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CONCURRENT DONSKER-VARADHAN AND HYDRODYNAMICAL LARGE DEVIATIONS

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We consider the weakly asymmetric exclusion process on the d-dimensional torus. We prove a large deviations principle for the time averaged empirical density and current in the joint limit in which both the time interval and the number of particles diverge. This result is obtained both by analyzing the variational convergence, as the number of particles diverges, of the Donsker-Varadhan functional for the empirical process and by considering the large time behavior of the hydrodynamical rate function. The large deviations asymptotic of the time averaged current is then deduced by contraction principle. The structure of the minimizers of this variational problem corresponds to the possible occurrence of dynamical phase transitions.

1. Introduction. Stochastic lattice gases, that describe the evolution of interacting random particles on a lattice of mesh $1/N$, have been an instrumental tool in the development of non-equilibrium statistical mechanics [4, 15, 21]. Their macroscopic behavior, usually referred to as hydrodynamic scaling limit, is described as follows. Given a microscopic realization of the process, the empirical density $\boldsymbol{\pi}_N$ is defined by counting locally the average number of particles while the empirical current $\boldsymbol{J}_N$ is defined by counting the net flow of particles. By the local conservation of the number of particles, $\boldsymbol{\pi}_N$ and $\boldsymbol{J}_N$ satisfy the continuity equation. The content of the hydrodynamical limit is the law of large numbers for the pair $(\boldsymbol{\pi}_N, \boldsymbol{J}_N)$ in the limit $N \to \infty$. For driven-diffusive systems the limiting evolution is given by

$$\begin{cases} \partial_t \boldsymbol{\rho} + \nabla \cdot \boldsymbol{j} = 0, \\ \boldsymbol{j} = -D(\boldsymbol{\rho}) \nabla \boldsymbol{\rho} + \sigma(\boldsymbol{\rho}) E, \end{cases} \tag{1.1}$$

where $E = E(x)$ is the applied external field, D is the diffusion matrix, and σ is the mobility. In particular, the density profile $\boldsymbol{\rho} = \boldsymbol{\rho}(t,x)$ solves the non-linear driven diffusive equation

$$\partial_t \boldsymbol{\rho} + \nabla \cdot \big(\sigma(\boldsymbol{\rho}) E\big) = \nabla \cdot \big(D(\boldsymbol{\rho}) \nabla \boldsymbol{\rho}\big). \tag{1.2}$$

We refer to [15] for the details on the derivation of (1.2), while the scaling limit of the empirical current leading to (1.1) is discussed in [3] in the case of the symmetric exclusion process.

The large deviations with respect to the hydrodynamic limit in the time window $[0,T]$ are characterized by the rate function

$$A_T(\boldsymbol{\rho}, \boldsymbol{j}) = \int_0^T dt \int dx \, \frac{\big|\boldsymbol{j} + D(\boldsymbol{\rho}) \nabla \boldsymbol{\rho} - \sigma(\boldsymbol{\rho}) E\big|^2}{4\, \sigma(\boldsymbol{\rho})}, \tag{1.3}$$

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that is at the base of the Macroscopic Fluctuations Theory and it is widely used in non-equilibrium statistical mechanics [4]. We refer to [15, 17] for the derivation of this rate function when the empirical current is disregarded.

A significant problem is the behavior of the average of empirical current over the time interval $[0,T]$ in the limit when $N \to \infty$ and then $T \to \infty$. By the hydrodynamical large deviations principle and contraction principle, this amounts to analyze the behavior as $T \to \infty$ of the minimizers to (1.3) with the constraint $\frac{1}{T}\int_0^T dt\, \boldsymbol{j} = J$. This problem has been initially raised in [7] while in [1] it has been pointed out that the minimizers can exhibit a non-trivial time dependent behavior. In [2, 8] it has been then shown that this is actually the case for the weakly asymmetric exclusion process and the Kipnis-Marchioro-Presutti model (KMP) [16] where, for suitable value of the parameters, traveling waves are more convenient than constant profiles.

By [3, Proposition 4.1], the limiting value as $T \to \infty$ of the minimum to $T^{-1} A_T$ with the constraint $\frac{1}{T}\int_0^T dt\, \boldsymbol{j} = J$ exists. Denote it by $I^{(2)}(J)$. Varadhan [22] proposed the following representation for $I^{(2)}$

$$I^{(2)}(J) = \inf\left\{\int dP \int dx\, \frac{\big|\boldsymbol{j}(t) + D(\boldsymbol{\rho}(t))\nabla\boldsymbol{\rho}(t) - \sigma(\boldsymbol{\rho}(t))E\big|^2}{4\sigma(\boldsymbol{\rho}(t))};\ \int dP\, \boldsymbol{j}(t) = J\right\} \tag{1.4}$$

where the infimum is carried out over the probabilities P invariant by time translations on the set of paths $(\boldsymbol{\rho}, \boldsymbol{j})$ satisfying the continuity equation $\partial_t \boldsymbol{\rho} + \nabla\cdot\boldsymbol{j} = 0$. Note that $I^{(2)}$ is convex and that, by the stationarity of P, the actual value of t on the right hand side of (1.4) is irrelevant.

The purpose of the present analysis is to prove the validity of the representation (1.4) in the context of the weakly asymmetric exclusion process for which $D = 1$ and $\sigma(\rho) = \rho(1-\rho)$. This will be achieved both when the limit $T \to \infty$ is carried out after the hydrodynamic limit $N \to \infty$ and when the limits are carried out in the opposite order. In fact, the representation (1.4) will be deduced by the contraction principle from a large deviation result at the level of the empirical processes that we next introduce.

Consider first the case in which the limit $N \to \infty$ is taken after $T \to \infty$. By the Donsker-Varadhan result, see e.g. [11, 23], as $T \to \infty$ the empirical process associated to the weakly asymmetric exclusion process satisfies a large deviation principle in which the affine rate function is the relative entropy per unit of time with respect to the stationary process. By projecting this functional to the stationary probabilities on the empirical density and current, and analyzing its variational convergence as $N \to \infty$ we deduce the desired large deviation principle with affine rate function given by

$$\boldsymbol{I}(P) = \int P(d\boldsymbol{\rho}, d\boldsymbol{j}) \int dx\, \frac{\big|\boldsymbol{j}(t) + D(\boldsymbol{\rho}(t))\nabla\boldsymbol{\rho}(t) - \sigma(\boldsymbol{\rho}(t))E\big|^2}{4\,\sigma(\boldsymbol{\rho}(t))}\,. \tag{1.5}$$

The main ingredient in this derivation is, as for hydrodynamical large deviations, the validity of local equilibrium with probability super-exponentially close to one as $N \to \infty$. Observe that the rate function in (1.4) is obtained from (1.5) by contraction. The proof for the case in which the limit $T \to \infty$ is taken after $N \to \infty$ is achieved by lifting the hydrodynamical rate function (1.3) to the set of stationary probabilities on density and current and analyzing its variational convergence as $T \to \infty$.

As a corollary of the analysis here presented, we also deduce the “level two” large deviations relative to the family of random probability measures $\frac{1}{T}\int_0^T dt\, \delta_{\boldsymbol{\pi}_N(t)}$ in the joint limit $N, T \to \infty$. Letting $\imath_t(\boldsymbol{\rho}, \boldsymbol{j}) = \boldsymbol{\rho}(t)$, the corresponding rate function is

$$\mathfrak{I}(\wp) = \inf\left\{\boldsymbol{I}(P)\,;\ P\circ\imath_t^{-1} = \wp\right\} \tag{1.6}$$

Since $\mathfrak{I}(\wp) = 0$ if and only if $\wp$ is a stationary measure for the flow associated to the hydrodynamic equation (1.2), this large deviations statement implies the hydrostatic limit for the weakly asymmetric exclusion process: in the limit $N \to \infty$ the empirical density constructed by sampling the particles according to the stationary measure converges to the unique stationary solution to (1.2).

We expect the large deviations principle stated in Theorem 2.1 and Corollaries 2.3, 2.4 to hold in great generality since the proof does not rely on particular features of the WASEP with periodic boundary conditions. In particular, an analogous result should be in force for zero-range processes (under suitable conditions on the rates), the KMP model (where substantial technical difficulties are expected), and dynamics in contact with boundary reservoirs.

The existence or not of a non trivial time-dependent behavior is a difficult task to be detected. For example, in the zero-range dynamics on a torus it is not simple to establish if and when this is the case. In the case of boundary driven zero-range dynamics we can instead rule out a time dependent behavior and the commutation of the limits can be verified by direct microscopic computations.

The main results of the article assert that we can exchange the order of the limit $T \to \infty$ and $N \to \infty$ in the large deviations principle for the current time average. This result does not imply that one obtains the same limit by taking N and $T \to +\infty$ simultaneously. This is an open and interesting question.

2. Notation and Results.

Microscopic dynamics. Denote by $\mathbb{T}^d = [0,1)^d$ the d-dimensional torus of length 1 and let dx be the corresponding Haar measure. Fix $N \geq 1$, and let $\mathbb{T}^d_N$ the discretization of $\mathbb{T}^d$: $\mathbb{T}^d_N = \mathbb{T}^d \cap (N^{-1}\mathbb{Z})^d$. The elements of $\mathbb{T}^d$ and $\mathbb{T}^d_N$ are represented by x and y. Let $\mathbb{B}_N$ be the set of ordered, nearest-neighbor pairs (x,y) in $\mathbb{T}^d_N$.

Denote by $\Sigma_N := \{0,1\}^{\mathbb{T}^d_N}$ the space of configurations. Elements of Σ_N are represented by η, so that $\eta_x = 1$, resp. 0, if site x is occupied, resp. vacant, for the configuration η. Fix E in $C^1(\mathbb{T}^d;\mathbb{R}^d)$, the space of continuously differentiable vector fields defined on $\mathbb{T}^d$. In some statements we assume that E is *orthogonally decomposable*: there are $U \in C^2(\mathbb{T}^d)$ and $\tilde{E} \in C^1(\mathbb{T}^d;\mathbb{R}^d)$ with vanishing divergence, $\nabla \cdot \tilde{E} = 0$, satisfying the pointwise orthogonality $\nabla U(x) \cdot \tilde{E}(x) = 0$, $x \in \mathbb{T}^d$, such that $E = -\nabla U + \tilde{E}$. We scrutinise this condition in Remark 2.2 below.

The weakly asymmetric exclusion process (WASEP) with external field E is the Markov process on Σ_N whose generator L_N acts on functions $f\colon \Sigma_N \to \mathbb{R}$ as

$$(L_N f)(\eta) \;=\; N^2 \sum_{(x,y)\in\mathbb{B}_N} \eta_x \,[1-\eta_y]\, e^{(1/2)\, E_N(x,y)} \big[f(\sigma^{x,y}\eta) - f(\eta) \big] \;. \tag{2.1}$$

In this formula, the configuration $\sigma^{x,y}\eta$ is obtained from η by exchanging the occupation variables η_x, η_y:

$$(\sigma^{x,y}\eta)_z \;:=\; \begin{cases} \eta_y & \text{if } \ z = x\,, \\ \eta_x & \text{if } \ z = y\,, \\ \eta_z & \text{if } \ z \neq x,y\,, \end{cases}$$

and $E_N(x,y)$ represents the line integral of E along the oriented segment from x to y:

$$E_N(x,y) \;=\; \int_x^y E \cdot d\ell \;=\; \int_0^1 E(x + r\,[y-x]) \cdot [y-x]\, dr\;, \tag{2.2}$$

where $a \cdot b$ is the inner product in $\mathbb{R}^d$. Note that $E_N\colon \mathbb{B}_N \to \mathbb{R}$ is antisymmetric and that E_N is of order $1/N$. It depends on N only because it is defined on $\mathbb{B}_N$.

Denote by $\Sigma_{N,K} = \{\eta \in \Sigma_N : \sum_{x\in\mathbb{T}_N^d} \eta_x = K\}$, $K = 0, \dots, N^d$, the set of configurations with K particles. The Markov process with generator L_N is irreducible in the finite state space $\Sigma_{N,K}$. It has therefore a unique stationary probability measure, denoted by $\mu_{N,K}$.

If the external field is orthogonally decomposable, the measure $\mu_{N,K}$ is the canonical measure of a non-homogeneous product measure provided the external field is suitably discretised. Apart from this special case, the stationary state $\mu_{N,K}$ is not explicitly known. This is not an obstruction, however, as we consider large deviations of time-averages in which the initial condition is not relevant.

Hereafter, $\mathcal{R}$ represents either $\mathbb{R}_+$ or $\mathbb{R}$. Denote by $D(\mathcal{R}, \Sigma_N)$, the set of right-continuous functions with left-limits from $\mathcal{R}$ to Σ_N, endowed with the Skorohod topology and the corresponding Borel σ-algebra. Elements of $D(\mathcal{R}, \Sigma_N)$ are represented by $\boldsymbol{\eta}$.

For a probability measure ν on Σ_N, denote by $\mathbb{P}_\nu^N$ the probability measure on $D(\mathbb{R}_+, \Sigma_N)$ induced by the Markovian dynamics associated to the generator L_N starting from ν. When the measure ν is concentrated on a configuration $\eta \in \Sigma_N$, $\nu = \delta_\eta$, we write $\mathbb{P}_\eta^N$ instead of $\mathbb{P}_{\delta_\eta}^N$. For $K = 0, \dots, N^d$, the stationary processes associated to the WASEP dynamics with K particles is denoted by $\mathbb{P}_{\mu_{N,K}}^N$ that we regard as a probability measure on $D(\mathbb{R}, \Sigma_{N,K})$ invariant with respect to time-translations. Expectation with respect to $\mathbb{P}_{\mu_{N,K}}^N$ is represented by $\mathbb{E}_{\mu_{N,K}}^N$.

Empirical density. Let $\mathcal{M}_+(\mathbb{T}^d)$ be the set of positive measures on $\mathbb{T}^d$ with total mass bounded by 1, endowed with the weak topology and the corresponding Borel σ-algebra. Let also $\mathcal{M}_m(\mathbb{T}^d)$, $m \in [0, 1]$, be the closed subset of $\mathcal{M}_+(\mathbb{T}^d)$ given by the measures whose total mass is equal to m.

The *empirical density* is the map $\pi_N \colon \Sigma_N \to \mathcal{M}_+(\mathbb{T}^d)$ defined by

$$\pi_N(\eta) \;:=\; \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \eta_x \, \delta_x \,, \tag{2.3}$$

where δ_x, $x \in \mathbb{T}^d$, is the point mass at x. For a continuous function $f \colon \mathbb{T}^d \to \mathbb{R}$ and a measure ν in $\mathcal{M}_+(\mathbb{T}^d)$, we represent by $\langle \nu \,,\, f\rangle$ the integral of f with respect to ν so that

$$\langle \pi_N \,,\, f\rangle \;=\; \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \eta_x \, f(x) \,. \tag{2.4}$$

We also call empirical density the map $\boldsymbol{\pi}_N \colon D(\mathcal{R}, \Sigma_N) \to D(\mathcal{R}, \mathcal{M}_+(\mathbb{T}^d))$ defined by

$$[\,\boldsymbol{\pi}_N(\boldsymbol{\eta})\,]\,(t) \;:=\; \pi_N(\boldsymbol{\eta}(t)) \;=\; \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \boldsymbol{\eta}_x(t)\, \delta_x \,, \quad t \in \mathcal{R} \,. \tag{2.5}$$

Empirical current. For an oriented bond $(x, y) \in \mathbb{B}_N$ and $s < t$, let $\mathcal{N}_{(s,t]}^{x,y}(\boldsymbol{\eta})$ be the number of jumps from x to y in the time interval $(s, t]$ of the path $\boldsymbol{\eta} \in D(\mathcal{R}, \Sigma_N)$:

$$\mathcal{N}_{(s,t]}^{x,y}(\boldsymbol{\eta}) \;=\; \sum_{s<r\le t} \boldsymbol{\eta}_x(r-)\,[1 - \boldsymbol{\eta}_y(r-)]\, \mathbf{1}\{\boldsymbol{\eta}(r) = \sigma^{x,y}\boldsymbol{\eta}(r-)\} \,. \tag{2.6}$$

Fix a trajectory $\boldsymbol{\eta} \in D(\mathcal{R}, \Sigma_N)$, and denote by $C(\mathbb{T}^d; \mathbb{R}^d)$ the space of continuous vector fields on $\mathbb{T}^d$. If $\boldsymbol{\eta}$ is a trajectory compatible with the WASEP dynamics, i.e. such that for each jump time t we have $\boldsymbol{\eta}(t) = \sigma^{x,y}\boldsymbol{\eta}(t-)$ for some $(x, y) \in \mathbb{B}_N$, we define the *integrated empirical current* $\boldsymbol{J}_N(\boldsymbol{\eta})$ as follows. Let $[\,\boldsymbol{J}_N(\boldsymbol{\eta})\,]\,(0) = 0$, and, for $t > 0$, let $[\,\boldsymbol{J}_N(\boldsymbol{\eta})\,]\,(t)$ be the linear functional on $C(\mathbb{T}^d; \mathbb{R}^d)$ defined by

$$\langle \boldsymbol{J}_N(\boldsymbol{\eta})\,(t) \,,\, F\rangle \;:=\; \frac{1}{N^d} \sum_{(x,y)\in\mathbb{B}_N} \mathcal{N}_{(0,t]}^{x,y}(\boldsymbol{\eta}) \int_x^y F \cdot d\ell \,, \quad F \in C(\mathbb{T}^d; \mathbb{R}^d) \,. \tag{2.7}$$

For $t < 0$, we replace in the previous formula $\mathcal{N}^{x,y}_{(0,t]}(\boldsymbol{\eta})$ by $-\mathcal{N}^{x,y}_{(t,0]}(\boldsymbol{\eta})$. If $F = (F_1, \dots, F_d)$, then for $t \geq 0$

$$\langle \boldsymbol{J}_N(\boldsymbol{\eta})\,(t)\,,\, F\rangle = \frac{1}{N^d} \sum_{j=1}^{d} \sum_{x\in\mathbb{T}^d_N} \frac{1}{N} \Big\{ \mathcal{N}^{x,x+\mathfrak{e}_j}_{(0,t]}(\boldsymbol{\eta}) - \mathcal{N}^{x+\mathfrak{e}_j,x}_{(0,t]}(\boldsymbol{\eta}) \Big\} \int_0^1 F_j(x+r\mathfrak{e}_j)\, dr\,,$$

where, $\mathfrak{e}_j = e_j/N$ and $\{e_1, \dots, e_d\}$ represents the canonical basis of $\mathbb{R}^d$.

Discrete vector fields. For technical issues, we need to define the empirical current also for paths $\boldsymbol{\eta}$ not coming from the WASEP dynamics. Consider an arbitrary path $\boldsymbol{\eta} \in D\big(\mathcal{R}; \Sigma_{N,K}\big)$. If for some $t \in \mathcal{R}$ it happens $\boldsymbol{\eta}(t) \neq \boldsymbol{\eta}(t-)$ some of the particles in the configuration $\boldsymbol{\eta}(t-)$ have rearranged themselves to construct the configuration $\boldsymbol{\eta}(t)$. The definition of the empirical current requires to decide the actual path taken by those particles.

A *discrete vector field* W is an antisymmetric function $W\colon \mathbb{B}_N \to \mathbb{R}$. The *discrete divergence* of the discrete vector field W is the function $\nabla_N \cdot W\colon \mathbb{T}^d_N \to \mathbb{R}$ defined by

$$(\nabla_N \cdot W)\,(x) := \sum_{y\colon\, (x,y)\in\mathbb{B}_N} W(x,y)\,.$$

Fix $0 \leq K \leq N^d$, and consider two configurations η, $\xi \in \Sigma_{N,K}$. Let $W_{\eta,\xi}$ be a discrete vector field which solves

$$(\nabla_N \cdot W_{\eta,\xi})\,(x) \;=\; \eta_x \,-\, \xi_x\,, \quad x \in \mathbb{T}^d_N\,. \tag{2.8}$$

Such a discrete vector field $W_{\eta,\xi}$ always exists. The configuration $\zeta = \eta - \xi$ belongs to $\{-1,0,1\}^{\mathbb{T}^d_N}$ and $\sum_{x\in\mathbb{T}^d_N} \zeta_x = 0$. To define $W_{\eta,\xi}$, one just needs to create nearest-neighbor flows from each $x \in \mathbb{T}^d_N$ such that $\zeta_x = 1$ to each $x' \in \mathbb{T}^d_N$ such that $\zeta_{x'} = -1$, and add all these flows.

Regarding η and ξ as positive measures on $\mathbb{T}^d_N$, both of mass K, there exists a constant C_0 such that for all $N \geq 1$, $0 \leq K \leq N^d$, η, $\xi \in \Sigma_{N,K}$ there exists a discrete vector field $W_{\eta,\xi}$ such that

$$\sum_{(x,y)\in\mathbb{B}_N} \big| W_{\eta,\xi}(x,y) \big| \;\leq\; C_0\, N^{d+1}\,, \tag{2.9}$$

Indeed, in the construction of $W_{\eta,\xi}$ we displace at most N^d particles along at most N sites. Actually, this bound is a particular case of the discrete Beckmann's problem, see e.g. [20].

Of course, equation (2.8) admits more than one solution and we do not claim that there is only one satisfying the previous bound. It turns out, however, that the scaling limit of the empirical current does not depend on the specific selection among those fulfilling (2.9). Hence, in the sequel and without further mention, we assume that, for each pair $(\eta,\xi) \in \Sigma^2_{N,K}$, a discrete vector field $W_{\eta,\xi}$ which matches (2.9) has been chosen.

When $\xi = \sigma^{x,y}\eta$ for some $(x,y) \in \mathbb{B}_N$ and $\eta_x = 1$, $\eta_y = 0$, we define $W_{\eta,\xi}$ as

$$W_{\eta,\xi}(x',y') \;=\; \begin{cases} 1 & \text{if } (x',y') = (x,y)\,, \\ -1 & \text{if } (x',y') = (y,x)\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.10}$$

This discrete vector field clearly satisfies (2.8) and (2.9).

Integrated currents for generic paths. Fix a generic path $\boldsymbol{\eta} \in D(\mathcal{R}; \Sigma_{N,K})$ and denote by τ_i its jump times. Let W_i be the discrete vector field given by

$$W_i \;=\; W_{\boldsymbol{\eta}(\tau_i-),\boldsymbol{\eta}(\tau_i)}\,. \tag{2.11}$$

For $t > 0$, the integrated empirical current of the path $\boldsymbol{\eta}$ is then defined by

$$\langle \boldsymbol{J}_N(\boldsymbol{\eta})(t), F\rangle = \frac{1}{2N^d} \sum_{i\colon \tau_i\in(0,t]} \sum_{(x,y)\in\mathbb{B}_N} W_i(x,y)\, F_N(x,y) \tag{2.12}$$

where $F_N\colon \mathbb{B}_N \to \mathbb{R}$ is the discrete vector field constructed from $F \in C(\mathbb{T}^d;\mathbb{R}^d)$ by (2.2). The factor $1/2$ has been introduced to avoid computing twice each jump. In view of (2.10), for trajectories $\boldsymbol{\eta}$ coming from the WASEP dynamics, this definition corresponds to the original one, given in (2.7).

As before, $\boldsymbol{J}_N(\boldsymbol{\eta})(0) = 0$ and for $t < 0$

$$\langle \boldsymbol{J}_N(\boldsymbol{\eta})(t), F\rangle = -\frac{1}{2N^d} \sum_{i\colon \tau_i\in[t,0)} \sum_{(x,y)\in\mathbb{B}_N} W_i(x,y) F_N(x,y).$$

Sobolev spaces. Let $h_n \in L^2(\mathbb{T}^d)$, $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$, be the orthonormal basis of $L^2(\mathbb{T}^d)$ given by $h_n(x) = \exp\{2\pi i(n\cdot x)\}$.

Denote by $\langle\cdot,\cdot\rangle$ the inner product in $L^2(\mathbb{T}^d)$, and by $\mathfrak{f}\colon \mathbb{Z}^d \to \mathbb{C}$ the Fourier coefficients of the function f in $L^2(\mathbb{T}^d)$:

$$\mathfrak{f}(n) := \langle f, h_n\rangle \;=\; \int_{\mathbb{T}^d} f(x)\, \overline{h_n(x)}\, dx\,, \quad n \in \mathbb{Z}^d\,,$$

where $\overline{z}$ represents the complex conjugate of $z \in \mathbb{C}$. Hence,

$$f \;=\; \sum_{n\in\mathbb{Z}^d} \mathfrak{f}(n)\, h_n$$

Denote by $\mathcal{H}_p$, $p \in \mathbb{R}$, the Hilbert space obtained by completing the space of smooth complex-valued functions on $\mathbb{T}^d$ endowed with the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_p$ defined by

$$\langle f\,,\, g\rangle_p \;=\; \langle (1-\Delta)^p f\,,\, g\rangle\,, \tag{2.13}$$

where Δ represents the Laplacian. An elementary computation yields that

$$\langle f\,,\, g\rangle_p \;:=\; \sum_{n\in\mathbb{Z}^d} \big(1 + 4\pi^2 |n|^2\big)^p\, \mathfrak{f}(n)\, \overline{\mathfrak{g}(n)}\,, \tag{2.14}$$

where $|n|^2 = |(n_1,\dots,n_d)|^2 = \sum_{1\le j\le d} n_j^2$. Denote by $\|\cdot\|_p$ the norm of $\mathcal{H}_p$: $\|f\|_p^2 = \langle f\,,\, f\rangle_p$. It is well known that $\mathcal{H}_{-p}$, is the dual of $\mathcal{H}_p$ relatively to the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ defined by

$$\langle j\,,\, g\rangle \;:=\; \sum_{n\in\mathbb{Z}^d} \mathfrak{j}(n)\, \overline{\mathfrak{g}(n)}\,, \quad j \in \mathcal{H}_{-p}\,, g \in \mathcal{H}_p\,. \tag{2.15}$$

Moreover, it follows from the definition that $\mathcal{H}_p \subset \mathcal{H}_{p'}$ for $p > p'$. Let $\mathcal{H}_p^d = \mathcal{H}_p \times \cdots \times \mathcal{H}_p$ that we consider endowed with the strong topology. We represent below by $\langle J\,,\, H\rangle$ the value at $H \in \mathcal{H}_p^d$ of a bounded linear functional J defined on $\mathcal{H}_p^d$.

Fix $p > d/2$. By the Sobolev embedding, $\mathcal{H}_p^d \subset C(\mathbb{T}^d;\mathbb{R}^d)$. In particular, by the definition of the empirical current, for each $t \in \mathcal{R}$, the functional $[\boldsymbol{J}_N(\boldsymbol{\eta})]\,(t)$ is bounded on $\mathcal{H}_p^d$. Therefore, for each $\boldsymbol{\eta} \in D(\mathcal{R};\Sigma_{N,K})$ and $t \in \mathcal{R}$ it belongs to dual of $\mathcal{H}_p^d$, that is, to $\mathcal{H}_{-p}^d$. Furthermore, it is easy to check that $\boldsymbol{J}_N$ is right-continuous and has left-limits. Hence, the empirical current $\boldsymbol{J}_N$ is a map from $D(\mathcal{R},\Sigma_{N,K})$ to $D(\mathcal{R},\mathcal{H}_{-p}^d)$.

