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Parity of the 8-regular partition function

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Abstract
We give a complete characterisation of the parity of $b_8(n)$, the number of 8-regular partitions of n. Namely, we prove that $b_8(n)$ is odd precisely when $24n + 7$ has the form $p^{4a+1}m^2$ with p prime and $p \nmid m$.

Keywords Congruences · Regular partitions · Dedekind eta-function · Modular forms

Mathematics Subject Classification 11P83 · 11F20

1 Introduction

Partition functions are very natural and elementary objects, being defined as the number of ways we can decompose a non-negative integer as the sum of positive integers, possibly with some constraints. A partition of n is a nonincreasing sequence $(\lambda_1, \dots, \lambda_s)$ of positive integers, possibly empty, such that $\lambda_1 + \cdots + \lambda_s = n$. A partition is called ℓ-regular if no λ_j is divisible by ℓ, and we denote the number of ℓ-regular partitions

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of n by $b_\ell(n)$. The generating functions of many partition functions have connections to the theory of modular forms, and in particular to the Dedekind eta-function $\eta(z)$ (see Sect. 2). For example, if p is prime the generating function of $b_{p^j}(n)$ is congruent modulo p to a power of $\eta(z)$ (see Lemma 2.1).

In [10], Serre proved that if r is an even positive integer, then $\eta^r(z)$ is lacunary, i.e. the density of the nonzero Fourier coefficients is zero, if and only if $r \in \{2, 4, 6, 8, 10, 14, 26\}$. All but one of the cases where Lemma 2.1 furnishes a relation between the arithmetic of $b_{p^j}(n)$ modulo p and an even lacunary power of $\eta(z)$ have appeared in the literature (see, for example, [1, 4, 7, 8]). The main result of our paper addresses the remaining case.

Theorem 1.1 *Let n be a positive integer and let $b_8(n)$ denote the number of 8-regular partitions of n. Then $b_8(n)$ is odd if and only if*

$$24n + 7 = p^{4a+1}m^2 \tag{1}$$

for some prime p and integers $m > 0$, $a \geq 0$ such that $p \nmid m$.

Curiously, the factorisation (1) also appears in [2, Theorem 1.4 (1)], which characterises the behaviour modulo 4 of certain overpartition functions (we are not aware of a direct connection between these overpartitions and 8-regular partitions). The parity of $b_8(n)$ was previously known only for n restricted to certain arithmetic progressions (see [3, Theorem 3.19]).

The paper is organised as follows: in Sect. 2, we recall some well-known results about modular forms and describe the main tool in our argument (Lemma 2.1). In Sect. 3, we characterise the parity of $b_8(n)$.

2 Preliminaries

It is well known that

$$\prod_{n=1}^{\infty}(1-q^n)^{-1} = \sum_{n=0}^{\infty} p(n)q^n,$$

where $p(n)$ is the number of partitions of n. We begin by establishing a connection between ℓ-regular partition functions and powers of the Dedekind eta-function

$$\eta(z) = q^{\frac{1}{24}} \prod_{m=1}^{\infty}(1-q^m),$$

where $q = e^{2\pi i z}$ and $\mathrm{Im}(z) > 0$.

Lemma 2.1 *Let p be a prime and let r and s be positive integers such that $r = (p^j - 1)s$ for some $j \geq 1$. Then*

$$\left(\sum_{n=0}^{\infty} b_{p^j}(n) q^{n+\frac{p^j-1}{24}}\right)^s \equiv \eta(z)^r \pmod{p}.$$

Proof We begin by noting that

$$\prod_{m=1}^{\infty}\left(\frac{1-q^{p^j m}}{1-q^m}\right) = \sum_{n=0}^{\infty} b_{p^j}(n) q^n.$$

Moreover, we have the congruence

$$(1-q^m)^r = \frac{(1-q^m)^{r+s}}{(1-q^m)^s} \equiv \frac{(1-q^{p^j m})^s}{(1-q^m)^s} \pmod{p}.$$

Combining these yields

$$\eta^r(z) = q^{\frac{r}{24}} \prod_{m=1}^{\infty}(1-q^m)^r \equiv \left(\sum_{n=0}^{\infty} b_{p^j}(n) q^{n+\frac{r}{24s}}\right)^s \pmod{p}$$

and our result follows.

