in C\Big([0,T];\mathcal{P}(\mathbb{R}^n)\Big)$, the aim is to prove that Φ is a contractive map with respect to $\overline{\mathcal{W}_1}$.

Step 3 Given $z \in \mathbb{R}^n$, let

$$\delta(z) \coloneqq \max_{t\in[0,T]} \Big|z^f(t,z) - z^g(t,z)\Big|.$$

By the triangular inequality, we have

$$\begin{aligned}\delta(z) &\leqslant \int_0^T \Big|\mathcal{F}[f_t](z^f(t,z)) - \mathcal{F}[g_t](z^g(t,z))\Big|\,\mathrm{d}t \\ &\leqslant \int_0^T \Big|\mathcal{F}[f_t](z^f(t,z)) - \mathcal{F}[f_t](z^g(t,z))\Big|\,\mathrm{d}t + \int_0^T |\mathcal{F}[f_t](z^g(t,z)) - \mathcal{F}[g_t](z^g(t,z))|\,\mathrm{d}t.\end{aligned}$$

Using (4.4) and (4.7), we obtain

$$\delta(z) \leqslant \mathrm{Lip}(F)T\delta(z) + \max\{\|F\|_\infty, \mathrm{Lip}(F)\}\int_0^T \mathcal{W}_1(f_t,g_t)\,\mathrm{d}t.$$

Hence, if $\mathrm{Lip}(F)T < 1$ we have that

$$\delta(z) \leqslant \frac{\max\{\|F\|_\infty, \mathrm{Lip}(F)\}}{1 - T\mathrm{Lip}(F)}\overline{\mathcal{W}_1}(f,g). \tag{4.10}$$

Step 4 We now pass to the proof of the contractivity of Φ. Let $\pi_t \in \mathcal{C}((\Phi f)_t, (\Phi g)_t)$ defined in this way:

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \alpha(z_1,z_2)\,\mathrm{d}\pi_t(z_1,z_2) = \int_{\mathbb{R}^n} \alpha(z^f(t,z), z^g(t,z))\,\mathrm{d}f_0(z),$$

for any function $\alpha \in C_b(\mathbb{R}^n \times \mathbb{R}^n)$, i.e. π_t is the push-forward of f_0 along the product flow defined by $z^f(t,z)$ and $z^g(t,z)$.

Using (4.10), we have

$$\int_{\mathbb{R}^{2n}} \min\{|z_1 - z_2|, 1\} \,\mathrm{d}\pi_t(z_1, z_2) = \int_{\mathbb{R}^n} \min\{|z^f(t,z) - z^g(t,z)|, 1\} \,\mathrm{d}f_0(z)$$
$$\leqslant \int_{\mathbb{R}^n} \delta(z) \,\mathrm{d}f_0(z) \leqslant \frac{\max\{\|F\|_\infty, \mathrm{Lip}(F)\}}{1 - T\mathrm{Lip}(F)} \overline{\mathcal{W}_1}(f, g)$$

and since the Wasserstein is an infimum over the couplings,

$$\mathcal{W}_1((\Phi f)_t, (\Phi g)_t) \leqslant \frac{\max\{\|F\|_\infty, \mathrm{Lip}(F)\}}{1 - T\mathrm{Lip}(F)} \overline{\mathcal{W}_1}(f, g).$$

Hence, from (4.9),

$$\overline{\mathcal{W}_1}(\Phi f, \Phi g) \leqslant \lambda(T) \overline{\mathcal{W}_1}(f, g),$$

where

$$\lambda(T) = \frac{T \max\{\|F\|_\infty, \mathrm{Lip}(F)\}}{1 - T\mathrm{Lip}(F)} < 1$$

for T small. □

To prove the second part of Theorem (4.1), we introduce a duality formulation of the Wasserstein distance. This point of view is much used in the theory of optimal transport and it will also be useful in the next Chapter.

Theorem 4.2 (Kantorovich duality). *Given two probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$, we have*

$$\mathcal{W}_1(\mu, \nu) = \sup\left\{ \int_{\mathbb{R}^n} \psi \,\mathrm{d}(\mu - \nu);\ \psi \quad \text{s. t.} \quad \sup_{x,y \in \mathbb{R}^n} \frac{|\psi(x) - \psi(y)|}{\min\{1, |x - y|\}} \leqslant 1 \right\}. \tag{4.11}$$

We refer to [60] and [61] for the proof of this result, we note only that from (4.11) it is clear that (4.5) holds.

Moreover, we introduce the “intermediate” dynamics that, for $i = 1, \ldots, N$ and $f \in C([0,T]; \mathcal{P}(\mathbb{R}^n))$, is given by

$$\dot{z}_i^f(t) = \mathcal{F}[f_t](z_i^f),$$

and the empirical measure

$$\nu_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{z_i^f(t)}.$$

The initial datum is $\nu_0^N = \mu_0^N$, i.e.

$$\{z_i^f(0)\}_{i=1}^N = \{z_i\}_{i=1}^N.$$

Proof of ii) of Theorem (4.1). We estimate $\mathcal{W}_1(f_t, \mu_t^N)$ using the triangular inequality

$$\mathcal{W}_1(f_t, \mu_t^N) \leqslant \mathcal{W}_1(f_t, \nu_t^N) + \mathcal{W}_1(\nu_t^N, \mu_t^N).$$

Given a function $\psi : \mathbb{R}^n \to \mathbb{R}$ which is Lipschitz with respect to the bounded distance in (4.11) and such that

$$\sup_{x,y \in \mathbb{R}^n} \frac{|\psi(x) - \psi(y)|}{\min\{1, |x - y|\}} \leqslant 1,$$

we have
$$\int \psi \,\mathrm{d}(f_t - \nu_t^N) = \int \psi(z^f(t,z))\,\mathrm{d}(f_0 - \mu_0)(z),$$
this because both f_t and $\nu^N t$ are the push-forward of (respectively) f_0 and μ_0 along the flow $z^f(t)$.

Since
$$|z^f(t,z) - z^f(t,z')| \leqslant |z - z'| + \max\{2\|F\|_\infty, L_F\} \int_0^t |z^f(s,z) - z^f(s,z')|\,\mathrm{d}s,$$
we have that
$$|z^f(t,z) - z^f(t,z')| \leqslant e^{\max\{2\|F\|_\infty, L_F\}t} \min\{1, |z - z'|\}.$$
Hence $z^f(t,z)$ is Lipschitz with respect to the bounded distance and so the function $\psi(z^f(t,z))$ is Lipschitz too with constant $e^{\max\{\|F\|_\infty, L_F\}t}$, hence
$$\int \psi\,\mathrm{d}(f_t - \nu_t^N) \leqslant e^{\max\{2\|F\|_\infty, L_F\}t} \mathcal{W}_1(f_0, \mu_0),$$
by the Kantorovich duality in (4.11).

From this and from the definition of Wasserstein distance we get
$$\mathcal{W}_1(f_t, \mu_t^N) \leqslant \int \min\{1, |z^f(t,z) - z_i(t,z)|\}\,\mathrm{d}\mu_0^N(z) + e^{\max\{2\|F\|_\infty, L_F\}t} \mathcal{W}_1(f_0, \mu_0^N).$$

We now estimate:
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|z_i^f(t) - z_i(t)| &\leqslant |\mathcal{F}[f_t](z_i^f(t)) - \mathcal{F}[\mu_t^N](z_i(t))| \\
&\leqslant |\mathcal{F}[f_t](z_i^f(t)) - \mathcal{F}[\mu_t^N](z_i^f(t))| + |\mathcal{F}[\mu_t^N](z_i^f(t)) - \mathcal{F}[\mu_t^N](z_i(t))| \\
&\leqslant \max\{2\|F\|_\infty, \mathrm{Lip}(F)\}\Big(\mathcal{W}_1(f_t, \mu_t^N) + \min\{1, |z_i^f(t) - z_i(t)|\}\Big),
\end{aligned}$$
where we have used again the Kantorovich duality in the first term.

By Gronwall's lemma, we get
$$\min\{1, |z_i^f(t) - z_i(t)|\} \leqslant \beta \int_0^t e^{\beta(t-s)} \mathcal{W}_1(f_s, \mu_s^N)\,\mathrm{d}s,$$
where $\beta \coloneqq \max\{2\|F\|_\infty, \mathrm{Lip}(F)\}$. We arrive at
$$\mathcal{W}_1(f_t, \mu_t^N) \leqslant \beta \int_0^t e^{\beta(t-s)} \mathcal{W}_1(f_s, \mu_s^N)\,\mathrm{d}s + e^{\beta t} \mathcal{W}_1(f_0, \mu_0^N).$$

Multiplying on both sides by $e^{-\beta t}$ and using again Gronwall's lemma, we obtain the thesis
$$\mathcal{W}_1(f_t, \mu_t^N) \leqslant e^{2\beta t} \mathcal{W}_1(f_0, \mu_0^N).$$
□

We cite that this approach by duality is also used in the proof of the mean-field limit given by W. Braun and W. K. Hepp in [10], where they work with the so-called bounded Lipschitz norm defined by
$$\|\mu\|_{BL} \coloneqq \sup_{\|\psi\|_\infty + \|\nabla\psi\|_\infty \leqslant 1} \int \psi\,\mathrm{d}\mu.$$

4.3 The Neunzert and Wick approach

In 1974, Helmut Neunzert and Joachim Wick in [47] gave the first proof of the mean-field limit, inspired by the work [63] of Hermann Weyl on equidistributed sequences modulo 1 of real numbers and using the discrepancy theory.

We are particularly interested in this result for two reasons: it is historically the first rigorous proof of the mean-field limit; moreover, its strategy is generalizable to cases with nonsmooth interaction, as in the proof of the mean-field limit for topological interactions given in the next Chapter.

The authors work in $\mathbb{R}^2$ and use the following notion of discrepancy distance

$$\mathcal{D}^*(\rho_1, \rho_2) = \sup_{z \in \mathbb{R}^2} \Big| \int_{R(z)} \mathrm{d}\rho_1 - \int_{R(z)} \mathrm{d}\rho_2 \Big|$$

where $R(z) = \{w \in \mathbb{R}^2 \text{ s.t.}\, w \leqslant z\}$ and $w \leqslant z$ is the product order (or component-wise order) on $\mathbb{R}^2$.

In this work F is assumed to be of bounded variation in the sense of Hardy-Krause: this means that F has one-dimensional bounded variation in each variable $F(z^1, \cdot)$ and $F(\cdot, z^2)$, $z = (z^1, z^2)$ and moreover that

$$V_2[F] = \sup_R \sum_{i,k} |\Delta_{R_{i,k}}(F)| < +\infty,$$

where $R = \bigcup_{i,k} R_{i,k}$ is the finite union of ordered intervals $R_{i,k} = (z_i^1, z_k^2)$ with $z_1^1 < \cdots < z_i^1$, $z_1^2 < \cdots < z_k^2$ and

$$\Delta_{R_{i,k}}(F) = F(z_{i+1}^1, z_{k+1}^2) + F(z_i^1, z_k^2) - F(z_{i+1}^1, z_k^2) - F(z_i^1, z_{k+1}^2).$$

Then $V_H(F)$ is defined as the sum of $V_2[F]$ and of the one-dimensional total variations.

