Expected Shortfall for the Makespan in Activity Networks With Fuzzy Durations

Gabriella Dellino, Carlo Meloni, Marco Pranzo, and Marcella Samà

***Abstract*—This article deals with the evaluation of the expected shortfall or the conditional value-at-risk for the makespan in scheduling problems represented as temporal activity networks where we assume that only a type-1 fuzzy representation for the activity-integer-valued durations is known to the scheduler. More precisely, we address the evaluation of the expected shortfall associated to a feasible schedule, and we extend the approach recently proposed for the case of interval-valued durations. We develop and analyze a suitable computational method to obtain the fuzzy evaluation of the expected shortfall of the makespan of a given schedule. The proposed method enables to use the expected shortfall as quality criterion for wide classes of scheduling approaches considering risk aversion in different practical contexts when only a fuzzy representation of activity durations is known.**

***Index Terms*—Activity networks, Conditional Value-at-Risk (CVaR), expected shortfall (ES), fuzzy intervals, makespan, project scheduling.**

I. Introduction

The makespan is the most popular performance measure in activity scheduling problems. Deterministic versions of scheduling problems have been studied extensively, but their practical applicability is often hindered by their disregard of uncertainties. In such problems, the activity duration times are usually assumed deterministic and known in advance. Even if this assumption may be valid when the effects of uncertainties are limited [1], it is not valid or acceptable for all real contexts [2].

The uncertainty on the activity durations can be represented as random variables, fuzzy numbers, or taking any value from given intervals. The latter as well as the deterministic case can be considered as a special case of a fuzzy model. Fuzzy scheduling is suitable when available data or experts opinions are insufficient to construct a more accurate viable probabilistic model.

As usual in scheduling practice, we consider—without loss of generality—the case in which the processing times (i.e., activity durations) are integers counting for a certain number of elementary units of time, and their uncertain values are represented by fuzzy numbers or intervals. It is worth noting that by appropriately choosing the time unit, this assumption does not exclude the possibility of also representing problems in which the durations are not originally indicated by integers. We assume that the realized duration of an activity is only known *a posteriori*, i.e., when the execution of the activity is completed. Since the value of an objective function depends on problem parameters, fuzziness on these parameters implies that the objective value has also a fuzzy representation.

To compare performances of different schedules, the expected shortfall (ES) of the objective (e.g., the makespan in a scheduling problem) has been considered as a suitable criterion for stochastic optimization (e.g., see [3], and [4] and [5] for scheduling problems). This criterion, introduced in the finance context where gained popularity, is advantageous due to its tendency of simultaneously minimizing both the variance and the expectation of a given performance measure while maintaining linearity whenever the expectation objective function can be represented by a linear expression. Usually, some scenario-based (or sample-based) method is adopted to estimate ES measures for scheduling problems, and this approach can require a relevant computational effort and expose to estimation accuracy issues [4], [5].

In this article, we address the evaluation of the ES of the makespan when the scheduling problem is represented by a temporal network [6]. At this aim, we consider scheduling models with a fixed and given set of precedence relations, but (independent) fuzzy processing times. More specifically, this research work assumes that nondeterministic durations are considered as classic or type-1 fuzzy sets [7], [8]. The objective is to evaluate the ES of the makespan, which is, in general, an NP-complete problem [9]. We extend the approach recently proposed in [4], by developing and analyzing a suitable computational method to obtain the fuzzy evaluation of the ES of the makespan of a given schedule. The proposed algorithms are pseudopolynomial in the general case, but they have been tested on a wide set of realistic instances for the experimental validation of their efficiency [4].

The evaluation of the ES of the makespan $\mathbf{C}_{\max}$ is an issue arising in several practical applications such as project or task bidding, risk evaluation and mitigation, due date setting or acceptance, order proposal and acceptance, and robust scheduling [3], [4], [10]–[12]. The goal of this research is twofold: 1) to propose a method for computing the ES of the makespan in the case of fuzzy durations; 2) to stimulate future research works devoted to the development and test of suitable optimization schemes and/or other useful performance measures.

