

ON THE JONES POLYNOMIAL MODULO PRIMES

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ABSTRACT. We derive an upper bound on the density of Jones polynomials of knots modulo a prime number p , within a sufficiently large degree range: $4/p^7$. As an application, we classify knot Jones polynomials modulo two of span up to eight.

1. INTRODUCTION

Describing the set of Jones polynomials of all knots is a difficult problem. In this note, we take a tiny step towards classifying Jones polynomials of knots with coefficients reduced modulo a prime number p .

Theorem 1. *For all $a, b \in \mathbb{Z}$ with $b - a \geq 7$, the set of Laurent polynomials with coefficients in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ within the degree range from a to b , that are realised as Jones polynomials of knots, has density at most $4/p^7$.*

As we will see, the bound $4/p^7$ is sharp in the special case $p = 2$, any $a \in \mathbb{Z}$, and $b = a + 8$.

Corollary 1. *For all $a \in \mathbb{Z}$, there are exactly 16 Jones polynomials of knots modulo two with minimal degree a and maximal degree $\leq a + 8$. All these Laurent polynomials are realised by finite connected sums of 54 prime knots with crossing number 12 or less.*

As Jones observed in his famous publication [4], for any knot K , the difference between the Jones polynomial $V_K(t)$ and 1 is divisible by $(t^3 - 1)(t - 1)$. The proof of Theorem 1 rests on the following refined statement, which does not seem to appear in the literature so far.

Theorem 2. *Let $h(t) = (t^3 - 1)(t - 1)(t^2 + 1)$ and*

$$f(t) = (t^2 - t + 1)h(t) = t^8 - 2t^7 + 3t^6 - 4t^5 + 4t^4 - 4t^3 + 3t^2 - 2t + 1.$$

For all knots K , there exists a unique polynomial $p(t) \in \mathbb{Z}[t]$ of degree at most seven, belonging to one of the four families below, so that $V_K(t) - p(t)$ is divisible by $f(t)$:

- (i) $1 + nh(t)$,
- (ii) $V_{3_1}(t) + nh(t)(2t - 1)$,
- (iii) $V_{5_1}(t) + nh(t)$,
- (iv) $V_{8_{21}}(t) + nh(t)(2t - 1)$.

All these families are parametrised by an integer n satisfying $2n = \pm 1 \pm 3^l$. The symbols $3_1, 5_1, 8_{21}$ refer to knots according to Rolfsen's notation [7].

The membership of a given knot K to one of these families, as well as the value $n \in \mathbb{Z}$, is determined by the pair of values $V_K(i)$, $V_K(\zeta_6)$. The explicit Jones polynomials appearing in Theorem 2 are

$$\begin{aligned} V_{3_1}(t) &= -t^4 + t^3 + t, \\ V_{5_1}(t) &= -t^7 + t^6 - t^5 + t^4 + t^2, \\ V_{8_{21}}(t) &= t^7 - 2t^6 + 2t^5 - 3t^4 + 3t^3 - 2t^2 + 2t. \end{aligned}$$

At this point, the reader might already guess that the first theorem is an easy consequence of the second. We will derive Theorems 1 and 2 in Sections 3 and 2, respectively. The corollary relies on the following curious fact: there exists a knot - 12n237 in knotinfo notation [5] - whose Jones polynomial is t^{12} , modulo two. This is explained in the fourth and last section.

2. LISTING POTENTIAL JONES POLYNOMIALS

The Jones polynomial $V_K(t) \in \mathbb{Z}[t^{\pm 1}]$ of a knot $K \subset S^3$ satisfies the following restrictions in the roots of unity $1, i, \zeta_3, \zeta_6$:

- (1) $V_K(1) = 1$,
- (2) $V'_K(1) = 0$,
- (3) $V_K(\zeta_3) = 1$,
- (4) $V_K(i) = \pm 1$,
- (5) $V_K(\zeta_6) = \pm(\sqrt{-3})^m$.

The exponent m in condition (5) coincides with the rank of the first homology of the double branched cover $M_2(K)$ with coefficients in $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$, and can also be interpreted as the dimension of the 3-coloring invariant of K , as described by Przytycki [6]. The sign in condition (4) is determined by the Arf invariant of K : $V_K(i) = (-1)^{\text{Arf}(K)}$. The first four conditions were already derived by Jones [4]. In terms of Vassiliev invariants, the first two conditions reflect the fact that knots admit no non-constant finite type invariants of order zero and one [1]. Interestingly, this implies that no monomial other than 1 is the Jones polynomial of a knot [3]. Here is a remarkable consequence of the first three conditions together: $V_K(t) - 1$ is divisible by $(t - 1)^2(t^2 + t + 1) = (t^3 - 1)(t - 1)$.