We introduce again the intermediate dynamic that, given $f \in C([0,T]; L^1(\mathbb{R}^2))$ and $i = 1, \dots, N$, is given by

$$\dot{z}_i^f(t) = \mathcal{F}[f_t](z_i^f),$$

and the empirical measure

$$\nu_t^N := \frac{1}{N} \sum_{k=1}^N \delta_{z_k^f(t)}.$$

The initial datum is $\nu_0^N = \mu_0^N$, i.e.

$$\{z_i^f(0)\}_{i=1}^N = \{z_i\}_{i=1}^N.$$

For functions F of bounded variation in the above sense, the following inequalities hold. They are widely used in the study of uniform distributions of sequences and discrepancy theory (see [41, 40, 36, 48]) and can be generalized to any dimension.

Proposition 4.1 (Koksma-Hlawka inequality)**.** *Given an interaction F of bounded variation in the Hardy-Krause sense and a probability measure $f \in C([0,T]; L^1(\mathbb{R}^2))$, we have that*

$$|\mathcal{F}[f_t](z) - \mathcal{F}[\mu_t^N](z)| \leqslant V_H(F) \mathcal{D}^*(f_t, \mu_t^N),$$

where μ_t^N is the empirical measure in (4.2). Moreover

$$|\mathcal{F}[\mu_t^N](z) - \mathcal{F}[\nu_t^N](z)| \leqslant V_H(F) \mathcal{D}^*(\mu_t^N, \nu_t^N) + 2\mathcal{D}^*(\nu_t^N, f_t).$$

Thanks to these properties, a version of the mean-field limit can be proved. Notice that we don't specify the convergence rate.

Theorem 4.3. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a function of bounded Hardy-Krause variation. Given a probability measure $f \in C([0,T]; L^1(\mathbb{R}^2))$ solution of the Vlasov equation, assume that $\mathcal{F}[f_t]$ is globally Lipschitz with Lipschitz constant L_F. Given the empirical measure μ_t^N related to (4.2), we have that*

$$\lim_{N\to+\infty} \mathcal{D}^*(f_t, \mu_t^N) = 0$$

if

$$\lim_{N\to+\infty} \mathcal{D}^*(f_0, \mu_0^N) = 0.$$

Proof of Theorem (4.3). We avoid giving all the details since a similar proof will be given in detail in the next Chapter. We have that

$$\mathcal{D}^*(f_t, \mu_t^N) \leqslant \mathcal{D}^*(f_t, \nu_t^N) + \mathcal{D}^*(\nu_t^N, \mu_t^N).$$

It can be proved that the first term goes to zero, since

$$\Big| \int_{z^f(t,R(z))} \mathrm{d}(f_0 - \mu_0^N) \Big| \to 0.$$

Concerning the second term, we use the following lemma which we don't prove (see [47] and Proposition (5.3) where an exact analog of this is proved).

Lemma 4.1. *Let*

$$\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{z_i} \ \ \text{and} \ \ \nu^N = \frac{1}{N}\sum_{i=1}^N \delta_{w_i}$$

be two empirical measures on $\mathbb{R}^2$ and take $\delta > 0$ such that $|z_i - w_i| \leqslant \delta$ for all $i = 1, \ldots, N$. It holds that, for any probability measure $f \in L^1(\mathbb{R}^2)$,

$$\mathcal{D}^*(\mu^N, \nu^N) \leqslant 2\sigma_f(\delta) + \mathcal{D}^*(\nu^N, f),$$

where

$$\sigma_f(\delta) = \sup_{z_2\in\mathbb{R}^2}\Big(\sup_{|z_1-z_2|<\delta} \Big| \int_{R(z_2)} f(z')\,\mathrm{d}z' - \int_{R(z_1)} f(z')\,\mathrm{d}z' \Big| \Big).$$

The proof follows by estimating the difference of the flows $|z_i^f(t,z) - z_i(t,z)|$ and using the Koksma-Hlawka inequalities (4.1) and Gronwall's lemma.

□

Besides what is briefly introduced here, there would be many interesting topics on mean-field validity problems such as the approach with BBGKY hierarchies, the quantum mean-field limit and the validity for the Vlasov-Poisson equation which are not covered here. We refer the reader to the many references on the subject ([17, 18, 29, 37, 53, 52, 54]).

We only mention that, although the theory for regular pairwise interactions is sufficiently well understood, going beyond it considering singular potentials, is a harder task. This is the case of the three-dimensional Vlasov-Poisson equation. In this equation, the potential $1/r$ is singular at the origin and does not belong to any L^p space. Although the mean-field limit for the Vlasov-Poisson equation remains an open problem, there has been important progress in recent years, see the works [34, 35] where the mean-field limit is proven for potentials with singularities "weaker than $1/r$" and also [42, 43].

However, in the case of the one-dimensional Vlasov-Poisson equation, the problem has been solved in [55, 56] and with a simpler proof in [33], being the force discontinuous, but not diverging. The analogy with the discontinuity of the Coulomb/Newton interaction in the one-dimensional case suggested the strategy for the proof of the mean-field limit for topological models which will be presented in the next Chapter.

Chapter 5

Topological interaction: mean-field limit for a Cucker-Smale type model

joint work with D. Benedetto and E. Caglioti ([5])

In this Chapter we present a mean-field limit result for the Cucker-Smale model with topological interaction. In topological models an agent reacts to the presence of another not according to the distance, but according to the proximity rank (see eq.s (5.1), (5.2), (5.3) below for a rigorous formulation). Due to this dependence on the rank, the interaction comes out of the two-body case, and present various problems in the kinetic treatment. In particular, in the case considered here, solutions of the kinetic equation are not weakly continuous with respect to the initial datum and there are also some difficulties in defining particle motion.

5.1 Topological Cucker-Smale model

In recent years, the conceptual and mathematical apparatus of kinetic equations has been used in the study of self-propelled particle systems of biological nature undergoing local interactions, as the motion of migrating cells [28], locust swarms [3] and fish schools [44]. Starting with the pioneering paper in [58], several models have been proposed to explain the evolution of these systems (see [59]). In the simplest [19, 20, 58], a bird is modeled as a self-propelling particle that interacts with its neighbors. The interaction is such that neighboring birds tend to align their velocities. For many of these models, the mean-field limit has often been used to obtain a kinetic description of the dynamics (see, for instance, [31, 14, 12, 13, 30, 4]).

A few years ago, supported by observational data ([2, 16, 1]), "topological" models for interaction were introduced: in these models, the strength of the interaction of an agent with another one is a function of the proximity rank of the latter with respect to the former.

The seminal paper [2] has been followed by several papers studying various aspects of this phenomenon see e.g. [9, 11, 27, 49, 50].

Mathematically, flocking of systems of topologically interacting particles has been investigated in [45, 51, 62] and as regards stochastic models with topological interaction also in [7, 8].

In [32], the author introduced the topological Cucker-Smale model which we are going to consider here. In addition to studying flocking, he proposes kinetic and fluid models derived from this mean-field topological interaction and a first mean-field limit result is proved for a smoothed version of the model in which the weak continuity in the initial datum is recovered.

In this Chapter instead, besides studying the well-posedness of the microscopic dynamics and the kinetic equation, we prove the mean-field limit for the effective dynamics with topological interaction, without hypotheses of further regularizations.

A Cucker-Smale type model for the motion of N agents, in the mean-field scaling, is the system

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} p_{ij}(v_j(t) - v_i(t)), \end{cases} \tag{5.1}$$

where the "communication weights" $\{p_{ij}\}_{i,j=1}^{N}$ are positive functions that take into account the interactions between agents. In classical models, p_{ij} depends only on the distance $|x_i - x_j|$ between the agents. In topological models the weights depend on the positions of the agents by their rank

$$p_{ij} \coloneqq K\big(M(x_i, |x_i - x_j|)\big), \tag{5.2}$$

where $K\colon [0,1] \to \mathbb{R}^+$ is a positive decreasing Lipschitz continuous function such that $\int_0^1 K(z)\,\mathrm{d}z = \gamma$ and, for $r > 0$, the function

$$M(x_i, r) \coloneqq \frac{1}{N}\sum_{k=1}^{N} \mathcal{X}\{|x_i - x_k| \leqslant r\} \tag{5.3}$$

counts the number of agents at distance less than or equal to r from x_i, normalized with N.

Here and after, $\mathcal{X}\{|x_i - x_k| \leqslant r\} = \mathcal{X}_{B_r(x_i)}(x_k)$ where $\mathcal{X}_A$ is the characteristic function of the set A and $B_r(x)$ denotes the closed ball of center x and radius r in $\mathbb{R}^d$. Note that in this case p_{ij} is a stepwise function of the positions of all the agents.

In the mean-field limit $N \to +\infty$, the one-agent distribution function $f_t = f(t,x,v)$ is expected to verify the equation

$$\partial_t f_t + v \cdot \nabla_x f_t + \nabla_v \cdot (W[Sf_t, f_t](x,v) f_t) = 0, \tag{5.4}$$

where, in this Chapter, instead of ρ we denote by $Sf_t(x) \coloneqq \int f_t(x,v)\,\mathrm{d}v$ the spatial distribution and where, given a probability density f in $\mathbb{R}^d \times \mathbb{R}^d$ and a probability density ρ in $\mathbb{R}^d$,

$$W[\rho, f](x,v) \coloneqq \int K\left(M[\rho](x, |x-y|)\right)(w - v) f(y,w)\,\mathrm{d}y\,\mathrm{d}w, \tag{5.5}$$

with

$$M[\rho](x,r) := \int_{|x'-x|\leqslant r} \rho(x')\ \mathrm{d}x'. \tag{5.6}$$

A weak formulation of this equation is given requiring that the solution f_t fulfills

$$\int \alpha(x,v)\,\mathrm{d}f_t(x,v) = \int \alpha\left(x^f(t,x,v), v^f(t,x,v)\right)\mathrm{d}f_0(x,v)$$

for any $\alpha \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$, where f_0 is the initial probability measure and $(x^f(t), v^t(f)) \equiv (x^f(t,x,v), v^f(t,x,v))$ is the flow defined by

$$\begin{cases} \dot{x}^f(t,x,v) = v^f(t,x,v) \\ \dot{v}^f(t,x,v) = W[Sf_t, f_t](x^f(t,x,v), v^f(t,x,v)) \\ x(0,x,v) = x, \quad v(0,x,v) = v. \end{cases} \tag{5.7}$$

In other words, f_t is the push-forward of f_0 along the flow generated by the velocity field, determined by f_t itself.

It is easy to verify that the empirical measure

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i^N(t)}\,\delta_{v_i^N(t)}$$

associated with the solution of (5.1), (5.2) and (5.3) is a weak solution of (5.4). Namely, $M[S\mu_t^N](x,r)$ is exactly $M(x,r)$ defined in (5.3) (from now on we use the more complete notation $M[S\mu_t^N](x,r)$). Thus, we can rewrite the agent evolution in (5.1) as

$$\begin{cases} \dot{x}_i^N(t) = v_i^N(t) \\ \dot{v}_i^N(t) = W[S\mu_t^N, \mu_t^N](x_i^N(t), v_i^N(t)). \end{cases} \tag{5.8}$$

As seen in the previous Chapter, in the Dobrushin approach to the mean-field limit, the result is achieved from this fact and from the weak continuity, with respect to the initial datum, of the weak solutions of (5.4).

We cannot use this approach in presence of topological interaction, since in general the solutions of (5.7) are not weakly continuous with respect to the initial datum (see section (5.3)).

