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Axiomatic Construction of Trees from Boundary Arcs

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Abstract. The aim of this article is to identify a set of independent axioms that characterize the end-point boundary Ω of a tree T and to develop a procedure to (re)construct T and its geometric features from Ω.

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1. Introduction

Considerable attention has been devoted to the reconstruction of large networks, e.g., expander graphs and Ramanujan graphs [5,8,9], starting from their geometry at large. Possibly infinite graphs with thin or no loops, for instance trees and more general Gromov hyperbolic graphs, real trees and dendrites have also been extensively studied in the literature in relation with their boundaries.

Given a tree T, it was shown in [1,2,6,10] how one-sided-maximal sequences of adjacent vertices are used to construct its topological boundary Ω_T, called the end-point boundary, or Poisson boundary, the minimal boundary where the Poisson transform reproduces all bounded functions harmonic with respect to the nearest-neighbor stochastic transition operator on T. If T is regarded as a hyperbolic graph, Ω_T is also called its Gromov boundary [7]. Indeed T is 0-hyperbolic: the three sides of any geodesic triangle intersect at a unique vertex, called their join.

Natural problems are to determine whether a topological space Ω can be realized as the boundary of a tree and to reconstruct a tree given its boundary. These general issues arise in integral geometry, for instance in

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the study of the horospherical fiber bundle $\mathcal{H}$ of a tree, or a more general Riemannian structure. In particular, homogeneous and semi-homogeneous trees have large automorphism groups, and significant results in the harmonic analysis of groups acting on $\mathcal{H}$ can be proved using techniques of integral geometry [3]. Therefore it is important to characterize the spaces that can arise as bases of horospherical fiber bundles of trees, even more so in the homogeneous or semi-homogeneous setting, and be able to reconstruct the trees and the bundles up to isomorphisms.

In general, the sole knowledge of the topology of a tree boundary Ω_T, namely of its open subsets, cannot determine the tree T up to isomorphisms, because non-isomorphic trees may have homeomorphic boundaries; for instance, collapsing two adjacent non-terminal vertices in a tree produces a non-isomorphic tree without modifying the boundary. Therefore, in order to reconstruct T, additional information is needed of Ω_T, for instance an appropriate base for its topology. We shall call boundary arc any subset of Ω_T consisting of all boundary points that lie on the same side with respect to some edge of T; the family $\mathcal{A}_T$ of all arcs is a base for the end-point topology of Ω_T.

On the other hand, no tree vertex that is flat, i.e., adjacent to exactly two other vertices, can be detected from any knowledge of the boundary, hence flat vertices must be excluded if reconstruction up to tree isomorphisms is sought (alternatively, arbitrary insertion or removal of flat vertices must be allowed).

This article studies the problem of determining whether a set Ω and a family $\mathcal{A}$ of its subsets can be realized as Ω_T, respectively $\mathcal{A}_T$ for a suitable tree T and possibly of reconstructing T explicitly. The reconstruction of the corresponding horospherical fiber bundle is discussed in [3]. We are grateful to Simon G. Gindikin for suggesting this viewpoint. The questions that we answer are:

- is a locally finite, flat-free tree T identified by $\mathcal{A}_T$ up to isomorphisms?
- can a reconstruction procedure of such T be described in terms of $\mathcal{A}_T$?
- can an explicit set of axioms be produced on the pair $(\Omega;\mathcal{A})$, where Ω is a set and $\mathcal{A}$ a family of its subsets, that allows a unique identification of a locally finite, flat-free tree $T(\Omega;\mathcal{A})$ and a canonical bijection $\iota\colon \Omega \to \Omega_{T(\Omega;\mathcal{A})}$ that induces a bijection $\mathcal{A} \to \mathcal{A}_{T(\Omega;\mathcal{A})}$?
- do such axioms hold for $(\Omega;\mathcal{A}) = (\Omega_T;\mathcal{A}_T)$ if T is any locally finite tree?
- can the join of three vertices in $T(\Omega;\mathcal{A})$ be described purely in terms of $(\Omega;\mathcal{A})$?

We shall give affirmative answers to all these questions. After some background in Sect. 2, we present the axioms on $(\Omega;\mathcal{A})$ in Sect. 3, show that they hold for $(\Omega_T;\mathcal{A}_T)$ whenever T is a locally finite tree (whence they are compatibile), prove their independence and some of their consequences, and describe how the tree $T(\Omega;A)$ can be constructed if they hold. In Sect. 4 we show that this procedure is a two-sided inverse of the construction given in [1] of the pair $(\Omega_T;\mathcal{A}_T)$ in terms of T. More precisely, we prove that if T is locally finite and flat-free then it is canonically isomorphic to $T(\Omega_T;\mathcal{A}_T)$;

conversely, any pair $(\Omega; \mathcal{A})$ that satisfies the axioms is canonically isomorphic to $(\Omega_{T(\Omega;\mathcal{A})}; \mathcal{A}_{T(\Omega;\mathcal{A})})$. Therefore the functors $(\Omega; \mathcal{A}) \mapsto T(\Omega; \mathcal{A})$ and $T \mapsto (\Omega_T; \mathcal{A}_T)$ are covariant and the inverse of each other. Section 5 provides directly in terms of arcs a description of some more geometry of $T(\Omega; \mathcal{A})$, namely the join of three vertices. Finally, because horospheres of vertices and of edges on trees have received considerable attention in the mathematical literature, Sect. 6 briefly alludes to horospheres of arcs.