Fix $p \ge 0$ and let J be a bounded linear functional on $\mathcal{H}_p^d$, i.e. $J \in \mathcal{H}_{-p}^d$. By Riesz representation theorem, there exists $G = G(J)$ in $\mathcal{H}_p^d$ such that $J(F) = \langle F\,,\, G\rangle_p$ for all F in $\mathcal{H}_p^d$, and

$\|J\|_{-p}^2 = \langle G\,,\,G\rangle_p$. Letting $G=(G_1,\dots,G_d)$ and defining $J_k(n)$ as $\{1+4\pi^2|n|^2\}^p\,\mathfrak{G}_k(n)$, where $\mathfrak{G}_k(n)$, $n\in\mathbb{Z}^d$, are the Fourier coefficients of G_k, $k=1,\dots,d$, by (2.14) we deduce that

$$J(F) \,=\, \sum_{k=1}^d \sum_{n\in\mathbb{Z}^d} \mathfrak{F}_k(n)\,\overline{J_k(n)}\,, \qquad F=(F_1,\dots,F_d)\,.$$

For $n\in\mathbb{Z}^d$, $j=1,\dots,d$, let $H^{j,n}\colon \mathbb{T}^d\to\mathbb{C}^d$ be the vector fields given by $H^{j,n} = (H^{j,n}_1,\dots,H^{j,n}_d)$ where $H^{j,n}_k = \delta_{j,k}\,h_n$. Taking $F=H^{j,n}$ in the previous displayed identity, one concludes that $\overline{J_k(n)} = J(H^{k,n})$, so that

$$J(F) \,=\, \sum_{k=1}^d \sum_{n\in\mathbb{Z}^d} \mathfrak{F}_k(n)\,J(H^{k,n})\,, \qquad \|J\|_{-p}^2 \,=\, \sum_{k=1}^d \sum_{n\in\mathbb{Z}^d} \frac{\big|\,J(H^{k,n})\,\big|^2}{\big(1+4\pi^2|n|^2\big)^p}\,. \tag{2.16}$$

Continuity equation. It follows from the conservation of mass that for each $x\in\mathbb{T}^d_N$, $t>0$, and each path $\boldsymbol{\eta}$ compatible with the WASEP dynamics

$$\boldsymbol{\eta}_x(t) \,-\, \boldsymbol{\eta}_x(0) \,=\, \sum_{j=1}^d \Big\{ \mathcal{N}^{x+\mathfrak{e}_j,x}_{(0,t]}(\boldsymbol{\eta}) \,-\, \mathcal{N}^{x,x+\mathfrak{e}_j}_{(0,t]}(\boldsymbol{\eta}) \,+\, \mathcal{N}^{x-\mathfrak{e}_j,x}_{(0,t]}(\boldsymbol{\eta}) \,-\, \mathcal{N}^{x,x-\mathfrak{e}_j}_{(0,t]}(\boldsymbol{\eta}) \Big\}\,.$$

Let f be a function in $C^\infty(\mathbb{T}^d)$ and recall the notation introduced in (2.4). Multiply the previous equation by $f(x)$, sum over $x\in\mathbb{T}^d_N$ and divide by N^d to get, after a summation by parts, that

$$\begin{aligned}&\langle [\,\boldsymbol{\pi}_N(\boldsymbol{\eta})\,]\,(t)\,,\,f\rangle \,-\, \langle [\,\boldsymbol{\pi}_N(\boldsymbol{\eta})\,]\,(0)\,,\,f\rangle \\ &\quad =\, \frac{1}{N^d}\sum_{j=1}^d \sum_{x\in\mathbb{T}^d_N} \big\{\,f(x+\mathfrak{e}_j)-f(x)\,\big\}\,\big\{\,\mathcal{N}^{x,x+\mathfrak{e}_j}_{(0,t]}(\boldsymbol{\eta}) \,-\, \mathcal{N}^{x+\mathfrak{e}_j,x}_{(0,t]}(\boldsymbol{\eta})\,\big\}\,.\end{aligned}$$

Since $f(x+\mathfrak{e}_j)-f(x) = \int_x^{x+\mathfrak{e}_j}(\nabla f)\cdot d\ell$, in view of the definition of the map $\boldsymbol{J}_N$,

$$\langle [\,\boldsymbol{\pi}_N(\boldsymbol{\eta})\,]\,(t)\,,\,f\rangle \,-\, \langle [\,\boldsymbol{\pi}_N(\boldsymbol{\eta})\,]\,(0)\,,\,f\rangle \,=\, \langle \boldsymbol{J}_N(\boldsymbol{\eta})\,(t)\,,\,\nabla f\rangle\,. \tag{2.17}$$

This is the microscopic version of the continuity equation. Observe that for paths $\boldsymbol{\eta}$ not coming from the WASEP dynamics, the definition of the integrated empirical current $\boldsymbol{J}_N$ has been engineered so that (2.17) always holds.

Hydrodynamical limit. Let $(\eta^N : N\ge 1)$, $\eta^N\in\Sigma_N$, be a sequence of configurations associated to a density profile $\rho\colon \mathbb{T}^d\to[0,1]$ in the sense that for each continuous function $f\colon\mathbb{T}^d\to\mathbb{R}$,

$$\lim_{N\to\infty}\langle \pi_N(\eta^N), f\rangle \,=\, \lim_{N\to\infty}\frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N} f(x)\,\eta^N_x \,=\, \int_{\mathbb{T}^d} f(x)\,\rho(x)\,dx\,.$$

It is proven in [3] that, as $N\to\infty$, $\big(\boldsymbol{\pi}_N,\boldsymbol{J}_N\big)$ converges in $\mathbb{P}^N_{\eta^N}$-probability to

$$\Big(\boldsymbol{\rho}(t,\cdot)\,dx\,,\,\int_0^t ds\,\boldsymbol{j}(s,\,\cdot\,)\Big)_{t\ge 0}\,,$$

where $(\boldsymbol{\rho},\boldsymbol{j})$ is the unique weak solution to the Cauchy problem

$$\begin{cases}\partial_t\boldsymbol{\rho} \,+\, \nabla\cdot\boldsymbol{j} \,=\, 0\,,\\ \boldsymbol{j} \,=\, -\nabla\boldsymbol{\rho} \,+\, \sigma(\boldsymbol{\rho})\,E\,,\\ \boldsymbol{\rho}(0,\,\cdot\,) \,=\, \rho(\,\cdot\,)\,,\end{cases} \tag{2.18}$$

in which $\sigma\colon [0,1]\to\mathbb{R}_+$, given by $\sigma(\rho)=\rho(1-\rho)$, is the mobility of the exclusion process.

If one considers only the empirical density and disregards the empirical current, the above result has been proven in [10, 17], see also [15, Ch. 10]. The case in which one considers also the empirical current is discussed in [3] for the SEP. The topology on the set of currents used in [3] is different from the one employed in the present paper. Actually, the proof of the tightness of the empirical current in [3] is incomplete but the arguments presented below in Section 4 fix this issue (in the topology here introduced). In view of the super-exponential estimates in [17] or [15, Ch. 10], the hydrodynamical limit extends directly to the WASEP dynamics.

Empirical process. Given $K=0,\dots,N^d$, $T>0$, and $\boldsymbol{\eta}\in D(\mathbb{R}_+,\Sigma_{N,K})$, let $\boldsymbol{\eta}^T\in D(\mathbb{R},\Sigma_{N,K})$ be the T-periodization of the trajectory $\boldsymbol{\eta}$, defined by

$$\boldsymbol{\eta}^T(t)\;=\;\boldsymbol{\eta}\Big(t-\Big\lfloor\frac{t}{T}\Big\rfloor T\Big)\,,$$

where $\lfloor a\rfloor$ represents the largest integer less than or equal to $a\in\mathbb{R}$. A probability measure on $D(\mathbb{R},\Sigma_{N,K})$ is stationary if it is invariant with respect to the group of time-translations $(\vartheta_t : t\in\mathbb{R})$, defined by

$$(\vartheta_t\boldsymbol{\eta})(s)\;:=\;\boldsymbol{\eta}(s-t)\,,\quad s\in\mathbb{R}\,.$$

Denote by $\mathcal{P}^{N,K}_{\text{stat}}$ the set of stationary probability measures on $D(\mathbb{R},\Sigma_{N,K})$ that we consider endowed with the topology induced by the weak convergence and the corresponding Borel σ-algebra. For a trajectory $\boldsymbol{\eta}\in D(\mathbb{R}_+,\Sigma_{N,K})$ and $T>0$, the *empirical process* $R_T(\boldsymbol{\eta})$ is the element in $\mathcal{P}^{N,K}_{\text{stat}}$ given by

$$R_T(\boldsymbol{\eta})\;:=\;\frac{1}{T}\int_0^T\delta_{\vartheta_t\boldsymbol{\eta}^T}\,dt\,. \tag{2.19}$$

Observe that the path $\boldsymbol{\eta}^T$ is not necessarily compatible with the WASEP dynamics; indeed at the times that are integral multiples of T there is a jump that is not - in general - coming from the WASEP dynamics. This is the reason for which we needed to define the empirical current for generic paths. However, in view of the bound (2.9), there exists a finite constant C_0, depending only on the space dimension d, such that

$$\sup_{t\in[0,T]}\Big|\,\langle\boldsymbol{J}_N(\boldsymbol{\eta}^T)(t)-\boldsymbol{J}_N(\boldsymbol{\eta})(t)\,,\,F\rangle\,\Big|\;\le\;C_0\,\|F\|_\infty \tag{2.20}$$

for all vector field F in $C(\mathbb{T}^d,\mathbb{R}^d)$. Indeed, by the definition of $\boldsymbol{\eta}^T$ and by (2.12), the left-hand side is equal to

$$\Big|\,\langle\boldsymbol{J}_N(\boldsymbol{\eta}^T)(T)-\boldsymbol{J}_N(\boldsymbol{\eta})(T)\,,\,F\rangle\,\Big|\;=\;\Big|\,\frac{1}{2N^d}\sum_{(x,y)\in\mathbb{B}_N}W(x,y)\,F_N(x,y)\,\Big|\,,$$

where $W=W_{\boldsymbol{\eta}(T-),\boldsymbol{\eta}(0)}$. Hence, by (2.9), the right-hand side of the previous identity is bounded by $C_0\|F\|_\infty$, as claimed in (2.20).

Fix $p>(d+2)/2$, and let $\mathcal{M}=\mathcal{M}_+(\mathbb{T}^d)\times\mathcal{H}^d_{-p}$. Denote by $\mathcal{S}$ the closed subset of $D(\mathbb{R},\mathcal{M})\equiv D\big(\mathbb{R};\mathcal{M}_+(\mathbb{T}^d)\big)\times D\big(\mathbb{R};\mathcal{H}^d_{-p}\big)$ given by the pairs $(\boldsymbol{\pi},\boldsymbol{J})$ which satisfy the continuity equation in the sense that for each $s<t$ and f in $C^\infty(\mathbb{T}^d)$,

$$\int_s^t ds_1\int_{s_1}^t ds_2\,\Big\{\langle\boldsymbol{\pi}(s_2)\,,\,f\rangle\;-\;\langle\boldsymbol{\pi}(s_1)\,,\,f\rangle\;-\;\langle\boldsymbol{J}(s_2)-\boldsymbol{J}(s_1)\,,\,\nabla f\rangle\Big\}\;=\;0\,. \tag{2.21}$$

We consider $\mathcal{S}$ endowed with the relative topology and the associated Borel σ-algebra. Denote by $\vartheta_t \colon \mathcal{S} \to \mathcal{S}$, $t \in \mathbb{R}$, the time-translation defined by

$$\vartheta_t(\boldsymbol{\pi}, \boldsymbol{J}) \;=\; (\vartheta_t \boldsymbol{\pi}, \vartheta_t \boldsymbol{J}) \,, \tag{2.22}$$

where $(\vartheta_t \boldsymbol{\pi})(s) = \boldsymbol{\pi}(s-t)$, and $(\vartheta_t \boldsymbol{J})(s) = \boldsymbol{J}(s-t) - \boldsymbol{J}(-t)$, $s \in \mathbb{R}$. Note that the time translations defined on $D(\mathbb{R}; \Sigma_{N,K})$ and $\mathcal{S}$ are compatible in the sense that $\vartheta_t \circ (\boldsymbol{\pi}_N, \boldsymbol{J}_N) = (\boldsymbol{\pi}_N, \boldsymbol{J}_N) \circ \vartheta_t$.

Given a path $(\boldsymbol{\pi}, \boldsymbol{J})$ in $\mathcal{S}$ and $T > 0$, denote by $(\boldsymbol{\pi}^T, \boldsymbol{J}^T)$ its T-periodization:

$$\begin{aligned} \boldsymbol{\pi}^T(t) \;&=\; \boldsymbol{\pi}\Big(t - \Big\lfloor \frac{t}{T} \Big\rfloor\Big) \,, \\ \boldsymbol{J}^T(t) \;&=\; \boldsymbol{J}\big(t - \lfloor \tfrac{t}{T} \rfloor\big) \;+\; \lfloor \tfrac{t}{T} \rfloor \big(\boldsymbol{J}(T) + A\big) \end{aligned} \tag{2.23}$$

where A is an element of $\mathcal{H}^d_{-p}$ satisfying $\nabla \cdot A \,+\, \boldsymbol{\pi}(T) - \boldsymbol{\pi}(0) = 0$. A straightforward computation shows that $(\boldsymbol{\pi}^T, \boldsymbol{J}^T)$ satisfies the continuity equation (2.21). We hereafter assume that the choice of A in the previous definition and of the discrete vector field W in (2.8) are compatible in the sense that whenever $\boldsymbol{\pi}(0) = N^{-d} \sum_{x \in \mathbb{T}^d_N} \eta_x$ and $\boldsymbol{\pi}(T) = N^{-d} \sum_{x \in \mathbb{T}^d_N} \xi_x$ for some $\eta, \xi \in \Sigma_N$ then

$$A = \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} \sum_{k=1}^{d} W_{\eta,\xi}(x, x + \mathfrak{e}_k)\, e_k \, \mathfrak{H}^1_{\llcorner [x, x+\mathfrak{e}_k]} \,,$$

where $\mathfrak{H}^1_{\llcorner [x, x+\mathfrak{e}_k]}$ is the restriction of the one-dimensional Hausdorff measure to the interval $[x, x+\mathfrak{e}_k]$. This compatibility implies that $(\,\boldsymbol{\pi}_N(\boldsymbol{\eta}^T)\,,\, \boldsymbol{J}_N(\boldsymbol{\eta}^T)\,) = (\boldsymbol{\pi}_N(\boldsymbol{\eta})^T, \boldsymbol{J}_N(\boldsymbol{\eta})^T)$, for each $\eta \in D(\mathbb{R}_+; \Sigma_{N,K})$.

Given $\boldsymbol{\eta} \in D(\mathbb{R}_+; \Sigma_{N,K})$, let finally $\mathfrak{R}_{T,N}(\boldsymbol{\eta})$ be the stationary probability (with respect to the map ϑ_t defined above) on $\mathcal{S}$ given by

$$\mathfrak{R}_{T,N}(\boldsymbol{\eta}) := \frac{1}{T} \int_0^T \delta_{\left(\boldsymbol{\pi}_N(\vartheta_t \boldsymbol{\eta}^T)\,,\, \boldsymbol{J}_N(\vartheta_t \boldsymbol{\eta}^T)\right)} \, dt = \frac{1}{T} \int_0^T \delta_{\vartheta_t\left(\boldsymbol{\pi}_N(\boldsymbol{\eta})^T\,,\, \boldsymbol{J}_N(\boldsymbol{\eta})^T\right)} \, dt \,. \tag{2.24}$$

Large deviations asymptotic. Our main result establishes the large deviations principle for $\mathfrak{R}_{T,N}(\boldsymbol{\eta})$ in the joint limit $T \to \infty$ and $N \to \infty$ when $\boldsymbol{\eta}$ is sampled according to the WASEP dynamics. We prove this result both when $T \to \infty$ before $N \to \infty$ and when $N \to \infty$ before $T \to \infty$. The corresponding rate function is independent of the limiting procedure.

The statement of this result requires further notation. Denote by $\mathcal{S}_m$, $m \in (0,1)$, the closed set of trajectories $(\boldsymbol{\pi}, \boldsymbol{J})$ in $\mathcal{S}$ such that $\boldsymbol{\pi}(t, \mathbb{T}^d) = m$ for all $t \in \mathbb{R}$. Recalling that $\sigma(\rho) = \rho(1-\rho)$ is the mobility of the exclusion process, let finally $\mathcal{S}_{m,\mathrm{ac}}$ be the subset of elements $(\boldsymbol{\pi}, \boldsymbol{J})$ in $\mathcal{S}_m$ such that

(a) $\boldsymbol{\pi} \in C(\mathbb{R}, \mathcal{M}_m(\mathbb{T}^d))$, $\boldsymbol{\pi}(t, dx) = \boldsymbol{\rho}(t,x)\, dx$ for some $\boldsymbol{\rho}$ such that $0 \le \boldsymbol{\rho}(t,x) \le 1$, and, for any $T > 0$,

$$\int_{-T}^{T} dt \int_{\mathbb{T}^d} dx \, \frac{|\,\nabla \boldsymbol{\rho}\,|^2}{\sigma(\boldsymbol{\rho})} \;<\; \infty \,;$$

(b) $\boldsymbol{J}$ belongs to $C(\mathbb{R}, \mathcal{H}^d_{-p})$, and $\boldsymbol{J}(t) = \int_0^t \boldsymbol{j}(s)\, ds$, $t \in \mathbb{R}$, for some $\boldsymbol{j}$ in $L^2_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^d, \sigma(\boldsymbol{\rho}(t,x))^{-1}\, dt\, dx; \mathbb{R}^d)$. Thus, for any $T > 0$

$$\int_{-T}^{T} dt \int_{\mathbb{T}^d} dx \, \frac{|\,\boldsymbol{j}\,|^2}{\sigma(\boldsymbol{\rho})} \;<\; \infty \,.$$

Let the *action* $A_{m,T}\colon \mathcal{S}\to[0,+\infty]$, $m\in(0,1)$, $T>0$, be defined by

$$(2.25)\qquad A_{m,T}(\boldsymbol{\pi},\boldsymbol{J}) = \begin{cases} \displaystyle\int_0^T dt\int_{\mathbb{T}^d} dx\,\frac{|\boldsymbol{j}+\nabla\boldsymbol{\rho}-\sigma(\boldsymbol{\rho})\,E|^2}{4\,\sigma(\boldsymbol{\rho})} & \text{if } (\boldsymbol{\pi},\boldsymbol{J})\in\mathcal{S}_{m,\mathrm{ac}}\,,\\ +\infty & \text{otherwise.}\end{cases}$$

By the arguments presented in [6, § 4], the functional $A_{m,T}$ is lower semicontinuous. Note that if $\boldsymbol{J}$ has a density $\boldsymbol{j}$ as in item (b) above, then the density of $\vartheta_t\boldsymbol{J}$ is given by $(\vartheta_t\boldsymbol{j})(s) := \boldsymbol{j}(s-t)$.

For $m\in(0,1)$, let $\mathcal{P}_{\mathrm{stat}}$, $\mathcal{P}_{\mathrm{stat},m}$, be the set of translation invariant probability measures on $\mathcal{S}$, $\mathcal{S}_m$, respectively. We consider $\mathcal{P}_{\mathrm{stat}}$ and $\mathcal{P}_{\mathrm{stat},m}$ endowed with the topology induced by the weak convergence and the corresponding Borel σ-algebra. Let $\boldsymbol{I}_m\colon\mathcal{P}_{\mathrm{stat}}\to[0,+\infty]$ be the functional defined by

$$(2.26)\qquad \boldsymbol{I}_m(P) := \frac{1}{T}\,E_P\big[\,A_{m,T}\,\big]\,,$$

observing that the right-hand side does not depend on $T>0$ by stationarity. Moreover, by using the continuity equation (2.21), the identity $\nabla\boldsymbol{\rho}/\sigma(\boldsymbol{\rho}) = \nabla h'(\boldsymbol{\rho})$, where $h(\rho) := \rho\log\rho+(1-\rho)\log(1-\rho)$ is the Bernoulli entropy, and the stationarity of P, integrating by parts we deduce that if $\boldsymbol{I}_m(P)<+\infty$ then for any $T>0$ (equivalently for some $T>0$)

$$(2.27)\qquad \frac{1}{T}\,E_P\Big[\int_{-T}^T dt\int_{\mathbb{T}^d} dx\,\Big(\frac{|\,\nabla\boldsymbol{\rho}\,|^2}{4\,\sigma(\boldsymbol{\rho})}+\frac{|\,\boldsymbol{j}\,|^2}{4\,\sigma(\boldsymbol{\rho})}\Big)\Big] < +\infty\,.$$

In the next statements and hereafter, by $\limsup_{T,N}$, we mean either $\limsup_N\limsup_T$ or $\limsup_T\limsup_N$. Analogously, $\liminf_{T,N}$ stands for either $\liminf_N\liminf_T$ or $\liminf_T\liminf_N$.

THEOREM 2.1. *Fix $m\in(0,1)$, $p>(d+2)/2$, and a sequence K_N such that $K_N/N^d\to m$. For each closed subset $\mathcal{F}$ of $\mathcal{P}_{\mathrm{stat}}$*

$$\limsup_{N,T\to\infty}\ \sup_{\eta\in\Sigma_{N,K_N}}\frac{1}{N^d}\frac{1}{T}\log\mathbb{P}^N_\eta\left[\mathfrak{R}_{T,N}\in\mathcal{F}\right] \le -\inf_{P\in\mathcal{F}}\boldsymbol{I}_m(P)\,.$$

Moreover, if E is orthogonally decomposable, then, for each open subset $\mathcal{G}$ of $\mathcal{P}_{\mathrm{stat}}$,

$$\liminf_{N,T\to\infty}\ \inf_{\eta\in\Sigma_{N,K_N}}\frac{1}{N^d}\frac{1}{T}\log\mathbb{P}^N_\eta\left[\mathfrak{R}_{T.N}\in\mathcal{G}\right] \ge -\inf_{P\in\mathcal{G}}\boldsymbol{I}_m(P)\,.$$

Finally, the functional $\boldsymbol{I}_m\colon\mathcal{P}_{\mathrm{stat}}\to[0,+\infty]$ is good and affine.

REMARK 2.2. Recall that “quasi-potential” is the name given to the rate functional of the large deviations principle for the empirical measure under the stationary state. In the lower bound, the technical condition that the external field E is orthogonally decomposable is only used to guarantee that the quasi-potential of the WASEP is bounded, an ingredient that enters in the proof of Lemma 4.12. Actually, under this assumption, the quasi-potential can be computed explicitly [5, 4]. We do believe that the quasi-potential of the WASEP is bounded even if the external field E is not orthogonally decomposable, but a proof is missing.

In the common terminology of large deviations, Theorem 2.1 corresponds to a level-three large deviation principle, for which the rate function has an explicit expression. The contraction principle permits to derive from this result large deviations principles for relevant observables.

Level two large deviations. Let $\mathcal{P}(\mathcal{M}_+(\mathbb{T}^d))$ be the space of probability measures on $\mathcal{M}_+(\mathbb{T}^d)$ endowed with the weak topology. Recalling (2.5), define $\wp_{T,N}$ as the map from $D(\mathbb{R}_+;\Sigma_{N,K})$ to $\mathcal{P}(\mathcal{M}_+(\mathbb{T}^d))$ by

$$\wp_{T,N}(\boldsymbol{\eta}) := \frac{1}{T}\int_0^T dt\, \delta_{\pi_N(\boldsymbol{\eta}(t))}\,,$$

i.e. $\wp_{T,N}$ is the time empirical measure associated to the path $\pi_N(\boldsymbol{\eta})$. Letting $\imath_t\colon \mathcal{S} \to \mathcal{M}_+(\mathbb{T}^d)$ be the map $(\boldsymbol{\pi},\boldsymbol{J}) \mapsto \boldsymbol{\pi}(t)$, then $\mathfrak{R}_{T,N}\circ \imath_t^{-1} = \wp_{T,N}$. Finally, for $m \in (0,1)$, denote by $\mathcal{I}_m\colon \mathcal{P}(\mathcal{M}_+(\mathbb{T}^d)) \to [0,+\infty]$ the functional given by

$$\mathcal{I}_m(\wp) = \inf\left\{\boldsymbol{I}_m(P) : P\in\mathcal{P}_{\mathrm{stat}}\,,\ P\circ \imath_t^{-1} = \wp\right\}. \tag{2.28}$$

Given $m \in (0,1)$, let $(\Phi_t^m : t \geq 0)$ be the flow induced by the hydrodynamic equation (2.18) on the set of densities with total mass equal to m. Namely, when $\int \rho\, dx = m$ we set $\Phi_t^m(\rho) = \boldsymbol{\rho}(t)$ where $(\boldsymbol{\rho},\boldsymbol{j})$ is the unique weak solution to (2.18). By identifying measures absolutely continuous with respect to dx with their densities, we regard $(\Phi_t^m : t\geq 0)$ as a flow on $\mathcal{M}_m(\mathbb{T}^d)$. The following result is obtained from Theorem 2.1 by the contraction principle and implies the *hydrostatic limit*: in the limit $N\to\infty$ the empirical density constructed by sampling the particles according to the stationary measure $\mu_{N,K}$ converges to the unique stationary solution to the hydrodynamic equation.

COROLLARY 2.3. *Fix $m\in(0,1)$ and a sequence K_N such that $K_N/N^d \to m$. For each closed subset $\mathcal{F}$ of $\mathcal{P}(\mathcal{M}_+(\mathbb{T}^d))$*

$$\limsup_{N,T\to\infty}\ \sup_{\eta\in\Sigma_{N,K_N}} \frac{1}{N^d}\frac{1}{T}\log \mathbb{P}_\eta^N\left[\wp_{T,N}\in\mathcal{F}\right] \leq -\inf_{\wp\in\mathcal{F}} \mathcal{I}_m(\wp)\,.$$

If E is orthogonally decomposable, then for each open subset $\mathcal{G}$ of $\mathcal{P}(\mathcal{M}_+(\mathbb{T}^d))$

$$\liminf_{N,T\to\infty}\ \inf_{\eta\in\Sigma_{N,K_N}} \frac{1}{N^d}\frac{1}{T}\log \mathbb{P}_\eta^N\left[\wp_{T,N}\in\mathcal{G}\right] \geq -\inf_{\wp\in\mathcal{G}} \mathcal{I}_m(\wp)\,.$$

Finally, the functional $\mathcal{I}_m\colon \mathcal{P}(\mathcal{M}_+(\mathbb{T}^d)) \to [0,+\infty]$ is good, convex, and vanishes only on the invariant probabilities for the flow Φ^m. In particular, if E is orthogonally decomposable, then $\mathcal{I}_m(\wp) = 0$ if and only if $\wp = \delta_{\bar\rho\, dx}$ where $\bar\rho$ is the unique stationary solution to the hydrodynamic equation with mass m.

Level one large deviations. For $T>0$ the *time-averaged empirical density* is the map $\pi_{T,N} : D(\mathbb{R}_+,\Sigma_N) \to \mathcal{M}_+(\mathbb{T}^d)$ defined by

$$\pi_{T,N}(\boldsymbol{\eta}) := \frac{1}{T}\int_0^T [\,\boldsymbol{\pi}_N(\boldsymbol{\eta})\,](t)\, dt\,. \tag{2.29}$$

Likewise, for $p > d/2$, the *time-averaged empirical current* is the map $J_{T,N}$ from the set $D(\mathbb{R}_+,\Sigma_{N,K})$ to $\mathcal{H}_{-p}^d$ defined by

$$J_{T,N}(\boldsymbol{\eta}) = \frac{1}{T}[\,\boldsymbol{J}_N(\boldsymbol{\eta})\,](T)\,, \tag{2.30}$$

which can also be written as

$$\langle J_{T,N}(\boldsymbol{\eta})\,,F\rangle = \frac{1}{T}\frac{1}{N^d}\sum_{(x,y)\in\mathbb{B}_N} \mathcal{N}_{(0,T]}^{x,y}(\boldsymbol{\eta})\int_x^y F\cdot d\ell\,,\quad F\in C^\infty(\mathbb{T}^d;\mathbb{R}^d)\,.$$

Note that

$$(\pi_{T,N}, J_{T,N}) = \int d\mathfrak{R}_{T,N}\Big(\boldsymbol{\pi}(t), \frac{1}{t}\boldsymbol{J}(t)\Big) + \frac{1}{T}\big(0, \mathcal{E}_{T,N}\big) \tag{2.31}$$

where the first term on right hand side does not depend on $t \neq 0$ by the stationarity of $\mathfrak{R}_{T,N}$ and

$$\mathcal{E}_{T,N}(\boldsymbol{\eta}) = \boldsymbol{J}_N(\boldsymbol{\eta}^T(T)) - \boldsymbol{J}_N(\boldsymbol{\eta}(T))\,.$$

By (2.20) and the Sobolev embedding, for $p > d/2$ we deduce that $\|\mathcal{E}_{T,N}(\boldsymbol{\eta})\|_{-p}$ is bounded uniformly in $\boldsymbol{\eta}$, T, and N. Hence the second term on the right hand side of (2.31) is irrelevant for large deviations in the asymptotics $T \to \infty$.