Even better, suppose $p(t) \in \mathbb{Z}[t^{\pm 1}]$ admits the same values as $V_K(t)$, for $t = 1, i, \zeta_3, \zeta_6$, and satisfies $p'(1) = 0$. Then the difference $V_K(t) - p(t)$ is divisible by the product of cyclotomic polynomials

$$\begin{aligned} f(t) &= (t - 1)^2(t^2 + t + 1)(t^2 + 1)(t^2 - t + 1) \\ &= t^8 - 2t^7 + 3t^6 - 4t^5 + 4t^4 - 4t^3 + 3t^2 - 2t + 1. \end{aligned}$$

Therefore, all we need in order to derive Theorem 2 is finding a suitable set of reference polynomials $p(t)$, with $p'(1) = 0$, covering all the possible values of knot Jones polynomials at $t = 1, i, \zeta_3, \zeta_6$. This is easy enough.

First, we observe that all the four families of polynomials listed in Theorem 2 satisfy $p(1) = 1$, $p'(1) = 0$, and $p(\zeta_3) = 1$. Here we use the fact that

$h(t) = (t^3 - 1)(t - 1)(t^2 + 1)$ has a double root at $t = 1$, and a single root at $t = \zeta_3$.

Next, we observe that all the polynomials of families (i) and (iv) listed in Theorem 2 satisfy $p(i) = 1$, and all the polynomials of families (ii) and (iii) satisfy $p(i) = -1$. Here we use that $h(t)$ also has a single root at $t = i$.

Last, we take care of the value $p(\zeta_6)$, which should cover all the complex numbers of the form $\pm(\sqrt{-3})^m$. The values of

$$p(t) = 1, V_{3_1}(t), V_{5_1}(t), V_{8_{21}}(t)$$

at $t = \zeta_6$ are $1, \sqrt{3}i, -1, \sqrt{3}i$, respectively. Furthermore, we have $h(\zeta_6) = 2$ and $h(\zeta_6)(2\zeta_6 - 1) = 2\sqrt{3}i$. This implies that the polynomials of families (i) and (iii) cover all the odd integers at $t = \zeta_6$, while the polynomials of families (ii) and (iv) cover all the odd multiples of $\sqrt{3}i$ at $t = \zeta_6$. Altogether, the four families listed in Theorem 2 cover all the possible combinations of values of knot Jones polynomial at $t = 1, i, \zeta_3, \zeta_6$, including the double root at $t = 1$. This finishes the proof of Theorem 2.

3. JONES POLYNOMIAL MODULO PRIMES

The goal of this section is to derive Theorem 1 by reducing Theorem 2 modulo a fixed prime number p . We use the notation $\bar{f}(t) \in \mathbb{F}_p[t^{\pm 1}]$ for the reduction of $f(t) \in \mathbb{Z}[t^{\pm 1}]$ modulo p . Theorem 2 remains valid modulo p , with the additional feature that the parameter n is in \mathbb{F}_p . From this, we deduce that the number of Jones polynomials of knots modulo p in the degree range $[0, 7]$ is at most $4p$. This is in accordance with the ratio $4/p^7$, since there are exactly p^8 polynomials modulo p in the degree range $[0, 7]$. We will refer to these $4p$ potential Jones polynomials as reference polynomials $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_{4p} \in \mathbb{F}_p[t^{\pm 1}]$.

Now suppose we are given a degree range $[a, b]$ with $b - a \geq 7$ and a knot K with Jones polynomial $\bar{V}_K(t)$ in that degree range. By Theorem 2, there exists a reference polynomial \bar{f}_i , so that $\bar{V}_K(t) - \bar{f}_i$ is divisible by

$$\bar{f}(t) = t^8 - 2t^7 + 3t^6 - 4t^5 + 4t^4 - 4t^3 + 3t^2 - 2t + 1 \in \mathbb{F}_p[t^{\pm 1}].$$

Denote the minimal and maximal degree of $\bar{V}_K(t) - \bar{f}_i$ by α and β , respectively. Then there exist unique coefficients

$$c_\alpha, c_{\alpha+1}, \dots, c_{\beta-8} \in \mathbb{F}_p,$$

satisfying the following equation:

$$\bar{V}_K(t) - \bar{f}_i = \bar{f}(t)(c_\alpha t^\alpha + c_{\alpha+1} t^{\alpha+1} + \dots + c_{\beta-8} t^{\beta-8}).$$