It follows from our axioms that $\mathcal{A}$ is (finite or) countable and all the coarsest and sub-coarsest partitions of Ω into elements of $\mathcal{A}$ are finite. More general axioms might allow to extend the results to trees with vertices of infinite valency and to **R**-trees, that will be considered in future papers.

2. Definitions and Background

In this article a *tree* T is a connected, locally finite, undirected graph without loops or self-loops, the union of the set V of its *vertices* and the set E of its *edges*, that are unordered pairs of distinct vertices. Two vertices v, v' are *adjacent*, or *neighbors*, and we write $v \sim v'$, if the unordered pair $[v, v']$ is an edge. The *valency* of a vertex is the number of its neighbors; vertices of valency 1 or 2 are called *terminal*, respectively *flat*. To avoid trivial exceptions we assume throughout that at least one vertex has valency 3 or more (thus every vertex has at least one neighbor and every edge contains at most one terminal vertex). Two distinct edges e, e' are adjacent, and we write $e \sim e'$, if they share one vertex. A *chain of vertices* $[v, v']$ is a finite, undirected sequence of consecutively adjacent vertices without repetitions, uniquely determined by its extremes (*endpoints*) v, v'. The integer *distance* between two vertices is the length of the chain between them. A *ray of vertices* is a maximal sequence of consecutively adjacent vertices without repetitions that starts at some vertex; a ray is finite if and only if it is a chain whose last vertex is terminal. Similar definitions hold for *chains*, *rays* and *distance of edges*.

A vertex v' *lies between* vertices v, v'' (extremes included) if it belongs to the chain $[v, v'']$; in this case, v, v', v'' (in any order) are *aligned*. The *join* of three vertices v, v', v'' (see [1, Proposition 1.2] and [3]) is the unique vertex that belongs to each of the chains $[v, v'], [v, v''], [v', v'']$. If v, v', v'' are not aligned, their join is an internal vertex of each of these chains; instead, if v' lies between v, v'' then the join is v'.

Two rays of vertices are *equivalent* if they merge after finite numbers of steps and coincide thereafter. Each equivalence class ω, called a *boundary point* of T, contains exactly one representative ray of vertices that starts at any given vertex. The *end-point boundary* (henceforth simply *boundary*) of T is the family Ω_T of such classes. Each class containing finite rays corresponds to a terminal vertex, and each terminal vertex is thus identified to a boundary point. One extends this equivalence relation by considering a ray of edges as equivalent to the ray of the intervening vertices; the relation is thereby extended to other rays of edges by transitivity. Thus each equivalence class contains rays of vertices and rays of edges, and so Ω_T can also be regarded as the set of equivalence classes of rays of edges.

A *boundary arc* $\Omega(v,v')$ is the set of equivalence classes in Ω_T with representative rays that contain distinct $v,v' \in V$ in this order. A boundary arc consists of a single point ω if and only if ω is (identified to) a terminal vertex. The family

$$\mathcal{A}_T = \{\Omega(v,v')\colon v,v' \in V \,\text{with}\, v \neq v'\}$$

is a base for the natural totally disconnected, compact topology of Ω_T, whilst T is equipped with the discrete topology. A *sector* is the set $S(v,v') \subset T$ that contains each vertex or edge such that the chain of vertices from v to it contains v'. The family $\{S(v,v')\cup\Omega(v,v')\colon v \neq v'\}$ of *closed sectors* generates a compact and totally disconnected topology on $T \cup \Omega_T$; it is not restrictive to assume $v \sim v'$ in these definitions. For distinct $e,e' \in E$ we define in a similar way the sector $S(e,e')$, the boundary arc $\Omega(e,e') \subset \Omega_T$ and the closed sector $S(e,e') \cup \Omega(e,e')$.

3. The Axioms

The elements of a family $\mathcal{A}$ of subsets of a given set Ω will be called *arcs* if the pair $(\Omega;\mathcal{A})$ fulfills the Axioms listed below. An *arc-partition* $P = (A_j)_j$ of a subset of Ω is an (unordered) decomposition thereof into the disjoint union of (one or several) arcs A_j. The symbol $\amalg$ will denote disjoint union, while the complement $\Omega \setminus A$ of A in Ω will be denoted by $\overline{A}$.

Axioms 1. *Given a (finite or) countable family $\mathcal{A}$ of subsets of a set Ω (which may itself be uncountable), we say that $(\Omega;\mathcal{A})$* satisfies the Axioms *if:*

I. $\Omega \notin \mathcal{A}$;
II. *if $A \in \mathcal{A}$ then $\overline{A} \in \mathcal{A}$;*
III. *if $A, A' \in \mathcal{A}$ are not disjoint, then $A \subseteq A'$, or $A' \subseteq A$, or $A \cup A' = \Omega$;*
IV. *for every distinct $\omega, \omega' \in \Omega$ there is an arc that contains exactly one of them;*
V. *if $A, A' \in \mathcal{A}$ are not disjoint then $A \cap A'$ admits a finite arc-partition;*
VI. *Ω does not admit an infinite arc-partition.*

To avoid pedantic exceptions we assume that Ω consists of at least 3 elements.

Theorem 3.1. *For every tree T, the pair $(\Omega_T;\mathcal{A}_T)$ satisfies the Axioms, which are therefore compatible.*

Proof. The family $\mathcal{A}_T$ is finite or countable because so is V. Since $\overline{\Omega(v,v')} = \Omega(v',v)$ if $v \sim v'$, Axioms I and II are obviously satisfied.