We can overcome this difficulty if the solution of (5.4) has a bounded density. To obtain our result, we were inspired by the work of Trocheris [55] in which the author uses the previously mentioned discrepancy theory techniques of Neunzert and Wick [47, 48] to prove the mean-field limit for the one-dimensional Vlasov-Poisson equation, where the interaction has a jump-type discontinuity at the origin.

5.2 Distances and weak convergence

In this Chapter we use the following definition of the 1-Wasserstein distance $\mathscr{W}$ between two probability measures ρ_1 and ρ_2 on $\mathbb{R}^d$:

$$\begin{aligned}\mathscr{W}(\rho_1,\rho_2) &= \sup_{\phi\in C_b(\mathbb{R}^d),\mathrm{Lip}(\phi)\leqslant 1} \int \phi(\mathrm{d}\rho_1 - \mathrm{d}\rho_2) \\ &= \sup_{\phi\in C_b^1(\mathbb{R}^d),\|\nabla\phi\|_\infty\leqslant 1} \int \phi(\mathrm{d}\rho_1 - \mathrm{d}\rho_2),\end{aligned}$$

where Lip(ϕ) is the Lipschitz constant of ϕ.

The counter of the number of particles in (5.6) is not continuous with respect to $\mathscr{W}$, so we work with the weaker topology induced by another distance, the discrepancy, defined as

$$\mathscr{D}(\rho_1, \rho_2) \coloneqq \sup_{x, r>0} \left| \int_{B_r(x)} \mathrm{d}\rho_1 - \int_{B_r(x)} \mathrm{d}\rho_2 \right| .$$

In the sequel, we also indicate by B_R the closed ball $B_R(0)$.

As discussed in the previous Chapter, the discrepancy distance is mostly used to quantify the uniformity of sequences of points (see [41, 26]), but its multidimensional version is used for the proof of the mean-field limit in [47] and is mentioned by Neunzert in his lecture notes[46].

By definition, it holds the following proposition.

Proposition 5.1 (Lipschitzianity of M with respect to $\mathscr{D}$)**.** *Let ρ_1 and ρ_2 be two probability measures on $\mathbb{R}^d$. Then, for any $x \in \mathbb{R}^d$ and $r > 0$,*

$$|M[\rho_1](x, r) - M[\rho_2](x, r)| \leqslant \mathscr{D}(\rho_1, \rho_2).$$

We can also define $\mathscr{D}$ in terms of regular functions. Let X be the subset of $C^1_b([0, +\infty); \mathbb{R})$, and define

$$\|\phi\|_X \coloneqq \int_0^{+\infty} |\phi'(r)| \,\mathrm{d}r.$$

Then

$$\mathscr{D}(\rho_1, \rho_2) = \sup_{\phi \in X \colon \|\phi\|_X \leqslant 1} \sup_x \int \phi\big(|x-y|\big)\big(\mathrm{d}\rho_1(y) - \mathrm{d}\rho_2(y)\big).$$

This assertion is an easy consequence of the following lemma.

Lemma 5.1. *Let g_1 and g_2 be two probability measures on $[0, +\infty)$. Then*

$$\sup_{r \geqslant 0} \left| \int_{[0,r]} \mathrm{d}g_1 - \int_{[0,r]} \mathrm{d}g_2 \right| = \sup_{\phi \in X \colon \|\phi\|_X \leqslant 1} \int_0^{+\infty} \phi \left(\mathrm{d}g_1 - \mathrm{d}g_2\right). \tag{5.9}$$

Proof. Fix $r > 0$, there exists $\phi_{r,\varepsilon} \in X$ with $\|\phi_{r,\varepsilon}\|_X = 1$ and such that $\phi_{r,\varepsilon}(s) = 1$ if $0 \leqslant s \leqslant r$ and $\phi_{r,\varepsilon}(s) = 0$ if $s \geqslant r + \varepsilon$. For any measure g,

$$\lim_{\varepsilon \to 0} \int_0^{+\infty} \big(\phi_{r,\varepsilon}(s) - \mathcal{X}\{s \in [0, r]\}\big) \,\mathrm{d}g(s) = 0,$$

then

$$\int_{[0,r]} (\mathrm{d}g_1 - \mathrm{d}g_2) = \lim_{\varepsilon \to 0} \int_0^{+\infty} \phi_{r,\varepsilon}(\mathrm{d}g_1 - \mathrm{d}g_2) \leqslant \sup_{\phi \in X \colon \|\phi\|_X \leqslant 1} \int_0^{+\infty} \phi(\mathrm{d}g_1 - \mathrm{d}g_2).$$

To prove the opposite inequality, we denote by G_1 and G_2 the distribution functions of g_1 and g_2:

$$G_i(r) \coloneqq \int_{[0,r]} \mathrm{d}g_i.$$

Then, integrating by parts,

$$\int_0^{+\infty} \phi(\mathrm{d}g_1 - \mathrm{d}g_2) = -\int_0^{+\infty} \phi'(r)\big(G_1(r) - G_2(r)\big)\,\mathrm{d}r \leqslant \|\phi\|_X \|G_1 - G_2\|_\infty.$$

We conclude the proof by noticing that $\|G_1 - G_2\|_\infty$ is exactly the left-hand-side of (5.9). □

For our purposes, we need the equivalence of $\mathscr{D}$ and $\mathscr{W}$ in the case in which one of the two measures has bounded density. We note that in the general case the equivalence is false, as can be easily checked by considering two Dirac measures δ_{x_1} and δ_{x_2}: $\mathscr{W}$ vanishes when $|x_1 - x_2| \to 0$, while $\mathscr{D}$ is one whenever $x_1 \neq x_2$. Nevertheless, using the covering principles as in [6], for measures on a compact set, it can be proved the continuity of the Wasserstein distance $\mathscr{W}$ with respect to the discrepancy distance $\mathscr{D}$. We refer to the appendix of [5] for a proof of this fact.

In the sequel, in the definition of $\mathscr{D}$ we choose functions in $\phi \in C([0, +\infty), \mathbb{R})$, with first derivative continuous up to a finite number of jumps. With abuse of notation, we keep calling this set of functions X. Let us expose some technical properties.

Given $\phi \in X$, we define some useful regularizations, $\phi^\pm$, ϕ_ε and ψ_ε, with $\varepsilon > 0$, as follows. Denoting by $\tilde{\phi}$ the function

$$\tilde{\phi}(r) \coloneqq \int_0^r |\phi'(s)|\,\mathrm{d}s,$$

we define

$$\phi^\pm(r) \coloneqq \begin{cases} \dfrac{1}{2}(\tilde{\phi}(r) \pm \phi(r)), & \text{if } r \geqslant 0, \\ \pm\dfrac{1}{2}\phi(0), & \text{if } r < 0, \end{cases}$$

and

$$\phi_\varepsilon(r) \coloneqq \phi^+(r + \varepsilon) - \phi^-(r - \varepsilon). \tag{5.10}$$

Finally, fixed a regular mollifier η supported in $(0, 1)$, we define

$$\psi_\varepsilon(r) \coloneqq \int_0^\varepsilon \eta_\varepsilon(s)\phi^+(r + s)\,\mathrm{d}s - \int_0^\varepsilon \eta_\varepsilon(s)\phi^-(r - s)\,\mathrm{d}s. \tag{5.11}$$

where $\eta_\varepsilon(s) \coloneqq \varepsilon^{-1}\eta(s/\varepsilon)$.

We summarize the properties of these regularizations in the following lemma, where we indicate with c any constant which does not depends on ϕ and ε.

Lemma 5.2. *i) $\phi^\pm$ are not decreasing. Moreover*

$$\int_0^{+\infty} (\phi^\pm)'(r)\,\mathrm{d}r \leqslant \|\phi\|_X \tag{5.12}$$

and $\phi(r) = \phi^+(r) - \phi^-(r)$ for $r \geqslant 0$.

ii) $\phi_\varepsilon \in X$, $\phi(r) \leqslant \phi_\varepsilon(r)$ and

$$\int_0^{+\infty} \big(\phi_\varepsilon(r) - \phi(r)\big)\,\mathrm{d}r \leqslant 2\varepsilon\|\phi\|_X. \tag{5.13}$$

iii) $\psi_\varepsilon(r) \geqslant \phi(r)$. *Moreover* ψ_ε *is a* C^1 *function in* X,

$$\|(\psi_\varepsilon)'\|_\infty \leqslant \frac{2}{\varepsilon}\|\eta\|_\infty\|\phi\|_X \tag{5.14}$$

and

$$\int_0^{+\infty} |\psi_\varepsilon(r) - \phi(r)|\,\mathrm{d}r \leqslant c\varepsilon\|\phi\|_X. \tag{5.15}$$

Proof. The proof is elementary, we only describe how to get the bounds in ii) and iii). Since $\phi = \phi^+ - \phi^-$, we rewrite the l.h.s. of (5.13) as

$$\begin{aligned}&\int_0^{+\infty} \big(\phi^+(r+\varepsilon) - \phi^+(r)\big) + \big(\phi^-(r) - \phi^-(r-\varepsilon)\big)\,\mathrm{d}r\\ &= \int_0^{+\infty}\left(\int_0^\varepsilon \big((\phi^+)'(r+\xi) + (\phi^-)'(r-\xi)\big)\,\mathrm{d}\xi\right)\mathrm{d}r \leqslant 2\varepsilon\|\phi\|_X.\end{aligned}$$

The estimate in (5.14) is immediate while, regarding (5.15), we rewrite $\psi_\varepsilon(r) - \phi(r)$ as

$$\begin{aligned}&\int_0^1 \eta(s)\big(\phi^+(r+\varepsilon s) - \phi^+(r) + \phi^-(r) - \phi^-(r-\varepsilon s)\big)\,\mathrm{d}s\\ &= \varepsilon\int_0^1 s\eta(s)\left(\int_0^1 (\phi^+)'(r+\varepsilon s\xi)\,\mathrm{d}\xi + \int_0^1 (\phi^-)'(r-\varepsilon s\xi)\,\mathrm{d}\xi\right)\mathrm{d}s.\end{aligned}$$

We conclude by integrating in r, switching the order of integration and using (5.12). □

Now we can prove the following proposition.

Proposition 5.2. *Let* ρ *and* ν *be two probability measures on* $\mathbb{R}^d$ *with support in a ball* B_R *and such that* $\rho \in L^\infty(\mathbb{R}^d)$. *Then*

$$\mathscr{D}(\nu,\rho) \leqslant C(\|\rho\|_\infty, R)\sqrt{\mathscr{W}(\nu,\rho)},$$

where C *is a constant that depends on the dimension* d, *as well as on* $\|\rho\|_\infty$ *and on* R.