Let $I_m \colon \mathcal{M} \to [0, +\infty]$, $m \in (0,1)$, be the functional defined by

$$I_m(\pi, J) \,:=\, \inf\big\{\, \boldsymbol{I}_m(P) : P \in \mathcal{P}_{\mathrm{stat}}\,,\, E_P[\,\boldsymbol{\pi}(t)\,] \,=\, \pi\,,\, E_P[\,\boldsymbol{J}(\,t\,)\,] \,=\, tJ\,\big\}\,,$$

which does not depend on $t \neq 0$. If the vector field J is not divergence free, the set on the right-hand side is empty. Indeed, by stationarity and the continuity equation (2.21), if the above constraints are satisfied, we deduce that for each smooth function f on $\mathbb{T}^d$ and $t > 0$

$$0 \,=\, E_P\Big[\int_0^t \langle \boldsymbol{\pi}(s) - \boldsymbol{\pi}(0)\,,\, f\rangle\, ds\,\Big] \,=\, E_P\Big[\int_0^t \langle \boldsymbol{J}(s)\,,\, \nabla f\rangle\, ds\Big] \,=\, \frac{t^2}{2}\,\langle J\,,\, \nabla f\rangle\,.$$

By the contraction principle, Theorem 2.1 implies the following statement.

COROLLARY 2.4. *Fix $m \in (0,1)$, $p > (d+2)/2$, and a sequence K_N so that $K_N/N^d \to m$. For each closed subset $\mathcal{F}$ of $\mathcal{M}$*

$$\limsup_{T,N\to\infty}\ \sup_{\eta\in\Sigma_{N,K_N}} \frac{1}{N^d}\frac{1}{T}\log \mathbb{P}^N_\eta\big[(\pi_{T,N}, J_{T,N}) \in \mathcal{F}\big] \,\le\, -\inf_{(\pi,J)\in\mathcal{F}} I_m(\pi, J)\,.$$

Moreover, if E is orthogonally decomposable, then for each open subset $\mathcal{G}$ of $\mathcal{M}$,

$$\liminf_{T,N\to\infty}\ \inf_{\eta\in\Sigma_{N,K_N}} \frac{1}{N^d}\frac{1}{T}\log \mathbb{P}^N_\eta\big[(\pi_{T,N}, J_{T,N}) \in \mathcal{G}\big] \,\ge\, -\inf_{(\pi,J)\in\mathcal{G}} I_m(\pi, J)\,.$$

Finally, the functional $I_m \colon \mathcal{M} \to [0, +\infty]$ is good and convex.

The projections of I_m on the two components can be further analyzed and computed explicitly under additional conditions which are satisfied, for instance, in the SEP case. Denote by $I^{(1)}_m \colon \mathcal{M}_+(\mathbb{T}^d) \to [0, +\infty]$ the projection of the functional I_m on the density, i.e.,

$$I^{(1)}_m(\pi) \,=\, \inf\big\{\, \boldsymbol{I}_m(P) : P \in \mathcal{P}_{\mathrm{stat},m}\,,\, E_P[\,\boldsymbol{\pi}(\,t\,)\,] \,=\, \pi\,\big\}\,. \tag{2.32}$$

It turns out that when the external field E is a gradient, so that the WASEP dynamics is reversible, then $I^{(1)}_m$ can be computed explicitly. Assume $E = -\nabla U$ for some $U \in C^2(\mathbb{T}^d)$ and let $\mathcal{V}_m \colon \mathcal{M}_m(\mathbb{T}^d) \to [0, +\infty]$ be the functional defined by

$$\mathcal{V}_m(\pi) := \begin{cases} \displaystyle\int_{\mathbb{T}^d} dx\, \frac{\big|\nabla\rho + \sigma(\rho)\nabla U\big|^2}{4\,\sigma(\rho)} & \text{if } \pi(\mathbb{T}^d) = m \text{ and } \pi(dx) = \rho\, dx, \\ +\infty & \text{otherwise.} \end{cases}$$

Let also $\mathrm{co}(\mathcal{V}_m)$ be the convex hull of $\mathcal{V}_m$ and observe that, in view of the concavity of σ, if $\nabla U = 0$ then $\mathrm{co}(\mathcal{V}_m) = \mathcal{V}_m$. The functional $\mathcal{V}_m$ can be seen as a non-linear version of the level two Donsker-Varadhan functional for reversible diffusions (sometimes called Fisher information). Indeed, in the case of independent particles $\sigma(\rho) = \rho$ and the functional $\mathcal{V}_m$ reduces to the Dirichlet form of the square root for the diffusion on $\mathbb{T}^d$ with generator $\Delta - \nabla U \cdot \nabla$.

THEOREM 2.5. *If* $E = -\nabla U$, *then* $I_m^{(1)} = \mathrm{co}(\mathcal{V}_m)$.

As discussed in the introduction, the projection of I_m on the second component is related to the possible occurrence of dynamical phase transitions for the current. For $p > (d+2)/2$, $m \in (0,1)$, denote by $I_m^{(2)} : \mathcal{H}_{-p}^d \to [0, +\infty]$ the projection of the functional I_m on the current, i.e.

$$I_m^{(2)}(J) \;=\; \inf \left\{ \boldsymbol{I}_m(P) : P \in \mathcal{P}_{\mathrm{stat},m} \,,\, E_P[\,\boldsymbol{J}(t)\,] \,=\, t\,J \,\right\}, \tag{2.33}$$

that corresponds to the Varadhan's proposal informally presented in (1.4). By the contraction principle, Corollary 2.4 implies that the time-averaged empirical current $J_{T,N}$ satisfies a large deviation principle with rate function $I_m^{(2)}$.

It has been pointed out in [2, 3, 8] that the variational problem (2.33) has a non-trivial solution when E is constant and large enough. Such behavior is interpreted as a dynamical phase transition. Strictly speaking, the problem (2.33) is not really considered in [2, 3, 8], but the analysis performed there implies the results summarized in the next statement. We restrict to the one-dimensional case with constant external field. Since $I_m^{(2)}(J) < +\infty$ implies $\nabla \cdot J = 0$, in the one-dimensional case, $I_m^{(2)}$ is finite only if $J(x) = j$ for some constant $j \in \mathbb{R}$.

THEOREM 2.6. *Let* $d = 1$, $m \in (0,1)$, *and* $E \geq 0$ *be constant.*

(i) *There exists* $E_0 > 0$ *such that if* $E \leq E_0$ *then for* $J \,=\, j$, $j \in \mathbb{R}$,

$$I_m^{(2)}(J) \;=\; \frac{(j - \sigma(m)E)^2}{4\,\sigma(m)} \,.$$

The optimal P *for the variational problem* (2.33) *is* $\delta_{(m,j)}$.

(ii) *There exists* $E_1 > E_0$ *such that if* $E \geq E_1$ *then for* $J = j$, $j \in \mathbb{R}$, *with* j *large enough*

$$I_m^{(2)}(J) \;<\; \frac{(j - \sigma(m)E)^2}{4\,\sigma(m)} \,.$$

Furthermore, taking the time average of a probability concentrated on a traveling wave provides a measure P *in* $\mathcal{P}_{\mathrm{stat},m}$ *such that* $E_P[\,\boldsymbol{J}(t)\,] = t\,J$, $\boldsymbol{I}_m(P) < \boldsymbol{I}_m(\delta_{(m,j)})$.

Regarding the higher dimensional case, we mention that the argument in [3, Prop. 5.1] implies that in the SEP case ($E = 0$) for J with vanishing divergence we have

$$I_m^{(1)}(J) = \inf_{\rho} \int_{\mathbb{T}^d} \frac{|J + \nabla \rho|^2}{4\sigma(\rho)}\, dx \,,$$

where the infimum is carried out over the density profiles ρ of mass m. In other words, the infimum in (2.33) is achieved for a probability measure of the form $P = \delta_{(\rho\, dx\,,\,J)}$ and no dynamical phase transition occurs.

3. Donsker-Varadhan large deviations principle. In this section, we recall the Donsker-Varadhan large deviations principle in the context of the WASEP dynamics with fixed number of particles.

Recall from (2.19) the definition of the empirical process $R_T(\boldsymbol{\eta})$. Referring to [23] for equivalent characterizations, we introduce the rate functional for the family $(R_T : T > 0)$ by a variational representation that will be most useful for our purposes. For $t > 0$, let $H_{N,K}(t, \cdot\,) : \mathcal{P}_{\mathrm{stat}}^{N,K} \to [0, +\infty]$ be the functional given by

$$H_{N,K}(t, \mathbb{Q}) \;=\; \sup_{\Phi} \int d\mathbb{Q}(\boldsymbol{\eta}) \Big[\Phi(\boldsymbol{\eta}) \;-\; \log \mathbb{E}^N_{\boldsymbol{\eta}(0)}\big(e^{\Phi}\big) \Big] \,,$$

where the supremum is carried over the bounded and continuous functions Φ on $D(\mathbb{R},\Sigma_{N,K})$ that are measurable with respect to $\sigma\{\boldsymbol{\eta}(s),\ s\in[0,t]\}$. Let $H_{N,K}\colon \mathcal{P}^{N,K}_{\rm stat}\to[0,+\infty]$ be the functional defined by

$$H_{N,K}(\mathbb{Q}) \,:=\, \sup_{t>0}\frac{1}{t}\,H_{N,K}(t,\mathbb{Q}) \,=\, \lim_{t\to\infty}\frac{1}{t}\,H_{N,K}(t,\mathbb{Q})\,, \tag{3.1}$$

where the second identity follows from the inequality before [23, Theorem 10.9]. By [23, Theorems 10.6 and 10.8], the functional $H_{N,K}$ is good and affine.

The classical Donsker-Varadhan theorem, see [11] or Theorems 11.6 and 12.5 in [23], states that, uniformly on the initial configuration $\eta\in\Sigma_{N,K}$, the family of probability measures $(\mathbb{P}^N_\eta\circ R_T^{-1}: T>0)$ satisfies a large deviations principle with rate function $H_{N,K}$.

THEOREM 3.1. *Fix N and K. For each closed set $\mathcal{F}$ and each open set $\mathcal{G}$ in $\mathcal{P}^{N,K}_{\rm stat}$,*

$$\limsup_{T\to\infty}\,\sup_{\eta\in\Sigma_{N,K}}\frac{1}{T}\log\mathbb{P}^N_\eta\big[R_T\in\mathcal{F}\big]\,\le\,-\inf_{\mathbb{Q}\in\mathcal{F}}H_{N,K}(\mathbb{Q})\,,$$

$$\liminf_{T\to\infty}\,\inf_{\eta\in\Sigma_{N,K}}\frac{1}{T}\log\mathbb{P}^N_\eta\big[R_T\in\mathcal{G}\big]\,\ge\,-\inf_{\mathbb{Q}\in\mathcal{G}}H_{N,K}(\mathbb{Q})\,.$$

The rate function $H_{N,K}(\mathbb{Q})$ can also be understood as the relative entropy *per unit of time* of the stationary probability $\mathbb{Q}$ with respect to the stationary process $\mathbb{P}^N_{\mu_{N,K}}$. Given $T_0<T_1$, denote by $i_{T_0,T_1}\colon D(\mathbb{R},\Sigma_{N,K})\to D([T_0,T_1],\Sigma_{N,K})$ the canonical projection. Given two probability measures $\mathbb{Q}^1,\mathbb{Q}^2$ on $D(\mathbb{R},\Sigma_{N,K})$, let $\mathbb{H}_{[T_0,T_1]}$ be the relative entropy between the marginal of $\mathbb{Q}^1$ relative to the time interval $[T_0,T_1]$ and the marginal of $\mathbb{Q}^2$ on the same interval:

$$\mathbb{H}_{[T_0,T_1]}(\mathbb{Q}^1|\mathbb{Q}^2)\,=\,\mathrm{Ent}\big(\mathbb{Q}^1_{[T_0,T_1]}\big|\mathbb{Q}^2_{[T_0,T_1]}\big):=\int\log\frac{d\,\mathbb{Q}^1_{[T_0,T_1]}}{d\,\mathbb{Q}^2_{[T_0,T_1]}}\,d\,\mathbb{Q}^1_{[T_0,T_1]}\,, \tag{3.2}$$

where $\mathbb{Q}^j_{[T_0,T_1]}=\mathbb{Q}^j\circ i^{-1}_{T_0,T_1}$, $j=1,\,2$. We also shorthand $\mathbb{H}_{[0,T]}$ by $\mathbb{H}^{(T)}$. By [13, Theorem 5.4.27], for each $\mathbb{Q}$ in $\mathcal{P}^{N,K}_{\rm stat}$

$$H_{N,K}(\mathbb{Q})\,=\,\lim_{T\to\infty}\frac{1}{T}\,\mathbb{H}^{(T)}\big(\,\mathbb{Q}\,\big|\,\mathbb{P}^N_{\mu_{N,K}}\,\big)\,=\,\sup_{T>0}\frac{1}{T}\,\mathbb{H}^{(T)}\big(\,\mathbb{Q}\,\big|\,\mathbb{P}^N_{\mu_{N,K}}\,\big)\,, \tag{3.3}$$

where the second identity follows by a super-additivity argument which stems from [23, Lemma 10.3]. Actually, Theorem 5.4.27 in [13] states that the empirical process $(R_T: T>0)$ satisfies a large deviations principle with good rate function given by $H^\star_{N,K}(\mathbb{Q}):=\lim_{T\to\infty}T^{-1}\,\mathbb{H}^{(T)}(\,\mathbb{Q}\,|\,\mathbb{P}^N_{\mu_{N,K}}\,)$. Since, by Theorem 3.1, $(R_T:T>0)$ also satisfies a large deviations principle with good rate function given by $H_{N,K}(\mathbb{Q})$, a simple argument using the lower semi-continuity of the functionals yields that $H_{N,K}(\mathbb{Q})=H^\star_{N,K}(\mathbb{Q})$ for all $\mathbb{Q}\in\mathcal{P}^{N,K}_{\rm stat}$.

Recall that we denote by $\mathcal{S}$ the set of trajectories $(\boldsymbol{\pi},\boldsymbol{J})$ which satisfy the continuity equation (2.21). By (2.17), the map $(\boldsymbol{\pi}_N,\boldsymbol{J}_N)\colon D(\mathbb{R},\Sigma_{N,K})\to D(\mathbb{R},\mathcal{M})$ takes values in $\mathcal{S}$. As already observed, $(\boldsymbol{\pi}_N,\boldsymbol{J}_N)(\vartheta_t\boldsymbol{\eta})=[\vartheta_t(\boldsymbol{\pi}_N,\boldsymbol{J}_N)](\boldsymbol{\eta})$ so that $(\boldsymbol{\pi}_N,\boldsymbol{J}_N)$ induces a map from the stationary probabilities on $D(\mathbb{R},\Sigma_{N,K})$ to the stationary probabilities on $\mathcal{S}$. More precisely, if $\mathbb{P}\in\mathcal{P}^{N,K}_{\rm stat}$, then $\mathbb{P}\circ(\boldsymbol{\pi}_N,\boldsymbol{J}_N)^{-1}$ belongs to $\mathcal{P}_{\rm stat}$. Let $\boldsymbol{I}_{N,K}:\mathcal{P}_{\rm stat}\to[0,\infty]$ be defined by

$$\boldsymbol{I}_{N,K}(P)\,=\,\inf\Big\{H_{N,K}(\mathbb{P})\colon\ \mathbb{P}\in\mathcal{P}^{N,K}_{\rm stat},\,\mathbb{P}\circ(\boldsymbol{\pi}_N,\boldsymbol{J}_N)^{-1}=P\Big\}\,. \tag{3.4}$$

Note that the set on the right-hand side is either empty (for example, if the P-measure of the set of piece-wise constant paths is not equal to 1) or it is a singleton because the map $(\boldsymbol{\pi}_N,\boldsymbol{J}_N)\colon D(\mathbb{R},\Sigma_{N,K})\to D(\mathbb{R},\mathcal{M})$ is injective.

COROLLARY 3.2. *Fix N and $K = 0,\dots,N^d$. The functional $\boldsymbol{I}_{N,K}\colon \mathcal{P}_{\rm stat} \to [0,+\infty]$ is affine and good. Moreover, for each closed $\mathcal{F}$ and each open $\mathcal{G}$ in $\mathcal{P}_{\rm stat}$,*

$$\limsup_{T\to\infty} \sup_{\eta\in\Sigma_{N,K}} \frac{1}{T} \log \mathbb{P}^N_\eta \big[\mathfrak{R}_{T,N} \in \mathcal{F} \big] \;\le\; - \inf_{P\in\mathcal{F}} \boldsymbol{I}_{N,K}(P)\,,$$

$$\liminf_{T\to\infty} \inf_{\eta\in\Sigma_{N,K}} \frac{1}{T} \log \mathbb{P}^N_\eta \big[\mathfrak{R}_{T,N} \in \mathcal{G} \big] \;\ge\; - \inf_{P\in\mathcal{G}} \boldsymbol{I}_{N,K}(P)\,.$$

PROOF. It is enough to show that the map $\mathcal{P}^{N,K}_{\rm stat} \ni \mathbb{Q} \mapsto \mathbb{Q}\circ(\boldsymbol{\pi}_N, \boldsymbol{J}_N)^{-1} \in \mathcal{P}_{\rm stat}$ is continuous. The statement then follows from Theorem 3.1 by the contraction principle.

Since the map $\boldsymbol{\pi}_N\colon D(\mathbb{R};\Sigma_{N,K}) \to D(\mathbb{R};\mathcal{M}_+(\mathbb{T}^d))$ introduced in (2.5) is continuous, we directly deduce the continuity of the map $\mathcal{P}^{N,K}_{\rm stat} \ni \mathbb{P} \mapsto \mathbb{P}\circ\boldsymbol{\pi}_N^{-1} \in \mathcal{P}_{\rm stat}\big(D(\mathbb{R};\mathcal{M}_+(\mathbb{T}^d))\big)$. In contrast, the map $\boldsymbol{\eta}\mapsto \boldsymbol{J}_N(\boldsymbol{\eta})$ is not continuous. Indeed, consider the sequence $\boldsymbol{\eta}^{(k)}$ in which $\boldsymbol{\eta}^{(k)}$ has a unique jump at time $1/k$ from $\boldsymbol{\eta}(0)$ to $\sigma^{x,y}\boldsymbol{\eta}(0)$ for some $(x,y)\in\mathbb{B}_N$. Then, $\boldsymbol{\eta}^{(k)}$ converges to the path $\boldsymbol{\eta}$ with a single jump at time $t=0$ but $\boldsymbol{J}_N(\boldsymbol{\eta}^{(k)})$ does not converge to $\boldsymbol{J}_N(\boldsymbol{\eta})$. In contrast, the map $\boldsymbol{\eta}\mapsto\boldsymbol{J}_N(\boldsymbol{\eta})$ is continuous if $\boldsymbol{\eta}$ does not have a jump at time $t=0$. Moreover, if $\mathbb{Q}$ is a stationary probability on $D\big(\mathbb{R};\Sigma_{N,K}\big)$, then the $\mathbb{Q}$-probability of the paths $\boldsymbol{\eta}$ which have a jump a time $t=0$ is necessarily zero. This implies that the map $\mathcal{P}^{N,K}_{\rm stat} \ni \mathbb{Q} \mapsto \mathbb{Q}\circ\boldsymbol{J}_N^{-1} \in \mathcal{P}_{\rm stat}\big(D(\mathbb{R};\mathcal{H}^d_{-p})\big)$ is continuous. □

4. Variational convergence of the Donsker-Varadhan functional..

Referring to [9] for an overview, we recall the definition of Γ-convergence. Fix a Polish space $\mathcal{X}$ and a sequence $(U_n : n\in\mathbb{N})$ of functionals on $\mathcal{X}$, $U_n\colon \mathcal{X}\to[0,+\infty]$. The sequence U_n is *equi-coercive* if for each $\ell\ge 0$ there exists a compact subset $\mathcal{K}_\ell$ of $\mathcal{X}$ such that $\{x\in\mathcal{X}\colon U_n(x)\le\ell\}\subset\mathcal{K}_\ell$ for any $n\in\mathbb{N}$. The sequence U_n Γ*-converges* to the functional $U\colon\mathcal{X}\to[0,+\infty]$, i.e. $U_n \xrightarrow{\Gamma} U$, if and only if the two following conditions are met:

(i) Γ*-liminf.* The functional U is a Γ-liminf for the sequence U_n: For each $x\in\mathcal{X}$ and each sequence $x_n\to x$, we have that $\liminf_n U_n(x_n)\ge U(x)$.
(ii) Γ*-limsup.* The functional U is a Γ-limsup for the sequence U_n: For each $x\in\mathcal{X}$ there exists a sequence $x_n\to x$ such that $\limsup_n U_n(x_n)\le U(x)$.

Recall the definition of the functionals $\boldsymbol{I}_m$, $\boldsymbol{I}_{N,K}$ introduced in (2.26) and (3.4), respectively. The main result of this section reads as follows.

THEOREM 4.1. *Fix $0<m<1$, $p>(d+2)/2$, and a sequence K_N so that $K_N/N^d\to m$. The sequence $(N^{-d}\boldsymbol{I}_{N,K_N} : N\ge 1)$ is equi-coercive. The functional $\boldsymbol{I}_m$ is a Γ-liminf for $N^{-d}\boldsymbol{I}_{N,K_N}$. If E is orthogonally decomposable, then the functional $\boldsymbol{I}_m$ is also a Γ-limsup for $N^{-d}\boldsymbol{I}_{N,K_N}$. Therefore, under this hypothesis on E, $N^{-d}\boldsymbol{I}_{N,K_N} \xrightarrow{\Gamma} \boldsymbol{I}_m$.*

The proof of this theorem is divided in three parts. In Subsection 4.1, we prove that the sequence $N^{-d}\boldsymbol{I}_{N,K_N}$ is equi-coercive. In Subsection 4.2, that $\boldsymbol{I}_m$ is a Γ-liminf, and, in Subsection 4.3, that $\boldsymbol{I}_m$ is a Γ-limsup provided E is decomposable. For the rest of this section, fix $m\in(0,1)$, $p>(d+2)/2$, and a sequence K_N such that $K_N/N^d\to m$.

4.1 Equi-coercivity. Set $P_N := \mathbb{P}^N_{\mu_{N,K_N}} \circ \big(\boldsymbol{\pi}_N, \boldsymbol{J}_N\big)^{-1} \in \mathcal{P}_{\rm stat}$. We first establish the exponential tightness of the sequence $(P_N : N\ge 1)\subset\mathcal{P}_{\rm stat}$.

PROPOSITION 4.2. *There exists a sequence $(\mathcal{K}_\ell : \ell\ge 1)$ of compact subsets of $\mathcal{S}$ such that*

$$\lim_{\ell\to+\infty} \limsup_{N\to+\infty} \frac{1}{N^d} \log P_N\big(\mathcal{K}_\ell^{\complement}\big) \;=\; -\infty\,.$$

PROOF. In view of Ascoli-Arzelà theorem, the compactness of $\mathcal{M}_+(\mathbb{T}^d)$, and the compact embedding $\mathcal{H}_{-p} \hookrightarrow \mathcal{H}_{-p'}$ for $p' > p$, the assertion of this proposition follows from the next three lemmata. $\square$

Let $D_{T,\delta} := \{(s,t) \in \mathbb{R}^2 : 0 \le s \le t \le T\,,\ |t-s| \le \delta\}$.

LEMMA 4.3. *For each $T > 0$, $\epsilon > 0$, and smooth $g\colon \mathbb{T}^d \to \mathbb{R}$,*

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}^N_{\mu_{N,K_N}} \Big[\sup_{(s,t)\in D_{T,\delta}} \big| \langle \boldsymbol{\pi}_N(t) - \boldsymbol{\pi}_N(s)\,,\, g\rangle \big| > \epsilon \Big] = -\infty\,.$$

LEMMA 4.4. *For each $T > 0$*

$$\lim_{A\to\infty} \limsup_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}^N_{\mu_{N,K_N}} \Big[\sup_{0\le t\le T} \big\| \boldsymbol{J}_N(t) \big\|^2_{-p} > A \Big] = -\infty\,.$$

LEMMA 4.5. *For each $\epsilon > 0$, $T > 0$, and smooth $H\colon \mathbb{T}^d \to \mathbb{R}^d$*

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}^N_{\mu_{N,K_N}} \Big[\sup_{(s,t)\in D_{T,\delta}} \big| \langle \boldsymbol{J}_N(t) - \boldsymbol{J}_N(s)\,,\, H\rangle \big| > \epsilon \Big] = -\infty\,.$$

Lemma 4.3 is a standard result in the large deviations theory of hydrodynamical limits, see e.g. [15, § 10.4]. Note that this result can be deduced from Lemma 4.5 by taking $H = \nabla g$ and using the continuity equation (2.17). On the other hand, the exponential tightness of the empirical current is stated in [3] but the proof presented there is incomplete. For this reason, we present below a detailed proof of Lemmata 4.4 and 4.5.

PROOF OF LEMMA 4.4. We use the notation and statements introduced in (2.13)-(2.16), and denote $\boldsymbol{J}_N(t)(H^{j,n})$ by $\langle \boldsymbol{J}_N(t)\,,\, H^{j,n}\rangle$. By (2.16),

$$\big\| \boldsymbol{J}_N(t) \big\|^2_{-p} = \sum_{j=1}^d \sum_{n\in\mathbb{Z}^d} \frac{1}{\gamma_n^p} \big| \langle \boldsymbol{J}_N(t)\,,\, H^{j,n}\rangle \big|^2\,, \qquad \gamma_n := 1 + 4\pi^2 |n|^2\,.$$

Let $\beta_n = \gamma_n^p / [c_p\,(1+|n|)^{2(p-1)}]$. Here, c_p is a constant such that $d\sum_n (\beta_n/\gamma_n^p) = 1$, that is, $c_p = d\sum_n (1+|n|)^{-2(p-1)}$. Note that this sum is finite because we assumed $p > 1 + (d/2)$. Introducing the supremum inside the sum yields that

$$\mathbb{P}^N_{\mu_{N,K_N}} \Big[\sup_{0\le t\le T} \sum_{j=1}^d \sum_{n\in\mathbb{Z}^d} \frac{1}{\gamma_n^p} \big| \langle \boldsymbol{J}_N(t)\,,\, H^{j,n}\rangle \big|^2 > A \Big]$$

$$\le \mathbb{P}^N_{\mu_{N,K_N}} \Big[\bigcup_{j=1}^d \bigcup_{n\in\mathbb{Z}^d} \Big\{ \frac{1}{\beta_n} \sup_{0\le t\le T} \big| \langle \boldsymbol{J}_N(t)\,,\, H^{j,n}\rangle \big|^2 > A \Big\} \Big]\,.$$

Fix $1 \le j \le d$, $n \in \mathbb{Z}^d$ and denote by $H^{j,n}_1$, $H^{j,n}_{-1}$ the real and the imaginary part of $H^{j,n}$, respectively. The previous expression is then bounded by

$$\sum_{b=\pm 1} \sum_{j=1}^d \sum_{n\in\mathbb{Z}^d} \mathbb{P}^N_{\mu_{N,K_N}} \Big[\sup_{0\le t\le T} \big| \langle \boldsymbol{J}_N(t)\,,\, H^{j,n}_b\rangle \big| > \sqrt{A\,\beta_n/2} \Big]\,.$$

We may remove the absolute value from the previous expression at the cost of an extra factor 2 in front of the sum and an estimation of $H^{j,n}_b$ and $-H^{j,n}_b$. We next bound the probability of the event $\{\sup_{0\le t\le T} \langle \boldsymbol{J}_N(t)\,,\, H^{j,n}_b\rangle > \sqrt{A\,\beta_n/2}\,\}$, the other one being similar.