Two distinct edges $[v_1,v_2], [v_3,v_4]$ can be connected by a chain; we can assume that $v_1,v_2,v_3,v_4 \in V$ are in sequential order, although v_2, v_3 might coincide. Then the four possibilities given by

$$\begin{aligned} A &\text{ equals } \Omega(v_1,v_2) \text{ or } \Omega(v_2,v_1),\\ A' &\text{ equals } \Omega(v_3,v_4) \text{ or } \Omega(v_4,v_3), \end{aligned} \tag{1}$$

yield the alternatives of Axiom III.

For distinct $\omega, \omega' \in \Omega_T$ there is $v_0 \in V$ and two rays, one in each equivalence class, that originate from v_0 and are otherwise disjoint. Any two consecutive vertices v_j, v_{j+1} in the ray from v_0 to ω yield an arc $\Omega(v_j, v_{j+1})$ that contains ω but not ω', whence Axiom IV.

If A, A' in (1) are not disjoint, then by Axiom III either $A \cup A' = \Omega_T$ or one is contained in the other. In the latter case Axiom V follows trivially. The former case is given by the configuration $A = \Omega(v_1, v_2)$, $A' = \Omega(v_4, v_3)$. Then $A \cap A'$ is the disjoint union of all $\Omega(w, u)$ for w belonging to the chain joining v_2, v_3, and u being a neighbor of w not belonging to this chain and different from v_1, v_4. Axiom V follows.

Suppose that $\Omega_T = \coprod_{j=0}^{\infty} \Omega(v_j, v_j')$ where $v_j \sim v_j'$ for every j, and v_j' can be assumed non-flat. In this disjoint union there can only be finitely many repetitions of each v_j', because T is locally finite. Therefore, since $T \cup \Omega$ is compact and T is discrete, the sequence v_j' has an accumulation point $\omega \in \Omega$, and we may assume $\omega \in \Omega(v_0, v_0')$. For every $j > 0$ we have $v_j' \notin S(v_0, v_0')$, otherwise $\Omega(v_j, v_j')$ would intersect $\Omega(v_0, v_0')$; but then ω cannot be an accumulation point of $\{v_j'\}$, hence Axiom VI must hold. □

Theorem 3.2. *The Axioms are independent.*

Proof. Let T be a tree. Then all Axioms except II hold for the pair $\big(\Omega_T; \mathcal{A}_T \cup \{\emptyset\}\big)$. Likewise, all Axioms except I hold for $\big(\Omega_T; \mathcal{A}_T \cup \{\emptyset, \Omega_T\}\big)$.

If we duplicate one $\omega \in \Omega_T$ into ω_1, ω_2, denote by Ω the resulting set, and $\mathcal{A}$ is correspondingly obtained from $\mathcal{A}_T$, then the pair $(\Omega; \mathcal{A})$ satisfies all Axioms except IV.

Let $\Omega' = \Omega_T \setminus \{\omega\}$, where ω is a non-terminal boundary point, and let $\mathcal{A}'$ be correspondingly obtained from $\mathcal{A}_T$. Let $v_0 \sim v_1 \sim \cdots$ be a ray of vertices in the class ω; the family of all boundary arcs $\Omega(v_k, u)$ such that u is a neighbor of v_k different from v_{k+1} and, if $k > 0$, also from v_{k-1}, is an infinite disjoint covering of Ω', so all Axioms except VI hold for $(\Omega', \mathcal{A}')$.

If T consists only of a non-terminal vertex v_0 adjacent to three terminal vertices v_1, v_2, v_3, then $\Omega_T = \{v_1, v_2, v_3\}$, while $\mathcal{A}_T$ consists of the three singletons and their complements. The pair $\big(\Omega_T; \mathcal{A}_T \setminus \big\{\{v_1\}, \overline{\{v_1\}}\big\}\big)$ fulfills all Axioms except V, which fails for $A = \{v_1, v_2\}$ and $A' = \{v_1, v_3\}$.

Consider a forest consisting of two disjoint trees T_1, T_2, with its boundary $\Omega = \Omega_{T_1} \cup \Omega_{T_2}$ equipped with the family $\mathcal{A}$ consisting of the elements of $\mathcal{A}_{T_1} \cup \mathcal{A}_{T_2}$ and their complements in Ω. If $A, A' \in \mathcal{A}_{T_1}$ with $A \subsetneq A'$, then $A', A \cup \Omega_{T_2} \in \mathcal{A}$ are not disjoint, their union $A' \cup \Omega_{T_2}$ is a proper subset of Ω and neither one is contained in the other. So all Axioms hold except III. □

Proposition 3.3. *The following properties are consequences of the Axioms:*

(i) *for every distinct $\omega, \omega' \in \Omega$ there are disjoint arcs $A \ni \omega$ and $A' \ni \omega'$;*
(ii) *an arc is minimal if and only if it is a singleton;*
(iii) *a non-empty finite intersection of arcs admits a finite arc-partition;*
(iv) *a finite union of arcs admits a finite arc-partition; any union of arcs admits an arc-partition;*

(v) *an intersection of arcs (in particular: an arc) does not admit an infinite arc-partition;*
(vi) *the union of a strictly increasing infinite sequence of arcs admits an infinite arc-partition, but is neither an arc nor all of* Ω*;*
(vii) *the intersection of a strictly decreasing infinite sequence of arcs is a singleton that is not an arc;*
(viii) *every singleton* $\{\omega\} \subset \Omega$ *is the intersection of a (finite or infinite) decreasing sequence of arcs (and possibly an arc itself);*
(ix) *any collection of arcs covering* Ω *contains a finite subcollection covering* Ω*.*

Proof. (i) Immediate by Axioms IV and II.