The polynomial $\bar{V}_K(t)$ is therefore determined by $\beta - \alpha - 7$ parameters in \mathbb{F}_p . However, since $\bar{V}_K(t)$ is in the degree range $[a, b]$, all the coefficients c_γ with $\gamma \notin [a, b - 8]$ are determined by \bar{f}_i alone. In other words, only the coefficients $c_\alpha, c_{\alpha+1}, \dots, c_{\beta-8}$ change if we vary $\bar{V}_K(t)$ in the given degree range. Since there are $4p$ reference polynomials \bar{f}_i , this allows for a maximum of $4p$ times $p^{\beta - \alpha - 7}$ potential Jones polynomials, out of a total of $p^{\beta - \alpha + 1}$ polynomials

with coefficients in \mathbb{F}_p in the degree range $[a, b]$. The resulting ratio is again $4/p^7$, as claimed.

For odd primes $p \geq 5$, the bound $4/p^7$ is never sharp, since the parameter n appearing in Theorem 2, case (i), satisfies $1 + 2n = \pm 3^l$. In particular, $2n$ cannot be $-1 \pmod{p}$, since 3^l cannot be zero \pmod{p} . The knot table at our disposition (knotinfo, up to 12 crossings [5]), is too small to draw any conclusion about the sharpness of the bound $4/p^7$ for $p = 3$. This leaves us with the case $p = 2$, which is most interesting and deserves its own section.

4. JONES POLYNOMIAL MODULO TWO

The list of $4p$ potential Jones polynomials in the degree range $[0, 7]$, called reference polynomials in the previous section, boils down to eight polynomials for $p = 2$. These are in fact realised by the following knots: the trivial knot O , 3_1 , 5_1 , 5_2 , 8_{21} , 9_{43} , 10_{140} , 10_{160} . The corresponding Jones polynomials $\pmod{2}$ are

$$1, t + t^3 + t^4, t^2 + t^4 + t^5 + t^6 + t^7, t + t^2 + t^4 + t^5 + t^6, \\ t^3 + t^4 + t^7, 1 + t + t^7, 1 + t + t^2 + t^3 + t^5 + t^6 + t^7, 1 + t^2 + t^3 + t^5 + t^6.$$

In order to prove Corollary 1, we need to find 16 knot Jones polynomials in the degree range $[a, a + 8]$, for all $a \in \mathbb{Z}$, which appears rather difficult. Luckily, a single knot comes at our rescue: $12n237$.

As mentioned above, no monomial other than 1 is the Jones polynomial of a knot. Indeed, no polynomial of the form $p(t) = at^n$, except 1, satisfies $p(1) = 1$ and $p'(1) = 0$. In contrast, the Jones polynomial of the knot $12n237$ is a non-trivial monomial modulo 2:

$$\bar{V}_{12n237}(t) = t^{12} \pmod{2}.$$

Remark. The connected sum of the knot $12n237$ with its mirror image has trivial Jones polynomial modulo 2. The existence of non-trivial knots with that property, even prime ones, was known before [2]. Likewise, for odd primes p , the monomial t^{12p} is a potential Jones polynomial modulo p , since $t^{12p} - 1$ is divisible by $f(t) = (t^2 - t + 1)(t^3 - 1)(t - 1)(t^2 + 1)$ in $\mathbb{F}_p[t^{\pm 1}]$. We do not know whether t^{12p} (modulo p) is the Jones polynomial of an actual knot.

Back to $p = 2$, suppose we find 16 Jones polynomials in a fixed degree range $[a, a + 8]$, realised by the knots K_1, K_2, \dots, K_{16} . Then, by adding k copies of the knot $12n237$ to the knots K_i , we obtain 16 Jones polynomials in the degree range $[a + 12k, a + 12k + 8]$. This also works for negative integers k , by adding $|k|$ copies of the mirror image of the knot $12n237$ to the K_i . Hence, in order to cover all degree ranges, it is sufficient to consider the cases $-9 \leq a \leq 2$. In fact, it is even enough to consider the cases $-4 \leq a \leq 2$, by the symmetry $V_K(t) = V_{K^*}(t^{-1})$ between the Jones polynomial of a knot K and its mirror image K^* . Based on Rolfsen's table [7] and knotinfo [5], we found 53 prime knots, plus the trivial knot O , which