Recall the notation introduced in (2.2) and let $B_{x,x+\mathfrak{e}_k}(\eta) = B_{x,x+\mathfrak{e}_k}(\eta, H_b^{j,n})$, $1 \le k \le d$, $x \in \mathbb{T}_N^d$, be given by

$$B_{x,x+\mathfrak{e}_k}(\eta) \;=\; N^2\, \eta_x \,[\, 1 - \eta_{x+\mathfrak{e}_k}\,]\, e^{(1/2)\, E_N(x,x+\mathfrak{e}_k)} \Big[e^{H_{b,N}^{j,n}(x,x+\mathfrak{e}_k)} \,-\, 1 \Big] \\ +\; N^2\, \eta_{x+\mathfrak{e}_k} \,[\, 1 - \eta_x\,]\, e^{(1/2)\, E_N(x+\mathfrak{e}_k,x)} \Big[e^{H_{b,N}^{j,n}(x+\mathfrak{e}_k,x)} \,-\, 1 \Big] \,,$$

By [15, Proposition A1.2.6], for each $\eta \in \Sigma_N$ the process

$$\mathbb{M}_N(t) \;:=\; \exp\Big\{ N^d \,\langle \boldsymbol{J}_N(t) \,,\, H_b^{j,n} \rangle \;-\; \int_0^t \sum_{k=1}^d \sum_{x\in\mathbb{T}_N^d} B_{x,x+e_k}(\boldsymbol{\eta}(s))\, ds \Big\}$$

is a mean-one $\mathbb{P}_\eta^N$-martingale.

Since $N\big|H_{b,N}^{j,n}(x,x+\mathfrak{e}_k)\big|$ is bounded uniformly in b,j,k,x,n,N, a Taylor expansion yields

$$\Big| \sum_{k=1}^d \sum_{x\in\mathbb{T}_N^d} B_{x,x+e_k}(\eta) \;-\; N^2 \sum_{k=1}^d \sum_{x\in\mathbb{T}_N^d} H_{b,N}^{j,n}(x,x+\mathfrak{e}_k)\,(\eta_x \,-\, \eta_{x+\mathfrak{e}_k}) \Big| \;\le\; C_1\, N^d \,,$$

for some constant C_1 independent of b,j,n,N. Summing by parts and using the inequality $\big|\partial_{x_k} H_b^{j,n}(x)\big| \le C_2|n|$ for some constant C_2 independent of b,j,k,x,n, we conclude that

$$\Big| \sum_{k=1}^d \sum_{x\in\mathbb{T}_N^d} B_{x,x+e_k}(\eta, H_b^{j,n}) \Big| \;\le\; C_0 N^d (1+|n|)$$

for some constant C_0 independent of b,j,n,N.

In view of the previous estimate, adding and subtracting the sum of the time-integrals of $B_{x,x+e_k}$ and taking exponentials, we get that

$$\mathbb{P}_{\mu_{N,K_N}}^N \Big[\sup_{0\le t\le T} \langle \boldsymbol{J}_N(t) \,,\, H_b^{j,n}\rangle \;>\; \sqrt{A\,\beta_n/2}\, \Big] \\ \le\; \mathbb{P}_{\mu_{N,K_N}}^N \Big[\sup_{0\le t\le T} \mathbb{M}_N(t) \;>\; e^{\sqrt{A\,\beta_n/2} N^d - C_0 T N^d(1+|n|)} \Big] \\ \le\; e^{-\sqrt{A\,\beta_n/2} N^d + C_0 T N^d(1+|n|)} \,,$$

where we used Doob's inequality in the last step and the fact that $\mathbb{M}_N(t)$ is a mean-one martingale.

We have thus shown that

$$\mathbb{P}_{\mu_{N,K_N}}^N \Big[\sup_{0\le t\le T} \big\| \boldsymbol{J}_N(t) \big\|_{\mathcal{H}_{-p}^d}^2 \;>\; A \Big] \;\le\; 4\, d \sum_{n\in\mathbb{Z}^d} e^{-N^d \big[\sqrt{A\,\beta_n/2} - C_0 T(1+|n|) \big]} \,.$$

By definition of β_n, there exists a positive constant c_0 such that $\beta_n \ge c_0(1+|n|)^2$. The statement follows. □

PROOF OF LEMMA 4.5. By a standard inclusion of events and the stationarity of $P_{\mu_{N,K_N}}^N$, it is enough to prove that for each $\epsilon > 0$ and smooth $H\colon \mathbb{T}^d \to \mathbb{R}^d$

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}_{\mu_{N,K_N}}^N \Big[\sup_{0\le t\le \delta} \big| \langle \boldsymbol{J}_N(t) \,,\, H\rangle \big| \;>\; \epsilon \Big] \;=\; -\infty \,, \tag{4.1}$$

where we used that $\boldsymbol{J}_N(0)=0$. As in the proof of the previous lemma, we can furthermore remove the modulus in the above bound.

Given a smooth vector-valued function $H\colon \mathbb{T}^d \to \mathbb{R}^d$ and $\ell > 0$, let $B^\ell_{x,x+\mathfrak{e}_k}(\eta) = B_{x,x+\mathfrak{e}_k}(\eta, \ell H)$, $x\in\mathbb{T}^d_N$, be given by

$$B^\ell_{x,x+\mathfrak{e}_k}(\eta) \;=\; N^2\, \eta_x\, (\,1-\eta_{x+\mathfrak{e}_k}\,)\, e^{(1/2)\,E_N(x,x+\mathfrak{e}_k)}\, \Big[e^{\ell H_N(x,x+\mathfrak{e}_k)} \,-\, 1\Big] \\ +\; N^2\, \eta_{x+\mathfrak{e}_k}\, [\,1-\eta_x\,]\, e^{(1/2)\,E_N(x+\mathfrak{e}_k,x)}\, \Big[e^{\ell H_N(x+\mathfrak{e}_k,x)} \,-\, 1\Big]\,.$$

By [15, App. 1, Prop. 2.6], for each $\eta\in\Sigma_N$, the process

$$\mathbb{M}^\ell_N(t) \;:=\; \exp\Big\{ N^d\,\ell\,\langle \boldsymbol{J}_N(t)\,,\, H\rangle \;-\; \int_0^t \sum_{k=1}^d \sum_{x\in\mathbb{T}^d_N} B^\ell_{x,x+e_k}(\boldsymbol{\eta}(s))\, ds \Big\}$$

is a mean-one $\mathbb{P}^N_\eta$-martingale.

The same computation of the previous lemma yields that there exists a constant $C_1 = C_1(H)$ such that

$$\Big| \sum_{k=1}^d \sum_{x\in\mathbb{T}^d_N} B^\ell_{x,x+\mathfrak{e}_k}(\eta) \Big| \;\le\; C_1\, N^d e^{C_1 \ell/N} (1+\ell^2)$$

for all N, ℓ and η. Therefore, by Doob's inequality,

$$\mathbb{P}^N_{\mu_{N,K_N}} \Big[\sup_{0\le t\le \delta} \langle \boldsymbol{J}_N(t)\,,\, H\rangle \;>\; \epsilon \Big] = \mathbb{P}^N_{\mu_{N,K_N}} \Big[\sup_{0\le t\le \delta} N^d\,\ell\,\langle \boldsymbol{J}_N(t)\,,\, H\rangle \;>\; N^d\,\ell\,\epsilon \Big] \\ \le P^N_{\mu_{N,K_N}} \Big[\sup_{0\le t\le\delta} \mathbb{M}^\ell_N(t) \;>\; \exp\big\{ N^d \big[\ell\,\epsilon - \delta\, C_1\, e^{C_1\ell/N}(1+\ell^2)\big]\big\}\Big] \\ \le \exp\Big\{ -\, N^d \big[\ell\,\epsilon - \delta\, C_1\, e^{C_1\ell/N}(1+\ell^2)\big]\Big\}\,,$$

which yields (4.1) by taking the limit $\ell\to\infty$ after the limits in N and δ. □

PROOF OF THEOREM 4.1. EQUI-COERCIVITY. For $\ell\ge 1$, let

$$\mathcal{E}_\ell \;:=\; \bigcup_{N\ge 1} \Big\{ P\in\mathcal{P}_{\rm stat}\colon\, \frac{1}{N^d} \boldsymbol{I}_{N,K_N}(P) \;\le\; \ell \Big\}\,.$$

In view of Ascoli-Arzelà theorem, the compactness of $\mathcal{M}_+(\mathbb{T}^d)$, and the compact embedding $\mathcal{H}_{-p} \hookrightarrow \mathcal{H}_{-p'}$ for $p'>p$, to show that the set $\mathcal{E}_\ell$ is pre-compact, it is enough to prove that for each $p>(d+2)/2$, $\epsilon>0$, $T>0$, and smooth functions $g\colon \mathbb{T}^d\to\mathbb{R}$, $H\colon \mathbb{T}^d\to\mathbb{R}^d$,

$$\lim_{\delta\to 0} \sup_{P\in\mathcal{E}_\ell} P\Big[\sup_{(s,t)\in D_{T,\delta}} \big|\,\langle \boldsymbol{\pi}(t) \,-\, \boldsymbol{\pi}(s)\,,\, g\rangle\big| \;>\; \epsilon \Big] \;=\; 0\,,$$

$$\lim_{A\to\infty} \sup_{P\in\mathcal{E}_\ell} P\Big[\sup_{0\le t\le T} \big\|\, \boldsymbol{J}(t)\, \big\|_{-p} \;>\; A \Big] \;=\; 0\,, \tag{4.2}$$

$$\lim_{\delta\to 0} \sup_{P\in\mathcal{E}_\ell} P\Big[\sup_{(s,t)\in D_{T,\delta}} \big|\,\langle \boldsymbol{J}(t) \,-\, \boldsymbol{J}(s)\,,\, H\rangle\big| \;>\; \epsilon \Big] \;=\; 0\,,$$

where $D_{T,\delta}$ has been introduced before the statement of Lemma 4.3.

To prove the first assertion in (4.2), fix $\epsilon>0$, $T>0$ and a smooth function $g\colon \mathbb{T}^d\to\mathbb{R}$. For $\delta>0$, let

$$\mathcal{B} \;=\; \mathcal{B}^{\delta,\epsilon}_{g,T} \;:=\; \Big\{ (\boldsymbol{\pi},\boldsymbol{J})\in\mathcal{S}\colon\, \sup_{(s,t)\in D_{T,\delta}} \big|\,\langle \boldsymbol{\pi}(t) \,-\, \boldsymbol{\pi}(s)\,,\, g\rangle\big| \;>\; \epsilon \Big\}\,.$$

Fix $P \in \mathcal{E}_\ell$. By definition of the set $\mathcal{E}_\ell$, there exists $N \ge 1$ such that $\boldsymbol{I}_{N,K_N}(P) \le \ell N^d$. Furthermore, by definition (3.4) of the rate function $\boldsymbol{I}_{N,K_N}$, $P = \mathbb{P} \circ (\boldsymbol{\pi}_N, \boldsymbol{J}_N)^{-1}$ for some $\mathbb{P} \in \mathcal{P}^{N,K}_{\rm stat}$, and $\boldsymbol{I}_{N,K_N}(P) = H_{N,K_N}(\mathbb{P})$.

Since the set $\mathcal{B}$ is measurable with respect to $\sigma\big\{(\boldsymbol{\pi}(t), \boldsymbol{J}(t)),\, t \in [0,T]\big\}$, by the definition (3.2) of the relative entropy $\mathbb{H}^{(T)}$ and by the entropy inequality (see e.g. [15, Proposition A1.8.2]),

$$P[\mathcal{B}] \;=\; \mathbb{P}\big[\,(\boldsymbol{\pi}_N, \boldsymbol{J}_N) \in \mathcal{B}\,\big] \;\le\; \frac{\log 2 \,+\, \mathbb{H}^{(T)}(\mathbb{P}\,|\,\mathbb{P}^N_{\mu_{N,K_N}})}{\log\Big(1 + \big(\mathbb{P}^N_{\mu_{N,K_N}}\big[\,(\boldsymbol{\pi}_N, \boldsymbol{J}_N) \in \mathcal{B}\,\big]\big)^{-1}\Big)}\,\cdot$$

By (3.3), and since $H_{N,K_N}(\mathbb{P}) = \boldsymbol{I}_{N,K_N}(P) \le \ell N^d$,

$$P[\mathcal{B}] \;\le\; \frac{\log 2 \,+\, T\,\ell\,N^d}{\log\Big(1 + \big(\mathbb{P}^N_{\mu_{N,K_N}}\big[\,(\boldsymbol{\pi}_N, \boldsymbol{J}_N) \in \mathcal{B}\,\big]\big)^{-1}\Big)}\,\cdot$$

This bound is uniform over $P \in \mathcal{E}_\ell$ provided we take the supremum over $N \ge 1$ on the right-hand side.

Fix $a > 0$, and let $\gamma = (\log 2 + T\,\ell)/a$. By Lemma 4.3, there exists $\delta_0 = \delta_0(T, g, \epsilon, \gamma)$ and $N_0 = N_0(T, g, \epsilon, \gamma)$ such that

$$\mathbb{P}^N_{\mu_{N,K_N}}\big[\,(\boldsymbol{\pi}_N, \boldsymbol{J}_N) \in \mathcal{B}^{\delta_0,\epsilon}_{g,T}\,\big] \;\le\; e^{-\gamma N^d}$$

for all $N \ge N_0$. By changing the value of δ_0 we may extend this inequality to all $N \ge 1$. In particular, by definition of γ,

$$\sup_{P \in \mathcal{E}_\ell} P\big[\,\mathcal{B}^{\delta_0,\epsilon}_{g,T}\,\big] \;\le\; \sup_{N\ge 1} \frac{\log 2 \,+\, T\,\ell\,N^d}{\gamma\,N^d} \;\le\; a\,.$$

As $\mathcal{B}^{\delta,\epsilon}_{g,T} \subset \mathcal{B}^{\delta_0,\epsilon}_{g,T}$ for $0 < \delta \le \delta_0$, the previous inequality holds for all $0 < \delta \le \delta_0$. Since $a > 0$ is arbitrary, this proves the first assertion of (4.2).

The second and third assertions in (4.2) are proven similarly, based on Lemmata 4.4 and 4.5. □

4.2 **The Γ-liminf.** Let $(P_N : N \ge 1)$ be a sequence of probability measures in $\mathcal{P}_{\rm stat}$ such that $\liminf_N N^{-d} \boldsymbol{I}_{N,K_N}(P_N) < \infty$. The following lemma lists properties of the cluster points of these sequences.

LEMMA 4.6. *Let $(P_N : N \ge 1)$ be a sequence of probability measures in $\mathcal{P}_{\rm stat}$ such that $\liminf_N N^{-d} \boldsymbol{I}_{N,K_N}(P_N) < +\infty$. Assume that $P_N \to P$ for some $P \in \mathcal{P}_{\rm stat}$. Then, P-almost surely $(\boldsymbol{\pi}, \boldsymbol{J})$ belongs to $\mathcal{S}_{m,\rm ac}$ and there exists a constant C_0 such that for all $T > 0$,*

$$\begin{aligned} &E_P\Big[\int_0^T dt \int_{\mathbb{T}^d} dx\, \frac{|\boldsymbol{j}(t,x)|^2}{\sigma(\boldsymbol{\rho}(t,x))} \;+\; \int_0^T dt \int_{\mathbb{T}^d} dx\, \frac{|\nabla \boldsymbol{\rho}(t,x)|^2}{\sigma(\boldsymbol{\rho}(t,x))}\,\Big] \\ &\qquad \le\; C_0\,(1 + T) \,+\, 2T \liminf_N \frac{1}{N^d} \boldsymbol{I}_{N,K_N}(P_N)\,, \end{aligned}$$

where $\boldsymbol{\pi}(t,dx) = \boldsymbol{\rho}(t,x)\,dx$ and $\boldsymbol{J}(t) \;=\; \int_0^t \boldsymbol{j}(s)\,ds$.

PROOF. By passing to a subsequence, we may assume $\liminf_N N^{-d} \boldsymbol{I}_{N,K_N}(P_N) = \lim_N N^{-d} \boldsymbol{I}_{N,K_N}(P_N)$. By (3.4) and (3.3), for every $T > 0$,

$$\boldsymbol{I}_{N,K_N}(P_N) \;\ge\; \frac{1}{T}\, \mathbb{H}^{(T)}\big(\,\mathbb{Q}_N \,\big|\, \mathbb{P}^N_{\mu_{N,K_N}}\,\big)\,,$$

where $\mathbb{Q}_N$ is the unique stationary probability on $D(\mathbb{R},\Sigma_{N,K})$ such that $\mathbb{Q}_N\circ(\boldsymbol{\pi}_N,\boldsymbol{J}_N)^{-1}=P_N$. In particular, for every $T>0$,

$$\limsup_N \frac{1}{N^d}\,\mathbb{H}^{(T)}\big(\mathbb{Q}_N\,\big|\,\mathbb{P}^N_{\mu_{N,K_N}}\big)\;\le\;T\,\lim_N \frac{1}{N^d}\boldsymbol{I}_{N,K_N}(P_N)\,. \tag{4.3}$$

By this bound, the marginal of $\mathbb{Q}_N$ in the time interval $[0,T]$ is absolutely continuous with respect to the marginal of $\mathbb{P}^N_{\mu_{N,K_N}}$ in the same interval. Moreover, for each continuous function $g\colon\mathbb{R}\times\mathbb{T}^d\to\mathbb{R}$ with support on $(0,T)\times\mathbb{T}^d$, P_N-almost surely,

$$\Big|\int_{\mathbb{R}}dt\int_{\mathbb{T}^d}\boldsymbol{\pi}(t,dx)\,g(t,x)\Big|\;\le\;\frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}\int_{\mathbb{R}}dt\,\big|\,g(t,x)\,\big|$$

because there is at most one particle per site. Since the left-hand side is a continuous function of $\boldsymbol{\pi}$ in the Skorohod topology, taking the limit $N\to\infty$ we deduce that P-almost surely

$$\Big|\int_{\mathbb{R}}dt\int_{\mathbb{T}^d}\boldsymbol{\pi}(t,dx)\,g(t,x)\Big|\;\le\;\int_{\mathbb{R}}dt\int_{\mathbb{T}^d}\big|\,g(t,x)\,\big|\,dx\,.$$

This implies that P-almost surely, for Lebesgue almost all t, the measure $\boldsymbol{\pi}(t,dx)$ is absolutely continuous with respect to the Lebesgue measure: $\boldsymbol{\pi}(t,dx)=\boldsymbol{\rho}(t,x)\,dx$ for some density $\boldsymbol{\rho}$ satisfying $0\le\boldsymbol{\rho}\le 1$.

On the other hand, as $\mathbb{Q}_N$ is absolutely continuous with respect to $\mathbb{P}^N_{\mu_{N,K_N}}$, $P_N[\boldsymbol{\pi}(t,\mathbb{T}^d)=K_N/N^d]=1$ for all $t\in\mathbb{R}$. As $K_N/N^d\to m$ and $P_N\to P$, $P[\boldsymbol{\pi}(t,\mathbb{T}^d)=m]=1$ for Lebesgue almost all $t\in\mathbb{R}$.

For a vector field F in $C^1(\mathbb{R}\times\mathbb{T}^d;\mathbb{R}^d)$ with compact support in $(0,T)\times\mathbb{T}^d$, $\epsilon>0$ and $a>0$, let

$$\begin{aligned}\mathcal{E}_{a,\epsilon}(F,\boldsymbol{\pi})\;&=\;\int_{\mathbb{R}}dt\,\big\langle\boldsymbol{\pi}(t)\,,\,\nabla\cdot F(t)\big\rangle\;-\;a\int_{\mathbb{R}}dt\int_{\mathbb{T}^d}dx\,\sigma(\boldsymbol{\pi}^\epsilon)\,|F|^2\,,\\ \mathcal{V}_{a,\epsilon}(F,\boldsymbol{\pi},\boldsymbol{J})\;&=\;\boldsymbol{J}(F)\;-\;a\int_{\mathbb{R}}dt\int_{\mathbb{T}^d}dx\,\sigma(\boldsymbol{\pi}^\epsilon)\,|F|^2\,.\end{aligned} \tag{4.4}$$

where $(\boldsymbol{\pi}^\epsilon)(t,x)=(2\epsilon)^{-d}\boldsymbol{\pi}(t,[x-\epsilon,x+\epsilon]^d)$ and $\boldsymbol{J}(F)=-\int_{\mathbb{R}}dt\,\langle\boldsymbol{J}(t),\partial_tF\rangle$.

Let $(F_j:j\ge1)$ be a family of vector fields in $C^1((0,T)\times\mathbb{T}^d;\mathbb{R}^d)$ with compact support and dense in $L^2([0,T]\times\mathbb{T}^d;\mathbb{R}^d)$. Assume that $F_1=0$. In view of Lemma A.5, the entropy bound (4.3), and a classical argument which allows to bound a maximum over a finite set in exponential estimates, there exist finite constants a and C_0 such that for all $k\ge1$,

$$\limsup_{\epsilon\to0}\limsup_{N\to\infty}E_{P_N}\Big[\max_{1\le j\le k}\,\mathcal{E}_{a,\epsilon}(F_j,\boldsymbol{\pi})\Big]\;\le\;A\,,$$

$$\limsup_{\epsilon\to0}\limsup_{N\to\infty}E_{P_N}\Big[\max_{1\le j\le k}\,\mathcal{V}_{a,\epsilon}(F_j,\boldsymbol{\pi},\boldsymbol{J})\Big]\;\le\;A\,,$$

where

$$A\;:=\;C_0\,(1+T)\;+\;T\,\lim_N\frac{1}{N^d}\boldsymbol{I}_{N,K_N}(P_N)\,.$$

Since P_N converges to P that is concentrated on measures which are absolutely continuous with respect to the Lebesgue measure, $\boldsymbol{\pi}(t,dx)=\boldsymbol{\rho}(t,x)\,dx$, and whose density $\boldsymbol{\rho}$ is bounded below by 0 and above by 1, taking the limit in N and ϵ yields

$$E_P\Big[\max_{1\le j\le k}\Big\{\boldsymbol{J}(F_j)\;-\;a\int_{\mathbb{R}}dt\int_{\mathbb{T}^d}dx\,\sigma(\boldsymbol{\rho})\,|F_j|^2\Big\}\Big]\;\le\;A\,,$$

$$E_P\Big[\max_{1\le j\le k}\Big\{\int_{\mathbb{R}}dt\int_{\mathbb{T}^d}dx\,\Big[\boldsymbol{\rho}\,\nabla\cdot F_j\;-\;a\,\sigma(\boldsymbol{\rho})\,|F_j|^2\Big]\Big\}\Big]\;\le\;A\,.$$

Each maximum is positive because $F_1 = 0$. By monotone convergence, taking the limit in k, we obtain a similar bound, where the maximum over $1 \le j \le k$ is replaced by the maximum over $j \ge 1$. Since the sequence F_j is dense in $L^2([0,T] \times \mathbb{T}^d; \mathbb{R}^d)$, by Riesz representation theorem, P-almost surely, $\boldsymbol{J}(t) = \int_0^t \boldsymbol{j}(s)\, ds$ for some $\boldsymbol{j}$ in $L^2([0,T] \times \mathbb{T}^d, \sigma(\boldsymbol{\rho})^{-1} dt\, dx; \mathbb{R}^d)$. These arguments also yield the bounds stated in the lemma.

We turn to the proof that P-almost surely $\boldsymbol{\pi} \in C\big(\mathbb{R}; \mathcal{M}_m(\mathbb{T}^d)\big)$. By the continuity equation, P-almost surely for all functions g in $C^1(\mathbb{T}^d)$ and $0 \le s < t \le T$,

$$\int_{\mathbb{T}^d} \boldsymbol{\pi}(t, dx)\, g(x) \;-\; \int_{\mathbb{T}^d} \boldsymbol{\pi}(s, dx)\, g(x) \;=\; \int_s^t dr \int_{\mathbb{T}^d} dx\, \boldsymbol{j}(r, x) \cdot (\nabla g)(x)\, .$$

Since $\boldsymbol{j}$ belongs to $L^2([0,T] \times \mathbb{T}^d, \sigma(\boldsymbol{\rho})^{-1} dt\, dx; \mathbb{R}^d)$, P-almost surely $\boldsymbol{\pi}$ belongs to $C([0,T], \mathcal{M}_m(\mathbb{T}^d))$, as claimed. □

Fix $T > 0$ and a continuous vector field $w \colon [0,T] \times \mathbb{T}^d \times \mathcal{M}_+(\mathbb{T}^d) \times D(\mathbb{R}; \mathcal{H}^d_{-p}) \to \mathbb{R}^d$ that is continuously differentiable in x and such that for each $(x, \pi) \in \mathbb{T}^d \times \mathcal{M}_+(\mathbb{T}^d)$ and $t \in [0,T]$ the map $[0,t] \times D(\mathbb{R}; \mathcal{H}^d_{-p}) \ni (s, \boldsymbol{J}) \to w(s, x, \pi, \boldsymbol{J})$ is measurable with respect to the Borel σ-algebra on $[0,t] \times D\big([0,t]; \mathcal{H}^d_{-p}\big)$. Let $G_w \colon \mathbb{R} \times \mathbb{T}^d \times \mathcal{S}_{m,\mathrm{ac}} \to \mathbb{R}^d$ be the progressively measurable map defined by

$$G_w(t, x, \boldsymbol{\pi}, \boldsymbol{J}) \;=\; w(t, x, \boldsymbol{\pi}(t), \boldsymbol{J})\, . \tag{4.5}$$

Finally, for $(\boldsymbol{\pi}, \boldsymbol{J}) \in \mathcal{S}_{m,\mathrm{ac}}$, let

$$V_{T,w}(\boldsymbol{\pi}, \boldsymbol{J}) \;=\; \frac{1}{T} \int_0^T dt \int_{\mathbb{T}^d} dx \Big\{ G_w \cdot \big[\, \boldsymbol{j} \,+\, \nabla \boldsymbol{\rho} \,-\, \sigma(\boldsymbol{\rho})\, E \,\big] \,-\, \sigma(\boldsymbol{\rho})\, |G_w|^2 \Big\}\, , \tag{4.6}$$

where $\boldsymbol{\pi}(t, dx) = \boldsymbol{\rho}(t,x)\, dx$, $\boldsymbol{J}(t) \;=\; \int_0^t \boldsymbol{j}(s)\, ds$, $t \in \mathbb{R}$.

LEMMA 4.7. *Let $(P_N \colon N \ge 1)$ be a sequence of probability measures in $\mathcal{P}_{\mathrm{stat}}$ such that $\liminf_N N^{-d} \boldsymbol{I}_{N,K_N}(P_N) < +\infty$. Assume that $P_N \to P$ for some $P \in \mathcal{P}_{\mathrm{stat}}$. Then, for each $T > 0$ and each function w as above,*

$$\liminf_{N \to \infty} \frac{1}{N^d}\, \boldsymbol{I}_{N,K_N}(P_N) \;\ge\; E_P\big[\, V_{T,w} \,\big]\, . \tag{4.7}$$

By Lemma 4.6, P–almost surely $(\boldsymbol{\pi}, \boldsymbol{J})$ belong to $\mathcal{S}_{m,\mathrm{ac}}$ so that the right hand side of (4.7) is well defined.

PROOF OF LEMMA 4.7. By passing to a subsequence, if needed, we may assume that $\liminf_N N^{-d} \boldsymbol{I}_{N,K_N}(P_N) = \lim_N N^{-d}\, \boldsymbol{I}_{N,K_N}(P_N)$. By definition of $\mathbb{H}_{N,K}(t, \mathbb{Q})$, (3.4) and (3.1), for each $T > 0$ and each bounded, continuous functions Φ on $D(\mathbb{R}, \Sigma_{N,K_N})$, measurable with respect to $\sigma\big\{\boldsymbol{\eta}(t),\, t \in [0,T]\big\}$,

$$\boldsymbol{I}_{N,K_N}(P_N) \;\ge\; \frac{1}{T} \int d\mathbb{Q}_N(\boldsymbol{\eta}) \Big\{ \Phi(\boldsymbol{\eta}) \,-\, \log \mathbb{E}^N_{\boldsymbol{\eta}(0)}\big[e^{\Phi}\big] \Big\}\, ,$$

where $\mathbb{Q}_N$ is the unique stationary probability measure on $D(\mathbb{R}, \Sigma_{N,K_N})$ such that $\mathbb{Q}_N \circ (\boldsymbol{\pi}_N, \boldsymbol{J}_N)^{-1} = P_N$.