II. ES of the Makespan

In many applications, such as project management and production scheduling, the decision maker may be risk-averse and may prefer solutions that do not just perform well "on average", but that also perform satisfactorily "in most cases" [10], [13]. In the literature, there is no universally accepted single risk measure. In fact, each risk measure has its own advantages and disadvantages [3]. Moreover, the choice of a risk measure also reflects a subjective preference of the decision maker in many real-world problems. Several scalar performance indicators have been used to characterize the makespan $\mathbf{C}_{\max}$ of a stochastic activity network. Notable examples include the following related measures:

1) *γ-quantile of makespan* $q_\gamma(\mathbf{C}_{\max}) = \inf\{t : \text{prob}(\mathbf{C}_{\max} \leq t) \geq \gamma\}$. Quantile measures, commonly used in finance and risk management, are gaining momentum also in scheduling [5], [11]. The γ-quantile of makespan represents the so-called Value-at-Risk

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Gabriella Dellino and Carlo Meloni are with the Polytechnic University of Bari, 70126 Bari, Italy (e-mail: gabriella.dellino@poliba.it; carlo.meloni@poliba.it).

Marco Pranzo is with the University of Siena, 53100 Siena, Italy (e-mail: marco.pranzo@unisi.it).

Marcella Samà is with the Roma Tre University, 00146 Roma, Italy (e-mail: marcella.sama@uniroma3.it).

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(VaR) of the makespan at level γ: $\mathrm{VaR}_\gamma(\mathbf{C}_{\max})$ of the distribution of the random variable $\mathbf{C}_{\max}$. In other words, $\mathrm{VaR}_\gamma(\mathbf{C}_{\max})$ represents a threshold that is exceeded in $(1-\gamma)100\%$ of all cases.

2) *ES at level* γ: $\mathrm{ES}_\gamma(\mathbf{C}_{\max}) = E(\mathbf{C}_{\max}|\mathbf{C}_{\max} \geq q_\gamma(\mathbf{C}_{\max}))$. The $\mathrm{ES}_\gamma(\mathbf{C}_{\max})$, also called γ-tail expectation or Conditional Value-at-Risk (CVaR) (at level γ) of the makespan $\mathrm{CVaR}_\gamma(\mathbf{C}_{\max})$, has been shown to be equivalent and related to $\mathrm{VaR}_\gamma(\mathbf{C}_{\max})$[3], [12]: $\mathrm{ES}_\gamma(\mathbf{C}_{\max}) = \mathrm{CVaR}_\gamma(\mathbf{C}_{\max}) = \frac{1}{1-\gamma}\int_\gamma^1 \mathrm{VaR}_\beta(\mathbf{C}_{\max})\,d\beta$. It represents the expected value of all cases exceeding the threshold $\mathrm{VaR}_\gamma(\mathbf{C}_{\max})$.

Considering the definitions, the following holds for all $\gamma \in [0,1]$: $\mathrm{VaR}_\gamma(\mathbf{C}_{\max}) \leq \mathrm{ES}_\gamma(\mathbf{C}_{\max})$.

If a decision maker is not only concerned with the frequency of undesirable outcomes, but also with their severity, ES_γ is recommended instead of VaR_γ [3], [5]. In this case, higher values of γ are chosen by decision makers who are more risk averse, and $\gamma = 0$ represents the risk-neutral choice. In fact, as γ tends to 1 (i.e., its upper extremum), the $\mathrm{ES}_\gamma(\mathbf{C}_{\max})$ tends to the worst case W; whereas when γ tends to 0, $\mathrm{ES}_\gamma(\mathbf{C}_{\max})$ tends to the expected value of the makespan $E(\mathbf{C}_{\max})$.

The features offered by ES or CVaR are also useful in planning and scheduling problems based on stochastic activity networks models [4], [5]. Compared to other sectors, where ES is more used, its restricted diffusion in scheduling is perhaps due to the computational difficulties connected to the adopted approaches that are often based on scenarios and sampling.