(ii) A singleton is minimal because no arc is empty, by Axioms I and II. Conversely, if an arc A contains distinct ω, ω', then by Axiom IV there is an arc A' containing ω but not ω', and $A \cap A'$ admits an arc-partition by Axiom III: its element containing ω is a proper subset of A, which is consequently not minimal.

(iii) Follows from Axiom V by iteration.

(iv) If $\{A_j\}_{j\geq 0}$ is a family of arcs, for each $j \geq 0$ set $B_j = A_j \cap \bigcap_{0\leq k<j} \overline{A_k}$ (we shall use this notation throughout this proof). Therefore $B_0 = A_0$, the B_j's are pairwise disjoint, and

$$\bigcup_j A_j = \coprod_j B_j. \tag{2}$$

By Axiom II and property (iii), each non-empty B_j admits a finite arc-partition, so we obtain an arc-partition of $\bigcup_j A_j$. If this union is finite then so is the arc-partition.

(v) We have that $\overline{\bigcap_{j=1}^n A_j} = \bigcup_{j=1}^n \overline{A_j}$ admits an arc-partition by Axiom II and property (iv). If $\bigcap_{j=1}^n A_j$ admitted an infinite arc-partition, then so would Ω, violating Axiom VI. The same argument holds if n equals 1 or is replaced by ∞.

(vi) If $\{A_j\}_{j=0}^\infty$ is strictly increasing then $B_j = A_j \cap \overline{A_{j-1}}$ is non-empty for every $j > 0$, so $A_\infty = \bigcup_{j=0}^\infty A_j$ admits an infinite arc-partition by (2). By Axiom VI, A_∞ cannot equal Ω; likewise, if A_∞ were an arc, including $\overline{A_\infty}$ into that arc-partition would contradict the same Axiom.

(vii) If $\{A_j\}_{j=0}^\infty$ is strictly decreasing, by Axiom II and property (vi) its intersection A_∞ is neither empty nor an arc, and $\overline{A_\infty}$ admits an infinite arc-partition. If A_∞ contained distinct ω, ω', then by Axiom IV there would be an arc D containing ω but not ω'. Denote by $\{C_k\}$ the collection of the arcs in the arc-partition of $A_0 \cap D$ given by Axiom V that would intersect A_∞. Then each C_k would be contained in A_0, D and intersect every A_j; moreover $C_k \cup A_j \subset A_0 \subsetneq \Omega$ and C_k would not contain A_j because it would exclude ω'. Therefore C_k would be contained in every A_j by Axiom III, hence in A_∞, and $\coprod_k C_k \subseteq A_\infty \cap D$. Every element of $A_\infty \cap D$ would belong to $A_0 \cap D$, hence to some C_k. Therefore $\coprod_k C_k = A_\infty \cap D$, an arc-partition. By the same token, exchanging ω, D with $\omega', \overline{D}$ we see that $A_\infty \cap \overline{D}$ would admit an arc-partition. Then $\Omega = \overline{A_\infty} \amalg (A_\infty \cap D) \amalg (A_\infty \cap \overline{D})$ would admit an infinite arc-partition, thus contradicting Axiom VI.

(viii) Any fixed $\omega \in \Omega$ is the intersection of all arcs of Axiom IV as ω' ranges in $\Omega \setminus \{\omega\}$. There are at most countably many distinct such arcs, so, labeling them as $A_0, A_1, \ldots$, we have $\{\omega\} = \bigcap_j A_j$. Set $D_0 = A_0$ and, for each j, inductively let D_{j+1} be the arc containing ω in an arc-partition of $A_{j+1} \cap D_j$ ensured by Axiom V. Thus $\{D_j\}$ is a (possibly finite) decreasing sequence of arcs and $\bigcap_j D_j = \{\omega\}$.

(ix) Assume $\bigcup_{j\geq 0} A_j = \Omega$. Every B_j as in (2) is either empty or (as already observed) it admits an arc-partition, and $\coprod_{j\geq 0} B_j = \Omega$. By Axiom VI, B_j is empty for $j > j_0$ large enough. Since $B_j \subseteq A_j$ for every j, then $\bigcup_{0\leq j\leq j_0} A_j \supseteq \coprod_{0\leq j\leq j_0} B_j = \Omega$. □

In the remainder of this section we shall assume that $(\Omega; \mathcal{A})$ satisfies the Axioms.

Definition 3.4. Given arc-partitions $P = (A_j)_j, P' = (A_{j'})_{j'}$ of $B \subseteq \Omega$, we say that P is *coarser* than P' (or P' is *finer* than P) if every $A'_{j'}$ is contained in some A_j (note that P is finite if P' is); this is a partial ordering on arc-partitions of B. An arc-partition of $B \in \mathcal{A}$ that is strictly finer only of the trivial, coarsest (i.e., maximal) arc-partition (B) is *sub-coarsest*.

By Axioms I and II every (unordered) pair $(A, \overline{A})$ for $A \in \mathcal{A}$ is a coarsest arc-partition of $B = \Omega$; we call it an *edge*. Instead, we call *vertex*: either any finite arc-partition of Ω that is *sub-coarsest*, being strictly finer only of some such pairs; or any arc that is a singleton (then a *terminal* vertex). We shall say that an arc *belongs* to an edge or a non-terminal vertex (and the edge or vertex *owns* the arc) if it belongs to the arc-partition; the arcs that belong to the terminal vertex corresponding to $\{\omega\} \in \mathcal{A}$ are $\{\omega\}, \overline{\{\omega\}}$.