degree range	knots
$[-4, 4]$	$O, 3_1, 3_1^*, 4_1, 6_1, 6_1^*, 6_3, 7_7, 7_7^*, 4_1\#4_1, 8_3, 8_{12}, 8_{17}, 9_{42}, 10_{136}, 10_{136}^*$
$[-3, 5]$	$O, 3_1, 4_1, 6_1, 6_2, 6_3, 7_7, 8_4, 8_8, 8_{20}, 9_{42}, 9_{44}, 10_{136}, 10_{146}, 10_{147}, 10_{163}$
$[-2, 6]$	$O, 3_1, 4_1, 5_2, 6_1, 6_2, 3_1\#4_1, 7_6, 3_1^*\#5_1, 8_1, 8_7, 8_{10}, 8_{20}, 9_{44}, 10_{160}, 10_{163}$
$[-1, 7]$	$O, 3_1, 5_1, 5_2, 6_2, 3_1\#4_1, 7_6, 8_6, 8_{11}, 8_{14}, 8_{20}, 8_{21}, 9_{43}, 10_{140}, 10_{160}, 11n173$
$[0, 8]$	$O, 3_1, 5_1, 5_2, 3_1\#3_1, 7_2, 7_4, 8_2, 8_5, 8_{19}, 8_{21}, 9_{43}, 10_{126}, 10_{140}, 10_{143}, 10_{160}$
$[1, 9]$	$3_1, 5_1, 5_2, 3_1\#3_1, 7_2, 7_3, 7_4, 7_5, 8_{19}, 8_{21}, 10_{133}, 10_{165}, 11n77, 11n99, 11n118, 4_1\#8_{21}$
$[2, 10]$	$5_1, 3_1\#3_1, 7_1, 7_3, 7_5, 3_1\#5_2, 8_{15}, 8_{19}, 8_{21}, 10_{124}, 10_{127}, 10_{128}, 10_{145}, 10_{165}, 11n63, 11n118$

TABLE 1. Jones polynomials of span ≤ 8

provide 16 Jones polynomials in all degree ranges of the form $[a, a + 8]$, $a \in \{-4, -3, -2, -1, 0, 1, 2\}$. These knots include all knots with crossing number ≤ 8 , except the knots $8_9, 8_{13}, 8_{16}, 8_{18}$ (whose Jones polynomials modulo 2 coincide with the ones of $4_1\#4_1, 8_4, 8_{10}, 8_{12}$, in this order), as well as the following knots:

$$9_{42}, 9_{43}, 9_{44}, 10_{124}, 10_{126}, 10_{127}, 10_{128}, 10_{133}, 10_{136}, 10_{140}, 10_{143}, 10_{145}, \\ 10_{146}, 10_{147}, 10_{160}, 10_{163}, 10_{165}, 11n63, 11n71, 11n99, 11n118, 11n173.$$

The table below indicates the degree range of their corresponding Jones polynomials modulo 2. Our convention here is chosen so that K has higher maximal degree than K^* . By taking suitable connected sums of these knots, together with the knot $12n237$ (making it a total of 54 prime knots), and all their mirror images, we find 16 Jones polynomials in every degree range of the form $[a, a + 8]$, as stated in Corollary 1. We do not know to what extent the latter can be generalised. For example, we found 64 knot Jones polynomials modulo two in the degree ranges $[-5, 5]$ and $[0, 10]$, all realised by knots with 12 or fewer crossings.

We invite the reader to answer the following concluding questions.

Question 1. *Let p be an odd prime. Is there a knot $K \subset S^3$ with*

$$\bar{V}_K(t) = t^{12p} \pmod{p}?$$

Question 2. *Does every degree range $[a, b]$ with $b - a \geq 7$ contain 2^{b-a-4} Jones polynomials modulo 2, as predicted by Theorem 1?*

Question 3. *Is every Laurent polynomial $p(t) \in \mathbb{Z}[t^{\pm 1}]$ satisfying conditions (1)-(5) the Jones polynomial of a knot?*

REFERENCES

- [1] J. S. Birman, X.-S. Lin: *Knot polynomials and Vassiliev's invariants*, Invent. Math. 111 (1993), no. 2, 225–270.
- [2] S. Eliahou, J. Fromentin: *A remarkable 20-crossing tangle*, J. Knot Theory Ramifications 26 (2017), no. 14.
- [3] S. Ganzell: *Local moves and restrictions on the Jones polynomial*, J. Knot Theory Ramifications 23 (2014), no. 2.
- [4] V. F. R. Jones: *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. (N.S.) 12 (1985), no. 1, 103–111.
- [5] C. Livingston, A. H. Moore: *KnotInfo: Table of Knot Invariants*, knot-info.math.indiana.edu, April 13, 2022.
- [6] J. H. Przytycki: *3-coloring and other elementary invariants of knots*, Knot theory (Warsaw, 1995), 275–295, Banach Center Publ. 42, Polish Acad. Sci. Inst. Math., Warsaw, 1998.
- [7] D. Rolfsen: *Knots and links*, Mathematics Lecture Series, No. 7, Publish or Perish, Inc., Berkeley, Calif., 1976.

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