III. Temporal Activity Networks With Fuzzy-Valued Durations

A *fuzzy temporal activity network* (FTAN) is a temporal activity network with fuzzy-valued durations. It can be defined by adopting and extending the notation used in [4]. An FTAN is described by the pair $(G, \mathbf{D})$, where $G = (N, A)$ is the directed acyclic graph representing the precedences, the set of nodes N is associated to events, the set A of arcs represents the activities, and $\mathbf{D} = (\mathbf{D}_1, \ldots, \mathbf{D}_m)$ is the vector of m type-1 fuzzy durations associated to the m arcs in A representing the activities. The network is directed, connected, and acyclic with single source and sink nodes.

In an FTAN, all quantities that depend on the activity durations have a fuzzy characterization. Therefore, the starting and completion time of any activity, and the makespan $\mathbf{C}_{\max}$ are all fuzzy quantities [14]–[16]. A fuzzy quantity is a fuzzy set of the real line $\mathbb{R}$. A fuzzy set M of the universe of values X is characterized by a membership function μ_M, which takes its value in interval [0,1]. For each element $x \in X$, $\mu_M(x)$ defines the degree to which x belongs to $M = \{x \in X; \mu_M(x) \in [0,1]\}$.

An α-level cut (or α-cut, for short) of M is the crisp set $M_\alpha = \{x \in X | \mu_M(x) \geq \alpha\}$. The support of M is the crisp set $\mathrm{supp}(M) = \{x \in X | \mu_M(x) > 0\}$.

The duration of all the activities is a fuzzy number that is defined as a bounded support fuzzy quantity whose α-cuts are closed intervals. More specifically, we consider integer fuzzy numbers that are characterized as follows.

1) The support is a closed integer interval denoted as $[\underline{M}, \overline{M}]$.
2) M is normal, i.e., there exists $\hat{x} \in [\underline{M}, \overline{M}] | \mu_M(\hat{x}) = 1$.
3) For any $x_1, x_2 \in [\underline{M}, \hat{x}]$, $\mu_M(x_1) \leq \mu_M(x_2)$ holds.
4) For any $x_1, x_2 \in [\hat{x}, \overline{M}]$, $\mu_M(x_1) \geq \mu_M(x_2)$ holds.

Given the fuzzy durations $\mathbf{D} = (\mathbf{D}_1, \ldots, \mathbf{D}_m)$, the extension principle of Zadeh [17] provides a powerful technique to extend a real continuous function of the activity durations [such as $\mathrm{ES}_\gamma(\mathbf{C}_{\max})$] to a fuzzy function $F(\mathbf{D})$ of the fuzzy durations $\mathbf{D}$. Moreover, the decomposition by α-cuts can be used to compute that fuzzy function by means of a decomposition method, due to a result of Nguyen [18]: $[F(\mathbf{D})]_\alpha = F([\mathbf{D}_1]_\alpha, \ldots, [\mathbf{D}_m]_\alpha)$. According to this method, the membership function $\mu_F(x)$ of $F(\mathbf{D})$ can be reconstructed from its α-cuts F_α as follows: $\mu_F(x) = \max\{\alpha : x \in F_\alpha\}$. Although this decomposition method is more general, we adopt a piecewise linear model for the membership function of the activity durations to simplify either information collection (i.e., data gathering or expert elicitation) and computational aspects, as often considered in the literature (e.g., see [7] and [19]). We use a representation based on the full breakpoints ordered sequence (an example is illustrated in Fig. 1, where also examples of α-cuts are reported), which is very general and can be easily adapted to the cases of popular models, such as triangular, trapezoid, and six-point approximated functions [20].

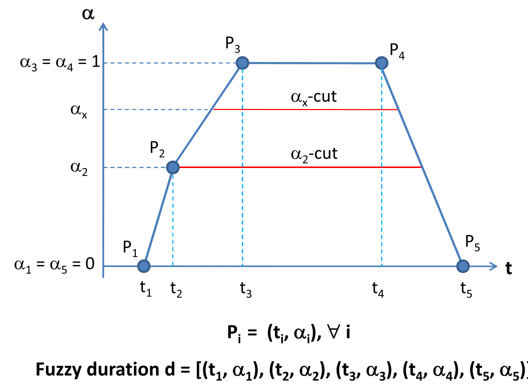


Fig. 1. Fuzzy duration representation.