Proof. Let ϕ be in X and consider ψ_ε as in (5.11). Fixed $x \in \mathbb{R}^d$, let Φ and Ψ_ε be the functions

$$\Phi(y) := \phi(|x-y|) \quad\text{and}\quad \Psi_\varepsilon(y) := \psi_\varepsilon(|x-y|).$$

Then, from iii) of Lemma (5.2),

$$\int \Phi\,\mathrm{d}\nu - \int \Phi\,\mathrm{d}\rho \leqslant \int \Psi_\varepsilon\,\mathrm{d}\nu - \int \Phi\,\mathrm{d}\rho = \int \Psi_\varepsilon\,\mathrm{d}(\nu-\rho) + \int (\Psi_\varepsilon - \Phi)\,\mathrm{d}\rho.$$

From (5.14) of Lemma (5.2), the first term is bounded by $\frac{c}{\varepsilon}\|\phi\|_X\mathscr{W}(\nu,\rho)$. Regarding the second term, denoting by σ_r the uniform measure on $\partial B_r(x)$, we have

$$\begin{aligned}\int (\Psi_\varepsilon - \Phi)\,\mathrm{d}\rho &\leqslant \|\rho\|_\infty \int_0^{+\infty} \mathrm{d}r\,(\psi_\varepsilon(r) - \phi(r)) \int_{\partial B_r(x)} \mathcal{X}\{z \in B_R\}\sigma(\mathrm{d}z)\\ &\leqslant c\varepsilon R^{d-1}\|\phi\|_X\|\rho\|_\infty,\end{aligned} \tag{5.16}$$

where in the last inequality we have used (5.15). Optimizing on ε and passing to the supremum in ϕ, we get the proof. □

Note that if μ^N is an empirical measure and ν a probability measure that does not give mass to the atoms of μ^N, $\mathscr{D}(\mu^N, \rho) \geqslant 1/N$. With this constraint, the discrepancy between two empirical measures is "small" if the measures are close in the sense specified in the following proposition.

Proposition 5.3. *Let*

$$\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i} \quad \textit{and} \quad \nu^N = \frac{1}{N}\sum_{i=1}^N \delta_{y_i}$$

be two empirical measures on $\mathbb{R}^d$ *and take* $\delta > 0$ *such that* $|x_i - y_i| \leqslant \delta$ *for all* $i = 1, \dots, N$. *Then, for any probability measure* $\rho \in L^\infty(\mathbb{R}^d)$ *supported on a ball* B_R,

$$\mathscr{D}(\mu^N, \nu^N) \leqslant cR^{d-1}\delta\|\rho\|_\infty + c\mathscr{D}(\mu^N, \rho).$$

Proof. Given $\phi \in X$ with $\|\phi\|_X \leqslant 1$, we construct ϕ_δ as in (5.10) and, fixed $x \in \mathbb{R}^d$, we consider $\Phi(y) := \phi(|x-y|)$, $\Phi_\delta(y) := \phi_\delta(|x-y|)$.

Since $|x - x_i| - \delta \leqslant |x - y_i| \leqslant |x - x_i| + \delta$, we have that

$$\Phi(y_i) = \phi^+(|x-y_i|) - \phi^-(|x-y_i|) \leqslant \Phi_\delta(x_i).$$

Then

$$\int \Phi \, \mathrm{d}(\nu^N - \mu^N) \leqslant \int (\Phi_\delta - \Phi)\,\mathrm{d}\mu^N = \int (\Phi_\delta - \Phi)\,\mathrm{d}(\mu^N - \rho) + \int (\Phi_\delta - \Phi)\,\mathrm{d}\rho.$$

Since $(\phi_\delta - \phi) \in X$, the first term is bounded by $c\mathscr{D}(\mu^N, \rho)$. Using (5.13) and reasoning as in (5.16) we estimate the second term with $c\delta R^{d-1}\|\rho\|_\infty$. ☐

5.3 Agent dynamics

One of the difficulties in handling (5.8) is that the dynamic is not continuous with respect to the initial datum. For instance, consider three agents $\{X_i\}_{i=1}^3$ on a line, such that

$$\begin{aligned} x_1(0) &= -1, \quad x_2(0) = \varepsilon, \quad x_3(0) = 1, \\ v_1(0) &= -1, \quad v_2(0) = 0, \quad v_3(0) = 1, \end{aligned} \tag{5.17}$$

with $\varepsilon \in (-1, 1)\backslash\{0\}$. Then $p_{i,j} = M(x_i, |x_i - x_j|)$ takes the values $1/3, 2/3, 1$. Suppose for simplicity that $K(2/3) = 3$ and $K(1) = 0$, then the equations for v_1 and v_3 read as

$$\begin{cases} \dot{v}_1(t) = v_2(t) - v_1(t) \\ \dot{v}_3(t) = v_2(t) - v_3(t), \end{cases}$$

while

$$\dot{v}_2(t) = \begin{cases} v_3(t) - v_2(t) & \text{if } \varepsilon \in (0,1) \\ v_1(t) - v_2(t) & \text{if } \varepsilon \in (-1,0). \end{cases}$$

It follows that

$$\begin{cases} v_1(t) = -(1 + e^{-2t})/2 \\ v_2(t) = -(1 - e^{-2t})/2 \\ v_3(t) = (-1 + 4e^{-t} - e^{-2t})/2 \end{cases}$$

if $\varepsilon \in (-1, 0)$, while

$$\begin{cases} v_1(t) = -(-1 + 4e^{-t} - e^{-2t})/2 \\ v_2(t) = (1 - e^{-2t})/2 \\ v_3(t) = (1 + e^{-2t})/2 \end{cases}$$

if $\varepsilon \in (0, 1)$, so that $\{x_i(t), v_i(t)\}_{i=1}^3$ is discontinuous in $\varepsilon = 0$. Note that the discontinuity of the trajectories in the phase space is easily translated in the weak discontinuity of the empirical measure at time t, with respect to the initial measure.

This discontinuity reflects the fact that, for data as in (5.17) with $\varepsilon = 0$, there is no a unique way to define the dynamics. Nevertheless, we can prove that the system (5.8) is well-posed for almost all initial data. To do so, let us define some subsets of the phase space

$$\left\{(X_N, V_N) \coloneqq (x_1, \dots, x_N, v_1, \dots, v_N) \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}\right\},$$

where $d \geqslant 1$ is the dimension of the configuration space of the agents.

Definition 5.1. $\mathcal{R}$ is the set of "the regular points", i.e. the set of points (X_N, V_N) such that for each triad of different indices it holds that $|x_i - x_k| \neq |x_j - x_k|$.

$\mathcal{S}$ is the "iso-rank" manifold, i.e. the set of points (X_N, V_N) such that there exists a triad of different indices i, j, k for which $|x_i - x_k| = |x_j - x_k|$, i.e. the agents i and j have the same rank with respect to the agent k.

$\mathcal{S}_r$ is the set of the "regular points" of the iso-rank manifold, i.e. the subset of points $(X_N, V_N) \in \mathcal{S}$ such that if $|x_i - x_k| = |x_j - x_k|$ then x_i, x_j, x_k are different and $(v_i - v_k) \cdot \hat{n}_{ik} \neq (v_j - v_k) \cdot \hat{n}_{jk}$, where $\hat{n}_{ab} \coloneqq (x_a - x_b)/|x_a - x_b|$.

We can define the dynamics locally in time, not only for initial data in $\mathcal{R}$, but also in $\mathcal{S}_r$. Namely, if initially the agents i and j have the same rank with respect to the agent k, we can redefine the force exerted on the agent k accordingly to the velocities: if $(v_i - v_k) \cdot \hat{n}_{ik} > (v_j - v_k) \cdot \hat{n}_{jk}$ we evaluate the rank as if $|x_i - x_k| > |x_j - x_k|$ for $t > 0$ and as if $|x_i - x_k| < |x_j - x_k|$ for $t < 0$. In other words, the different speeds of change of the distances among the agents allow the dynamics to leave $\mathcal{S}$ instantaneously.

We discuss the existence of the dynamics, so redefined.

Lemma 5.3. *If $(X_N, V_N) \in \mathcal{R} \cup \mathcal{S}_r$, there exists $\tau > 0$ such that the system (5.8) has a unique solution for $t \in (-\tau, \tau)$, with initial datum (X_N, V_N). Moreover the solution is locally Lipschitz in t and in (X_N, V_N).*

We omit the proof.

In $\mathcal{R}$ the solution is regular, so we can compute the determinant of the Jacobian of the flow $J(t) \equiv J(X_N, V_N, t)$. It verifies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} J(t) = -\left(\frac{d}{N} \sum_{i,j: i \neq j} p_{ij}\right) J(t) = -dN\gamma_N J(t), \tag{5.18}$$

where

$$\gamma_N \coloneqq \frac{1}{N}\sum_{n=2}^{N} K\left(n/N\right).$$

Thus, volumes of the phase space are shrunk in time at a constant rate, therefore their measure cannot vanish in finite time. This implies the following fact, of which we omit the proof.

Lemma 5.4. *The subset of initial data $(X_N, V_N) \in \mathcal{R}$ such that the trajectory, at a first time in the future or in the past, intersects $\mathcal{S}\backslash\mathcal{S}_r$, has Lebesgue measure zero. Namely, $\mathcal{S}\backslash\mathcal{S}_r$ has dimension $2Nd-2$.*

This lemma guarantees that, except for a subset of Lebesgue measure zero, we can prolong the dynamics with initial data in $\mathcal{R}$ also after a crossing in $\mathcal{S}$. To define the dynamics for all times, we need to control the number of crossings.

Lemma 5.5. *The subset of initial data $(X_N, V_N) \in \mathcal{R}$ such that the trajectory intersects $\mathcal{S}_r$ infinitely many times in finite time, has Lebesgue measure zero.*

Proof. Fix $T>0$ and suppose to take $(X_N, V_N) \in \mathcal{R}$ such that the solution $\left(X^N(t), V^N(t)\right) = (x_1(t), \ldots, x_N(t), v_1(t), \ldots, v_N(t))$ with initial data (X_N, V_N) intersects $\mathcal{S}_r$ a finite number of times in $[0, T-\varepsilon)$ and infinitely many times in $[0,T)$. The number of particles is finite, so we can assume that there exists a triad of indices such that $|x_i - x_k| = |x_j - x_k|$ infinitely many times. Since the velocities v_i are bounded by a constant, as follows by simple considerations (see also Theorem (5.1)), from the equation we have that $|x_i - x_k|$ and $|x_j - x_k|$ are C^1 functions, with time derivatives uniformly Lipschitz, if $|x_i - x_k|$ and $|x_j - x_k|$ remain far from 0. Then, as $t \to T$, either $|x_i - x_k| \to 0$ or $(v_i - v_k)\cdot \hat{n}_{ik}$ and $(v_j - v_k)\cdot\hat{n}_{jk}$ converge to the same limit. In both cases, the trajectory reaches $\mathcal{S}$ at a point that is not in $\mathcal{S}_r$. We conclude the proof by observing that the initial point with these properties lives in a subset of dimension $2Nd-1$. □

From these lemmas and other few considerations, we obtain the following theorem.

Theorem 5.1. *Except for a set of measure zero, given $(X_N, V_N) \in \mathbb{R}^{Nd}\times\mathbb{R}^{Nd}$, there exists a unique global solution*

$$\left(X^N(t, X_N, V_N), V^N(t, X_N, V_N)\right) \in C^1(\mathbb{R}^+, \mathbb{R}^{2dN}) \times C(\mathbb{R}^+, \mathbb{R}^{2dN})$$

with initial datum (X_N, V_N).

Moreover, given $R_x > 0$ and $R_v > 0$, we have that

$$|x_i^N(t)| \leqslant R_x + tR_v, \; |v_i^N(t)| \leqslant R_v$$

for any i, if $|x_i| \leqslant R_x$ and $|v_i| \leqslant R_v$. Therefore $v_i^N(t, X_N, V_N)$ has Lipschitz constant bounded by $2R_vK(0)$.

Proof. The proof follows easily from Lemma (5.3), Lemma (5.4) and Lemma (5.5).