Recalling (2.6), let Φ be given by

$$\Phi(\boldsymbol{\eta}) = \sum_{(x,y) \in \mathbb{B}_N} \Big\{ \int_0^T \phi^{x,y}(t)\, \mathcal{N}^{x,y}_{(t,t+dt]}(\boldsymbol{\eta}) \,-\, N^2 \int_0^T \boldsymbol{\eta}_x(t)[1 - \boldsymbol{\eta}_y(t)]\big[e^{\phi^{x,y}(t)} - 1\big]\, dt \Big\}\, ,$$

where

$$\phi^{x,y}(t) \;=\; \int_x^y w(t,\cdot\,,\boldsymbol{\pi}_N(t),\boldsymbol{J}_N)\cdot d\ell\,.$$

By Lemma A.1, $\mathbb{E}^N_\eta\big[e^{\Phi}\big]=1$ for each $\eta\in\Sigma_{N,K}$ and by Lemma A.6,

$$\frac{1}{T}\lim_{N\to\infty}\frac{1}{N^d}\,E_{\mathbb{Q}_N}[\Phi] \;=\; E_P\big[\,V_{T,w}\,\big]\,,$$

which completes the proof. □

PROOF OF THEOREM 4.1. Γ-LIMINF. Fix P in $\mathcal{P}_{\rm stat}$ and a sequence $(P_N:N\ge 1)$, $P_N\in\mathcal{P}_{\rm stat}$, such that $P_N\to P$, $\liminf_N N^{-d}\boldsymbol{I}_{N,K_N}(P_N)<+\infty$. By Lemma 4.6, we may assume that P-almost surely $(\boldsymbol{\pi},\boldsymbol{J})$ belongs to $\mathcal{S}_{m,\rm ac}$ and that there exists a constant C_1 such that for all $T>0$,

$$E_P\Big[\int_0^T dt\int_{\mathbb{T}^d}dx\,\frac{|\boldsymbol{j}(t,x)|^2}{\sigma(\boldsymbol{\rho}(t,x))}\;+\;\int_0^T dt\int_{\mathbb{T}^d}dx\,\frac{|\nabla\boldsymbol{\rho}(t,x)|^2}{\sigma(\boldsymbol{\rho}(t,x))}\,\Big]\;\le\;C_1\,(1+T)\,, \tag{4.8}$$

where $\boldsymbol{\pi}(t,dx)=\boldsymbol{\rho}(t,x)\,dx$, $\boldsymbol{J}(t)\;=\;\int_0^t\boldsymbol{j}(s)\,ds$.

By Lemma 4.7, it is enough to show that for some $T>0$,

$$\boldsymbol{I}_m(P)\;\le\;\sup_w E_P\big[\,V_{T,w}\big]\,, \tag{4.9}$$

where the supremum is carried over all continuous vector fields $w\colon[0,T]\times\mathbb{T}^d\times\mathcal{M}_+(\mathbb{T}^d)\times D(\mathbb{R};\mathcal{H}^d_{-p})\to\mathbb{R}^d$ satisfying the assumption enunciated above (4.5).

Fix $T>0$ and a bounded and continuous function $f\colon\mathbb{T}^d\times\mathcal{H}_1\times L^2(\mathbb{T}^d;\mathbb{R}^d)\to\mathbb{R}^d$ that is continuously differentiable in x. For $\delta>0$, let $f_\delta\colon(0,T)\times\mathbb{T}^d\times\mathcal{M}_+(\mathbb{T}^d)\times D\big(\mathbb{R};\mathcal{H}^d_{-p}\big)\to\mathbb{R}^d$ be given by

$$f_\delta(t,x,\pi,\boldsymbol{J})\;=\;\chi_\delta(t)\,f\big(x,\pi_\delta,\boldsymbol{j}_\delta(t)\big)\,.$$

Here χ_δ, $0<\delta<1$, stands for a sequence of continuous functions, bounded below by 0 and above by 1, whose support is contained in $[\delta,T]$, and which converges in L^1 to the indicator functions of the set $[0,T]$. Moreover, $\pi_\delta(x)=\langle\pi,\kappa_\delta(x-\cdot)\rangle$ where $\kappa_\delta\colon\mathbb{T}^d\to\mathbb{R}_+$ is a smooth approximation of the identity, and

$$\boldsymbol{j}_\delta(t,x)\;=\;\int_{-\infty}^t ds\;a'_\delta(t-s)\,\langle\boldsymbol{J}(s),\imath_\delta(x-\cdot)\rangle\,, \tag{4.10}$$

where $a_\delta\colon\mathbb{R}\to\mathbb{R}_+$ is a smooth approximation of identity with compact support in $(0,\delta)$, a'_δ the derivative of a_δ, and $\imath_\delta\colon\mathbb{T}^d\to\mathbb{R}^d$ another smooth approximation of the identity. Observe that $\boldsymbol{j}_\delta(t)$ depends on $\boldsymbol{J}(s)$ only for $s\in(t-\delta,t)$. Hence, since $\chi_\delta(t)=0$ for $t\le\delta$, the function $f_\delta(t,x,\pi,\cdot)$ depends on $\boldsymbol{J}(s)$ only for $s\in[0,t]$. This is a requirement of the test functions w introduced above (4.5). Since f_δ satisfies the conditions presented above (4.5) for each $\delta>0$, we deduce

$$\sup_w E_P\big[\,V_{T,w}\big]\;\ge\;\limsup_{\delta\to 0}E_P\big[\,V_{T,f_\delta}\big]\,.$$

Let $H_f\colon[0,T]\times\mathbb{T}^d\times L^2\big([0,T];\mathcal{H}_1\big)\times L^2([0,T]\times\mathbb{T}^d;\mathbb{R}^d)\to\mathbb{R}^d$ be defined by

$$H_f(t,x,\boldsymbol{\rho},\boldsymbol{j})\;=\;f(x,\boldsymbol{\rho}(t),\boldsymbol{j}(t))\,.$$

By (4.8), P–almost surely, $\boldsymbol{\pi}_\delta(t)\to\boldsymbol{\rho}(t)$ and $\boldsymbol{j}_\delta(t)\to\boldsymbol{j}(t)$ for Lebesgue almost all $t\in(0,T)$. Since f is bounded, by dominated convergence,

$$\lim_{\delta\to 0}E_P\Big[\int_0^T dt\int_{\mathbb{T}^d}dx\,\Big\{\,G_{f_\delta}(t,x,\boldsymbol{\pi},\boldsymbol{J})\;-\;H_f(t,x,\boldsymbol{\rho},\boldsymbol{j})\Big\}^2\,\Big]\;=\;0\,, \tag{4.11}$$

where G_{f_δ} has been introduced in (4.5).

Denote by $W_{T,f}$ the right-hand side of (4.6) when G_w is replaced by H_f. By (4.8), (4.11) and the dominated convergence theorem,

$$\lim_{\delta\to 0} E_P\big[\, V_{T,f_\delta}\big] \;=\; E_P\big[\, W_{T,f}\big]\,.$$

In view of the previous estimates, it remains to show that

$$\sup_f E_P\big[\, W_{T,f}\big] \;\geq\; \boldsymbol{I}_m(P)\,. \tag{4.12}$$

Let $\hat f\colon \mathbb{T}^d\times\mathcal{H}_1\times L^2(\mathbb{T}^d;\mathbb{R}^d)\to\mathbb{R}^d$ be given by

$$\hat f(x,\rho,j) \;=\; \frac{j(x)+(\nabla\rho)(x)-\sigma(\rho(x))\,E(x)}{2\,\sigma(\rho(x))}$$

and note that $\boldsymbol{I}_m(P)=E_P[W_{T,\hat f}]$, which is finite in view of (4.8). Using this bound and approximating the function $\hat f$ by a sequence of bounded and continuous functions that are continuously differentiable in x, we obtain (4.12) by the dominated convergence theorem. □

4.3 The Γ-limsup. Given $P\in\mathcal{P}_{\rm stat}$, we shall construct a sequence $(P_N : N\geq 1)$, such that $P_N\to P$ and

$$\limsup_{N\to\infty} \boldsymbol{I}_{N,K_N}(P_N) \;\leq\; \boldsymbol{I}_m(P)\,.$$

We carry this out first for P satisfying certain regularity assumptions, and then use density arguments to extend the result to any P with finite rate function, $\boldsymbol{I}_m(P)<+\infty$.

Fix $T>0$. Recalling (2.22), a path $(\boldsymbol{\pi},\boldsymbol{J})$ in $\mathcal{S}_m$ is T-periodic if $\vartheta_T(\boldsymbol{\pi},\boldsymbol{J})=(\boldsymbol{\pi},\boldsymbol{J})$. An element P in $\mathcal{P}_{\rm stat}$ is said to be T-*holonomic* if there exists a T-periodic path $(\boldsymbol{\pi},\boldsymbol{J})$ such that

$$P \;=\; \frac{1}{T}\int_0^T \delta_{\vartheta_s(\boldsymbol{\pi},\boldsymbol{J})}\, ds\,. \tag{4.13}$$

An element of $\mathcal{P}_{\rm stat}$ is *holonomic* if it is T-holonomic for some $T>0$.

Fix $T>0$ and a T-periodic path $(\boldsymbol{\pi},\boldsymbol{J})$ in $\mathcal{S}_{m,\rm ac}$. Denote by $(\boldsymbol{\rho},\boldsymbol{j})$ the densities so that $\boldsymbol{\pi}(t,dx)=\boldsymbol{\rho}(t,x)\,dx$, $\boldsymbol{J}(t)=\int_0^t ds\,\boldsymbol{j}(s)$. Assume that $\boldsymbol{\rho}$, $\boldsymbol{j}$ are smooth and that there exists $\delta>0$ such that $\delta\leq\boldsymbol{\rho}(t,x)\leq 1-\delta$ for all (t,x). Denote by $F\colon\mathbb{R}\times\mathbb{T}^d\to\mathbb{R}^d$ the smooth, T-periodic vector field defined by

$$F \;=\; \frac{\boldsymbol{j}\,+\,\nabla\boldsymbol{\rho}\,-\,\sigma(\boldsymbol{\rho})E}{\sigma(\boldsymbol{\rho})}\,.$$

As the path $(\boldsymbol{\rho},\boldsymbol{J})$ satisfies the continuity equation (2.21), $\partial_t\boldsymbol{\rho}+\nabla\cdot\boldsymbol{j}=0$, by definition of F,

$$\partial_t\boldsymbol{\rho} \;=\; \Delta\boldsymbol{\rho} \,-\, \nabla\cdot\big(\sigma(\boldsymbol{\rho})\big[\,E+F(t)\,\big]\big)\,. \tag{4.14}$$

Let finally $P\in\mathcal{P}_{\rm stat}$ be the T-holonomic probability corresponding, as in (4.13), to the T-periodic path $(\boldsymbol{\pi},\boldsymbol{J})$.

Let L_N^F be the time-dependent generator of a perturbed WASEP defined by

$$(L_N^F f)(\eta) \;=\; N^2 \sum_{(x,y)\in\mathbb{B}_N} \eta_x\,[1-\eta_y]\, e^{(1/2)[\,E_N(x,y)+F_N(t,x,y)]}\big[\,f(\sigma^{x,y}\eta)-f(\eta)\big]\,, \tag{4.15}$$

where $F_N(t,x,y)$ represents the line integral of $F(t,\cdot)$ along the oriented segment from x to y introduced in (2.2). Denote by $(\boldsymbol{\eta}^F(t) : t \geq 0)$ the continuous-time, time-inhomogeneous Markov chain whose generator is L_N^F. By the hydrodynamic limit of the time dependent WASEP dynamics, see e.g. [15, Section 10.5], the empirical density $\boldsymbol{\pi}_N(\boldsymbol{\eta}^F)$ associated to the process $\boldsymbol{\eta}^F$ evolves, in the limit $N \to \infty$, according to the solution of the PDE (4.14). This explains the introduction of the process $\boldsymbol{\eta}^F$.

Let $(\xi_k : k \geq 0)$ be the discrete-time, Σ_N-valued Markov chain given by $\xi_k = \boldsymbol{\eta}^F(kT)$. Since F is T-periodic, ξ_k is time-homogeneous. As it is irreducible on each set $\Sigma_{N,K}$, ξ_k has a unique stationary state, denoted by $\mu_{N,K}^F$. Let finally, $\mathbb{P}_{N,K}^F$ be the law of $\boldsymbol{\eta}^F$ when the initial condition is sampled according to $\mu_{N,K}^F$. Note that $\mathbb{P}_{N,K}^F$ is invariant by T-translations: $\mathbb{P}_{N,K}^F \circ \vartheta_T^{-1} = \mathbb{P}_{N,K}^F$. Since this measure, defined on $D(\mathbb{R}_+, \Sigma_N)$, is invariant by T-translations, we may extend it to $D(\mathbb{R}, \Sigma_N)$.

Let P_N be the measure on $\mathcal{S}$ given by

$$P_N \,=\, \Big(\frac{1}{T}\int_0^T \mathbb{P}_{N,K_N}^F \circ \vartheta_t^{-1}\, dt\Big) \circ (\boldsymbol{\pi}_N, \boldsymbol{J}_N)^{-1}\,, \tag{4.16}$$

that, by construction, belongs to $\mathcal{P}_{\mathrm{stat}}$.

PROPOSITION 4.8. *For each T-periodic path $(\boldsymbol{\pi}, \boldsymbol{J})$ in $\mathcal{S}_{m,\mathrm{ac}}$ with smooth densities $(\boldsymbol{\rho}, \boldsymbol{j})$ with $\boldsymbol{\rho}$ bounded away from zero and one, the sequence $(P_N : N \geq 1)$ introduced in* (4.16) *converges to the T-holonomic probability P given by* (4.13) *and*

$$\limsup_{N\to\infty} \frac{1}{N^d} \boldsymbol{I}_{N,K_N}(P_N) \,\leq\, \boldsymbol{I}_m(P)\,.$$

The proof of this proposition relies on the following lemma.

LEMMA 4.9. *In the setting of Proposition 4.8, the sequence of probability measures on $D(\mathbb{R}; \mathcal{M}_+(\mathbb{T}^d))$ given by $\mathbb{P}_{N,K_N}^F \circ \boldsymbol{\pi}_N^{-1}$ converges to $\delta_{\boldsymbol{\pi}}$.*

PROOF. By the smoothness of the external field F, Lemma 4.3 holds also when $\mathbb{P}_{\mu_{N,K_N}}^N$ is replaced by $\mathbb{P}_{N,K_N}^F$. By the compactness of $\mathcal{M}_+(\mathbb{T}^d)$, this implies the pre-compactness of the family $\big(\mathbb{P}_{N,K_N}^F \circ \boldsymbol{\pi}_N^{-1} : N \geq 1\big)$. Let $\mathcal{P}^F$ be a cluster point of this sequence. By the T-periodicity of $\mathbb{P}_{N,K_N}^F$ and the hydrodynamic limit for the time dependent WASEP with generator (4.15), $\mathcal{P}^F$ is T periodic and $\mathcal{P}^F$ almost surely $\boldsymbol{\pi}(t,dx) = \boldsymbol{\rho}^F(t,x)dx$ for some density $\boldsymbol{\rho}^F$ of mass m that solves (4.14). By the uniqueness of T-periodic solutions to (4.14) and the $L^1(\mathbb{T}^d)$ convergence to this unique solution, as stated in Theorem 7.1, $\boldsymbol{\rho}^F = \boldsymbol{\rho}$. Hence $\mathcal{P}^F = \delta_{\boldsymbol{\pi}}$ as claimed. □

PROOF OF PROPOSITION 4.8. By the smoothness of the external field F, Lemmata 4.3, 4.4, and 4.5 holds also when $\mathbb{P}_{\mu_{N,K_N}}^N$ is replaced by $\mathbb{P}_{N,K_N}^F$. This implies the pre-compactness of the sequence of probabilities on $\mathcal{S}$ given by $\big\{\mathbb{P}_{N,K_N}^F \circ (\boldsymbol{\pi}_N, \boldsymbol{J}_N)^{-1}\big\}$. Let now P_0 be a cluster point of this sequence. By the hydrodynamic limit for the perturbed WASEP, P_0-almost surely, $(\boldsymbol{\pi}, \boldsymbol{J})$ belongs to $\mathcal{S}_{m,\mathrm{ac}}$ and the corresponding densities $(\boldsymbol{\rho}, \boldsymbol{j})$ is a T-periodic weak solution to the hydrodynamic equation

$$\begin{cases} \partial_t \boldsymbol{\rho} \,+\, \nabla\cdot \boldsymbol{j} \,=\, 0\,, \\ \boldsymbol{j} \,=\, -\nabla\boldsymbol{\rho} \,+\, \sigma(\boldsymbol{\rho})\,[\,E+F\,]\,. \end{cases}$$

By Lemma 4.9, $P_0 = \delta_{(\boldsymbol{\pi},\boldsymbol{J})}$. Taking time-averages, we deduce $P_N \to P$.

We turn to the second claim of the proposition. By (3.4),

$$\boldsymbol{I}_{N,K_N}(P_N) \;=\; H_{N,K_N}\Big(\frac{1}{T}\int_0^T \mathbb{P}^F_{N,K_N}\circ\vartheta_t^{-1}\,dt\Big)\,. \tag{4.17}$$

Fix $\ell\in\mathbb{N}$. By (3.3) and the convexity of the relative entropy, the right-hand side of the previous equation is equal to

$$\begin{aligned}&\lim_{\ell\to\infty}\frac{1}{\ell T}\,\mathbb{H}^{(\ell T)}\Big(\frac{1}{T}\int_0^T \mathbb{P}^F_{N,K_N}\circ\vartheta_t^{-1}\,dt\,\Big|\,\mathbb{P}^N_{\mu_{N,K_N}}\Big)\\ &\quad\le\;\lim_{\ell\to\infty}\frac{1}{\ell T^2}\int_0^T \mathbb{H}^{(\ell T)}\Big(\mathbb{P}^F_{N,K_N}\circ\vartheta_t^{-1}\,\Big|\,\mathbb{P}^N_{\mu_{N,K_N}}\Big)\,dt\,.\end{aligned}$$

Since $\mathbb{P}^N_{\mu_{N,K_N}}$ is translation invariant, for fixed ℓ and $0\le t\le T$, by definition of the translations $(\vartheta_s : s\in\mathbb{R})$, introduced in (2.22), recalling (3.2),

$$\begin{aligned}\mathbb{H}^{(\ell T)}\Big(\mathbb{P}^F_{N,K_N}\circ\vartheta_t^{-1}\,\Big|\,\mathbb{P}^N_{\mu_{N,K_N}}\Big)\;&=\;\mathbb{H}_{[0\,,\,\ell T]}\Big(\mathbb{P}^F_{N,K_N}\circ\vartheta_t^{-1}\,\Big|\,\mathbb{P}^N_{\mu_{N,K_N}}\circ\vartheta_t^{-1}\Big)\\ &=\;\mathbb{H}_{[-t\,,\,\ell T-t]}\Big(\mathbb{P}^F_{N,K_N}\,\Big|\,\mathbb{P}^N_{\mu_{N,K_N}}\Big)\,.\end{aligned}$$

As $\mathbb{P}^F_{N,K_N}$ is T-translation invariant and $\mathbb{P}^N_{\mu_{N,K_N}}$ is translation invariant, the dynamical contribution to the relative entropy of the time interval $[-t,0]$ corresponds to the one of the time interval $[-t+\ell T,\ell T]$. Hence, the previous expression is equal to

$$\mathbb{H}^{(\ell T)}\Big(\mathbb{P}^F_{N,K_N}\,\Big|\,\mathbb{P}^N_{\mu_{N,K_N}}\Big)\;-\;\mathrm{Ent}\big(\mu^F_{N,K_N}\,\big|\,\mu_{N,K_N}\big)\;+\;\mathrm{Ent}\Big(\mathbb{P}^F_{N,K_N}\circ\imath_{-t}^{-1}\,\Big|\,\mu_{N,K_N}\Big)\,,$$

where $\mathbb{P}^F_{N,K_N}\circ\imath_s^{-1}$ is the marginal at time s of $\mathbb{P}^F_{N,K_N}$.

Using again that $\mathbb{P}^F_{N,K_N}$ is T-translation invariant and $\mathbb{P}^N_{\mu_{N,K_N}}$ is translation invariant,

$$\begin{aligned}&\mathbb{H}^{(\ell T)}\Big(\mathbb{P}^F_{N,K_N}\,\Big|\,\mathbb{P}^N_{\mu_{N,K_N}}\Big)\;-\;\mathrm{Ent}\big(\mu^F_{N,K_N}\,\big|\,\mu_{N,K_N}\big)\\ &\quad=\;\ell\,\mathbb{H}^{(T)}\Big(\mathbb{P}^F_{N,K_N}\,\Big|\,\mathbb{P}^N_{\mu_{N,K_N}}\Big)\;-\;\ell\,\mathrm{Ent}\big(\mu^F_{N,K_N}\,\big|\,\mu_{N,K_N}\big)\,.\end{aligned}$$

Putting together the previous estimates and letting $\ell\to\infty$ yields that

$$\boldsymbol{I}_{N,K_N}(P_N)\;\le\;\frac{1}{T}\Big\{\mathbb{H}^{(T)}\Big(\mathbb{P}^F_{N,K_N}\,\Big|\,\mathbb{P}^N_{\mu_{N,K_N}}\Big)\;-\;\mathrm{Ent}\big(\mu^F_{N,K_N}\,\big|\,\mu_{N,K_N}\big)\Big\}\,.$$

By Lemma 4.9 and the large deviations lower bound in hydrodynamical limits, see e.g. [15, Lemma 10.5.4],

$$\limsup_{N\to\infty}\frac{1}{N^d}\Big\{\mathbb{H}^{(T)}\Big(\mathbb{P}^F_{N,K_N}\,\Big|\,\mathbb{P}^N_{\mu_{N,K_N}}\Big)-\mathrm{Ent}\,\big(\mu^F_{N,K_N}\big|\mu_{N,K_N}\big)\Big\}\;\le\;A_{T,m}(\boldsymbol{\pi},\boldsymbol{J})\,.$$

Therefore,

$$\limsup_{N\to\infty}\frac{1}{N^d}\,\boldsymbol{I}_{N,K_N}(P_N)\;\le\;\frac{1}{T}\,A_{T,m}(\boldsymbol{\pi},\boldsymbol{J})\,.$$

The right-hand side is equal to $\boldsymbol{I}_m(P)$ in view of the equations (2.26), (4.13), and the T-periodicity of the path $(\boldsymbol{\pi},\boldsymbol{J})$. □

To complete the proof of the Γ-limsup, we show that any P in $\mathcal{P}_{\rm stat}$ can be approximated by convex combinations of holonomic probability measures supported by smooth paths bounded away from zero and one and that the corresponding rate function converges to $\boldsymbol{I}_m(P)$. Denote by $\mathcal{P}^\epsilon_{{\rm stat},m}$, $\epsilon>0$, the subset of $\mathcal{P}_{{\rm stat},m}$ formed by the stationary measures P such that P-almost surely $(\boldsymbol{\pi},\boldsymbol{J})$ belongs to $\mathcal{S}_{m,{\rm ac}}$ with smooth densities $(\boldsymbol{\rho},\boldsymbol{j})$ such that $\epsilon\le\boldsymbol{\rho}(t,x)\le 1-\epsilon$ for all (t,x).

THEOREM 4.10. *Assume that E is orthogonally decomposable and fix $P \in \mathcal{P}_{\rm stat}$ such that $\boldsymbol{I}_m(P) < +\infty$. There exist a sequence $(\epsilon_n : n \geq 1)$, a triangular array $(\alpha_{n,i}\,, 1 \leq i \leq n\,, n \geq 1)$ with $\alpha_{n,i} \geq 0$, $\sum_i \alpha_{n,i} = 1$, and a triangular array $(P_{n,i}\,, 1 \leq i \leq n\,, n \geq 1)$ of holonomic measures belonging to $\mathcal{P}^{\epsilon_n}_{\rm stat,m}$ such that by setting $P_n := \sum_i \alpha_{n,i} P_{n,i}$ we have $P_n \to P$ and $\boldsymbol{I}_m(P_n) \to \boldsymbol{I}_m(P)$.*

Postponing the proof of this statement, we first conclude the Γ-convergence of the Donsker-Varadhan functional.

PROOF OF THEOREM 4.1. Γ-LIMSUP. The statement follows, by a diagonal argument, from Proposition 4.8 and Theorem 4.10. □

We turn to the proof of Theorem 4.10. It relies on two lemmata.

LEMMA 4.11. *Fix P satisfying* (2.27). *There exists a sequence $(P_n \colon n \geq 1)$ converging to P and such that P_n belongs to $\mathcal{P}^{\epsilon_n}_{\rm stat,m}$ for some $\epsilon_n > 0$, and*

$$\limsup_{n\to\infty} \boldsymbol{I}_m(P_n) \;=\; \boldsymbol{I}_m(P)\,.$$

PROOF. Fix a smooth probability density $\phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}_+$ whose support is contained in $[-1,1] \times [-1,1]^d$, so that

$$\int_{\mathbb{R}\times\mathbb{R}^d} \phi(t,x)\, dt\, dx \;=\; 1\,.$$

Let $\phi_\epsilon(t,x) = \epsilon^{-(d+1)}\phi(t/\epsilon, x/\epsilon)$, $\epsilon > 0$. For a trajectory $(\boldsymbol{\pi}, \boldsymbol{J})$ in $\mathcal{S}_{m,{\rm ac}}$, whose density is represented by $(\boldsymbol{\rho}, \boldsymbol{j})$, let

$$\boldsymbol{\rho}_\epsilon \;:=\; (1-\epsilon)\,(\boldsymbol{\rho} * \phi_\epsilon) \;+\; \epsilon\, m\,, \quad \boldsymbol{j}_\epsilon \;:=\; (1-\epsilon)\,(\boldsymbol{j} * \phi_\epsilon)\,,$$

where $*$ denotes space-time convolution and $0 < \epsilon < 1$. Observe that $(\boldsymbol{\rho}_\epsilon, \boldsymbol{j}_\epsilon)$ satisfy the continuity equation.

Denote by Ψ_ϵ the map $(\boldsymbol{\rho}, \boldsymbol{j}) \mapsto (\boldsymbol{\rho}_\epsilon, \boldsymbol{j}_\epsilon) =: \Psi_\epsilon(\boldsymbol{\rho}, \boldsymbol{j})$ and set $P^\epsilon = P \circ \Psi_\epsilon^{-1}$. Then, for each $\epsilon > 0$, the probability P^ϵ belongs to $\mathcal{P}^{\delta}_{\rm stat,m}$ for some $\delta = \delta(\epsilon) > 0$ and $P^\epsilon \to P$ as $\epsilon \to 0$.

It remains to show that $\lim_\epsilon \boldsymbol{I}_m(P^\epsilon) = \boldsymbol{I}_m(P)$. As

$$\boldsymbol{I}_m(P^\epsilon) \;=\; E_P\Big[\, \frac{1}{T} \int_0^T dt \int_{\mathbb{T}^d} dx\, \frac{|\boldsymbol{j}_\epsilon + \nabla\boldsymbol{\rho}_\epsilon - \sigma(\boldsymbol{\rho}_\epsilon)\, E|^2}{4\,\sigma(\boldsymbol{\rho}_\epsilon)} \,\Big]\,,$$

and since the sequence

$$\frac{|\boldsymbol{j}_\epsilon + \nabla\boldsymbol{\rho}_\epsilon - \sigma(\boldsymbol{\rho}_\epsilon)\, E|^2}{4\,\sigma(\boldsymbol{\rho}_\epsilon)} \tag{4.18}$$

converges $(dP\, dt\, dx)$-almost surely to the same expression without the subscript ϵ, it is enough to prove that the sequence (4.18) is uniformly integrable.