IV. Evaluation of the ES of the Makespan in FTANs

In this section, a methodology devoted to compute an evaluation of the ES of the makespan in FTANs is presented. First, Section IV-A summarizes a recent method proposed for activity networks with interval valued durations, which can be considered as special cases of FTANs. Section IV-B describes the proposed approach for the extension to the case of fuzzy-valued durations.

A. Method for the Case of Crisp Interval Durations

The case of crisp (i.e., ordinary) interval durations can be considered as a special case of FTAN. For this special case, in [4], different algorithms have been proposed and experimentally validated. All these algorithms are based on a counting approach to evaluate the ES_γ for the makespan. The counting approach works backward, starting from the pessimistic makespan (i.e., the worst possible makespan W), it counts backward how many configurations lead to each possible makespan value. This counting process is implemented as an iterative process (schematically reported in Fig. 2), which continues until sufficient information has been gathered to compute the ES at a desired probability level γ. The algorithm scheme contains a preliminary phase in which the input instance is analyzed to determine the total number of possible configurations, the target number of configurations associated with the level of probability γ, and the worst case makespan value W to start the evaluation. The computation at each following iteration refers to a specific possible makespan value indicated as level L (initially $L = W$). The counting step at a level L is done by considering the possible series and parallel reduction on a specific critical subgraph [4]. If all the considered critical subgraphs completely reduce by series and parallel transformations, then the proposed approach computes the exact ES_γ, otherwise a tight approximation is obtained on the basis of

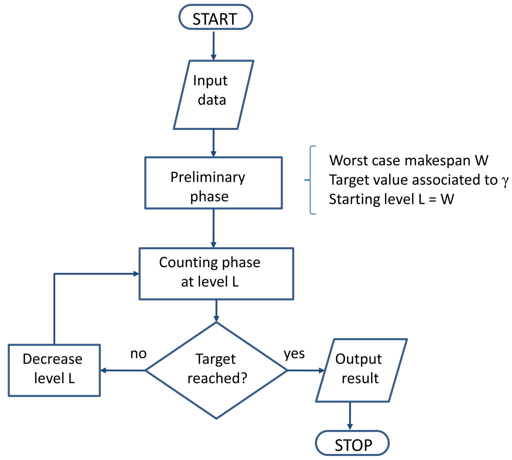


Fig. 2. Scheme of the basic algorithms for the case of crisp interval durations.

lower and upper bounds (indicated as LB_γ and UB_γ) for the ES_γ. On the basis of the computational results reported in [4], in this work, we adopt the algorithmic variant named ES_MinS, which is able to determine a fast and extremely good estimation of ES of the makespan. Regarding the computational complexity, ES_MinS is $O(\Gamma^2\, m^2)$, where m is the number of arcs of the network, and Γ is the amount of uncertainty of the activity network represented by the size (in terms of number of integers) of the time interval $[\underline{C}_{\max}, \overline{C}_{\max}]$, where $\underline{C}_{\max}$ ($\overline{C}_{\max}$) is the makespan when all the activities durations are at their minimum (maximum) value.