Note that when $(r, s) = (14, 2)$, Lemma 2.1 yields

$$\eta^{14}(z) \equiv \left(\sum_{n=0}^{\infty} b_8(n) q^{n+\frac{7}{24}}\right)^2 \equiv \sum_{n=0}^{\infty} b_8(n) q^{2n+\frac{7}{12}} \pmod{2}. \tag{2}$$

For $r \in \{2, 4, 6, 8, 10, 14, 26\}$, Serre [10] proved that η^r can be written as a linear combination of modular forms with complex multiplication (CM) associated to Hecke characters on $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$ (for the definition of modular forms with CM and their relations with Hecke characters, see [9, Sect. 3]). More precisely, η^r is a scalar multiple of a CM form when $r \in \{2, 4, 6, 8\}$, and a linear combinations of two or four forms when $r \in \{10, 14\}$ or $r = 26$, respectively.

We end this section by recalling properties of the Fourier coefficients of a Hecke eigenform. Let N and k be positive integers. We denote by $\mathcal{S}_k(N, \chi)$ the $\mathbb{C}$-vector space of cusp forms of weight k that are invariant under the action of $\Gamma_1(N)$ and on which $\Gamma_0(N)$ acts via the Dirichlet character χ modulo N (for more details see for instance [5, Sect. 4.3, p. 119]). Let $f \in \mathcal{S}_k(N, \chi)$ be a normalised eigenform with q-expansion

$$f(z) = \sum_{n=1}^{\infty} a(n) q^n.$$

The Fourier coefficients $a(n)$ are multiplicative, i.e.

$$a(nm) = a(n)a(m) \tag{3}$$

when $\gcd(n, m) = 1$, and we have the recursion

$$a(p^j) = a(p)a(p^{j-1}) - \chi(p)p^{k-1}a(p^{j-2})$$

for p prime and $j \geq 2$ (see for instance [5, Proposition 5.8.5]). It follows that for all $j \geq 2$, we have

$$a(p^j) = \sum_{r=0}^{\lfloor j/2 \rfloor} (-1)^r \binom{j-r}{r} \chi(p)^r p^{(k-1)r} a(p)^{j-2r}. \tag{4}$$

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In this section, we prove Theorem 1.1. Let χ be the Dirichlet character given by

$$\chi(n) = \begin{cases} (-1)^{(n-1)/2} & \text{if } \gcd(n, 6) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\varphi_{K,c_\pm} \in \mathcal{S}_7(144, \chi)$ be the normalised eigenforms associated to the Hecke characters $c_\pm$ on $K = \mathbb{Q}(\sqrt{-3})$ of conductor $\mathfrak{f} = 4\sqrt{-3}\mathcal{O}_K$ and defined as follows. Let $\mathfrak{a}$ be an ideal in $\mathcal{O}_K = \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ coprime to $\mathfrak{f}$ and let $\alpha = x + y\sqrt{-3}$ $(x, y \in \mathbb{Z})$ be the unique generator of $\mathfrak{a}$ with $x \equiv 1 \pmod 3 \equiv 1 \pmod 4$). Letting

$$c_\pm(\mathfrak{a}) = (-1)^{(x \mp y - 1)/2}\alpha^6,$$

we have

$$\varphi_{K,c_\pm}(z) = \sum_{\mathfrak{a}} c_\pm(\mathfrak{a})q^{\mathrm{Norm}(\mathfrak{a})} = \sum_{n=1}^{\infty} a_\pm(n)q^n, \tag{5}$$

where in the first sum $\mathfrak{a}$ runs over the nonzero ideals of $\mathcal{O}_K$ coprime to $\mathfrak{f}$ (the Galois orbit $\{\varphi_{K,c_+}, \varphi_{K,c_-}\}$ is listed with label 144.7.g.d on [6]). By [10, Sect. 2.6], we have

$$\eta^{14}(12z) = q^7 - 14q^{19} + 77q^{31} - \cdots = \frac{1}{720\sqrt{-3}}(\varphi_{K,c_+}(z) - \varphi_{K,c_-}(z)). \tag{6}$$