Since P satisfies (2.27), by [6, Lemma 5.3] there exist increasing convex functions Υ_1, $\Upsilon_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{r\to\infty} \Upsilon_a(r)/r = \infty$, $a = 1, 2$, and

$$E_P\Big[\, \int_0^T dt \int_{\mathbb{T}^d} dx\, \Big\{\, \Upsilon_1\Big(\frac{|\boldsymbol{j}|^2}{\sigma(\boldsymbol{\rho})}\Big) + \Upsilon_2\Big(\frac{|\nabla\boldsymbol{\rho}|^2}{\sigma(\boldsymbol{\rho})}\Big) \,\Big\}\, \Big] \;<\; +\infty\,. \tag{4.19}$$

Moreover, the uniform integrability of the sequence (4.18) follows from the bound

$$\limsup_{\epsilon\to 0} E_P\Big[\, \int_0^T dt \int_{\mathbb{T}^d} dx\, \Big\{\, \Upsilon_1\Big(\frac{|\boldsymbol{j}_\epsilon|^2}{\sigma(\boldsymbol{\rho}_\epsilon)}\Big) + \Upsilon_2\Big(\frac{|\nabla\boldsymbol{\rho}_\epsilon|^2}{\sigma(\boldsymbol{\rho}_\epsilon)}\Big) \,\Big\}\, \Big] \;<\; +\infty\,. \tag{4.20}$$

Note that

$$|\boldsymbol{j}_\epsilon|^2 \,=\, [1-\epsilon]^2 \big|\boldsymbol{j}*\phi_\epsilon\big|^2 \,\le\, \big[(1-\epsilon)\,\sigma(\boldsymbol{\rho}*\phi_\epsilon) \,+\, \epsilon\,\sigma(m)\big] \Big\{ [1-\epsilon]\,\frac{|\,\boldsymbol{j}*\phi_\epsilon\,|^2}{\sigma(\boldsymbol{\rho}*\phi_\epsilon)} \Big\}\,.$$

By the concavity of σ, the first term on the right-hand side is bounded by $\sigma(\boldsymbol{\rho}_\epsilon)$. In conclusion,

$$\frac{|\boldsymbol{j}_\epsilon|^2}{\sigma(\boldsymbol{\rho}_\epsilon)} \,\le\, [1-\epsilon]\,\frac{|\,\boldsymbol{j}*\phi_\epsilon\,|^2}{\sigma(\boldsymbol{\rho}*\phi_\epsilon)} \,\le\, \frac{|\,\boldsymbol{j}*\phi_\epsilon\,|^2}{\sigma(\boldsymbol{\rho}*\phi_\epsilon)}\,\cdot$$

On the other hand, by concavity of σ and Cauchy-Schwarz inequality,

$$\begin{aligned}
&\frac{|\boldsymbol{j}*\phi_\epsilon|^2}{\sigma(\boldsymbol{\rho}*\phi_\epsilon)}(t,x)\\
&\le\, \frac{1}{[\sigma(\boldsymbol{\rho})*\phi_\epsilon](t,x)}\Big(\int_{\mathbb{R}} ds \int_{\mathbb{R}^d} dy\,\phi_\epsilon(t-s,x-y)\,\frac{j(s,y)}{\sqrt{\sigma(\boldsymbol{\rho}(s,y))}}\sqrt{\sigma(\boldsymbol{\rho}(s,y))}\Big)^2\\
&\quad\le\, \Big(\phi_\epsilon\,*\,\frac{|\boldsymbol{j}|^2}{\sigma(\boldsymbol{\rho})}\Big)(t,x)\,.
\end{aligned}$$

Whence, as Υ_1 is increasing,

$$E_P\Big[\int_0^T dt\int_{\mathbb{T}^d} dx\,\Upsilon_1\Big(\frac{|\boldsymbol{j}_\epsilon|^2}{\sigma(\boldsymbol{\rho}_\epsilon)}\Big)\Big] \le\, E_P\Big[\int_0^T dt\int_{\mathbb{T}^d} dx\,\Upsilon_1\Big(\phi_\epsilon\,*\,\frac{|\boldsymbol{j}|^2}{\sigma(\boldsymbol{\rho})}\Big)\Big]\,.$$

Since Υ_1 is convex, integrating the convolution, we deduce that the previous expression is bounded by

$$E_P\Big[\int_0^T dt\int_{\mathbb{T}^d} dx\,\phi_\epsilon\,*\,\Upsilon_1\Big(\frac{|\boldsymbol{j}|^2}{\sigma(\boldsymbol{\rho})}\Big)\Big] \,=\, E_P\Big[\int_0^T dt\int_{\mathbb{T}^d} dx\,\Upsilon_1\Big(\frac{|\boldsymbol{j}|^2}{\sigma(\boldsymbol{\rho})}\Big)\Big]\,.$$

Since the previous argument for $\boldsymbol{j}$ applies to $\nabla\boldsymbol{\rho}$, the bound (4.20) follows from (4.19). □

LEMMA 4.12. *Assume that E is orthogonally decomposable. There exists $T_0>0$ and $C_0>0$ such that the following holds. For any $\pi_0,\pi_1\in\mathcal{M}_m(\mathbb{T}^d)$ there exists $\bar{T}\le T_0$ and a path $\big(\boldsymbol{\pi}(t),\boldsymbol{J}(t)\big)$, $t\in[0,\bar{T}]$, with $\boldsymbol{\pi}(0)=\pi_0$, $\boldsymbol{\pi}(\bar{T})=\pi_1$, satisfying the continuity equation* (2.21) *for each $0\le s<t\le\bar{T}$ and*

$$A_{m,\bar{T}}(\bar{\boldsymbol{\pi}},\bar{\boldsymbol{J}})\le C_0\,. \tag{4.21}$$

PROOF. In the case of the symmetric exclusion process, this statement is proven in [3, Lemma 4.7] with the following strategy. Start from π_0 and follow the hydrodynamic equation for a long but fixed time interval $[0,T_1]$ so that $\boldsymbol{\pi}(T_1)$ lies in small neighborhood of the stationary solution with mass m. Then interpolate, in the time interval $[T_1,T_1+1]$ from $\boldsymbol{\pi}(T_1)$ to a suitable $\hat{\pi}$ that is still close to the stationary solution. Finally, from $\hat{\pi}$ use the optimal path for the escape problem to reach π_1. Provided that the *quasi-potential* is bounded, this argument applies also to the WASEP case. As discussed in [4, § V.C] and [5], if the external field is orthogonally decomposable then the quasi-potential can be computed explicitly and it is indeed bounded. □

PROOF OF THEOREM 4.10. Fix $P\in\mathcal{P}_{\mathrm{stat},m}$. By Lemma 4.11 we can assume that $P\in\mathcal{P}^\epsilon_{\mathrm{stat},m}$ for some $\epsilon>0$. Since P can be written as a convex combination of ergodic probabilities and $\boldsymbol{I}_m$ is affine, it suffices to show that for each ergodic $P\in\mathcal{P}^\epsilon_{\mathrm{stat},m}$ with $\boldsymbol{I}_m(P)<+\infty$ there exists a sequence of holonomic measures P_T in $\mathcal{P}^\epsilon_{\mathrm{stat},m}$ converging to P and such that $\lim_T \boldsymbol{I}_m(P_T)=\boldsymbol{I}_m(P)$.

Recalling that the T-periodization of paths in $\mathcal{S}$ as been defined in (2.23), set

$$\mathcal{A}_P := \Big\{ (\boldsymbol{\pi}, \boldsymbol{J}) \in \mathcal{S}_m \colon \ \frac{1}{T}\int_0^T \delta_{\vartheta_t(\boldsymbol{\pi}^T, \boldsymbol{J}^T)}\, dt \to P \\ \text{and } \lim_{T\to+\infty} \frac{1}{T} A_{T,m}(\boldsymbol{\pi}, \boldsymbol{J}) \to E_P\big[A_{1,m}\big] \Big\}.$$

Since P is ergodic, by the Birkhoff ergodic theorem $P(\mathcal{A}_P) = 1$. Pick an element $(\boldsymbol{\pi}^*, \boldsymbol{J}^*) \in \mathcal{A}_P$. By definition, the T-holonomic probability associated to the T-periodization of $(\boldsymbol{\pi}^*, \boldsymbol{J}^*)$ converges to P but, in general, its rate function does not since when T-periodizing paths we may insert jumps. By using Lemma 4.12, we now show how the path $(\boldsymbol{\pi}^*, \boldsymbol{J}^*)$ can be modified to accomplish our needs.

Given $T^* > 0$, let $\pi_0 = \boldsymbol{\pi}^*(T^*)$ and $\pi_1 = \boldsymbol{\pi}^*(0)$. Let also $(\bar{\boldsymbol{\pi}}(u), \bar{\boldsymbol{J}}(u))$, $u \in [0, \bar{T}]$ the path provided by Lemma 4.12 satisfying $\bar{\boldsymbol{\pi}}(0) = \pi_0$, $\bar{\boldsymbol{\pi}}(\bar{T}) = \pi_1$.

Set $T := T^* + \bar{T}$ and let $(\boldsymbol{\pi}(u), \boldsymbol{J}(u))$, $u \in [0, T]$ be the path defined by

$$\big(\boldsymbol{\pi}(u), \boldsymbol{J}(u)\big) = \begin{cases} \big(\boldsymbol{\pi}^*(u), \boldsymbol{J}^*(u)\big) & \text{if } u \in [0, T^*], \\ \big(\bar{\boldsymbol{\pi}}(u - T^*), \boldsymbol{J}(T^*) + \bar{\boldsymbol{J}}(u - T^*)\big)\big) & \text{if } u \in (T^*, T]. \end{cases}$$

Observe that $\boldsymbol{\pi}(0) = \boldsymbol{\pi}(T)$ and extend $(\boldsymbol{\pi}, \boldsymbol{J})$ to the path $(\boldsymbol{\pi}^T, \boldsymbol{J}^T)$ defined on $\mathbb{R}$ by periodicity. By construction, $t \mapsto \boldsymbol{\pi}^T(t)$ is continuous and denote by P_T the T-holonomic measure associated to $(\boldsymbol{\pi}^T, \boldsymbol{J}^T)$ as in (4.13). Since $\bar{T} \le T_0$ for some fixed T_0, $P_T \to P$ as $T \to \infty$. Moreover, by construction and by Lemma 4.12,

$$\begin{aligned} \boldsymbol{I}_m(P_T) &= \frac{1}{T} A_{m,T}(\boldsymbol{\pi}, \boldsymbol{J}) = \frac{1}{T} A_{m,T^*}(\boldsymbol{\pi}^*, \boldsymbol{J}^*) + \frac{1}{T} A_{m,\bar{T}}(\bar{\boldsymbol{\pi}}, \bar{\boldsymbol{J}}) \\ &\le \frac{1}{T} A_{m,T^*}(\boldsymbol{\pi}^*, \boldsymbol{J}^*) + \frac{1}{T} C_0 \end{aligned}$$

so that, since $(\boldsymbol{\pi}^*, \boldsymbol{J}^*)$ belongs to $\mathcal{A}_P$, $\limsup_{T\to\infty} \boldsymbol{I}_m(P_T) \le \boldsymbol{I}_m(P)$. As $\boldsymbol{I}_m$ is lower semicontinuous, actually, $\lim_{T\to\infty} \boldsymbol{I}_m(P_T) = \boldsymbol{I}_m(P)$, as claimed. □

5. Long time behavior of the hydrodynamical rate function . In this section, we consider the asymptotic in which we take first the limit as $N \to \infty$ and then $T \to \infty$. The former limit is essentially the content of the large deviations from the hydrodynamical scaling limit in which we emphasize that the corresponding T-dependent rate function still depends on the initial condition. To analyze the limit as $T \to \infty$ we first lift this rate function to the set of translation invariant probabilities on $\mathcal{S}$ and then analyze its variational convergence, showing in particular that the limit is independent of the initial condition.

Hereafter, fix $m \in (0, 1)$ and a sequence K_N such that $N^{-d} K_N \to m$.

5.1. *Hydrodynamical large deviations.* Recall that the sequence $\{\eta^N, N \ge 1\}$, $\eta^N \in \Sigma_{N,K_N}$ is *associated* to a measurable density $\rho\colon \mathbb{T}^d \to [0, 1]$ satisfying $\int \rho\, dx = m$ (hereafter of total mass m) if and only if $\pi_N(\eta^N) \to \rho(x)\, dx$ in the topology of $\mathcal{M}_+(\mathbb{T}^d)$.

Recalling that $\mathcal{M} = \mathcal{M}_+(\mathbb{T}^d) \times \mathcal{H}^d_{-p}$, let $\mathcal{S}^T$, $T > 0$, be the subset of the paths in $D([0, T]; \mathcal{M})$ which satisfy the continuity equation (2.21) for any $0 < s < t < T$. Let also $\mathcal{S}^T_m$ be the subset of $\mathcal{S}^T$ given by the elements $(\boldsymbol{\pi}, \boldsymbol{J})$ which satisfy $\boldsymbol{\pi}(t, \mathbb{T}^d) = m$ for all $0 \le t \le T$.

Finally, given a measurable function $\rho\colon \mathbb{T}^d \to [0, 1]$ of total mass m, that plays the role of the initial datum, let $\mathcal{S}^T_{m,\mathrm{ac},\rho}$ be the subset of elements $(\boldsymbol{\pi}, \boldsymbol{J})$ in $\mathcal{S}^T_m$ such that (a') $\boldsymbol{\pi} \in C([0,T], \mathcal{M}_m(\mathbb{T}^d))$, and $\boldsymbol{\pi}(t,dx) = \boldsymbol{\rho}(t,x)\,dx$ for some $\boldsymbol{\rho}$ such that $0 \le \boldsymbol{\rho}(t,x) \le 1$. Moreover,

$$\int_0^T dt \int_{\mathbb{T}^d} dx\, \frac{|\nabla \boldsymbol{\rho}|^2}{\sigma(\boldsymbol{\rho})} \,<\, \infty\,;$$

(b') $\boldsymbol{J} \in C([0,T], \mathcal{H}^d_{-p})$, and $\boldsymbol{J}(t) = \int_0^t \boldsymbol{j}(s)\,ds,\ t \in [0,T]$, for some $\boldsymbol{j}$ in $L^2\big([0,T] \times \mathbb{T}^d, \sigma(\boldsymbol{\rho}(t,x))^{-1}\,dt\,dx; \mathbb{R}^d\big)$. Thus,

$$\int_0^T dt \int_{\mathbb{T}^d} dx\, \frac{|\,\boldsymbol{j}\,|^2}{\sigma(\boldsymbol{\rho})} \,<\, \infty\,;$$

(c') $\boldsymbol{\pi}(0,dx) = \rho(x)\,dx$.

Note that conditions (a') and (b') are the same of conditions (a) and (b) below equation (2.21) apart from the fact that the here the path $\big(\boldsymbol{\rho}(t), \boldsymbol{j}(t)\big)$ is defined only for $t \in [0,T]$.

Let the *action* $A_{m,T,\rho} : \mathcal{S}^T \to [0,+\infty]$, be defined by

$$A_{m,T,\rho}(\boldsymbol{\pi}, \boldsymbol{J}) \,=\, \begin{cases} \displaystyle\int_0^T dt \int_{\mathbb{T}^d} dx\, \frac{|\boldsymbol{j} + \nabla \boldsymbol{\rho} - \sigma(\boldsymbol{\rho})\,E|^2}{4\,\sigma(\boldsymbol{\rho})} & \text{if } (\boldsymbol{\pi}, \boldsymbol{J}) \in \mathcal{S}_{T,m,\mathrm{ac},\rho}, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.1}$$

The large deviation principle with respect to the hydrodynamical limit for the WASEP dynamics can be stated as follows. Here, we understand that the empirical density and current $(\boldsymbol{\pi}_N, \boldsymbol{J}_N)$ is defined as a map from $D\big([0,T]; \Sigma_{N,K_N}\big)$ to $\mathcal{S}^T$.

THEOREM 5.1. *Fix $T > 0$, $m > 0$ and a density profile $\rho : \mathbb{T}^d \to [0,1]$ of total mass m. For each sequence $(\eta^N : N \ge 1)$ associated to ρ, each closed set $\mathcal{F} \subset \mathcal{S}^T$, and each open set $\mathcal{G} \subset \mathcal{S}^T$,*

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}^N_{\eta^N} \left[(\boldsymbol{\pi}_N, \boldsymbol{J}_N) \in \mathcal{F} \right] \,\le\, - \inf_{(\boldsymbol{\pi},\boldsymbol{J}) \in \mathcal{F}} A_{m,T,\rho}(\boldsymbol{\pi}, \boldsymbol{J})\,,$$

$$\liminf_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}^N_{\eta^N} \left[(\boldsymbol{\pi}_N, \boldsymbol{J}_N) \in \mathcal{G} \right] \,\ge\, - \inf_{(\boldsymbol{\pi},\boldsymbol{J}) \in \mathcal{G}} A_{m,T,\rho}(\boldsymbol{\pi}, \boldsymbol{J})\,.$$

Moreover, $A_{m,T,\rho} \colon \mathcal{S}^T \to [0,+\infty]$ is a good rate function.

If one considers only the empirical density and disregards the empirical current, the above result has been proven in [17] in case of SEP, see also [15, Ch. 10]. This result has been extended to WASEP in [6]. Relying on the super-exponential estimates proven in [17], the case in which one considers the empirical current is discussed in [3] for the SEP. However, the topology on the set of currents there introduced is different from the one used in the present paper and the proof of the exponential tightness is incomplete. The issue of the exponential tightness of the empirical current is fixed in the present paper (in the topology here introduced). Indeed, Lemmata 4.3, 4.4, and 4.5 which hold also in the present setting, yield the exponential tightness of the sequence $\big(\mathbb{P}^N_{\eta^N} \circ (\boldsymbol{\pi}_N, \boldsymbol{J}_N)^{-1} \colon N \ge 1\big)$ thus completing, together with [3], the proof of the above result for the SEP. The extension to WASEP requires, for the lower bound, a density argument that has been carried out in detail in [6, Thm. 5.1] and can be adapted to include the current.

Recall, from (2.23), the definition of the T-periodization of a path $(\boldsymbol{\pi}, \boldsymbol{J}) \in \mathcal{S}_m$, which depends only on the restriction of the path $(\boldsymbol{\pi}, \boldsymbol{J})$ to the time interval $[0,T]$. Let $\chi_T \colon \mathcal{S}^T \to \mathcal{P}_{\mathrm{stat}}$ be the continuous map defined by

$$\chi_T(\boldsymbol{\pi}, \boldsymbol{J}) := \frac{1}{T} \int_0^T \delta_{\vartheta_s(\boldsymbol{\pi}^T, \boldsymbol{J}^T)}\, ds\,,$$

where the translation ϑ_s acting on $\mathcal{S}$ has been introduced above (2.26). Namely, $\chi_T(\boldsymbol{\pi},\boldsymbol{J})$ is the T-holonomic measure associated to the T-periodic path $(\boldsymbol{\pi}^T,\boldsymbol{J}^T)$ obtained by T-periodizing the path $(\boldsymbol{\pi},\boldsymbol{J})\in\mathcal{S}^T$.

Recall the definition of the empirical process $R_T(\boldsymbol{\eta})$ introduced in (2.19). Since $(\,\boldsymbol{\pi}_N(\boldsymbol{\eta}^T),\boldsymbol{J}_N(\boldsymbol{\eta}^T)\,) = (\boldsymbol{\pi}_N(\boldsymbol{\eta})^T,\boldsymbol{J}_N(\boldsymbol{\eta})^T)$, for each $\boldsymbol{\eta}\in D\big([0,T];\Sigma_{N,K_N}\big)$,

$$
\begin{aligned}
\mathfrak{R}_{T,N}(\boldsymbol{\eta}) &= R_T(\boldsymbol{\eta})\circ(\boldsymbol{\pi}_N,\boldsymbol{J}_N)^{-1} = \frac{1}{T}\int_0^T \delta_{\vartheta_s\boldsymbol{\eta}^T}\circ(\boldsymbol{\pi}_N,\boldsymbol{J}_N)^{-1}\,ds \\
&= \frac{1}{T}\int_0^T \delta_{\vartheta_s\left(\boldsymbol{\pi}_N(\boldsymbol{\eta})^T,\boldsymbol{J}_N(\boldsymbol{\eta})^T\right)}\,ds \;=\; \chi_T\big(\boldsymbol{\pi}_N(\boldsymbol{\eta}),\boldsymbol{J}_N(\boldsymbol{\eta})\big)\,.
\end{aligned}
\tag{5.2}
$$

Observe that the image of $\mathcal{S}^T$ by χ_T corresponds to the set of T-holonomic measures. For ρ of total mass m, let $\boldsymbol{I}_{m,T,\rho}\colon\mathcal{P}_{\rm stat}\to[0,+\infty]$ be the functional defined by

$$
\boldsymbol{I}_{m,T,\rho}(P) \,:=\, \inf\big\{A_{m,T,\rho}(\boldsymbol{\pi},\boldsymbol{J})\,,\;(\boldsymbol{\pi},\boldsymbol{J})\in\chi_T^{-1}\big(P\big)\big\}\,,
\tag{5.3}
$$

where we adopted the convention that $\inf\varnothing=+\infty$. In particular, $\boldsymbol{I}_{m,T,\rho}(P)<+\infty$ only for T-holonomic measures P. Moreover, $\boldsymbol{I}_{m,T,\rho}(P)<+\infty$ only if the T-periodic path $(\boldsymbol{\pi},\boldsymbol{J})\in\mathcal{S}$ associated to P satisfies the following condition. There there exists $s\in[0,T]$ such that the restriction of $\vartheta_s(\boldsymbol{\pi},\boldsymbol{J})$ to $[0,T]$ belongs to $\mathcal{S}^T_{m,{\rm ac},\rho}$. In particular $\boldsymbol{\pi}(t)=\rho$ for some $t\in\mathbb{R}$.

In view of the identity (5.2), Theorem 5.1, by the contraction principle, yields the following statement.

COROLLARY 5.2. *Fix $T>0$, $m>0$ and a density profile $\rho:\mathbb{T}^d\to[0,1]$ of total mass m. For each sequence $(\eta^N: N\ge 1)$ associated to ρ, each closed set $\mathcal{F}\subset\mathcal{P}_{\rm stat}$, and each open set $\mathcal{G}\subset\mathcal{P}_{\rm stat}$,*

$$
\limsup_{N\to\infty}\frac{1}{N^d}\log\mathbb{P}^N_{\eta^N}\big[\mathfrak{R}_{T,N}\in\mathcal{F}\big] \;\le\; -\inf_{P\in\mathcal{F}}\boldsymbol{I}_{m,T,\rho}(P)\,,
$$

$$
\liminf_{N\to+\infty}\frac{1}{N^d}\log\mathbb{P}^N_{\eta^N}\big[\mathfrak{R}_{T,N}\in\mathcal{G}\big] \;\ge\; -\inf_{P\in\mathcal{G}}\boldsymbol{I}_{m,T,\rho}(P)\,.
$$

Moreover, $\boldsymbol{I}_{m,T,\rho}\colon\mathcal{P}_{\rm stat}\to[0,+\infty]$ is a good rate function.

5.2. *Variational convergence of the hydrodynamical rate function.* The main result of this section reads as follows.

THEOREM 5.3. *Fix $m\in(0,1)$ and a density profile $\rho:\mathbb{T}^d\to[0,1]$ of total mass m. As $T\to+\infty$, the sequence $T^{-1}\boldsymbol{I}_{m,T,\rho}$ is equi-coercive uniformly in ρ, and Γ-converges uniformly in ρ to the functional $\boldsymbol{I}_m$ introduced in* (2.26). *That is:*

(i) *For each $\ell>0$ there exists a compact $\mathcal{K}_\ell\subset\mathcal{P}_{\rm stat}$ such that for any $T>1$ and any ρ, $\big\{P: T^{-1}\boldsymbol{I}_{m,T,\rho}(P)\le\ell\big\}\subset\mathcal{K}_\ell$;*

(ii) *For any $P\in\mathcal{P}_{\rm stat}$, any sequence of density profiles $\rho_T:\mathbb{T}^d\to[0,1]$ of total mass m and any sequence $P_T\to P$,*

$$
\liminf_{T\to\infty}\frac{1}{T}\boldsymbol{I}_{m,T,\rho_T}(P_T)\ge\boldsymbol{I}_m(P)\,;
$$

(iii) *If the external field E is orthogonally decomposable, then for any $P\in\mathcal{P}_{\rm stat}$ and any sequence of density profiles $\rho_T:\mathbb{T}^d\to[0,1]$ of total mass m, there exists a sequence $P_T\to P$ such that*

$$
\limsup_{T\to\infty}\frac{1}{T}\boldsymbol{I}_{m,T,\rho_T}(P_T)\le\boldsymbol{I}_m(P)\,.
$$

PROOF. The proof is divided in three parts.

Equicoercivity. In view of the compactness of $\mathcal{M}_m(\mathbb{T}^d)$, the compact embedding of $L^2(\mathbb{T}^d;\mathbb{R}^d)$ into $\mathcal{H}^d_{-p}$ for $p>0$, Ascoli-Arzelà theorem, and Chebyshev inequality, it is enough to prove the following bounds. For each $T_0>0$ and each smooth vector field $H\colon \mathbb{T}^d\to\mathbb{R}^d$ there exists constants $C_0=C_0(T_0)$ and $C_1=C_1(T_0,H)$ such that for any $T\geq 1$ and $\delta\in(0,1)$

$$E_P\Big[\sup_{t\in[-T_0,T_0]}\big\|\boldsymbol{J}(t)\big\|^2_{L^2(\mathbb{T}^d;\mathbb{R}^d)}\Big]\leq C_0\,\Big[\frac{1}{T}\boldsymbol{I}_{m,T,\rho}(P)+1\Big] \tag{5.4}$$

$$E_P\Bigg[\sup_{\substack{t,s\in[-T_0,T_0]\\ |t-s|<\delta}}\big|\langle\boldsymbol{J}(t),H\rangle-\langle\boldsymbol{J}(s),H\rangle\big|^2\Bigg]\leq C_1\delta\,\Big[\frac{1}{T}\boldsymbol{I}_{m,T,\rho}(P)+1\Big]. \tag{5.5}$$

As already remarked right after the statement of Lemma 4.5, by choosing $H=\nabla g$, the bound (5.5) provides indeed also a control on the continuity modulus of the map $t\mapsto\boldsymbol{\pi}(t)$.

By the stationarity of P and the argument below (2.26), if $\boldsymbol{I}_{m,T,\rho}(P)<+\infty$, then there exists a constant C depending only on E such that

$$\frac{1}{2T}\,E_P\Big[\int_{-T}^{T}dt\int_{\mathbb{T}^d}dx\,\frac{|\boldsymbol{j}\,|^2}{4\,\sigma(\boldsymbol{\rho})}\Big]\leq C\Big[\frac{1}{T}\,\boldsymbol{I}_{m,T,\rho}(P)+1\Big]\,,$$

where $\boldsymbol{J}(t)=\int_0^t\boldsymbol{j}(s)\,ds$. By Cauchy-Schwarz inequality and condition (b') on the current $\boldsymbol{J}$ stated at the beginning of this section,

$$\sup_{t\in[-T_0,T_0]}\big\|\boldsymbol{J}(t)\big\|^2_{L^2(\mathbb{T}^d;\mathbb{R}^d)}\leq T_0\int_{-T_0}^{T_0}dt\int dx\big|\boldsymbol{j}(t,x)\big|^2\leq T_0\int_{-T_0}^{T_0}dt\int dx\frac{\big|\boldsymbol{j}(t,x)\big|^2}{4\sigma(\boldsymbol{\rho}(t,x))}\,.$$

Since $\boldsymbol{I}_{m,T,\rho}(P)<+\infty$ implies that P is a T-holonomic probability measure,

$$E_P\Big[\int_{-T_0}^{T_0}dt\int dx\frac{\big|\boldsymbol{j}(t,x)\big|^2}{4\sigma(\boldsymbol{\rho}(t,x))}\Big]=\frac{T_0}{T}E_P\Big[\int_{-T}^{T}dt\int dx\frac{\big|\boldsymbol{j}(t,x)\big|^2}{4\sigma(\boldsymbol{\rho}(t,x))}\Big]$$

which completes the proof of (5.4).

For $s<t$, Cauchy-Schwarz inequality and condition (b') yield

$$\big|\langle\boldsymbol{J}(t),H\rangle-\langle\boldsymbol{J}(s),H\rangle\big|^2\leq(t-s)\|H\|^2_{L^2(\mathbb{T}^d;\mathbb{R}^d)}\int_s^t du\int dx\,\big|\boldsymbol{j}(u,x)\big|^2\,,$$

so that (5.5) is obtained by the same argument as before.