B. Decomposition Approach for the Case of Fuzzy-Valued Durations

According to the α-cuts decomposition method described in Section III, we follow the approximated approach proposed in [15] and [21] to calculate the makespan of a series–parallel fuzzy PERT network, and extend it designing a method for the evaluation of ES_γ of the makespan of a more general FTAN. To this aim, the basic algorithms for ES_γ evaluation involving ordinary (i.e., nonfuzzy) intervals introduced in Section IV-A can be extended to solve the fuzzy cases, by the decomposition of the membership functions of the activity durations into a finite number of α-cuts. In the proposed method, schematically illustrated in Fig. 3, the basic algorithms can be applied on the instances associated to the selected α-levels to obtain the corresponding α-cuts of the desired fuzzy ES_γ evaluation. This finite number of α-cuts can be used to obtain an approximated reconstruction of the membership function $\mu_F(x)$ of ES_γ. Applying the general reconstruction rule described in Section III (i.e., $\mu_F(x) = \max\{\alpha : x \in F_\alpha\}$) to a finite set of α-cuts produces a stepped function that can be interpolated with a piecewise linear function using the extreme points of the α-cuts as breakpoints. More specifically, in the α-cuts decomposition phase, for each selected level α, each fuzzy duration is cut at level α. This phase produces a set of activity networks with interval-valued durations, each of which can be solved (as described in Section IV-A) in the α-cuts crisp instances solution phase. Then, in the successive Fuzzy ES_γ Reconstruction step, an approximation of the fuzzy membership function of ES_γ, is determined from their α-cuts (e.g., see [15] and [21]).

This method is simple to implement but it could be intractable if ran for too many cuts and can be carried out only on a selection of suitably chosen level-cuts. Thus, an issue comes from the selection of the relevant α-cuts. Possible solutions include: 1) to choose α-cuts arbitrarily, e.g., according to a precision degree fixed by the user; 2) to use a given number of α-cuts at fixed α levels. Since we consider fuzzy durations represented by piecewise linear functions, we adopt a more

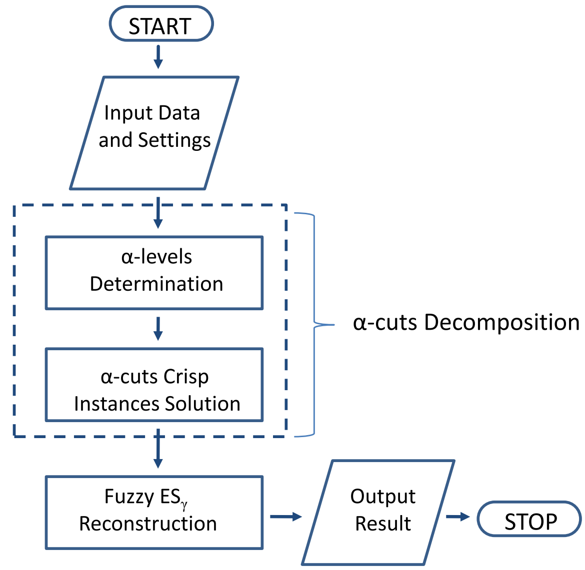
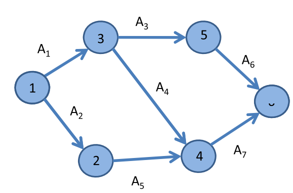


Fig. 3. Algorithmic scheme for the case of FTANs.



(a)

Activity	Fuzzy Duration d_i
A_1	(9,0),(10,0.6),(11,1),(12,1),(13,0)
A_2	(7,0),(8,1),(9,1),(10,0)
A_3	(8,0),(9,1),(10,1),(11,0)
A_4	(5,0),(6,1),(7,1),(8,0.6),(9,0)
A_5	(7,0),(9,1),(11,0)
A_6	(7,0),(9,1),(11,0)
A_7	(9,0),(10,1),(11,1),(12,0.6),(13,0)

(b)

Fig. 4. Illustative example. (a) Structure of the activity network. (b) Membership functions of the fuzzy activity durations.

suitable choice of the levels α allowing for an accurate computation of the fuzzy quantities of interest. In fact, the resulting fuzzy quantities will be described by piecewise linear functions too. Hence, the relevant α-cuts will be those corresponding to the breakpoints (i.e., of the right or left parts) of these fuzzy intervals. Assuming these levels known (otherwise they can be easily determined in a preprocessing phase), an exact (approximate) interval-based procedure applied to the breakpoint values would compute the actual (approximate) fuzzy ES values. In the proposed method, the α-cuts decomposition phase has a preliminary step devoted to determine the α-levels to adopt on the basis of either a specific input or default setting. We can then apply the method illustrated in Section IV-A, which solves the ES_γ evaluation in the case of interval-valued durations to know the corresponding α-cuts of ES_γ. The fuzzy ES_γ is then reconstructed from its α-cuts and returned as output. Considering the number K of α-cuts used in the adopted decomposition, the overall complexity of the algorithm is $O(K\Gamma^2\, m^2)$. In particular, associating the α-cuts to breakpoints (which is adopted as default scheme in our method) yields to a complexity $O(K_B\Gamma^2\, m^2)$, where K_B is the overall number of the different α values in breakpoints in the durations contained in the FTAN.