The a-priori bound on the support follows from (5.18) and by noticing that

$$\frac{\mathrm{d}}{\mathrm{d}t}|v_i^N(t)|^2 = -2\sum_{j\neq i} p_{ij}\left(|v_i^N(t)|^2 - v_i^N(t)\cdot v_j^N(t)\right)$$

is null or negative if $|v_i^N|^2$ is maximum in i.

□

5.4 The mean-field equation in L^∞

In this section we show how to get an existence and uniqueness result for bounded weak solutions of equation (5.4). We start by stating some elementary facts.

Lemma 5.6. *Let $\rho \in L^\infty(\mathbb{R}^d)$ be a probability density.*

i) Given $r_1, r_2 > 0$,

$$|M[\rho](x, r_1) - M[\rho](x, r_2)| \leqslant c\|\rho\|_\infty \left| r_1^d - r_2^d \right|.$$

ii) Given $x_1, x_2 \in \mathbb{R}^d$ and $r > 0$,

$$|M[\rho](x_1, r) - M[\rho](x_2, r)| \leqslant c\|\rho\|_\infty r^{d-1}|x_1 - x_2|.$$

Proof. The proof of the first assertion is immediate. For the second, we use the following splitting

$$\begin{aligned}\mathcal{X}\{|x_1 - y| < r\} - \mathcal{X}\{|x_2 - y| < r\} = \mathcal{X}\{|x_1 - y| < r\}\mathcal{X}\{|x_2 - y| \geqslant r\} \\ - \mathcal{X}\{|x_2 - y| < r\}\mathcal{X}\{|x_1 - y| \geqslant r\}\end{aligned}$$

and we note that, if $|x_1 - x_2| \geqslant r$,

$$\int_{|x_1 - y| < r} \mathcal{X}\{|x_2 - y| \geqslant r\} \,\mathrm{d}y \leqslant cr^d \leqslant cr^{d-1}|x_1 - x_2|,$$

while, if $|x_1 - x_2| < r$,

$$\begin{aligned}\int_{|x_1 - y| < r} \mathcal{X}\{|x_2 - y| \geqslant r\} \,\mathrm{d}y &\leqslant \int \mathcal{X}\left\{r - |x_1 - x_2| < |x_1 - y| < r\right\} \mathrm{d}y \\ &= cr^d \left(1 - (1 - |x_1 - x_2|/r)^d\right) \leqslant cdr^{d-1}|x_1 - x_2|.\end{aligned}$$

□

In the following, we denote by $\mathcal{B}_r$ the closed ball of center 0 and radius r in $L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and by $C_w\left([0, +\infty); L^\infty(\mathbb{R}^d \times \mathbb{R}^d)\right)$ the set of families of bounded probability densities $\{f_t\}_{t \geqslant 0}$ which are weakly continuous in time in the sense of measures.

Lemma 5.7. *Let $\{f_t\}_{t \geqslant 0}$ be a family of probability densities such that $\{f_t\} \in C_w\left([0, +\infty); \mathcal{B}_{r(t)}\right)$, with $r(t)$ a continuous nondecreasing function. Suppose that*

$$\textit{supp}(f_t) \subset B_{R_x(t)} \times B_{R_v(t)}, \tag{5.19}$$

where $R_v(t)$ and $R_x(t)$ are two continuous non-decreasing functions. Then, for any initial datum $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, there exists a unique global solution of (5.7).

Proof. From the classical Cauchy-Lipschitz theory, we only have to verify that $W[Sf_t, f_t](x, v)$ is bounded on compact sets, locally Lipschitz and continuous in t.

Recalling (5.5), the boundedness on compact sets follows from

$$|W[Sf_t, f_t](x, v)| \leqslant \|K\|_\infty \left(R_v(t) + |v|\right).$$

Since from i) and ii) of Lemma (5.6)

$$\begin{aligned}|M[Sf_t](x_1, |x_1 - y|) - M[Sf_t](x_2, |x_2 - y|)|\\ \leqslant c\|Sf_t\|_\infty (|x_1| + |x_2| + |y|)^{d-1}|x_1 - x_2|,\end{aligned}$$

we have that, if (x_1, v_1) and (x_2, v_2) belong to a compact subset of $\mathbb{R}^d \times \mathbb{R}^d$,

$$|W[Sf_t, f_t](x_1, v_1) - W[Sf_t, f_t](x_2, v_2)| \leqslant C(|x_1 - x_2| + |v_1 - v_2|),$$

where C depends on R_x, R_v and the diameter of the compact set.

In order to prove that $W[Sf_t, f_t](x, v)$ is continuous in t, we first observe that from the Lipschitzianity of K and Propositions (5.1) and (5.2), since $\mathscr{W}(Sf_t, Sf_s) \leqslant \mathscr{W}(f_t, f_s)$, we have $K\left(M[Sf_t](x, |x - y|)\right)$ is continuous in t. Since $K(M[Sf_t](x, |x - y|))$ is Lipschitz in y, also

$$\int K\left(M[Sf_t](x, |x - y|)\right)(v - w)\left(f_t(y, w) - f_s(y, w)\right) \mathrm{d}y \, \mathrm{d}w$$

vanishes when $\mathscr{W}(f_t, f_s) \to 0$. ☐

Now we can prove the main theorem of this section.

Theorem 5.2. *Let $f_0(x, v) \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be a probability density such that supp$(f_0) \subset B_{R_x} \times B_{R_v}$. Given $T > 0$, there exists a unique weak solution $f \in C_w\left([0, T]; L^\infty(\mathbb{R}^d \times \mathbb{R}^d)\right)$ of the topological Cucker-Smale equation. Moreover*

$$\textit{supp}(f_t) \subset B_{R_x + tR_v} \times B_{R_v}. \tag{5.20}$$

Proof. We first note that, if the solution exists, (5.20) follows from an argument similar to the one used in the discrete case (see Theorem (5.1)).

We now prove the existence. As in Lemma (5.7), consider a family of probability densities $\{g_t\}_{t\geqslant 0} \in C_w\left([0, T]; \mathcal{B}_M\right)$, with $M := \|f_0\|_\infty e^{d\gamma T}$ and such that (5.19) holds with $R_x(t) = R_x + tR_v$ and $R_v(t) = R_v$. The push-forward of f_0 along the flow generated by g_t, denoted by $\tilde{g}_t$, is weakly continuous in t, uniformly in g_t, with $t \in [0, T]$. Moreover, the determinant of the Jacobian of the flow $J(t) = J(t, x, v)$ verifies

$$\frac{\mathrm{d}}{\mathrm{d}t} J(t) = -J(t) d\gamma.$$

So the push-forward $\tilde{g}_t$ is bounded by $\|f_0\|_\infty e^{d\gamma t}$.

With a standard construction we can prove that, for T sufficiently small, the map $\{g_t\} \mapsto \{\tilde{g}_t\}$ is a contraction in $C_w\left([0, T]; \mathcal{B}_M\right)$, with the distance defined by the supremum on time of the Wasserstein distance; in this way we prove local existence and uniqueness. Using the a-priori estimate on the supremum and the support, we get the global result. ☐

5.5 The mean-field limit

In this section we prove the main result regarding the mean-field limit for the topological Cucker-Smale equation. In the sequel, f_t is the fixed global solution of eq. (5.7) as in Theorem (5.2), with initial datum f_0, and μ_t^N is the global solution of equation (5.8) in the sense of Theorem (5.1), with initial datum

$$\mu_0^N = \frac{1}{N}\sum_{i=0}^{N} \delta_{x_i}\delta_{v_i}.$$

We assume that f_0 and μ_0^N are supported in $B_{R_x} \times B_{R_v}$. Fixed T, we indicate by $C(T)$ any constant that depends only on T, R_x, R_v and $\|f_0\|_\infty$.

To get the result, we compare the N-agent dynamics with the "intermediate" dynamics given by

$$\begin{cases} \dot{x}_i^f(t) = v_i^f(t) \\ \dot{v}_i^f(t) = W[Sf_t, \nu_t^N](x_i^f, v_i^f), \end{cases}$$

where

$$\nu_t^N \coloneqq \frac{1}{N}\sum_{k=1}^{N} \delta_{x_k^f(t)}\,\delta_{v_k^f(t)}$$

is the empirical measure. The initial datum is $\nu_0^N = \mu_0^N$, i.e.

$$\{(x_i^f(0), v_i^f(0))\}_{i=1}^N = \{(x_i, v_i)\}_{i=1}^N.$$

Proposition 5.4. *Given $T > 0$, it holds that*

i) For $t \in [0, T]$,

$$\mathscr{W}(f_t, \nu_t^N) \leqslant C(T)\mathscr{W}(f_0, \mu_0^N). \tag{5.21}$$

ii) For $t \in [0, T]$, the distance

$$\delta(t) \coloneqq \max_{i=1,\dots,N}\left(|x_i^f(t) - x_i^N(t)| + |v_i^f(t) - v_i^N(t)|\right)$$

verifies

$$\delta(t) \leqslant C(T)\sqrt{\mathscr{W}(f_0, \mu_0^N)}. \tag{5.22}$$

Proof. Since f_t is bounded, $K\left(M[Sf_t](x, |x-y|)\right)$ is locally Lipschitz in x and y (see i) and ii) of Lemma (5.6)) and then $W[Sf_t, \nu](x, v)$ is weakly continuous in ν in the sense that

$$\sup_{x,v} |W[S_f, \nu_1](x, v) - W[S_f, \nu_2](x, v)| \leqslant C(T)\mathscr{W}(\nu_1, \nu_2).$$

It is straightforward to prove that the solution ν_t of the system

$$\begin{cases} \dot{x}_t = v_t \\ \dot{v}_t = W[Sf_t, \nu_t](x_t, v_t) \\ \nu_t = \text{ push-forward of } \nu_0 \text{ along the flow } (x_t, v_t) \end{cases}$$

is continuous in $\mathscr{W}$ with respect to the initial datum ν_0. Taking $\nu_0 = f_0$ and $\nu_0 = \mu_0^N$ we get the proof of i).