Γ-liminf. Denote by $\mathcal{S}^T_{m,\mathrm{ac}}$ the subset of the paths $(\boldsymbol{\pi},\boldsymbol{J})\in\mathcal{S}^T$ satisfying only conditions (a'), (b') and let $A_{m,T}$ be the action defined in (5.1) with the constraint $(\boldsymbol{\pi},\boldsymbol{J})\in\mathcal{S}^T_{m,\mathrm{ac},\rho}$ replaced by $(\boldsymbol{\pi},\boldsymbol{J})\in\mathcal{S}^T_{m,\mathrm{ac}}$. Accordingly, let $\boldsymbol{I}_{m,T}\colon\mathcal{P}_{\mathrm{stat}}\to[0,+\infty]$ be the functional defined by

$$\boldsymbol{I}_{m,T}(P):=\inf\big\{A_{m,T}(\boldsymbol{\pi},\boldsymbol{J})\,,\,(\boldsymbol{\pi},\boldsymbol{J})\in\chi_T^{-1}(P)\big\}\,.$$

By the translation invariance of $A_{m,T}$, if $\chi_T^{-1}(P)$ is not empty (i.e., if P is T-holonomic) then

$$\boldsymbol{I}_{m,T}(P)=E_P\big[A_{m,T}\big]\,.$$

Hence, in view of the translation invariance of P,

$$\boldsymbol{I}_{m,T,\rho}(P)\geq\boldsymbol{I}_{m,T}(P)\;=\;E_P\big[A_{m,T}\big]\;=\;TE_P\big[A_{m,1}\big]\,.$$

Let $(\rho_T \colon T > 0)$ be an arbitrary sequence and $\big(P_T \colon T > 0\big) \subset \mathcal{P}_{\mathrm{stat}}$ be a sequence converging to P. By (2.26), the previous displayed bound and the lower semi-continuity of $A_{m,1}$,

$$\liminf_{T\to\infty} \frac{1}{T} \boldsymbol{I}_{m,T,\rho_T}(P_T) \geq \liminf_{T\to\infty} E_{P_T}\big[A_{m,1}\big] \geq E_P\big[A_{m,1}\big] = \boldsymbol{I}_m(P)\ .$$

Γ-*limsup.* By Theorem 4.10, it suffices to consider the case in which P is an S-holonomic measure with smooth density. More precisely, we may assume that

$$P = \frac{1}{S} \int_0^S \delta_{\vartheta_s(\boldsymbol{\pi}^*, \boldsymbol{J}^*)}\, ds \tag{5.6}$$

for some $S > 0$, where the S-periodic path $(\boldsymbol{\pi}^*, \boldsymbol{J}^*)$ has smooth densities $(\boldsymbol{\rho}^*, \boldsymbol{j}^*)$ with $\boldsymbol{\rho}^*$ bounded away from 0 and 1. Given the sequence $(\rho_T,\, T > 0) \subset \mathcal{M}_m(\mathbb{T}^d)$, let $\big(\bar{\boldsymbol{\pi}}(t), \bar{\boldsymbol{J}}(t)\big)$, $t \in [0, \bar{T}]$ be the path provided by Lemma 4.12 with $\pi_0 = \rho_T\, dx$ and $\pi_1 = \boldsymbol{\pi}^*(0)$. Let also $\big(\boldsymbol{\pi}(t), \boldsymbol{J}(t)\big)$, $t \in [0, +\infty)$ be the path defined by

$$\big(\boldsymbol{\pi}(t), \boldsymbol{J}(t)\big) := \begin{cases} \big(\bar{\boldsymbol{\pi}}(t)\,,\, \bar{\boldsymbol{J}}(t)\big) & \text{if } t \in [0, \bar{T}], \\ \big(\boldsymbol{\pi}^*(t - \bar{T})\,,\, \bar{\boldsymbol{J}}(\bar{T}) + \boldsymbol{J}^*(t - \bar{T})\big) & \text{if } t > \bar{T}. \end{cases}$$

Note that, although not explicit in the notation, the path $(\boldsymbol{\pi}, \boldsymbol{J})$ depends on T via the sequence ρ_T. Denote finally by $\big(\boldsymbol{\pi}^T, \boldsymbol{J}^T\big)$ the T-periodization, as defined in (2.23), of $\big(\boldsymbol{\pi}, \boldsymbol{J}\big)$ and by P_T the associated T-holonomic probability, i.e.,

$$P_T = \frac{1}{T} \int_0^T \delta_{\vartheta_s(\boldsymbol{\pi}^T, \boldsymbol{J}^T)}\, ds\ .$$

Since $\bar{T}/T \to 0$, as $T \to \infty$ the sequence P_T converges to P given by (5.6). Moreover, in view of (5.3) and (4.21),

$$\boldsymbol{I}_{m,T,\rho_T}(P_T) = A_{m,\bar{T},\rho_T}\big(\bar{\boldsymbol{\pi}}, \bar{\boldsymbol{J}}\big) + A_{m,T-\bar{T}}(\boldsymbol{\pi}^*, \boldsymbol{J}^*) \leq C_0 + A_{m,T-\bar{T}}(\boldsymbol{\pi}^*, \boldsymbol{J}^*)\ .$$

Hence, by the S-periodicity of $(\boldsymbol{\pi}^*, \boldsymbol{J}^*)$,

$$\limsup_{T\to\infty} \frac{1}{T} \boldsymbol{I}_{m,T,\rho_T}(P_T) = \frac{1}{S} A_{m,S}(\boldsymbol{\pi}^*, \boldsymbol{J}^*) = \frac{1}{S} E_P\big[A_{m,S}\big] = \boldsymbol{I}_m(P)\ ,$$

which concludes the proof of Theorem 5.3. □

6. Large deviations and projections. In this section, relying on the variational convergence proven before, we discuss the large deviations asymptotics in the joint limit $N \to \infty$, $T \to \infty$. In particular, we conclude the proof of Theorem 2.1 and discuss the corresponding projections.

PROOF OF THEOREM 2.1. We start by considering the case in which we first perform the limit as $T \to \infty$ and then take limit as $N \to \infty$. The asymptotic as $T \to \infty$ follows directly from the Donsker-Varadhan large deviation principle for the empirical process, see Corollary 3.2. By [3, Lemma 4.1] or [19, Corollary 4.3] the limit as $N \to \infty$ is accomplished by the Γ-convergence of the family $\big(N^{-d} \boldsymbol{I}_{N,K_N},\, N \geq 1\big)$ that has been proven in Theorem 4.1. Actually, the statements in [3, 19] give the upper bound only for compact sets. However, the goodness of the functional $\boldsymbol{I}_{N,K}$ together with the equi-coercivity in Theorem 4.1 allow to deduce the upper bound for closed sets.

The proof of the statement when the limit as $T \to \infty$ is carried out after the limit as $N \to \infty$ is accomplished by the similar argument. Indeed, the asymptotic as $N \to \infty$ follows directly from the hydrodynamical large deviations, see Corollary 5.2, while the Γ-convergence of the family $\big(T^{-1} \boldsymbol{I}_{m,T,\rho},\, T \geq 1\big)$ has been proven, uniformly with respect to ρ, in Theorem 5.3. □

We now discuss the level two projection and the hydrostatic limit.

PROOF OF COROLLARY 2.3. Recall (2.28) and that $\wp_{T,N} = \mathfrak{R}_{T,N} \circ \imath_t^{-1}$ where $\imath_t \colon \mathcal{S} \to \mathcal{M}_+(\mathbb{T}^d)$ is the map $(\boldsymbol{\pi}, \boldsymbol{J}) \mapsto \boldsymbol{\pi}(t)$. Note that i_t is not continuous since we are using the Skorohod topology. However, the map $\mathcal{P}_{\mathrm{stat}} \ni P \mapsto P \circ \imath_t^{-1} \in \mathcal{P}\big(\mathcal{M}_+(\mathbb{T}^d)\big)$ is continuous since, by stationary, the P-probability of a jump at a time t is zero. The large deviations asymptotic thus follows from Theorem 2.1 by the contraction principle.

We now show that the zero level set of $\mathcal{I}_m$ is equal to the set of invariant probability measures for the flow Φ^m associated to the hydrodynamic evolution (2.18). By the goodness of $\boldsymbol{I}_m$, if $\wp$ lies in the zero level set of $\mathcal{I}_m$ then the infimum in (2.28) is achieved, i.e. there exists $P \in \mathcal{P}_{\text{stat}}$ satisfying $\boldsymbol{I}_m(P) = 0$ and $\wp = P \circ \imath_t^{-1}$. As follows from (2.26) and (2.25) $\boldsymbol{I}_m(P) = 0$ implies that P almost surely $(\boldsymbol{\pi}, \boldsymbol{J})$ have densities $(\boldsymbol{\rho}, \boldsymbol{j})$ that satisfy $\boldsymbol{j} = -\nabla\boldsymbol{\rho} + \sigma(\boldsymbol{\rho})E$. Hence, the marginal of P on the first variable is concentrated on the set of $\boldsymbol{\pi}$ whose density $\boldsymbol{\rho}$ solves (1.2) with $D = 1$. By stationarity of P, this implies that $P \circ \imath_t^{-1}$ is an invariant probability of Φ^m as claimed.

It remains to show that if E is orthogonally decomposable then $\mathcal{I}_m(\wp) = 0$ implies $\wp = \delta_{\bar\rho\, dx}$ where $\bar\rho$ is the unique stationary solution to (1.2) with $D = 1$. As already remarked, if E is orthogonally decomposable then the quasi-potential of the WASEP dynamics can be explicitly computed and it is a Lypuanov functional for the hydrodynamic evolution. The argument in [5, Thm. 7.7] then implies that there exists a unique stationary solution of mass m to the hydrodynamic equation that is globally attractive, hence a unique stationary probability for the flow Φ^m that is concentrated on the stationary solution. □

We consider now the level one projection.

PROOF OF COROLLARY 2.4. Let $\psi \colon \mathcal{P}_{\mathrm{stat}} \to \mathcal{M}_+(\mathbb{T}^d) \times \mathcal{H}^d_{-p}$ be the map defined by

$$\psi(P) := \Big(E_P\big[\boldsymbol{\pi}(t)\big]\,,\ \frac{1}{t}E_P\big[\boldsymbol{J}(t)\big]\Big)$$

where we understand that ψ is defined only for the probabilities P such that for any $t \in \mathbb{R}$ we have $E_P\big[\|\boldsymbol{J}(t)\|_{\mathcal{H}^d_{-p}}\big] < +\infty$. Note that ψ does not depend on $t \neq 0$ by the stationarity of P.

Recall the definitions of $\pi_{T,N}$, $J_{T,N}$, and $\mathfrak{R}_{T,N}$, in (2.29), (2.30), and (2.24). Recall also the relation (2.31) according to which for each $\boldsymbol{\eta} \in D(\mathbb{R}_+, \Sigma_N)$ and $t \in \mathbb{R}$, $t \neq 0$, we have

$$\psi\big(\mathfrak{R}_{T,N}\big) = \int \mathfrak{R}_{T,N}(d\boldsymbol{\pi}, d\boldsymbol{J})\, \Big(\boldsymbol{\pi}(t), \frac{\boldsymbol{J}(t)}{t}\Big) = \big(\pi_{T,N}, J_{T,N}\big) - \frac{1}{T}\big(0, \mathcal{E}_{T,N}\big)\,. \tag{6.1}$$

Since by (2.20) the error term $\mathcal{E}_{T,N}$ is uniformly bounded in N and T, it is irrelevant in the large deviations asymptotics for $T \to +\infty$. We can therefore deduce the large deviations for the pair $\big(\pi_{T,N}, J_{T,N}\big)$ from the large deviations for $\psi\big(\mathfrak{R}_{T,N}\big)$.

Since ψ is not continuous the result does not follow directly from Theorem 2.1 and the contraction principle. However, in the terminology of [12, § 4.2.2], it possible to approximate ψ by a sequence of continuous functions and construct exponentially good approximations of the family $\big(\mathbb{P}^N_\eta \circ \big(\pi_{T,N}, J_{T,N}\big)^{-1} \colon T > 0,\, N \geq 1\big)$. We obtain in this way the result and observe that the rate functional is given by $I_m(\pi, J) = \inf\{\mathbf{I}_m(P) : P \in \mathcal{P}_{\mathrm{stat}}, \psi(P) = (\pi, J)\}$. □

The next result concerns the projection on the density for the level one large deviations functional.

PROOF OF THEOREM 2.5. In the case $E = -\nabla U$, developing the square in formula (2.25) we have that the cross term

$$\int_0^T dt \int_{\mathbb{T}^d} dx \, \frac{\boldsymbol{j} \cdot (\nabla \boldsymbol{\rho} + \sigma(\boldsymbol{\rho}) \nabla U)}{2\sigma(\boldsymbol{\rho})}$$

after an integration by parts and using the continuity equation coincides with

$$\frac{1}{2} \int_{\mathbb{T}^d} dx \, [h(\boldsymbol{\rho}(T)) - h(\boldsymbol{\rho}(0)) - (\boldsymbol{\rho}(T) - \boldsymbol{\rho}(0)) \, U] \,, \tag{6.2}$$

where $h(\rho) = \rho \log \rho + (1 - \rho) \log(1 - \rho)$. By stationarity the expected value of (6.2) with respect to any $P \in \mathcal{P}_{\mathrm{stat}}$ is zero. We have therefore that when $E = -\nabla U$

$$\boldsymbol{I}_m(P) = \frac{1}{T} \, E_P \Big[\int_0^T dt \int_{\mathbb{T}^d} dx \, \Big(\frac{|\, \nabla \boldsymbol{\rho} + \sigma(\boldsymbol{\rho}) \nabla U \,|^2}{4 \, \sigma(\boldsymbol{\rho})} + \frac{|\, \boldsymbol{j} \,|^2}{4 \, \sigma(\boldsymbol{\rho})} \Big) \Big] \,. \tag{6.3}$$

Consider a $P \in \mathcal{P}_{\mathrm{stat}}$ and call $\wp \in \mathcal{P}\big(\mathcal{M}_m(\mathbb{T}^d)\big)$ its 1-marginal $\wp = P \circ \imath_t^{-1}$ (see notation above formula (2.28)). Let $A \colon \mathcal{M}_+(\mathbb{T}^d) \to \mathcal{S}$ be the map that associate to $\pi \in \mathcal{M}_+(\mathbb{T}^d)$ the element $(\boldsymbol{\pi}, \boldsymbol{J}) \in \mathcal{S}$ defined by $\boldsymbol{\pi}(t) = \pi$ and $\boldsymbol{J}(t) = 0$ for any $t \in [0, T]$. Finally let us define $\tilde{P} \in \mathcal{P}_{\mathrm{stat}}$ as $\wp \circ A^{-1}$.

Since the second term in the right hand side of (6.3) is non negative and since $P \circ \imath_t^{-1} = \tilde{P} \circ \imath_t^{-1} = \wp$, we deduce

$$\boldsymbol{I}_m(P) \geq \boldsymbol{I}_m(\tilde{P}) = E_\wp \Big[\int_{\mathbb{T}^d} dx \, \Big(\frac{|\, \nabla \rho + \sigma(\rho) \nabla U \,|^2}{4 \, \sigma(\rho)} \Big) \Big] = E_\wp \, [\mathcal{V}_m] \,.$$

We have therefore that

$$I_m^{(1)}(\pi) = \inf \Big\{ E_\wp(\mathcal{V}_m) \;\; \wp \in \mathcal{P}\Big(\mathcal{M}_m(\mathbb{T}^d)\Big) \; E_\wp(\pi') = \pi \Big\} = \mathrm{co}(\mathcal{V}_m)(\pi) \,,$$

the last equality follows since in the middle we have one of the possible definitions of convex hull. Since σ is concave, in the case $\nabla U = 0$ we have that $\mathcal{V}_m$ is convex and therefore $\mathrm{co}(\mathcal{V}_m) = \mathcal{V}_m$. □

Finally, we give a sketch of the proof of Theorem 2.6. This is based on analysis in [2, 3, 8] and we just show how to deduce the result based on the arguments there.

PROOF OF THEOREM 2.6. Let us call $(\boldsymbol{\pi}^*, \boldsymbol{J}^*)$ the element of $\mathcal{S}_m$ defined by $\boldsymbol{\pi}^*(t) = m$ and $\boldsymbol{J}^*(t) = jt$. The result is obtained by the analysis of the action functional $A_{m,T}$ (2.25). In the case (i) for $E \leq E_0$ by [2, 3, 8] we have that for any $(\boldsymbol{\pi}, \boldsymbol{J}) \in \mathcal{S}_m$ such that $\boldsymbol{J}(T) = jT$ it holds $A_{m,T}(\boldsymbol{\pi}^*, \boldsymbol{J}^*) \leq A_{m,T}(\boldsymbol{\pi}, \boldsymbol{J})$ and this allows to deduce that $\delta_{(\boldsymbol{\pi}^*, \boldsymbol{J}^*)}$ is the minimizer in (2.33).

In the case (ii) for $E > E_1$ it is possible to construct [2, 3, 8] a time dependent $(\boldsymbol{\pi}, \boldsymbol{J}) \in \mathcal{S}_m$, that has indeed the structure of a traveling wave, such that $\boldsymbol{J}(T) = jT$ and $A_{m,T}(\boldsymbol{\pi}^*, \boldsymbol{J}^*) > A_{m,T}(\boldsymbol{\pi}, \boldsymbol{J})$. Considering $P \in \mathcal{P}_{\mathrm{stat}}$ defined by $P = \frac{1}{T} \int_0^T dt \delta_{\theta_t(\boldsymbol{\pi}, \boldsymbol{J})}$ we have therefore that $\boldsymbol{I}_m(P) < \boldsymbol{I}_m(\delta_{(\boldsymbol{\pi}^*, \boldsymbol{J}^*)})$. □

7. Uniqueness of periodic solutions.

Fix $T > 0$. Throughout this section, $F : \mathbb{R} \times \mathbb{T}^d \to \mathbb{R}^d$ is a smooth, T-periodic vector field. We investigate in this section the asymptotic behavior of solutions to the Cauchy problem

$$\begin{cases} \partial_t \boldsymbol{u} \;=\; \Delta \boldsymbol{u} \,+\, \nabla \cdot [\, \sigma(\boldsymbol{u}) \, F \,] \\ \boldsymbol{u}(0, \cdot) = u_0(\cdot) \,, \end{cases} \tag{7.1}$$

where the initial condition $u_0 : \mathbb{T}^d \to [0,1]$ is such that $0 \le u_0(x) \le 1$ for all $x \in \mathbb{T}^d$.

Existence of weak solutions is provided by the hydrodynamic limit of WASEP. This argument shows that the solution takes value in the interval $[0,1]$. These bounds can be derived also from the maximum principle and the observation that $\sigma(1) = \sigma(0) = 0$. By parabolic regularity, a weak solution is smooth in $(0,\infty) \times \mathbb{T}^d$. Uniqueness is derived as in [14, Lemma 7.2]. The proof of this lemma yields that the $L^1(\mathbb{T}^d)$ norm of the difference of two weak solutions does not increase in time. The main result of this section strengthen this lemma and asserts that the $L^1(\mathbb{T}^d)$ distance of two different weak solutions decreases in time. It reads as follows.

THEOREM 7.1. *Let $F : \mathbb{R} \times \mathbb{T}^d \to \mathbb{R}^d$ be a smooth, T-periodic vector field. For each $m \in [0,1]$, the equation*

$$\partial_t \boldsymbol{u} \;=\; \Delta \boldsymbol{u} \,+\, \nabla \cdot [\, \sigma(\boldsymbol{u})\, F\,]\,.$$

admits a unique T-periodic solution $\boldsymbol{u} : \mathbb{R} \times \mathbb{T}^d \to [0,1]$ such that $\int_{\mathbb{T}^d} \boldsymbol{u}(t,x)\, dx = m$ and $0 \le \boldsymbol{u}(t,x) \le 1$ for all t. This solution is represented by $\boldsymbol{u}^{(m)}$. Moreover, for each $u_0 : \mathbb{T}^d \to [0,1]$ such that $\int_{\mathbb{T}^d} u_0(x)\, dx = m$, $0 \le u_0(x) \le 1$ for all $x \in \mathbb{T}^d$, the unique weak solution of (7.1) *converges to $\boldsymbol{u}^{(m)}$ in $L^1(\mathbb{T}^d)$ as $t \to \infty$.*

The proof of this result relies on a method of coupling of two diffusions due to Lindvall and Rogers [18].

7.1. *Coupling diffusions.* Let $G : \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}^d$ be a smooth vector field uniformly bounded: there exists $C_0 < \infty$ such that $\sup_{(t,x)\in \mathbb{R}_+\times\mathbb{T}^d} \| \, G(t,x) \, \| \le C_0$.

Denote by $\mathcal{L}_t$ the time-dependent generator

$$\mathcal{L}_t f \;=\; \Delta f \,+\, \nabla f \cdot G_t\,, \quad f \,\in\, C^2(\mathbb{T}^d)\,. \tag{7.2}$$

Let $(Z^x_t : t \ge 0)$, $x \in \mathbb{T}^d$, be the $\mathbb{T}^d$-valued, continuous-time Markov process whose generator is $\mathcal{L}_t$ and which starts from x.

Recall that a coupling between Z^x_t and Z^y_t is a process $(\widetilde{Z}^x_t, \widetilde{Z}^y_t)$ whose first (resp. second) coordinate evolves as Z^x_t (resp. Z^y_t). The coupling time, denoted by $\tau^Z_{x,y}$, is the first time at which the processes meet:

$$\tau^Z_{x,y} \;:=\; \inf\big\{\, t > 0 : \widetilde{Z}^x_t \,=\, \widetilde{Z}^y_t \,\big\}\,.$$

Next result relies on the Lindvall–Rogers coupling [18].

PROPOSITION 7.2. *There exist constants $A < \infty$ and $\lambda > 0$, which depends only on $\sup_{(t,x)\in \mathbb{R}_+\times\mathbb{T}^d} \| \, G(t,x) \, \|$, and, for each x, $y \in \mathbb{T}^d$, a coupling between Z^x_t, Z^y_t, such that*

$$\sup_{x,y\in\mathbb{T}^d} P\big[\, \tau^Z_{x,y} \ge t \,\big] \;\le\; A\, e^{-\lambda t}$$

for all $t \ge 0$.

Let $\mathbb{P}_x$ be the probability measure on $C(\mathbb{R}_+, \mathbb{T}^d)$ induced by the diffusion associated to the generator $\mathcal{L}_t$ starting from $x \in \mathbb{T}^d$. Expectation with respect to $\mathbb{P}_x$ is represented by $\mathbb{E}_x$.

COROLLARY 7.3. *There exist constants $A < \infty$ and $\lambda > 0$, which depends only on $\sup_{(t,x)\in \mathbb{R}_+\times\mathbb{T}^d} \| \, G(t,x) \, \|$, such that*

$$\sup_{x,y\in\mathbb{T}^d} \Big|\, \mathbb{E}_x\big[\, f(Z_t) \,\big] \,-\, \mathbb{E}_y\big[\, f(Z_t) \,\big] \,\Big| \;\le\; A\, e^{-\lambda t}\, \|f\|_\infty$$

for every $f \in C(\mathbb{T}^d)$.

PROOF. Since the difference may be written as

$$\Big| E\big[f(\widetilde{Z}_t^x) - f(\widetilde{Z}_t^y) \big] \Big| \le 2\, \| f\|_\infty\, P\big[\tau_{x,y}^Z \ge t \big] , \tag{7.3}$$

the assertion is an elementary consequence of Proposition 7.2. □

PROOF OF PROPOSITION 7.2. Denote by $(W_t : t \ge 0)$, the standard Brownian motion on $\mathbb{R}^d$. Let $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be the $\mathbb{T}^d$-periodic vector field whose restriction to $\mathbb{T}^d$ coincides with G. Denote by X_t^x the solutions of the SDE

$$\begin{cases} dX_t^x \,=\, c(t, X_t^x)\, dt \,+\, dW_t \,, \\ X_0^x = x \,, \end{cases}$$

where $c(t,x) = (1/2)\, b(t,x)$. For each $x \in \mathbb{R}^d$, X_t^x is a diffusion on $\mathbb{R}^d$ whose time-dependent generator, denoted by $\mathcal{A}_t$, is given by

$$\mathcal{A}_t f \,=\, (1/2)\, \Delta f \,+\, \nabla f \cdot c_t \,, \quad f \in C_0^2(\mathbb{R}^d) \,,$$

where $C_0^2(\mathbb{R}^d)$ stands for the twice continuously differentiable functions with compact support. We replaced $b(t,x)$ by $c(t,x)$ in order to have a simple relation between the generators $\mathcal{A}_t$ and $\mathcal{L}_t$.

Fix x, $y \in \mathbb{T}^d$. Lindvall and Rogers [18] provide a coupling between X_t^x and X_t^y, represented by $(\widetilde{X}_t^x, \widetilde{X}_t^y)$, such that, before hitting the origin, $D_t := \|\widetilde{X}_t^x - \widetilde{X}_t^y\|$ evolves as

$$dD_t \,=\, 2\, dB_t \,+\, \Big\langle \frac{\widetilde{X}_t^x - \widetilde{X}_t^y}{D_t} \,,\, c(t, \widetilde{X}_t^x) - c(t, \widetilde{X}_t^y) \Big\rangle\, dt \,, \tag{7.4}$$

where $\langle \cdot \,,\, \cdot \rangle$ stands for the scalar product in $\mathbb{R}^d$, and B_t is a one-dimensional Brownian motion. Note that the drift term is bounded.

Denote by $(\widetilde{Z}_t^x, \widetilde{Z}_t^y)$ the projection of the process $(\widetilde{X}_t^x, \widetilde{X}_t^y)$ on $\mathbb{T}^d \times \mathbb{T}^d$. Each coordinate of the pair $(\widetilde{Z}_t^x, \widetilde{Z}_t^y)$ is a Markov process whose generator is equal to $(1/2)\mathcal{L}_t$. Hence, $(\widetilde{Z}_{2t}^x, \widetilde{Z}_{2t}^y)$ is a coupling to Z_t^x, Z_t^y, and, to prove the proposition, it is enough to show that there exist constants $A < \infty$ and $\lambda > 0$, which depends only on $\sup_{(t,x)\in\mathbb{R}_+\times\mathbb{T}^d} \| G(t,x) \|$, such that

$$\sup_{x,y\in\mathbb{T}^d} P\big[\tau_{x,y}^{\widetilde{Z}} \ge t \big] \,\le\, (1/2)\, A\, e^{-\lambda t} \tag{7.5}$$

for all $t \ge 0$, where $\tau_{x,y}^{\widetilde{Z}}$ is the first time the processes $(\widetilde{Z}_t^x, \widetilde{Z}_t^y)$ meet.

By construction, before hitting 0, $\widetilde{D}_t := \|\widetilde{Z}_t^x - \widetilde{Z}_t^y\|$ evolves as D_t, except when D_t attains the maximal distance between two points in $\mathbb{T}^d$, that is, when D_t hits $L := \sqrt{d}/2$, in which case $\widetilde{D}_t$ is reflected, while D_t evolves according to (7.4).

Let $M := 2 \sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d} \| c(t,x) \| = \sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d} \| b(t,x) \|$. By definition of b, $M = \sup_{(t,x)\in\mathbb{R}_+\times\mathbb{T}^d} \| G(t,x) \|$. Moreover, for all z such that $\|z\| = 1$, $|\langle z \,,\, c(t, \widetilde{X}_t^x) - c(t, \widetilde{X}_t^y) \rangle | \le M$.

Let $\widehat{D}_t$ the diffusion on $[0, L+1]$ which is absorbed at the origin, reflected at $L+1$ and which evolves according to the SDE

$$d\widehat{D}_t \,=\, 2\, dB_t \,+\, M\, dt \,.$$

By the previous bound on the drift term of D_t, we may couple $\widetilde{D}_t$ and $\widehat{D}_t$ in such a way that $\widetilde{D}_t \le \widehat{D}_t$ for all $t \ge 0$, almost surely, provided $\widetilde{D}_0 \le \widehat{D}_0$. In particular, $\widetilde{D}_0$ hits the origin before $\widehat{D}_0$. Therefore, the coupling time of $(\widetilde{Z}_t^x, \widetilde{Z}_t^y)$ is bounded above by the absorption time of $\widehat{D}_t$, represented by H_0^r, where r stands for the initial state. An elementary computation yields that there exists a finite constant T_0, depending only on M and L such that

$$\sup_{r\in[0,L+1]} E[\,H_0^r\,] \;\le\; T_0 \quad \text{so that} \quad \sup_{r\in[0,L+1]} P[\,H_0^r > 2T_0\,] \;\le\; \frac{1}{2}\,\cdot$$

In consequence,

$$\sup_{x,y\in\mathbb{T}^d} P\big[\,\tau_{x,y}^{\widetilde{Z}} \;\ge\; 2T_0\,\big] \;\le\; \frac{1}{2}\,\cdot$$

To complete the proof of (7.5) [and the one of the proposition], it remains to apply the Markov property. □

7.2. *Asymptotic behavior of linear parabolic equations.* Let $G:\mathbb{R}_+\times\mathbb{T}^d\to\mathbb{R}^d$ be a smooth vector field satisfying the hypotheses stated in the previous subsection.