V. Illustrative Example

This section offers an example of the proposed method. To this aim, we consider an illustrative case study, which is represented in Fig. 4. More specifically, the figure reports the structure of the network [see part (a)], and the membership functions of the fuzzy durations in the part (b). Note that, in Fig. 4, we represent the fuzzy durations according to the notation introduced in Section III based on the full breakpoints ordered

TABLE I
CRISP INSTANCES ASSOCIATED TO THE α-CUTS IN THE SCHEMES UNI-5 (ALL), AND BP (IN BOLD)

Activity	$\boldsymbol{\alpha = 0}$	$\alpha = 0.2$	$\alpha = 0.4$	$\boldsymbol{\alpha = 0.6}$	$\alpha = 0.8$	$\boldsymbol{\alpha = 1}$
A_1	**[9,13]**	[10,12]	[10,12]	**[10,12]**	[11,12]	**[11,12]**
A_2	**[7,10]**	[8,9]	[8,9]	**[8,9]**	[8,9]	**[8,9]**
A_3	**[8,11]**	[9,10]	[9,10]	**[9,10]**	[9,10]	**[9,10]**
A_4	**[5,9]**	[6,8]	[6,8]	**[6,8]**	[6,7]	**[6,7]**
A_5	**[7,11]**	[8,10]	[8,10]	**[9,9]**	[9,9]	**[9,9]**
A_6	**[7,11]**	[8,10]	[8,10]	**[9,9]**	[9,9]	**[9,9]**
A_7	**[9,13]**	[10,12]	[10,12]	**[10,12]**	[10,11]	**[10,11]**

TABLE II
CVAR VALUES ASSOCIATED TO α-CUTS IN THE SCHEMES UNI-5 (ALL), AND BP (IN BOLD)

γ	$\boldsymbol{\alpha = 0}$	$\alpha = 0.2$	$\alpha = 0.4$	$\boldsymbol{\alpha = 0.6}$	$\alpha = 0.8$	$\boldsymbol{\alpha = 1}$
0.99	**35**	32	32	**32**	31	**31**
0.95	**[33.72,34.35]**	32	32	**31.67**	31	**31**
0.90	**[32.56,34.18]**	31.86	31.86	**31.36**	31	**31**

sequence. The activity on arcs (AoA) network contains six nodes and the (arcs) activities A_i, with $i = 1, \ldots, 7$. Each activity has its own fuzzy duration. Some durations, as in the case of A_5 and A_6, have a simple triangular membership function while more general models are associated to other activities. In this example, we consider three different values for the probability level γ of the ES of the makespan, namely: $\gamma \in \{0.90, 0.95, 0.99\}$. As for the choice of the α levels to use in the adopted decomposition scheme, we consider two different approaches. The first one, named UNI-5, uses α values uniformly sampled in the interval $[0, 1]$ with a step value of 0.2 (i.e., the values: 0, 0.2, 0.4, 0.6, 0.8, and 1). The second approach, named BP, uses all the different α values corresponding to the breakpoints of all the fuzzy durations d_i of the activities received as input [i.e., in the considered example, these values are 0, 0.6, and 1, as indicated in Fig. 4(b)]. Then, the α-cuts are generated according the considered specific decomposition plan (i.e., UNI-5 and BP), and for each considered α-level, an instance with crisp (integer) interval durations is generated as reported in Table I. More specifically, the crisp instances associated to different α-levels are reported in columns, where α-cuts belonging to the Scheme BP are in bold. The rows contain all the intervals (possibly reduced to a single value) associated to each activity indicated as first row's entry.All these instances are solved applying the algorithm ES_MinS described in Section IV-A running in the following computational environment: Intel i9 (4.4 GHz clock) with 32 GB of RAM. The code is written in Python 3.8 and run on a single thread.