In order to estimate $\delta(t)$, we need to evaluate, for $0 \leqslant s \leqslant t$ and for $i = 1, \dots, N$, the difference $|\dot{v}_i^f(s) - \dot{v}_i^N(s)|$ given by

$$|W[Sf_s, \nu_s^N](x_i^f, v_i^f) - W[S\mu_s^N, \mu_s^N](x_i^N, v_i^N)|.$$

We estimate this quantity with the sum of three terms:

$$\begin{aligned}
&(a)\ \ |W[Sf_s, \nu_s^N](x_i^f, v_i^f) - W[Sf_s, \nu_s^N](x_i^N, v_i^N)|,\\
&(b)\ \ |W[Sf_s, \nu_s^N](x_i^N, v_i^N) - W[Sf_s, \mu_s^N](x_i^N, v_i^N)|,\\
&(c)\ \ |W[Sf_s, \mu_s^N](x_i^N, v_i^N) - W[S\mu_s^N, \mu_s^N](x_i^N, v_i^N)|.
\end{aligned}$$

Since $K\left(M[Sf_s](x, |x-y|)\right)$ is Lipschitz in x, from the definition of W it is easy to prove that (a) is bounded by

$$\left(c\,\mathrm{Lip}(K)\|Sf_s\|_\infty R_x^{d-1}(s) R_v + c\|K\|_\infty\right)\delta(s)$$

and that (b) is estimated by

$$c\,\mathrm{Lip}(K)\|Sf_s\|_\infty R_x^{d-1}(s) R_v \delta(s).$$

Note that $\|Sf_s\|_\infty \leqslant cR_v^d\|f_s\|_\infty$. From Proposition (5.1) we have that (c) is bounded by

$$c\mathrm{Lip}(K) R_v \mathscr{D}(Sf_s, S\mu_s^N).$$

Since

$$\mathscr{D}(Sf_s, S\mu_s^N) \leqslant \mathscr{D}(Sf_s, S\nu_s^N) + \mathscr{D}(S\nu_s^N, S\mu_s^N),$$

by Proposition (5.3) with $\rho = Sf_s$, $\mu^N = S\nu_s^N$ and $\nu^N = S\mu_s^N$, we get

$$\mathscr{D}(S\nu_s^N, S\mu_s^N) \leqslant c\delta(s) + c\mathscr{D}(Sf_s, S\nu_s^N).$$

Writing $\delta(t)$ in terms of the time integral of $\delta(s)$ and the difference of the interaction terms and using the Gronwall lemma, we readily get the estimate

$$\delta(t) \leqslant C(T)\int_0^t \mathscr{D}(Sf_s, S\nu_s^N)\,\mathrm{d}s,$$

valid for $0 \leqslant t \leqslant T$. We conclude the proof by using Proposition (5.2), equation (5.21) and the fact that $\mathscr{W}(Sf_s, S\nu_s^N) \leqslant \mathscr{W}(f_s, \nu_s^N)$. □

Theorem 5.3. *Fixed $T > 0$, let f_t be a solution of eq. (5.7) as in Theorem (5.2) with initial datum f_0 and let μ_t^N be a solution of equation (5.8) in the sense of Theorem (5.1) with initial datum μ_0^N. Then, for $0 \leqslant t \leqslant T$,*

$$\mathscr{W}(f_t, \mu_t^N) \leqslant C(T)\mathit{max}\left\{\mathscr{W}(f_0, \mu_0^N), \sqrt{\mathscr{W}(f_0, \mu_0^N)}\right\}.$$

Proof. By the triangular inequality,

$$\mathscr{W}(f_t, \mu_t^N) \leqslant \mathscr{W}(f_t, \nu_t^N) + \mathscr{W}(\nu_t^N, \mu_t^N).$$

From (5.21), using that $\mathscr{W}(\nu_t^N, \mu_t^N) \leqslant \delta(t)$ and (5.22), we get the thesis. □

Chapter 6

Propagation of chaos for a jump process with topological interactions

joint work with P. Degond and M. Pulvirenti ([22])

In this Chapter we consider a system of particles that interact through a jump process. The jump intensities are of topological type, being functions of the proximity rank of the particles. We show that, in the large number of particles limit and under minimal smoothness assumptions on the data, the model converges to a kinetic equation which was rigorously derived in the earlier work [21] under more stringent regularity assumptions. We do this by showing that the total variation distance between the two processes tends to zero as the number of particles tends to infinity, with an error typical of the law of large numbers.

6.1 Presentation of the model and main results

In [7] the authors introduced the following stochastic model. We consider a N-particle system in $\mathbb{R}^d$, $d = 1, 2, 3\dots$ (or in $\mathbb{T}^d$ the d-dimensional torus). Each particle, say particle i, has a position x_i and velocity v_i. The configuration of the system is denoted by

$$Z_N = \{z_i\}_{i=1}^N = \{(x_i, v_i)\}_{i=1}^N = (X_N, V_N).$$

Given the particle i, we order the remaining particles $j_1, j_2, \cdots j_{N-1}$ according to their distance from i, namely by the following relation

$$|x_i - x_{j_h}| \leqslant |x_i - x_{j_{h+1}}|, \qquad h = 1, 2 \cdots N-1.$$

The rank $R(i, k)$ of particle $k = j_h$ (with respect to i) is h. Note that, if $B_r(x)$ denotes the closed ball of center $x \in \mathbb{R}^d$ and radius $r > 0$, we have

$$R(i, k) = \sum_{\substack{1 \leqslant h \leqslant N \\ h \neq i}} \mathcal{X}_{B_{|x_i - x_k|}(x_i)}(x_h),$$

where $\mathcal{X}_A$ is the characteristic function of the set A.

Considering a nonincreasing Lipschitz continuous function

$$K : [0,1] \to \mathbb{R}^+ \quad \text{s.t.} \quad \int_0^1 K(r)\,\mathrm{d}r = 1,$$

we introduce the transition probabilities

$$\pi_{i,j}^N = \frac{K(r(i,j))}{\sum_{s=1}^{N-1} K(\frac{s}{N-1})}, \tag{6.1}$$

where $r(i,j)$ is the normalized rank:

$$r(i,j) = \frac{R(i,j)}{N-1} \in \left\{ \frac{1}{N-1}, \frac{2}{N-1}, \dots \right\}.$$

This is similar to the function $M(x_i, |x_i - x_j|)$ introduced in (5.3) of the previous Chapter, but with a different normalization.

Thanks to the normalization in (6.1), we have that $\sum_j \pi_{i,j}^N = 1$. We can also rewrite $\pi_{i,j}^N$ as

$$\pi_{i,j}^N = \alpha_N K\Big(r(i,j)\Big), \tag{6.2}$$

where

$$\alpha_N = \frac{1}{(N-1)(1-e_K(N))} \tag{6.3}$$

and $e_K(N)$ is the error given by the Riemann sums

$$e_K(N) = \int_0^1 K(x)\,\mathrm{d}x - \frac{1}{N-1}\sum_s K\Big(\frac{s}{N-1}\Big). \tag{6.4}$$

We are now in position to introduce a stochastic process describing alignment via a topological interaction. The particles go freely: $x_i + v_i t$. At some random time dictated by a Poisson process of intensity N, choose a particle (say i) with probability $\frac{1}{N}$ and a partner particle, say j, with probability $\pi_{i,j}$. Then perform the transition

$$(v_i, v_j) \to (v_j, v_j).$$

After that the system goes freely with the new velocities and so on.

The process is described by the following Markov generator given, for any $\Phi \in C_b^1(\mathbb{R}^{2dN})$, by

$$L_N\Phi(X_N, V_N) = \sum_{i=1}^N v_i \cdot \nabla_{x_i}\Phi(X_N, V_N) + \sum_{i=1}^N \sum_{\substack{1\leqslant j\leqslant N \\ i\neq j}} \pi_{i,j}^N \big[\Phi(X_N, V_N^i(v_j)) - \Phi(X_N, V_N)\big], \tag{6.5}$$

where $V_N^i(v_j) = (v_1 \dots v_{i-1}, v_j, v_{i+1} \dots v_N)$ if $V_N = (v_1 \dots v_{i-1}, v_i, v_{i+1} \dots v_N)$.

Note that $\pi_{i,j}^N$ depends not only on N but also on the whole spatial configuration X_N. Therefore the law of the process $W^N(t) = W^N(t, Z_N)$ is driven by the following evolution equation

$$\partial_t \int W^N(t)\Phi = \int W^N(t) \sum_{i=1}^N v_i \cdot \nabla_{x_i} \Phi + \int W^N(t, Z_N) \sum_{i=1}^N \sum_{\substack{1 \leqslant j \leqslant N \\ i \neq j}} \pi_{i,j}^N \big[\Phi(X_N, V_N^i(v_j)) - \Phi(X_N, V_N)\big], \quad (6.6)$$

for any test function Φ.

We assume that the initial measure $W^N(0)$ factorizes, namely $W^N(0) = f_0^{\otimes N}$ where f_0 is the initial datum for the limit kinetic equation we are going to establish. Note also that $W^N(t, Z_N)$, for $t \geqslant 0$, is symmetric in the exchange of particles.

The strong form of equation (6.6) is

$$\Big(\partial_t + \sum_{i=1}^N v_i \cdot \nabla_{x_i}\Big) W^N(t) = -N W^N(t) + \mathcal{L}_N W^N(t)$$

where

$$\mathcal{L}_N W^N(t, X_N, V_N) = \sum_{i=1}^N \sum_{\substack{1 \leqslant j \leqslant N \\ i \neq j}} \int du \, \pi_{i,j}^N \, W^N(t, X_N, V_N^{(i)}(u)) \delta(v_i - v_j).$$

6.1.1 Heuristic derivation

We now want to derive the kinetic equation we expect to be valid in the limit $N \to \infty$. Setting $\Phi(Z_N) = \varphi(z_1)$ in (6.6), we obtain

$$\partial_t \int f_1^N \varphi = \int f_1^N v \cdot \nabla_x \varphi - \int f_1^N \varphi + \int W^N \sum_{j \neq 1} \pi_{i,j}^N \varphi(x_1, v_j). \quad (6.7)$$

Here f_1^N denotes the one-particle marginal of the measure W^N. We recall that the s-particle marginals are defined by

$$f_s^N(Z_s) = \int W^N(Z_s, z_{s+1} \cdots z_N) dz_{s+1} \cdots dz_N, \qquad s = 1, 2 \cdots N \quad (6.8)$$

and are the distribution of the first s particles (or of any group of s tagged particles).

In order to describe the system in terms of a single kinetic equation, we expect that chaos propagates. Actually since W^N is initially factorizing, although the dynamics creates correlations, we hope that, due to the weakness of the interaction, factorization still holds approximately also at any positive time t, namely

$$f_s^N \approx f_1^{\otimes s}.$$

In this case the law of large numbers does hold, that is

$$\frac{1}{N} \sum_j \delta(z - z_j) \approx f_1^N(t, z)$$

for W^N- almost all $Z_N = \{z_1 \cdots z_N\}$. Then

$$\pi^N_{i,j} \approx \frac{1}{N-1} K\Big(\frac{1}{N-1}\sum_k \mathcal{X}_{B_{|x_i-x_j|}(x_i)}(x_k)\Big)$$
$$\approx \frac{1}{N-1} K\Big(M[\rho_1^N](x_1, |x_1-x_2|)\Big)$$

where, given a measure $\rho \in \mathcal{P}(\mathbb{R}^d)$,

$$M[\rho](x,R) = \int_{B_R(x)} \rho(y)\,\mathrm{d}y, \tag{6.9}$$

and $\rho_1^N(x) = \int \mathrm{d}v f_1^N(x,v)\,\mathrm{d}v$ is the spatial density. Motivated by this remark and similarly to the previous Chapter, from now on we use the following notation

$$M[X^N](x_i, |x_i - x_j|) = r(i,j) = \frac{1}{N-1}\sum_k \mathcal{X}_{B_{|x_i-x_j|}(x_i)}(x_k).$$

Here M stands for 'mass' and the notation introduced is justified by the law of large numbers.

In conclusion we expect that, by (6.7), in the limit $N \to \infty$, $f_1^N \to f$ and $f_2^N \to f^{\otimes 2}$, where f solves

$$\partial_t \int f\varphi = \int f v\cdot\nabla_x\varphi - \int f\varphi + \int f(z_1)f(z_2)\varphi(x_1,v_2)K\Big(M[\rho](x_1,|x_1-x_2|)\Big)$$

which is the weak form of the equation

$$\Big(\partial_t + v\cdot\nabla_x\Big) f(t,x,v) = -f(t,x,v) + \rho(t,x)\int K\Big(M[\rho](x,|x-y|)\Big)\, f(t,y,v)\,\mathrm{d}y. \tag{6.10}$$

We remark that existence and uniqueness of global solutions in $L^1(\mathbb{R}^{2d})$ for the kinetic equation (6.10) can be proved by using a standard Banach fixed-point argument.