A non-terminal vertex $(A_1, \ldots, A_k)$ *bounds* exactly the $k \geq 3$ distinct edges $(A_j, \overline{A_j})$ for $j = 1, \ldots, k$, whereas a terminal vertex $\{\omega\}$ bounds the edge $(\{\omega\}, \overline{\{\omega\}})$ only. Distinct vertices are *adjacent* if they bound the same edge, that *connects* them; this happens exactly if one vertex owns an arc A while the other owns $\overline{A}$, the connecting edge being $(A, \overline{A})$. The *valency* of a vertex is the number of edges that it bounds, whence it is either 1 or larger than 2 (no vertex of valency 2 is produced with this procedure, cf. Remark 4.2). Two distinct edges are *adjacent* if they are bound by a same non-terminal vertex.

Lemma 3.5.

(a) *every $A \in \mathcal{A}$ that is not a singleton admits a unique sub-coarsest arc-partition P, and P is finite; every arc $C \subsetneq A$ is contained in one of the elements of P;*
(b) *given non-disjoint $A, B \in \mathcal{A}$, there exists a unique coarsest arc-partition P of $A \cap B$, and P is finite; every arc $C \subseteq A \cap B$ is contained in one of the elements of P;*
(c) *any sub-coarsest arc-partition of Ω is finite (then a non-terminal vertex); two that share an element coincide.*

Proof. (a) By Axiom IV there is an arc A' intersecting A but not containing it. Putting together the finite arc-partitions of $A \cap A'$ and of $A \cap \overline{A'}$ ensured by Axioms V and II, we get the existence of a finite proper arc-partition of A. Thus there are sub-coarsest arc-partitions of A that are finite; let $P = (A_j)_j$ be one.

Taken an arc $C \subsetneq A$, we may assume that it intersects A_0. One of C, A_0 contains the other, because their union is contained in $A \subsetneq \Omega$, by Axioms III and I. If C contained A_0 properly, then it would intersect at least another A_j; indeed it would contain every A_j that it intersected, again by Axiom III. Yet C could not contain every A_j because $C \subsetneq A$. By replacing in P all the A_j's contained in C with C itself we would obtain a proper, strictly coarser arc-partition of A. Therefore $C \subseteq A_0$.

Let $P' = (A'_{j'})_{j'}$ be another sub-coarsest arc-partition of A. By the previous part, every $A'_{j'}$ is contained in an A_j, hence P is coarser than P'. Exchanging roles we see that $P = P'$.

(b) There is a finite arc-partition of $A \cap B$ by Axiom V. The rest of this part follows as in part (a).

(c) If $(A_j)_j, (A'_{j'})_{j'}$ are sub-coarsest arc-partitions of Ω and $A_0 = A'_0$, then, by removing this first arc from both, we obtain two sub-coarsest arc-partitions of $\overline{A_0}$ that must coincide by part (a). □

Corollary 3.6. *Every $A \in \mathcal{A}$ belongs to exactly one edge and one vertex.*

Every edge $(A, \overline{A})$ connects exactly two distinct vertices, one or both non-terminal (therefore the valency of a vertex equals the number of its adjacent vertices). Unless A is a singleton, one is $(A_1, \ldots, A_k, \overline{A})$, where $(A_1, \ldots, A_k)$ is the sub-coarsest arc-partition of A; likewise, unless $\overline{A}$ is a s-ingleton, the other is $(A, A'_1, \ldots, A'_{k'})$, where $(A'_1, \ldots, A'_{k'})$ is the sub-coarsest arc-partition of $\overline{A}$.

The only neighbor of a terminal vertex $\{\omega\}$ is the sub-coarsest arc-partition of Ω obtained putting together $\{\omega\}$ and the sub-coarsest arc-partition of $\overline{\{\omega\}}$.

Proof. Given $A \in \mathcal{A}$, by Axiom II $(A, \overline{A})$ is an edge, the only one that owns A. If $\overline{A}$ is a singleton then it is a terminal vertex that owns A, and no other vertex can. Else, putting A together with the sub-coarsest arc-partition of $\overline{A}$ ensured by Lemma 3.5(a) gives a non-terminal vertex that owns A, with uniqueness given by Lemma 3.5(c).

The rest of the statement follows again from Lemma 3.5(a),(c). □

Definition 3.7. We say that $A \in \mathcal{A}$ is *linked* to $A' \in \mathcal{A}$, and write $A \supsetneqq A'$, if A' is strictly contained in A and is maximal (under inclusion) with this property; equivalently, by Lemma 3.5(a), if A' is an element of the sub-coarsest arc-partition of A. Thus two vertices are adjacent if and only if one arc of one is linked to one arc of the other; for instance $A \supsetneqq A_j$ for $j = 1, \ldots, k$ and $\overline{A} \supsetneqq A'_{j'}$ for $j' = 1, \ldots, k'$ if all are given as in Corollary 3.6.

A finite sequence $A_0 \supsetneqq \cdots \supsetneqq A_n$ of arcs will be called a *(descending) chain of arcs*. A *chain of edges* is a minimal finite sequence of edges $e_0 \sim e_1 \sim \cdots \sim e_n$.

Proposition 3.8. *Given distinct edges $(A, \overline{A}), (B, \overline{B})$, up to exchanging A with $\overline{A}$ and/or B with $\overline{B}$ we can assume $A \supsetneq B$; then the two edges are adjacent exactly if $\overline{A}, B$ belong to the same (non-terminal) vertex, which happens if and only if $A \gtrsim B$.*

Therefore every chain of arcs $A_0 \gtrsim \cdots \gtrsim A_n$ gives a chain of edges $(A_0, \overline{A_0}) \sim \cdots \sim (A_n, \overline{A_n})$. Conversely, if $(A_0, \overline{A_0}) \sim \cdots \sim (A_n, \overline{A_n})$ is a chain of edges, then, up to exchanging some A_j with $\overline{A_j}$, we may assume that $A_0 \gtrsim \cdots \gtrsim A_n$ is a chain of arcs. In particular, if $n > 1$ the extreme edges $(A_0, \overline{A_0}), (A_n, \overline{A_n})$ are distinct.