In Table II, the results associated to different α-levels are reported in columns, where those included in the Scheme BP are, again, in bold. Table II summarizes, in its rows (associated to the probability level γ), the results obtained considering the different α-values. In this way, the rows contain the representation of the CVaR in terms of α-cuts, which can be used to obtain its membership function, as illustrated in Figs. 5–7. Considering the case with $\gamma = 0.99$, Fig. 5 reports the approximated fuzzy reconstruction of the CVaR for both decomposition Schemes BP (in red) and UNI-5 (in blue). When, for a point, it is necessary to make membership of the function unambiguous, that point is indicated in bold (i.e., the point is empty if it does not belong to the function).

Similarly, Figs. 6 and 7 show the results for the case $\gamma = 0.95$ and $\gamma = 0.90$, respectively. In the latter cases, when $\alpha = 0$, the algorithm provides an approximate result obtained as the average of a lower bound

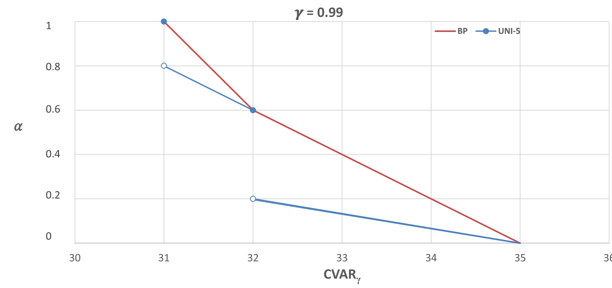


Fig. 5. Illustrative example: Approximated membership functions of the CVaR when $\gamma = 0.99$.

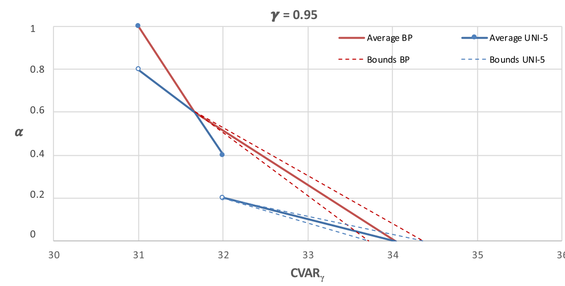


Fig. 6. Illustrative example: Approximated membership functions of the CVaR when $\gamma = 0.95$.

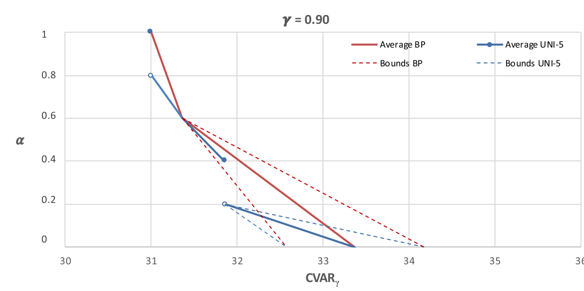


Fig. 7. Illustrative example: Approximated membership functions of the CVaR when $\gamma = 0.90$.

and an upper bound of the CVaR reported in Table II. Considering this evidence, Figs. 6 and 7 show the results reporting both the average values (solid lines) and the bounds explicitly (dotted lines). The latter method better accounts for the uncertainty associated with the assessment of the CVaR, even if, in the considered cases, this concerns only the evaluations related to lower α values. The results highlight how denser sampling of α can lead to a more accurate fuzzy representation of CVaR. However, this should be evaluated considering also the comparison in terms of computation times. The average computation time required for this example is 0.05 s, and the requirements of the UNI-5 Scheme results on average 1.78 times higher.