PROPOSITION 7.4. *Fix two probability densities w_1, w_2 on $\mathbb{T}^d$, $w_j\colon\mathbb{T}^d\to\mathbb{R}_+$, $\int_{\mathbb{T}^d} w_j(x)\,dx = 1$, $j=1,\,2$. Denote by $\boldsymbol{w}_j\colon\mathbb{R}_+\times\mathbb{T}^d\to\mathbb{R}_+$ the unique weak solution of the linear parabolic equation*

$$\begin{cases}\partial_t \boldsymbol{w}_j \;=\; \Delta \boldsymbol{w}_j \;-\; \nabla\cdot[\,\boldsymbol{w}_j\, G\,] \\ \boldsymbol{w}_j(0,\cdot) = w_j(\cdot)\end{cases} \tag{7.6}$$

Then, there are $A<\infty$ and $\lambda>0$, which depends only on $\sup_{(t,x)\in\mathbb{R}_+\times\mathbb{T}^d}\|\,G(t,x)\,\|$, such that

$$\int \big|\,\boldsymbol{w}_2(t,x) \;-\; \boldsymbol{w}_1(t,x)\,\big|\,dx \;\le\; A\,e^{-\lambda t}$$

for all $t\ge 0$.

PROOF. Recall the definition of the diffusions $(Z_t^x : t\ge 0)$, $x\in\mathbb{T}^d$, introduced in the previous subsection. Denote its transition probability by $p_t(x,dy) = p_t(x,y)dy$ so that

$$\mathbb{E}_x[\,f(Z_t)\,] \;=\; \int p_t(x,y)\,f(y)\,dy$$

for all functions $f\in C(\mathbb{T}^d)$.

Since $\mathcal{L}_t$ is the generator of the diffusion Z_t, for every function f in $C^2(\mathbb{T}^d)$ and $t>0$,

$$\mathbb{E}_x[\,f(Z_t)\,] \;=\; f(x) \;+\; \int_0^t \mathbb{E}_x[\,(\mathcal{L}_s f)(Z_s)\,]\,ds\,.$$

integrating both sides of this identity with respect to $w_j(x)dx$ and integrating by parts yields that $\boldsymbol{v}_j(t,x) := \int_{\mathbb{T}^d} w_j(y) p_t(y,x)dy$ solves (7.6) with initial condition $\boldsymbol{v}_j(0,x) = w_j(x)$. By the uniqueness of weak solutions,

$$\boldsymbol{w}_j(t,x) \;=\; \int_{\mathbb{T}^d} w_j(y)\,p_t(y,x)\,dy\,.$$

Since $\int \boldsymbol{w}_j(t,x)\,f(x)\,dx = \int w_j(x)\,\mathbb{E}_x[\,f(Z_t)\,]\,dx$ for every continuous function $f:\mathbb{T}^d\to\mathbb{R}$, as $w_j(x)$ is a probability density,

$$\begin{aligned}&\int_{\mathbb{T}^d} \boldsymbol{w}_2(t,x)\,f(x)\,dx \;-\; \int_{\mathbb{T}^d} \boldsymbol{w}_1(t,x)\,f(x)\,dx \\ &\quad =\; \int_{\mathbb{T}^d} dx\,w_1(x) \int_{\mathbb{T}^d} dy\,w_2(y)\,\Big\{\mathbb{E}_x\big[\,f(Z_t)\,\big] \;-\; \mathbb{E}_y\big[\,f(Z_t)\,\big]\Big\}\,.\end{aligned}$$

Therefore, by (7.3), for every $t \ge 0$,

$$\sup_f \Big| \int \boldsymbol{w}_2(t,x)\, f(x)\, dx \;-\; \int \boldsymbol{w}_1(t,x)\, f(x)\, dx \Big| \;\le\; 2 \sup_{x,y\in\mathbb{T}^d} P\big[\, \tau^Z_{x,y} \ge t \,\big]\,,$$

where the supremum is carried over all continuous function f such that $\|f\|_\infty \le 1$. Hence,

$$\int \big| \boldsymbol{w}_2(t,x) \,-\, \boldsymbol{w}_1(t,x) \big|\, dx \;\le\; 2 \sup_{x,y\in\mathbb{T}^d} P\big[\, \tau^Z_{x,y} \ge t \,\big]\,,$$

and the assertion of the proposition follows from Proposition 7.2. □

We turn to Theorem 7.1 whose proof relies on the following estimate.

PROPOSITION 7.5. *There exist constants $A < \infty$ and $\lambda > 0$, which depends only on $\sup_{(t,x)\in[0,T]\times\mathbb{T}^d} \| F(t,x) \|$, with the following property. Fix $0 < m < 1$ and $u_j \colon \mathbb{T}^d \to [0,1]$, $j = 1,\, 2$, such that $\int_{\mathbb{T}^d} u_j(x)\, dx = m$. Denote by $\boldsymbol{u}_j \colon \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}_+$ the unique weak solution of* (7.1) *with initial condition u_j. Then,*

$$\int \big| \boldsymbol{u}_2(t,x) \,-\, \boldsymbol{u}_1(t,x) \big|\, dx \;\le\; A\, e^{-\lambda t}$$

for all $t \ge 0$.

PROOF. Let $\boldsymbol{v}(t,x) = \boldsymbol{u}_2(t,x) - \boldsymbol{u}_1(t,x)$, so that $\int_{\mathbb{T}^d} \boldsymbol{v}(t,x)\, dx = 0$ for all $t \ge 0$. Since $\sigma(b) - \sigma(a) = (b-a)(1-a-b)$, $\boldsymbol{v}(t,x)$ solves the linear equation

$$\partial_t \boldsymbol{w} \;=\; \Delta \boldsymbol{w} \,+\, \nabla\cdot [\, \boldsymbol{w}\, G\,]\,, \tag{7.7}$$

where G is the vector field $G = (1 - \boldsymbol{u}_1 - \boldsymbol{u}_2)F$.

Let $v_0 \colon \mathbb{T}^d \to \mathbb{R}$ be given by $v_0(x) = u_2(x) - u_1(x)$. Denote by v^+, v^-, the positive, negative part of v_0, respectively. Note that $\int_{\mathbb{T}^d} v^+(x)dx = \int_{\mathbb{T}^d} v^-(x)dx =: m' \in [0,m]$. If $m' = 0$, $0 = v_0(x) = u_2(x) - u_1(x)$, and there is nothing to prove. Assume that $m' > 0$ and let $w_2(x) = v^+(x)/m'$, $w_1(x) = v^-(x)/m'$ so that w_j is the density of a probability measure on $\mathbb{T}^d$.

Denote by $\boldsymbol{w}_j(t,x)$ the solution of (7.7) with initial condition $w_j(x)$. By linearity $m'[\boldsymbol{w}_2(t,x) - \boldsymbol{w}_1(t,x)]$ solves (7.7) with initial condition $m'[w_2(x) - w_1(x)] = v_0(x)$. Since $\boldsymbol{v}(t,x)$ solves the same Cauchy problem, $\boldsymbol{v}(t,x) = m'[\boldsymbol{w}_2(t,x) - \boldsymbol{w}_1(t,x)]$. Thus, as $m' \le 1$,

$$\int \big| \boldsymbol{u}_2(t,x) \,-\, \boldsymbol{u}_1(t,x) \big|\, dx \;=\; \int \big| \boldsymbol{v}(t,x) \big|\, dx \;\le\; \int \big| \boldsymbol{w}_2(t,x) \,-\, \boldsymbol{w}_1(t,x) \big|\, dx\,.$$

To complete the proof, it remains to recall the statement of Proposition 7.4, and to observe that $\sup_{(t,x)\in\mathbb{R}_+\times\mathbb{T}^d} \| G(t,x) \|| \le \| F \|_\infty$ because $\boldsymbol{u}_j(t,x)$ takes value in the interval $[0,1]$. □

From this result we can deduce the first assertion of Theorem 7.1.

COROLLARY 7.6. *For each $m \in (0,1)$, the equation* (7.1) *admits a unique T-periodic solution $\boldsymbol{u} : \mathbb{R}\times\mathbb{T}^d \to [0,1]$.*

PROOF. Fix $m \in (0,1)$ and denote by $L^1_m(\mathbb{T}^d)$ the closed subspace of $L^1(\mathbb{T}^d)$ defined by $L^1_m(\mathbb{T}^d) = \{ u \in L^1(\mathbb{T}^d) : \int_{\mathbb{T}^d} u(x)\, dx = m\,,\, 0 \le u(x) \le 1\,\}$.

Define the operator $\mathfrak{P} : L^1_m(\mathbb{T}^d) \to L^1_m(\mathbb{T}^d)$ given by $\mathfrak{P}(u) = \boldsymbol{u}(T,\cdot)$, where $\boldsymbol{u}(t,x)$ is the weak solution of (7.1) with initial condition $u(\cdot)$. Let $u_0 : \mathbb{T}^d \to [0,1]$ be given by $u_0(x) = m$

for all x, and set $u_{j+1} = \mathfrak{P} u_j$, $j \ge 0$. We claim that the sequence $(u_j : j \ge 1)$ is Cauchy in $L^1(\mathbb{T}^d)$. Fix n, $j \ge 1$. Since $\mathfrak{P}_{n+j} u = \mathfrak{P}_n \mathfrak{P}_j u$, by Proposition 7.5,

$$\big\| \mathfrak{P}_{n+j} u - \mathfrak{P}_n u \big\|_1 = \big\| \mathfrak{P}_n [\mathfrak{P}_j u - u] \big\|_1 \le A e^{-\lambda n T} .$$

Denote by w the limit in L^1 of the sequence u_j, and observe that $\mathfrak{P} w = w$. This proves that the solution of equation (7.1) with initial condition w is T-periodic. By Proposition 7.5, such T-periodic solution is unique. □

PROOF OF THEOREM 7.1. Fix $m \in [0,1]$. As the result is trivial for $m = 0$ or 1, we may assume that $0 < m < 1$. In this range, the assertions of the theorem corresponds to the ones of Proposition 7.5 and Corollary 7.6. □

APPENDIX A: DYNAMICAL BOUNDS

We present in this section some estimates used in the article. Let

$$c_{x,y}(\eta) = \eta_x [1 - \eta_y] e^{(1/2) E_N(x,y)} , \quad (x,y) \in \mathbb{B}_N$$

and recall the notation introduced in (2.18).

LEMMA A.1. *Given a set of bounded functions $\phi^{x,y}_s : D([0,\infty), \Sigma_N) \to \mathbb{R}$, $(x,y) \in \mathbb{B}_N$, progressively measurable, the process*

$$\text{(A.1)} \qquad \mathbb{M}^\phi_t = \exp \sum_{(x,y) \in \mathbb{B}_N} \Big\{ \int_0^t \phi^{x,y}_s \, \mathcal{N}^{x,y}_{(s,s+ds]}(\boldsymbol{\eta}) - N^2 \int_0^t c_{x,y}(\boldsymbol{\eta}(s)) \{ e^{\phi^{x,y}_s} - 1 \} \, ds \Big\}$$

is a mean one, positive martingale with respect to $\mathbb{P}^N_\eta$ for any configuration $\eta \in \Sigma_N$.

The proof of this lemma is similar to the one of Proposition A.2.6 in [15] and left to the reader. This martingale corresponds to the Radon-Nikodym derivative (restricted to the interval $[0,t]$) of the law of a jump process with rates $c_{x,y}(\eta) e^{\phi^{x,y}}$ with respect $\mathbb{P}^N_\eta$.

Recall that the symmetric simple exclusion process is the Markov chain on Σ_N whose generator is L_N, introduced in (2.1), with $E \equiv 0$. Denote by ν_α, $0 \le \alpha \le 1$, the Bernoulli product measure on Σ_N with density α, and by $\mathbb{P}^0_{\nu_\alpha}$ the probability measure on $D(\mathbb{R}_+, \Sigma_N)$ induced by the symmetric simple exclusion process starting from ν_α. Expectation with respect to $\mathbb{P}^0_{\nu_\alpha}$ is represented by $\mathbb{E}^0_{\nu_\alpha}$.

LEMMA A.2. *For all $1 < p < \infty$ and $E \in C^1(\mathbb{T}^d; \mathbb{R}^d)$, there exists a constant C_p such that*

$$\log \mathbb{E}^0_{\nu_{1/2}} \Big[\Big(\frac{d\mathbb{P}^N_{\mu_{N,K}}}{d\mathbb{P}^0_{\nu_{1/2}}} \Big|_{[0,T]} \Big)^p \Big] \le C_p (1 + T) N^d$$

for all $T > 0$, $N \ge 1$, $0 \le K \le N^d$.

PROOF. The proof reduces to a standard computation of exponential martingales. To emphasize the dependence of the measure $\mathbb{P}^N_{\mu_{N,K}}$ on the external field E, in this proof, we represent the measure $\mathbb{P}^N_{\mu_{N,K}}$ by $\mathbb{P}^E_{\mu_{N,K}}$. Clearly,

$$\mathbb{E}^0_{\nu_{1/2}} \Big[\Big(\frac{d\mathbb{P}^E_{\mu_{N,K}}}{d\mathbb{P}^0_{\nu_{1/2}}} \Big|_{[0,T]} \Big)^p \Big] = \mathbb{E}^0_{\mu_{N,K}} \Big[\Big(\frac{d\mu_{N,K}}{d\nu_{1/2}} \Big)^{p-1} \Big(\frac{d\mathbb{P}^E_{\mu_{N,K}}}{d\mathbb{P}^0_{\mu_{N,K}}} \Big|_{[0,T]} \Big)^p \Big] .$$

As $\nu_{1/2}(\eta) = (1/2)^{N^d}$, this expression is bounded by

$$2^{(p-1)N^d}\, \mathbb{E}^0_{\mu_{N,K}} \Big[\Big(\frac{d\mathbb{P}^E_{\mu_{N,K}}}{d\mathbb{P}^0_{\mu_{N,K}}} \Big|_{[0,T]} \Big)^p \Big] .$$

On the other hand,

$$\mathbb{E}^0_{\mu_{N,K}} \Big[\Big(\frac{d\mathbb{P}^E_{\mu_{N,K}}}{d\mathbb{P}^0_{\mu_{N,K}}} \Big|_{[0,T]} \Big)^p \Big] \;=\; \mathbb{E}^{pE}_{\mu_{N,K}} \Big[\Big(\frac{d\mathbb{P}^E_{\mu_{N,K}}}{d\mathbb{P}^0_{\mu_{N,K}}} \Big|_{[0,T]} \Big)^p \, \frac{d\mathbb{P}^0_{\mu_{N,K}}}{d\mathbb{P}^{pE}_{\mu_{N,K}}} \Big|_{[0,T]} \Big] .$$

Note that in the last expectation the external field is pE. A direct computation based on the explicit formula for the Radon-Nikodym derivatives provided by Lemma A.1 yields that

$$\Big\| \Big(\frac{d\mathbb{P}^E_{\mu_{N,K}}}{d\mathbb{P}^0_{\mu_{N,K}}} \Big|_{[0,T]} \Big)^p \, \frac{d\mathbb{P}^0_{\mu_{N,K}}}{d\mathbb{P}^{pE}_{\mu_{N,K}}} \Big|_{[0,T]} \Big\|_\infty \;\le\; e^{C_p\, T\, N^d}$$

for some finite constant C_p, see [5, Lemma 4.5]. □

Until the end of the appendix, fix $T > 0$, $m \in (0,1)$ and a sequence $(K_N : N \ge 1)$ such that $K_N/N^d \to m$. Consider a progressively measurable, continuous function $w\colon \mathbb{R} \times \mathbb{T}^d \times \mathcal{M}_+(\mathbb{T}^d) \times D(\mathbb{R}; \mathcal{H}^d_{-p}) \to \mathbb{R}^d$ with support on $[0,T] \times \mathbb{T}^d \times \mathcal{M}_+(\mathbb{T}^d) \times D(\mathbb{R}; \mathcal{H}^d_{-p})$. Recall from (4.5) the definition of the progressively measurable function $G_w\colon \mathbb{R} \times \mathbb{T}^d \times \mathcal{S}_{m,\mathrm{ac}} \to \mathbb{R}^d$. For $\epsilon > 0$ and a cylinder function Ψ, let

$$F^{w,\Psi}_{N,\epsilon}(t,\boldsymbol{\eta}) \;=\; \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} G_w\Big(t\,,\, x\,,\, \boldsymbol{\pi}_N\,,\, \boldsymbol{J}_N\Big)\, \big\{\, (\tau_x \Psi)(\eta(t)) - \widehat{\Psi}((\boldsymbol{\pi}^\epsilon_N)(t,x)) \,\big\} ,$$

where $(\boldsymbol{\pi}^\epsilon_N)(t,x) = (2\epsilon)^{-d} \boldsymbol{\pi}_N(t, [x-\epsilon, x+\epsilon]^d)$, and $\widehat{\Psi} : [0,1] \to \mathbb{R}$ is the function given by

$$\widehat{\Psi}(\alpha) \;=\; E_{\nu_\alpha}\big[\, \Psi \,\big] .$$

LEMMA A.3. *For all $\delta > 0$*

$$\lim_{\epsilon \to 0} \limsup_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}^N_{\mu_{N,K_N}} \Big[\Big| \int_0^T F^{w,\Psi}_{N,\epsilon}(t,\boldsymbol{\eta})\, dt \Big| > \delta \Big] \;=\; -\infty .$$

PROOF. First, we claim that for all $\delta > 0$,

$$\lim_{\epsilon \to 0} \limsup_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}^0_{\nu_{1/2}} \Big[\Big| \int_0^T F^{w,\Psi}_{N,\epsilon}(t,\boldsymbol{\eta})\, dt \Big| > \delta \Big] \;=\; -\infty .$$

We refer to [15, Theorem 10.3.1] for the proof in the case in which w does not depend on π and $\boldsymbol{J}$. The arguments to include this dependence are tedious, but straightforward and left to the reader. The extension to the measure $\mathbb{P}^N_{\mu_{N,K_N}}$ follows from Schwarz inequality and Lemma A.2. □

Next result is a consequence of the entropy inequality, [15, Proposition A1.8.2], and the previous lemma.

COROLLARY A.4. *Let $(\mathbb{Q}_N : N \ge 1)$ be a sequence of probability measures in $\mathcal{P}^{N,K_N}_{\mathrm{stat}}$. Assume that there exists a finite constant C_0 such that for all $S > 0$,*

$$\limsup_{N\to\infty} \frac{1}{N^d}\, \mathbb{H}^{(S)}\big(\mathbb{Q}_N \,|\, \mathbb{P}^N_{\mu_{N,K_N}} \big) \;\le\; C_0\, S .$$

Then, for all $\delta > 0$

$$\lim_{\epsilon \to 0} \limsup_{N\to\infty} \mathbb{Q}_N \Big[\Big| \int_0^T F^{w,\Psi}_{N,\epsilon}(t,\boldsymbol{\eta})\, dt \Big| > \delta \Big] \;=\; 0 .$$

Fix a vector field F in $C^1(\mathbb{R}\times\mathbb{T}^d;\mathbb{R}^d)$ with compact support in $(0,T)\times\mathbb{T}^d$, $a>0$, and recall the definition of $\mathcal{E}_{a,\epsilon}(F,\boldsymbol{\pi})$, $\mathcal{V}_{a,\epsilon}(F,\boldsymbol{\pi},\boldsymbol{J})$, $\epsilon>0$ in (4.4).

LEMMA A.5. *There exist finite, positive constants a and C_0 such that for all vector fields F in $C^1(\mathbb{R}\times\mathbb{T}^d;\mathbb{R}^d)$ with compact support in $(0,T)\times\mathbb{T}^d$,*

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\frac{1}{N^d}\log\mathbb{E}^N_{\mu_{N,K_N}}\Big[\exp\big\{N^d\,\mathcal{E}_{a,\epsilon}(F,\boldsymbol{\pi}_N)\big\}\Big]\;\le\;C_0\,(1+T)\,,$$

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\frac{1}{N^d}\log\mathbb{E}^N_{\mu_{N,K_N}}\Big[\exp\big\{N^d\,\mathcal{V}_{a,\epsilon}(F,\boldsymbol{\pi}_N,\boldsymbol{J}_N)\big\}\Big]\;\le\;C_0\,(1+T)\,.$$

PROOF. We claim that there exists a finite constant a such that for any $T>0$ and any F in $C^1(\mathbb{R}\times\mathbb{T}^d;\mathbb{R}^d)$ with compact support in $(0,T)\times\mathbb{T}^d$,

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\frac{1}{N^d}\log\mathbb{E}^0_{\nu_{1/2}}\Big[\exp N^d\big\{\,\mathcal{E}_{a,\epsilon}(F,\boldsymbol{\pi}_N)\,\big\}\Big]\;\le\;0\,.$$

This statement is proven in [6, Section 3.2] (see the proof of the bound presented in the last displayed equation at page 2367). To deduce the first assertion of the lemma from this result, it suffices to apply Schwarz inequality, to recall the statement of Lemma A.2, and to observe that $2\mathcal{E}_{a,\epsilon}(F,\boldsymbol{\pi}_N)=\mathcal{E}_{a/2,\epsilon}(2F,\boldsymbol{\pi}_N)$.

We turn to the second assertion of the lemma. By Lemma A.1,

$$\mathbb{E}^N_{\mu_{N,K_N}}\Big[\exp\Big\{\,2\,N^d\boldsymbol{J}_N(F)\;-\;\boldsymbol{W}_N(T)\,\Big\}\Big]\;=\;1\,,$$

provided

$$\boldsymbol{W}_N(T)\;=\;N^2\sum_{(x,y)\in\mathbb{B}_N}\int_0^T\boldsymbol{\eta}_x(s)\,[1-\boldsymbol{\eta}_y(s)]\,\{e^{2\,F_N(s,x,y)}-1\}\,ds\,.$$

Recalling (4.4), by adding and subtracting $(1/2N^d)\boldsymbol{W}_N(T)$ and applying Schwarz inequality we get

$$\begin{aligned}&\mathbb{E}^N_{\mu_{N,K_N}}\Big[\,\exp N^d\Big\{\,\boldsymbol{J}_N(F)\;-\;a\int_{\mathbb{R}}ds\int_{\mathbb{T}^d}dx\,\sigma(\boldsymbol{\pi}^\epsilon_N)\,|F|^2\Big\}\Big]\\&\quad\le\;\mathbb{E}^N_{\mu_{N,K_N}}\Big[\,\exp\Big\{\,\boldsymbol{W}_N(T)\;-\;2\,N^d a\int_{\mathbb{R}}ds\int_{\mathbb{T}^d}dx\,\sigma(\boldsymbol{\pi}^\epsilon_N)\,|F|^2\Big\}\Big]^{1/2}\,.\end{aligned}$$

Expanding the exponential $\exp\{2\,F_N(s,x,y)\}$ which appears in the definition of $\boldsymbol{W}_N(T)$, summing by parts, using Lemma A.3, and the first part of the proof yields the desired bound. □

Consider a continuous function $w\colon\,[0,T]\times\mathbb{T}^d\times\mathcal{M}_+(\mathbb{T}^d)\times D(\mathbb{R};\mathcal{H}^d_{-p})\to\mathbb{R}^d$ that is continuously differentiable in x and such that for each $(x,\pi)\in\mathbb{T}^d\times\mathcal{M}_+(\mathbb{T}^d)$ and $t\in[0,T]$ the map $[0,t]\times D(\mathbb{R};\mathcal{H}^d_{-p})\ni(s,\boldsymbol{J})\to w(s,x,\pi,\boldsymbol{J})$ is measurable with respect to the Borel σ-algebra on $[0,t]\times D([0,t];\mathcal{H}^d_{-p})$. Let $\phi^{x,y}\colon\,[0,T]\times D([0,t],\Sigma_N)\to\mathbb{R}$, $(x,y)\in\mathbb{B}_N$, be given by

$$\phi^{x,y}(t)\;=\;\int_x^y w(t,\cdot,\boldsymbol{\pi}_N(t),\boldsymbol{J}_N)\cdot d\ell\,,$$

and let $\mathbb{M}^\phi_T$ be the martingale introduced in (A.1), LEMMA A.6. *Let* $(\mathbb{Q}_N : N \geq 1)$ *be a sequence of probability measures in* $\mathcal{P}_{\rm stat}^{N,K_N}$ *such that* $\mathbb{Q}_N \circ (\boldsymbol{\pi}_N, \boldsymbol{J}_N)^{-1} \to P$ *for some* $P \in \mathcal{P}_{\rm stat}$ *satisfying* (2.27). *Assume that there exists a finite constant* C_0 *such that for all* $S > 0$,

$$\limsup_{N\to\infty} \frac{1}{N^d}\, \mathbb{H}^{(S)}\big(\, \mathbb{Q}_N \,|\, \mathbb{P}^N_{\mu_{N,K_N}} \,\big) \;\le\; C_0\, S\,.$$

Then, for each w *as above,*

$$\frac{1}{T} \lim_{N\to\infty} \frac{1}{N^d}\, E_{\mathbb{Q}_N}\big[\, \log \mathbb{M}^{\phi}_T \,\big] \;=\; E_P\big[\, V_{T,w}\,\big]\,,$$

where $V_{T,w}$ *has been introduced in* (4.6)

PROOF. On the one hand, by definition of $\phi^{x,y}$ and of the current $\boldsymbol{J}_N$, and since $\mathbb{Q}_N \circ (\boldsymbol{\pi}_N, \boldsymbol{J}_N)^{-1} \to P$ for some measure $P \in \mathcal{P}_{\rm stat}$ satisfying (2.27),

$$\lim_{N\to\infty} \frac{1}{N^d}\, E_{\mathbb{Q}_N}\Big[\, \sum_{(x,y)\in\mathbb{B}_N} \int_0^T \phi^{x,y}_s\, \mathcal{N}^{x,y}_{(s,s+ds]}\,\Big] \;=\; E_P\Big[\, \int_0^T ds \int_{\mathbb{T}^d} dx\, G_w \cdot \boldsymbol{j}\,\Big]\,.$$

On the other hand, a straightforward computation yields that

$$N^2 \sum_{(x,y)\in\mathbb{B}_N} c_{x,y}(\eta)\{e^{\phi^{x,y}} - 1\} \;=\; N^d\, \langle \pi_N \,,\, \nabla\cdot G_w\rangle$$

$$+\; \frac{1}{2} \sum_{j=1}^d \sum_{x\in\mathbb{T}^d_N} [\eta_{x+\mathfrak{e}_j} - \eta_x]^2\, w_j(x)\, [w_j(x) + E_j(x)] \;+\; o(N^d)\,.$$

Therefore, by Corollary A.4 and since $\mathbb{Q}_N \circ (\boldsymbol{\pi}_N, \boldsymbol{J}_N)^{-1} \to P$ for some measure $P \in \mathcal{P}_{\rm stat}$ satisfying (2.27),

$$\lim_{N\to\infty} \frac{1}{N^d}\, E_{\mathbb{Q}_N}\Big[\, N^2 \sum_{(x,y)\in\mathbb{B}_N} \int_0^T ds\, c_{x,y}(\boldsymbol{\eta}(s))\{e^{\phi^{x,y}_s} - 1\}\,\Big]$$

$$=\; E_P\Big[\int_0^T dt \int_{\mathbb{T}^d} dx \Big\{\, G_w \cdot \big[\, -\, \nabla\boldsymbol{\rho} \,+\, \sigma(\boldsymbol{\rho})\, E\,\big] \,+\, \sigma(\boldsymbol{\rho})\, |G_w|^2 \,\Big\}\Big]\,.$$

The assertion of the lemma follows from the two previous estimates. □

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