Once known f, we can construct the one-particle nonlinear process given by the generator

$$L_1^{(1)}\phi(x,v) = (v\cdot\nabla_x - 1)\phi(x,v) + \int f(y,w)\phi(x,w)K\Big(M[\rho](x,|x-y|)\Big)\,\mathrm{d}y\,\mathrm{d}w.$$

We also introduce the N-particle process given by N independent copies of the above process. Its generator is

$$L_N^{(1)}\Phi(Z_N) = V_N\cdot\nabla_{X_N}\Phi(Z_N)$$
$$+\sum_i\Big[\int \Phi(X_N, V_N^i(w_i))K\Big(M[\rho](x_i,|x_i-y_i|)\Big) f(y_i,w_i)\,\mathrm{d}y_i\,\mathrm{d}w_i - \Phi(X_N,V_N)\Big]. \tag{6.11}$$

6.1.2 Motivations and main result

We want to prove propagation of chaos for the N-particle process described by (6.5). Propagation of chaos consists in preparing a system of N particles with initial configurations i.i.d with a given law f_0 and show that, considering any group of fixed s particles between the N ones, this independence (chaos) is also recovered for future times for the fixed s-group when $N \to \infty$. This is expressed mathematically by saying that the s-particle marginal $f_s^N(t)$ introduced in (6.8) approximates $f^{\otimes s}(t)$ for positive times, where $f(t)$ is the solution with initial datum f_0 of the kinetic equation (6.10).

The work [22] presented here is strongly aligned with [7, 8, 21] where kinetic models are derived for topological interaction models based on jump processes. More precisely, [21] proves propagation of chaos and provides a rigorous proof of the model introduced before and formally derived in [7]. On the other hand, [8] formally derives a kinetic model for a more singular interaction. The mathematical validity of this formal result is still open.

The proof of [21] makes the limiting assumption that the interaction strength is an analytic function of the normalized rank and is based on the BBGKY hierarchy. Indeed, the BBGKY hierarchies are a powerful approach but in this case the nonbinary nature of the topological interaction does not allow to derive this hierarchical structure, unless the interaction function K is real analytic and so expandable in series.

Here we want to provide a different derivation of the limit kinetic equation, using the classic probabilistic coupling technique. In general, given two stochastic processes X and Y, a coupling is a realization of a new process on a product probability space that has as marginal distributions those of X and Y.

The advantage of the coupling method over the BBGKY hierarchy is that it only requires the interaction strength to be Lipschitz continuous, a much more general and natural assumption than that of [21].

Theorem 6.1. *Let $f \in C([0,T]; L^1(\mathbb{R}^{2d}))$ solution of the kinetic equation (6.10) with initial datum $f_0 \in L^1(\mathbb{R}^{2d})$. Assume that the interaction function K is Lipschitz-continuous and consider the N-particle dynamics such that $W_N(0) = f_0^{\otimes N}$.*

If f_s^N denotes the s-marginal as defined in (6.8), for $t \in [0,T]$ and $s \in \{1, \ldots, N\}$, it holds that

$$\|f_s^N(t) - f^{\otimes s}(t)\|_{L^1(\mathbb{R}^{2ds})} \leqslant s \frac{e^{C_K T}}{\sqrt{N-1}}, \tag{6.12}$$

where C_K is a constant depending on the Lipschitz constant of K.

The topological character of the interaction bring us naturally to work with norms of strong type and in particular with the L^1/Total variation distance (coherently with [5] and the previous Chapter where the similar Discrepancy distance has been used to prove the validity of the mean-field limit for the deterministic Cucker-Smale model with topological interactions introduced in [32]).

Indeed, given two measures ρ_1 and ρ_2, from (6.9) we have

$$|M[\rho_1](x,r) - M[\rho_2](x,r)| \leqslant \|\rho_1 - \rho_2\|_{TV}$$

where, given $(X, \mathcal{A})$ a measurable space and two measures μ and ν over X, the total variation distance is defined as

$$\|\mu - \nu\|_{TV} = \sup_{A \in \mathcal{A}} |\mu(A) - \nu(A)|.$$

In the sequel, we use the equivalence between the L^1 distance and the Total variation for regular measures and the characterization of the TV distance given by the Wasserstein distance

$$\|\mu - \nu\|_{TV} = \inf_{\pi \in \mathcal{C}(\mu,\nu)} \int_{X \times X} d(x, y)\, \mathrm{d}\pi(x, y),$$

where $\mathcal{C}(\mu, \nu)$ is the set of all couplings, i.e. measures on the product space with marginals respectively μ and ν in the first and second variables, and $d(a, b) = 1 - \delta_{a,b}$ is the discrete distance (see [60]).

6.2 Proof of the result

6.2.1 Coupling and strategy of the proof

We introduce, as a coupling between (6.5) and (6.11), the process $t \to (Z_N(t); \Sigma_N(t))$ on the product space $\mathbb{R}^{2dN} \times \mathbb{R}^{2dN}$, where $\Sigma_N(t) = (Y_N(t), W_N(t))$. The generator of the new process is

$$Q_N = Q_0 + \widetilde{Q}_N,$$

where

$$Q_0 \Phi(Z_N; \Sigma_N) = (V_N \cdot \nabla_{X_N} + W_N \cdot \nabla_{Y_N}) \Phi(Z_N; \Sigma_N) \tag{6.13}$$

is the free-stream operator, while

$$\widetilde{Q}_N \Phi(Z_N; \Sigma_N) = \sum_{i=1}^{N} \sum_{j \neq i} \lambda_{i,j} [\Phi(X_N, V_N^i(v_j); Y_N, W_N^{(i)}(w_j)) - \Phi(Z_N; \Sigma_N)] \tag{6.14a}$$

$$+ \sum_{i=1}^{N} \sum_{j \neq i} [\pi_{i,j}^N(X_N) - \lambda_{i,j}][\Phi(X_N, V_N^i(v_j); \Sigma_N) - \Phi(Z_N; \Sigma_N)] \tag{6.14b}$$

$$+ \sum_{i=1}^{N} \sum_{j \neq i} [\pi^f(y_i, y_j) - \lambda_{i,j}][\Phi(Z_N; Y_N, W_N^{(i)}(w_j)) - \Phi(Z_N; \Sigma_N)] \tag{6.14c}$$

$$+ \sum_{i=1}^{N} \int \mathrm{d}u\, \mathcal{E}_i^N(u) [\Phi(Z_N; Y_N, W_N^{(i)}(u)) - \Phi(Z_N; \Sigma_N)] \tag{6.14d}$$

tends to penalize the discrepancies that can occur over time between Z_N and Σ_N.

Indeed, in (6.14a) the process jumps jointly on both variables with a rate given by

$$\lambda_{i,j}(X_N; y_i, y_j) \coloneqq \min\{\pi_{i,j}^N(X_N), \pi^f(y_i, y_j)\}, \tag{6.15}$$

where

$$\pi^f(y_i, y_j) \coloneqq \alpha_N K\Big(M[\rho](y_i, |y_i - y_j|)\Big). \tag{6.16}$$

In (6.14b) and (6.14c) the jumps occur only for one of the pair, with a transition probability given by the error between $\lambda_{i,j}$ and π^N or π^f. Finally, in (6.14d),

$$\mathcal{E}_i^N(u) = \int K\Big(M[\rho](y_i, |y_i - y|)\Big) f(y,u)\,\mathrm{d}y - \sum_{j\neq i} \pi^f(y_i, y_j)\delta(u - w_j)$$

is the last error due to the approximation of the limit kinetic equation by the N-particle dynamics with transition probabilities given by π^f and will be treated using the law of large numbers.

We remark that, since $\int K(x)\,\mathrm{d}x = 1$, formally we have[1],

$$\begin{aligned}\int K\Big(M[\rho](x, |x-y|)\Big)\rho(y)\,\mathrm{d}y &= \int_0^{+\infty} \mathrm{d}r K(M[\rho](x,r)) \int_{|x-y|=r} \rho(y)\,\mathrm{d}\mathcal{H}^{n-1}(dy) \\ &= \int_0^{+\infty} \mathrm{d}r K(M[\rho](x,r)) \frac{\mathrm{d}}{\mathrm{d}r}[M[\rho](x,r)] = \int K(x)\,\mathrm{d}x = 1.\end{aligned}$$

From this fact, it follows that Q_N is a coupling of the two previously described processes, i.e. we recover, considering test functions depending only Z_N and Σ_N respectively, the two processes as the two marginals.

We want to prove that f and f_1^N (defined as in (6.8)) agree asymptotically in the limit $N \to +\infty$. To do this we consider $R^N(t) = R^N(t, Z_N, \Sigma_N)$ the law at time t for the coupled process. As initial distribution at time 0 we assume

$$R^N(0) = f_0^{\otimes N}(Z_N)\delta(Z_N - \Sigma_N). \tag{6.17}$$

Let $D_N(t)$ be the average fraction of particles having different positions or velocities, i.e. using the symmetry of the law,

$$D_N(t) = \int \mathrm{d}R^N(t)\Big[\frac{1}{N}\sum_{i=1}^N d(z_i, \sigma_i)\Big] = \int \mathrm{d}R^N(t) d(z_1, \sigma_1), \tag{6.18}$$

where $z_i = (x_i, v_i)$, $\sigma_i = (y_i, w_i)$ and $d(a,b) = 1 - \delta_{a,b}$ is the discrete distance.

The aim is to show that $D_N(t) \to 0$. This means the following: initially the coupled system has all the pairs of particles overlapping. The dynamics creates discrepancies and the average number of separated pairs is exactly D_N which is also the Total Variation distance ($L^1(x,v)$ in our case) between f_1^N and f.

Notice that the convergence of the s-marginals f_s^N to $f^{\otimes s}$ claimed in (6.12) is easily recovered by the fact that

$$\begin{aligned}\|f_s^N(t) - f^{\otimes s}(t)\|_{TV} &\leqslant \int \delta(Z_s, \Sigma_s)\,\mathrm{d}R^N(t, Z_N, \Sigma_N) \\ &\leqslant \sum_{i=1}^s \int d(z_i, \sigma_i)\,\mathrm{d}R^N(t, Z_N, \Sigma_N) = sD_N(t)\end{aligned}$$

where $\delta(a,b)$ denotes the discrete distance on the space $\mathbb{R}^{2ds} \times \mathbb{R}^{2ds}$.

[1]In general, the formula is true for $\rho \in L^1(\mathbb{R}^d)$ and it is a consequence of the coarea formula (see [25, Thm 3.12, p. 140]).

6.2.2 Convergence estimates

Let S_t^N be the semigroup defined by the free-stream generator Q_0 in (6.13). To estimate $D_N(t)$ we apply the Duhamel formula in (6.18) and we get

$$\int \mathrm{d}R^N(t)d(z_1,\sigma_1) = \int \mathrm{d}R^N(0)d\Big(S_t^N(z_1,\sigma_1)\Big) + \int_0^t \mathrm{d}\tau \int \mathrm{d}R^N(\tau)\,\widetilde{Q}_N d\Big(S_{t-\tau}^N(z_1,\sigma_1)\Big), \tag{6.19}$$

where $\widetilde{Q}_N$ is defined in (6.14).