Next, by (2), we have

$$\eta^{14}(12z) \equiv \sum_{n=0}^{\infty} b_8(n)q^{24n+7} \pmod 2.$$

It follows that

$$b_8(n) \equiv \frac{a_+(24n+7) - a_-(24n+7)}{720\sqrt{-3}} \pmod 2. \tag{7}$$

Lemma 3.1 *Let p be a prime such that $p \equiv 5$ or $11 \pmod{12}$. Then*

$$\begin{cases} a_+(p^j) = a_-(p^j) \equiv 1 \pmod 2 & \text{if } j \text{ is even,} \\ a_+(p^j) = a_-(p^j) = 0 & \text{if } j \text{ is odd.} \end{cases}$$

Proof Since $p \equiv 2 \pmod 3$, p is inert in $\mathcal{O}_K$. This implies that $\mathcal{O}_K$ has no ideal of norm p^j if j is odd, while if j is even, there is a unique ideal $p^{j/2}\mathcal{O}_K$ of norm p^j. Our result follows.

Lemma 3.2 *Let p be a prime such that $p \equiv 1 \pmod{12}$. Then*

$$a_+(p^j) = a_-(p^j) \equiv \begin{cases} (-1)^{j/2} \pmod 4 & \text{if } j \text{ is even,} \\ 2 \pmod 4 & \text{if } j \equiv 1 \pmod 4, \\ 0 \pmod 4 & \text{if } j \equiv 3 \pmod 4. \end{cases}$$

Proof Write $p = z^2 + 3w^2$ with $z, w \in \mathbb{Z}$ and $z \equiv 1 \pmod 3$. Since $z^2 + 3w^2 \equiv 1 \pmod 4$, we have that z is odd and w is even. This implies that $z + w \equiv z - w \pmod 4$, and hence $(-1)^{(z+w-1)/2} = (-1)^{(z-w-1)/2}$. It follows from this and (4) that $a_+(p^j) = a_-(p^j)$ for all $j \geq 1$. Next, since $(z \pm w\sqrt{-3})$ are the ideals of $\mathcal{O}_K$ above p, by (5), we have

$$a_\pm(p) = (-1)^{(z+w-1)/2}(2z^6 - 90z^4w^2 + 270z^2w^4 - 54w^6) \equiv 2 \pmod 4.$$

Combining this with (4), we find

$$a_\pm(p^j) \equiv \sum_{r=0}^{\lfloor j/2 \rfloor} (-1)^r \binom{j-r}{r} 2^{j-2r} \pmod 4$$

from which our lemma follows.

Lemma 3.3 *Let p be a prime such that $p \equiv 7 \pmod{12}$. If j is even, then*

$$a_+(p^j) = a_-(p^j) \equiv \begin{cases} 9 \pmod{16} & \text{if } p \equiv 19 \pmod{24} \text{ and } j \equiv 2 \pmod 4, \\ 1 \pmod{16} & \text{otherwise.} \end{cases}$$

If j is odd, then

$$a_+(p^j) = -a_-(p^j) = t\sqrt{-3}$$

with

$$t \equiv \begin{cases} 8 \pmod{16} & \text{if } p \equiv 7 \pmod{24} \text{ and } j \equiv 1 \pmod 4, \\ 0 \pmod{16} & \text{otherwise.} \end{cases}$$