For any $A, A' \in \mathcal{A}$ with $A \supsetneq A'$ there exists a unique descending chain of arcs $A = A_0 \gtrsim \cdots \gtrsim A_n = A'$. Correspondingly, any two edges are connected by a unique chain of edges.

Proof. The first statement follows from Axiom III and Lemma 3.5(b).

If $(A_0, \overline{A_0}) \sim \cdots \sim (A_n, \overline{A_n})$ is a chain of edges with $n \geq 1$, we have noticed that we may assume $A_0 \gtrsim A_1$. Suppose by induction that $A_0 \gtrsim \cdots \gtrsim A_j$ for $j < n$. Up to exchanging A_{j+1} and $\overline{A_{j+1}}$, by Axiom III either $A_j \subseteq A_{j+1}$ or $A_j \supseteq A_{j+1}$; in either case $\overline{A_j}, \overline{A_{j+1}}$ intersect. The vertex bounding both $(A_{j-1}, \overline{A_{j-1}}), (A_j, \overline{A_j})$ must contain $\overline{A_{j-1}}, A_j$ by the induction hypothesis. By Lemma 3.5(c) the vertex bounding $(A_j, \overline{A_j}), (A_{j+1}, \overline{A_{j+1}})$ cannot contain A_j, hence it contains $\overline{A_j}$, so it cannot contain $\overline{A_{j+1}}$, thus it must contain A_{j+1}. Whence $\overline{A_j}, A_{j+1}$ are disjoint, that is $A_j \supsetneq A_{j+1}$ and $A_j \gtrsim A_{j+1}$, which completes the induction step.

Given $A, A' \in \mathcal{A}$ with $A \supsetneq A'$, by Lemma 3.5(a) A' is contained in exactly one of the elements, say A_1, of the sub-coarsest arc-partition of $A = A_0$; thus $A_0 \gtrsim A_1 \supset A'$. By continuing in this way as far as possible we build a sequence of arcs $A = A_0 \gtrsim A_1 \gtrsim \cdots \supset A'$, which is unique, and must be finite by Proposition 3.3(vii) because A' is non-empty by Axioms I and II. Thus A' must be the last arc A_n of the sequence. □

Collecting the previous results we have:

Theorem 3.9. *The graph whose vertices, edges, and relations thereof were introduced in Definition 3.4 is connected, locally finite, and does not contain any loops or self-loops, therefore it is a tree, which in addition has no flat vertices.*

Such tree will be denoted by $T(\Omega; \mathcal{A})$.

4. Two-Sided Inverse

Theorem 4.1. *If T is a locally finite tree without flat vertices then there is a canonical tree isomorphism $\eta\colon T \to T(\Omega_T; \mathcal{A}_T)$.*

Proof. By Theorem 3.1, $(\Omega_T; \mathcal{A}_T)$ satisfies the Axioms. Let η map the edge $[v, v']$ of T to the edge $(\Omega(v, v'), \Omega(v', v))$ in $T(\Omega_T; \mathcal{A}_T)$; thus η is surjective. To show that it is injective, let $[w, w']$ be a different edge. Then v, v' and w, w'

belong to a chain of vertices, and, up to relabeling, one of $\Omega(v,v'),\Omega(w,w')$ is contained in the other, with strict inclusion because there are no flat vertices.

We may assume that v,v',w,w' belong to that chain in this order. Then $[v,v'],[w,w']$ are adjacent, i.e., $v'=w$, exactly if there is no boundary arc strictly contained in $\Omega(v,v')$ and strictly containing $\Omega(w,w')$, again by the fact that there are no flat vertices. Therefore η preserves adjacency and is a tree isomorphism. □

Remark 4.2. Flat vertices cannot be detected from any knowledge of the boundary, hence they either must be excluded if reconstruction of a tree up to isomorphisms is sought, or else their arbitrary insertion and removal must be allowed.

Theorem 4.3. *If the pair $(\Omega;\mathcal{A})$ satisfies the Axioms, then there is a canonical bijection $\iota\colon\Omega\to\Omega_{T(\Omega;\mathcal{A})}$ that induces a bijection $\mathcal{A}\to\mathcal{A}_{T(\Omega;\mathcal{A})}$.*

Proof. By Proposition 3.3(viii), each boundary point $\omega\in\Omega$ is the intersection of a strictly decreasing (finite or infinite) sequence in $\mathcal{A}$ starting at some A_0. By adding intermediate arcs if necessary, by Proposition 3.8 this can be completed to the maximal sequence $A_0\supsetneqq A_1\supsetneqq\cdots\supset\{\omega\}$, so $(A_0,\overline{A_0})\sim(A_1,\overline{A_1})\sim\cdots$ is a finite or infinite ray of edges in $T(\Omega;\mathcal{A})$ (according if $\{\omega\}$ is a terminal vertex or not). Let $\iota(\omega)\in\Omega_{T(\Omega;\mathcal{A})}$ be the equivalence class of this ray.

To show that ι is well defined, assume $A'_0\supsetneqq A'_1\supsetneqq\cdots\supset\{\omega\}$ is another maximal such sequence. By Axiom III every arc A_j for large enough j must be contained in an arc A'_k and conversely. By maximality, up to an index shift these sequences definitively coincide, hence the same happens for the corresponding maximal sequences of edges $(A_0,\overline{A_0})\sim(A_1,\overline{A_1})\sim\cdots$ and $(A'_0,\overline{A'_0})\sim(A'_1,\overline{A'_1})\sim\cdots$, that are consequently equivalent.