The first term in (6.19) is negligible: indeed, from (6.17), we have

$$\int \mathrm{d}R^N(0)d\Big(S_t^N(z_1,\sigma_1)\Big) = \int \mathrm{d}f_0^{\otimes N}(Z_N)d\Big(S_t^N(z_1,z_1)\Big) \equiv 0.$$

Concerning the second term in (6.19), we define

$$\bar{z}_1 = (x_1+v_1(t-\tau),v_1), \quad \bar{z}_1^{(j)} = (x_1+v_1(t-\tau),v_j)$$

and $\bar{X}_N = (x_1+v_1(t-\tau),\dots,x_N+v_N(t-\tau))$; similarly for $\bar{\sigma}$, $\bar{\sigma}^{(j)}$ and $\bar{Y}_N$.

By (6.14) we get

$$\int \mathrm{d}R^N(\tau)\,\widetilde{Q}_N d\Big(S_{t-\tau}^N(z_1,\sigma_1)\Big) = A_1(\tau)+A_2(\tau)+A_3(\tau),$$

where

$$A_1(\tau) = \sum_{j\neq 1}\int \mathrm{d}R^N(\tau)\lambda_{1,j}(\bar{X}_N;\bar{y}_1,\bar{y}_j)[d(\bar{z}_1^{(j)};\bar{\sigma}_1^{(j)}) - d(\bar{z}_1;\bar{\sigma}_1)]$$

is due to the term of the generator $\widetilde{Q}_N$ where the velocities of the particles jump simultaneously;

$$A_2(\tau) = \sum_{j\neq 1}\int \mathrm{d}R^N(\tau)(\pi_{1,j}^N(\bar{X}_N)-\lambda_{1,j})[d(\bar{z}_1^{(j)};\bar{\sigma}_1) - d(\bar{z}_1;\bar{\sigma}_1)] + \sum_{j\neq 1}\int \mathrm{d}R^N(\tau)(\pi^f(\bar{y}_1,\bar{y}_j)-\lambda_{1,j})[d(\bar{z}_1;\bar{\sigma}_1^{(j)}) - d(\bar{z}_1;\bar{\sigma}_1)]$$

is due to the terms of the generator where only one of the two coupled processes jump and

$$A_3(\tau) = \int \mathrm{d}R^N(\tau)\int \mathrm{d}u\,\bar{\mathcal{E}}_1^N(u)[d(\bar{z}_1;\bar{\sigma}_1^{(u)}) - d(\bar{z}_1;\bar{\sigma}_1)]$$

is due to the remainder term. Here $\bar{\mathcal{E}}_1^N(u)$ is $\mathcal{E}_1^N(u)$ evaluated along the moving frame of the free transport.

Here, we have used that $d(z_1,\sigma_1)$ depends only on the configurations of the first particle; hence, the only nonzero contribution in the sum over i is given for $i=1$.

Concerning $A_1(\tau)$, it follows from (6.3) and (6.4) that

$$|e_K(N)| \leqslant \frac{\mathrm{Lip}(K)}{N-1}$$

and that, for $N > 2\text{Lip(K)} + 1$,

$$\alpha_N \leqslant \frac{4e^{\frac{\text{Lip}(K)}{N-1}}}{N-1},$$

using the inequality $1/(1-x) \leqslant 4e^x$ for $x \in (0, 1/2)$. Therefore, from (6.15) we get

$$\lambda_{1,j} \leqslant \alpha_N \|K\|_\infty \leqslant \frac{4\sqrt{e}\,\text{Lip}(K)}{N-1}.$$

By the symmetry of R^N and denoting $C_K \coloneqq 8\sqrt{e}\,\text{Lip}(K)$,

$$A_1(\tau) \leqslant \frac{C_K}{2(N-1)} \sum_{j\neq 1} \int \mathrm{d}R^N(\tau)[d(z_j, \sigma_j) + d(z_1, \sigma_1)] \leqslant C_K D_N(\tau), \tag{6.20}$$

since $d(\bar{z}_1^{(j)}; \bar{\sigma}_1^{(j)}) \leqslant d(z_j, \sigma_j) + d(z_1; \sigma_1)$. Indeed the right-hand side is vanishing iff $z_1 = \sigma_1$ and $z_j = \sigma_j$ and, in this case, also the left-hand side is clearly vanishing.

We now give a bound on $A_2(\tau)$. Since $\lambda_{1,j}$ is the minimum between $\pi^N_{1,j}$ and $\pi^f_{i,j}$, we have

$$|A_2(\tau)| \leqslant \sum_{j\neq 1} \int \mathrm{d}R^N(\tau)|\pi^N_{1,j}(\bar{X}_N) - \pi^f_{1,j}(\bar{y}_1, \bar{y}_j)|. \tag{6.21}$$

From (6.2) and (6.16),

$$|\pi^N_{1,j}(\bar{X}_N) - \pi^f_{1,j}(\bar{y}_1, \bar{y}_j)| \leqslant \alpha_N \text{Lip}(K)|M[\bar{X}_N](\bar{x}_1, |\bar{x}_1 - \bar{x}_j|) - M[\rho](\bar{y}_1, |\bar{y}_1 - \bar{y}_j|)|.$$

From now on we use the shorthand notation $M[\bar{X}_N](\bar{B}^x_{1,j}) = M[\bar{X}_N](\bar{x}_1, |\bar{x}_1 - \bar{x}_j|)$ and $M[\rho](\bar{B}^y_{1,j}) = M[\rho](\bar{y}_1, |\bar{y}_1 - \bar{y}_j|)|$, where we have introduced the balls

$$\bar{B}^x_{1,j} = B_{|\bar{x}_1 - \bar{x}_j|}(\bar{x}_1) \quad \text{and} \quad \bar{B}^y_{1,j} = B_{|\bar{y}_1 - \bar{y}_j|}(\bar{y}_1).$$

By the triangular inequality

$$\begin{aligned}|M[\bar{X}_N](\bar{B}^x_{1,j}) - M[\rho](\bar{B}^y_{1,j})| &\leqslant |M[\bar{X}_N](\bar{B}^x_{1,j}) - M[\bar{X}_N](\bar{B}^y_{1,j})| \\ &+ |M[\bar{X}_N](\bar{B}^y_{1,j}) - M[\bar{Y}_N](\bar{B}^y_{1,j})| + |M[\bar{Y}_N](\bar{B}^y_{1,j}) - M[\rho](\bar{B}^y_{1,j})|.\end{aligned}$$

Hence we divide the estimate (6.21) respectively in three terms:

$$|A_2(\tau)| \leqslant T_1(\tau) + T_2(\tau) + T_3(\tau).$$

In $T_1(\tau)$ we are considering particles with spatial configuration given by X_N and we want to estimate the discrepancy of the configuration over two different balls $\bar{B}^x_{1,j}$ and $\bar{B}^y_{1,j}$. Since $\bar{B}^x_{1,j} = \bar{B}^y_{1,j}$ iff $z_1 = \sigma_1$ and $z_j = \sigma_j$, using that $M[\bar{X}_N] \in [0, 1]$, we have

$$|M[\bar{X}_N](\bar{B}^x_{1,j}) - M[\bar{X}_N](\bar{B}^y_{1,j})| \leqslant d(z_1, \sigma_1) + d(z_j, \sigma_j).$$

Therefore, by the symmetry of R^N,

$$\begin{aligned}T_1(\tau) &\leqslant \alpha_N \text{Lip}(K) \sum_{j\neq 1} \int \mathrm{d}R^N(\tau)[d(z_1, \sigma_1) + d(z_j, \sigma_j)] \\ &\leqslant C_K D_N(\tau).\end{aligned}$$

Regarding $T_2(\tau)$, we are considering the discrepancy of two different configurations over the same ball $\bar{B}^y_{1,j}$. Since

$$|M[\bar{X}_N](\bar{B}^y_{1,j}) - M[\bar{Y}_N](\bar{B}^y_{1,j})| \leqslant \frac{1}{N}\sum_{i=1}^{N} d(z_i,\sigma_i),$$

using again the symmetry of the law, we get

$$T_2(\tau) \leqslant \alpha_N \mathrm{Lip}(K)\sum_{j\neq 1}\int \mathrm{d}R^N(\tau) d(z_1,\sigma_1) \leqslant C_K D_N(\tau).$$

The last estimate on $T_3(\tau)$ is a consequence of the law of large numbers. After a change of variable, using the symmetry of the law R^N and the fact that this last term depends only on the Y_N configuration, we have that

$$T_3(\tau) = \alpha_N \mathrm{Lip}(K)\sum_{j\neq 1}\int \mathrm{d}\rho^{\otimes N}(\tau)|M[Y_N](B^y_{1,j}) - M[\rho](B^y_{1,j})|,$$

where $B^y_{1,j} = B_{|y_1-y_j|}(y_1)$. By Cauchy-Schwartz,

$$\begin{aligned}
&\Big|\int \mathrm{d}\rho^{\otimes N}(\tau)|M[Y_N](B^y_{1,j}) - M[\rho](B^y_{1,j})|\Big|^2 \\
&\leqslant \int \mathrm{d}\rho^{\otimes N}(\tau)\Big|\frac{1}{N-1}\sum_{h\neq 1}\Big[\mathcal{X}_{B^y_{1,j}}(y_h) - M[\rho](B^y_{1,j})\Big]\Big|^2 \\
&\leqslant \sum_{h_1,h_2\neq 1}\int \frac{\mathrm{d}\rho^{\otimes N}(\tau)}{(N-1)^2}\Big[\mathcal{X}_{B^y_{1,j}}(y_{h_1}) - M[\rho](B^y_{1,j})\Big]\Big[\mathcal{X}_{B^y_{1,j}}(y_{h_2}) - M[\rho](B^y_{1,j})\Big].
\end{aligned}$$

Thanks to the independence of the limit process, we get that the only nonzero contributions are given when $h_1 = h_2$ and this happens only for $N-1$ terms. Hence

$$T_3(\tau) \leqslant \frac{C_K}{\sqrt{N-1}}.$$

Collecting the estimates on T_1, T_2 and T_3, we obtain that

$$A_2(\tau) \leqslant C_K\Big(D_N(\tau) + \frac{1}{\sqrt{N-1}}\Big). \tag{6.22}$$

We conclude the proof estimating $A_3(\tau)$. Since this term depends only on the independent Y_N configuration

$$\begin{aligned}
|A_3(\tau)| \leqslant &\int \frac{\mathrm{d}f^{\otimes N}(\tau)}{N-1}\sum_{j\neq 1}\Big|\int K\Big(M[\rho](B_{|\bar{y}_1-y|}(\bar{y}_1))\Big)\rho(y)\,\mathrm{d}y - K(M[\rho](\bar{B}^y_{1,j}))\Big| \\
&+ \frac{1}{N-1}\int \mathrm{d}f^{\otimes N}(\tau)\frac{e_K(N)}{1-e_K(N)}\sum_{j\neq 1}K(M[\rho](\bar{B}^y_{1,j})),
\end{aligned}$$

where we added and subtracted the term $\sum_j K(M[\rho](\bar{B}^y_{1,j}))/(N-1)$.

Applying again the law of large numbers on the first term and estimating the second term thanks to

$$\frac{e_K(N)}{1 - e_K(N)} \leqslant \frac{C_K}{N-1},$$

we arrive at

$$|A_3(\tau)| \leqslant \frac{C_K}{\sqrt{N-1}}. \tag{6.23}$$

Collecting the estimates in (6.20), (6.22) and (6.23) and using Gronwall's lemma, we conclude the proof of the theorem.

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