Proof Write $p = z^2 + 3w^2$ with z even, w odd and $z \equiv 1 \pmod 3$. Replacing w with $-w$ if necessary, we may ensure that $z + w \equiv 1 \pmod 4$ and hence $z - w \equiv 3 \pmod 4$. Then by (5), we have

$$a_+(p) = -(z + w\sqrt{-3})^6 + (z - w\sqrt{-3})^6 = -a_-(p)$$

which implies

$$\frac{a_+(p)}{\sqrt{-3}} = -12z^5w + 120z^3w^3 - 108zw^5 \equiv -108zw^5 \equiv 4z \pmod{16}.$$

If j is odd, by (4), we obtain $a_+(p^j) = -a_-(p^j)$ and

$$\frac{a_+(p^j)}{\sqrt{-3}} = \frac{1}{\sqrt{-3}} \sum_{r=0}^{\frac{j-1}{2}} \binom{j-r}{r} p^{6r} a_+(p)^{j-2r} \equiv 2z(j+1)p^{3(j-1)} \pmod{16}.$$

If j is even, by (4), we have $a_+(p^j) = a_-(p^j)$ and

$$a_+(p^j) = \sum_{r=0}^{j/2} \binom{j-r}{r} p^{6r} a_+(p)^{j-2r} \equiv p^{3j} \pmod{16}.$$

The result follows by noting that $z \equiv 0 \pmod 4$ if and only if $p \equiv 19 \pmod{24}$.

Proof of Theorem 1.1 Write $24n + 7 = \prod_p p^{\alpha_p}$. Then, by (3), we have

$$a_\pm(24n + 7) = \prod_p a_\pm(p^{\alpha_p}). \tag{8}$$

By Lemmas 3.1, 3.2, 3.3, we find that $a_+(p^j) = a_-(p^j)$ if $p \not\equiv 7 \pmod{12}$ or if $p \equiv 7 \pmod{12}$ and j is even, while $a_+(p^j) = -a_-(p^j)$ if $p \equiv 7 \pmod{12}$ and j is odd. Therefore, we can write

$$a_+(24n + 7) - a_-(24n + 7) = (1 - (-1)^\gamma) \prod_p a_+(p^{\alpha_p}), \tag{9}$$

where

$$\gamma = \sum_{\substack{p \equiv 7 (12) \\ \alpha_p \text{ odd}}} 1.$$

If γ is even, it follows from (7) that $b_8(n)$ is even. Assume from now onwards that γ is odd. By Lemma 3.3, for each prime $p \equiv 7 \pmod{12}$ with α_p odd, we have $a_+(p^{\alpha_p}) = t\sqrt{-3}$ with $8 \mid t$. It follows that if $\gamma \geq 3$, then

$$\frac{a_+(24n+7) - a_-(24n+7)}{\sqrt{-3}} \equiv 0 \pmod{1024} \tag{10}$$

and $b_8(n)$ is even by (7).

Finally, assume $\gamma = 1$ and denote by p' the prime divisor of $24n+7$ with $p' \equiv 7 \pmod{12}$ and $\alpha_{p'}$ odd. Then $1 - (-1)^\gamma = 2$ and by Lemma 3.3, we deduce that $\frac{a_+(24n+7)-a_-(24n+7)}{\sqrt{-3}} \equiv 0 \pmod{32}$ if $p' \equiv 19 \pmod{24}$ or if $p' \equiv 7 \pmod{24}$ and $\alpha_{p'} \equiv 3 \pmod{4}$. If $p' \equiv 7 \pmod{24}$ and $\alpha_{p'} \equiv 1 \pmod{4}$, then by Lemma 3.3, we have

$$\frac{a_+(24n+7) - a_-(24n+7)}{\sqrt{-3}} \equiv 16 \prod_{p \neq p'} a_+(p^{\alpha_p}) \pmod{32}. \tag{11}$$

We observe that, by Lemmas 3.1, 3.2, 3.3, the product in (11) is even if and only if there is a prime $p \neq p'$ with α_p odd. This concludes the proof of the theorem by (7).

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