For every element of $\Omega_{T(\Omega;\mathcal{A})}$ let us choose a ray of edges $\{(A_j,\overline{A_j})\}_j$ in $T(\Omega;\mathcal{A})$ that represents it. By Proposition 3.8 we can assume that the sequence of arcs $A_0\supsetneqq A_1\supsetneqq\cdots$ is maximal. If it is infinite, then there exists $\omega\in\Omega$ such that $\bigcap_j A_j=\{\omega\}$ by Proposition 3.3(vii); else, the intersection is an arc that must be a singleton by Axiom IV. In either case, by definition, $\iota(\omega)$ is the equivalence class of the ray $\{(A_j,\overline{A_j})\}_j$, therefore ι is surjective.

The map ι is also injective, again by Axiom IV. Indeed, two maximal sequences of arcs $A_0\supsetneqq A_1\supsetneqq\cdots\supset\{\omega\}$ and $A'_0\supsetneqq A'_1\supsetneqq\cdots\supset\{\omega'\}$ are definitively disjoint if $\omega\neq\omega'$, so $A_j\cup A'_j\neq\Omega$ for large enough j, hence the sequences induce inequivalent rays $\{(A_j,\overline{A_j})\}_j,\{(A'_j,\overline{A'_j})\}_j$.

Redefine $\iota\colon\mathcal{A}\to\mathcal{A}_{T(\Omega;\mathcal{A})}$ as a map on arcs (rather than on points of Ω): for $A\in\mathcal{A}$ set $\iota(A)=\Omega(v,v')$ where v,v' are the unique adjacent vertices given by Corollary 3.6 that own $A,\overline{A}$, respectively. Every point in $\iota(A)$ has a representative ray of edges in $T(\Omega;\mathcal{A})$ that contains $[v,v'],[v',v'']$ in this order, where $v''\sim v'$, and by Proposition 3.8 corresponds to a maximal sequence of arcs $A_0=A\supsetneqq A_1=\overline{A}\supsetneqq\cdots$. Note that $\iota(\overline{A})=\Omega(v',v)=\overline{\Omega(v,v')}=\overline{\iota(A)}$.

If $\omega \in A$, there is a maximal sequence of arcs $A_0 = A \supsetneq A_1 \supsetneq \cdots \supset \{\omega\}$ obtained by inductively choosing, for each j, the arc A_{j+1} as the element containing ω of the sub-coarsest arc-partition of A_j; then $(A_0, \overline{A_0}) \sim (A_1, \overline{A_1}) \sim \cdots$ is a ray of edges in the equivalence class $\iota(\omega)$. On the other hand, this ray is also a representative of an equivalence class in $\iota(A)$, because it contains $[v, v'], [v', v'']$ in this order, where $v'' \sim v'$; therefore $\iota(\omega) \in \iota(A)$. Conversely, if $\omega \notin A$, then $\iota(\omega) \in \iota(\overline{A}) = \overline{\iota(A)}$, therefore the two definitions of ι are compatible. □

It follows from Theorems 4.1 and 4.3 that the process of obtaining a tree from $(\Omega; \mathcal{A})$ given by Theorem 3.9 is the two-sided inverse of the process of building the boundary of a (flat-free) tree and the family of its boundary arcs.

If Ω is equipped with the topology generated by $\mathcal{A}$, then we have shown:

Corollary 4.4. *The map ι of Theorem 4.3 is a homeomorphism, and $T(\Omega; \mathcal{A}) \cup \Omega$ may be endowed with the topology induced by the bijective extension $\tilde{\iota}\colon T(\Omega; \mathcal{A}) \cup \Omega \to T(\Omega; \mathcal{A}) \cup \Omega_{T(\Omega;\mathcal{A})}$ given by the identity map on $T(\Omega; \mathcal{A})$. Thus Ω is identified to the boundary of $T(\Omega; \mathcal{A})$ also in a topological sense, and, if one extends the map η of Theorem 4.1 to $\tilde{\eta}\colon T \cup \Omega_T \to T(\Omega_T; \mathcal{A}_T) \cup \Omega_T$ defining it as the identity on all non-terminal boundary points, then $\tilde{\eta}$ is a homeomorphism.*

Axiom V means that $\mathcal{A}$ is a base for the topology of Ω; Axiom IV yields its separation; by Axiom II each arc is a closed and open set, therefore Ω is totally disconnected; by Proposition 3.3(ix), Axiom VI ensures that Ω is compact.

5. Joins of Vertices in Terms of Arcs

Assume henceforth that the pair $(\Omega; \mathcal{A})$ satisfies the Axioms.

Definition 5.1. If v, v' are distinct vertices of $T(\Omega; \mathcal{A})$, let v^+ be the unique neighbor of v that is closest to v' and set $A_{vv'} = A_{vv^+} = \iota^{-1}(\Omega(v, v^+)) \in \mathcal{A}$.