VI. COMPUTATIONAL STUDY

The algorithm for the case with ordinary intervals has been experimentally analyzed and validated in [4] while the computational complexity of the extension to type-1 fuzzy activity durations has been characterized in Section IV-B. An experimental study is conducted to test the approach proposed in this article in the computational environment described in Section V. The test sets have been generated starting by the cases available in PSPLIB [22], a widely used instance library for resource-constrained project scheduling problems (RCPSP),

TABLE III
RESULTS ON SET J90-PSPLIB FUZZY INSTANCES FOR SCHEMES UNI-5 (ALL), AND BP (IN BOLD)

	γ=0.90		γ=0.95		γ=0.99	
α	$\widehat{MRE}$ (%)	T. (s)	$\widehat{MRE}$ (%)	T. (s)	$\widehat{MRE}$ (%)	T. (s)
1.00	**0.70**	**0.06**	**0.51**	**0.05**	**0.21**	**0.03**
0.80	1.25	0.12	1.00	0.10	0.53	0.06
0.60	**3.21**	**0.46**	**2.79**	**0.40**	**1.82**	**0.28**
0.40	3.36	0.50	2.98	0.45	2.01	0.32
0.20	3.90	0.66	3.52	0.59	2.63	0.45
0.00	**5.69**	**1.25**	**5.28**	**1.19**	**4.26**	**1.00**
UNI-5	3.02	3.05	2.68	2.78	1.91	2.14
BP	**3.20**	**1.77**	**2.86**	**1.64**	**2.10**	**1.31**

considered to be representative of realistic scheduling problems. In particular, we consider the AoA version of J90 RCPSP Single Mode Data Set composed of 480 instances having 90 activities each. In these instances, all the activities are considered uncertain and represented by six-point-approximated membership functions. The latter are randomly generated, assuming, for each of them, the support M equals the interval $[d, 1.5 * d]$ where d is the original duration considered in the set J90. This makes the instances even more difficult because a large amount of uncertainty is introduced.

Table III summarizes (averaging over the 480 instances) the results obtained for each cut in the decomposition schemes (by rows) and for the different γ values (by columns). The computation times (T.) are reported in seconds. The procedure, in general, is fast and requires computation efforts increasing as the sampling density increases (i.e., they are higher in the UNI-5 Scheme). In addition, computation times increase as the γ values decrease because the algorithms have to consider an increasing number of configurations in their counting procedures. However, this is not a serious defect as there is generally greater interest in estimating CVaR for high γ values. The UNI-5 Scheme has double the samples of BP, but the ratio of the related overall computation times obtained in the two schemes remains lower than 1.72, as shown in the last two rows of Table III. However, the actual value of the ratio of the computation times does not remain constant for the different γ values, but it varies according to the structure of the networks. The columns indicated as $\widehat{\text{MRE}}$ contain the averages of an excess estimate of the relative error for the ES_γ: $\widehat{\text{MRE}} = \frac{UB_\gamma - LB_\gamma}{UB_\gamma + LB_\gamma}$. The procedure, in general, is more accurate as α increases. The last rows of Table III show that the UNI-5 Scheme offers a better accuracy but requires an higher computational effort than BP. The overall speed and quality of the proposed method makes it an enabling tool for the use of CVaR as an analysis criterion in fuzzy scheduling problems while the tradeoff between accuracy and computation effort indicates a possible research direction regarding the strategies for choosing the decomposition scheme in terms of both structure and size of the α sample set.

VII. CONCLUSION

This article addresses the evaluation of the CVaR of the makespan associated to a feasible schedule represented as an activity network. Its main contribution is the extension of a state-of-the-art algorithmic approach recently proposed for the case of interval-valued durations of the activities as a computational method to obtain the fuzzy evaluation of the CVaR of the makespan of a given schedule when only a type-1 fuzzy representation of activity durations is known. The method is based on the classic α-cuts decomposition, resulting an overall fast procedure, which can enable to use the CVaR as an analysis criterion for fuzzy scheduling approaches in different applications.

Further research directions include the following.

1) The application of the proposed methodology in real contexts.
2) The improvement in the performance of the algorithm devoted to solve crisp-interval instances.
3) The development of effective strategies to select the α-cuts in the decomposition scheme.
4) The extension of the proposed approach to compute the CVaR for the makespan in activity networks considering type-2 fuzzy [8] uncertain activity durations.

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