Proposition 5.2. (i) *If v, v' are distinct vertices of $T(\Omega; \mathcal{A})$ then $A_{vv'}, A_{v'v}$ are the unique arcs belonging to v, v', respectively, whose union is Ω. Moreover $A_{vv'}$ is the unique arc belonging to v that is linked to one belonging to v^+.*

(ii) *Distinct vertices v, v' are adjacent if and only if $\overline{A_{vv'}} = A_{v'v}$. In any case, the arc $\overline{A_{vv'}}$ belongs to v^+ and equals A_{v^+v}.*

(iii) *If v, v', v'' are pairwise distinct vertices, then v' lies between v, v'' if and only if $A_{v'v} \neq A_{v'v''}$; either condition implies $A_{vv'} = A_{vv''} \supsetneq A_{v'v''}$ and $A_{v''v'} = A_{v''v} \supsetneq A_{v'v}$. Therefore, v, v', v'' are not aligned if and only if*

$$A_{vv'} = A_{vv''}, \qquad A_{v'v} = A_{v'v''}, \qquad A_{v''v} = A_{v''v'}.$$

Proof. (i) If v'^- is the neighbor of v' that is closest to v, then $\Omega(v^+, v) = \Omega(v', v) \subseteq \Omega(v', v'^-) = A_{v'v}$; since $A_{vv'} = \Omega(v, v^+) = \overline{\Omega(v^+, v)}$, we obtain $A_{vv'} \cup A_{v'v} = \Omega$. No other pair of arcs belonging to v, v', respectively, can have the same property because either at least one of the two vertices is a partition of at least three arcs, or both are terminal and not adjacent. The other claim follows from Corollary 3.6.

(ii) By Definition 3.7 the inclusion $\overline{A_{vv'}} \subseteq A_{v'v}$ is an equality if and only if v, v' are adjacent.

(iii) Assume v' lies strictly between v, v''. Then v^+ is also the neighbor of v closest to v'', so $A_{vv'} = A_{vv^+} = A_{vv''}$. Moreover $A_{vv''}, A_{v'v''}$ belong to the same descending chain of arcs in this strict order, thus $A_{vv''} \supsetneq A_{v'v''}$. The relations $A_{v''v'} = A_{v''v} \supsetneq A_{v'v}$ follow similarly. If v'^-, v'^+ are the neighbors of v' that are closest to v, v'' respectively, then $v'^- \neq v'^+$, hence $A_{v'v} = A_{v'v'^-} \neq A_{v'v'^+} = A_{v'v''}$.

Instead, if v' does not lie between v, v'' then $v'^- = v'^+$, hence $A_{v'v} = A_{v'v''}$. □

Corollary 5.3. *If the vertices v, v', v'' are not aligned, then their join is the unique vertex u that owns pairwise distinct arcs A, A', A'' containing $\overline{A_{vv'}}, \overline{A_{v'v}}, \overline{A_{v''v}}$ respectively (by Proposition 5.2(iii) this is overall a symmetric condition in v, v', v''). Explicitly,*

$$A = A_{uv}, \qquad A' = A_{uv'}, \qquad A'' = A_{uv''}. \tag{3}$$

Proof. If u is the join of v, v', v'', then it lies strictly between v, v', therefore $A_{uv} \supset \overline{A_{vu}} = \overline{A_{vv'}}$ and $A_{uv} \neq A_{uv'}$ by Proposition 5.2(i),(iii); permuting v, v', v'' and defining A, A', A'' by (3) we obtain the stated property of u.

By Proposition 5.2(i), the arcs belonging to v and respectively containing $\overline{A_{v'v}}, \overline{A_{v''v}}$ are $A_{vv'}, A_{vv''}$, but these two coincide by Proposition 5.2(iii) because v does not lie between v', v''. Therefore a vertex u with the stated property cannot equal any of v, v', v''. For such u, since $A \supset \overline{A_{vv'}}$ and since $A, A_{vv'}$ belong to u, v respectively, by the uniqueness in Proposition 5.2(i) we have $A_{uv} = A$ and $A_{vu} = A_{vv'}$; permuting v, v', v'' along with A, A', A'' we gather that $A_{uv'} = A'$ and $A_{uv''} = A''$, as well as $A_{v'u} = A_{v'v}$ and $A_{v''u} = A_{v''v}$. Thus, by Proposition 5.2(iii), $A \neq A'$ implies that u lies between v, v'; analogously it lies between v, v'' and between v', v'', whence it is the join of v, v', v''. □

6. Horospheres of Arcs

In the environment of trees, horospheres of vertices were introduced in [1], horospheres of edges in [4] and horospherical fiber bundles (for vertices and for edges) in the monograph [3], which includes a comprehensive study of related notions and further references. Horospherical fiber bundles of arcs may be introduced as follows.

Given two arcs $A \supsetneq A'$, we define their *distance* $d(A, A')$ as the length n of the chain $A = A_0 \gtrapprox \cdots \gtrapprox A_n = A'$ that connects them according to Proposition 3.8; of course $d(A, A) = 0$. (This notion is easily extended to a symmetric pseudo-distance between any two arcs, but we do not need it here.) Given $\omega \in \Omega$ let $\mathcal{A}_\omega = \{A \in \mathcal{A} \colon \omega \in A\}$. If $A, A' \in \mathcal{A}_\omega$, by Axiom V there is a common sub-arc $B \in \mathcal{A}_\omega$. We define the *horospherical offset* of A, A' with respect to ω by $d(A', B) - d(A, B)$ (independent of the choice of B). If it vanishes we say that A, A' are ω-*equivalent*: the resulting equivalence classes in $\mathcal{A}_\omega$, that may be called *arc-horospheres tangent at* ω, are in one-to-one correspondence with horospheres of vertices and horospheres of edges, since each arc belongs to exactly one vertex and exactly one edge.

The methods of Sect. 4 lead to a characterization of the spaces that arise as bases of horospherical fiber bundles of a tree and to a procedure for reconstructing a tree up to isomorphisms from each of its horospherical fiber bundles [3].

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